Stability of Stochastic Differential Equations in Infinite Dimensions

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by

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Abstract

In engineering, physics and economics, many dynamical systems involving with stochastic components and random noise are often modeled by stochastic models. The stochastic effects of these models are often used to describe the uncertainty about the operating systems. Motivated by the development of analysis and theory of stochastic processes, as well as the studies of natural sciences, the theory of stochastic differential equations in infinite dimensional spaces evolves gradually into a branch of modern analysis. Many qualitative properties of such systems have been studied in the past few decades, among which, investigation of stability of such systems is often regarded as the first characteristic of the dynamical systems or models.

In general, this thesis is mainly concerned with the studies of the stability property of stochastic differential equations in Hilbert spaces. Chapter 1 is an introduction to a brief history of stochastic differential equations in infinite dimensions, together with an overview of the studies. Chapter 2 is a presentation of preliminaries to some basic stochastic analysis. In Chapter 3, we study the stability in distribution of mild solutions to stochastic delay differential equations with Poisson jumps. Firstly, we use approximation of strong solutions to pass on the stability of strong solutions to the mild ones. Then, by constructing a suitable metric between the transition probability functions of mild solutions, we obtain the desired stability result under some suitable conditions. In Chapter 4, we investigate the stochastic partial delay differential equations with Markovian switching and Poisson jumps. By estimating the coefficients of energy equality, both the exponential stability and almost sure exponential stability of energy solutions to the equations are obtained. In Chapter 5, we study the relationship among strong, weak and mild solutions to the stochastic functional differential equations of neutral type. Finally, in Chapter 6, we study the asymptotic stability of two types of equations, impulsive stochastic delay differential equations with Poisson jumps and stochastic evolution equations with Poisson jumps. By employing the fixed point theorem, we derive the desired stability results under some criteria.

This thesis is dedicated to my dearest mother, Li Yuanhui, her support, encouragement and constant love have sustained me throughout my life, and to the memory of my beloved father, Zhou Jianxi, he will be forever missed.

Contents

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General Notations

$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{R}_+$	nonnegative real numbers
$\mathrm{Re}\lambda$	real part of $\lambda \in \mathbb{C}$
$\mathrm{Im}\lambda$	imaginary part of $\lambda \in \mathbb{C}$
A	linear operator
$\mathcal{D}(A)$	domain of A
$\mathcal{R}(A)$	range of A
$\mathcal{B}(X)$	Borel σ-field of X
$\mathcal{L}(X)$	the set of all bounded linear operators on X
$\mathcal{L}(X, Y)$	the set of all bounded linear operators from X into Y
$\mathcal{L}_1(X, Y)$	the space of all nuclear operators from X into Y
$\mathcal{L}_2(X, Y)$	the space of all Hilbert-Schmidt operators from X into Y
$C(X, Y)$	the space of all continuous functions from X to Y
i.f.f.	if and only if
a.s.	almost surely
a.e.	almost everywhere

Other notations will be explained when they first appear.

Chapter 1

Introduction

1.1 Brief history

Material on the brief history is based on Da Prato and Zabczyk (1992), Huang and Yan (2000), Liu (2006) and Luo et al. (1999).

Infinite dimensional analysis as a branch of mathematical sciences was formed in the late 19th and early 20th centuries. Motivated by problems in mathematical physics, the early research in this field was carried out by V. Volterra, R. Gâteaux, P. Lévy and M. Fréchet, among others. Nevertheless, the most fruitful direction in this field is the infinite dimensional integration theory initiated by N. Wiener and A. N. Kolomogorov, which is closely related to the developments of the theory of stochastic processes. In 1923 N. Wiener constructed a probability measure on the space of all continuous functions (i.e. the Wiener measure), which provided an ideal mathematical model for Brownian motion.

The next generation of stochastic integration was laid out by A. N. Kolmogorov. In the most remarkable of his papers on probability theory, Kolmogorov (1931), he introduced a class of stochastic process which have since been called Markov processes. To study their probabilistic parameter (i.e. the transition probabilities), he proposed using differential equations, which shows that for processes with a finite-dimensional phase space, the transition probabilities satisfy second-order parabolic partial differential equations. Thus, it revealed the deep connection between theories of differential equations and stochastic processes. Following the work of Kolmogorov, K. Itô made an outstanding contribution to the development of stochastic analysis. The stochastic analysis created by K. Itô (also independently by I. I. Gihman) in the 1940s is essentially an infinitesimal analysis for trajectories of stochastic processes. By virtue of Itô stochastic differential equations one can construct diffusion processes via direct probabilistic methods and treat them as functionals of Brownian paths (i.e. the Wiener functionals). This affords a possibility of using probabilistic methods to investigate deterministic differential equations and many other pure analytical problems.

Itô stochastic equations was introduced in the 1940s by K. Itô (Itô (1942)) and in a different form by I. I. Gihman (Gihman (1947)). First results on infinite dimensional Itô equations started to appear in the mid 1960s, which was motivated by internal development of analysis and theory of stochastic processes as well as the needs of describing random phenomena studied in natural sciences. For example, physics (wave propagation in random media and turbulence), chemistry, biological sciences (population biology) and in control theory (early control theoretic application). In 1966 and 1967, Daleckii (Daleckii (1966)) and Gross (Gross (1967)) introduced Hilbert space valued Wiener processes and, more generally, Hilbert space valued diffusion processes as a tool to investigate Dirichlet problems and some classes of parabolic equations for functions which depend on infinitely many variable. In 1978, P. Malliavin (Malliavin (1978)) created the stochastic calculus of variation (known as Malliavin calculus) by successfully extending the gradient, divergence and Ornstein-Uhlenbeck operators to infinite dimensional cases. An infinite dimensional version of an Ornstein-Uhlenbeck process was also introduced for a stochastic study of regularity of fundamental solutions of deterministic parabolic equations (c.f. Stroock (1981)). In the late 1960s and early

1970s, in the study of conditional distributions of finite dimensional processes, Zakai (Zakai (1969)) studied stochastic parabolic type equations in the form of a linear stochastic equation; while Fujisaki, Kallianpur and Kunita (Fujisaki et al. (1972)) studied them in the form of the so-called non-linear filtering equation. Another source of inspiration was provided by stochastic flow defined by ordinary stochastic equations. Such flows are in fact processes with values in an infinite dimensional space of continuous or ever more regular mappings acting in a Euclidian space. They are solutions of corresponding backward and forward stochastic Kolomogorov-like equations; c.f. Carverhill and Elworthy (1983), Krylov and Rozovskii (1981), Kunita (1990).

Stochastic evolution equations in infinite dimensions are natural generalizations of stochastic ordinary differential equations. Their theory has motivations coming both from mathematics and the natural sciences: physics, chemistry and biology. Several motivating examples of stochastic evolution equations have been presented in Da Prato and Zabczyk (1992), such as examples of purely mathematical motivations (lifts of diffusion processes, Markovian lifting of stochastic delay equations and Zakai's equation); examples from physics (random motion of a string, stochastic equation of the free field and equation of stochastic quantization); examples from chemistry (reaction diffusion equation); and examples from biology (the cable equation arising in neurophysiology and equation of population genetics). In the 1970s and 1980s, under various sets of conditions, basic theoretical results on existence and uniqueness of solutions have been obtained (c.f. Bensoussan and Temam (1972, 1973) and Dawson (1972, 1975)). In particular, Pardoux (Pardoux (1975)) obtained fundamental results on stochastic non-linear partial differential equations of monotone type and Viot (Viot (1974)) obtained basic results on weak solutions. However, it is worth mentioning the existence and uniqueness of solutions are are still of great interest today.

The study of stability originates in mechanics, which can be traced back in the early 17th century. A principle called Torricelli's principle was already introduced which says that if a system of interconnected heavy bodies is in equilibrium, the center of gravity is at the lowest point. The principle was applied to the study of general motion including, but not limited to, mechanical motion. In fact, any time process in nature can be thought of as motion and to study stability is actually to study the effect of perturbations to motion. A system or process is said to be stable if such perturbations does not essentially change it.

The fundamental theory of stability was established by A. M. Lyapunov who published what is now widely known as the Lyapunov's direct method for stability analysis in his celebrated memoir 'The general problem of stability of motion' in 1892. Since then, Lyapunov's direct method has greatly stimulated the research on stability of motion, and further developments have been made possible through the efforts of scientists all over the world during the past 120 years. Nowadays, Lyapunov's stability theory is an indispensable tool for the study of all systems whether they are finite or infinite, linear or nonlinear, time-invariant or time varying, continuous or discrete. It is widely used in system analysis and control for various systems from electrical systems and mechanical systems to economic system and solar systems etc.

To illustrate the stability ideas, let us introduce the following example from Liu (2006). We consider solution $Y_t(y_0) = f(t, Y_t)dt$, $t \geq 0$, to a deterministic differential equation on the Hilbert space H,

$$\begin{cases} dY_t = f(t, Y_t)dt, \quad t \geq 0, \\ \\ Y_0 = y_0 \in H, \end{cases} \tag{1.1.1}$$

where $f(\cdot, \cdot)$ is some given function. Let $\hat{Y}_t$, $t \geq 0$, be a particular solution of Equation (1.1.1); the corresponding system is thought of as describing a process without perturbations. The system associated with other solution $Y_t(y_0)$ is regarded as a perturbed one. When one talks about stability, or stability in the sense of Lyapunov, of the solution $\hat{Y}_t(y_0)$, it means that the norm $\|Y_t - \hat{Y}_t\|_H$ could be made small enough if some reasonable conditions are imposed, for instance, that the initial perturbation scale $\|Y_0 - \hat{Y}_0\|_H$ is very small or time t is large enough. In practice, it is enough to investigate the stability problem for the null solution of some relevant system. Indeed, let $X_t = Y_t - \hat{Y}_t$, then the equation (1.1.1) could be change into

$$\begin{aligned} dX_t = dY_t - d\hat{Y}_t &= (f(t, Y_t) - f(t, \hat{Y}_t))dt \\ &= (f(t, X_t + \hat{Y}_t) - f(t, \hat{Y}_t))dt \\ &:= F(t, X_t)dt, \end{aligned} \tag{1.1.2}$$

where $F(t, 0) = 0$, $t \geq 0$. Therefore, several definitions of stability for the null solution of Equation (1.1.2) can be established as follows.

Definition 1.1 *The null solution of Equation (1.1.2) is said to be stable if for arbitrary given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $\|x_0\|_H < \delta$, then*

$$\|X_t(x_0)\|_H < \epsilon$$

for all $t \geq 0$.

Definition 1.2 *The null solution of Equation (1.1.2) is said to be asymptotically stable if it is stable and there exists $\delta > 0$ such that $\|x_0\|_H < \delta$ guarantees*

$$\lim_{t \to \infty} \|X_t(x_0)\|_H = 0.$$

Definition 1.3 *The null solution of Equation (1.1.2) is said to be exponentially stable if it is asymptotically stable and there exist numbers $\alpha > 0$, and $\beta > 0$ such that*

$$\|X_t(x_0)\|_H \leq \beta \|x_0\|_H e^{-\alpha t}$$

for all $t \geq 0$.

It is worth mentioning that for the stability of stochastic systems, there are at least three times as many definitions as there are for deterministic ones. This is because in a stochastic setting, there are three basic type of convergence; convergence in probability, convergence in mean and convergence in almost sure (sample path, probability one) sense. The above deterministic stability definitions can be translated into a stochastic setting by properly interpreting the notion of convergence (c.f. Kozin (1965) and Arnold (1974)). The relevant stability definitions of stochastic systems will be introduced and studied in the following chapters respectively.

1.2 Overview of the studies

This thesis mainly concentrates on stability of stochastic differential equations in infinite dimensional spaces, mainly Hilbert spaces. We attempt to investigate stability properties such as stability in distribution, exponential stability, almost sure exponential stability and asymptotic stability in mean square. Also, various types of stochastic differential equations have been considered such as stochastic delay differential equations with Poisson jumps, stochastic delay differential equations with Poisson jumps and Markovian switching, impulsive stochastic delay differential equations with jumps and neutral stochastic functional differential equations.

In Chapter 2, we recall some basic concepts of the theory of stochastic differential equations in infinite dimensional spaces, mainly Hilbert spaces. This chapter begins with some basic definitions and preliminaries of stochastic integration and stochastic differential equations in infinite dimensional spaces. In this way, such notions as Q-Wiener processes, stochastic integral with respect to Wiener processes, stochastic integral with respect to compensated Poisson random measures, strong and mild solutions to stochastic differential equations will be appropriately reviewed in order to help readers gain required knowledge to understand the following chapters. In addition, some of important mathematical tools like Burkholder-Davis-Gundy inequalities are stated in the latter part of this chapter. The book, Liu (2006) contributes to the development of this thesis as the main source of reference but the reader can find most of theses mathematical concepts in many fine books related to stochastic analysis such as Da Prato and Zabczyk (1992), Kuo (2006), Métivier (1982) and Øksendal (1995).

Stochastic evolution equations and stochastic partial differential equations driven by Wiener processes have been studied by many researchers. However, there have not been very much studies of stochastic partial differential equations driven by jump processes. Chapter 3 of this thesis is devoted to an investigation of the stability of the mild solution to stochastic delay differential equations with Poisson jumps. We introduce a proper approximating strong solution system of mild solution to pass on stability of strong solutions to mild ones. The main result is that by constructing a suitable metric between the transition probability functions of mild solutions, we are able to give sufficient conditions for stability in distribution of mild solution.

Many practical systems may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections and abrupt environmental disturbances. The hybrid systems driven by continuous-time Markov chains have been used to model such systems. The hybrid systems combine a part of the state that takes values continuously and another part of the state that takes discrete values. Such hybrid system have been considered for the modeling of electric power systems by Wilsky and Levy (Wilsky and Levy (1979)) as well as for the control of a solar thermal central receiver by Sworder and Rogers (Sworder and Rogers (1983)). In 1987, Athans (1987) suggested that the hybrid systems would become a basic framework in posing and solving control-related issues in Battle Management

Command, Control and Communications (BM/C^3) systems. Moreover, one of the important classes of hybrid systems is the jump linear systems

$$\dot{x}(t) = A(r(t)x(t))$$

where one part of the state $x(t)$ takes values in $\mathbb{R}_+$ while the other part of the state $r(t)$ is a Markov chain taking values in $S = \{1, 2, \cdots, \mathbb{N}\}$. One of the most important issues in the study of hybrid systems is the automatic control, with consequent emphasis being placed on the analysis of stability (c.f. Basak et al. (1996), Ji and Chizeck (1990), Mao (1999) and Mao et al. (2000).

If the stochastic system in Chapter 3 experiences abrupt changes in their structure and parameters, and we use the continuous-time Markov chain to model these abrupt changes, we then need to deal with stochastic partial differential equation with Markovian switching and Poisson jumps. Therefore, in Chapter 4, motivated by the hybrid systems, we proceed to the study of stochastic partial differential equations with Markovian switching and Poisson jumps. By using the energy equality, we obtain the exponential stability and almost sure exponential stability of the energy solution to the equations which we are interested under some suitable conditions.

Chapter 5 is a study of the solutions of stochastic neutral functional differential equations in infinite dimensions. By introducing the fundamental solutions (Green's operator) which firstly introduced in Liu (2008), we established the variational constants formula of mild solution to the neutral stochastic system. We then discuss the relationship among strong, weak and mild solutions.

Besides the abrupt changes modeled by continuous-time Markov chain, there are many real-world systems and natural processes that display some kind of dynamic behavior in a style of both continuous and discrete characteristics. For instance, in the field of medicine, biology, electronics and economic, many evolution processes are characterized by abrupt changes of states at certain time instants. This is the familiar impulsive phenomenon. Often, sudden and sharp changes occur instantaneously, in the form of impulsive, which cannot be well described by using pure continuous or pure discrete models. Taking the environmental disturbances into account, impulsive stochastic systems arise naturally from a wide variety of applications.

Therefore, in Chapter 6, we firstly study the impulsive stochastic partial differential equations, followed by the stochastic evolution equations with Poisson jumps. By applying the fixed point theorem, we obtain the asymptotic stability in mean square for both equations under some suitable criteria. Finally, a summary of this thesis is presented in Chapter 7.

Chapter 2

Stochastic differential equations in infinite dimensions

This thesis deals with the stochastic differential equations in Hilbert spaces. More specifically, we study the solutions of stochastic differential equations, for which we need the notion of the stochastic integral.

The aim of this chapter is to introduce the concepts of 'stochastic integral' and 'stochastic differential equation' in a manner that is general enough to allow us to study stochastic differential equations driven by Wiener processes or Lévy processes. In Section 2.1, we introduce some basic definitions and preliminaries in stochastic analysis. Wiener processes and the stochastic integral with respect to Wiener processes are defined in Section 2.2. To be able to study equations driven by jump processes, we introduce Lévy processes and the stochastic integral with respect to compensated Poisson random measures in Section 2.3. In section 2.4, we describe the notions of strong and mild solutions of stochastic differential equations in infinite dimensions and mention some preliminary results. Proofs of the results presented in this chapter are not given as they are widely available in the existing literature (c.f. Section 2.5).

2.1 Notations, definitions and preliminaries

A *measurable space* is a pair $(\Omega, \mathcal{F})$ where Ω is a set and $\mathcal{F}$ is a σ-field, also called a σ-algebra, of subsets of Ω. This means that the family $\mathcal{F}$ contains the set Ω and is closed under the operation of taking complements and countable unions of it elements. If $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ are two measurable spaces, then a mapping X from Ω into E such that the set $\{\omega \in \Omega : X(\omega) \in A\} = \{X \in A\}$ belongs to $\mathcal{F}$ for arbitrary $A \in \mathcal{E}$ is called a *measurable mapping* or *random variable* from $(\Omega, \mathcal{F})$ into $(E, \mathcal{E})$. A random variable is called *simple* if it takes on only a finite number of values.

In this thesis, we shall only be concerned with the case when E is a complete and separable metric space. In this case, $\mathcal{E} = \mathcal{B}(E)$, the *Borel σ-field* of E which is the smallest σ-field containing all closed (or open) subsets of E. An E-valued random variable is a mapping $X : \Omega \to E$ which is measurable from $(\Omega, \mathcal{F})$ into $(E, \mathcal{B}(E))$. If E is a separable Banach or Hilbert space, we shall denote its norm by $\|\cdot\|_E$ and its topological dual by E^*.

A *probability measure* on a measurable space $(\Omega, \mathcal{F})$ is a σ-additive function $\mathbb{P}$ from $\mathcal{F}$ into $[0, 1]$ such that $\mathbb{P}(\Omega) = 1$. The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, we set

$$\bar{\mathcal{F}} = \{A \subset \Omega : \exists\, B, C \in \mathcal{F}; \quad B \subset A \subset C, \quad \mathbb{P}(B) = \mathbb{P}(C)\}.$$

Then $\bar{\mathcal{F}}$ is a σ-field and is called the *completion* of $\mathcal{F}$. If $\mathcal{F} = \bar{\mathcal{F}}$, then the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be complete.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A family $\{\mathcal{F}_t\}_{t \geq 0}$, for which $\mathcal{F}_t$ are sub-σ-fields of $\mathcal{F}$ and form an increasing family of σ-field, is called a *filtration* if $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $s \leq t$. With the $\{\mathcal{F}_t\}_{t \geq 0}$, one can associate two other filtrations by setting σ-fields: $\mathcal{F}_{t-} = \bigvee_{s<t} \mathcal{F}_s$ if $t > 0$, $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ if $t \geq 0$, where $\bigvee_{s<t} \mathcal{F}_s$ is the smallest σ-filed containing $\bigcup_{s<t} \mathcal{F}_s$. The σ-field $\mathcal{F}_{0-}$ is not defined and,

by convention, we put $\mathcal{F}_{0-} = \mathcal{F}_0$, and also $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. An increasing family $\{\mathcal{F}_t\}_{t\geq 0}$ is said to be *right-continuous* if for every $t \geq 0$, $\mathcal{F}_{t+} = \mathcal{F}_t$.

From now on, we shall always work on a given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions, i.e. $\{\mathcal{F}_t\}_{t\geq 0}$ is an increasing and right-continuous family of sub-σ-fields of $\mathcal{F}$ and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$.

If X is a random variable from $(\Omega, \mathcal{F})$ into $(E, \mathcal{E})$ and $\mathbb{P}$ a probability measure on Ω, then by $\mathscr{L}(X)(\cdot)$ we denote the image of $\mathbb{P}$ under the mapping X:

$$\mathscr{L}(X)(A) = \mathbb{P}\{\omega \in \Omega : X(\omega) \in A\}, \quad \forall\, A \in \mathcal{E}.$$

The measurable $\mathscr{L}(X)$ is called the *distribution* or *law of X*.

Definition 2.1 ($\{\mathcal{F}_t\}$-stopping time) *A nonnegative random variable $\tau(\omega) : \Omega \to \overline{\mathbb{R}_+} = [0, \infty]$ is called an $\{\mathcal{F}_t\}$-stopping time if $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for arbitrary $t \geq 0$.*

Assume now E is a separable Banach space with norm $\|\cdot\|_E$ and X is an E-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. We can define the integral $\int_\Omega X d\mathbb{P}$ of X with respect to the probability measure $\mathbb{P}$ by a standard limit argument (c.f. Da Prato and Zabczyk (1992), Section 1.1).

Definition 2.2 (Bochner integrable) *A random variable X is said to be Bochner integrable or simply integrable if*

$$\int_\Omega \|X(\omega)\|_E \mathbb{P}(d\omega) < \infty.$$

Definition 2.3 (Bochner's integral) *Let the random variable X be integrable, there exists a sequence $\{X_n\}$ of simple random variables such that the sequence $\{\|X(\omega) - X_n(\omega)\|_E\}$ decreases to 0 as $n \to \infty$, for all $\omega \in \Omega$. It follows that*

$$\begin{aligned}
&\left\| \int_\Omega X_n(\omega)\mathbb{P}(d\omega) - \int_\Omega X_m(\omega)\mathbb{P}(d\omega) \right\|_E \\
\leq\ & \int_\Omega \|X(\omega) - X_n(\omega)\|_E \mathbb{P}(d\omega) + \int_\Omega \|X(\omega) - X_m(\omega)\|_E \mathbb{P}(d\omega) \\
\downarrow\ & 0 \ \ as \ \ n, m \to \infty.
\end{aligned}$$

Therefore the integral of X can be defined by

$$\int_\Omega X(\omega)\mathbb{P}(d\omega) = \lim_{n\to\infty} \int_\Omega X_n(\omega)\mathbb{P}(d\omega).$$

The integral $\int_\Omega X d\mathbb{P}$ will be often denoted by $\mathbb{E}(X)$. Then integral defined in this way is called Bochner's integral.

We denote by $L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$ the set of all equivalence classes of E-valued random variables with respect to the equivalence relation of almost sure equality. In the same way as for real random variables, one can be checked that $L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$, equipped with the norm $\|X\|_{L^1} = \mathbb{E}\|X\|_E$, is a Banach space. In a similar manner, one can define $L^p(\Omega, \mathcal{F}, \mathbb{P}; E)$, for arbitrary $p > 1$ with norms $\|X\|_{L^p} = (\mathbb{E}\|X\|_E^p)^{1/p}$, $p \in (1, \infty)$, and $\|X\|_{L^\infty} =$ ess. $\sup_{\omega\in\Omega} \|X(\omega)\|_E$. If Ω is an interval $[0, T]$, $\mathcal{F} = \mathcal{B}([0, T])$, $0 \leq T < \infty$, and $\mathbb{P}$ is the Lebesgue measure on $[0, T]$, we also write $L^p(0, T; E)$ or simply $L^p(0, T)$, for the spaces defined above when no confusion arises.

Operator-valued random variables and their integrals are often of great interest to us. Let H and K be two separable Hilbert spaces and we denote by $\|\cdot\|_H$ and $\|\cdot\|_K$ their norms and by $\langle\cdot,\cdot\rangle_H$, $\langle\cdot,\cdot\rangle_K$ their inner products, respectively. We denote by $\mathcal{L}(K, H)$ the set of all linear bounded operators from K into H, equipped with the usual operator norm $\|\cdot\|$. We always use the same symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion may arise. The set $\mathcal{L}(K, H)$ is a linear space and equipped with the operator norm, becomes a Banach space. However if both spaces are infinite dimensional, then space $\mathcal{L}(K,H)$ is not a separable space. A direct consequence of this inseparability is that Bochner's integral definition can not be applied to $\mathcal{L}(K,H)$-valued functions. To overcome these difficulties, we introduce the following weaker concept of measurability.

A mapping $\Phi(\cdot)$ from Ω into $\mathcal{L}(K,H)$ is said to be *strongly measurable* if for arbitrary $k \in K$, $\Phi(\cdot)k$ is measurable as a mapping from $(\Omega,\mathcal{F})$ into $(H,\mathcal{B}(H))$. Let $\mathcal{F}(\mathcal{L}(K,H))$ be the smallest σ-field of subsets of $\mathcal{L}(K,H)$ containing all sets of the form

$$\{\Phi \in \mathcal{L}(K,H) : \Phi k \in A\}, \quad k \in K, \quad A \in \mathcal{B}(H),$$

then $\Phi : \Omega \to \mathcal{L}(K,H)$ is a strongly measurable mapping from $(\Omega,\mathcal{F})$ into the space $(\mathcal{L}(K,H),\mathcal{F}(\mathcal{L}(K,H)))$. Elements of $\mathcal{F}(\mathcal{L}(K,H))$ are called *strongly measurable*. Mapping Φ is said to be *Bochner integrable* with respect to the measure $\mathbb{P}$ if for arbitrary k, the mapping $\Phi(\cdot)k$ is Bochner integrable and there exists a bounded linear operator $\Psi \in \mathcal{L}(K,H)$ such that

$$\int_\Omega \Phi(\omega)k\mathbb{P}(d\omega) = \Psi k, \quad k \in K.$$

The operator Ψ is then denoted as

$$\Psi = \int_\Omega \Phi(\omega)\mathbb{P}(d\omega)$$

and called the *strong Bochner integral* of Φ. This integral has many of the properties of the Lebesgue integral. For instance, it can be shown that if K and H are both separable, then $\|\Phi(\cdot)\|$ is a measurable function and

$$\|\Psi\| \leq \int_\Omega \|\Phi(\omega)\|\mathbb{P}(d\omega).$$

The following operator spaces are of fundamental importance. An element $A \in \mathcal{L}(H,H)$ is called symmetric if $\langle Au,v\rangle_H = \langle u,Av\rangle_H$ for all $u,v \in H$. In addition, $A \in \mathcal{L}(H,H)$ is called nonnegative if $\langle Au,v\rangle_H \geq 0$ for all $u \in H$.

Definition 2.4 (Nuclear operator) *An element $A \in \mathcal{L}(K, H)$ is said to be a nuclear operator if there exists a sequence $\{a_j\}_{j\in\mathbb{N}}$ in H and a sequence $\{b_j\}_{j\in\mathbb{N}}$ in K such that*

$$Ax = \sum_{j=1}^{\infty} a_j \langle b_j, x\rangle_K \text{ for all } x \in K$$

and

$$\sum_{j=1}^{\infty} \|a_j\|_H \cdot \|b_j\|_K < \infty.$$

The space of all nuclear operators from K to H is denoted by $\mathcal{L}_1(K, H)$. If $K = H$, $A \in \mathcal{L}_1(K, H)$ is nonnegative and symmetric, then A is called trace class. The space $\mathcal{L}_1(K, H)$ endowed with the norm

$$\|A\|_{\mathcal{L}_1} := \inf\left\{ \sum_{j=1}^{\infty} \|a_j\|_H \cdot \|b_j\|_K : Ax = \sum_{j=1}^{\infty} a_j \langle b_j, x\rangle_K,\ x \in K \right\}$$

is a Banach space (c.f. Da Prato and Zabczyk (1992), Appendix C).

Let $A \in \mathcal{L}(H, H)$ and let e_k, $k \in \mathbb{N}$, be an orthonormal basis of H. Then we define

$$\text{Tr } A := \sum_{k=1}^{\infty} \langle Ae_k, e_k\rangle_H$$

if the series is convergent. If $A \in \mathcal{L}(H, H)$ then Tr A is well-defined independently of the choice of the orthonormal basis e_k, $k \in \mathbb{N}$ (c.f. Da Prato and Zabczyk (1992), Appendix C).

Definition 2.5 (Hilbert-Schmidt operator) *A bounded linear operator $A : K \to H$ is called Hilbert-Schmidt if*

$$\sum_{k=1}^{\infty} \|Ae_k\|_H^2 < \infty$$

where e_k, $k \in \mathbb{N}$, is an orthonormal basis of K.

The space of all Hilbert-Schmidt operators from K to H is denoted by $\mathcal{L}_2(K, H)$.

The definition of Hilbert-Schmidt operator and the number

$$\|A\|_{\mathcal{L}_2}^2 := \sum_{k=1}^{\infty} \|Ae_k\|_H^2$$

are independent of the choice of orthonormal basis e_k, $k \in \mathbb{N}$. Moreover, $\|A\|_{\mathcal{L}_2} = \|A^*\|_{\mathcal{L}_2}$. Let $S, T \in \mathcal{L}_2(K, H)$ and e_k, $k \in \mathbb{N}$, be an orthonormal basis of K, $\mathcal{L}_2(K, H)$ is a separable Hilbert space with inner product

$$\langle T, S\rangle_{\mathcal{L}_2} := \sum_{k=1}^{\infty} \langle Se_k, Te_k\rangle_H$$

(c.f. Da Prato and Zabczyk (1992), Appendix C). It can be proved that the spaces $\mathcal{L}_1(K, H)$ and $\mathcal{L}_2(K, H)$ are strongly measurable subsets of $\mathcal{L}(K, H)$ (c.f. Da Prato and Zabczyk (1992), Section 1.2).

Assume that E is a separable Banach space with norm $\|\cdot\|_E$ and let $\mathcal{B}(E)$ be the σ-field of its Borel subsets. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. An arbitrary family $X = \{X(t)\}_{t\geq 0}$ of E-valued random variable $X(t)$, $t \geq 0$, defined on Ω is called a *stochastic process*. Sometimes, we also write $X(t, \omega) = X(t) = X_t(\omega)$ for all $t \geq 0$ and $\omega \in \Omega$. The functions $X(\cdot, \omega)$ are called the *trajectories* of X.

To this end, we introduce several definitions of regularity for a process X on $[0, T)$, where T could be finite or infinite.

(a) X is *stochastically continuous* at $t_0 \in [0, T)$ if, for all $\varphi > 0$ and $\delta > 0$, there exists $\rho > 0$ such that $\mathbb{P}\{\|X(t) - X(t_0)\|_E \geq \varphi\} \leq \delta, \quad \forall\, t \in [t_0 - \rho, t_0 + \rho] \cap [0, T)$;

(b) X is *stochastically continuous* in $[0, T)$ if it is stochastically continuous at every point of $[0, T)$;

(c) X is *continuous with probability one (or continuous)* if its trajectories $X(\cdot, \omega)$ are continuous almost surely;

(d) X is *càdlàg (right-continuous and left limit)* if it is right-continuous and for almost all $\omega \in \Omega$ the left limit $X(t-, \omega) = \lim_{s \to t} X(s, \omega)$ exists and is finite for all $t > 0$.

(e) X is *mean square continuous* at $t_0 \in [0, T)$ if $\lim_{t \to t_0} \mathbb{E}(\|X(t) - X(t_0)\|_E^2) = 0$.

(f) X is *mean square continuous* in $[0, T)$ if it is mean square continuous at every point of $[0, T)$.

Definition 2.6 *A stochastic process X on $[0, T)$, where T could be finite or process, is said to be*

- ***measurable*** *if the mapping $X(\cdot, \cdot) : [0, T) \times \Omega \to E$ is $\mathcal{B}([0, T)) \times \mathcal{F}$-measurable (all stochastic processes considered in this thesis will be assumed to be measurable);*
- ***$\{\mathcal{F}_t\}$-adapted*** *if, for every $t \in [0, T)$, $X(t)$ is measurable with respect to $\mathcal{F}_t$;*
- ***progressively measurable*** *with respect to $\{\mathcal{F}_t\}$ if, for every $t \in [0, T)$, the mapping $[0, t] \times \Omega \to E$, $(s, \omega) \to X(s, \omega)$, is $\mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable.*

The following σ-field $\mathcal{P}_\infty$ of subsets of $[0, \infty) \times \Omega$ plays an essential role in the construction of the stochastic integrals with respect to martingales. That is, $\mathcal{P}_\infty$ is defined as the σ-field generated by sets of the form:

$$(s, t] \times F, \quad 0 \le s < t < \infty, \quad F \in \mathcal{F}_s \text{ and } \{0\} \times F, \quad F \in \mathcal{F}_0.$$

This σ-field is called *predictable* and its elements are called *predictable sets.* The restriction of the σ-field $\mathcal{P}_\infty$ to $[0, T] \times \Omega$, $0 \le T < \infty$, will be denoted by $\mathcal{P}_T$. An arbitrary measurable mapping from $([0, \infty) \times \Omega, \mathcal{P}_\infty)$ or $([0, T] \times \Omega, \mathcal{P}_T)$ into $(E, \mathcal{B}(E))$ is called *predictable process.* A predictable process is necessarily an adapted one. We have the following relationship among the various processes:

predictable processes $\subset$ progressive processes $\subset$ adapted processes.

Theorem 2.1

(i) An adapted process $\Phi(t)$, $t \in [0,T)$ with values in $(\mathcal{L}(K,H),\ \mathcal{F}(\mathcal{L}(K,H)))$ such that for arbitrary $k \in K$ and $h \in H$ the process $\langle \Phi(t)k, h\rangle_H$, $t \in [0,T)$ has left continuous trajectories, is predictable ($\mathcal{L}(K,H)$ and $\mathcal{F}(\mathcal{L}(k,H))$ are defined on Page 13 and 14);

(ii) Assume that $\Phi(t)$, $t \in [0,T)$, is an adapted and stochastically continuous process on the interval $[0,T)$. Then the process Φ has a predictable version on $[0,T)$.

The proof can be found in Da Prato and Zabczyk (1992), Proposition 3.6. ■

Let E be a separable Banach space with norm $\|\cdot\|_E$ and $M = M(t)$, $t \in (0,T]$, where T could be finite or infinite, an E-valued stochastic process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in[0,T)}, \mathbb{P})$. If $\mathbb{E}\|M(t)\|_E < \infty$ for all $t \in [0,T)$, then the process is called *integrable*. An integrable and adapted E-valued process $M(t)$, $t \in [0,T)$, is said to ba a *martingale* with respect to $\{\mathcal{F}\}_{t\in[0,T)}$ if

$$\mathbb{E}(M(t)|\mathcal{F}_s) = M(s) \quad \mathbb{P}-\ \text{a.s.} \tag{2.1.1}$$

for arbitrary $t \geq s$, $t, s \in [0,T)$. If $\mathbb{E}\|M(t)\|_E^2 < \infty$ for all $t \in [0,T)$ then $M(t)$ is called *square integrable*.

We also recall that a real-valued integrable and adapted process $M(t)$, $t \in [0,T)$, is said to be a *submartingale* (resp. a *supermartingale*) with respect to $\{\mathcal{F}_t\}_{t\in[0,T)}$ if

$$\mathbb{E}(M(t)|\mathcal{F}_s) \geq M(s) \quad (\text{ resp. } \mathbb{E}(M(t)|\mathcal{F}_s) \leq M(s)), \quad \mathbb{P}-\text{a.s.}$$

for any $s \leq t$, $s, t \in [0,T)$.

Let $[0,T]$, $0 \leq T < \infty$, be a sub-interval of $[0,\infty)$. A continuous E-valued stochastic process $M(t)$, $t \in [0,T]$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, is a *continuous square integrable martingale* with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ if it is a martingale with almost surely continuous trajectories and satisfies, $\sup_{t\in[0,T]} \mathbb{E}\|M(t)\|_E^2 < \infty$. Let us denote by $\mathcal{M}_T^2(E)$ the space of all E-valued continuous, square integrable martingales M. The space $\mathcal{M}_T^2(E)$, equipped with the norm $\|M\|_{\mathcal{M}_T^2(E)} = \left(\mathbb{E}\sup_{t\in[0,T]} \|M(t)\|_E^2\right)^{1/2}$, is a Banach space (c.f. Da Prato and Zabczyk (1992), Proposition 3.9).

Denote by $\mathcal{L}_1 = \mathcal{L}_1(K) = \mathcal{L}_1(K,K)$ the space of all nuclear operators from the separable Hilbert space K into itself, equipped with the usual nuclear norm. Then $\mathcal{L}_1$ is a separable Banach space. An $\mathcal{L}_1$-valued process $V(\cdot)$ is said to be *increasing* if the operators $V(t)$, $t \in [0,T]$, are nonnegative, denoted by $V(t) \geq 0$, i.e., for any $k \in K$, $\langle V(t)k, k\rangle_K \geq 0$, $t \in [0,T]$, and $0 \leq V(s) - V(t)$ if $0 \leq t \leq s \leq T$. An $\mathcal{L}_1$-valued continuous, adapted and increasing process $V(t)$ such that $V(0) = 0$ is said to be a *tensor quadratic variation process* of the martingale $M(t) \in \mathcal{M}_T^2(K)$ if and only if for arbitrary $a, b \in K$, the process

$$\langle M(t), a\rangle_K \langle M(t), b\rangle_K - \langle V(t)a, b\rangle_K, \quad t \in [0,T],$$

is a continuous $\mathcal{F}_t$-martingale, or equivalently, if and only if the process

$$M(t) \otimes M(t) - V(t), \quad t \in [0,T],$$

is a continuous $\mathcal{F}_t$ -martingale, where $(a \otimes b)k := a\langle b, k\rangle_K$ for any $k \in K$ and $a, b \in K$. One can show that the process $V(t)$ is uniquely determined and can be denoted therefore by $\ll M(t) \gg$, $t \in [0,T]$ (c.f. Da Prato and Zabczyk (1992), Proposition 3.12).

On the other hand, if $M(t) \in \mathcal{M}_T^2(K)$ then there exists a real-valued, increasing, continuous process which is uniquely determined up to probability one, denoted by $[M(t)]$ with $[M(0)] = 0$, called the *quadratic variation* of $M(t)$, such that $\|M(t)\|_K^2 - [M(t)]$ is an $\mathcal{F}_t$-martingale (c.f. Da Prato and Zabczyk (1992), Page 79).

With regard to the relation between $\ll M(t) \gg$ and $[M(t)]$ of $M(t)$, we have the following:

Theorem 2.2 *For arbitrary $M(t) \in \mathcal{M}_T^2(K)$, there exists a unique predictable, positive symmetric element $Q_M(\omega, t)$ or simply $Q(\omega, t)$, of $\mathcal{L}_1(K)$ such that*

$$\ll M(t) \gg = \int_0^t Q_M(\omega, s) d[M(s)]$$

for all $t \in [0, T]$. In particular, we also call the K-valued stochastic process $M(t)$, $t \geq 0$, a $Q_M(\omega, t)$-martingale process.

The proof can be found in Métivier (1982), Theorem 21.6. ■

In a similar manner, one can define the so-called *cross quadratic variation* for any $M(t) \in \mathcal{M}_T^2(K)$, $N(t) \in M_T^2(K)$ as a unique continuous process, denoted by $\ll M(t), N(t) \gg$, such that $M(t) \otimes N(t) - \ll M(t), N(t) \gg$, $t \in [0, T]$, is a continuous $\mathcal{F}_t$-martingale (c.f. Da Prato and Zabczyk (1992), Page 80).

2.2 Wiener processes and the stochastic integral with respect to Wiener processes

Let K be a real separable Hilbert space with norm $\|\cdot\|_K$ and inner product $\langle \cdot, \cdot \rangle_K$, respectively. A probability measure $\mathcal{N}$ on $(K, \mathcal{B}(K))$ is called *Gaussian* if for arbitrary $u \in K$, there exist numbers $\mu \in \mathbb{R}$ and $\sigma \geq 0$ such that

$$\mathcal{N}\{x \in K : \langle u, x \rangle_K \in A\} = N(\mu, \sigma^2)(A), \quad A \in \mathcal{B}(\mathbb{R}),$$

where $N(\mu, \sigma^2)$ is the usual one dimensional normal distribution with mean μ and variance σ^2. It follows from Da Prato and Zabczyk (1992), Lemma 2.14 that if $\mathcal{N}$ is Gaussian, then there exist an element $m \in K$ and a symmetric nonnegative continuous operator Q such that:

$$\int_K \langle k, x \rangle_K \mathcal{N}(dx) = \langle m, k \rangle_K, \quad \forall\, k \in K,$$

$$\int_K \langle k_1, x\rangle_K \langle k_2, x\rangle_K \mathcal{N}(dx) - \langle m, k_1\rangle_K \langle m, k_2\rangle_K = \langle Qk_1, k_2\rangle_K, \quad \forall\, k_1, k_2 \in K,$$

and, furthermore the characteristic function

$$\widehat{\mathcal{N}}(\lambda) = \int_K e^{i\langle \lambda, x\rangle_K} \mathcal{N}(dx) = e^{i\langle \lambda, m\rangle_K - \frac{1}{2}\langle Q\lambda, \lambda\rangle_K}, \quad \lambda \in K.$$

Therefore, the measure $\mathcal{N}$ is uniquely determined by m and Q and denoted also by $\mathcal{N}(m, Q)$. In particular, in this case we call m the *mean* and Q the *covariance operator* of $\mathcal{N}$. Note that, let $\mathcal{N}$ be a Gaussian probability measure with mean 0 and covariance Q, then Q is a trace class operator (c.f. Da Prato and Zabczyk (1992), Proposition 2.15).

Recall that we always assume the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero. We consider two real-valued separable Hilbert spaces H and K, and a symmetric nonnegative operator $Q \in \mathcal{L}(K)$ with Tr $Q < \infty$. Then there exists a complete orthonormal system e_k, $k \in \mathbb{N}$ in K, and a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \cdots$.

Definition 2.7 (K-valued Q-Wiener process) *A K-valued stochastic process $W_Q(t)$, $t \geq 0$, is called a Q-Wiener process if*

(i) $W_Q(0) = 0$;

(ii) $W_Q(t)$ *has continuous trajectories;*

(iii) $W_Q(t)$ *has independent increments;*

(iv) $\mathscr{L}(W_Q(t) - W_Q(s)) = \mathcal{N}(0, (t-s)Q)$, $t \geq s \geq 0$, *i.e.* $\mathbb{E}(W_Q(t)) = 0$ *and* $Cov(W_Q(t) - W_Q(s)) = (t-s)Q$, *where* $\mathscr{L}(X)$ *denotes the distribution of* X *(c.f. Page 12) and* $Cov(X)$ *denotes the covariance operator of* $X \in H$. *(c.f. Da Prato and Zabczyk (1992), Page 26).*

Proposition 2.1 *Assume that $W_Q(t)$ is a Q-Wiener process with Tr $Q < \infty$.*

Then $W_Q(t)$ is a continuous martingale relative to $\{\mathcal{F}\}_{t\geq 0}$ and we have the following representation of $W_Q(t)$:

$$W_Q(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)e_j \tag{2.2.1}$$

where

$$\beta_j(t) = \frac{1}{\sqrt{\lambda_j}}\langle W_Q(t), e_j\rangle, \quad j = 1, 2, \cdots, \tag{2.2.2}$$

are real-valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$ and the series in (2.2.1) is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

The proof can be found in Da Prato and Zabczyk (1992), Proposition 4.1. ■

Theorem 2.3 *For an arbitrary symmetric nonnegative trace class operator Q on the real separable Hilbert space K, there exists a Q-Wiener process $W_Q(t)$, $t \geq 0$. Moreover, the series (2.2.1) is uniformly convergent on $[0, T]$ almost surely for arbitrary $T \geq 0$.*

The proof can be found in Da Prato and Zabczyk (1992), Proposition 4.2 and Theorem 4.3. ■

Note that the tensor quadratic variation of a Q-Wiener process in K with Tr $Q < \infty$, is given by the formula $\ll W_Q(t) \gg = tQ$, $t \geq 0$ from the following theorem.

Theorem 2.4 *A continuous martingale $M(t) \in \mathcal{M}_T^2(K)$, $M(0) = 0$, is a Q-Wiener process on $[0, T]$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and with increments $M(t) - M(s)$, $0 \leq s \leq t \leq T$, independent of $\mathcal{F}_s$, for $s \in [0, T]$, if and only if $\ll M(t) \gg = tQ$, $t \in [0, T]$.*

The proof can be found in Da Prato and Zabczyk (1992), Theorem 4.4. ■

The stochastic integral $\int_0^t \Phi(t)dW_Q(s)$ may be defined in the following ways. Let us fix $T < \infty$. An $\mathcal{L}(K, H)$-valued process $\Phi(t)$, $t \in [0, T]$, taking only a finite number of values is said to be *elementary* if there exists a sequence $0 = t_0 < t_1 < \cdots < t_k = T$ and a sequence $\Phi_0, \Phi_1, \cdots, \Phi_{k-1}$ of $\mathcal{L}(K, H)$-valued random variables taking only a finite number of values such that Φ_m are $\mathcal{F}_{t_m}$-measurable and

$$\Phi(t) = \Phi_m, \text{ for } t \in (t_m, t_{m+1}],\ m = 0, 1, \cdots, k-1.$$

For elementary processes Φ one defines the stochastic integral by the formula

$$\int_0^t \Phi(s) dW_Q(s) = \sum_{m=0}^{k-1} \Phi_m (W_Q(t_{m+1} \wedge t) - W_Q(t_m \wedge t)), \quad t \in [0, T] \quad (2.2.3)$$

We introduce the subspace $K_0 = Q^{1/2}(K)$ of K which, endowed with the inner product

$$\langle u, v \rangle_{K_0} = \langle Q^{-1/2} u, Q^{-1/2} v \rangle_K, \quad u, v \in K_0, \quad (2.2.4)$$

is a Hilbert space, where $Q^{1/2}$ is a positive square root of $Q \in \mathcal{L}(H, H)$ and $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$ (For definition of positive square root and pseudo-inverse, please refer to Kreyszig (1989), Page 476 and Prévôt and Röckner (2007), Appendix C).

In the construction of the stochastic integral for more general processes an important role will be played by the space of all Hilbert-Schmidt operator $\mathcal{L}_2^0(K_0, H)$ from K_0 into H. The space $\mathcal{L}_2^0(K_0, H)$ is also a separable Hilbert space, equipped with the norm

$$\|\Psi\|^2_{\mathcal{L}_2^0} = \mathrm{Tr}\big((\Psi Q^{1/2})(\Psi Q^{1/2})^*\big) \quad \text{for any } \Psi \in \mathcal{L}_2^0(K_0, H). \quad (2.2.5)$$

For arbitrarily given $T \geq 0$, let $\Phi(t)$, $t \in [0, T]$, be an $\mathcal{F}_t$-adapted, $\mathcal{L}_2^0(K_0, H)$-valued process. We define the following norms for arbitrary $t \in [0, T]$,

$$
\begin{aligned}
|\Phi|_t &:= \left\{ \mathbb{E} \int_0^t \|\Phi(s)\|^2_{\mathcal{L}_2^0} ds \right\}^{\frac{1}{2}} \\
&= \left\{ \mathbb{E} \int_0^t \mathrm{Tr}\big(\Phi(s)Q^{1/2}\big)\big(\Phi(s)Q^{1/2}\big)^* ds \right\}^{\frac{1}{2}}. \qquad (2.2.6)
\end{aligned}
$$

In particular, if $\Phi(t) \in \mathcal{L}_2^0(K_0, H)$, $t \in [0, T]$, is an $\mathcal{F}_t$-adapted, $\mathcal{L}(K, H)$-valued process, (2.2.6) turns out to be

$$
|\Phi|_t = \left\{ \mathbb{E} \int_0^t \mathrm{Tr}\big(\Phi(s)Q\Phi(s)^*\big) ds \right\}^{\frac{1}{2}}.
$$

Proposition 2.2 *For arbitrary $T \geq 0$, if a process Φ is elementary and $|\Phi|_T < \infty$, then the process $\int_0^t \Phi(s)dW_Q(s)$ is a continuous, square integrable H-valued martingale on $[0, T]$ and*

$$
\mathbb{E}\left\| \int_0^t \Phi(s)dW_Q(s) \right\|_H^2 = |\Phi|_t^2, \quad t \in [0, T]. \qquad (2.2.7)
$$

The proof can be found in Da Prato and Zabczyk (1992), Proposition 4.5. ■

Note that the stochastic integral is an isometric transformation from the space of all elementary processes equipped with the norm $|\cdot|_T$ into the space $\mathcal{M}_T^2(H)$ of H-valued martingales.

For arbitrary $T \geq 0$, from (c.f. Da Prato and Zabczyk (1992), Chapter 4), we are able to extend the definition of stochastic integral $\int_0^t \Phi(s)dW_Q(s)$, $t \geq 0$ to all $\mathcal{L}_2^0(K_0, H)$-valued predictable process Φ such that $|\Phi|_T < \infty$. Note that they form a Hilbert space. We denote all $\mathcal{L}_2^0(K_0, H)$-valued predictable processes Φ such that $|\Phi|_T < \infty$ by $\mathcal{W}^2([0, T]; \mathcal{L}_2^0)$. By Da Prato and Zabczyk (1992), Proposition 4.7, elementary processes form a dense set in $\mathcal{W}^2([0, T]; \mathcal{L}_2^0)$. By Proposition 2.2, the stochastic integral $\int_0^t \Phi(s)dW_Q(s)$, is an isometric transformation from that dense set into $\mathcal{M}_T^2(H)$, therefore the definition of the integral can be immediately extended to all elements of $\mathcal{W}^2([0, T]; \mathcal{L}_2^0)$. As a final step we extend the definition of the stochastic integral to $\mathcal{L}_2^0(K_0, H)$-valued predictable processes satisfying the even weaker condition

$$\mathbb{P}\left\{ \int_0^T \|\Phi(s)\|^2_{\mathcal{L}_2^0} ds < \infty,\, 0 \leq t \leq T \right\} = 1. \tag{2.2.8}$$

Please refer to Da Prato and Zabczyk (1992), Page 94-96 for the details.

We introduce the following properties of the stochastic integral.

Theorem 2.5 *Assume that $\Phi \in \mathcal{W}^2([0,T]; H)$, then $\int_0^t \Phi(s)dW_Q(s)$ is a continuous square integrable martingale, and its quadratic variation is of the form*

$$\ll \int_0^t \Phi(s)dW_Q(s) \gg = \int_0^t Q_\Phi(s)ds,$$

where

$$Q_\Phi(t) = \left(\Phi(t)Q^{1/2}\right)\left(\Phi(t)Q^{1/2}\right)^*, \quad t \in [0,T].$$

The proof can be found in Da Prato and Zabczyk (1992), Theorem 4.12. ■

Theorem 2.6 (Burkholder-Davis-Gundy inequality) *For arbitrary $p > 0$, there exists a constant $C = C_p > 0$, dependent only on p, such that for any $T \geq 0$,*

$$\mathbb{E}\left\{ \sup_{0\leq t\leq T} \int_0^t \left\|\Phi(s)dW_Q(s)\right\|_H^p \right\} \leq C_p\, \mathbb{E}\left\{ \int_0^T \|\Phi(s)\|^2_{\mathcal{L}_2^0} ds \right\}^{p/2}.$$

The proof can be found in Da Prato and Zabczyk (1992), Lemma 7.2. ■

Assume that A is a linear operator, generally unbounded, on H and $T(t)$, $t \geq 0$, a strongly continuous semigroup of bounded linear operators with infinitesimal generator A. Suppose $\Phi(t) \in \mathcal{W}^2([0,T]; \mathcal{L}_2^0)$, $t \in [0,T]$, is an $\mathcal{L}_2^0(K_0, H)$-valued process such that the stochastic integral

$$\int_0^t T(t-s)\Phi(s)dW_Q(s) = W_A^\Phi(t), \quad t \in [0,T],$$

is well defined, then the process $W_A^\Phi(t)$ is called the *stochastic convolution* of Φ. In general, the stochastic convolution is no longer a martingale. However, we have the following result which could be regarded as an infinite dimensional version of Burkholder-Davis-Gundy type of inequality for stochastic convolutions.

Theorem 2.7 *Let $p > 2$, $T \geq 0$ and assume $\Phi(s) \in \mathcal{W}^2([0,T]; \mathcal{L}_2^0)$ is an $\mathcal{L}_2^0(K_0, H)$-valued process such that $\mathbb{E}\Big(\int_0^T \|\Phi(s)\|_{\mathcal{L}_2^0}^p ds\Big) < \infty$. Then there exists a constant $C = C_{p,T} > 0$, dependent on p and T, such that*

$$\mathbb{E} \sup_{t\in[0,T]} \left\| \int_0^t T(t-s)\Phi(s)dW_Q(s)\right\|_H^p \leq C_{p,T}\, \mathbb{E}\Big(\int_0^T \|\Phi(s)\|_{\mathcal{L}_2^0}^p ds\Big).$$

The proof can be found in Tubaro (1984), Theorem 1. ■

In Theorem 2.7, the case $p = 2$ is hold if A is assumed to generate a contraction C_0-semigroup $T(t)$, i.e., $\|T(t)\| \leq e^{\mu t}$, $t \geq 0$, for some $\mu \in \mathbb{R}$ (c.f. Liu (2006), Theorem 1.2.4).

Definition 2.8 (Fréchet differentiability) *Let E be a Banach space or Hilbert space with norm denoted by $\|\cdot\|_E$ and E^* be the topological dual of E with norm denoted by $\|\cdot\|_{E^*}$. Let $\Psi : E \to \mathbb{R}$ be Borel measurable function. The function Ψ is said to be Fréchet differentiable at $x \in E$ if for each $y \in E$,*

$$\Psi(x+y) - \Psi(x) = D\Psi(x)(y) + o(y),$$

where $D\Psi : E \to E^*$ and $o : E \to \mathbb{R}$ is function such that $\frac{o(x)}{\|x\|_E} \to 0$ as $\|x\|_E \to 0$. In this case, $D\Psi(x) \in E^*$ is called the Fréchet derivative of Ψ at $x \in E$. The function Φ is said to be continuously Fréchet differentiable at $x \in E$ if its Fréchet derivative $D\Psi : E \to E^*$ is continuous under the operator norm $\|\cdot\|_{E^*}$. The function Ψ is said to be twice Fréchet differentiable at $x \in E$ if its Fréchet derivative $D\Psi : E \to \mathbb{R}$ exists and there exists $D^2\Psi(x) : E \times E \to \mathbb{R}$ such that for each y, $z \in E$, $D^2\Psi(x)(\cdot, z)$, $D^2\Psi(x)(y, \cdot) \in E^*$ and

$$D\Psi(x+y)(z) - D\Psi(x)(z) = D^2\Psi(x)(y,z) + o(y,z).$$

Here, $o : E \times E \to \mathbb{R}$ is such that $\frac{o(y,z)}{\|y\|_E} \to 0$ as $\|y\|_E \to 0$ and $\frac{o(y,z)}{\|z\|_E} \to 0$ as $\|z\|_E \to 0$. In this case, the bounded bilinear functional $D^2\Psi(x) : E \times E \to \mathbb{R}$ is the second Fréchet derivative of Ψ at $x \in E$.

The function $\Psi : E \to \mathbb{R}$ is said to be Fréchet differentiable (respectively, twice Fréchet differentiable) if Ψ is Fréchet differentiable (twice Fréchet differentiable) at every $x \in E$.

As another important tool, we mention the following infinite dimensional version of the classic Itô's formula which will play an essential role in our study. For $T > 0$, suppose that $V(t,x) : [0,T] \times H \to \mathbb{R}$ is a continuous function with properties:

(i) $V(t,x)$ is differentiable in t and $V_t'(t,x)$ is continuous on $[0,T] \times H$;

(ii) $V(t,x)$ is twice Fréchet differentiable in x, $V_x'(t,x) \in H$ and $V_{xx}''(t,x) \in \mathcal{L}(H)$ are continuous on $[0,T] \times H$.

Assume that $\Phi(t) \in \mathcal{W}^2([0,T]; \mathcal{L}_2^0)$ is an $\mathcal{L}_2^0(K_0, H)$-valued process, $\phi(t)$ is an H-valued continuous, Bochner integrable process on $[0,T]$, and X_0 is an $\mathcal{F}_0$-measurable, H-valued random variable. Then the following H-valued process

$$X(t) = X_0 + \int_0^t \phi(s)ds + \int_0^t \Phi(s)dW_Q(s), \quad t \in [0,T], \tag{2.2.9}$$

is well defined.

Theorem 2.8 (Itô's formula) *Suppose the above conditions (i) and (ii) hold, then for all $t \in [0,T]$, $Z(t) = V(t, X(t))$ has the stochastic differential*

$$\begin{aligned} dZ(t) \;=\; & \Big\{ V_t'(t, X(t)) + \langle V_x'(t, X(t)), \phi(t) \rangle_H \\ & + \frac{1}{2} Tr\Big[V_{xx}''(t, X(t)) \big(\Phi(t) Q^{1/2}\big) \big(\Phi(t) Q^{1/2}\big)^* \Big] \Big\} dt \\ & + \langle V_x'(t, X(t)), \Phi(t) dW_Q(t) \rangle_H . \end{aligned}$$

The proof can be found in Da Prato and Zabczyk (1992), Theorem 4.17. ∎

2.3 Lévy processes and the stochastic integral with respect to compensated Poisson random measures

The Lévy process, which include the Poisson process and Brownian motion as special cases, were the first class of stochastic processes to be studied in the modern spirit by the French mathematician Paul Lévy. They still provide prototypic examples for Markov processes as well as for semimartingales. In this section, we will give a brief introduction to Lévy process, the stochastic integral with respect to compensated Poisson random measures and its relevant properties. We shall define a Lévy process on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$ satisfying the usual hypotheses with values in $(K, \mathcal{B}(K))$.

Definition 2.9 (Lévy process) *An process $Y = (Y(t))_{t\geq 0}$ with state space $(K, \mathcal{B}(K))$, is an $\mathcal{F}_t$-Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$ if*

(i) Y is adapted (to $\{\mathcal{F}_t\}_{t\geq 0}$);

(ii) $Y(0) = 0$ a.s.;

(iii) Y has increments independent of the past, i.e. $Y(t) - Y(s)$ is independent of $\mathcal{F}_s$ if $0 \leq s < t$;

(iv) Y has stationary increments, that is $Y(t) - Y(s)$ has the same distribution as $Y(t-s)$, $0 \leq s < t$;

(v) Y is stochastically continuous;

(vi) Y is càdlàg.

Definition 2.10 (Counting process) *Let $(T_n)_{n\geq 0}$ be a strictly increasing sequence of random variables with values in $\mathbb{R}_+$, such that $T_0 = 0$. Let*

$$\begin{aligned} \mathbf{1}_{t\geq T_n}(\omega) &= 1 \text{ if } t \geq T_n(\omega) \\ &= 0 \text{ if } t < T_n(\omega) \end{aligned}$$

$(N_t)_{t\geq 0}$ with $N_t = \sum_{n\geq 1} \mathbf{1}_{t\geq T_n}$ is called the counting process associated to $(T_n)_{n\geq 0}$.

$T = \sup_n T_n$ is the explosion time of $(N_t)_{t\geq 0}$. If $T = \infty$ almost surely then $(N_t)_{t\geq 0}$ is a counting process without explosion. The counting process $(N_t)_{t\geq 0}$ is adapted i.f.f. $(T_n)_{n\geq 0}$ are stopping times (c.f. Protter (2004), Chapter I, Theorem 22).

A σ-finite measure ν on $K - \{0\}$ is called a Lévy measure if

$$\int_{K-\{0\}} (\|y\|_K^2 \wedge 1)\nu(dy) < \infty.$$

An alternative convention is to define the Lévy measure on the whole of K via the assignment $\nu(\{0\}) = 0$.

Let $Y(t-) := \lim_{s\uparrow t} Y(s)$, the left limit at t, we define $\triangle Y(t) = Y(t) - Y(t-)$, the jump of Y at time t. We call that a Lévy process has bounded jumps if there exists a constant $C > 0$ with $\sup_{t\geq 0} \|\triangle Y(t)\|_K < C$.

We can obtain a counting Poisson random measure N on $(K - \{0\})$ through

$$N(t, \Lambda) := \#\{0 \leq s \leq t : \triangle Y(s) \in \Lambda\} = \sum_{0\leq s\leq t} \mathbf{1}_\Lambda(\triangle Y(s)), \quad t \geq 0,$$

almost surely for any $\Lambda \in \mathcal{B}(K - \{0\})$ with $0 \notin \bar{\Lambda}$, the closure of Λ in K. Here $\mathcal{B}(K - \{0\})$ denotes the Borel σ-filed of $K - \{0\}$.

Definition 2.11 (Compensated Poisson random measure) *We call the random measure $\widetilde{N}(t, dy) := N(t, dy) - t\nu(dy)$ the compensated Poisson random measure of the Lévy process Y.*

Let $\Lambda \in \mathcal{B}(K - \{0\})$ with $0 \notin \bar{\Lambda}$, the closure of Λ in K. If $f : \Lambda \to K$ is measurable, we may define

$$\int_\Lambda f(y)N(t,dy) = \sum_{0 \le s \le t} f((\triangle Y(s))\mathbf{1}_\Lambda(\triangle Y(s))$$

as a random finite sum. We denote ν_Λ the restriction of the measure ν to Λ, still denoted by ν, so that ν_Λ is finite. If $f \in L^2(\Lambda, \nu_\Lambda; K)$, we could define

$$\int_\Lambda f(y)\widetilde{N}(t,dy) = \int_\Lambda f(y)N(t,dy) - t\int_\Lambda f(y)\nu(dy).$$

We have the following result:

Theorem 2.9 *If $f \in L^2(\Lambda, \nu_\Lambda; K)$, and for any $t \ge 0$, $\Lambda \in \mathcal{B}(K - \{0\})$, then*

$$\mathbb{E}\left[\left\| \int_\Lambda f(y)\widetilde{N}(t,dy)\right\|_K^2\right] = t\int_\Lambda \|f(y)\|_K^2 \nu(dy).$$

The proof can be found in Albeverio and Rüdiger (2005), Theorem 3.25. ∎

Theorem 2.10 (Lévy-Itô decomposition) *Let $(Y(t))_{t\ge 0}$ be a Lévy process on $(K, \mathcal{B}(K))$, suppose $\widetilde{N}(t,dy) := N(t,dy) - t\nu(dy)$, then for all $c \in (0,\infty]$, there is $\alpha_c \in K$ such that for all $t \ge 0$,*

$$Y(t) = \alpha_c t + W_Q(t) + \int_{\|y\|_K \le c} y\widetilde{N}(t,dy) + \int_{\|y\|_K \ge c} yN(t,dy), \tag{2.3.1}$$

where W_Q is an K-valued Wiener process which is independent of N.

The proof can be found in Albeverio and Rüdiger (2005), Theorem 4.1. ∎

In (2.3.1),

$$\int_{\|y\|_K \le c} y\widetilde{N}(t,dy) = \lim_{n\to\infty} \int_{\frac{1}{n} < \|y\|_K < c} y\widetilde{N}(t,dy)$$

where the limit is taken in L^2-sense, and it is a square-integrable martingale. The convenient parameter $c \in (0,\infty]$ allows us to specify the 'small' and 'large' jump by $\|y\|_K < c$ and $\|y\|_K \ge c$, respectively. If we want to put 'small' and 'large' jumps on the same footing we let $c = \infty$ or $c \to 0$ so that the term involving 'small' or 'large' jumps is absent in Equation (2.3.1). In many situations, the term in Equation (2.3.1) involving 'large' jumps maybe handled by using an interlacing technique (c.f. Applebaum (2004)). In the remainder of this thesis, for the sake of simplicity, we proceed by omitting this term and concentrate on the study of the equation driven by continuous noise interspersed with 'small' jumps.

Remark 2.1 (Lévy-Khintchine formula) *Let p_t be the law of $Y(t)$ for each $t \geq 0$; then $(p_t, t \geq 0)$ is a weakly continuous convolution semigroup of probability measures on K. We have the Lévy-Khintchine formula (c.f. Protter (2004), Theorem 44) which yields for all $t \geq 0$, $u \in K$,*

$$\mathbb{E}\left(e^{i\langle u, Y(t)\rangle_K}\right) = e^{t\eta(u)},$$

where

$$\eta(u) = i\langle b, u\rangle_K - \frac{1}{2}\langle u, Qu\rangle_K + \int_{K-\{0\}} \left[e^{i\langle u,y\rangle_K} - 1 - i\langle u, y\rangle_K \cdot \mathbf{1}_{\|y\|_K<1}(y)\right]\nu(dy), \tag{2.3.2}$$

where $b \in K$, Q is a positive, self-adjoint, trace class operator on K and ν is a Lévy measure on $K - \{0\}$. Here we use $\mathbf{1}_E$ to denote the characteristic function on set $E \subset K$. We call the triplet (b, Q, ν) the characteristics of the process Y, and the mapping η the characteristic exponent of Y.

We will now study the stochastic integration of predictable processes against compensated Poisson random measures. As usual the stochastic integral is first defined for simple functions by an isometry. These simple functions are dense in a certain space on which the stochastic integral can be defined by L^2-limits.

Definition 2.12 *Let $\mathcal{O} \in \mathcal{B}(K - \{0\})$ with $0 \notin \bar{\mathcal{O}}$, the closure of $\mathcal{O}$ in K and Let $\nu_{\mathcal{O}}$ denote the restriction of the measure ν to $\mathcal{O}$, still denoted by ν, so that ν is finite on $\mathcal{O}$. Fix $0 < T < \infty$ and let $\mathcal{P}$ denote the smallest σ-field on $[0, T] \times \Omega$*

with respect to which all mappings $L : [0, T] \times \mathcal{O} \times \Omega \to H$ satisfying (i) and (ii) are measurable:

(i) for each $0 \le t \le T$ the mapping $(y, \omega) \to L(t, y, \omega)$ is $\mathcal{B}(\mathcal{O}) \times \mathcal{F}_t$-measurable;

(ii) for each $y \in \mathcal{O}$, the mapping $t \to L(t, y, \omega)$ is left-continuous.

We call $\mathcal{P}$ the predictable σ-algebra. A $\mathcal{P}$-measurable mapping $L : [0, T] \times \mathcal{O} \times \Omega$ is then said to be predictable.

Note that, by (i), if L is predictable then the process $t \to L(t, y, \cdot)$ is adapted, for each $y \in \mathcal{O}$. If L satisfies (i) and is left-continuous then it is clearly predictable.

Fix $T > 0$. Let $\mathcal{P}^2([0, T] \times \mathcal{O}; H)$ to be the space of all predictable mappings $L : [0, T] \times \mathcal{O} \times \Omega \to H$ for which

$$\int_0^T \int_{\mathcal{O}} \mathbb{E}\|L(t, y)\|_H^2 \nu(dy) dt < \infty.$$

Define $S(T, \mathcal{O})$ to be the space of all simple processes in $\mathcal{P}^2([0, T] \times \mathcal{O}; H)$. L is simple if, for some $m, n \in \mathbb{N}$, there exists $0 = t_0 < t_1 < t_2 < \cdots < t_{m+1} = T$ and disjoint Borel subsets A_0, A_1, $\cdots$, A_n of $\mathcal{O}$ with each $\nu(A_i) < \infty$ such that

$$L := \sum_{i=0}^{m} \sum_{j=0}^{n} L_{ij} \mathbf{1}_{(t_i, t_{i+1}]} \mathbf{1}_{A_j},$$

where each L_{ij} is a bounded $\mathcal{F}_{t_i}$-measurable random variable. $S(T, \mathcal{O})$ is dense in $\mathcal{P}^2([0, T] \times \mathcal{O}; H)$ (c.f. Applebaum (2004), Section 4.1). To generalize the construction of stochastic integrals with compensated Poisson random measures, for each $L \in S(T, \mathcal{O})$, $0 \le t \le T$ define

$$I_t(L) := \sum_{i=0}^{m} \sum_{j=0}^{n} L_{ij} \widetilde{N}((t_i, t_{i+1}], A_j). \tag{2.3.3}$$

In Applebaum (2006), Section 3.2, it is shown that each $I_t(L)$, given by (2.3.3) extends to an isometry from $\mathcal{P}^2([0, T] \times \mathcal{O}; H)$ to $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ and we write $\int_0^T \int_{\mathcal{O}} L(s, y) \widetilde{N}(ds, dy) := I_t(L)$, for each $0 \le t \le T$, $L \in \mathcal{P}^2([0, T] \times \mathcal{O}; H)$. The process $(I_t, t \ge 0)$ is a square integrable martingale.

Proposition 2.3 *The process $(I_t(L))_{t\geq 0}$ is an H-valued square-integrable centred martingale (martingales with property of having mean zero are said to be centred, c.f. Applebaum (2004), p.73). Furthermore,*

$$\mathbb{E}\left(\left\| \int_0^T \int_{\mathcal{O}} L(t,y)\widetilde{N}(dt,dy) \right\|_H^2\right) = \int_0^T \int_{\mathcal{O}} \mathbb{E}\|L(t,y)\|_H^2 \nu(dy)dt.$$

The proof of can be found in Applebaum (2004), Section 4.2. ■

The following stochastic Fubini theorem which was presented in Applebaum (2006) in a slightly different form is fundamental to our study. Let $\mathcal{P}_T = \mathcal{P}([0,T] \times \Omega)$ denote the predictable σ-field and $(Z, \mathcal{Z}, \mu)$ be a finite measure space. Let $\mathcal{O} \in \mathcal{B}(K - \{0\})$ and $\mathcal{H}_2(T, \mathcal{O}, Z)$ be the real Hilbert space of all $\mathcal{P}_T \times \mathcal{B}(\mathcal{O}) \times \mathcal{Z}$-measurable functions G from $[0,T] \times \Omega \times \mathcal{O} \times Z \to H$ for which

$$\int_Z \int_0^T \int_{\mathcal{O}} \mathbb{E}\|G(s,y,z)\|_H^2 \nu(dy) ds \mu(dz) < \infty.$$

The space $S(T, \mathcal{O}, Z)$ is dense in $\mathcal{H}_2(T, \mathcal{O}, Z)$, where $G \in S(T, \mathcal{O}, Z)$ if

$$G = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \sum_{k=0}^{N_3} G_{ijk} \mathbf{1}_{(t_i, t_{i+1}]} \mathbf{1}_{A_j} \mathbf{1}_{B_k},$$

where N_1, N_2, $N_3 \in \mathbb{N}$, $A_0, \ldots, A_{N_1}$ are disjoint sets in $\mathcal{B}(\mathcal{O})$, $0 = t_0 < t_1 < \cdots < t_{N_1+1} = T$, $B_0, \ldots, B_{N_3}$ is a partition of Z, wherein each $B_k \in \mathcal{Z}$ and each G_{ijk} is a bounded $\mathcal{F}_{t_i}$-measurable random variable with values in H.

Theorem 2.11 (Fubini Theorem) *If $G \in \mathcal{H}_2(T, \mathcal{O}, Z)$, then for each $0 \leq t \leq T$,*

$$\int_Z \left(\int_0^t \int_{\mathcal{O}} G(s,y,z)\widetilde{N}(ds,dy) \right) \mu(dz) = \int_0^t \int_{\mathcal{O}} \left(\int_Z G(s,y,z)\mu(dz) \right) \widetilde{N}(ds,dy)$$

almost surely.

The proof can be found in Applebaum (2006), Theorem 5. ■

We introduce the following infinite dimensional version of the classic Itô's formula from Luo and Liu (2008), which will play a key role in our study.

Let $k > 0$ and $D := D([-k,0];H)$ denote the family of all right-continuous functions with left-hand limits φ from $[-k,0]$ to H. The space $D([-k,0];H)$ is assumed to be equipped with the norm $\|\varphi\|_D = \sup_{-k\le\theta\le 0}\|\varphi(\theta)\|_H$. $D^b_{\mathcal{F}_0}([-k,0];H)$ denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $D([-k,0];H)$-valued random variables. Moreover, $\mathbb{Z} \in \mathcal{B}(K-\{0\})$ with $0 \notin \bar{\mathbb{Z}}$, the closure of $\mathbb{Z}$ in K and $\mathcal{B}(K-\{0\})$ denotes the Borel σ-filed of $K-\{0\}$.

Consider the following semiliner stochastic functional differential equation driven by Lévy process on H: for any $t > 0$,

$$\begin{aligned} X(t) &= X(0) + \int_0^t [AX(s) + F(X_s)]ds + \int_0^t G(X_s)dW_Q(s) \\ &\quad + \int_0^t \int_{\mathbb{Z}} L(X_s, y)\widetilde{N}(ds, dy) \\ X_0(\cdot) &= \xi \in D^b_{\mathcal{F}_0}([-k,0];H), \end{aligned}$$

where A with $\mathcal{D}(A)$ is the infinitesimal generator of C_0-semigroup $T(t)$, $X_t(\theta) := X(t+\theta)$, $\theta \in [-k,0]$. The mappings $F : D([-k,0];H) \to H$, $G : D([-k,H]) \to \mathcal{L}(K,H)$ and $L : D([-k,0];H) \times K \to H$ are properly defined measurable functions such that the associated integrals make sense.

We denote by $C^2(H;\mathbb{R}_+)$ the family of all real-valued nonnegative functions $V(x)$ on H which are continuously twice differentiable with respect to x.

Theorem 2.12 (Itô's formula) *Suppose $V \in C^2(H;\mathbb{R}_+)$. Then for any $\varphi \in D([-k,0];H)$, $Z(t) = V(X(t))$ has the following stochastic differential*

$$\begin{aligned} dZ(t) &= (\mathbf{L}V)(X(t))dt + \langle V'_x(X(t)), G(X(t))dW_Q(t)\rangle_H \\ &\quad + \int_{\mathbb{Z}} [V(X(t) + L(X(t),y)) - V(X(t))]\widetilde{N}(t,dy) \qquad (2.3.4) \end{aligned}$$

where for all $\varphi(0) \in \mathcal{D}(A)$,

$$\begin{aligned} (\mathbf{L}V)(\varphi) &= \langle V'_x(\varphi(0)), A\varphi(0) + F(\varphi)\rangle_H + \frac{1}{2}\ Tr\ [V''_{xx}(\varphi(0))G(\varphi)QG^*(\varphi)] \\ &\quad + \int_{\mathbb{Z}} [V(\varphi(0) + L(\varphi,y)) - V(\varphi(0)) - \langle V'_x(\varphi(0), L(\varphi,y))\rangle_H]\nu(dy). \end{aligned}$$

2.4 Stochastic differential equations

Generally, we are concerned with two ways of giving a rigorous meaning to solution of stochastic differential equations in infinite dimensional spaces, that is, the variational one (c.f. Pardoux (1975)) and the semigroup one (c.f. Da Prato and Zabczyk (1992)). Correspondingly, as in the case of deterministic evolution equations, we have two notions of *strong* and *mild* solutions.

2.4.1 Variational approach and strong solutions

Let H is a Hilbert space with norm $\|\cdot\|_H$ and a corresponding inner product $\langle\cdot,\cdot\rangle_H$. Assume that $V\subset H$ is a linear subspace that is dense in H. Assume that V has its own norm and that V is a real reflexive Banach space (i.e. $V^{**}=(V^*)^*=V$) with $\|\cdot\|_V$. Assume that the injection $V\hookrightarrow H$ is continuous, i.e. $\|v\|_H\le C\|v\|_V$, $\forall v\in V$. [For example, $H=L^2(0,1)$ and $V=L^p(0,1)$ with $p>2$.] There is a canonical map $T:H^*\to V^*$ that is simply the restriction to V of continuous linear functionals φ on H, i.e., $\langle T\varphi,v\rangle_{V,V^*}=\langle\varphi,v\rangle_{H^*,H}$, $\forall v\in V$. Identifying H^* with H and using T as a canonical embedding from H^* into V^*, one usually writes

$$V\hookrightarrow H\equiv H^*\hookrightarrow V^*, \tag{2.4.1}$$

where all the injections are continuous and dense. Note that the inner product $\langle\cdot,\cdot\rangle_{V,V^*}$ and $\langle\cdot,\cdot\rangle_H$ coincide whenever both make sense, i.e., $\langle f,v\rangle_{V,V^*}=\langle f,v\rangle_H$, $\forall f\in H$ and $v\in V$. If V turns out to be a Hilbert space with its inner product $\langle\cdot,\cdot\rangle_V$ associated to the norm $\|\cdot\|_V$, the common habit is to identify H^* with H, to write (2.4.1), and not to identify V^* with V. For more details and an instructive example please refer to Brezis (2011), Page 135-138.

Unless otherwise specified, we always denote by $\|\cdot\|_V$, $\|\cdot\|_H$ and $\|\cdot\|_{V^*}$ the norms in V, H and V^* respectively; by $\langle\cdot,\cdot\rangle_{V,V^*}$ the dual product between V and

V^*, and by $\langle\cdot,\cdot\rangle_H$ the inner product in H. Recall that K is a separable Hilbert space with norm $\|\cdot\|_K$ and we assume $W_Q(t)$, $t\geq 0$, is a K-valued Q-Wiener process with covariance operator $Q\in\mathcal{L}_1(K)$. Here $W_Q(t)$, $t\geq 0$, is supposed to be defined on some complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ with respect to which $\{W_Q(t)\}_{t\geq 0}$ is a continuous martingale.

Consider the following nonlinear stochastic differential equation in V^*:

$$X(t)=X_0+\int_0^t A(s,X(s))ds+\int_0^t B(s,X(s))dW_Q(s), \tag{2.4.2}$$

where $A(t,\cdot):V\to V^*$ and $B(t,\cdot):V\to\mathcal{L}(K,H)$, are two families of measurable nonlinear operators satisfying that $t\in[0,T]\to A(t,x)\in V^*$, $t\in[0,T]\to B(t,x)\in\mathcal{L}(K,H)$ are Lebesgue measurable for any $x\in V$, $T\geq 0$.

Definition 2.13 *For arbitrary given numbers $T\geq 0$, $p>1$ and $X_0\in H$, a stochastic process $X(t)$, $0\leq t\leq T$, is said to be a strong solution of Equation (2.4.2) if the following conditions are satisfied:*

(i) For any $0\leq t\leq T$, $X(t)$ is a V-valued $\mathcal{F}_t$-measurable random variable;

(ii) $X(t)\in M^p(0,T;V)$, where $M^p(0,T;V)$ denotes the space of all V-valued processes $X(t)$, $t\in[0,T]$ which are measurable from $[0,T]\times\Omega$ into V and satisfy

$$\int_0^T \mathbb{E}\|X(t)\|_V^p dt<\infty;$$

(iii) Equation (2.4.2) in V^* is satisfied for every $t\in[0,T]$ with probability one.

In order to obtain the existence and uniqueness of Equation (2.4.2), we shall impose the following assumptions on $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$.

Assumption 2.1 *There exist constants $\alpha>0$, $p>1$ and $\lambda,\gamma\in\mathbb{R}$ such that*

(i) (Coercivity)

$$2\langle v,A(t,v)\rangle_{V,V^*}+\|B(t,v)\|^2_{\mathcal{L}_2^0}\leq -\alpha\|v\|_V^p+\lambda\|v\|_H^2+\gamma,\quad \forall\, v\in V,\quad 0\leq t\leq T,$$

where $\|\cdot\|_{\mathcal{L}_2^0}$ *denotes the Hilbert-Schmidt norm*

$$\|B(t,v)\|_{\mathcal{L}_2^0}^2 = Tr(B(t,v)QB(t,v)^*);$$

(ii) (Growth) There exists a constant $c > 0$ *such that*

$$\|A(t,v)\|_{V^*} \le c(1 + \|v\|_V^{p-1}), \quad \forall\, v \in V, \quad 0 \le t \le T;$$

(iii) (Monotonicity)

$$-\lambda\|u-v\|_H^2 + 2\langle u-v, A(t,u) - A(t,v)\rangle_{V,V^*} \;\le\; \|B(t,u) - B(t,v)\|_{\mathcal{L}_2^0}^2 \qquad \forall\, u,v \in V, \quad 0 \le t \le T;$$

(iv) (Continuity) The map $\theta \in \mathbb{R} \to \langle w, A(t,u+\theta v)\rangle_{V,V^*} \in \mathbb{R}$ *is continuous for arbitrary* u, v, $w \in V$ *and* $0 \le t \le T$*;*

(v) (Lipschitz) There exists constant $L > 0$ *such that*

$$\|B(t,u) - B(t,v)\|_{\mathcal{L}_2^0} \le L\|u-v\|_V \quad \forall\, u,v \in V, \quad 0 \le t \le T.$$

Theorem 2.13 *Under the Assumption 2.1 above, Equation (2.4.2) has a unique* $\{\mathcal{F}_t\}$*-progressively measurable strong solution* $X(t)$, $0 \le t \le T$, *which satisfies:*

$$X(\cdot,\omega) \in M^p(0,T;V) \quad \forall\, T \ge 0,$$

and $X(\cdot,\omega) \in C([0,T];H)$ *almost surely where* $C([0,T];H)$ *denotes the space of all continuous functions from* $[0,T]$ *into* H.

The proof can be found in Pardoux (1975). ■

Theorem 2.14 (Itô's formula) *Let* $X(t) \in M^p(0,T;V)$, $p > 1$, *be a continuous process with values in* V^*. *Suppose there exist* $X_0 \in H$, $\phi(\cdot) \in M^q(0,T;V^*)$,

$1/p + 1/q = 1$, *and* $\Phi(\cdot) \in \mathcal{W}^2([0,T];\mathcal{L}_2^0)$ *such that*

$$X(t) = X_0 + \int_0^t \phi(s)ds + \int_0^t \Phi(s)dW_Q(s), \quad t \in [0,T].$$

Then $X(\cdot) \in C([0,T];H)$ *almost surely and moreover*

$$\begin{aligned} \|X(t)\|_H^2 &= \|X_0\|_H^2 + 2\int_0^t \langle X(s), \phi(s)\rangle_{V,V^*} ds + 2\int_0^t \langle X(s), \Phi(s)dW_Q(s)\rangle_H \\ &\quad + \int_0^t Tr[\Phi(s)Q\Phi(s)^*]ds \end{aligned}$$

for arbitrary $0 \le t \le T$.

c.f. Liu (2006), Theorem 1.3.3. ∎

2.4.2 Semigroup approach and mild solutions

In most situations, one finds that the concept of strong solutions is too limited to include important examples. There is a weaker concept, mild solutions, which is found to be more appropriate for practical purposes.

Consider the following semilinear stochastic differential equation on $[0,T]$, $T \ge 0$,

$$\begin{cases} dX(t) = [AX(t) + F(t, X(t))]dt + G(t, X(t))dW_Q(t), \\ X(0) = X_0 \in H, \end{cases} \tag{2.4.3}$$

where A, generally unbounded, is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \ge 0$, of bounded linear operators on the Hilbert space H.

Assumption 2.2 *The coefficients* $F(\cdot,\cdot)$ *and* $G(\cdot,\cdot)$ *are two nonlinear measurable mappings from* $[0,T] \times H \to H$ *and* $[0,T] \times H \to \mathcal{L}(K,H)$*, respectively, satisfying the following Lipschitz continuity conditions:*

$$\|F(t,x) - F(t,y)\|_H \le \alpha(T)\|x-y\|_H, \quad \alpha(T) > 0, \quad x, y \in H, \quad t \in [0,T],$$

$$\|G(t,x) - G(t,y)\|_{\mathcal{L}_2^0} \le \beta(T)\|x-y\|_H, \quad \beta(T) > 0, \quad x, y \in H, \quad t \in [0,T].$$

Definition 2.14 (Mild solution) *A stochastic process* $X(t)$*,* $t \in [0, T]$*,* $T \geq 0$*, defined on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ *is called a mild solution of Equation (2.4.3) if*

(i) $X(t)$ *is adapted to* $\mathcal{F}_t$*,* $t \geq 0$*;*

(ii) For arbitrary $0 \leq t \leq T$*,* $\mathbb{P}\{\omega : \int_0^t \|X(s, \omega)\|_H^2 ds < \infty\} = 1$ *and*

$$X(t) = T(t)X_0 + \int_0^t T(t-s)F(s, X(s))ds + \int_0^t T(t-s)G(s, X(s))dW_Q(s), \tag{2.4.4}$$

for any $X_0 \in H$ *almost surely.*

Definition 2.15 (Strong solution) *A stochastic process* $X(t)$*,* $t \in [0, T]$*,* $T \geq 0$*, defined on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ *is called a strong solution of Equation (2.4.3) if*

(i) $X(t) \in \mathcal{D}(A)$*,* $0 \leq t \leq T$*, almost surely and is adapted to* $\mathcal{F}_t$*,* $t \in [0, T]$*,* $T \geq 0$*;*

(ii) $X(t)$ *is continuous in* $t \in [0, T]$*,* $T \geq 0$ *almost surely. For arbitrary* $0 \leq t \leq T$*,* $\mathbb{P}\left\{\omega : \int_0^t \|X(s, \omega)\|_H^2 ds < \infty\right\} = 1$ *and*

$$X(t) = X_0 + \int_0^t [AX(s) + F(s, X(s))]ds + \int_0^t G(s, X(s))dW_Q(s),$$

for any $X_0 \in \mathcal{D}(A)$ *almost surely.*

By a straightforward argument, it is possible to establish the following result.

Proposition 2.4 *Assume the Assumption 2.2 holds, then there exists at most one mild solution of Equation (2.4.3). In other words, under the Assumption 2.2 the mild solution of Equation (2.4.3) is unique.*

The proof can be found in Ichikawa (1982), Proposition 2.2 and Theorem 2.1. ■

The following stochastic version of the classic Fubini theorem will be frequently used in the thesis and its proof can be found in Da Prato and Zabczyk (1992).

Proposition 2.5 (Fubini theorem) *Let* $G : [0,T] \times [0,T] \times \Omega \to (\mathcal{L}(K,H), \mathcal{F}(\mathcal{L}(K,H)))$*,* $T \geq 0$*, be strongly measurable in the sense of Section 2.1 such that* $G(s,t)$ *is* $\{\mathcal{F}_t\}$*-measurable for each* $s \geq 0$ *with*

$$\int_0^T \int_0^T \|G(s,t)\|^2 ds dt < \infty \quad a.s.$$

then

$$\int_0^T \int_0^T G(s,t) dW_Q(t) ds = \int_0^T \int_0^T G(s,t) ds dW_Q(t) \quad a.s..$$

The proof can be found in Da Prato and Zabczyk (1992), Theorem 4.18. ■

The following result gives sufficient conditions for a mild solution to be also a strong solution.

Proposition 2.6 *Suppose that the following conditions hold;*

(i) $X_0 \in \mathcal{D}(A)$*,* $T(t-s)F(s,x) \in \mathcal{D}(A)$*,* $T(t-s)G(s,x)k \in \mathcal{D}(A)$ *for each* $x \in H$*,* $k \in K$ *and* $t \geq s$*;*

(ii) $\|AT(t-s)F(s,x)\|_H \leq f(t-s)\|x\|_H, \quad f(\cdot) \in L^1(0,T;\mathbb{R}_+)$*;*

(iii) $\|AT(t-s)G(s,x)\|_{\mathcal{L}_2^0} \leq g(t-s)\|x\|_H, \quad g(\cdot) \in L^2(0,T;\mathbb{R}_+)$.

Then a mild solution $X(t)$*,* $t \in [0,T]$*,* $T \geq 0$ *of Equation (2.4.3) is also a strong solution with* $X(t) \in \mathcal{D}(A)$*,* $t \in [0,T]$*,* $T \geq 0$*, in the sense of Definition 2.15.*

The proof can be found in Luo (2006), Proposition 1.3.5. ■

By the standard Picard iteration procedure or a probabilistic fixed-point theorem type of argument, we can establish an existence theorem for mild solution of Equation (2.4.3) in the following form.

Theorem 2.15 *Suppose Assumption 2.2 holds. Let* $X_0 \in H$ *be an arbitrarily given* $\mathcal{F}_0$*-measurable random variable with* $\mathbb{E}\|X_0\|_H^p < \infty$ *for some integer* $p \geq 2$*. Then there exists a unique mild solution of Equation (2.4.3) in the space* $C(0,T;L^p(\Omega,\mathcal{F},\mathbb{P};H))$*.*

The proof can be found in Ichikawa (1982), Theorem 2.1. ■

As we pointed out in Section 2.2, the stochastic convolution in Equation (2.4.4) is no longer a martingale. A remarkable consequence of this fact is that we cannot employ Itô's formula for mild solutions directly on most occasions of our arguments. We can handle this problem, however, by introducing approximating systems with strong solutions to which Itô's formula can be well applied and by using a limiting argument. In particular, by virtue of Proposition 2.6, we may obtain an approximation result of mild solutions, which will play an important role in the subsequent stability analysis.

Let E be a real or complex Banach space. A linear operator $A : \mathcal{D}(A) \subset E \to E$ is called closed if its graph $\mathscr{G}_A = \{(x, y) \in E \times E : x \in \mathcal{D}(A), Ax = y\}$, is closed in $E \times E$.

Definition 2.16 (Resolvent set and resolvent) *If $A : \mathcal{D}(A) \subset E \to E$ is a closed linear operator on E and I is the identity operator on $\mathcal{D}(A)$, then $\rho(A)$ the resolvent set of A is defined by*

$$\rho(A) = \{\lambda \in \mathbb{C}, \lambda I - A \text{ is one-to-one and onto.}\}$$

If $\lambda \in \rho(A)$, then we set $R(\lambda, A) := (\lambda I - A)^{-1}$, and call $R(\lambda, A)$ the resolvent of A. By the closed graph theorem, $R(\lambda, A)$ is bounded.

To this end, we introduce an approximating system of Equation (2.4.3) as follows:

$$\begin{aligned} dX(t) &= AX(t)dt + R(n)F(t, X(t))dt + R(n)G(t, X(t))dW_Q(t), \\ X(0) &= R(n)X_0, \quad X_0 \in H, \end{aligned} \tag{2.4.5}$$

where $n \in \rho(A)$, the resolvent set of A and $R(n) := nR(n, A)$, $R(n, A)$ is the resolvent of A.

Proposition 2.7 *Let X_0 be an arbitrarily given random variable in H with $\mathbb{E}\|X_0\|_H^p < \infty$ for some integer $p > 2$. Suppose the nonlinear terms $F(\cdot, \cdot)$, $G(\cdot, \cdot)$ in Equation (2.4.3) satisfy the Lipschitz condition (Assumption 2.2). Then, for each $n \in \rho(A)$, the stochastic differential equation (2.4.5) has a unique strong solution $X(t, n) \in \mathcal{D}(A)$, which lies in $L^p(\Omega, \mathcal{F}, \mathbb{P}; C(0, T; H))$ for all $T > 0$ and $p > 2$. Moreover, there exists a subsequence, denoted by $X^n(t)$, such that for arbitrary $T > 0$, $X^n(t) \to X(t)$ almost surely as $n \to \infty$, uniformly with respect to $[0, T]$.*

The proof can be found in Liu (2006), Proposition 1.3.6. ■

It is worth pointing out that, in general, we cannot conclude directly from Theorem 2.15 that the mild solution of Equation (2.4.3) has continuous paths, a fact which makes it justifiable to consider asymptotic stability of its sample paths. However, Proposition 2.7 allows us to have a modification with continuous sample paths of the mild solution of Equation (2.4.3). In particular, unless otherwise stated, we will always suppose the mild solution considered have continuous sample paths in the sequel.

2.5 Notes and remarks

The results of this chapter are not new. For finite dimensional stochastic integration and differential equations, c.f. Applebaum (2004) and Protter (2004). The Hilbert space theory is developed in Da Prato and Zabczyk (1992) (for Wiener processes) and Kallianpur and Xiong (1995), Métivier and Pellaumail (1980) and Peszat and Zabczyk (2007) (for more general processes). Materials in Section 2.1, 2.2 and 2.4 are classical and taken mainly from Da Prato and Zabczyk (1992) and Liu (2006). Materials in Section 2.3 are taken mainly from Albeverio and Rüdiger (2005) and Applebaum (2006).

Chapter 3

Stability in distribution of mild solutions to stochastic delay differential equations with jumps

3.1 Introduction

In the past few decades, stochastic differential equations (SDEs) in a separable Hilbert space have been studied extensively by many researchers. Many quantitative and qualitative results such as the existence, uniqueness, stability and invariant measures have been established. For instance, in their book Da Prato and Zabczyk (Da Prato and Zabczyk (1992)) established a systematic theory of the existence and uniqueness for infinite dimensional systems; the almost sure stability and the mean square stability were considered in Caraballo and Real (1994), Caraballo and Liu (1999), Chow (1982), Govindan (2003, 2005), Haussmann (1978), Ichikawa (1982, 1983), Liu and Mandrekar (1997), Liu and Truman (2002), Liu (2006), Luo and Liu (2008), Taniguchi (2007) and references therein.

Stochastic differential equations are well known to model problems from many areas of science and engineering. Quite often the future state of such systems depends not only the present state but also on its past history (delay), leading to stochastic partial/functional differential equations (SPDEs/SFDEs).

Strictly speaking a delay differential equation is a specific example of a functional differential equation, in which the functional part of the differential equation is the evaluation of a functional on the past of the process. However, we will freely interchange the terminology 'delay differential equation' and 'functional differential equation'.

In particular, stochastic partial differential equations with finite or infinite delays seem very important as models of biological, chemical, physical and economical systems. The corresponding stability properties of such systems have attracted a great deal of attention. For example, Liu and Truman (2002) obtained several criteria for the asymptotic exponential stability of a class of Hilbert space-valued, non-autonomous stochastic evolution equations with variable delays by introducing a proper approximating strong solution system. Caraballo and Liu (1999) and Govindan (2003) studied a semilinear stochastic partial differential equation with variable delays. The former gave sufficient conditions for the exponential stability in the p-th mean of mild solutions by using the properties of the stochastic convolution, while the latter obtained the exponential stability in mean and asymptotic stability in probability of its sample paths by employing a comparison principle. Taniguchi (2007) and Wan and Duan (2008) considered the almost sure exponential stability of the energy solutions to the non-linear stochastic functional partial differential equation with finite delays by energy equality method.

However, by the development of practical needs, stochastic partial differential equations driven by jump processes began to draw attentions. For example, there exists an extensive literature dealing with stochastic differential equations with discontinuous paths incurred by Lévy process, for instance, c.f. monographs Applebaum (2004), Bertoin (1996), Protter (2004) and references therein. These equations are used as in the study of queues, insurance risks, dams and more recently in mathematical finance. On the other hand, some recent research in automatic control such as Boukas and Liu (2002) and Ji and Chizeck (1990) have been devoted to stochastic differential equations with Markovian jumps. As a popular and important topic, the stability property of stochastic differential equations has always lain at the center of our understanding concerning stochastic models described by these equations. Dong and Xu (2007) proved the global existence and uniqueness of the strong, weak and mild solutions and the existence of invariant measure for one-dimensional Burgers equation in $[0, 1]$ with a random perturbation of the body forces in the form of Poisson and Brownian motion. Later the uniqueness of invariant measure is given in Dong (2008). Röckner and Zhang (2007) established the existence and uniqueness for solutions of stochastic evolution equations driven both by Brownian motion and by Poisson point processes via successive approximations. In addition, a large deviation principle is obtained for stochastic evolution equations driven by additive Lévy noise. Svishchuk and Kazmerchuk (2002) made a first attempt to study the pth-moment exponential stability of solutions of linear Itô stochastic delay differential equations associated with Poisson jumps and Markovian switching, which was motivated by some practical applications in mathematical finance. Quite recently, Luo and Liu (2008) considered a strong solutions approximation approach for mild solutions of stochastic functional differential equations with Markovian switching driven by Lévy martingales in Hilbert space. In addition, the sufficient conditions for the moment exponential stability and almost sure exponential stability of equations have been established by the Razumikhin-Lyapunov type function methods and comparison principles.

As far as we notices, most of these papers above are concerned with the stability of the trivial solution either in probability or moment. i.e. the solution will tend to zero in probability or in moment. However, such stability is sometimes too strong while in many practical situations it is useful to know whether or not the probability distribution of the solution will converge to some distribution but not necessarily to zero. The classical example is the Ornstein-Uhlenbeck process

$$dX(t) = -\alpha X(t)dt + \sigma dB(t),$$

where the distribution of the solution $X(t)$ will converge to the normal distribution $N(0, \sigma^2/2\alpha)$. Gushchin and Küchler (2000) discussed the stochastic differential delay equation

$$dX(t) = \left(\int_{[-\tau,0]} X(t+s)\nu(ds) \right) dt + dB(t)$$

on $t \geq 0$ with initial data $X_0 = \xi \in C([-\tau, 0]; \mathbb{R})$, where ν is a finite measure on a finite interval $[-\tau, 0]$, $\tau \geq 0$. Introduce the delay equation

$$\dot{x}(t) = \int_{[-\tau,0]} x(t+s)\nu(ds)$$

on $t \geq 0$ with initial data $x(0) = 1$ and $x(u) = 0$, $u \in [-\tau, 0]$. Gushchin and Küchler (2000) showed that the distribution of $X(t)$ converges to $N(0, \int_0^\infty x^2(s)ds)$ as $t \to \infty$ if and only if $\int_0^\infty x^2(s)ds < \infty$. The convergence described in these examples is called stability in distribution and the limit distribution is known as the stationary distribution. For the finite dimensional case, Basak et al. (1996) discussed such stability for a semi-linear stochastic differential equation with Markovian switching of the form

$$dX(t) = A(r(t))X(t)dt + \sigma(X(t), r(t))dW(t),$$

where $r(t)$ is a continuous time Markov chain taking values in a finite set $\{1, 2, \cdots, N\}$, $A(i)$, $i = 1, 2, \cdots, N$, are $d \times d$ matrices. Also, $\sigma(\cdot, \cdot)$ is $d \times d$ matrix and $W(\cdot)$ is a standard d-dimensional Brownian motion. Later, by using the Lyapunov function methods, Yuan and Mao (2003) generalized the results from Basak et al. (1996) to a nonlinear stochastic differential equation with Markovian switching

$$dX(t) = f(X(t), r(t))dt + g(X(t), r(t))dB(t),$$

where $r(t)$ is a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ and $B(t)$ is an m-dimensional Brownian motion. $f : \mathbb{R}^n \times S \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times S \to \mathbb{R}^{n \times m}$. On the other hand, Yuan et al. (2003) investigated the stability in distribution for a more general stochastic delay differential equation

$$dX(t) = f(X(t), X(t-\tau), r(t))dt + g(X(t), X(t-\tau), r(t))dB(t),$$

where $\tau > 0$, $f : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^{n \times m}$. Moreover, Bao et al. (2009a) derived sufficient conditions for stability in distribution and generalized some results of Basak et al. (1996) and Yuan et al. (2003) to cover a class of much more general neutral stochastic differential delay equations with Markovian switching

$$d[X(t) - G(X(t-\tau))] = f(X(t), X(t-\tau), r(t))dt + g(X(t), X(t-\tau), r(t))dB(t),$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^{n \times m}$.

For infinite dimensional case, Bao et al. (2010) investigated the following semi-linear stochastic partial differential equations in a separable Hilbert space H:

$$dX(t) = [AX(t) + FX(t)]dt + G(X(t))dW(t),$$

where $W(t)$ is a Hilbert space valued Wiener process, A is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$. $F : H \to H$ and $G : H \to \mathcal{L}(K, H)$. By introducing a suitable metric between the transition probability functions of mild solutions, they derived sufficient conditions for stability in distribution of mild solutions. While, Bao et al. (2009b) also generalized the stability in distribution results to a stochastic partial differential delay equation with jumps.

In this chapter we will be interested in the stability in distribution property with infinite dimensional stochastic delay differential equations in Hilbert spaces of the form:

$$\begin{aligned} dX(t) &= \Big[AX(t) + F\Big(\int_{-r}^{0} X(t+\theta)d\theta\Big)\Big]dt + G\Big(\int_{-r}^{0} X(t+\theta)d\theta\Big)dW_Q(t) \\ &\quad + \int_{\mathbb{Z}} L\Big(\int_{-r}^{0} X(t+\theta)d\theta, u\Big)\widetilde{N}(dt, du) \end{aligned} \tag{3.1.1}$$

See Section 3.2 for details of the equation.

It is worth mentioning that in comparison with stochastic stability in finite dimensions, the theory for infinite dimensions case is much more complicated. The standard solution (strong solution) concept turns out to be too strong for most stochastic partial differential equations in which we are especially interested. Therefore, as a natural generalization of this aspect, mild solution (c.f. Da Prato and Zabczyk (1992) for its definition and relevant properties) is introduced which is more useful from both practical and theoretical purposes. For the model we considered, since the stochastic convolution is no longer a martingale, therefore for the treatment of mild solutions, a lot of standard tools in stochastic calculus like Itô's formula or Doob's theorem could not be employed any longer or directly in most of arguments. To overcome this difficulty, a version of a Burkholder type of inequality for stochastic convolution driven by a compensated Poisson random measures was formulated for our stability purpose. Another difficulty was that the approaches by the traditional Lyapunov functions in finite dimensions (c.f. Yuan et al. (2003)) are not available to deal with the stability in distribution of mild solutions to our model. The key solution to such difficulty was to introduce a proper approximating strong solution system and carry out a limiting type of argument to pass on stability of strong solutions to mild ones. Moreover, a suitable metric between the transition probability functions of mild solutions was constructed to give sufficient conditions for stability in distribution of mild solutions. Besides, comparing to the model in Bao et al. (2009b), we are interested in studying a different form of delay term, i.e. continuous time delay, which is more realistic in practical needs. In addition, we improved the sufficient conditions of the stability in distribution of mild solutions in Bao et al. (2009b) by improving the estimations of Burkholder type of inequality for stochastic convolution driven by a compensated Poisson random measures.

The structure of rest chapter is organized as follows. In Section 3.2, we shall recall some basic definitions and preliminary results of stochastic delay differential equations with jumps. We then derive the sufficient conditions for existence and uniqueness of mild solutions in Section 3.3 and construct a proper approximating strong solution system in Section 3.4. In Section 3.5 we investigate the stability in distribution of the studied equations. Finally, we will give an illustrative example to demonstrate the applicability of our theory in Section 3.6.

3.2 Stochastic delay differential equations with jumps

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let H, K be two real separable Hilbert spaces and we denote by $\langle\cdot,\cdot\rangle_H$, $\langle\cdot,\cdot\rangle_K$ their inner products and by $\|\cdot\|_H$, $\|\cdot\|_K$ their norms, respectively. We denote by $\mathcal{L}(K,H)$ the set of all linear bounded operators from K into H, which is equipped with the usual operator norm $\|\cdot\|$. Let $r>0$ and $D := D([-r,0];H)$ denote the family of all right-continuous functions with left-hand limits φ from $[-r,0]$ to H. The space $D([-r,0];H)$ is assumed to be equipped with the norm $\|\varphi\|_D = \sup_{-r\leq\theta\leq 0}\|\varphi(\theta)\|_H$. $D^b_{\mathcal{F}_0}([-r,0];H)$ denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, D-valued random variable. For all $t\geq 0$, $X_t = \{X(t+\theta) : -r\leq\theta\leq 0\}$ is regarded as a D-valued stochastic process.

We recall two kinds of stochastic integrals which appear in our studied equations (c.f. Section 2.2 and Section 2.3).

1. The stochastic integral with respect to Wiener process $\{W_Q(t)\}_{t\geq 0}$:
   Let $\Phi : (0,\infty) \to \mathcal{L}_2^0$ be a predictable, $\mathcal{F}_t$-adapted process such that $\int_0^t \mathbb{E}\|\Phi(s)\|_{\mathcal{L}_2^0}^2 < \infty$, $\forall\, t > 0$. Then, we can define the H-valued stochastic integral $\int_0^t \Phi(s)dW_Q(s)$.

2. The stochastic integral with respect to compensated Poisson random measures $\widetilde{N}(dt,du)$: Let $p = (p(t)), t \in D_p$, be a stationary $\mathcal{F}_t$-Poisson point process with characteristic measure λ, where D_p is a countable subset of $(0,\infty)$. Denoted by $N(dt,du)$ the Poisson counting measure associated with p, i.e. $N(t,\mathbb{Z}) = \sum_{s\in D_p, s\leq t} \mathbf{1}_{\mathbb{Z}}(p(s))$, where $\mathbb{Z} \in \mathcal{B}(K-\{0\})$ with $0 \notin \bar{\mathbb{Z}}$, the closure of $\mathbb{Z}$ in K and $\mathcal{B}(K-\{0\})$ denotes the Borel σ-filed of $K-\{0\}$, c.f. Section 2.3, Page 29. Let $\widetilde{N}(dt,du) := N(dt,du) - dt\lambda(du)$ be the compensated Poisson measure that is independent of $W_Q(t)$. Denoted by $\mathcal{P}^2([0,T]\times\mathbb{Z};H)$ the space of all predictable mappings $L : [0,T]\times\mathbb{Z}\times\Omega \to H$ for which $\int_0^T \int_{\mathbb{Z}} \mathbb{E}\|L(t,u)\|_H^2 dt\lambda(du) < \infty$. We may then define the H-valued stochastic integral $\int_0^T \int_{\mathbb{Z}} L(t,u)\widetilde{N}(dt,du)$.

Let $T(t)$, $t \geq 0$, be some C_0-semigroup of bounded linear operator over H which has its infinitesimal generator A, generally unbounded with domain $\mathcal{D}(A) \subset H$. Consider the following stochastic delay differential equation with jumps: for any $t \in [0,T]$, $T \geq 0$ and arbitrary given $r > 0$,

$$\begin{aligned}
dX(t) &= \Big[AX(t) + F\Big(\int_{-r}^0 X(t+\theta)d\theta\Big)\Big]dt + G\Big(\int_{-r}^0 X(t+\theta)d\theta\Big)dW_Q(t) \\
&\quad + \int_{\mathbb{Z}} L\Big(\int_{-r}^0 X(t+\theta)d\theta, u\Big)\widetilde{N}(dt,du) \\
X(t) &= \xi(t) \in D^b_{\mathcal{F}_0}([-r,0];H), -r \leq t \leq 0, \qquad (3.2.1)
\end{aligned}$$

where the mappings

$$F : H \to H, \quad G : H \to \mathcal{L}(K, H), \quad L : H \times \mathbb{Z} \to H$$

are Borel measurable.

In order to establish the existence and uniqueness of solutions of Equation (3.2.1), we shall impose the following assumptions.

Assumption 3.1 *Assume $A : \mathcal{D}(A) \subset H \to H$ generally unbounded, is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, of contraction.*

The mappings $F : H \to H$, $G : H \to \mathcal{L}(K, H)$ and $L : H \times \mathbb{Z} \to H$ satisfy the following Lipschitz continuity condition and linear growth conditions:

Assumption 3.2 *There exists a constant $K_1 > 0$ such that arbitrary x_1, $x_2 \in H$,*

$$\|F(x_1) - F(x_2)\|_H^2 + \|G(x_1) - G(x_2)\|_{\mathcal{L}_2^0}^2 \leq K_1 \|x_1 - x_2\|_H^2 \tag{3.2.2}$$

and

$$\int_{\mathbb{Z}} \|L(x_1, u) - L(x_2, u)\|_H^2 \lambda(du) \leq K_1 \|x_1 - x_2\|_H^2. \tag{3.2.3}$$

Moreover, there exists a constant $K_2 > 0$ such that for arbitrary $x \in H$,

$$\|F(x)\|_H^2 + \|G(x)\|_{\mathcal{L}_2^0}^2 \leq K_2(1 + \|x\|_H^2), \tag{3.2.4}$$

and

$$\int_{\mathbb{Z}} \|L(x, u)\|_H^2 \lambda(du) \leq K_2(1 + \|x\|_H^2). \tag{3.2.5}$$

For convenience of the reader, we recall two kinds of solutions to Equation (3.2.1) as follows (c.f. Luo and Liu (2008)).

Definition 3.1 (Strong Solution) *A stochastic process $X(t)$, $t \in [0, T]$, $T \geq 0$, is called a strong solution of Equation (3.2.1) if* *(i)* $X(t)$ *is adapted to* $\mathcal{F}_t$ *and has càdlàg path on* $t \geq 0$ *almost surely and*

(ii) $X(t) \in \mathcal{D}(A)$ *on* $[0,T] \times \Omega$ *with* $\int_0^T \|AX(t)\|_H dt < \infty$ *almost surely and for all* $t \in [0,T]$

$$\begin{aligned} X(t) &= \xi(0) + \int_0^t \Big[AX(s) + F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)\Big]ds \\ &\quad + \int_0^t G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s) \\ &\quad + \int_0^t \int_{\mathbb{Z}} L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\widetilde{N}(ds,du), \end{aligned} \tag{3.2.6}$$

for any $X_0(\cdot) = \xi(\cdot) \in D^b_{\mathcal{F}_0}([-r,0];H)$, $-r \leq t \leq 0$.

In general, this concept is rather strong and a weaker one described below is more appropriate for practical purposes.

Definition 3.2 (Mild Solution) *A stochastic process* $X(t)$, $t \in [0,T]$, $T \geq 0$, *is called a mild solution of Equation (3.2.1) if*

(i) $X(t)$ *is adapted to* $\mathcal{F}_t$ *and has càdlàg path on* $t \geq 0$ *almost surely and*

(ii) for arbitrary $t \in [0,T]$, $\mathbb{P}\{\omega : \int_0^t \|X(t)\|_H^2 ds < \infty\} = 1$ *and almost surely*

$$\begin{aligned} X(t) &= T(t)\xi(0) + \int_0^t T(t-s)F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds \\ &\quad + \int_0^t T(t-s)G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s) \\ &\quad + \int_0^t \int_{\mathbb{Z}} T(t-s)L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\widetilde{N}(ds,du) \end{aligned} \tag{3.2.7}$$

for any $X_0(\cdot) = \xi(\cdot) \in D^b_{\mathcal{F}_0}([-r,0];H)$, $-r \leq t \leq 0$.

As a direct application of the properties of semigroup theory, it may be easily proved that:

Proposition 3.1 *For arbitrary* $\xi(\cdot) \in D^b_{\mathcal{F}_0}([-r,0];H)$ *with* $\xi(\theta) \in \mathcal{D}(A)$, $\theta \in [-r,0]$, *assume that* $X(t) \in \mathcal{D}(A)$, $t \in [0,T]$, $T \geq 0$, *is a strong solution of Equation (3.2.1); then it is also a mild solution of Equation (3.2.1).*

Proof. Let $v(s,h) = T(t-s)h$ for $h \in H$ and $0 \le s \le t \le T$, then we have $v_s'(s,h) = -T(t-s)Ah$, $v_h'(s,h) = T(t-s)$ and $v_h''(s,h) = 0$. By using Itô's formula, it can be deduced that for any $0 \le t \le T$,

$$\begin{aligned} v(t, X(t)) - v(0, \xi(0)) &= X(t) - T(t)\xi(0) \\ &= \int_0^t (-T(t-s)AX(s))ds \\ &\quad + \int_0^t T(t-s)\Big[AX(s) + F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)\Big]ds \\ &\quad + \int_0^t T(t-s)G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s) \\ &\quad + \int_0^t \int_{\mathbb{Z}} T(t-s)L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\widetilde{N}(ds,du), \end{aligned}$$

that is,

$$\begin{aligned} X(t) &= T(t)\xi(0) + \int_0^t T(t-s)F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds \\ &\quad + \int_0^t T(t-s)G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s) \\ &\quad + \int_0^t \int_{\mathbb{Z}} T(t-s)L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\widetilde{N}(ds,du) \end{aligned}$$

■

Note that the converse statement of Proposition 3.1 is generally not true, i.e. a mild solution of Equation (3.2.1) is not necessarily a strong one. The following result gives sufficient conditions for a mild solution to be also a strong solution, which is quite useful in our stability analysis.

Proposition 3.2 *Suppose that the following conditions hold: for arbitrary $x \in H$, $t \ge 0$,*

(i) $\xi(\cdot) \in D^b_{\mathcal{F}_0}([-r,0];H)$ *with* $\xi(\theta) \in \mathcal{D}(A)$ *for any* $\theta \in [-r,0]$*;*

(ii) $T(t)F(x) \in \mathcal{D}(A)$, $T(t)G(x)k \in \mathcal{D}(A)$, $T(t)L(x,u) \in \mathcal{D}(A)$ *for any* $k \in K$*, and* $u \in K$*;*

(iii) $\|AT(t)F(x)\|_H \le z_1(t)\|x\|_H, \quad z_1(\cdot) \in L^1(0,T;\mathbb{R}_+)$*;*

(iv) $\|AT(t)G(x)\|^2_{\mathcal{L}_2^0} \le z_2(t)\|x\|^2_H, \quad z_2(\cdot) \in L^2(0,T;\mathbb{R}_+)$*;*

(v) $\int_{\mathbb{Z}} \|AT(t)L(x,u)\|^2_H \lambda(du) \le z_3(t)\|x\|^2_H, \quad z_3(\cdot) \in L^2(0,T;\mathbb{R}_+)$*.*

Then a mild solution $X(t)$, $t \in [0,T]$ of Equation (3.2.1) with initial datum $\xi \in D^b_{\mathcal{F}_0}([-r,0];H)$ is also a strong solution such that $X(t) \in \mathcal{D}(A)$, $t \in [0,T]$, almost surely.

Proof. It suffices to prove that the mild solution $X(t)$, $t \in [0,T]$, takes values in $\mathcal{D}(A)$ and satisfies (3.2.6). By the above conditions, we have almost surely

$$\int_0^T \int_0^t \Big\| AT(t-s)F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)\Big\|_H dsdt < \infty,$$

$$\begin{aligned}\int_0^T \int_0^t \operatorname{tr}\Big[\Big(AT(t-s)G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)\Big) \times \\ Q\Big(AT(t-s)G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)\Big)^*\Big] dsdt < \infty,\end{aligned}$$

and

$$\int_0^T \int_0^t \int_{\mathbb{Z}} \Big\| AT(t-s)L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\Big\|_H^2 \lambda(du)dsdt < \infty.$$

Thus by Fubini theorem and property of semigroup, we have

$$\begin{aligned} & \int_0^t \int_0^v AT(v-s)F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dsdv \\ = & \int_0^t \int_s^t AT(v-s)F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dvds \\ = & \int_0^t T(t-s)F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds - \int_0^t F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds. \end{aligned}$$

On the other hand, by the Fubini type of theorems for Q-Wiener processes (c.f. Proposition 2.5) and for compensated Poisson random measures (c.f. Theorem 2.11), we have

$$
\begin{aligned}
&\int_0^t \int_0^v AT(v-s)G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s)dv \\
= &\int_0^t \int_s^t AT(v-s)G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dvdW_Q(s) \\
= &\int_0^t T(t-s)G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s) - \int_0^t G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s).
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \int_0^v \int_{\mathbb{Z}} AT(v-s)L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\widetilde{N}(ds,du)dv \\
= &\int_0^t \int_s^t \int_{\mathbb{Z}} AT(v-s)L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)dv\widetilde{N}(ds,du) \\
= &\int_0^t \int_{\mathbb{Z}} T(v-s)L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\widetilde{N}(ds,du) \\
&\quad -\int_0^t \int_{\mathbb{Z}} L\Big(\int_{-r}^0 X(v+\theta)d\theta, u\Big)\widetilde{N}(ds,du).
\end{aligned}
$$

Hence $AX(t)$ is integrable almost surely and

$$
\begin{aligned}
&\int_0^t AX(v)dv \\
= \ &T(t)\xi(0) - \xi(0) + \int_0^t T(t-s)F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds \\
&\quad -\int_0^t F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds + \int_0^t T(t-s)G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s) \\
&\quad -\int_0^t G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s) \\
&\quad +\int_0^t \int_{\mathbb{Z}} T(t-s)L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\widetilde{N}(ds,du) \\
&\quad -\int_0^t \int_{\mathbb{Z}} L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\widetilde{N}(ds,du) \\
= \ &X(t) - \xi(0) - \int_0^t F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds - \int_0^t G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s) \\
&\quad -\int_0^t \int_{\mathbb{Z}} L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\widetilde{N}(ds,du).
\end{aligned}
$$

In other words, $X(t) \in \mathcal{D}(A)$, $t \in [0,T]$ is a strong solution of Equation (3.2.1).

■

At the moment, we assume that $A : \mathcal{D}(A) \subset H \to H$, generally unbounded, is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$ on H satisfying $\|T(t)\| \leq e^{\alpha t}$ from some number $\alpha \in \mathbb{R}$. It is well known that (c.f. Liu (2006), Proposition 2.1.4) in this case, there exists a positive constant α such that

$$\langle Ax, x\rangle_H \leq \alpha \|x\|_H^2. \tag{3.2.8}$$

for any $x \in \mathcal{D}(A)$. We are also interested in the following stochastic convolution with respect to compensated Poisson random measures.

$$W_T^{\widetilde{N}}(t) = \int_0^t \int_{\mathbb{Z}} T(t-s)L(s,u)\widetilde{N}(ds,du),$$

defined for any fixed $t \in [0,T]$. In particular, we establish below a special case of Burkholder type of inequality for stochastic convolutions driven by the compensated Poisson random measures $\widetilde{N}(\cdot,\cdot)$.

Lemma 3.1 *Let $\alpha > 0$, $T > 0$ and assume (3.2.3) and (3.2.8) hold. Then for any $L(\cdot,\cdot) \in \mathcal{P}^2([0,T]\times\mathbb{Z};H)$, there exits some positive constant $C_{\alpha,T}$, dependent on α and T, such that*

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big\|\int_0^t \int_{\mathbb{Z}} T(t-s)L(s,u)\widetilde{N}(ds,du)\Big\|_H^2\Big] \\ \leq\ & C_{\alpha,T}\,\mathbb{E}\int_0^T \int_{\mathbb{Z}} \|L(s,u)\|_H^2 \lambda(du)ds. \end{aligned} \tag{3.2.9}$$

Proof. **Step 1** Firstly, we introduce the following approximating systems:

$$dX(t) = AX(t) + \int_{\mathbb{Z}} R(n)L(t,u)\widetilde{N}(dt,du), \quad 0 \leq t \leq T \tag{3.2.10}$$

with initial value $X(0) = 0$, where $n \in \rho(A)$, the resolvent set of A and $R(n) = nR(n,A)$, $R(n,A)$ is the resolvent of A.

Taking (3.2.3) into account, it can be derived that Equation (3.2.10) admits a unique mild solution $X(t)$, $0 \leq t \leq T$.

By Pazy (1983), Theorem 2.4 that if $L(\cdot,\cdot) \in \mathcal{P}^2([0,T]\times\mathbb{Z};\mathcal{D}(A))$, then for any $0 \leq s \leq t \leq T$, we have $AT(t-s)R(n)L(\cdot,\cdot) \in \mathcal{D}(A)$. Hence, we can show that the mild solution $X(t)$, $0 \leq t \leq T$, is indeed a strong one on $[0, T]$. From Proposition 3.2, we know that the process $X(t)$ also satisfies that for any $t \in [0, T]$,

$$X(t) = \int_0^t AX(s)ds + \int_0^t \int_{\mathbb{Z}} L(s, u)\widetilde{N}(ds, du).$$

Secondly, applying Itô's formula to function $\|x\|_H^2$ and strong solution $X(t)$, $0 \leq t \leq T$, of Equation (3.2.10) yields that

$$\begin{aligned} \|X(t)\|_H^2 &= 2\int_0^t \langle AX(s), X(s)\rangle_H ds + \int_0^t \int_{\mathbb{Z}} \|L(s, u)\|_H^2 \lambda(du)ds \\ &\quad + \int_0^t \int_{\mathbb{Z}} [2\langle X(s-), L(s, u)\rangle_H + \|L(s, u)\|_H^2]\widetilde{N}(ds, du). \end{aligned} \tag{3.2.11}$$

Let us denote

$$M(t) := \int_0^t \int_{\mathbb{Z}} [2\langle X(s-), L(s, u)\rangle_H + \|L(s, u)\|_H^2]\widetilde{N}(ds, du)$$

and $[M(t)]$ the corresponding quadratic variation.

By Burkholder-Davis-Gundy inequality (c.f. Protter (2004), Theorem 48), it follows that there is a constant $\overline{C} > 0$ such that

$$\mathbb{E}\left(\sup_{0\leq s\leq t} \|M(s)\|_H^2\right) \leq \overline{C}\,\mathbb{E}[M(t)]^{\frac{1}{2}}. \tag{3.2.12}$$

In what follows, we compute

$$\begin{aligned} [M(t)]^{\frac{1}{2}} &= \left\{ \sum_{s\in D_p, 0\leq s\leq t} (2\langle X(s), L(s, p(s))\rangle_H + \|L(s, p(s))\|_H^2)^2 \right\}^{\frac{1}{2}} \\ &\leq \sqrt{2}\left\{ \sum_{s\in D_p, 0\leq s\leq t} \|L(s, p(s))\|_H^4 \right\}^{\frac{1}{2}} \\ &\quad + 2\sqrt{2}\left\{ \sum_{s\in D_p, 0\leq s\leq t} \|X(s)\|_H^2 \|L(s, p(s))\|_H^2 \right\}^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\le \sqrt{2} \sum_{s\in D_p, 0\le s\le t} \|L(s,p(s))\|_H^2 \\
&\qquad + 2\sqrt{2} \sup_{0\le s\le t} \|X(s)\|_H \Big\{ \sum_{s\in D_p, 0\le s\le t} \|L(s,p(s))\|_H^2 \Big\}^{\frac{1}{2}} \\
&\le \frac{1}{2\overline{C}} \sup_{0\le s\le t} \|X(s)\|_H^2 + (\sqrt{2}+4\overline{C}) \sum_{s\in D_p, 0\le s\le t} \|L(s,p(s))\|_H^2,
\end{aligned}
\tag{3.2.13}
$$

where $\overline{C}$ is the positive constant appearing in the right-hand side of Equation (3.2.12).

Consequently,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T} \|X(t)\|_H^2 \Big) \\
&\le 4\alpha \mathbb{E} \int_0^T \|X(t)\|_H^2 dt \\
&\qquad + 2[1+(\sqrt{2}+4\overline{C})\overline{C}]\, \mathbb{E} \int_0^T \int_{\mathbb{Z}} \|L(t,u)\|_H^2 \,\lambda(du) dt \\
&\le 4\alpha \int_0^T \mathbb{E}\Big(\sup_{0\le s\le t} \|X(s)\|_H^2 \Big) dt \\
&\qquad + 2[1+(\sqrt{2}+4\overline{C})\overline{C}]\, \mathbb{E} \int_0^T \int_{\mathbb{Z}} \|L(t,u)\|_H^2 \,\lambda(du) dt
\end{aligned}
\tag{3.2.14}
$$

which, combining with Gronwall's inequality, immediately implies that

$$
\mathbb{E}\Big(\sup_{0\le t\le T} \|X(t)\|_H^2 \Big) \le C_{\alpha,T}\, \mathbb{E} \int_0^T \int_{\mathbb{Z}} \|L(t,u)\|_H^2 \,\lambda(du) dt,
$$

where

$$
C_{\alpha,T} = 2[1+(\sqrt{2}+4\overline{C})\overline{C}] e^{4\alpha T} > 0.
$$

Step 2: By a straightforward application of the dominated convergence theorem, we have $L_n = nR(n,A)L(t,u) \to L(\cdot,\cdot)$ in $\mathcal{P}^2([0,T]\times\mathbb{Z}; H)$, as $n\to\infty$. Moreover, by Proposition 3.2, we have that $L_n \in \mathcal{P}^2([0,T]\times\mathbb{Z}; \mathcal{D}(A))$. Defining

$$
X_n(t) = \int_0^t \int_{\mathbb{Z}} T(t-s) L_n(s,u) \widetilde{N}(ds,du),
$$

we have that $X_n(t) \to X(t)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ for any $t \in [0,T]$ as $n \to \infty$ according to Proposition 2.4 in Luo and Liu (2008). On the other hand, we can apply the inequality (3.2.9) to the difference $X_n(t) - X_m(t)$ with $L(\cdot, \cdot)$ replaced by the difference $L_n - L_m$ from which we deduce that

$$\mathbb{E}\Big(\sup_{0 \le s \le t} \|X(s) - X_n(s)\|_H^2 \Big) \to 0, \quad \text{as} \quad n \to \infty,$$

and hence (3.2.9) is true for any $L \in \mathcal{P}^2([0,T] \times \mathbb{Z}; \mathcal{D}(A))$. The proof is now complete.

∎

3.3 Existence and uniqueness

Carrying out a fixed-point theorem type of procedure, we can follow the arguments in Ichikawa (1982) to establish an existence and uniqueness theorem for mild solutions of Equation (3.2.1).

Theorem 3.1 *Assume the conditions in Assumption 3.1 and 3.2 hold and let $\xi \in D^b_{\mathcal{F}_0}([-r,0]; H)$ be an arbitrarily given initial datum. Then there exists a unique mild solution of Equation (3.2.1).*

Proof. Denote by $\mathcal{H}_2$ the Banach space of all $\mathcal{F}$-adapted processes $Y(t,\omega)$: $[-r,T] \to H$, which are almost surely right-continuous functions with left-hand limits in t for fixed $\omega \in \Omega$ with $\|Y\|_2 < \infty$, where

$$\|Y\|_2 := \left(\mathbb{E}\Big[\sup_{0 \le t \le T} \|Y(t)\|_H^2 \Big] \right)^{1/2}. \tag{3.3.1}$$

Moreover, $Y(t,\omega) = \xi(t)$ for $t \in [-r,0]$.

For any $Y \in \mathcal{H}_2$, define a mapping $\mathcal{K}$ on $\mathcal{H}_2$:

$$\begin{aligned}
\mathcal{K}(Y)(t) &= T(t)\xi(0) + \int_0^t T(t-s)F\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)ds \\
&\quad + \int_0^t T(t-s)G\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)dW_Q(s) \\
&\quad + \int_0^t \int_{\mathbb{Z}} T(t-s)L\Big(\int_{-r}^0 Y(s+\theta)d\theta,\Big)\widetilde{N}(ds,du) \\
&:= \sum_{i=1}^4 I_i(t). \qquad (3.3.2)
\end{aligned}$$

Then we prove the existence of mild solution to Equation (3.2.1) by finding a fixed point for the map $\mathcal{K}$. Next we will show by using Banach fixed point theorem that $\mathcal{K}$ has a unique fixed point. We divide the subsequent proof into three steps.

Step 1. For arbitrary $Y \in \mathcal{H}_2$, $\mathcal{K}(Y)(t)$ is mean square continuous on the interval $[0,T]$. Let $Y \in \mathcal{H}_2$ and $|h|$ be sufficiently small. Then for any fixed $Y \in \mathcal{H}_2$, we have that

$$\mathbb{E}\|\mathcal{K}(Y)(t_1+h) - \mathcal{K}(Y)(t_1)\|_H^2 \le 4\sum_{i=1}^4 \mathbb{E}\|I_i(t_1+h) - I_i(t_1)\|_H^2.$$

By the properties of C_0-semigroup, $T(t+s) = T(t)T(s)$, $t,s \ge 0$, and strong continuity of $T(t)$, it is easily to obtain that

$$\begin{aligned}
\mathbb{E}\|I_1(t_1+h) - I_1(t_1)\|_H^2 &= \mathbb{E}\|T(t_1+h)\xi(0) - T(t_1)\xi(0)\|_H^2 \\
&= \mathbb{E}\|T(t_1)[T(h)-I]\xi(0)\|_H^2 \\
&\to 0 \quad \text{as} \quad h \to 0. \qquad (3.3.3)
\end{aligned}$$

For $i = 2$, by Lebesgue's dominated convergence theorem and the strongly continuity of $T(t)$, it can be shown that

$$\begin{aligned}
&\mathbb{E}\|I_2(t_1+h) - I_2(t_1)\|_H^2 \\
= &\ \mathbb{E}\Big\|\int_0^{t_1+h} T(t_1+h-s)F\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)ds \\
&\quad - \int_0^{t_1} T(t_1-s)F\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)ds\Big\|_H^2
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}\Big\| \int_0^{t_1} T(t_1+h-s)F\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)ds \\
&\qquad + \int_{t_1}^{t_1+h} T(t_1+h-s)F\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)ds \\
&\qquad - \int_0^{t_1} T(t_1-s)F\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)ds\Big\|_H^2 \\
&= \mathbb{E}\Big\| \int_0^{t_1} [T(t_1+h-s)-T(t_1-s)]F\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)ds \\
&\qquad + \int_{t_1}^{t_1+h} T(t_1+h-s)F\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)ds\Big\|_H^2 \\
&\leq 2\mathbb{E}\int_0^{t_1} \Big\|[T(t_1+h-s)-T(t_1-s)]F\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)\Big\|_H^2 ds \\
&\qquad +2\mathbb{E}\int_{t_1}^{t_1+h} \Big\|T(t_1+h-s)F\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)\Big\|_H^2 ds \\
&\to 0 \quad \text{as} \quad h \to 0; \qquad (3.3.4)
\end{aligned}$$

Similarly, for $i = 3$, we deduce that

$$\begin{aligned}
&\mathbb{E}\|I_3(t_1+h) - I_3(t_1)\|_H^2 \\
&= \mathbb{E}\Big\| \int_0^{t_1+h} T(t_1+h-s)G\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)dW_Q(s) \\
&\qquad - \int_0^{t_1} T(t_1-s)G\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)dW_Q(s)\Big\|_H^2 \\
&= \mathbb{E}\Big\| \int_0^{t_1} T(t_1+h-s)G\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)dW_Q(s) \\
&\qquad + \int_{t_1}^{t_1+h} T(t_1+h-s)G\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)dW_Q(s) \\
&\qquad - \int_0^{t_1} T(t_1-s)G\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)dW_Q(s)\Big\|_H^2 \\
&\leq 2\,\mathbb{E}\Big\| \int_0^{t_1} [T(t_1+h-s)-T(t_1-s)]G\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)dW_Q(s)\Big\|_H^2 \\
&\qquad + 2\,\mathbb{E}\Big\| \int_{t_1}^{t_1+h} T(t_1+h-s)G\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)dW_Q(s)\Big\|_H^2 \\
&\leq 2\int_0^{t_1} \mathbb{E}\Big\|[T(t_1+h-s)-T(t_1-s)]G\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)\Big\|_{\mathcal{L}_2^0}^2 ds \\
&\qquad + 2\int_{t_1}^{t_1+h} \mathbb{E}\Big\|T(t_1+h-s)G\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big)\Big\|_{\mathcal{L}_2^0}^2 ds \\
&\to 0 \quad \text{as} \quad h \to 0. \qquad (3.3.5)
\end{aligned}$$

and for $i = 4$, we have that

$$
\begin{aligned}
& \mathbb{E}\|I_4(t_1+h) - I_4(t_1)\|_H^2 \\
= \; & \mathbb{E}\Big\| \int_0^{t_1+h} \int_{\mathbb{Z}} T(t_1+h-s) L\Big(\int_{-r}^0 Y(s+\theta)d\theta, u\Big) \widetilde{N}(ds, du) \\
& \quad - \int_0^{t_1} \int_{\mathbb{Z}} T(t_1-s) L\Big(\int_{-r}^0 Y(s+\theta)d\theta, u\Big) \widetilde{N}(ds, du)\Big\|_H^2 \\
\leq \; & 2\mathbb{E}\Big\| \int_0^{t_1} \int_{\mathbb{Z}} [T(t_1+h-s) - T(t_1-s)] L\Big(\int_{-r}^0 Y(s+\theta)d\theta, u\Big) \widetilde{N}(ds, du)\Big\|_H^2 \\
& \quad + 2\mathbb{E}\Big\| \int_{t_1}^{t_1+h} \int_{\mathbb{Z}} T(t_1+h-s) L\Big(\int_{-r}^0 Y(s+\theta)d\theta, u\Big) \widetilde{N}(ds, du)\Big\|_H^2 \\
= \; & 2\int_0^{t_1} \int_{\mathbb{Z}} \mathbb{E}\Big\| [T(t_1+h-s) - T(t_1-s)] L\Big(\int_{-r}^0 Y(s+\theta)d\theta, u\Big)\Big\|_H^2 \lambda(du)ds \\
& \quad + 2\int_{t_1}^{t_1+h} \int_{\mathbb{Z}} \mathbb{E}\Big\| T(t_1+h-s) L\Big(\int_{-r}^0 Y(s+\theta)d\theta, u\Big)\Big\|_H^2 \lambda(du)ds \\
\to \; & 0 \quad \text{as} \quad h \to 0.
\end{aligned}
\tag{3.3.6}
$$

Therefore,

$$
\mathbb{E}\|I_i(t_1+h) - I_i(t_1)\|_H^2 \to 0, \quad i = 1, 2, 3, 4, \quad \text{as} \quad r \to 0,
$$

which means $\mathcal{K}(Y)(t)$ is mean square continuous on $[0, T]$, $T \geq 0$.

Step 2. We show that $\mathcal{K}$ maps $\mathcal{H}_2$ into $\mathcal{H}_2$. Let $Y \in \mathcal{H}_2$. Then we have that

$$
\begin{aligned}
& \mathbb{E}\|\mathcal{K}(Y)(t)\|_H^2 \\
\leq \; & 4\mathbb{E}\Big[\sup_{0\leq t\leq T} \|T(t)\xi(0)\|_H^2 \Big] \\
& \quad + 4\mathbb{E}\Big[\sup_{0\leq t\leq T} \Big\| \int_0^t T(t-s) F\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big) ds\Big\|_H^2 \Big] \\
& \quad + 4\mathbb{E}\Big[\sup_{0\leq t\leq T} \Big\| \int_0^t T(t-s) G\Big(\int_{-r}^0 Y(s+\theta)d\theta\Big) dW(s)\Big\|_H^2 \Big] \\
& \quad + 4\mathbb{E}\Big[\sup_{0\leq t\leq T} \Big\| \int_0^t \int_{\mathbb{Z}} T(t-s) L\Big(\int_{-r}^0 Y(s+\theta)d\theta, u\Big) \widetilde{N}(ds, du)\Big\|_H^2 \Big] \\
= \; & 4(J_1 + J_2 + J_3 + J_4).
\end{aligned}
\tag{3.3.7}
$$

In view of $\xi \in D^b_{\mathcal{F}_0}([-r,0];H)$, we hence obtain by Assumption 3.1 that

$$J_1 = \mathbb{E}\bigg[\sup_{0\le t\le T}\|T(t)\xi(0)\|_H^2\bigg] \le \mathbb{E}\|\xi\|_D^2 < \infty.$$

By condition (3.2.4) and using the Hölder inequality twice, we get that

$$\begin{aligned}
J_2 &\le \mathbb{E}\bigg[\sup_{0\le t\le T}\bigg(\int_0^t \bigg\|T(t-s)F\bigg(\int_{-r}^0 Y(s+\theta)d\theta\bigg)\bigg\|_H ds\bigg)^2\bigg] \\
&\le \mathbb{E}\bigg[\sup_{0\le t\le T}\bigg(\int_0^t \|T(t-s)\|^2 ds\bigg)\bigg(\int_0^t \bigg\|F\bigg(\int_{-r}^0 Y(s+\theta)d\theta\bigg)\bigg\|_H^2 ds\bigg)\bigg] \\
&\le TK_2\int_0^T \mathbb{E}\bigg[\bigg(1+\bigg\|\int_{-r}^0 Y(s+\theta)d\theta\bigg\|_H^2\bigg)\bigg]ds \\
&\le TK_2\bigg[T+\int_0^T \mathbb{E}\bigg\|\int_{-r}^0 Y(s+\theta)d\theta\bigg\|_H^2 ds\bigg] \\
&\le TK_2\bigg[T+\int_0^T \mathbb{E}\bigg(\int_{-r}^0 \|Y(s+\theta)\cdot 1\|_H d\theta\bigg)^2 ds\bigg] \\
&\le TK_2\bigg[T+\int_0^T \mathbb{E}\bigg(\int_{-r}^0 \|Y(s+\theta)\|_H^2 d\theta\bigg)\bigg(\int_{-r}^0 1^2 d\theta\bigg)ds\bigg] \\
&\le TK_2\bigg[T+r\int_0^T\int_{-r}^0 \mathbb{E}\|Y(s+\theta)\|_H^2 d\theta ds\bigg] \\
&\le TK_2\bigg[T+r\int_0^{(T+\theta)}\int_{-r}^0 \mathbb{E}\|Y(u)\|_H^2 d\theta du\bigg] \\
&\le TK_2\bigg[T+r\bigg(\int_{-r}^0\int_{-r}^0 \mathbb{E}\|Y(u)\|_H^2 d\theta dt + \int_0^T\int_{-r}^0 \mathbb{E}\|Y(u)\|_H^2 d\theta du\bigg)\bigg] \\
&\le TK_2\bigg[T+r\bigg(\int_{-r}^0\int_{-r}^0 \mathbb{E}\bigg[\sup_{-r\le u\le 0}\|Y(u)\|_H^2\bigg]d\theta dt \\
&\qquad\qquad + \int_0^T\int_{-r}^0 \mathbb{E}\bigg[\sup_{0\le u\le T}\|Y(u)\|_H^2\bigg]d\theta du\bigg)\bigg] \\
&\le TK_2[T+r(r^2\|\xi\|_D^2 + rT\|Y\|_2^2)] \\
&\le TK_2[T+r^3\|\xi\|_D^2 + r^2T\|Y\|_2^2].
\end{aligned}$$

While, by the virtue of Theorem 2.7 Burkholder-Davis-Gundy type of inequality for stochastic convolutions and condition (3.2.4), we obtain that there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
J_3 &\leq C_1 \mathbb{E}\Big(\int_0^T \Big\| G\Big(\int_{-r}^0 Y(s+\theta) d\theta \Big) \Big\|_H^2 ds \Big) \\
&\leq C_1 K_2 \int_0^T \mathbb{E}\Big(1 + \Big\| \int_{-r}^0 Y(s+\theta) d\theta \Big\|_H^2 ds \Big) \\
&\leq C_1 K_2 \Big(T + \int_0^T \mathbb{E} \Big\| \int_{-r}^0 Y(s+\theta) d\theta \Big\|_H^2 ds \Big) \\
&\leq C_1 K_2 \Big[T + \int_0^T \mathbb{E} \Big(\int_{-r}^0 \|Y(s+\theta) \cdot 1\|_H d\theta \Big)^2 ds \Big] \\
&\leq C_1 K_2 \Big[T + \int_0^T \mathbb{E} \Big(\int_{-r}^0 \|Y(s+\theta)\|_H^2 d\theta \Big) \Big(\int_{-r}^0 1^2 d\theta \Big) ds \Big] \\
&\leq C_1 K_2 \Big[T + r \int_0^T \int_{-r}^0 \mathbb{E} \|Y(s+\theta)\|_H^2 d\theta ds \Big] \\
&\leq C_1 K_2 \Big[T + r \int_0^{(T+\theta)} \int_{-r}^0 \mathbb{E} \|Y(u)\|_H^2 d\theta du \Big] \\
&\leq C_1 K_2 \Big[T + r \Big(\int_{-r}^0 \int_{-r}^0 \mathbb{E} \|Y(u)\|_H^2 d\theta dt + \int_0^T \int_{-r}^0 \mathbb{E} \|Y(u)\|_H^2 d\theta du \Big) \Big] \\
&\leq C_1 K_2 \Big[T + r \Big(\int_{-r}^0 \int_{-r}^0 \mathbb{E} \Big[\sup_{-r \leq u \leq 0} \|Y(u)\|_H^2 \Big] d\theta dt \\
&\qquad\qquad + \int_0^T \int_{-r}^0 \mathbb{E} \Big[\sup_{0 \leq u \leq T} \|Y(u)\|_H^2 \Big] d\theta du \Big) \Big] \\
&\leq C_1 K_2 [T + r(r^2 \|\xi\|_D^2 + rT \|Y\|_2^2)] \\
&\leq C_1 K_2 (T + r^3 \mathbb{E} \|\xi\|_D^2 + r^2 T \|Y\|_2^2).
\end{aligned}$$

Similarly, by Lemma 3.1 and condition (3.2.5), there exists a constant $C_2 > 0$ satisfying

$$\begin{aligned}
J_4 &\leq C_2 \mathbb{E}\Big(\int_0^T \int_{\mathbb{Z}} \Big\| L\Big(\int_{-r}^0 Y(s+\theta) d\theta, u \Big) \Big\|_H^2 \lambda(du) ds \Big) \\
&\leq C_2 K_2 \int_0^T \mathbb{E}\Big(1 + \Big\| \int_{-r}^0 Y(s+\theta) d\theta \Big\|_H^2 ds \Big) \\
&\leq C_2 K_2 \Big(T + \int_0^T \mathbb{E} \Big\| \int_{-r}^0 Y(s+\theta) d\theta \Big\|_H^2 ds \Big) \\
&\leq C_2 K_2 \Big[T + \int_0^T \mathbb{E} \Big(\int_{-r}^0 \|Y(s+\theta) \cdot 1\|_H d\theta \Big)^2 ds \Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq C_2K_2\Big[T+\int_0^T\mathbb{E}\Big(\int_{-r}^0\|Y(s+\theta)\|_H^2d\theta\Big)\Big(\int_{-r}^0 1^2d\theta\Big)ds\Big]\\
&\leq C_2K_2\Big[T+r\int_0^T\int_{-r}^0\mathbb{E}\|Y(s+\theta)\|_H^2d\theta ds\Big]\\
&\leq C_2K_2\Big[T+r\int_0^{(T+\theta)}\int_{-r}^0\mathbb{E}\|Y(u)\|_H^2d\theta du\Big]\\
&\leq C_2K_2\Big[T+r\Big(\int_{-r}^0\int_{-r}^0\mathbb{E}\|Y(u)\|_H^2d\theta dt+\int_0^T\int_{-r}^0\mathbb{E}\|Y(u)\|_H^2d\theta du\Big)\Big]\\
&\leq C_2K_2\Big[T+r\Big(\int_{-r}^0\int_{-r}^0\mathbb{E}\Big[\sup_{-r\leq u\leq 0}\|Y(u)\|_H^2\Big]d\theta dt\\
&\qquad\qquad+\int_0^T\int_{-r}^0\mathbb{E}\Big[\sup_{0\leq u\leq T}\|Y(u)\|_H^2\Big]d\theta du\Big)\Big]\\
&\leq C_2K_2[T+r(r^2\|\xi\|_D^2+rT\|Y\|_2^2)]\\
&\leq C_2K_2(T+r^3\mathbb{E}\|\xi\|_D^2+r^2T\|Y\|_2^2).
\end{aligned}
$$

In consequence, $\mathcal{K}$ maps $\mathcal{H}_2$ into $\mathcal{H}_2$.

Step 3. It remains to verify that $\mathcal{K}$ is a contraction on $\mathcal{H}_2$. Suppose that Y_1, $Y_2\in\mathcal{H}_2$, then for any fixed $t\in[0,T]$,

$$
\begin{aligned}
&\|\mathcal{K}(Y_1)-\mathcal{K}(Y_2)\|_2^2\\
\leq\ & 3\mathbb{E}\Big\{\sup_{0\leq t\leq T}\Big\|\int_0^t T(t-s)\Big[F\Big(\int_{-r}^0 Y_1(s+\theta)d\theta\Big)-F\Big(\int_{-r}^0 Y_2(s+\theta)d\theta\Big)\Big]ds\Big\|_H^2\Big\}\\
&+3\mathbb{E}\Big\{\sup_{0\leq t\leq T}\Big\|\int_0^t T(t-s)\Big[G\Big(\int_{-r}^0 Y_1(s+\theta)d\theta\Big)\\
&\qquad\qquad -G\Big(\int_{-r}^0 Y_2(s+\theta)d\theta\Big)\Big]dW_Q(s)\Big\|_H^2\Big\}\\
&+3\mathbb{E}\Big\{\sup_{0\leq t\leq T}\Big\|\int_0^t\int_{\mathbb{Z}} T(t-s)\Big[L\Big(\int_{-r}^0 Y_1(s+\theta)d\theta,u\Big)\\
&\qquad\qquad -L\Big(\int_{-r}^0 Y_2(s+\theta)d\theta,u\Big)\Big]\widetilde{N}(ds,du)\Big\|_H^2\Big\}\\
:=\ & S_1+S_2+S_3. \qquad (3.3.8)
\end{aligned}
$$

On one hand, by Hölder inequality and condition (3.2.2), it follows that

$$\begin{aligned}
S_1 &= 3\mathbb{E}\Big\{\sup_{0\le t\le T}\Big\|\int_0^t T(t-s)\Big[F\Big(\int_{-r}^0 Y_1(s+\theta)d\theta\Big)-F\Big(\int_{-r}^0 Y_2(s+\theta)d\theta\Big)\Big]ds\Big\|_H^2\Big\} \\
&\le 3\mathbb{E}\Big[\sup_{0\le t\le T}\Big(\int_0^t \|T(t-s)\|^2 ds\Big) \\
&\qquad \times\Big(\int_0^t \Big\|F\Big(\int_{-r}^0 Y_1(s+\theta)d\theta\Big)-F\Big(\int_{-r}^0 Y_2(s+\theta)d\theta\Big)\Big\|_H^2 ds\Big)\Big] \\
&\le 3TK_1\int_0^T \mathbb{E}\Big\|\int_{-r}^0 Y_1(s+\theta)d\theta-\int_{-r}^0 Y_2(s+\theta)d\theta\Big\|_H^2 ds \\
&\le 3rTK_1\int_0^T \mathbb{E}\int_{-r}^0 \|Y_1(s+\theta)-Y_2(s+\theta)\|_H^2 d\theta ds \\
&\le 3rTK_1\int_0^T\int_{-r}^0 \mathbb{E}\Big(\sup_{0\le u\le T}\|Y_1(u)-Y_2(u)\|_H^2\Big)d\theta ds \\
&\le 3r^2T^2K_1\|Y_1-Y_2\|_2^2. \qquad (3.3.9)
\end{aligned}$$

On the other hand, by using Theorem 2.7 and condition (3.2.2), we have for some constants $C_3>0$,

$$\begin{aligned}
S_2 &= 3\mathbb{E}\Big\{\sup_{0\le t\le T}\Big\|\int_0^t T(t-s)\Big[G\Big(\int_{-r}^0 Y_1(s+\theta)d\theta\Big) \\
&\qquad -G\Big(\int_{-r}^0 Y_2(s+\theta)d\theta\Big)\Big]dW_Q(s)\Big\|_H^2\Big\} \\
&\le 3C_3\mathbb{E}\int_0^T \Big\|G\Big(\int_{-r}^0 Y_1(s+\theta)d\theta\Big)-G\Big(\int_{-r}^0 Y_2(s+\theta)d\theta\Big)\Big\|_H^2 dW_Q(s) \\
&\le 3C_3K_1\int_0^T \mathbb{E}\Big\|\int_{-r}^0 Y_1(s+\theta)d\theta-\int_{-r}^0 Y_2(s+\theta)d\theta\Big\|_H^2 ds \\
&\le 3C_3r^2TK_1\|Y_1-Y_2\|_2^2. \qquad (3.3.10)
\end{aligned}$$

Moreover, from Lemma 3.1 and condition (3.2.3), we can show that for some $C_4>0$,

$$\begin{aligned}
S_3 &= 3\mathbb{E}\Big\{\sup_{0\le t\le T}\Big\|\int_0^t\int_{\mathbb{Z}} T(t-s)\Big[L\Big(\int_{-r}^0 Y_1(s+\theta)d\theta,u\Big) \\
&\qquad -L\Big(\int_{-r}^0 Y_2(s+\theta)d\theta,u\Big)\Big]\widetilde{N}(ds,du)\Big\|_H^2\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\leq 3C_4\mathbb{E}\int_0^T\int_{\mathbb{Z}}\Big\|L\Big(\int_{-r}^0 Y_1(s+\theta)d\theta,u\Big)-L\Big(\int_{-r}^0 Y_2(s+\theta)d\theta,u\Big)\Big\|_H^2\lambda(du)ds\\
&\leq 3C_4K_1\int_0^T\int_{\mathbb{Z}}\mathbb{E}\Big\|\int_{-r}^0 Y_1(s+\theta)d\theta-\int_{-r}^0 Y_2(s+\theta)d\theta\Big\|_H^2\lambda(du)ds\\
&\leq 3C_4r^2TK_1\|Y_1-Y_2\|_2^2. \qquad (3.3.11)
\end{aligned}
$$

Hence, substituting (3.3.9)- (3.3.11) into (3.3.8) implies that

$$
\|\mathcal{K}(Y_1)-\mathcal{K}(Y_2)\|_2^2 \leq \Theta(T)\|Y_1-Y_2\|_2^2 \qquad (3.3.12)
$$

with

$$
\Theta(T) = 3r^2T^2K_1+3C_3r^2TK_1+3C_4r^2TK_1 < 1, \qquad (3.3.13)
$$

then we can take a suitable $0 < T_1 < T$ sufficient small such that $\Theta(T_1) < 1$, and hence $\mathcal{K}$ is a contraction on $H_2^{T_1}$ ($H_2^{T_1}$ denotes $\mathcal{H}_2$ with T substituted by T_1). Thus, by the well-know Banach fixed point theorem we obtain a unique fixed point $Y^* \in \mathcal{H}_2^{T_1}$ for operator $\mathcal{K}$, and hence Equation (3.2.7) is a mild solution of Equation (3.2.1). This procedure can be repeated to extend the solution to the entire interval $[0,T]$ in finitely many similar steps, thereby completing the proof for the existence and uniqueness of mild solutions on the whole interval $[0,T]$.

■

3.4 Approximation system

The following version of Itô's formula Theorem 3.2 has been introduced in Luo and Liu (2008), which plays an important role in our stability analysis. Let $C^2(H;\mathbb{R}_+)$ denotes the space of all real-valued nonnegative functions V on H with properties:

1. $V(x)$ is twice (Fréchet) differentiable in x and
2. $V_x'(x)$ and $V_{xx}''(x)$ are both continuous in H and $\mathcal{L}(H)=\mathcal{L}(H,H)$, respec-

tively.

Theorem 3.2 *Suppose $V \in C^2(H;\mathbb{R}_+)$, let $X(t)$, $t \geq 0$ be a strong solution of Equation (3.2.1), then with $t \geq 0$,*

$$\begin{aligned} & V(X(t)) \\ = \ & V(\xi) + \int_0^t \mathcal{L}V\Big(X(s), \int_{-r}^0 X(s+\theta)d\theta\Big)ds \\ & + \int_0^t \Big\langle V_x'(X(s)), G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s)\Big\rangle_H \\ & + \int_0^t \int_{\mathbb{Z}} \Big[V\Big(X(s) + L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\Big) - V(X(s))\Big]\widetilde{N}(ds,du) \end{aligned}$$

where $\forall\, x, y \in \mathcal{D}(A)$, the domain of operator A,

$$\begin{aligned} \mathcal{L}V(x,y) &= \langle V_x'(x), Ax + F(y)\rangle_H + \frac{1}{2}\ Tr\ (V_{xx}''(x)G(y)QG^*(y)) \\ & \quad + \int_{\mathbb{Z}} [V(x + L(y,u)) - V(x) - \langle V_x'(x), L(y,u)\rangle_H]\lambda(du). \end{aligned}$$

Since the mild solutions involve with semigroup, hence they are not martingales. We cannot deal with mild solutions directly in most arguments by the Itô's formula.

For any $t \geq 0$, we introduce the following approximating system:

$$\begin{aligned} dX(t) &= AX(t) + R(n)F\Big(\int_{-r}^0 X(t+\theta)d\theta\Big)dt \\ & \quad + R(n)G\Big(\int_{-r}^0 X(t+\theta)d\theta\Big)dW_Q(t) \\ & \quad + \int_{\mathbb{Z}} R(n)L\Big(\int_{-r}^0 X(t+\theta)d\theta, u\Big)\widetilde{N}(dt,du) \qquad (3.4.1) \\ X(t) &= R(n)\xi(t) \in \mathcal{D}(A), \quad -r \leq t \leq 0. \end{aligned}$$

Here $n \in \rho(A)$, the resolvent set of A and $R(n) = nR(n,A)$, $R(n,A)$ is the resolvent of A. Similar to operator $\mathcal{L}$ defined in Theorem 3.2, the operator $\mathcal{L}_n$

associated with (3.4.1), for any $x, y \in \mathcal{D}(A)$, can be defined as follows:

$$\begin{aligned}\mathcal{L}_n V(x,y) &= \langle V'_x(x), Ax + R(n)F(y)\rangle_H \\ &+ \frac{1}{2}\,\mathrm{Tr}\,[V''_{xx}(x)R(n)G(y)Q(R(n)G(y))^*] \\ &+ \int_{\mathbb{Z}} [V(x+R(n)L(y,u)) - V(x) - \langle V'_x(x), R(n)L(y,u)\rangle_H]\lambda(du).\end{aligned}$$

Theorem 3.3 *Let $\xi \in D^b_{\mathcal{F}_0}([-r,0];H)$ be an arbitrarily given initial datum and assume that the conditions (3.2.2)-(3.2.5) hold. Then the stochastic differential equation (3.4.1) has a unique strong solution $X_n(t) \in \mathcal{D}(A)$, which lies in $C(0,T;$ $L^2(\Omega,\mathcal{F},\mathbb{P};H))$ for all $T > 0$. Moreover, $X_n(t)$ converges to the mild solution $X(t)$ of Equation (3.2.1) almost surely in $C(0,T;L^2(\Omega,\mathcal{F},\mathbb{P};H))$ as $n \to \infty$.*

Proof. The existence of a unique strong solution $X_n(t)$ of the kind we desire is an immediate consequence of Proposition 3.2 and Theorem 3.1 on noting the fact that $AR(n) = AnR(n,A) = n - n^2R(n,A)$ are bounded operators (c.f. Liu (2006), Proposition 1.3.6). To prove the remainder of the proposition, let us consider

$$\begin{aligned}&X(t) - X_n(t) \\ = \;& T(t)[\xi(0) - R(n)\xi(0)] \\ &+ \int_0^t T(t-s)\Big[F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big) - R(n)F\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big]ds \\ &+ \int_0^t T(t-s)\Big[G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big) - R(n)G\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big]dW_Q(s) \\ &+ \int_0^t\int_{\mathbb{Z}} T(t-s)\Big[L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big) \\ &\qquad\qquad - R(n)L\Big(\int_{-r}^0 X_n(s+\theta)d\theta, u\Big)\Big]\widetilde{N}(ds,du) \quad (3.4.2)\end{aligned}$$

for any $t \geq 0$.

Since $|a+b+c+d|^2 \leq 4^2(|a|^2+|b|^2+|c|^2+|d|^2)$, we thus have that for any $T \geq 0$,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}\|X(t)-X_n(t)\|_H^2\Big]\\
\le\;& 4^2\mathbb{E}\Big[\sup_{0\le t\le T}\Big\|\int_0^t T(t-s)R(n)\Big[F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)\\
&-F\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big]ds\Big\|_H^2\Big]\\
&+4^2\mathbb{E}\Big[\sup_{0\le t\le T}\Big\|\int_0^t T(t-s)R(n)\Big[G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)\\
&-G\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big]dW_Q(s)\Big\|_H^2\Big]\\
&+4^2\mathbb{E}\Big[\sup_{0\le t\le T}\Big\|\int_0^t\int_{\mathbb{Z}} T(t-s)R(n)\Big[L\Big(\int_{-r}^0 X(s+\theta)d\theta,u\Big)\\
&-L\Big(\int_{-r}^0 X_n(s+\theta)d\theta,u\Big)\Big]\widetilde{N}(ds,du)\Big\|_H^2\Big]\\
&+4^2\mathbb{E}\Big\{\sup_{0\le t\le T}\Big\|\Big[T(t)\xi(0)-T(t)R(n)\xi(0)\Big]\\
&+\int_0^t T(t-s)[I-R(n)]F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds\\
&+\int_0^t T(t-s)[I-R(n)]G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s)\\
&+\int_0^t\int_{\mathbb{Z}} T(t-s)[I-R(n)]L\Big(\int_{-r}^0 X(s+\theta)d\theta,u\Big)\widetilde{N}(ds,du)\Big\|_H^2\Big\}\\
&:=16[N_1+N_2+N_3+N_4]. \qquad (3.4.3)
\end{aligned}
$$

Note that $\|R(n)\|\le 2$ for $n>0$ large enough. The Lipschitz continuity conditions in Assumption 3.2 and Hölder inequality imply that

$$
\begin{aligned}
N_1 \;=\;& \mathbb{E}\Big[\sup_{0\le t\le T}\Big\|\int_0^t T(t-s)R(n)\Big[F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)\\
&-F\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big]ds\Big\|_H^2\Big]\\
\le\;& \mathbb{E}\Big\{\sup_{0\le t\le T}\Big[\int_0^t\|T(t-s)R(n)\|^2ds\Big]\\
&\times\Big[\int_0^t\Big\|F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)-F\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big\|_H^2ds\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\le 4T\int_0^T \mathbb{E}\Big\|F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big) - F\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big\|_H^2 ds \\
&\le 4TK_1\int_0^T \mathbb{E}\Big\|\int_{-r}^0 X(s+\theta)d\theta - \int_{-r}^0 X_n(s+\theta)d\theta\Big\|_H^2 ds \\
&\le 4rTK_1\int_0^T\int_{-r}^0 \mathbb{E}\Big\|X(s+\theta) - X_n(s+\theta)\Big\|_H^2 d\theta ds \\
&\le 4r^2K_1\mathbb{E}\int_0^T \sup_{0\le u\le s}\|X(u)-X_n(u)\|_H^2 du \\
&\quad +4rT^2K_1\mathbb{E}\sup_{-r\le\theta\le 0}\|X(\theta)-X_n(\theta)\|_H^2. \qquad (3.4.4)
\end{aligned}
$$

where $K_1 > 0$ is the Lipschitz constant in Assumption 3.2. On the other hand, by virtue of the Burkholder-Davis-Gundy type of inequality for stochastic convolutions in Tubaro (1984), we have for $n > 0$ large enough that there exists a number $B_1(T) > 0$ such that

$$
\begin{aligned}
N_2 &= \mathbb{E}\Big[\sup_{0\le t\le T}\Big\|\int_0^t T(t-s)R(n)\Big[G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big) \\
&\qquad -G\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big]dW_Q(s)\Big\|_H^2\Big] \\
&\le r^2TB_1(T)K_1\int_0^T \mathbb{E}\sup_{0\le u\le s}\|X(u)-X_n(u)\|_H^2 du \\
&\qquad +rT^2B_1(T)K_1\mathbb{E}\sup_{-r\le\theta\le 0}\|X(\theta)-X_n(\theta)\|_H^2. \qquad (3.4.5)
\end{aligned}
$$

and by Lemma 3.1 and (3.2.3), it follows that there exists a number $B_2(T) > 0$ such that

$$
\begin{aligned}
N_3 &= \mathbb{E}\Big[\sup_{0\le t\le T}\Big\|\int_0^t\int_{\mathbb{Z}} T(t-s)R(n)\Big[L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big) \\
&\qquad -L\Big(\int_{-r}^0 X_n(s+\theta)d\theta, u\Big)\Big]\widetilde{N}(ds,du)\Big\|_H^2\Big] \\
&\le r^2TB_2(T)K_1\int_0^T \mathbb{E}\sup_{0\le u\le s}\|X(u)-X_n(u)\|_H^2 du \\
&\qquad +rT^2B_2(T)K_1\mathbb{E}\sup_{-r\le\theta\le 0}\|X(\theta)-X_n(\theta)\|_H^2. \qquad (3.4.6)
\end{aligned}
$$

Also, it can be shown that

$$
\begin{aligned}
N_4 \quad \leq \quad & 16\Bigg\{\mathbb{E}\sup_{0\leq t\leq T}\Bigg\|\Big[T(t)\xi(0)-T(t)R(n)\xi(0)\Big]\Bigg\|_H^2 \\
& +\mathbb{E}\sup_{0\leq t\leq T}\Bigg\|\int_0^t T(t-s)[I-R(n)]F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds\Bigg\|_H^2 \\
& +\mathbb{E}\sup_{0\leq t\leq T}\Bigg\|\int_0^t T(t-s)[I-R(n)]G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s)\Bigg\|_H^2 \\
& +\mathbb{E}\sup_{0\leq t\leq T}\Bigg\|\int_0^t\int_{\mathbb{Z}} T(t-s)[I-R(n)]L\Big(\int_{-r}^0 X(s+\theta)d\theta,u\Big)\widetilde{N}(ds,du)\Bigg\|_H^2\Bigg\}.
\end{aligned} \tag{3.4.7}
$$

By using the dominated convergence theorem and $\|I-R(n)\|\to 0$ as $n\to\infty$, we can obtain

$$
\mathbb{E}\sup_{0\leq t\leq T}\|T(t)[\xi(0)-R(n)]\xi(0)\|_H^2 \leq \mathbb{E}\|[I-R(n)]\xi(0)\|_H^2 \to 0, \text{ as } n\to\infty, \tag{3.4.8}
$$

and

$$
\begin{aligned}
& \mathbb{E}\sup_{0\leq t\leq T}\Bigg\|\int_0^t T(t-s)[I-R(n)]\Big[F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds\Bigg\|_H^2 \\
\leq \quad & \int_0^T \mathbb{E}\Bigg\|[I-R(n)]\Big[F\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds\Bigg\|_H^2 \to 0, \text{ as } n\to\infty.
\end{aligned} \tag{3.4.9}
$$

In a similar manner, by using the Burkholder-Davis-Gundy type of inequality for stochastic convolutions in Tubaro (1984), it can be deduced that there exists a number $B_3(T)>0$ such that

$$
\begin{aligned}
& \mathbb{E}\sup_{0\leq t\leq T}\Bigg\|\int_0^t T(t-s)[I-R(n)]G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)dW_Q(s)\Bigg\|_H^2 \\
\leq \quad & B_3(T)\int_0^T \mathbb{E}\Bigg\|[I-R(n)]G\Big(\int_{-r}^0 X(s+\theta)d\theta\Big)ds\Bigg\|_H^2 \to 0, \text{ as } n\to\infty.
\end{aligned} \tag{3.4.10}
$$

and by Lemma 3.1 and the dominated convergence theorem, we deduce that there exists a number $B_4(T) > 0$ such that

$$\begin{aligned}
&\mathbb{E} \sup_{0\le t\le T} \Big\| \int_0^t T(t-s)[I-R(n)]L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)\widetilde{N}(ds,du)\Big\|_H^2 \\
\le\ & B_4(T)\int_0^T \mathbb{E}\Big\|[I-R(n)]L\Big(\int_{-r}^0 X(s+\theta)d\theta, u\Big)ds\Big\|_H^2 \to 0, \text{ as } n\to\infty.
\end{aligned} \tag{3.4.11}$$

Hence, $N_4 \to 0$ as $n \to \infty$. Combining with (3.4.3)-(3.4.11), we can get that there exists numbers $B(T) > 0$ and $\varepsilon(0)$ such that

$$\mathbb{E}\Big[\sup_{0\le t\le T} \|X(t) - X_n(t)\|_H^2\Big] \le B(T)\int_0^T \mathbb{E}\Big[\sup_{0\le u\le s} \|X(u) - X_n(u)\|_H^2\Big] + \varepsilon(n),$$

where $\lim_{n\to\infty} \varepsilon(n) = 0$. By the Gronwall inequality,

$$\mathbb{E}\Big[\sup_{0\le t\le T} \|X(t) - X_n(t)\|_H^2\Big] \le \varepsilon(n)e^{B(T)T} \to 0, \quad \text{as } n\to\infty. \tag{3.4.12}$$

Then the desired assertion has been proved.

■

It can be observed that the mild solution $X^\xi(t)$ of Equation (3.2.1) is not a Markov process. However, it can be shown that $X_t^\xi, t \ge 0$ is a time-homogeneous strong Markov process as in Mohammed (1984). The Markov property will be used in the proof of inequality (3.5.18).

3.5 Stability in distribution

In this section, we concern the stability in distribution of mild solutions to Equation (3.2.1). Now, we recall the definition.

Let $p(t, \xi, d\zeta)$ denote the transition probability of the process $y(t)$ with the initial state $y(0) = \xi$. Denoted by $P(t, \xi, \Gamma)$ the probability of event $y(t) \in \Gamma$ given initial condition $y(0) = \xi$, i.e. with $\Gamma \in \mathcal{B}(H)$, which denotes the Borel

σ-algebra of H,

$$P(t, \xi, \Gamma) = \int_\Gamma p(t, \xi, d\zeta).$$

Denoted by $X^\xi(t)$ the mild solution to Equation (3.2.1) with the initial datum $\xi \in D^b_{\mathcal{F}_0}([-r, 0]; H)$. Correspondingly, $X^\xi_t = \{X^\xi(t+\theta), -r \le \theta \le 0\}$.

Definition 3.3 (Stability in Distribution) *The process $X^\xi_t, t \ge 0$ is said to be stable in distribution if there exits a probability measure $\pi(\cdot)$ on $D([-r, 0]; H)$ such that its transition probability $p(t, \xi, d\zeta)$ converges weakly to $\pi(d\zeta)$ as $t \to \infty$ for every $\xi \in D^b_{\mathcal{F}_0}([-r, 0]; H)$. In this case, Equation (3.2.1) is said to be stable in distribution.*

Since $X^\xi_t, t \ge 0$ is a Markov process, using the Kolmogorov-Chapman equation, it can be shown that the stability in distribution of $X^\xi_t, t \ge 0$ implies the existence of a unique invariant probability measure for $X^\xi_t, t \ge 0$, c.f. Yuan et al. (2003) and Bao et al. (2009a,b).

We obtain the result of stability in distribution Theorem 3.4 through the following four Lemmas.

Lemma 3.2 *Let the conditions of Theorem 3.3 hold. Assume there exist constants $\lambda_1 > \lambda_2 \ge 0$ and $\beta \ge 0$ such that, for any $x, y \in \mathcal{D}(A)$,*

$$\begin{aligned} &2\langle x, Ax + F(y)\rangle_H + \|G(y)\|^2_{\mathcal{L}^2_0} + \int_{\mathbb{Z}} \|L(y, u)\|^2_H \lambda(du) \\ &\quad \le -\lambda_1 \|x\|^2_H + \lambda_2 \|y\|^2_H + \beta. \end{aligned} \tag{3.5.1}$$

Then

$$\sup_{0 \le t < \infty} \mathbb{E}\|X^\xi_t\|^2_D < \infty \quad \forall\, \xi \in D^b_{\mathcal{F}_0}([-r, 0]; H). \tag{3.5.2}$$

Proof. For simplicity, we denote $X^\xi(t)$ by $X(t)$. Firstly, we show that, for any $t \ge 0$,

$$\mathbb{E}\|X(t)\|^2_H < \infty. \tag{3.5.3}$$

For any $t \geq 0$, $\mu > 0$, using (3.5.1) and applying the Itô's formula to the function $V(t,x) = e^{\mu t}V(x) = e^{\mu t}\|x\|_H^2$ and the strong solution $X_n(t)$ of Equation (3.4.1), we have

$$
\begin{aligned}
& \mathbb{E}e^{\mu t}\|X_n(t)\|_H^2 \\
= \;& \mathbb{E}\|X_n(0)\|_H^2 + \int_0^t \mathbb{E}e^{\mu s}\Big[\mu\|X_n(s)\|_H^2 + \mathcal{L}_n V\Big(X_n(s), \int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big]ds \\
= \;& \mathbb{E}\|X_n(0)\|_H^2 + \int_0^t \mathbb{E}e^{\mu s}\Big[\mu\|X_n(s)\|_H^2 - \mathcal{L}V\Big(X_n(s), \int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big]ds \\
& + \int_0^t \mathbb{E}e^{\mu s}\Big[\mathcal{L}_n V\Big(X_n(s), \int_{-r}^0 X_n(s+\theta)d\theta\Big) \\
& \qquad\qquad\qquad\qquad - \mathcal{L}V\Big(X_n(s), \int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big]ds \\
\leq \;& \mathbb{E}\|X_n(0)\|_H^2 + \int_0^t \mathbb{E}e^{\mu s}\Big[\mu\|X_n(s)\|_H^2 - \lambda_1\|X_n(s)\|_H^2 \\
& \qquad\qquad\qquad\qquad + \lambda_2\Big\|\int_{-r}^0 X_n(s+\theta)d\theta\Big\|_H^2 + \beta\Big]ds \\
& + \int_0^t \mathbb{E}e^{\mu s}\Big[\mathcal{L}_n V\Big(X_n(s), \int_{-r}^0 X_n(s+\theta)d\theta\Big) \\
& \qquad\qquad\qquad\qquad - \mathcal{L}V\Big(X_n(s), \int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big]ds \\
\leq \;& \mathbb{E}\|X_n(0)\|_H^2 + \int_0^t \mathbb{E}e^{\mu s}[(\mu - \lambda_1)\|X_n(s)\|_H^2]ds \\
& + r\lambda_2 \int_0^t \mathbb{E}e^{\mu s}\int_{-r}^0 \|X_n(s+\theta)\|_H^2 d\theta ds + \frac{\beta e^{\mu t}}{\mu} \\
& + \int_0^t \mathbb{E}e^{\mu s}\Big[\mathcal{L}_n V\Big(X_n(s), \int_{-r}^0 X_n(s+\theta)d\theta\Big) \\
& \qquad\qquad\qquad\qquad - \mathcal{L}V\Big(X_n(s), \int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big]ds \\
\leq \;& \mathbb{E}\|X_n(0)\|_H^2 + \int_0^t \mathbb{E}e^{\mu s}\Big[\Big(\mu - \lambda_1 + r\lambda_2\Big(\frac{e^{\mu r}-1}{\mu}\Big)\Big)\|X_n(s)\|_H^2\Big]ds \\
& + r\lambda_2\Big(\frac{e^{\mu r}-1}{\mu}\Big)\int_{-r}^0 \|X_n(s)\|_H^2 ds \\
& + \frac{\beta e^{\mu t}}{\mu} + \int_0^t 2e^{\mu s}\mathbb{E}\Big\langle X_n(s), (R(n)-I)F\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big\rangle_H ds \\
& + \int_0^t \mathbb{E}e^{\mu s}\Big\|R(n)G\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big\|_{\mathcal{L}_2^0}^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^t \mathbb{E}e^{\mu s}\Big\|G\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)\Big\|_{\mathcal{L}_2^0}^2 ds \\
&+\int_0^t\int_{\mathbb{Z}} \mathbb{E}e^{\mu s}\Big\|R(n)L\Big(\int_{-r}^0 X_n(s+\theta)d\theta,u\Big)\Big\|_H^2 \lambda(du)ds \\
&-\int_0^t\int_{\mathbb{Z}} \mathbb{E}e^{\mu s}\Big\|L\Big(\int_{-r}^0 X_n(s+\theta)d\theta,u\Big)\Big\|_H^2 \lambda(du)ds
\end{aligned}
$$

Note that for $\lambda_1 > r\lambda_2 \geq 0$, the equation

$$
x^2 = \lambda_1 x - r\lambda_2(e^{xr}-1),
$$

admits a unique positive root denoted by ρ. Letting $\mu=\rho$ and using Theorem 3.3 and the dominated convergence theorem, we get

$$
\begin{aligned}
\mathbb{E}\|X(t)\|_H^2 &\leq \mathbb{E}e^{\rho t}\|X(t)\|_H^2 \\
&\leq \mathbb{E}\|\xi(0)\|_H^2 + r\lambda_2(\frac{e^{\rho r}-1}{\rho})\int_{-r}^0 \mathbb{E}\|\xi(s)\|_H^2 d\theta ds + \frac{\beta}{\rho}.
\end{aligned}
$$

Thus, together with $\xi \in D^b_{\mathcal{F}_0}([-r,0];H)$, we have $\mathbb{E}\|X(t)\|_H^2 < \infty$.

Next, we intend to show that, for any $t \geq 0$,

$$
\sup_{0<t<\infty} \mathbb{E}\|X_t\|_D^2 = \sup_{0\leq t<\infty} \mathbb{E}\Big[\sup_{-r\leq\theta<0} \|X(t+\theta)\|_H^2\Big] < \infty. \tag{3.5.4}
$$

Again, applying Itô's formula to the function $V(x) = \|x\|_H^2$ and the strong solution $X_n(t)$ of Equation (3.4.1), for any $t \geq r$ and $\theta \in [-r,0]$, we obtain

$$
\begin{aligned}
&\|X_n(t+\theta)\|_H^2 \\
=\ &\|X_n(t-r)\|_H^2 + \int_{t-r}^{t+\theta} \mathcal{L}_n V\Big(X_n(s), \int_{-r}^0 X_n(s+\theta)d\theta\Big)ds \\
&+2\int_{t-r}^{t+\theta}\Big\langle X_n(s), R(n)G\Big(\int_{-r}^0 X_n(s+\theta)d\theta\Big)dW_Q(s)\Big\rangle_H \\
&+\int_{t-r}^{t+\theta}\int_{\mathbb{Z}}\Big[\Big\|R(n)L\Big(\int_{-r}^0 X_n(s+\theta)d\theta,u\Big)\Big\|_H^2 \\
&\qquad +2\Big\langle X_n(s), R(n)L\Big(\int_{-r}^0 X_n(s+\theta)d\theta,u\Big)\Big\rangle_H^2\Big]\widetilde{N}(ds,du)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \|X_n(t-r)\|_H^2 + \lambda_1 \int_{t-r}^{t+\theta} \|X_n(s)\|_H^2 ds + \lambda_2 \int_{t-r}^{t+\theta} \Big\| \int_{-r}^{0} X_n(s+\theta)d\theta \Big\|_H^2 ds \\
& + \beta r + 2\int_{t-r}^{t+\theta} \Big\langle X_n(s), R(n)G\Big(\int_{-r}^{0} X_n(s+\theta)d\theta\Big)dW_Q(s)\Big\rangle_H \\
& + \int_{t-r}^{t+\theta}\int_{\mathbb{Z}} \Big[\Big\|R(n)L\Big(\int_{-r}^{0} X_n(s+\theta)d\theta, u\Big)\Big\|_H^2 \\
& \qquad + 2\Big\langle X_n(s), R(n)L\Big(\int_{-r}^{0} X_n(s+\theta)d\theta, u\Big)\Big\rangle_H\Big]\widetilde{N}(ds,du) \\
& + 2\int_{t-r}^{t+\theta} \Big\langle X_n(s), (R(n)-I)F\Big(\int_{-r}^{0} X_n(s+\theta)d\theta\Big)\Big\rangle_H ds \\
& + \int_{t-r}^{t+\theta} \Big\|R(n)G\Big(\int_{-r}^{0} X_n(s+\theta)d\theta\Big)\Big\|_{\mathcal{L}_0^2}^2 ds \\
& - \int_{t-r}^{t+\theta} \Big\|G\Big(\int_{-r}^{0} X_n(s+\theta)d\theta\Big)\Big\|_{\mathcal{L}_0^2}^2 ds \\
& + \int_0^t \int_{t-r}^{t+\theta} \Big\|R(n)L\Big(\int_{-r}^{0} X_n(s+\theta)d\theta\Big)\Big\|_H^2 \lambda(du)ds \\
& - \int_0^t \int_{t-r}^{t+\theta} \Big\|L\Big(\int_{-r}^{0} X_n(s+\theta)d\theta\Big)\Big\|_H^2 \lambda(du)ds.
\end{aligned}
$$

By Theorem 3.3 together with the dominated convergence theorem, we thus get

$$
\begin{aligned}
& \mathbb{E}\Big[\sup_{-r\leq\theta<0} \|X(t+\theta)\|_H^2\Big] \\
\leq\ & \mathbb{E}\|X(t-r)\|_H^2 + \lambda_1 \int_{t-r}^{t} \mathbb{E}\|X(s)\|_H^2 ds + \lambda_2 \int_{t-r}^{t} \Big\| \int_{-r}^{0} X_n(s+\theta)d\theta \Big\|_H^2 ds \\
& +\beta r + 2\mathbb{E}\Big[\sup_{-r\leq\theta<0}\int_{t-r}^{t+\theta} \Big\langle X(s), G\Big(\int_{-r}^{0} X(s+\theta)d\theta\Big)dW_Q(s)\Big\rangle_H\Big] \\
& +\mathbb{E}\Big\{\sup_{-r\leq\theta<0}\int_{t-r}^{t+\theta}\int_{\mathbb{Z}} \Big[\Big\|L\Big(\int_{-r}^{0} X(s+\theta)d\theta, u\Big)\Big\|_H^2 \\
& \qquad +2\Big\langle X(s), L\Big(\int_{-r}^{0} X(s+\theta)d\theta, u\Big)\Big\rangle_H\Big]\widetilde{N}(ds,du)\Big\}. \qquad (3.5.5)
\end{aligned}
$$

Now, by virtue of Burkholder-Davis-Gundy's inequality (c.f. Ichikawa (1982) Proposition 1.6), we derive for some positive constant K_1 such that

$$
\begin{aligned}
& 2\mathbb{E}\Big[\sup_{-r\leq\theta<0}\int_{t-r}^{t+\theta} \Big\langle X(s), G\Big(\int_{-r}^{0} X(s+\theta)d\theta\Big)dW_Q(s)\Big\rangle_H\Big] \\
\leq\ & K_1\mathbb{E}\Big(\sup_{-r\leq\theta<0}\int_{t-r}^{t+\theta} \|X(t+\theta)\|_H^2 \int_{t-r}^{t} \Big\|G\Big(\int_{-r}^{0} X(s+\theta)d\theta\Big)\Big\|_{\mathcal{L}_0^2}^2 ds\Big)^{\frac{1}{2}},
\end{aligned}
$$

which, immediately yields for certain $K_2 > 0$,

$$
\begin{aligned}
& 2\mathbb{E}\Big[\sup_{-r\le\theta<0}\int_{t-r}^{t+\theta}\Big\langle X(s), G\Big(\int_{-r}^{0}X(s+\theta)d\theta\Big)dW_Q(s)\Big\rangle_H\Big] \\
\le\;& \frac{1}{2}\mathbb{E}\Big[\sup_{-r\le\theta<0}\|X(s+\theta)\|_H^2\Big] + K_2\int_0^t \mathbb{E}\Big\|G\Big(\int_{-r}^{0}X(s+\theta)d\theta\Big)\Big\|_{\mathcal{L}_0^2}^2 ds.
\end{aligned}
\tag{3.5.6}
$$

Next, we shall estimate the last term of (3.5.5). The method used is similar to that of the proof Röckner and Zhang (2007) Proposition 3.1. Let

$$
\begin{aligned}
M(t,\theta) \;=\;& \int_{t-r}^{t+\theta}\int_{\mathbb{Z}}\Big[\Big\|L\Big(\int_{-r}^{0}X(s+\theta)d\theta, u\Big)\Big\|_H^2 \\
& +2\Big\langle X(s), L\Big(\int_{-r}^{0}X(s+\theta)d\theta, u\Big)\Big\rangle_H\Big]\widetilde{N}(ds,du)
\end{aligned}
$$

and $[M(t,0)]$ denotes the quadratic variation of the process $M(t,0)$. Then, from Burkholder-Davis-Gundy's inequality (c.f. Applebaum (2004)), there exists a positive constant K_3 such that

$$
\mathbb{E}\Big[\sup_{-r\le\theta<0}|M(t,\theta)|\Big] \le K_3\mathbb{E}\Big([M(t,0)]^{1/2}\Big). \tag{3.5.7}
$$

By the definition of quadratic variation,

$$
\begin{aligned}
& [M(t,0)]^{1/2} \\
=\;& \Big\{\sum_{s\in D_p,\, t-r\le s\le t}\Big(\Big\|L\Big(\int_{-r}^{0}X(s+\theta)d\theta, p(s)\Big)\Big\|_H^2 \\
& \qquad + 2\Big\langle X(s), L\Big(\int_{-r}^{0}X(s+\theta)d\theta, p(s)\Big)\Big\rangle_H\Big)^2\Big\}^{1/2} \\
\le\;& \sqrt{2}\Big\{\sum_{s\in D_p,\, t-r\le s\le t}\Big\|L\Big(\int_{-r}^{0}X(s+\theta)d\theta, p(s)\Big)\Big\|_H^4\Big\}^{1/2} \\
& \qquad + \sqrt{2}\Big\{4\sum_{s\in D_p,\, t-r\le s\le t}\Big|\Big\langle X(s), L\Big(\int_{-r}^{0}X(s+\theta)d\theta, p(s)\Big)\Big\rangle_H\Big|^4\Big\}^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sqrt{2}\Big\{\sum_{s\in D_p,\,t-r\leq s\leq t}\Big\|L\Big(\int_{-r}^{0}X(s+\theta)d\theta,p(s)\Big)\Big\|_H^4\Big\}^{1/2}\\
&\qquad +2\sqrt{2}\Big\{\sum_{s\in D_p,\,t-r\leq s\leq t}\|X(s)\|_H^2\Big\|L\Big(\int_{-r}^{0}X(s+\theta)d\theta,p(s)\Big)\Big\|_H^2\Big\}^{1/2}\\
&\leq \sqrt{2}\sum_{s\in D_p,\,t-r\leq s\leq t}\Big\|L\Big(\int_{-r}^{0}X(s+\theta)d\theta,p(s)\Big)\Big\|_H^2\\
&\quad +2\sqrt{2}\sup_{-r\leq\theta<0}\|X(t+\theta)\|_H\Big\{\sum_{s\in D_p,\,t-r\leq s\leq t}\Big\|L\Big(\int_{-r}^{0}X(s+\theta)d\theta,p(s)\Big)\Big\|_H^2\Big\}^{1/2}\\
&\leq \frac{1}{4K_3}\sup_{-r\leq\theta<0}\|X(t+\theta)\|_H^2\\
&\qquad +(8K_3+\sqrt{2})\sum_{s\in D_p,\,t-r\leq s\leq t}\Big\|L\Big(\int_{-r}^{0}X(s+\theta)d\theta,p(s)\Big)\Big\|_H^2.
\end{aligned}
$$

Hence, in (3.5.7)

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{-r\leq\theta<0}|M(t,\theta)|\Big]\\
&\leq \frac{1}{4}\mathbb{E}\Big[\sup_{-r\leq\theta<0}\|X(t+\theta)\|_H^2\Big]\\
&\qquad +K_3(8K_3+\sqrt{2})\mathbb{E}\Big[\sum_{s\in D_p,\,t-r\leq s\leq t}\Big\|L\Big(\int_{-r}^{0}X(s+\theta)d\theta,p(s)\Big)\Big\|_H^2\Big]\\
&\leq \frac{1}{4}\mathbb{E}\Big[\sup_{-r\leq\theta<0}\|X(t+\theta)\|_H^2\Big]\\
&\qquad +K_3(8K_3+\sqrt{2})\mathbb{E}\Big[\int_{t-r}^{t}\int_{\mathbb{Z}}\Big\|L\Big(\int_{-r}^{0}X(s+\theta)d\theta,p(s)\Big)\Big\|_H^2\Big] \qquad (3.5.8)
\end{aligned}
$$

Substituting (3.5.3), (3.5.6), (3.5.8) into (3.5.5) and combining (3.2.5), directly we have that (3.5.4) holds. Therefore, the desired assertion (3.5.2) follows.

∎

By the well-known Chebyshev inequality, for any positive number l, we have

$$
P(\|X_t^\xi\|_D\geq l)\leq\frac{\mathbb{E}\|X_t^\xi\|_D^2}{l^2}.
$$

Let $l\to\infty$, (3.5.2) implies that the right-hand side tends to 0. We assume, for any $\varepsilon>0$, there is a compact subset $\mathcal{K}=\mathcal{K}(\xi,\varepsilon)$ of $D([-r,0];H)$ such that

$P(t,\xi,\mathcal{K}) \geq 1-\varepsilon$. That is, the family $\{p(t,\xi,d\zeta) : t \geq 0\}$ is tight.

In what follows, we consider of two mild solutions that start from different initial conditions, namely

$$\begin{aligned}
& X^{\xi}(t) - X^{\eta}(t) \\
= \ & T(t)\xi(0) - T(t)\eta(0) \\
& + \int_0^t T(t-s)\Big[F\Big(\int_{-r}^0 X^{\xi}(s+\theta)d\theta\Big) - F\Big(\int_{-r}^0 X^{\eta}(s+\theta)d\theta\Big)\Big]ds \\
& + \int_0^t (T-s)\Big[G\Big(\int_{-r}^0 X^{\xi}(s+\theta)d\theta\Big) - G\Big(\int_{-r}^0 X^{\eta}(s+\theta)d\theta\Big)\Big]dW_Q(t) \\
& + \int_0^t \int_{\mathbb{Z}} T(t-s)\Big[L\Big(\int_{-r}^0 X^{\xi}(s+\theta)d\theta, u\Big) \\
& \qquad\qquad -L\Big(\int_{-r}^0 X^{\eta}(s+\theta)d\theta, u\Big)\Big]\widetilde{N}(ds,du).
\end{aligned} \tag{3.5.9}$$

Furthermore, we introduce an approximating system in correspondence with (3.5.9) in the following form:

$$\begin{aligned}
& d[X^{\xi}(t) - X^{\eta}(t)] \\
= \ & \Big\{A(X^{\xi}(t) - X^{\eta}(t)) \\
& +R(n)\Big[F\Big(\int_{-r}^0 X^{\xi}(s+\theta)d\theta\Big) - F\Big(\int_{-r}^0 X^{\eta}(s+\theta)d\theta\Big)\Big]\Big\}dt \\
& +R(n)\Big[G\Big(\int_{-r}^0 X^{\xi}(s+\theta)d\theta\Big) - G\Big(\int_{-r}^0 X^{\eta}(s+\theta)d\theta\Big)\Big]dW_Q(t) \\
& + \int_{\mathbb{Z}} R(n)\Big[L\Big(\int_{-r}^0 X^{\xi}(s+\theta)d\theta, u\Big) \\
& \qquad\qquad -L\Big(\int_{-r}^0 X^{\eta}(s+\theta)d\theta, u\Big)\Big]\widetilde{N}(ds,du)
\end{aligned} \tag{3.5.10}$$

where $n \in \rho(A)$, the resolvent set of A and $R(n) = nR(n,A)$, $R(n,A)$ is the resolvent of A.

For given $U \in C^2(H;\mathbb{R}_+)$, define an operator $\mathcal{L}_n U : H^4 \to \mathbb{R}$ associated with (3.5.10) by for any x, y, z_1 and $z_2 \in \mathcal{D}(A)$

$$\begin{aligned}
& \mathcal{L}_n U(x,y,z_1,z_2) \\
= \ & \langle U'_x(x-y), A(x-y) + R(n)[F(z_1) - F(z_2)]\rangle_H \\
& + \frac{1}{2} \text{ Tr } (U''_{xx}(x-y))R(n)[G(z_1) - G(z_2)]Q(R(n)[G(z_1) - G(z_2)])^* \\
& + \int_{\mathbb{Z}} \Big[U(x+y+R(n)[L(x,u) - L(y,u)]) - u(x-y) \\
& \quad - \langle U'_x(x-y), R(n)[L(z_1,u) - L(z_2,u)]\rangle_H \Big] \lambda(du).
\end{aligned}$$

Lemma 3.3 *Suppose the conditions of Theorem 3.3 hold. Assume also that there are constants $\lambda_3 > \lambda_4 \geq 0$ such that, for any x, y, z_1, and $z_2 \in \mathcal{D}(A)$,*

$$\begin{aligned}
& 2\langle x-y, A(x-y) + F(z_1) - F(z_2)\rangle_H + \|G(z_1) - G(z_2)\|^2_{\mathcal{L}_2^0} \\
& \quad + \int_{\mathbb{Z}} \|L(z_1,u) - L(z_2,u)\|^2_H \lambda(du) \\
\leq \ & -\lambda_3\|x-y\|^2_H + \lambda_4\|z_1 - z_2\|^2_H. \qquad (3.5.11)
\end{aligned}$$

Then, for any compact subset $\mathcal{K}$ of $D([-r,0];H)$,

$$\lim_{t\to\infty} \mathbb{E}\|X^\xi_t - X^\eta_t\|^2_D = 0 \quad \textit{uniformly in} \quad \xi,\eta \in \mathcal{K}. \qquad (3.5.12)$$

Proof. For integer $N \in \mathbb{R}_+$ and the strong solution $X^\xi_n(t) - X^\eta_n(t)$ to approximating system (3.5.10), we define stopping time as follows:

$$\tau_N = \inf\{t \geq 0 : \|X^\xi_n(t) - X^\eta_n(t)\|_H > N\}.$$

Clearly, $\tau_N \to \infty$ almost surely as $N \to \infty$. Let $T_N = \tau_N \wedge t$. Using the Itô's formula to the strong solution $X^\xi_n(t) - X^\eta_n(t)$ of (3.5.10), for any $t \geq 0$ and $\lambda > 0$, we derive

$$
\begin{aligned}
& \mathbb{E}e^{\lambda t}\|X_n^\xi(T_N)-X_n^\eta(T_N)\|_H^2 \\
= \;& \mathbb{E}\|R(n)\xi-R(n)\eta\|_H^2+\int_0^{T_N}\mathbb{E}e^{\lambda s}\Big[\lambda\|X_n^\xi(s)-X_n^\eta(s)\|_H^2 \\
& +\mathcal{L}_n\Big(X_n^\xi(s),X_n^\eta(s),\int_{-r}^0 X_n^\xi(s+\theta)d\theta,\int_{-r}^0 X_n^\eta(s+\theta)d\theta\Big)\Big]ds \\
= \;& \mathbb{E}\|R(n)\xi-R(n)\eta\|_H^2 \\
& +\int_0^{T_N}\mathbb{E}e^{\lambda s}\Big[\lambda\|X_n^\xi(s)-X_n^\eta(s)\|_H^2 \\
& +\mathcal{L}\Big(X_n^\xi(s),X_n^\eta(s),\int_{-r}^0 X_n^\xi(s+\theta)d\theta,\int_{-r}^0 X_n^\eta(s+\theta)d\theta\Big)\Big] \\
& +\int_0^{T_N}\mathbb{E}e^{\lambda s}\Big[\mathcal{L}_n\Big(X_n^\xi(s),X_n^\eta(s),\int_{-r}^0 X_n^\xi(s+\theta)d\theta,\int_{-r}^0 X_n^\eta(s+\theta)d\theta\Big) \\
& -\mathcal{L}\Big(X_n^\xi(s),X_n^\eta(s),\int_{-r}^0 X_n^\xi(s+\theta)d\theta,\int_{-r}^0 X_n^\eta(s+\theta)d\theta\Big)\Big]ds \\
\le \;& \mathbb{E}\|R(n)\xi-R(n)\eta\|_H^2+\int_0^{T_N}\mathbb{E}e^{\lambda s}\Big[\lambda\|X_n^\xi(T_N)-X_n^\eta(T_N)\|_H^2 \\
& -\lambda_3\|X_n^\xi(s)-X_n^\eta(s)\|_H^2+\lambda_4\Big\|\int_{-r}^0 X_n^\xi(s+\theta)-\int_{-r}^0 X_n^\eta(s+\theta)\Big\|_H^2\Big]ds \\
& +\int_0^{T_N}\mathbb{E}e^{\lambda s}\Big[\mathcal{L}_n\Big(X_n^\xi(s),X_n^\eta(s),\int_{-r}^0 X_n^\xi(s+\theta)d\theta,\int_{-r}^0 X_n^\eta(s+\theta)d\theta\Big) \\
& -\mathcal{L}\Big(X_n^\xi(s),X_n^\eta(s),\int_{-r}^0 X_n^\xi(s+\theta)d\theta,\int_{-r}^0 X_n^\eta(s+\theta)d\theta\Big)\Big]ds \\
\le \;& \mathbb{E}\|R(n)\xi-R(n)\eta\|_H^2+\int_0^{T_N}\mathbb{E}e^{\lambda s}(\lambda-\lambda_3)\|X_n^\xi(s)-X_n^\eta(s)\|_H^2 ds \\
& +\int_0^{T_N}\mathbb{E}e^{\lambda s}\lambda_4\Big\|\int_{-r}^0 X_n^\xi(s+\theta)d\theta-\int_{-r}^0 X_n^\eta(s+\theta)d\theta\Big\|_H^2 ds \\
& +\int_0^{T_N}\mathbb{E}e^{\lambda s}\Big[\mathcal{L}_n\Big(X_n^\xi(s),X_n^\eta(s),\int_{-r}^0 X_n^\xi(s+\theta)d\theta,\int_{-r}^0 X_n^\eta(s+\theta)d\theta\Big) \\
& -\mathcal{L}\Big(X_n^\xi(s),X_n^\eta(s),\int_{-r}^0 X_n^\xi(s+\theta)d\theta,\int_{-r}^0 X_n^\eta(s+\theta)d\theta\Big)\Big]ds \\
\le \;& \mathbb{E}\|R(n)\xi-R(n)\eta\|_H^2+r\lambda_4\Big(\frac{e^{\lambda r}-1}{\lambda}\Big)\int_{-r}^0\mathbb{E}\|X_n^\xi(s)-X_n^\eta(s)\|_H^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\lambda-\lambda_3+r\lambda_4\Big(\frac{e^{\lambda r}-1}{\lambda}\Big)\Big)\mathbb{E}\int_0^{T_N} e^{\lambda s}\|X_n^{\xi}(s)-X_n^{\eta}(s)\|_H^2 ds \\
&+\int_0^{T_N}\mathbb{E}e^{\lambda s}\Big[\mathcal{L}_n\Big(X_n^{\xi}(s),X_n^{\eta}(s),\int_{-r}^0 X_n^{\xi}(s+\theta)d\theta,\int_{-r}^0 X_n^{\eta}(s+\theta)d\theta\Big) \\
&-\mathcal{L}\Big(X_n^{\xi}(s),X_n^{\eta}(s),\int_{-r}^0 X_n^{\xi}(s+\theta)d\theta,\int_{-r}^0 X_n^{\eta}(s+\theta)d\theta\Big)\Big]ds \\
\leq\ & \mathbb{E}\|R(n)\xi-R(n)\eta\|_H^2+r\lambda_4\Big(\frac{e^{\lambda r}-1}{\lambda}\Big)\int_{-r}^0\mathbb{E}\|X_n^{\xi}(s)-X_n^{\eta}(s)\|_H^2 ds \\
&+\Big(\lambda-\lambda_3+r\lambda_4\Big(\frac{e^{\lambda r}-1}{\lambda}\Big)\Big)\mathbb{E}\int_0^{T_N} e^{\lambda s}\|X_n^{\xi}(s)-X_n^{\eta}(s)\|_H^2 ds \\
&+2\mathbb{E}\int_0^{T_N} e^{\lambda s}\Big\langle X_n^{\xi}(s)-X_n^{\eta}(s),(R(n)-I)\Big[F\Big(\int_{-r}^0 X_n^{\xi}(s+\theta)d\theta\Big) \\
&\qquad\qquad -F\Big(\int_{-r}^0 X_n^{\eta}(s+\theta)d\theta\Big)\Big]\Big\rangle_H ds \\
&+\mathbb{E}\int_0^{T_N} e^{\lambda s}\Big\|R(n)G\Big(\int_{-r}^0 X_n^{\xi}(s+\theta)d\theta\Big)-G\Big(\int_{-r}^0 X_n^{\eta}(s+\theta)d\theta\Big)\Big\|_{\mathcal{L}_2^0}^2 ds \\
&-\mathbb{E}\int_0^{T_N} e^{\lambda s}\Big\|G\Big(\int_{-r}^0 X_n^{\xi}(s+\theta)d\theta\Big)-G\Big(\int_{-r}^0 X_n^{\eta}(s+\theta)d\theta\Big)\Big\|_{\mathcal{L}_2^0}^2 ds \\
&+\mathbb{E}\int_0^{T_N}\int_{\mathbb{Z}} e^{\lambda s}\Big\|R(n)\Big[L\Big(\int_{-r}^0 X_n^{\xi}(s+\theta)d\theta,u\Big) \\
&\qquad\qquad -L\Big(\int_{-r}^0 X_n^{\eta}(s+\theta)d\theta\Big)\Big]\Big\|_H^2\lambda(du)ds \\
&-\mathbb{E}\int_0^{T_N}\int_{\mathbb{Z}} e^{\lambda s}\Big\|\Big[L\Big(\int_{-r}^0 X_n^{\xi}(s+\theta)d\theta,u\Big) \\
&\qquad\qquad -L\Big(\int_{-r}^0 X_n^{\eta}(s+\theta)d\theta\Big)\Big]\Big\|_H^2\lambda(du)ds. \qquad (3.5.13)
\end{aligned}
$$

Note that for $\lambda_3 > r\lambda_4 \geq 0$, the equation

$$x^2=\lambda_3 x-r\lambda_4(e^{xr}-1),$$

admits a unique positive root denoted by δ. In (3.5.13), letting $\lambda=\delta$ and using Theorem 3.3 and the dominated convergence theorem, it thus follows:

$$\mathbb{E}e^{\delta t}\|X^{\xi}(t)-X^{\eta}(t)\|_H^2\leq\Big(1+r^2\lambda_4\Big(\frac{e^{\delta r}-1}{\delta}\Big)\Big)\mathbb{E}\|\xi-\eta\|_D^2.$$

That is,

$$\mathbb{E}\|X^{\xi}(t)-X^{\eta}(t)\|_H^2 \le \Big(1+r^2\lambda_4\Big(\frac{e^{\delta r}-1}{\delta}\Big)\Big)\mathbb{E}\|\xi-\eta\|_D^2 e^{-\delta t}.$$

Hence, for any $\varepsilon>0$, there exists a $\delta>0$ such that for $\|\xi-\eta\|_D<\delta$

$$\mathbb{E}\|X^{\xi}(t)-X^{\eta}(t)\|_H^2 \le \frac{\epsilon}{9} \tag{3.5.14}$$

and

$$\lim_{t\to\infty}\mathbb{E}\|X^{\xi}(t)-X^{\eta}(t)\|_H^2 = 0. \tag{3.5.15}$$

Since $\mathcal{K}$ is compact, there exist $\xi_1,\ \xi_2,\ \xi_3$ such that $\mathcal{K}\subseteq\cup_{j=1}^{k}\rho(\xi_i,\delta)$, where $\rho(\xi_i,\delta)=\{\xi\in D([-r,0];H):\|\xi-\xi_i\|_D<\delta\}$. By (3.5.15), there exists a $T_1>0$ such that for $t\ge T_1$ and $1\le u,v\le k$,

$$\mathbb{E}\|X^{\xi_u}(t)-X^{\xi_v}(t)\|_H^2 \le \frac{\varepsilon}{9}. \tag{3.5.16}$$

For any ξ, $\eta\in\mathcal{K}$, we can find l, m such that $\xi\in\rho(\xi_l,\delta)$, $\eta\in\rho(\xi_m,\delta)$. By (3.5.14) and (3.5.16), we derive that for all $t\ge T_1$,

$$\begin{aligned}\mathbb{E}\|X^{\xi}(t)-X^{\eta}(t)\|_H^2 &\le 3\Big(\mathbb{E}\|X^{\xi}(t)-X^{\xi_l}(t)\|_H^2+\mathbb{E}\|X^{\eta}(t)-X^{\xi_m}(t)\|_H^2\\ &\qquad +\mathbb{E}\|X^{\xi_l}(t)-X^{\xi_m}(t)\|_H^2\Big)\\ &\le \varepsilon.\end{aligned}$$

Consequently, for any compact subset $\mathcal{K}$ of $D([-r,0];H)$,

$$\lim_{t\to\infty}\mathbb{E}\|X^{\xi}(t)-X^{\eta}(t)\|_H^2=0 \quad \text{uniformly in} \quad \xi,\eta\in\mathcal{K}.$$

■

Under the conditions of Lemma 3.3, we can show that for any $\varepsilon>0$ and any compact subset $\mathcal{K}$ of $D([-r,0];H)$, there is a $T=T(\varepsilon,\mathcal{K})>0$ such that for any

ξ, $\eta \in \mathcal{K}$,

$$\mathbb{P}(\|X^{\xi}(t) - X^{\eta}(t)\|_H < \varepsilon) \geq 1 - \varepsilon, \quad \forall\, t \geq T.$$

Actually, it is sufficient to show that there exists $T = T(\varepsilon, \mathcal{K}) > 0$ such that for any $t \geq T$

$$\mathbb{P}(\|X^{\xi}(t) - X^{\eta}(t)\|_H \geq \varepsilon) \leq \varepsilon.$$

By the Chebyshev inequality,

$$\mathbb{P}(\|X^{\xi}(t) - X^{\eta}(t)\|_H \geq \varepsilon) \leq \frac{\mathbb{E}\|X^{\xi}(t) - X^{\eta}(t)\|_H^2}{\varepsilon^2}.$$

However, by virtue of Lemma 3.3, it follows that there exists $T = T(\varepsilon, \mathcal{K}) > 0$ such that for any $t \geq T$

$$\mathbb{E}\|X^{\xi}(t) - X^{\eta}(t)\|_H^2 \leq \varepsilon^3,$$

then the required assertion follows.

Let $\mathcal{P}(D([-r,0];H))$ denote all probability measures on $D([-r,0];H)$. For $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(D([-r,0];H))$ define metric d_L as follows:

$$d_L(\mathbb{P}_1, \mathbb{P}_2) = \sup_{f \in L} \left| \int_H f(x)\mathbb{P}_1(dx) - \int_H f(x)\mathbb{P}_2(dx) \right|$$

and

$$L = \{f : \mathcal{P}(D([-r,0];H)) \to \mathbb{R} : |f(x) - f(y)| \leq \|x - y\|_D \text{ and } |f(\cdot)| \leq 1\}.$$

By the definition of stability in distribution, we need to show that there exists a probability measure $\pi(\cdot)$ such that for any $\xi \in D^b_{\mathcal{F}_0}([-r,0];H)$, the transition probabilities $\{p(t, \xi, \cdot) : t \geq 0\}$ converge weakly to $\pi(\cdot)$. It is known that the weak convergence of probability measures is equivalent to a metric concept (c.f. Ikeda and Watanabe (1989), Proposition 2.5). Meanwhile, according to Theorem 5.4 in Chen (2004), $\mathcal{P}(D[-\tau,0];H)$ is a complete metric space under metric d_L.

Lemma 3.4 *Let (3.5.2) and (3.5.12) hold. Then, $\{p(t,\xi,\cdot) : t \geq 0\}$ is Cauchy in the space $\mathcal{P}(D([-r,0];H))$ for any $\xi \in D^b_{\mathcal{F}_0}([-r,0];H)$.*

Proof. Fix $\xi \in D^b_{\mathcal{F}_0}([-r,0];H)$. We need to show that, for any $\varepsilon > 0$, there exists a $T > 0$ such that

$$d_L(p(t+s,\xi,\cdot), p(t,\xi,\cdot)) \leq \varepsilon, \quad \forall\, t \geq T,\ s > 0,$$

which is equivalent to show that for any $f \in L$

$$\sup_{f\in L} |\mathbb{E}f(X^{\xi}_{t+s}) - \mathbb{E}f(X^{\xi}_{t})| \leq \varepsilon, \quad \forall\, t \geq T,\ s > 0. \tag{3.5.17}$$

We thus compute for any $f \in L$ and $t,\ s > 0$

$$\begin{aligned} |\mathbb{E}f(X^{\xi}_{t+s}) - \mathbb{E}f(X^{\xi}_{t})| &= |\mathbb{E}[\mathbb{E}f(X^{\xi}_{t+s})|\mathcal{F}_s] - \mathbb{E}f(X^{\xi}_{t})| \\ &= \left| \int_H \mathbb{E}f(X^{\zeta}_{t}) p(s,\xi,d\zeta) - \mathbb{E}f(X^{\xi}_{t}) \right| \\ &\leq \int_H \mathbb{E}|f(X^{\zeta}_{t}) - \mathbb{E}f(X^{\xi}_{t})| p(s,\xi,d\zeta), \end{aligned} \tag{3.5.18}$$

where for the first equality we used the property of conditional expectation, while the second equality we used the Markov property of X^{ξ}_{t}. From Page 80, there exists a compact subset $\mathcal{K}$ of $D([-r,0];H)$ for any $\varepsilon > 0$ such that

$$p(s,\xi,\mathcal{K}) > 1 - \frac{\varepsilon}{8}. \tag{3.5.19}$$

Using (3.5.17) and (3.5.18), we obtain

$$\begin{aligned} |\mathbb{E}f(X^{\xi}_{t+s}) - \mathbb{E}f(X^{\xi}_{t})| &\leq \int_H \mathbb{E}|f(X^{\zeta}_{t}) - f(X^{\xi}_{t})| p(s,\xi,d\zeta) \\ = &\int_{\mathcal{K}} \mathbb{E}|f(X^{\zeta}_{t}) - f(X^{\xi}_{t})| p(s,\xi,d\zeta) + \int_{H-\mathcal{K}} \mathbb{E}|f(X^{\zeta}_{t}) - f(X^{\xi}_{t})| p(s,\xi,d\zeta) \\ \leq &\int_{\mathcal{K}} \mathbb{E}|f(X^{\zeta}_{t}) - f(X^{\xi}_{t})| p(s,\xi,d\zeta) + \frac{\varepsilon}{4}. \end{aligned} \tag{3.5.20}$$

Furthermore, from (3.5.12), we derive that there is a $T > 0$ for the given $\varepsilon > 0$

such that

$$\sup_{f\in L}|\mathbb{E}f(X_t^\zeta)-\mathbb{E}f(X_t^\xi)|\le\frac{3\varepsilon}{4},\quad\forall\, t\ge T,\tag{3.5.21}$$

which, in addition to (3.5.20), implies

$$|\mathbb{E}f(X_{t+s}^\xi)-\mathbb{E}f(X_t^\xi)|\le\varepsilon,\quad\forall\, t\ge T,\ s>0.$$

Since $f\in L$ is arbitrary, the desired inequality (3.5.17) is obtained.

■

Lemma 3.5 *Let (3.5.12) hold. Then for any compact subset $\mathcal{K}$ of $D([-r,0];H)$,*

$$\lim_{t\to\infty}d_L(p(t,\xi,\cdot),p(t,\zeta,\cdot))=0\quad\textit{uniformly in}\quad\xi,\zeta\in\mathcal{K}.$$

Proof. We need to show that, for any $\varepsilon>0$ and ξ, $\zeta\in\mathcal{K}$, there is a $T>0$ such that

$$d_L(p(t,\xi,\cdot),p(t,\zeta,\cdot))\le\varepsilon,\quad\forall\, t\ge T,$$

which is equivalent to, for any ξ, $\zeta\in\mathcal{K}$,

$$\sup_{f\in L}|\mathbb{E}f(X_t^\xi)-\mathbb{E}f(X_t^\zeta)|\le\varepsilon,\quad\forall\, t\ge T.$$

As a matter of fact, for any $f\in L$,

$$|\mathbb{E}f(X_t^\xi)-\mathbb{E}f(X_t^\zeta)|\le\mathbb{E}(2\wedge\|X_t^\xi-X_t^\zeta\|_D).$$

From (3.5.12), for any ξ, $\zeta\in\mathcal{K}$, there exists a $T>0$ satisfying

$$\mathbb{E}\|X_t^\xi-X_t^\zeta\|_D^2\le\varepsilon^2,\quad\forall\, t\ge T.$$

Since $f\in L$ is arbitrary, we obtain that

$$\sup_{f\in L}|\mathbb{E}f(X_t^\xi)-\mathbb{E}f(X_t^\zeta)|\le\varepsilon,\quad\forall\, t\ge T.$$

The desired result is now proved.

∎

We are now in a position to present our main result.

Theorem 3.4 *Under the conditions of Lemmas 3.2 and 3.3, the mild solution $X^{\xi}(t)$ to Equation (3.2.1) is stable in distribution.*

Proof. By the definition of stability in distribution, we need to show that there exists a probability measure $\pi(\cdot)$ such that for any $\xi \in D^b_{\mathcal{F}_0}([-r,0];H)$, the transition probabilities $\{p(t,\xi,\cdot) : t \geq 0\}$ converge weakly to $\pi(\cdot)$. As we know, the weak convergence of probability measures is equivalent to a metric concept, we then need to show that, for any $\xi \in D^b_{\mathcal{F}_0}([-r,0];H)$,

$$\lim_{t\to\infty} d_L(p(t,\xi,\cdot),\pi(\cdot)) = 0.$$

By Lemma 3.4, $\{p(t,0,\cdot) : t \geq 0\}$ is Cauchy in the space $\mathcal{P}(D([-r,0];H))$ with metric d_L. Since $\mathcal{P}(D([-r,0];H))$ is a complete metric space under metric d_L (c.f. Theorem 5.4 in Chen (2004)), there is a unique probability measure $\pi(\cdot) \in \mathcal{P}(D([-r,0];H))$ such that

$$\lim_{t\to\infty} d_L(p(t,0,\cdot),\pi(\cdot)) = 0. \tag{3.5.22}$$

Moreover,

$$\lim_{t\to\infty} d_L(p(t,\xi,\cdot),\pi(\cdot)) \leq \lim_{t\to\infty} d_L(p(t,\xi,\cdot),p(t,0,\cdot)) + \lim_{t\to\infty} d_L(p(t,0,\cdot),\pi(\cdot)).$$

Therefore, apply Lemma 3.5 yields that

$$\lim_{t\to\infty} d_L(p(t,\xi,\cdot),\pi(\cdot)) = 0.$$

∎

3.6 An illustrative example

Consider the stochastic process $Z(t,x)$ with Passion jumps describe by

$$
\begin{aligned}
dZ(t,x) = & \left[a\frac{\partial^2}{\partial x^2} + b\int_{-r}^{0} Z(t+\theta,x)d\theta\right]dt + f\left(\int_{-r}^{0} Z(t+\theta,x)d\theta\right)dB(t) \\
& + \int_1^\infty \int_{-r}^{0} Z(t+\theta,x)d\theta y\widetilde{N}(dt,dy),\ t\geq 0,\ a>0,\ 0<x<\pi \\
& Z(t,0) = Z(t,\pi) = 0,\ t\geq 0;\ Z(\theta,x) = \phi(\theta,x),\ 0\leq x\leq \pi, \theta\in[0,\pi] \\
& \phi(\theta,\cdot)\in H = L^2(0,\pi),\ \phi(\cdot,x)\in C([-r,0];\mathbb{R}),
\end{aligned}
$$

where $B(t)$, $t\geq 0$, is a real standard Brownian motion and $\widetilde{N}(\cdot,\cdot)$ is a compensated Poisson random measure on $[1,\infty]\times\mathbb{R}_+$ with parameter $\lambda(dy)dt$ such that $\int_1^\infty y^2\lambda(dy)<\infty$.

Assume moveover that $B(t)$ is independent of $\widetilde{N}(\cdot,\cdot)$, f is a real Lipschitz continuous function on $L^2(0,\pi)$ satisfying for $u,v\in L^2(0,\pi)$

$$|f(u)|\leq c(\|u\|_H+1),\ |f(u)-f(v)|\leq k\|u-v\|_H,$$

with some positive constant c, k. In this example, we take $H = L^2(0,\pi)$ and $A = a(\partial^2/\partial x^2)$ with domain

$$\mathcal{D}(A) = \left\{u\in H = L^2(0,\pi) : \frac{\partial u}{\partial x}, \frac{\partial^2}{\partial x^2}\in L^2(0,\pi), u(0) = u(\pi) = 0\right\}.$$

It can be shown that for arbitrary $u\in\mathcal{D}(A)$

$$\langle u, Au\rangle \leq -a\|u\|_H^2.$$

Moreover, for any $u\in\mathcal{D}(A)$,

$$
\begin{aligned}
& 2\langle u, Au+bv\rangle_H + |f(v)|^2 + \int_1^\infty \|uy\|_H^2\lambda(dy) \\
& \leq -\left(2a-1+\int_1^\infty y^2\lambda(dy)\right)\|u\|_H^2 + (b^2+2c^2)\|v\|_H^2 + 2c^2.
\end{aligned}
$$

Similarly, for $u, v \in \mathcal{D}(A)$,

$$\begin{aligned}
&2\langle u-v, A(u-v)+b(z_1+z_2)\rangle_H |f(z_1)-f(z_2)|^2 \\
&\quad + \int_1^\infty \|(u-v)y\|_H^2 \lambda(dy) \\
&\leq -\left(2a-1+\int_1^\infty y^2\lambda(dy)\right)\|u-v\|_H^2 + (k^2+b^2)\|z_1-z_2\|_H^2.
\end{aligned}$$

Therefore, if $2a > 1 + k^2 + b^2 - \int_1^\infty y^2\lambda(dy)$ is hold, by Theorem 3.4, we then immediately observe that the mild solution process $Z(t,x)$ is stable in distribution.

3.7 Conclusion

To sum up, in this chapter we have focussed on stability in distribution of mild solutions to stochastic delay differential equations with Poisson jumps in Hilbert spaces. Thus, we have first proved that the existence and uniqueness to our SDDEs with jumps. However, in order to obtain the stability in distribution, we cannot directly deal with the mild solutions because the mild solutions are no longer martingales. Therefore, convergence property of strong solutions to mild solutions has been established by approximation of strong solutions. Finally, we have obtained some sufficient conditions for the stability in distribution to our SDDEs and an example is given to demonstrate the applicability of our work. As a consequence, we certainly generalized the stability results of finite dimensions such as in Basak et al. (1996) and Yuan and Mao (2003) to infinite dimensional cases.

Chapter 4

Exponential stability of energy solutions to stochastic partial differential equations with Markovian switching and jumps

4.1 Introduction

So far, there exists an extensive literature dealing with stochastic partial differential equations in a separable Hilbert space. Researchers have obtained fruitful results on the existence, uniqueness and the stability behavior; e.g. Caraballo et al. (2002b), Kwiecinska (2002), Leha et al. (1999), Maslowski (1995) and references therein. In particular, the exponential stability of the strong solutions and mild solutions has been studied frequently. Various methods have been applied to obtain the exponential stability by different researchers, for instance, the method of coercivity condition has been used in Caraballo and Liu (1999); the Lyapunov method has been applied in Liu and Mao (1998) and the method of estimate of solutions has been employed in Taniguchi (1995).

Whereafter, the stochastic partial differential equations with delays have been discussed by several researchers; for example, Caraballo et al. (2000, 2002a) and Taniguchi (1995). The exponential stability of these type of equations was also discussed, for instance, in Taniguchi (1998), the exponential stability of the semi-linear stochastic delay evolution equations was obtained by the estimate of the mild solutions, while in Liu (1998), the result was derived by using the Lyapunov functionals. In addition, stochastic delay differential equations with Markovian switching also have been studied. These can be regarded as the result of several stochastic differential delay equations switching among each other according to the movement of a Markov chain. For example, the exponential stability of these equations have been presented in Mao (1999), Mao et al. (2000), Mao and Yuan (2006) and the almost surely asymptotic stability have been discussed in Mao et al. (2008). On the other hand, some researchers have studied another type of stochastic delay differential equations which driven by jump processes. For example, the existence and uniqueness of stochastic evolution equations of jumps was established in Röckner and Zhang (2007).

However, in recent studies, researchers start to show interests in a type of stochastic differential equations driven both by Markovian switching and Poisson jumps. For example, Svishchuk and Kazmerchuk (2002) are the first to study the pth moment exponential stability of solutions of linear Itô stochastic delay differential equations with Poisson jump and Markovian switching. Luo (2006) established the comparison principle for the nonlinear Itô stochastic differential delay equations with Poisson jump and Markovian switching, using this comparison principle, stability criteria including stability in probability, asymptotic stability in probability, stability in the pth mean, asymptotic stability in the pth mean and the pth moment exponential stability of such equations have been obtained. Later, Luo and Liu (2008) have generalized and improved the results of Svishchuk and Kazmerchuk (2002) using Razumikhin-Lyapunov type function methods and comparison principles as a special case of their theory. To the best of our knowledge to date, there is not much literature dealing with these kind of equations, especially for infinite dimensional case.

It is noticed that the above literature discussed the exponential stability behavior of strong or mild solution to the different kinds of stochastic differential equations. However, by estimating the coefficients functions in the stochastic energy equality, the exponential stability and almost sure exponential stability of energy solutions can be obtained. For example, Taniguchi (2007) discussed the following stochastic delay differential equation with finite delays:

$$dX(t) = A[(t, X(t)) + f(t)]dt + h(t, X(t - \rho(t)))dt + g(t, X(t - \tau(t)))dW(t)$$

where $\rho, \tau : [0, \infty) \to [0, r]$ be differentiable functions with $\rho'(t) \leq M < 1$, $\tau'(t) \leq M < 1$ with $M \geq 0$ and $r > 0$ is a constant; the mappings A, f, h and G are similar to Equation (4.3.1). Then, Wan and Duan (2008) also discussed the same kind of equations and obtained the exponential stability under some suitable conditions. Compared to Taniguchi (2007), the advantage in Wan and Duan (2008) is that $\rho(t)$ and $\tau(t)$ are no longer required to be differentiable functions. Later, Hou et al. (2010) generalized the results from Taniguchi (2007) and Wan and Duan (2008) to cover a class of more general stochastic partial delay differential equations with jumps.

In this chapter, we consider the exponential stability and almost sure exponential stability of energy solutions to stochastic partial functional differential equations with Markovian switching and Poisson jumps of the form:

$$\begin{aligned} dX(t) &= [A(t, X(t)) + \bar{F}(t, X_t, r(t))]dt + \bar{G}(t, X_t, r(t))dW_Q(t) \\ &\quad + \int_{\mathbb{Z}} \bar{L}(t, X_t, r(t), u)\widetilde{N}(dt, du), \quad t \geq 0, \end{aligned} \tag{4.1.1}$$

See Section 4.2 for details of this equation.

As it is well known, in the case without delay, Lyapunov's method is powerful and in general available to obtain sufficient conditions for the stability of solutions. However, the construction of Lyapunov functionals turns out to be more difficult for functional differential equations such as differential equations with memory, even with constant delays, since the history of the process must be taken into account. The advantage of this study is that we do not make use of general methods such as traditional Lyapunov methods, Itô's formula methods and so forth, unlike earlier studies. As mentioned, it is well known that the construction of the Lyapunov functionals is very difficult in stability analysis, thus we use the estimate of the coefficients functions in the stochastic energy equality to overcoming it. In addition, for the case of $L = 0$ or $\lambda = 0$, Equation (4.1.1) has been studied by Taniguchi (2007) and Wan and Duan (2008); for the case of $r = 0$, Equation (4.1.1) has been studied by Hou et al. (2010). We derive sufficient conditions for the exponential stability by the energy equality method and improve the existing results to cover a class of more general stochastic partial delay differential equations with Markovian switching and Poisson jumps. Moreover, unlike Taniguchi (2007) we need not require the functions $\rho_1(t)$, $\rho_2(t)$ and $\rho_3(t)$ to be differentiable.

The contents of this chapter are as follows. In Section 4.2 we give preliminaries of stochastic functional differential equations with Markovian switching and Poisson jumps together with some basic definitions. In Section 4.3 we consider the existence of energy solution (c.f. Definition 4.1). In Section 4.4 we discuss the exponential stability theorems of the energy solution to Equation (4.3.1) by using the energy equality, while the almost sure exponential stability is discussed in Section 4.5. In Section 4.6, we present an example which illustrates the main theorem in this Chapter.

4.2 Stochastic partial differential equations with Markovian switching and jumps

Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let V, H and K be separable Hilbert spaces such that V is continuously, densely embedded in H. Identifying H with its dual, we have the following relation (c.f.Brezis (2011), Page 135-138):

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

where V^* is the dual of V and the injections '$\hookrightarrow$' are continuous and dense. Let $\mathcal{L}(K, H)$ be the space of all bounded linear operators from K to H. We denote by $\|\cdot\|_V$, $\|\cdot\|_{V^*}$, $\|\cdot\|_H$, $\|\cdot\|_K$ and $\|\cdot\|$ the norms in V, V^*, H, K and $\mathcal{L}(K, H)$, respectively; by $\langle \cdot, \cdot \rangle_{V,V^*}$ the duality product between V and V^* and $\langle \cdot, \cdot \rangle_H$ the inner product of H. Furthermore, assume that for $\lambda > 0$,

$$\lambda \|v\|_H^2 \leq \|v\|_V^2, \quad \lambda > 0, \quad v \in V. \tag{4.2.1}$$

Let $k > 0$ and $D := D([-k, 0]; H)$ denote the family of all right-continuous functions with left-hand limits φ from $[-k, 0]$ to H. The space $D([-k, 0]; H)$ is assumed to be equipped with the norm $\|\varphi\|_D = \sup_{-k\leq\theta\leq 0} \|\varphi(\theta)\|_H$. $D^b_{\mathcal{F}_0}([-k, 0]; H)$ denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $D([-k, 0]; H)$-valued random variables. Given $k > 0$ and $T > 0$, we denote by $M^2(-k, T; V)$ the space of all V-valued processes $\{X(t)\}_{t\in[-k,T]}$ which is $\mathcal{F}_t$ measurable and satisfies $\int_{-k}^{T} \mathbb{E}\|X(t)\|_H^2 dt < \infty$.

Let $\{r(t), t \in \mathbb{R}_+\}$, $\mathbb{R}_+ = [0, \infty)$, be a right-continuous Markov chain on the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$

with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+h) = j\,|r(t) = i\} = \begin{cases} \gamma_{ij}\,h + o(h), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\,h + o(h), & \text{if } i = j, \end{cases}$$

for any $t \geq 0$ and small $h > 0$. Here $\gamma_{ij} \geq 0$ is the rate of transition from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. It is well know that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of jumps in any finite sub-interval of $\mathbb{R}_+$.

Let $W_Q(t)$ be a Wiener process and $\int_0^t \Phi(s)dW_Q(s)$ be the stochastic integral with respect to $W_Q(t)$, which is a continuous square-integrable martingale (c.f. Section 2.2). Let $\widetilde{N}(dt,du) := N(dt,du) - dt\lambda(du)$ be the compensated Poisson random measures and $\int_0^T\int_{\mathbb{Z}} L(t,u)\widetilde{N}(dt,du)$ the stochastic integral with respect to $\widetilde{N}(dt,du)$ which is a centered square-integrable martingale, where $\mathbb{Z} \in \mathcal{B}(K - \{0\})$ with $0 \notin \bar{\mathbb{Z}}$, the closure of $\mathbb{Z}$ in K and $\mathcal{B}(K-\{0\})$ denotes the Borel σ-filed of $K - \{0\}$ (c.f. Section 2.3). We always assume in this chapter that W_Q, $r(\cdot)$ and $\widetilde{N}$ are independent of the $\mathcal{F}_0$ and of each other.

We introduce the following stochastic functional partial differential equations with Markovian switching and Poisson jumps:

$$\begin{aligned} dX(t) &= [A(t,X(t)) + \bar{F}(t,X_t,r(t))]dt + \bar{G}(t,X_t,r(t))dW_Q(t) \\ &\quad + \int_{\mathbb{Z}} \bar{L}(t,X_t,r(t),u)\widetilde{N}(dt,du), \quad t \geq 0, \end{aligned} \tag{4.2.2}$$

with the initial datum $X(\theta) = \xi(\theta) \in L^2(\Omega, D([-k,0];H))$, $r(0) = r_0$, $\theta \in [-k,0]$, where $A : [0,\infty] \times V \to V^*$, $\bar{F} : [-k,\infty) \times D \times \mathbb{S} \to V^*$, $\bar{G} : [-k,\infty) \times D \times \mathbb{S} \to \mathcal{L}(K,H)$ and $\bar{L} : [-k,\infty) \times D \times \mathbb{S} \times \mathbb{Z} \to H$.

Now, we introduce the definition of energy solution to Equation (4.2.2) following from Taniguchi (2007).

Definition 4.1 (Energy solution) *An $\mathcal{F}_t$-adapted stochastic process $X(t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be the energy solution to Equation (4.2.2) if the following* *conditions are satisfied:*

(i) $X(t) \in M^2(-k,T;V) \cap L^2(\Omega; D(-k,T;H)), \quad T>0,$

(ii) the following equation holds in V^* *almost surely, for* $t \in [0,T)$,

$$\begin{aligned} X(t) &= X(0) + \int_0^t \big[A\big(s,X(s)\big) + \bar{F}(s,X_s,r(s))\big]ds \\ &\quad + \int_0^t \bar{G}(s,X_s,r(s))dW_Q(s) \\ &\quad + \int_0^t \int_{\mathbb{Z}} \bar{L}(s,X_s,r(s),u)\widetilde{N}(ds,du), \quad t \geq 0, \\ X(t) &= \xi(t), \quad t \in [-\tau,0]; \end{aligned} \tag{4.2.3}$$

(iii) the following stochastic energy equality holds:

$$\begin{aligned} &\|X(t)\|_H^2 \\ = \;& \|X(0)\|_H^2 + \int_0^t \sum_{j=1}^N \gamma_{ij}\|X(s)\|_H^2 ds + \int_0^t \|\bar{G}(s,X_s,r(s))\|_{\mathcal{L}_2^0}^2 ds \\ &+2\int_0^t \langle X(s), A(s,X(s)) + \bar{F}(s,X_s,r(s))\rangle_{V,V^*} ds \\ &+2\int_0^t \langle X(s), \bar{G}(s,X_s,r(s))dW_Q(s)\rangle_H \\ &+\int_0^t \int_{\mathbb{Z}} \|\bar{L}(s,X_s,r(s),u)\|_H^2 \lambda(du)ds \\ &+\int_0^t \int_{\mathbb{Z}} [\|\bar{L}(s,X_s,r(s),u)\|_H^2 + 2\langle X_{s-}, \bar{L}(s,X_s,r(s),u)\rangle_H]\widetilde{N}(ds,du). \end{aligned} \tag{4.2.4}$$

For convenience, we need to introduce the following two concepts from Taniguchi (2007). Assume the term $F(t,0,i) \equiv 0$, $G(t,0,i) \equiv 0$ and $L(t,0,i,u) \equiv 0$ for any $i \in S$, $u \in \mathbb{Z}$, then Equation (4.2.2) has a trivial solution $X(t,0) \equiv 0$ only if $\xi = 0$.

Definition 4.2 *The trivial solution of Equation (4.2.2) or the equation (4.2.2) is said to be exponentially stable in mean square if there exist* $\eta > 0$ *and* $B =$

$B(X(0)) > 0$, *such that*

$$\mathbb{E}\|X(t)\|_H^2 \le Be^{-\eta t}, \quad t \ge 0.$$

Definition 4.3 *The trivial solution of Equation (4.2.2) or the equation (4.2.2) is said to be exponentially stable almost surely if there exist positive constants* $B = \delta(\epsilon)$, $\eta > 0$, *a subset* $\Omega_0 \subset \Omega$ *with* $\mathbb{P}(\Omega_0) = 0$, *and for each* $\omega \in \Omega - \Omega_0$, *there exists a positive random number* $T(\omega)$ *such that*

$$\|X(t)\|_H^2 \le Be^{-\eta t}, \quad t \ge T(\omega).$$

4.3 Existence of energy solutions

In this section we consider the existence of the energy solutions to the following stochastic partial functional differential equation with Markovian switching and Poisson jumps.

For $k > 0$, let $\rho_1(t)$, $\rho_2(t)$ and $\rho_3(t)$ be continuous functions from $[0, \infty] \to [0, k]$, assume that

$$A : [0, \infty) \times V \to V^* \text{ with } A(t, 0) = 0,$$

$$F : [0, \infty) \times H \times \mathbb{S} \to H,$$

$$G : [0, \infty) \times H \times \mathbb{S} \to \mathcal{L}(K, H)$$

and

$$L : [0, \infty] \times H \times \mathbb{Z} \times \mathbb{S} \to H$$

are Lebesgue measurable.

We investigate the following stochastic partial differential equation with Markovian switching and jumps:

$$
\begin{aligned}
dX(t) &= A\Big(t, X(t)\Big)dt + F\Big(t, X(t-\rho_1(t)), r(t)\Big)dt \\
&\quad + G\Big(t, X(t-\rho_2(t)), r(t)\Big)dW_Q(t), \\
&\quad + \int_{\mathbb{Z}} L\Big(t, X(t-\rho_3(t)), r(t), u\Big)\widetilde{N}(dt, du) \quad t \geq 0,
\end{aligned} \tag{4.3.1}
$$

with the initial condition

$$
X(t) = \xi(t) \in L^2(\Omega, D([-k, 0]; H)), \quad t \in [-k, 0] \tag{4.3.2}
$$

and $r(0) = r_0$, r_0 is an $\mathbb{S}$-valued, $\mathcal{F}_{t_0}$-measurable random variable.

Set $\bar{F}(t, \phi, r(t)) = F(t, \phi(-\rho_1(t)), r(t))$, $\bar{G}(t, \phi, r(t)) = G(t, \phi(-\rho_2(t), r(t))$ and $\bar{L}(t, \phi, r(t)) = L(t, \phi(-\rho_3(t), r(t)))$ for any $\phi \in D$. Then Equation (4.3.1) can be regarded as a stochastic partial functional differential equation (4.2.2). Furthermore, we impose the following conditions:

Conditions 4.1 *(Monotonicity and coercivity) There is a pair of constants $\alpha > 0$ and $\lambda_1 \in \mathbb{R}$ such that for a.e. $t \in (0, T)$ and for all x, $y \in V$ and $i \in \mathbb{S}$*

$$
-2\langle A(t, x) - A(t, y), x - y\rangle_{V,V^*} + \lambda_1 \|x - y\|_H^2 \geq \alpha \|x - y\|_V^2.
$$

Conditions 4.2 *(Measurability) For all $x \in V$, the map $t \in (0, T) \to A(t, x) \in V^*$ is measurable.*

Conditions 4.3 *(Hemicontinuity) The map $\xi \in \mathbb{R} \to \langle A(t, u + \xi v), w\rangle \in \mathbb{R}$ is continuous for all u, v, $w \in V$ and a.e. $t \in (0, T)$.*

Conditions 4.4 *(Boundedness) There exists a constant $c > 0$ such that for any $x \in V$ and a.e $t \in (0, T)$,*

$$
\|A(t, x)\|_{V^*} \leq c\|x\|_V.
$$

Conditions 4.5 *(Lipschitz condition and linear growth condition) There exists a pair of positive constants c_1, c_2 satisfying that for all $x, y \in V$, and $i \in \mathbb{S}$*

$$\begin{aligned}
&\|F(t,x,i) - F(t,y,i)\|_{V^*}^2 + \|G(t,x,i) - G(t,y,i)\|_{\mathcal{L}_2^0}^2 \\
&\quad + \|L(t,x,i,u) - L(t,y,i,u)\|_H^2 \\
\leq\ & c_1\|x-y\|_D^2
\end{aligned}$$

and

$$\|F(t,x,i)\|_{V^*}^2 + \|G(t,x,i)\|_{\mathcal{L}_2^0}^2 + \|L(t,x,i,u)\|_H^2 \leq c_2(1 + \|x\|_D^2).$$

Theorem 4.1 *Under Conditions 4.1- 4.5, Equation (4.3.1) with the given initial data (4.3.2) has a unique energy solution $X(t)$ on $t \in [-k, T]$. Moreover, it holds that*

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}\|X(t)\|_H^2 &= 2\mathbb{E}\Big\langle X(t), A\Big(t, X(t)\Big) + F\Big(t, X(t-\rho_1(t)), r(t)\Big)\Big\rangle_{V,V^*} \\
&\quad + \mathbb{E}\Big\|G\Big(t, X(t-\rho_2(t)), r(t)\Big)\Big\|_{\mathcal{L}_2^0}^2 + \mathbb{E}\sum_{j=1}^{N}\gamma_{ij}\|X(t)\|_H^2 \\
&\quad + \mathbb{E}\int_{\mathbb{Z}}\Big\|L\Big(t, X(t-\rho_3(t)), r(t), u\Big)\Big\|_H^2 \lambda(du) \qquad (4.3.3)
\end{aligned}$$

Proof. Let $T > 0$ be arbitrary number. It is sufficient to show that Equation (4.3.1) has a unique solution on $[-k, T]$. Recall that almost every sample path of $r(\cdot)$ is a right-continuous step function with a finite number of simple jumps on $[0, T]$. Also, it is known (c.f. Skorokhod (1989)) that there is a sequence $\{\tau_k\}_{k\geq 0}$, $0 = \tau_0 < \tau_1 < \cdots < \tau_k \to \infty$ of stopping times such that $r(t)$ is a constant on every interval $[\tau_k, \tau_{k+1})$, that is, for every $k \geq 0$,

$$r(t) = r(\tau_k) \quad \text{on} \quad \tau_k \leq t < \tau_{k+1}.$$

We first consider Equation (4.3.1) on $t \in [0, \tau_1 \wedge T]$ which becomes

$$
\begin{aligned}
dX(t) &= \Big[A\Big(t, X(t)\Big) + F\Big(t, X(t-\rho_1(s)), r(0)\Big)\Big]dt \\
&\quad + G\Big(t, X(t-\rho_2(s)), r(0)\Big)dW_Q(t) \\
&\quad + L\Big(t, X(t-\rho_3(t)), r(0), u\Big)\widetilde{N}(dt, du) \qquad (4.3.4)
\end{aligned}
$$

with initial data $X(0) = \xi$. This is a stochastic partial differential equation without Markovian switching. By following the argument from Proposition 3.1 and Theorem 3.2 in Röckner and Zhang (2007), it can be shown that Equation (4.3.4) has a unique solution on $[-k, \tau_1 \wedge T]$.

We next consider Equation (4.3.1) on $t \in [\tau_1 \wedge T, \tau_2 \wedge T]$ which becomes

$$
\begin{aligned}
dX(t) &= \Big[A\Big(t, X(t)\Big) + F\Big(t, X(t-\rho_1(s)), r(\tau_1 \wedge T)\Big)\Big]dt \\
&\quad + G\Big(t, X(t-\rho_2(s)), r(\tau_1 \wedge T)\Big)dW_Q(t) \\
&\quad + L\Big(t, X(t-\rho_3(t)), r(\tau_1 \wedge T), u\Big)\widetilde{N}(dt, du) \qquad (4.3.5)
\end{aligned}
$$

with initial data $X(t \wedge \tau_1)$ given by the solution of Equation (4.3.1). Again, by following the argument from Proposition 3.1 and Theorem 3.2 in Röckner and Zhang (2007), we know Equation (4.3.1) has a unique solution on $[\tau_1 \wedge T - k, \tau_2 \wedge T]$. Repeating this procedure, it can be shown that Equation (4.3.1) has a unique solution $X(t)$ on $t \in [-k, T]$. Since T is arbitrary, the existence and uniqueness have been proved.

Under Conditions 4.1- 4.5, the Equation (4.3.3) is proved in Hou et al. (2010), Theorem 2.1 and Röckner and Zhang (2007), Theorem 3.2.

■

4.4 Exponential stability in mean square

In this section we consider the exponential stability theorem in mean square and almost sure exponential stability theorem of an energy solution to Equation (4.3.1). First we present some sufficient conditions for an solution $X(t)$ to Equation (4.3.1) to be exponentially stable in mean square.

Theorem 4.2 *Suppose that Conditions 4.2 - 4.4 holds. Moreover, Equation (4.3.1) satisfies the following conditions:*

Conditions 4.6 *There exists constants $a_1 > 0$ and continuous, integrable functions $\alpha_1(t) > 0$ such that for a.e. $t \in [0, \infty)$ and for all x, $y \in V$,*

$$-2\langle A(t,x) - A(t,y), x - y\rangle_{V,V^*} + \alpha_1(t)\|x - y\|_H^2 \geq a_1\|x - y\|_V^2; \tag{4.4.1}$$

Conditions 4.7 *There exist integrable functions α_2, $\beta_2 : [0, \infty) \to \mathbb{R}_+$ such that for certain constant $a_2 \geq 0$, $x \in H$ and $i \in \mathbb{S}$,*

$$\|F(t,x,i)\|_H^2 \leq \Big(a_2 + \alpha_2(t)\Big)\|x\|_H^2 + \beta_1(t); \tag{4.4.2}$$

Conditions 4.8 *There exist integrable functions α_3, $\beta_3 : [0, \infty) \to \mathbb{R}_+$ such that for certain constant $a_3 \geq 0$, $x \in H$ and $i \in \mathbb{S}$,*

$$\|G(t,x,i)\|_{\mathcal{L}_2^0}^2 \leq \Big(a_3 + \alpha_3(t)\Big)\|x\|_H^2 + \beta_2(t); \tag{4.4.3}$$

Conditions 4.9 *There exist integrable functions α_4, $\beta_4 : [0, \infty) \to \mathbb{R}_+$ such that for certain constant $a_4 \geq 0$, $x \in H$ and $i \in \mathbb{S}$,*

$$\int_{\mathbb{Z}} \|L(t,x,u,i)\|_H^2 \lambda(du) \leq \Big(a_4 + \alpha_4(t)\Big)\|x\|_H^2 + \beta_3(t); \tag{4.4.4}$$

Conditions 4.10 *There exist $\sigma > 0$ such that*

$$\int_0^\infty e^{\sigma t}\beta_k(t)dt < \infty, \quad k = 1, 2, 3; \tag{4.4.5}$$

Conditions 4.11 *For given λ defined by (4.2.1),*

$$\lambda a_1 > 2\sqrt{a_2} + a_3 + a_4. \tag{4.4.6}$$

Then for any energy solution $X(t)$ to Equation (4.3.1), there exist $\delta \in (0, \sigma)$ and $B \geq 1$ such that

$$\mathbb{E}\|X(t)\|_H^2 \leq Be^{-\delta t}, \quad t \geq 0. \tag{4.4.7}$$

In other words the energy solution $X(t)$ to Equation (4.3.1) is exponentially stale in mean square.

Proof. By Theorem 4.1 we have unique energy solution $X(t)$ to Equation (4.3.1). Now by elemental inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ for all $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we can take γ obeying

$$a_1\lambda > \gamma + \frac{a_2}{\gamma} + a_3 + a_4 > 2\sqrt{a_2} + a_3 + a_4.$$

Furthermore, there exists a $\delta \in (0, \sigma)$ sufficiently small such that

$$a_1\lambda > \delta + \gamma + e^{\delta k}\frac{a_2}{\gamma} + e^{\delta k}a_3 + e^{\delta k}a_4. \tag{4.4.8}$$

Now, in view of (4.4.1) and (4.2.1), we derive that, for any $x \in V$ and $t \geq 0$,

$$\begin{aligned} 2\langle x, A(t,x)\rangle_{V,V^*} &\leq -\delta_1\|x\|_V^2 + \alpha_1(t)\|x\|_H^2 \\ &\leq [-\lambda\delta_1 + \alpha_1(t)]\|x\|_H^2. \end{aligned} \tag{4.4.9}$$

For convenience, we denote

$$\mu(t) = \alpha_1(t) + e^{\delta k}\frac{\alpha_2(t)}{\gamma} + e^{\delta k}\alpha_3(t) + e^{\delta k}\alpha_4(t),$$

$$\beta(t) = \frac{\beta_1(t)}{\gamma} + \beta_2(t) + \beta_3(t).$$

Since $\alpha_i(t)$, $i = 1, 2, 3, 4$ is integrable, together with Condition 4.10, this yields that

$$R_1 = \int_0^\infty \mu(s)ds < \infty, \quad R_2 = \int_0^\infty \beta(s)ds \leq R_3 = \int_0^\infty e^{\sigma s}\beta(s)ds < \infty. \tag{4.4.10}$$

Let us define the following function

$$K(t) = \mathbb{E}\|X(t)\|_H^2 e^{\delta t}\exp\bigg(-\int_0^t[\mu(s)+e^{\delta s}\beta(s)]ds\bigg), \text{ if } \ t\geq 0, \quad (4.4.11)$$

$$K(t) = \mathbb{E}\|X(t)\|_H^2 e^{\delta t}, \text{ if } \ -k\leq t\leq 0. \quad (4.4.12)$$

$K(t)$ is continuous on $[-k,\infty)$ and we have,

$$\begin{aligned}\frac{dK(t)}{dt} &= e^{\delta t}\exp\bigg(-\int_0^t[\mu(s)+e^{\delta s}\beta(s)]ds\bigg)\bigg\{\delta\mathbb{E}\|X(t)\|_H^2\\ &\quad -[\mu(t)+e^{\delta t}\beta(t)]\mathbb{E}\|X(t)\|_H^2+2\mathbb{E}\langle X(t),A(t,X(t))\rangle_{V,V^*}\\ &\quad +2\mathbb{E}\Big\langle X(t),F\big(t,X(t-\rho_1(s)),r(t)\big)\Big\rangle_{V,V^*}\\ &\quad +\mathbb{E}\Big\|G\big(t,X(t-\rho_2(t)),r(t)\big)\Big\|_{\mathcal{L}_2^0}^2+\mathbb{E}\sum_{j=1}^N\gamma_{ij}\|X(t)\|_H^2\\ &\quad +\mathbb{E}\int_{\mathbb{Z}}\Big\|L\big(t-X(t-\rho_3(t),r(t),u)\big)\Big\|_H^2\lambda(du)\bigg\}. \end{aligned} \quad (4.4.13)$$

From Conditions 4.7 - 4.9 and (4.4.9), it immediately follows that for $t\geq 0$,

$$\begin{aligned}\frac{dK(t)}{dt} &\leq e^{\delta t}\exp\Big(-\int_0^t[\mu(s)+e^{\delta s}\beta(s)]ds\Big)\Big\{\delta\mathbb{E}\|X(t)\|_H^2\\ &\quad -[\mu(t)+e^{\delta t}\beta(t)]\mathbb{E}\|X(t)\|_H^2+[-\lambda\delta_1+\alpha_1(t)]\mathbb{E}\|X(t)\|_H^2\\ &\quad +\gamma_2\mathbb{E}\|X(t)\|_H^2+\frac{\mathbb{E}\Big\|F\big(t,X(t-\rho_1(t)),r(t)\big)\Big\|_H^2}{\gamma_2}\\ &\quad +\mathbb{E}\Big\|G\big(t,X(t,X(t-\rho_2(t)),r(t)\big)\Big\|_{\mathcal{L}_2^0}^2+\mathbb{E}\sum_{j=1}^N\gamma_{ij}\|X(t)\|_H^2\\ &\quad +\mathbb{E}\int_{\mathbb{Z}}\Big\|L\big(t,X(t-\rho_3(t)),r(t),u\big)\Big\|_H^2\lambda(du)\Big\}\\ &\leq e^{\delta t}\exp\Big(-\int_0^t[\mu(s)+e^{\delta s}\beta(s)]ds\Big)\Big\{\delta\mathbb{E}\|X(t)\|_H^2\\ &\quad -[\mu(t)+e^{\delta t}\beta(t)]\mathbb{E}\|X(t)\|_H^2+[-\lambda a_1+\alpha_1(t)]\mathbb{E}\|X(t)\|_H^2\\ &\quad +\gamma\mathbb{E}\|X(t)\|_H^2+\frac{a_2+\alpha_2(t)}{\gamma}\|X(t-\rho_1(t))\|_H^2+\beta_1(t)\\ &\quad +\big(a_3+\alpha_3(t)\big)\|X(t-\rho_2(t))\|_H^2+\beta_2(t)\\ &\quad +\big(a_4+\alpha_4(t)\big)\|X(t-\rho_3(t))\|_H^2+\beta_3(t)\Big\}\end{aligned}$$

$$
\begin{aligned}
&\leq [\delta-\mu(t)+\alpha_1(t)-\lambda a_1+\gamma]K(t)+e^{\delta t}\beta(t)-e^{\delta t}\beta(t)K(t)\\
&\quad+e^{\delta t}\exp\Big(-\int_0^t[\mu(s)+e^{\delta s}\beta(s)]ds\Big)\frac{a_2+\alpha_2(t)}{\gamma}\mathbb{E}\|X(t-\rho_1(t))\|_H^2\\
&\quad+e^{\delta t}\exp\Big(-\int_0^t[\mu(s)+e^{\delta s}\beta(s)]ds\Big)\big(a_3+\alpha_3(t)\big)\mathbb{E}\|X(t-\rho_2(t))\|_H^2\\
&\quad+e^{\delta t}\exp\Big(-\int_0^t[\mu(s)+e^{\delta s}\beta(s)]ds\Big)\big(a_4+\alpha_4(t)\big)\mathbb{E}\|X(t-\rho_3(t))\|_H^2.
\end{aligned}
\tag{4.4.14}
$$

In what follows, we justify that for any $t \geq 0$

$$
K(t) \leq M := 1+\sup_{-k\leq t\leq 0}\mathbb{E}\|X(t)\|_H^2. \tag{4.4.15}
$$

We show (4.4.15) by contradiction. If (4.4.15) is not true, then there must exists some $t_1 > 0$ such that for all $\varphi > 0$,

$$
K(t) < M, \quad 0 \leq t < t_1, \quad K(t_1) = M, \quad K(t) > M, \quad t_1 < t \leq t_1+\varphi. \tag{4.4.16}
$$

Since $dK(t)/dt$ exists by (4.4.13), in addition to (4.4.16), immediately implies that

$$
\frac{d}{dt}K(t_1) \geq 0. \tag{4.4.17}
$$

Substituting (4.4.16) into (4.4.14) and combining $M \geq 1$ from (4.4.15) to note that the term $e^{\delta t_1}\beta(t) - e^{\delta_1 t}\beta(t)K(t) \leq 0$, it yields

$$
\begin{aligned}
\frac{d}{dt}K(t_1) &\leq [\delta-\mu(t_1)+\alpha_1(t_1)-\lambda a_1+\gamma]K(t_1)\\
&\quad+e^{\delta t_1}\exp\Big(-\int_0^{t_1}[\mu(s)+e^{\delta s\beta(s)}]ds\Big)\frac{a_2+\alpha_2(t_1)}{\gamma}\mathbb{E}\|X(t_1-\rho_1(t_1))\|_H^2\\
&\quad+e^{\delta t_1}\exp\Big(-\int_0^{t_1}[\mu(s)+e^{\delta s\beta(s)}]ds\Big)\big(a_3+\alpha_3(t_1)\big)\mathbb{E}\|X(t_1-\rho_2(t_1))\|_H^2\\
&\quad+e^{\delta t_1}\exp\Big(-\int_0^{t_1}[\mu(s)+e^{\delta s\beta(s)}]ds\Big)\big(a_4+\alpha_4(t_1)\big)\mathbb{E}\|X(t_1-\rho_2(t_1))\|_H^2.
\end{aligned}
\tag{4.4.18}
$$

Next, we consider the following different cases to derive the desired assertion.

Case 1: Suppose that $t_1-\rho_1(t_1)\geq 0$, $t_1-\rho_2(t_1)\geq 0$ and $t_1-\rho_3(t_1)\geq 0$. Then we have $t_1>t_1-\rho_1(t_1)\geq 0$, $t_1>t_1-\rho_2(t_1)\geq 0$ and $t_1>t_1-\rho_3(t_1)\geq 0$, which implies that $K(t_1-\rho_1(t_1))<M$, $K(t_1-\rho_2(t_1))<M$ and $K(t_1-\rho_3(t_1))<M$. By (4.4.11) the definition of $K(t)$, we then have from (4.4.8) and (4.4.16) that

$$\begin{aligned}
\frac{d}{dt}K(t_1) &\leq [\delta-\mu(t_1)+\alpha_1(t_1)-\lambda a+\gamma]K(t_1)\\
&\quad +e^{\delta\rho_1(t_1)}\exp\Big(-\int_{t_1-\rho_1(t_1)}^{t_1}[\mu(s)+e^{\delta s}\beta(s)]ds\Big)\frac{a_2+\alpha_2(t_1)}{\gamma}K(t_1-\rho_1(t_1))\\
&\quad +e^{\delta\rho_2(t_1)}\exp\Big(-\int_{t_1-\rho_2(t_1)}^{t_1}[\mu(s)+e^{\delta s}\beta(s)]ds\Big)\big(a_3+\alpha_3(t_1)\big)K(t_1-\rho_2(t_1))\\
&\quad +e^{\delta\rho_2(t_1)}\exp\Big(-\int_{t_1-\rho_3(t_1)}^{t_1}[\mu(s)+e^{\delta s}\beta(s)]ds\Big)\big(a_4+\alpha_4(t_1)\big)K(t_1-\rho_2(t_1))\\
&\leq [\delta-\mu(t_1)+\alpha_1(t_1)-\lambda a_1+\gamma]M+e^{\delta k}\frac{a_2+\alpha_2(t_1)}{\gamma}M\\
&\quad +e^{\delta k}(a_3+\alpha_3(t_1))M+e^{\delta k}(a_4+\alpha_4(t_1))M\\
&\leq [\delta-\lambda a_1+\gamma+e^{\delta k}\frac{a_2}{\gamma}+e^{\delta k}a_3+e^{\delta k}a_4]M\\
&< 0,
\end{aligned}$$

which contradicts with (4.4.17). That is, the desired assertion (4.4.15) must hold.

Case 2: Suppose that $t_1-\rho_1(t_1)\leq 0$, $t_1-\rho_2(t_1)\leq 0$ and $t_1-\rho_3(t_1)\leq 0$. Then $-k\leq t_1-\rho_1(t_1)\leq 0$, $-k\leq t_1-\rho_2(t_1)\leq 0$ and $-k\leq t_1-\rho_3(t_1)\leq 0$. Taking into account (4.4.12) the definition of $K(t)$, it follows that

$$\begin{aligned}
\frac{d}{dt}K(t_1) &\leq [\delta-\mu(t_1)+\alpha_1(t_1)-\lambda a_1+\gamma]K(t_1)\\
&\quad +e^{\delta\rho_1(t_1)}\exp\Big(-\int_0^{t_1}[\mu(s)+e^{\delta s}\beta(s)]ds\Big)\frac{a_2+\alpha_2(t_1)}{\gamma}K(t_1-\rho_1(t_1))\\
&\quad +e^{\delta\rho_2(t_1)}\exp\Big(-\int_0^{t_1}[\mu(s)+e^{\delta s}\beta(s)]ds\Big)\big(a_3+\alpha_3(t_1)\big)K(t_1-\rho_2(t_1))\\
&\quad +e^{\delta\rho_3(t_1)}\exp\Big(-\int_0^{t_1}[\mu(s)+e^{\delta s}\beta(s)]ds\Big)\big(a_4+\alpha_4(t_1)\big)K(t_1-\rho_3(t_1))
\end{aligned} \tag{4.4.19}$$

Putting (4.4.8), (4.4.15) and (4.4.16) into (4.4.19) gets

$$\frac{d}{dt}K(t_1) \leq [\delta - \lambda a_1 + \gamma + e^{\delta k}\frac{a_2}{\gamma} + e^{\delta k}a_3 + e^{\delta k}a_4]M < 0,$$

by contradiction, we must have (4.4.15) holds for any $t \geq 0$.

For other cases, suppose that:

Case 3: $t_1 - \rho_1(t_1) \leq 0$, $t_1 - \rho_2(t_1) \geq 0$ and $t_1 - \rho_3(t_1) \geq 0$.

Case 4: $t_1 - \rho_1(t_1) \leq 0$, $t_1 - \rho_2(t_1) \leq 0$ and $t_1 - \rho_3(t_1) \geq 0$.

Case 5: $t_1 - \rho_1(t_1) \leq 0$, $t_1 - \rho_2(t_1) \geq 0$ and $t_1 - \rho_3(t_1) \leq 0$.

Case 6: $t_1 - \rho_1(t_1) \geq 0$, $t_1 - \rho_2(t_1) \leq 0$ and $t_1 - \rho_3(t_1) \leq 0$.

Case 7: $t_1 - \rho_1(t_1) \geq 0$, $t_1 - \rho_2(t_1) \leq 0$ and $t_1 - \rho_3(t_1) \geq 0$.

Case 8: $t_1 - \rho_1(t_1) \geq 0$, $t_1 - \rho_2(t_1) \geq 0$ and $t_1 - \rho_3(t_1) \leq 0$.

In the same way as case 1 and 2, we get

$$\frac{d}{dt}K(t_1) \leq [\delta - \lambda a_1 + \gamma + e^{\delta k}\frac{a_2}{\gamma} + e^{\delta k}a_3 + e^{\delta k}a_4]M < 0.$$

Thus, one obtains

$$\frac{d}{dt}K(t_1) < 0,$$

which contradicts with (4.4.17). By contradiction, we have (4.4.15) holds true for any $t \geq 0$. Thus we obtain from (4.4.15) that

$$\mathbb{E}\|X(t)\|_H^2 \leq Me^{-\delta t}\exp\Big(\int_0^t [\mu(s) + e^{\delta s}\beta(s)]ds\Big),$$

which, combining (4.4.10), immediately implies that with $B = e^{R_1+R_3}$

$$\mathbb{E}\|X(t)\|_H^2 \leq MBe^{-\delta t}, \quad t \geq 0.$$

Consequently, the exponential stability in mean square of energy solution to Equation (4.3.1) follows.

■

4.5 Almost sure exponential stability

In this section, we consider the almost sure exponential stability of energy solution to Equation (4.3.1). First of all, we recall the Borel-Cantelli lemma from Kuo (2006) which play a key role in the analysis for almost sure exponential stability of energy solution to Equation (4.3.1). Then we recall an important result Lemma 4.2 from Röckner and Zhang (2007) which deals with the term of jumps in Equation (4.3.1).

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events in some probability space. Consider the event A given by $A = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$. It can be shown that $\omega \in A$ if and only if $\omega \in A_n$ for infinitely many n's. Thus we can think of event A as the event that A_n's occur infinitely often. We will use the following notation:

$$\{A_n \text{ i.o.}\} = \{A_n \text{ infinity often }\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Lemma 4.1 *(Borel-Cantelli lemma) Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. Then $\mathbb{P}\{A_n \text{ i.o.}\} = 0$.*

The complement of $\{A_n \text{ i.o.}\}$ is the event $\{A_n \text{ f.o.}\} = \{A_n \text{ finitely often }\}$ that A_n's occur finitely often.

Lemma 4.2 *Let $X(t) \in M^2(-k, T; V) \cap L^2(\Omega; D(-k, T; H))$ and $T > 0$. Then for any $t \geq 0$, there exists a constant $C > 0$ such that*

$$\begin{aligned} &\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \int_{\mathbb{Z}} [\|L(l, X(l-\delta(l)), u)\|_H^2 \right. \\ &\qquad \left. + 2\langle X(l-), L(l, X(l-\delta(l)), u)\rangle_H] \widetilde{N}(dl, du) \right| \\ \leq\ & C\mathbb{E} \int_0^t \int_{\mathbb{Z}} \|L(s, X(s-\delta(s)), u)\|_H^2 \lambda(du) ds + \frac{1}{4} \mathbb{E} \sup_{0 \leq s \leq t} \|X(s)\|_H^2. \end{aligned} \tag{4.5.1}$$

Theorem 4.3 *Assume that all the conditions of Theorem 4.2 are satisfied. Furthermore, we impose the following condition:*

$$\text{Both } \alpha_i(t), i = 1, 2, 3, 4 \text{ and } e^{\delta t}\beta_k(t), k = 1, 2, 3 \text{ are bounded functions}\ . \tag{4.5.2}$$

Then there exists $T(\omega) > 0$ such that for all $t > T(\omega)$,

$$\|X(t)\|_H^2 \le e^{\delta/2} e^{-\delta t/2}$$

with probability one.

In other words, the energy solution $X(t)$ to Equation (4.3.1) is almost sure exponentially stable.

Proof. Let N_1, N_2 and N_3 be positive integers such at

$$N_1 - \rho_1(N_1) \ge N_1 - k \ge 1, \quad N_2 - \rho_2(N_2) \ge N_2 - k \ge 1$$

and

$$N_3 - \rho_3(N_3) \ge N_3 - k \ge 1.$$

Set $N = \max\{N_1, N_2, N_3\}$ and $I_N = [N, N+1]$.

Furthermore, we denote

$$\begin{aligned}
\alpha(t) &= 1 + \alpha_1(t) + \gamma + \frac{a_2 + \alpha_2(t)}{\gamma} e^{\delta k} + 33\big(a_3 + \alpha_3(t)\big)e^{\delta k} \\
&\quad + (C+1)(a_4 + \alpha_4(t))e^{\delta k}, \\
\beta(t) &= \frac{\beta_1(t)}{\gamma} + 33\beta_2(t) + (C+1)\beta_3(t),
\end{aligned}$$

where C is the given constant in Lemma 4.2.

Since $X(t)$ is the energy solution to equation (4.3.1), it follows that for any $t \in [N, N+1]$,

$$\begin{aligned}
\|X(t)\|_H^2 &= \|X(N)\|_H^2 + \int_N^t \sum_{j=1}^N \gamma_{ij} \|X(s)\|_H^2 ds \\
&\quad + \int_N^t \big\|G\big(s, X(s-\rho_2(s)), r(s)\big)\big\|_{\mathcal{L}_2^0}^2 ds \\
&\quad + 2\int_N^t \Big\langle X(s), A\big(s, X(s)\big) + F\big(s, X(s-\rho_1(s)), r(s)\big)\Big\rangle_{V,V^*} ds \\
&\quad + 2\int_N^t \Big\langle X(s), G\big(s, X(s-\rho_2(s)), r(s)\big) dW(s)\Big\rangle_H
\end{aligned}$$

$$
\begin{aligned}
&+\int_N^t\int_{\mathbb{Z}}\|L(s,X(s-\rho_3(s)),r(s),u)\|_H^2 ds\lambda(du)\\
&+\int_N^t\int_{\mathbb{Z}}\Big[\|L(s,X(s-\rho_3(s),r(s),u))\|_H^2\\
&\quad+2\langle X(s-),L(s,X(s-\rho_3(s)),r(s),u)\rangle_H\Big]\widetilde{N}(ds,du).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\mathbb{E}\sup_{t\in I_N}\|X(t)\|_H^2 \leq\ & \mathbb{E}\|X(N)\|_H^2+\mathbb{E}\sup_{t\in I_N}\int_N^t\|X(s)\|_H^2 ds\\
&+2\mathbb{E}\sup_{t\in I_N}\int_N^t\langle X(s),A\big(s,X(s)\big)\rangle_{V,V^*}ds\\
&+2\mathbb{E}\sup_{t\in I_N}\int_N^t\langle X(s),F\big(s,X(s-\rho_1(s)),r(s)\big)\rangle_H ds\\
&+\int_N^{N+1}\mathbb{E}\|G\big(s,X(s-\rho_2(s)),r(s)\|_{\mathcal{L}_2^0}^2 ds\\
&+2\mathbb{E}\sup_{t\in I_N}\int_N^t\Big\langle X(s),G\big(s,X(s-\rho_2(s)),r(s)\big)dW(s)\Big\rangle_{V,V^*}\\
&+\int_N^{N+1}\int_{\mathbb{Z}}\mathbb{E}\|L(s,X(s-\rho_3(s),r(s),u))\|_H^2\lambda(du)ds\\
&\mathbb{E}\sup_{t\in I_N}\int_N^t\int_{\mathbb{Z}}\Big[\|L(s,X(s-\rho_3(s),r(s),u))\|_H^2\\
&\quad+2\langle X(s-),L(s,X(s-\rho_3(s)),r(s),u)\rangle_H\Big]\widetilde{N}(ds,du).
\end{aligned}
\tag{4.5.3}
$$

From (4.4.9), it can be shown that

$$
2\mathbb{E}\sup_{t\in I_N}\int_N^t\langle X(s),A\big(s,X(s)\big)\rangle_{V,V^*}ds\leq\int_N^{N+1}\alpha_1(s)\mathbb{E}\|X(s)\|_H^2 ds. \tag{4.5.4}
$$

On the other hand, derived by virtue of Condition 4.7,

$$
\begin{aligned}
&2\mathbb{E}\sup_{t\in I_N}\int_N^t\Big\langle X(s),F\big(s,X(s-\rho_1(s)),r(s)\big)\Big\rangle_{V,V^*}ds\\
\leq\ &\int_N^{N+1}\left[\gamma\mathbb{E}\|X(s)\|_H^2+\frac{\mathbb{E}\|F\big(s,X(s-\rho_1(s)),r(s)\big)\|_H^2}{\gamma}\right]ds\\
\leq\ &\int_N^{N+1}\left[\gamma\mathbb{E}\|X(s)\|_H^2+\frac{a_2+\alpha_2(t)}{\gamma}\mathbb{E}\|X(s-\rho_1(s))\|_H^2+\frac{\beta_1(s)}{\gamma}\right]ds.
\end{aligned}
\tag{4.5.5}
$$

Now, according to Burkhölder-Davis-Gundy inequality (Da Prato and Zabczyk (1992)), it can be estimated that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in I_N}\Big|2\int_N^t \Big\langle X(s), G\big(s, X(s-\rho_2(s)), r(s)\big)\Big\rangle_H\Big|\Big] \\
\leq\ & 2\mathbb{E}\Big[\int_N^{N+1} \|X(s)\|_H \|G(s, X(s-\rho_2(s)), r(s)\big)\|_{\mathcal{L}_2^0} ds\Big] \\
\leq\ & K_1\mathbb{E}\Big[\int_N^{N+1} \|X(s)\|_H^2 \|G(s, X(s-\rho_2(s)), r(s)\big)\|_{\mathcal{L}_2^0}^2 ds\Big]^{1/2} \\
\leq\ & K_1\mathbb{E}\Big[\sup_{t\in I_N}\|X(t)\|_H\Big(\int_N^{N+1} \|G(s, X(s-\rho_2(s)), r(s)\big)\|_{\mathcal{L}_2^0}^2\Big)^{1/2}\Big] \\
\leq\ & \frac{1}{2}\mathbb{E}\Big[\sup_{t\in I_N}\|X(t)\|_H^2\Big] + K_2\int_N^{N+1} \mathbb{E}\|G(s, X(s-\rho_2(s)), r(s)\big)\|_{\mathcal{L}_2^0}^2 ds,
\end{aligned}$$

where K_1 and K_2 denote some proper positive constants. Taking $K_2 = 32$ for the convenience of estimation later, the Burkholder-Davis-Gundy inequality yields holds:

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in I_N}\Big|2\int_N^t \Big\langle X(s), G\big(s, X(s-\rho_2(s)), r(s)\big)\Big\rangle_H\Big|\Big] \\
\leq\ & \frac{1}{2}\mathbb{E}\Big[\sup_{t\in I_N}\|X(t)\|_H^2\Big] + 32\int_N^{N+1} \mathbb{E}\|G(s, X(s-\rho_2(s)), r(s)\big)\|_{\mathcal{L}_2^0}^2 ds.
\end{aligned} \tag{4.5.6}$$

Applying Lemma 4.2, for any $t \geq 0$ and certain positive constant C we can yield that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in I_N}\int_N^t\int_{\mathbb{Z}}\Big[\|L(s, X(s-\rho_3(s), r(s), u))\|_H^2 \\
&\quad +2\langle X(s-), L(s, X(s-\rho_3(s)), r(s), u)\rangle_H\Big]\widetilde{N}(ds, du) \\
\leq\ & C\mathbb{E}\int_N^{N+1}\int_{\mathbb{Z}}\|L(s, X(s-\rho_3(s), r(s), u))\|_H^2\lambda(du)ds + \frac{1}{4}\mathbb{E}\sup_{t\in I_N}\|X(t)\|_H^2.
\end{aligned} \tag{4.5.7}$$

Substituting (4.5.4), (4.5.6) and (4.5.7) into (4.5.3), we have

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in I_N}\|X(t)\|_H^2 \\
\leq\ & \mathbb{E}\sup_{t\in I_N}\|X(t)\|_H^2+\mathbb{E}\sup_{t\in I_N}\int_N^t\|X(s)\|_H^2ds+\int_N^{N+1}\alpha_1(s)\mathbb{E}\|X(s)\|_H^2ds \\
&+\int_N^{N+1}\left[\gamma\mathbb{E}\|X(s)\|_H^2+\frac{a_2+\alpha_2(t)}{\gamma}\mathbb{E}\|X(s-\rho_1(s))\|_H^2+\beta_1(s)\right]ds \\
&+\int_N^{N+1}\mathbb{E}\|G(s,X(s-\rho_2(s),r(s)))\|_{\mathcal{L}_2^0}^2ds \\
&+\frac{1}{2}\mathbb{E}\sup_{t\in I_N}\|X(s)\|_H^2+32\int_N^{N+1}\mathbb{E}\|G(s,X(s-\rho_2(s)),r(s))\|_{\mathcal{L}_2^0}^2ds \\
&+\int_N^{N+1}\int_{\mathbb{Z}}\mathbb{E}\|L(s,X(s-\rho_3(s),r(s),u))\|_H^2\lambda(du)ds \\
&+C\mathbb{E}\int_{\mathbb{Z}}\|L(s,X(s-\rho_3(s)),r(s),u)\|_H^2\lambda(du)ds+\frac{1}{4}\mathbb{E}\sup_{t\in I_N}\|X(t)\|_H^2.
\end{aligned}
\tag{4.5.8}
$$

Now, combining Condition 4.8 and 4.9, multiplying both sides of (4.5.8) by 4 gives

$$
\begin{aligned}
&4\mathbb{E}\sup_{t\in I_N}\|X(t)\|_H^2 \\
\leq\ & 4\mathbb{E}\|X(N)\|_H^2+4\int_N^{N+1}(1+\alpha_1(s))\mathbb{E}\|X(s)\|_H^2ds \\
&+4\int_N^{N+1}\left[\gamma\mathbb{E}\|X(s)\|_H^2+\frac{a_2+\alpha_2(t)}{\gamma}\mathbb{E}\|X(s-\rho_1(s))\|_H^2+\beta_1(s)\right]ds \\
&+4\int_N^{N+1}[(a_3+\alpha_3(s))\mathbb{E}\|X(s-\rho_2(s))\|_H^2+\beta_2(s)]ds \\
&+2\mathbb{E}\sup_{t\in I_N}\|X(t)\|_H^2+128\int_N^{N+1}[(a_3+\alpha_3(s))\mathbb{E}\|X(s-\rho_2(s))\|_H^2+\beta_2(s)]ds \\
&+4\int_N^{N+1}[(a_4+\alpha_4(s))\mathbb{E}\|X(s-\rho_3(s))\|_H^2+\beta_3(s)]ds \\
&+4C\mathbb{E}\int_N^{N+1}[(a_4+\alpha_4(s))\mathbb{E}\|X(s-\rho_3(s))\|_H^2+\beta_3(s)]ds+\mathbb{E}\sup_{t\in I_N}\|X(t)\|_H^2.
\end{aligned}
\tag{4.5.9}
$$

This together with Theorem 4.2, the energy solution $X(t)$ to equation (4.3.1) satisfies $\mathbb{E}\|X(t)\|_H^2\leq\overline{M}e^{-\delta t}$, immediately yields the following:

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in I_N}\|X(t)\|_H^2 \\
\leq\;& 4\mathbb{E}\|X(N)\|_H^2 + 4\int_N^{N+1}(1+\alpha_1(s))\mathbb{E}\|X(s)\|_H^2 ds \\
&+4\int_N^{N+1}\left[\gamma\mathbb{E}\|X(s)\|_H^2 + \frac{a_2+\alpha_2(t)}{\gamma}\mathbb{E}\|X(s-\rho_1(s))\|_H^2 + \beta_1(s)\right]ds \\
&+132\int_N^{N+1}[(a_3+\alpha_3(s))\mathbb{E}\|s-\rho_2(s)\|_H^2+\beta_2(s)]ds \\
&+4(1+C)\mathbb{E}\int_N^{N+1}[(a_4+\alpha_4(s))\mathbb{E}\|X(s-\rho_3(s))\|_H^2+\beta_3(s)]ds \\
\leq\;& 4\overline{M}e^{-\delta N}+4\int_N^{N+1}\left[\frac{\beta_1(s)}{\gamma}+33\beta_2(s)+(1+C)\beta_3(s)\right]ds \\
&+4\int_N^{N+1}\Bigg[(1+\alpha_1(s)+\gamma)\mathbb{E}\|X(s)\|_H^2+\frac{a_2+\alpha_2(s)}{\gamma}\mathbb{E}\|X(s-\rho_1(s))\|_H^2 \\
&+(a_3+\alpha_3(s))\mathbb{E}\|X(s-\rho_2(s))\|_H^2+(a_4+\alpha_4(s))\mathbb{E}\|X(s-\rho_3(s))\|_H^2\Bigg]ds \\
\leq\;& 4\overline{M}e^{-\delta N}+4\int_N^{N+1}\beta(s)ds+4\int_N^{N+1}\Bigg[(1+\alpha_1(s)+\gamma)\overline{M}e^{-\delta s} \\
&+\frac{a_2+\alpha_2(s)}{\gamma}e^{\delta r}\overline{M}e^{-\delta s}+(a_3+\alpha_3(s))e^{\delta r}\overline{M}e^{-\delta s}+(a_4+\alpha_4(s))e^{\delta r}\overline{M}e^{-\delta s}\Bigg]ds \\
\leq\;& 4\overline{M}e^{-\delta N}+4\int_N^{N+1}[\beta(s)+\alpha(s)\overline{M}e^{-\delta s}]ds \\
\leq\;& 4\overline{M}e^{-\delta N}+4\int_N^{N+1}\overline{M}e^{-\delta s}[\alpha(s)+e^{\delta s}]ds \qquad (4.5.10)
\end{aligned}
$$

It follows from Condition (4.5.2) that there exists a certain constant $B_1 > 0$ satisfying

$$\alpha(t)+e^{\delta k}\beta(t)\leq B_1.$$

This, in addition to (4.5.10), yields that

$$
\begin{aligned}
\mathbb{E}\sup_{t\in I_N}\|X(t)\|_H^2 \;&\leq\; 4\overline{M}e^{-\delta N}+4\overline{M}B_1\int_N^{N+1}e^{-\delta s}ds \\
&\leq\; 4\overline{M}e^{-\delta N}\Big(1+\frac{B_1}{\delta}\Big).
\end{aligned}
$$

By the Chebyshev inequality, for any fixed positive real number ϵ_N, then we get that

$$P\Big\{\sup_{t\in I_N}\|X(t)\|_H^2 > \epsilon_N^2\Big\} \le \frac{\mathbb{E}\sup_{t\in I_N}\|X(t)\|_H^2}{\epsilon_N^2} \le \frac{4\overline{M}e^{-\delta N}\Big(1+\frac{B_1}{\delta}\Big)}{\epsilon_N^2}.$$

Since ϵ_N is arbitrary, letting $\epsilon_N^2 = e^{-\delta N/2}$, obtain

$$P\Big\{\sup_{t\in I_N}\|X(t)\|_H^2 > \epsilon_N^2\Big\} \le 4\overline{M}e^{-\delta N/2}\Big(1+\frac{B_1}{\delta}\Big).$$

Since $\sum_N 4\overline{M}e^{-\delta N/2}(1+\frac{B_1}{\delta}) \le \infty$, by Lemma 4.1 the Borel-Cantelli lemma, we have

$$\mathbb{P}\Big\{\sup_{t\in I_N}\|X(t)\|_H^2 > e^{-\delta N/2}\ \text{ i.o.}\Big\} = 0.$$

Take the complement of the event $\Big\{\sup_{t\in I_N}\|X(t)\|_H^2 > e^{-\delta N/2}\ \text{ i.o.}\Big\}$ to get

$$\mathbb{P}\Big\{\sup_{t\in I_N}\|X(t)\|_H^2 > e^{-\delta N/2}\ \text{ f.o.}\Big\} = 1.$$

Hence there exists an event $\Omega - \Omega_0$ such that $\mathbb{P}(\Omega-\Omega_0) = 1$ with $\mathbb{P}(\Omega_0) = 0$ and for each $\omega \in \Omega - \Omega_0$, there exists a positive integer $T(\omega)$ such that

$$\sup_{t\in I_N}\|X(t)\|_H^2 < e^{-\delta N/2}, \quad \forall\, N > T(\omega).$$

Consequently, we have the following result

$$\|X(t)\|_H^2 \le \sup_{t\in I_N}\|X(t)\|_H^2 < e^{-\delta N/2} \le e^{-\delta/2(t-1)},$$

i.e.

$$\|X(t)\|_H^2 \le e^{\delta/2}e^{-\delta t/2}.$$

In other words, the energy solution to Equation (4.3.1) is almost surely exponentially stable.

■

4.6 An illustrative example

In this section, we construct an example to illustrate the main results. Assume $B(t), t \geq 0$, is a real standard Brownian motion and $\widetilde{N}(\cdot,\cdot)$ is a compensated Poisson random measure on $[1,\infty]$ with parameter $\lambda(dy)dt$ such that $\int_1^\infty y^2\lambda(dy) < \infty$. Let $\rho_1(t)$, $\rho_2(t)$ and $\rho_3(t)$ be non-differentiable function defined by

$$\rho_1(t) = \frac{1}{1+|\sin t|}, \quad \rho_2(t) = \frac{1}{1+|\cos t|}, \quad \rho_3(t) = |\sin t|.$$

Consider the following stochastic partial differential equation with with finite memory $\rho_1(t)$, $\rho_2(t)$ and $\rho_3(t)$ with Poisson jump:

$$\begin{aligned}
dY(t,x) &= \mu\frac{\partial^2}{\partial x^2}Y(t,x)dt + \Big[(b_1+k_1(t))Y\big(t-\rho_1(t),x\big) + e^{-kt}p\Big]dt \\
&\quad + (b_2+k_2(t))bY\big(t-\rho_2(t),x\big)dB(t) \\
&\quad + \int_1^\infty (b_3+k_3(t))Y(t-\delta(t),x)y\widetilde{N}(dt,dy), \\
&\qquad\qquad\qquad\qquad t\geq 0, \quad \mu>0, \quad x\in(0,\pi), \\
Y(t,0) &= Y(t,\pi) = 0, \quad t\geq 0; \\
Y(t,x) &= \phi(t,x), \quad x\in[0,\pi],\, t\in[-1,0], \\
\phi(t,\cdot) &= L^2(0,\pi); \quad \phi(\cdot,x) = C([-1,0];\mathbb{R}).
\end{aligned} \tag{4.6.1}$$

where $k_1,\ k_2,k_3 : [0,\infty) \to \mathbb{R}_+$ are continuous functions, b_1, b_2, b_3 and k are positive real numbers. We can set this problem in our formulation by taking $H = L^2(0,\pi)$, $K = \mathbb{R}$ and $V = H_0^1(0,\pi)$, which is a Hilbert space (c.f. Brezis (2011), Page 287).

Let $A = \mu\frac{\partial^2}{\partial x^2}$ be the one dimensional Laplace operator with domain $\mathcal{D}(A) = H_0^1(0,\pi)\cap H^2(0,\pi)$, where

$$H_0^1(0,\pi) = \Big\{u\in L^2(0,\pi) : \frac{\partial u}{\partial x}\in L^2(0,\pi), u(0)=u(\pi)=0\Big\},$$

and

$$H^2(0,\pi) = \Big\{ u \in L^2(0,\pi) : \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in L^2(0,\pi) \Big\}.$$

In this example, the norm and inner product of H and V are defined by

$$\|\xi\|_H^2 = \int_0^\pi \xi^2(x)dx, \ \langle \xi, \eta \rangle_H = \int_0^\pi \xi(x)\eta(x)dx, \ \text{ for any } \xi, \eta \in H,$$

and

$$\|u\|_V^2 = \int_0^\pi \Big(\frac{\partial u}{\partial x}\Big)^2 dx, \ \langle u, v \rangle_V = \int_0^\pi \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dx, \ \text{ for any } \ u, v \in V.$$

It is known that for arbitrary $u \in V$

$$\langle u, Au \rangle_{V,V^*} \le -\mu \|u\|_H^2.$$

We can also show that for $p \in H$ with $\|p\|_H < \infty$ that

$$\begin{aligned} & \|(b_1 + k_1(t))Y\big(t - \rho_1(t), x\big) + e^{-kt}p\|_H^2 \\ \le \ & 4(b_1^2 + k_1^2(t))\|Y(t - \rho_1(t), x)\|_H^2 + 2e^{-2kt}\|p\|^2, \end{aligned}$$

$$\|(b_2 + k_2(t))Y\big(t - \rho_2(t), x\big)\|_{\mathcal{L}_2^0}^2 \le 4(b_2^2 + k_2^2(t))\|Y(t - \rho_2, x)\|_H^2,$$

$$\begin{aligned} & \int_1^\infty \|(b_3 + k_3(t))Y(t - \rho_3(t), x)y\|_H^2 \lambda(dy) \\ \le \ & 2(b_3^2 + k_3^2(t)) \int_1^\infty y^2 \lambda(dy) \|Y(t - \rho_3(t), x)\|_H^2. \end{aligned}$$

Note that, $\lambda = 1$, $a_1 = 2\mu$, $a_2 = 4b_1^2$, $a_3 = 4b_2^2$, $a_4 = 2b_3^2 \int_1^\infty y^2\lambda(dy)$, $\alpha_2(t) = 4k_1^2(t)$, $\alpha_3(t) = 4k_2^2(t)$, $\alpha_4(t) = 2\int_1^\infty y^2(dy)k_3^2(t)$, $\beta_1(t) = 2e^{-2kt}\|p\|_H^2$, $\beta_2(t) = 0$, $\beta_3(t) = 0$. Let

$$\mu > 2b_1 + 2b_2^2 + b_3^2 \int_1^\infty y^2 \lambda(dy),$$

and take $\alpha_1(t) = 1$, $\sigma = k$. By Theorem 4.2 and 4.3, the energy solution $X(t)$ to equation (4.6.1) is exponentially stable and almost surely exponentially stable.

4.7 Conclusion

The stochastic partial differential equation driven both by Markovian switching and Poisson jumps for modeling systems in many science subjects gives more generalized formula of the SDDEs model (3.2.1). We treated this type of SPDEs with Markovian switching and Poisson jumps within a variational formulation and the concept of energy solution. Therefore, we have first proved that this type of equation admits a unique energy solution. Then, estimation of the coefficients functions in the stochastic energy equality has been established to show the exponential stability and almost sure exponential stability of our equations. Finally, an applicable example is provided for illustration.

Chapter 5

The fundamental solution of stochastic neutral functional differential equations

5.1 Introduction

Many dynamical systems not only depend on present and past states but also involve derivatives with delays. This kind of system is often described by neutral functional differential equations (NFDEs). NFDEs and their stability have been studied by many authors. For example, Hale (1977), Hale and Verduyn Lunel (1993) (and reference therein). In general, an n-dimensional NFDE can take the following form. For $\tau > 0$ and $T > 0$,

$$\begin{aligned} \frac{d}{dt}[x(t) - D(x_t)] &= f(x_t), \quad 0 \le t \le T; \\ x(t) &= \xi(t) \in C([-\tau, 0]; \mathbb{R}^n), \quad t \in [-\tau, 0], \end{aligned} \tag{5.1.1}$$

where D and f are functionals from $C([-\tau, 0]; \mathbb{R}^n)$ to $\mathbb{R}^n$.

Taking the environmental disturbances into account led to a stochastic neutral functional differential equations (SNFDEs) which was motivated by chemical engineering systems as well as the theory of aeroelasticity. For instance, Kolmanovskii and Myshkis (1992) and Mao (1997) have studied the following n-dimensional SNFDEs: let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ denotes the family of continuous function φ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau \le \theta \le 0} \|\varphi(\theta)\|_{\mathbb{R}^n}$. Let $B(t)$ be an m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual condition. For each $t \ge 0$, denote by $L^2_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_t$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\phi = \{\phi(\theta) : -\tau < \theta \le 0\}$ such that $\sup_{-\tau \le \theta \le 0} \mathbb{E}\|\phi(\theta)\|^2_{\mathbb{R}^n} < \infty$. For $\tau > 0$,

$$\begin{aligned} d[x(t) - D(x_t)] &= f(t, x_t)dt + g(t, x_t)dB(t), \quad t \ge 0; \\ x(t) &= \xi(t) \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n), \quad t \in [-\tau, 0], \end{aligned} \tag{5.1.2}$$

where $D : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$, $f : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$, and $g : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}_n) \to \mathbb{R}^{n\times m}$ are all continuous functionals.

Kolmanovskii and Myshkis (1992) has established the theory of existence and uniqueness of the solution to Equation (5.1.2) and investigated the stability and asymptotic stability of the equations. Furthermore, Mao (1995) and Mao (1996) have studied the exponential stability and the almost sure exponential stability by the Razumikhin argument. So far, the theory of SNFDEs in finite dimensional spaces have been extensively studied in the literature, many important results have been reported in Bao and Hou (2010), Huang and Deng (2008), Kolmanovskii and Myshkis (1992), Kolmanovskii et al. (2003), Liu and Xia (1999), Mao (1995, 1996, 1997) and Ren and Xia (2009).

However, SNFDEs in infinite dimensional spaces began to receive a great deal of attention recently. Meanwhile, various results on the existence and uniqueness and the stability of SNFDEs have been obtained in Boufoussi and Hajji (2010a,b), Caraballo et al. (2007), Chen (2009), Govindan (2005) and Govindan (2009). In Bao and Hou (2010), under a non-Lipschitz condition with the Lipschitz condition being considered as a special case and a weakended linear growth condition, the existence and uniqueness of mild solutions to a class of stochastic neutral partial functional differential equations (SNPFDEs) on the Hilbert space H is investigated. The following form of SNPFDE is studied in Bao and Hou (2010): for $\tau > 0$,

$$d[X(t) + G(t, X_t)] = [AX(t) + F(t, X_t)]dt + L(t, X_t)dW(t), t \geq 0 \qquad (5.1.3)$$

with the initial condition $X(t) = \xi(t) \in C^b_{\mathcal{F}_0}([-\tau, 0]; H)$, which denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $C([-\tau, 0]; H)$-valued random variables. $C([-\tau, 0]; H)$ denotes the family of continuous function φ from $[-\tau, 0]$ to H with norm $\sup_{-\tau \leq \theta \leq 0} \|\varphi(\theta)\|_H$. The mapping G, $F : \mathbb{R}_+ \times C([-\tau, 0]; H) \to H$ and $L : \mathbb{R}_+ \times C([-\tau, 0]; H) \to \mathcal{L}(K, H)$ are measurable, respectively.

In this chapter, we are concerned with a stochastic neutral functional differential equations (SNFDEs) in Hilbert spaces, with the mild solutions to such equations can be represented by the so-called fundamental solutions. The fundamental solutions (Green's operator) was firstly introduced and constructed in Liu (2008)) for a class of stochastic retarded evolution equations, c.f. Liu (2009, 2010, 2011) and Section 5.2. Our main objective is to investigate the solution processes (strong, weak and mild) of Equation (5.2.1) by obtaining some necessary criteria. In particular, we generalized the results in Liu (2008) (the case when $D = 0$ in Equation (5.2.1)) to a neutral type.

The contents of this chapter are as follows. In Section 5.2 we recall some preliminaries on stochastic functional differential equations of neutral types with some basic definitions. In Section 5.3 we investigate the relationships between strong, weak and mild solutions to Equation (5.2.1).

5.2 Stochastic functional differential equations of neutral type

Let H and K be two separable Hilbert spaces with norm $\|\cdot\|_H$ and $\|\cdot\|_K$, inner products $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_K$. We use $\mathcal{L}(K,H)$ to denote the space of all bounded linear operators from K into H with domain K. Every operator norm is simply denoted by $\|\cdot\|$ when there is no danger of confusion. The adjoint space of H which consists of all bounded linear functionals on H is denoted by H^*. Unless otherwise stated, in this chapter we always identity H with its adjoint space H^* according to the well-known Riesz's representation theorem. For a closed linear operator A on a dense domain $\mathcal{D}(A)\subset H$ into H, its adjoint operator is denoted by A^*.

For any fixed constant $r>0$ and the Hilbert space H, we denote by $L_r^2 = L^2([-r,0];H)$ the usual Hilbert space of all H-valued equivalence classes of measurable functions which are square integrable on $[-r,0]$. Let $\mathcal{H}$ denote the Hilbert space $H\times L_r^2$, with the norm

$$\|\phi\|_{\mathcal{H}} = \sqrt{\|\phi_0\|_H^2 + \|\phi_1\|_{L_r^2}^2}, \text{ for all } \phi=(\phi_0,\phi_1)\in\mathcal{H}.$$

Note that ϕ_0 is not necessarily equal to $\phi_1(0)$ here.

Let $W_Q(t)$ be a Wiener process and $\int_0^t \Phi(s)dW_Q(s)$ be the stochastic integral with respect to $W_Q(t)$, which is a continuous square-integrable martingale. Recall that $\mathcal{W}^2([0,T];\mathcal{L}_2^0)$ denote all $\mathcal{L}_2^0(K_0,H)$-valued predictable processes Φ such that $|\Phi|_T<\infty$ (c.f. Section 2.2).

In this chapter, we shall consider a class of stochastic neutral functional differential equations on the Hilbert space H which are defined by:

for any $T > 0$,

$$d\Big[y(t) - \int_{-r}^{0} D(\theta)y(t+\theta)d\theta\Big] = Ay(t)dt + \int_{-r}^{0} d\eta(\theta)y(t+\theta)dt + B(t)dW_Q(t), \quad t \in (0,T],$$

$$y(0) = \phi_0,\ y_0 = \phi_1,\ \phi = (\phi_0, \phi_1) \in \mathcal{H}, \tag{5.2.1}$$

where $A : H \to H$ with domain $\mathcal{D}(A) \subset H$ is the infinitesimal generator of a C_0-semigroup $\{T(t) : t \geq 0\}$ on H, $B(t) \in \mathcal{W}^2([0,T]; \mathcal{L}_2^0)$; and $y_t(\theta) := y(t+\theta)$ for any $\theta \in [-r, 0]$ and $t \geq 0$. Here η is the Stieltjes measure given by

$$\eta(\tau) = -\sum_{i=1}^{m} \chi_{(-\infty,-r_i]}(\tau)A_i - \int_{\tau}^{0} F(\theta)d\theta,\ \tau \in [-r, 0].$$

It is assumed that $0 < r_1 < r_2 < \cdots < r_m \leq r$, $A_i \in \mathcal{L}(H)$, $i = 1, \cdots, m$, the family of all bounded, linear operators on H and $D(\cdot)$, $F(\cdot) \in L^2([-r, 0]; \mathcal{L}(H))$, the space of all $\mathcal{L}(H)$-valued equivalent class of square integrable functions on $[-r, 0]$. The delay term $\int_{-r}^{0} d\eta(\theta)y(t+\theta)$ can be written by

$$\sum_{i=1}^{m} A_i y(t - r_i) + \int_{-r}^{0} F(\theta)y(t+\theta)d\theta.$$

To this end, we further assume that for each $i = 1, \cdots, m$ and $\theta \in [-r, 0]$, $\mathcal{R}(D(\theta)) \subset \mathcal{D}(A)$ such that $AD(\cdot) \in L^2([-r, 0]; \mathcal{L}(H))$.

Consider the corresponding deterministic system of Equation (5.2.1) on H,

$$\frac{d}{dt}\Big[y(t) - \int_{-r}^{0} D(\theta)y(t+\theta)d\theta\Big] = Ay(t) + \int_{-r}^{0} d\eta(\theta)y(t+\theta),\ t \in (0,T],$$
$$y(0) = \phi_0,\ y_0(\cdot) = \phi_1(\cdot),\ \phi = (\phi_0, \phi_1) \in \mathcal{H}. \tag{5.2.2}$$

and its corresponding integral equation:

$$\begin{aligned} y(t,\phi) = {} & \int_{-r}^{0} D(\theta)y(t+\theta,\phi)d\theta + T(t)\Big[\phi_0 - \int_{-r}^{0} D(\theta)\phi_1(\theta)d\theta\Big] \\ & + \int_{0}^{t} T(t-s)\Big[\int_{-r}^{0} d\eta(\theta)y(s+\theta,\phi) + \int_{-r}^{0} AD(\theta)y(s+\theta,\phi)d\theta\Big]ds, \quad \forall\, t > 0, \end{aligned}$$

$$y(0,\phi) = \phi_0,\ \ y_0(\cdot,\phi) = \phi_1(\cdot),\ \phi = (\phi_0, \phi_1) \in \mathcal{H}. \tag{5.2.3}$$

For simplicity, we denote $y(t, \phi)$ and $y_t(\cdot, \phi)$ by $y(t)$ and $y_t(\cdot)$ respectively, in the sequel.

We introduce the following results: the existence and uniqueness of mild solution to Equation (5.2.3) - Proposition 5.1 (c.f. Liu (2009), Theorem 2.1); Green's operator - Equation (5.2.4) and (5.2.5) (c.f. Liu (2009), Page 10) and variation of constants formula of mild solutions - Proposition 5.2 (c.f. Liu (2009), Theorem 2.2).

Proposition 5.1 *For arbitrary $T \geq 0$, $\phi = (\phi_0, \phi_1) \in \mathcal{H}$, (i) there exists a unique solution $y(t, \phi) \in L^2([-r, T]; H)$ of Equation (5.2.3); (ii) for arbitrary $t \in [0, T]$, $\|y(t, \phi)\|_H \leq Ce^{\gamma t}\|\phi\|_{\mathcal{H}}$ almost everywhere for some constants $\gamma \in \mathbb{R}$ and $C > 0$.*

The solution $y(t, \phi)$ of Equation (5.2.3) is called a *mild solution* of Equation (5.2.2). For any $h \in H$, let $\phi_0 = h$, $\phi_1(\theta) = 0$ for $\theta \in [-r, 0]$ and $\phi = (h, 0)$, the so-called *Green's operator* or *fundamental solution* $G(t) : (-\infty, \infty) \to \mathcal{L}(H)$, $t \in \mathbb{R}$, of Equation (5.2.3) with such an initial datum can be defined by

$$G(t)h = \begin{cases} y(t, \phi), & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{5.2.4}$$

The term (5.2.4) implies that $G(t)$ is a unique solution of the equation

$$G(t) = \begin{cases} T(t) + \int_{-r}^{0} D(\theta)G(t+\theta)d\theta + \int_0^t T(t-s)\Big[\int_{-r}^{0} d\eta(\theta)G(s+\theta) \\ \qquad\qquad + \int_{-r}^{0} AD(\theta)G(s+\theta)d\theta\Big]ds, & \text{if } t \geq 0, \\ \mathbf{O}, & \text{if } t < 0. \end{cases} \tag{5.2.5}$$

where $G_t(\theta) = G(t+\theta)$, $\theta \in [-r, 0]$, and $\mathbf{O}$ denotes the null operator on H. It turns out that $G(t)$, $t \geq 0$, is strongly continuous one-parameter family of bounded linear operator on H such that $\|G(t)\| \leq Ce^{\gamma t}$, $t \geq 0$, for some constants $C > 0$ and $\gamma \in \mathbb{R}$.

It is possible to find an explicit representation for the solution $y(t,\phi)$ of Equation (5.2.3) if we restrict the initial data of Equation (5.2.3) to some proper subset of H. Let $W^{1,2}([-r,0];H)$ denote the Sobolev space of H-valued functions $x(\cdot)$ on $[-r,0]$ such that $x(\cdot)$ and its weak derivative belong to $L^2([-r,0];H)$. We define

$$\mathcal{W}^{1,2} = H \times W^{1,2}([-r,0];H).$$

The following *variation of constants formula* provides a representation for solutions of Equation (5.2.3) in terms of the fundamental solution $G(t) \in \mathcal{L}(H)$.

Proposition 5.2 *For arbitrary $\phi = (\phi_0, \phi_1) \in \mathcal{W}^{1,2}$, the solution $y(t,\phi)$ of Equation (5.2.3) can be represented almost everywhere by*

$$y(t) = G(t)\phi_0 - V(t,0)\phi_1(0) + \int_{-r}^{0} U(t,\theta)\phi_1(\theta)d\theta + \int_{-r}^{0} V(t,\theta)\phi_1'(\theta)d\theta.$$

where for any $t \geq 0$, the kernels

$$\begin{aligned} U(t,\theta) &= \int_{-r}^{\theta} G(t-\theta+\tau)d\eta(\tau) \in L^2([-r,0];\mathcal{L}(H)), \quad \theta \in [-r,0], \\ V(t,\theta) &= \int_{-r}^{\theta} G(t-\theta+\tau)D(\tau)d\tau \in L^2([-r,0];\mathcal{L}(H)), \quad \theta \in [-r,0]. \end{aligned}$$

For details of the proof of Proposition 5.1 and 5.2, c.f. Liu (2009), Theorem 2.1 and 2.2.

By applying the fundamental solutions (Green's operator) to our stochastic systems of neutral type, we are able to define the strong, weak and mild solutions for Equation (5.2.1) as follows.

Definition 5.1 (Strong solution) *A stochastic process $y(t)$, $t \in [-r,T]$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is called a strong solution of Equation (5.2.1) if*

(i) $y(t) \in \mathcal{D}(A)$, $t \in [0,T]$, almost surely and it is adapted to $\mathcal{F}_t$, $t \in [0,T]$;

(ii) $y(t)$ has continuous paths on $t \in [0,T]$ in H and it satisfies

$$
\begin{aligned}
y(t) - \int_{-r}^{0} D(\theta)y(t+\theta)d\theta &= \phi_0 - \int_{-r}^{0} D(\theta)\phi_1(\theta)d\theta \\
&+ \int_0^t \left(Ay(s) + \int_{-r}^{0} d\eta(\theta)y(s+\theta) \right) ds + \int_0^t B(s)dW_Q(s) \\
y(0) = \phi_0,\ y_0 = \phi_1, & \qquad (5.2.6)
\end{aligned}
$$

for arbitrary $\phi = (\phi_0, \phi_1) \in \mathcal{H}$ almost surely.

In some situations, ones find that the strong solution concept is too strong to include important examples. Therefore, a weaker concept, weak or mild solutions, which turns out to be very useful in applications.

Definition 5.2 (Weak solution) *A stochastic process $y(t)$, $t \in [-r,T]$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is called a weak solution of Equation (5.2.1) if*

(i) $y(t)$ is adapted to $\mathcal{F}_t$, $t \in [0,T]$ and the trajectories of $y(\cdot)$ are almost surely Bochner integrable such that for arbitrary $0 \le t \le T$,

$$
P\left\{ \omega : \int_0^t \|y(s,\omega)\|_H^2 ds < \infty \right\} = 1;
$$

(ii) For arbitrary $h \in \mathcal{D}(A^*)$ and $t \in [0,T]$,

$$
\begin{aligned}
\left\langle h, y(t) - \int_{-r}^{0} D(\theta)y(t+\theta)d\theta \right\rangle_H &= \left\langle h, \phi_0 - \int_{-r}^{0} D(\theta)\phi_1(\theta)d\theta \right\rangle_H \\
&+ \int_0^t \langle A^*h, y(s)\rangle_H ds + \left\langle h, \int_0^t \int_{-r}^{0} d\eta(\theta)y(s+\theta)ds \right\rangle_H \\
&+ \left\langle h, \int_0^t B(s)dW_Q(s) \right\rangle_H \qquad (5.2.7)
\end{aligned}
$$

almost surely with $y(0) = \phi_0$, $y_0 = \phi_1$, and $\phi = (\phi_0, \phi_1) \in \mathcal{H}$.

Definition 5.3 (Mild solution) *A stochastic process $y(t)$, $t \in [-r,T]$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is called a mild solution of Equation (5.2.1) if* *(i) $y(t)$ is adapted to $\mathcal{F}_t$, $t \in [0,T]$ and for arbitrary $0 \le t \le T$,*

$$P\left\{\omega : \int_0^t \|y(s,\omega)\|_H^2 ds < \infty\right\} = 1;$$

(ii) For arbitrary $t \in [0,T]$ and $\phi \in \mathcal{H}$,

$$\begin{aligned} y(t) &= G(t)\phi_0 - V(t,0)\phi_1(0) + \int_{-r}^0 U(t,\theta)\phi_1(\theta)d\theta + \int_{-r}^0 V(t,\theta)\phi_1'(\theta)d\theta \\ &\quad + \int_0^t G(t-s)B(s)dW_Q(s), \quad t \ge 0. \end{aligned} \tag{5.2.8}$$

almost surely where for any $t \ge 0$, the kernels

$$U(t,\theta) = \int_{-r}^{\theta} G(t-\theta+\tau)d\eta(\tau) \in L^2([-r,0];\mathcal{L}(H)), \quad \theta \in [-r,0],$$

$$V(t,\theta) = \int_{-r}^{\theta} G(t-\theta+\tau)D(\tau)d\tau \in L^2([-r,0];\mathcal{L}(H)), \quad \theta \in [-r,0].$$

5.3 Main results

In this section, we shall investigate the relationships among the strong, weak and mild solutions of Equation (5.2.1). Firstly, we introduce the following lemma from Liu (2009) to show that the strong solution is actually a mild one.

Lemma 5.1 *Let $h \in \mathcal{D}(A)$, then $\int_0^t G(s)Ahds \in H$ for almost all $t \in \mathbb{R}_+$, and the relation*

$$\begin{aligned} \int_0^t G(s)Ahds &= G(t)h - h - \int_0^t \int_{-r}^0 G(s+\theta)d\eta(\theta)hds \\ &\quad - \int_{-r}^0 G(t+\theta)D(\theta)hd\theta, \quad t \in \mathbb{R}_+, \quad h \in \mathcal{D}(A), \end{aligned} \tag{5.3.1}$$

holds almost everywhere.

Theorem 5.1 *Suppose that $y(t)$, $t \in [-r,T]$, is a strong solution of Equation (5.2.1), then it is also a mild solution.*

Proof. By Lemma 5.1, rearrange and differentiate (5.3.1) both sides, we have

$$\frac{d}{dt}G(t)h = G(t)Ah + \int_{-r}^{0} G(t+\theta)d\eta(\theta)h - \frac{d}{dt}\int_{-r}^{0} G(t+\theta)D(\theta)hd\theta. \quad (5.3.2)$$

Let $v(s,h) = G(t-s)h$, then by (5.3.2), we have

$$v'_s(s,h) = -G(t-s)Ah - \int_{-r}^{0} G(t-s+\theta)d\eta(\theta)h - \frac{d}{dt}\int_{-r}^{0} G(t-s+\theta)D(\theta)hd\theta,$$

$$v'_h(s,h) = G(t-s),$$

$$v''_{hh}(s,h) = 0.$$

By using Ito's formula, it can be deduced that for any $0 \le t \le T$,

$$\begin{aligned}
& v(t,y(t)) - v(0,\phi_0) \\
= \ & y(t) - G(t)\phi_0 \\
= \ & \int_0^t \Big[-G(t-s)Ay(s) - \int_{-r}^{0} G(t-s+\theta)d\eta(\theta)y(s) \\
& \qquad - \frac{d}{dt}\int_{-r}^{0} G(t-s+\theta)D(\theta)y(s)d\theta \Big] ds \\
& + \int_0^t G(t-s)\Big(Ay(s) + \int_{-r}^{0} d\eta(\theta)y(s+\theta) + \int_{-r}^{0} D(\theta)y(s+\theta)d\theta \Big) ds \\
& + \int_0^t G(t-s)dW_Q(s), \qquad (5.3.3)
\end{aligned}$$

that is,

$$\begin{aligned}
y(t) \ = \ & G(t)\phi_0 - \int_0^t \int_{-r}^{0} G(t-s+\theta)d\eta(\theta)y(s)ds + \int_0^t \int_{-r}^{0} G(t-s)d\eta(\theta)y(s+\theta)ds \\
& - \int_0^t \frac{d}{dt}\int_{-r}^{0} G(t-s+\theta)D(\theta)y(s)d\theta ds + \int_0^t \int_{-r}^{0} G(t-s)D(\theta)y(s+\theta)d\theta ds \\
& + \int_0^t G(t-s)dW_Q(s). \qquad (5.3.4)
\end{aligned}$$

However, by using Fubini's theorem and the fact that $G(t) = \mathbf{O}$ for any $t < 0$, we have for $t \ge 0$,

$$
\begin{aligned}
&\int_0^t \int_{-r}^0 G(t-s)d\eta(\theta)y(s+\theta)ds - \int_0^t \int_{-r}^0 G(t-s+\theta)d\eta(\theta)y(s)ds \\
= &\int_{-r}^0 \int_0^t G(t-s)y(s+\theta)dsd\eta(\theta) - \int_{-r}^0 \int_0^t G(t-s+\theta)y(s)dsd\eta(\theta) \\
= &\int_{-r}^0 \int_{-r}^{t+\theta} G(t-s+\theta)y(s)dsd\eta(\theta) - \int_{-r}^0 \int_0^t G(t-s+\theta)y(s)dsd\eta(\theta) \\
= &\int_{-r}^0 \int_{-r}^{\theta} G(t-s+\theta)\phi_1(s)dsd\eta(\theta) \qquad (5.3.5)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \int_{-r}^0 G(t-s)D(\theta)y(s+\theta)d\theta ds - \int_0^t \frac{d}{dt}\int_{-r}^0 G(t-s+\theta)D(\theta)y(s)d\theta ds \\
= &\int_{-r}^0 \int_{-r}^{t+\theta} G(t-s+\theta)D(\theta)y(s)dsd\theta - \int_{-r}^0 \frac{d}{dt}\int_0^t G(t-s+\theta)D(\theta)y(s)dsd\theta \\
= &\int_{-r}^0 \frac{d}{dt}\int_{-r}^{\theta} G(t-s+\theta)D(\theta)\phi_1(s)dsd\theta \\
= &-\int_{-r}^0 \frac{d}{ds}\int_{-r}^{\theta} G(t-s+\theta)D(\theta)\phi_1(s)dsd\theta \qquad (5.3.6)
\end{aligned}
$$

On the other hand, expanding the mild solution, Equation (5.2.8), we have:

$$
\begin{aligned}
y(t) = \; &G(t)\phi_0 + \int_0^t G(t-s)B(s)dW_Q(s) - \int_{-r}^0 G(t+\tau)D(\tau)d\tau\phi_1(0) \\
&+ \int_{-r}^0 \int_{-r}^{\theta} G(t-\theta+\tau)d\eta(\tau)\phi_1(\theta)d\theta \\
&+ \int_{-r}^0 \int_{-r}^{\theta} G(t-\theta+\tau)D(\tau)d\tau d\phi_1(\theta)d\theta. \qquad (5.3.7)
\end{aligned}
$$

Applying differentiation by parts to the last term of (5.3.7) yields,

$$
\begin{aligned}
&\int_{-r}^0 \int_{-r}^{\theta} G(t-\theta+\tau)D(\tau)d\tau d\phi_1(\theta)d\theta \\
= &\left[\int_{-r}^{\theta} G(t-\theta+\tau)D(\tau))d\tau\phi_1(\theta)\right]_{-r}^0 - \int_{-r}^0 \frac{d}{d\theta}\int_{-r}^{\theta} G(t-\theta+\tau)D(\tau)\phi_1(\theta)d\tau d\theta \\
= &\int_{-r}^0 G(t+\tau)D(\tau)d\tau\phi_1(0) - \int_{-r}^0 \frac{d}{d\theta}\int_{-r}^{\theta} G(t-\theta+\tau)D(\tau)\phi_1(\theta)d\tau d\theta \qquad (5.3.8)
\end{aligned}
$$

Hence, we have:

$$
\begin{aligned}
y(t) &= G(t)\phi_0 + \int_{-r}^{0}\int_{-r}^{\theta} G(t-\theta+\tau)d\eta(\tau)\phi_1(\theta)d\theta \\
&\quad - \int_{-r}^{0}\frac{d}{d\theta}\int_{-r}^{\theta} G(t-\theta+\tau)D(\tau)\phi_1(\theta)d\tau d\theta \\
&\quad + \int_{0}^{t} G(t-s)B(s)dW_Q(s)
\end{aligned}
\tag{5.3.9}
$$

Substitute Equation (5.3.5) and (5.3.6) into (5.3.4), we have the same result as Equation (5.3.9). This shows that $y(t)$, $t \in [0, T]$, is also a mild solution in the sense of Definition 5.3.

■

Next, we wish to examine the relationship between the mild solution and the weak one. Again, we introduce the following lemma from Liu (2009).

Lemma 5.2 *Let $h \in D(A^*)$, then $\int_0^t G^*(s)A^*hds \in H$ for almost all $t \in \mathbb{R}_+$, and the relation*

$$
\begin{aligned}
\int_0^t G^*(s)A^*hds &= G^*(t)h - h - \int_0^t\int_{-r}^{0} G^*(s+\theta)d\eta^*(\theta)hds \\
&\quad - \int_{-r}^{0} G^*(t+\theta)D^*(\theta)hd\theta, \quad t \in \mathbb{R}_+,\ h \in D(A^*).
\end{aligned}
$$

holds almost everywhere.

We shall show that the mild solution of Equation (5.2.1) is actually a weak one. Precisely, we have the result:

Proposition 5.3 *Suppose that $B(t) \in \mathcal{W}^2([0, T]; \mathcal{L}_2^0)$, then Equation (5.2.1) has a unique weak solution. Moreover, the solution is represented by*

$$
\begin{aligned}
y(t) &= G(t)\phi_0 - V(t,0)\phi_1(0) + \int_{-r}^{0} U(t,\theta)\phi_1(\theta)d\theta + \int_{-r}^{0} V(t,\theta)\phi_1'(\theta)d\theta \\
&\quad + \int_0^t G(t-s)B(s)dW_Q(s), \quad T \geq 0.
\end{aligned}
$$

almost surely for $\phi = (\phi_0, \phi_1) \in \mathcal{W}^{1,2}$*, where for any* $t \in [0, T]$*, the kernels*

$$U(t,\theta) = \int_{-r}^{\theta} G(t-\theta+\tau)d\eta(\tau) \in L^2([-r,0];\mathcal{L}(H)), \quad \theta \in [-r,0],$$

$$V(t,\theta) = \int_{-r}^{\theta} G(t-\theta+\tau)D(\tau)d\tau \in L^2([-r,0];\mathcal{L}(H)), \quad \theta \in [-r,0].$$

and $y(t) = \phi_1(t)$, $t \in [-r, 0)$.

Similar to the prove of Proposition 4.1 in Liu (2008), to prove this result, first note that a process $y(t)$ is a weak solution of Equation (5.2.1) if and only if the process $\tilde{y}(t)$ given by the formula

$$\begin{aligned} \tilde{y}(t) &= y(t) - G(t)\phi_0 \quad \text{for } t \in [0,T], \text{ and} \\ \tilde{y}(t) &= y(t) - \phi_1(t) \quad \text{for } t \in [-r,0), \end{aligned} \tag{5.3.10}$$

is a weak solution to the equation

$$d\left[\tilde{y}(t) - \int_{-r}^{0} D(\theta)\tilde{y}(t+\theta)d\theta\right] = A\tilde{y}(t)dt + \int_{-r}^{0} d\eta(\theta)\tilde{y}(t+\theta)dt + B(t)dW_Q(t),\ t \in (0,T]. \tag{5.3.11}$$

and $\tilde{y}(t) = 0$ for $t \in [-r, 0]$.

Proof. Let us put

$$\tilde{y}(t) = \int_0^t G(t-s)B(s)dW_Q(s),\ t \ge 0, \text{ and } \tilde{y}(t) = 0 \text{ for } t \in [-r,0], \tag{5.3.12}$$

then ones can deduce by using the well-known stochastic Fubini's theorem (c.f. Da Prato and Zabczyk (1992)) and Lemma 5.2 that for any $h \in D(A^*)$,

$$\begin{aligned} &\int_0^t \langle A^*h, \tilde{y}(s)\rangle_H ds \\ =& \left\langle A^*h, \int_0^t \tilde{y}(s)ds \right\rangle_H \\ =& \left\langle A^*h, \int_0^t \int_0^s G(s-u)B(u)dW_Q(u)ds \right\rangle_H \end{aligned}$$

$$
\begin{aligned}
&= \left\langle A^*h, \int_0^t \int_u^t G(s-u)dsB(u)dW_Q(u) \right\rangle_H \\
&= \left\langle \int_u^t G^*(s-u)A^*hds, \int_0^t B(u)dW_Q(u) \right\rangle_H \\
&= \int_0^t \left\langle \int_u^t B^*(u)G^*(s-u)A^*hds, dW_Q(u) \right\rangle_H \\
&= \int_0^t \left\langle \int_u^t \frac{d}{ds}B^*(u)G^*(s-u)hds \right. \\
&\qquad - \int_u^t \frac{d}{ds}B^*(u)\int_{-r}^0 G^*(s-u+\theta)D^*(\theta)hd\theta ds \\
&\qquad\quad \left. - \int_u^t B^*(u)\int_{-r}^0 G^*(s-u+\theta)d\eta^*(\theta)hds, dW_Q(u) \right\rangle_H \\
&= \int_0^t \left\langle \int_u^t \frac{d}{ds}B^*(u)G^*(s-u)hds, dW_Q(u) \right\rangle_H \\
&\qquad - \int_0^t \left\langle \int_u^t \frac{d}{ds}B^*(u)\int_{-r}^0 G^*(s-u+\theta)D^*(\theta)hd\theta ds, dW_Q(u) \right\rangle_H \\
&\qquad\quad - \int_0^t \left\langle \int_u^t B^*(u)\int_{-r}^0 G^*(s-u+\theta)d\eta^*(\theta)hds, dW_Q(u) \right\rangle_H \\
&= \int_0^t \left\langle B^*(u)G^*(t-u)h, dW_Q(u) \right\rangle_H - \int_0^t \left\langle B^*(u)h, dW_Q(u) \right\rangle_H \\
&\qquad - \int_0^t \left\langle B^*(u)\int_{-r}^0 G^*(t-u+\theta)D^*(\theta)hd\theta ds, dW_Q(u) \right\rangle_H \\
&\qquad\quad - \left\langle h, \int_0^t \int_u^t \int_{-r}^0 d\eta(\theta)G(s-u+\theta)B(u)ds, dW_Q(u) \right\rangle_H \\
&= \left\langle h, \tilde{y}(t)_H - \int_0^t B(u)dW_Q(u) \right\rangle_H \\
&\qquad - \left\langle \int_{-r}^0 B^*(u)G^*(s-u+\theta)d\eta^*(\theta)hds, \int_0^t dW_Q(u) \right\rangle_H \\
&\qquad\quad \left\langle h, \int_0^t \int_0^{s+\theta} \int_{-r}^0 d\eta(\theta)G(s-u+\theta)B(u)dW_Q(u)ds \right\rangle_H \\
&= \left\langle h, \tilde{y}(t)_H - \int_0^t B(u)dW_Q(u) \right\rangle_H \\
&\qquad - \left\langle h, \int_0^t \int_{-r}^0 D(\theta)G(t-u+\theta)B(u)d\theta dW_Q(u) \right\rangle_H \\
&\qquad\quad - \left\langle h, \int_0^t \int_{-r}^0 d\eta(\theta)\int_0^{s+\theta} G(s-u+\theta)B(u)dW_Q(u)ds \right\rangle_H
\end{aligned}
$$

$$
\begin{aligned}
&= \left\langle h, \tilde{y}(t)\right\rangle_H - \left\langle h, \int_0^t B(u)dW_Q(u)\right\rangle - \left\langle h, \int_0^t \int_{-r}^0 d\eta(\theta)\tilde{y}(s+\theta)ds\right\rangle_H \\
&\qquad -\left\langle h, \int_{-r}^0 D(\theta)\int_0^t G(t-u+\theta)B(u)dW_Q(u)d\theta\right\rangle_H \\
&= \left\langle h, \tilde{y}(t)\right\rangle_H - \left\langle h, \int_0^t B(u)dW_Q(u)\right\rangle - \left\langle h, \int_0^t \int_{-r}^0 d\eta(\theta)\tilde{y}(s+\theta)ds\right\rangle_H \\
&\qquad -\left\langle h, \int_{-r}^0 D(\theta)\tilde{y}(t+\theta)d\theta\right\rangle_H,
\end{aligned}
$$

which means

$$
\begin{aligned}
&\left\langle h, \tilde{y}(t) - \int_{-r}^0 D(\theta)\tilde{y}(t+\theta)d\theta\right\rangle_H \\
= &\int_0^t \langle A^*h, \tilde{y}(s)\rangle_H ds + \left\langle h, \int_0^t \int_{-r}^0 d\eta(\theta)\tilde{y}(s+\theta)ds\right\rangle_H \\
&+\left\langle h, \int_0^t B(s)dW_Q(s)\right\rangle_H.
\end{aligned}
$$

This shows that $\tilde{y}(t)$ is a weak solution of Equation (5.3.11). Next, we shall show that every weak solution $\tilde{y}(t)$ of (5.3.11) is the form of Equation (5.3.12). To this end, we first show that for arbitrary $z(t) \in C([0,T];\mathcal{D}(A^*))$, the following relation holds

$$
\begin{aligned}
\langle z(t), \tilde{y}(t)\rangle_H &= \int_0^t \langle z'(s) + A^*z(s), \tilde{y}(s)\rangle_H ds \\
&\quad + \int_0^t \langle z(s), \int_{-r}^0 d\eta(\theta)\tilde{y}(s+\theta)\rangle_H ds \\
&\quad + \int_0^t \langle z(s), B(s)dW_Q(s)\rangle_H. \qquad (5.3.13)
\end{aligned}
$$

Indeed, consider functions of the form $z(t) = z_0\varphi(t)$, $t \in [0,T]$, where $\varphi \in C([0,T];\mathbb{C})$ and $z_0 \in \mathcal{D}(A^*)$. Let us put

$$
J_{z_0}(t) = \int_0^t \langle A^*z_0, \tilde{y}(s)\rangle_H ds + \int_0^t \langle z_0, \int_{-r}^0 d\eta(\theta)\tilde{y}(s+\theta)\rangle_H ds + \int_0^t \langle z_0, B(s)dW_Q(s)\rangle_H.
$$

Applying Itô's formula to the processes $J_{z_0}(t)\varphi(t)$ we get

$$
d(J_{z_0}(t)\varphi(t)) = \varphi(t)dJ_{z_0}(t) + \varphi'(t)J_{z_0}(t)dt.
$$

In particular,

$$\begin{aligned} J_{z_0}(t)\varphi(t) &= \int_0^t \langle z_0\varphi(s), B(s)dW_Q(s)\rangle_H + \int_0^t \langle z_0\varphi(s), \int_{-r}^0 d\eta(\theta)\tilde{y}(s+\theta)\rangle_H ds \\ &\quad \int_0^t [\langle A^* z_0\varphi(s), \tilde{y}(s)\rangle_H + \varphi'(s)\langle z_0, \tilde{y}(s)\rangle_H] ds. \end{aligned}$$

Since $J_{z_0}(\cdot) = \langle z_0, \tilde{y}(\cdot)\rangle_H$ almost surely. Thus, (5.3.13) is proved for the function $z(t) = z_0\varphi(t)$. Since these functions form a dense set in $C([0,T];\mathcal{D}(A^*))$, the equality (5.3.13) is thus true for all $z(t) \in C([0,T];\mathcal{D}(A^*))$. Now applying (5.3.13) to the function $z(s) = G^*(t-s)z_0$, $0 \le s \le t \le T$, we have

$$\langle z_0, \tilde{y}(t)\rangle_H = \left\langle z_0, \int_0^t G(t-s)B(s)dW_Q(s)\right\rangle_H$$

and since $\mathcal{D}(A^*)$ is dense in H, we find that $\tilde{y}(t) = \int_0^t G(t-s)B(s)dW_Q(s)$, $t \in [0,T]$. The proof is complete.

■

Finally, we wish to find suitable conditions under which the mild solution of Equation (5.2.1) is a strong one. To begin with, we introduce the following lemma from Liu (2009) again.

Lemma 5.3 *Let $h \in H$, then $\int_0^t G(s)hds \in \mathcal{D}(A)$ for almost all $t \in \mathbb{R}_+$, and the relation*

$$\begin{aligned} A\int_0^t G(s)hds &= G(t)h - h - \int_0^t \int_{-r}^0 d\eta(v)G(s+v)hds \\ &\quad - \int_{-r}^0 D(v)G(t+v)hdv, \quad t \in \mathbb{R}_+, \quad h \in H, \end{aligned}$$

holds almost everywhere.

Theorem 5.2 *Suppose that $\phi_0 \in \mathcal{D}(A)$ and $\phi_1 \in W^{1,2}([-r,0];H)$, for any $k \in K$,*

(a) $\mathcal{R}(F(\theta)) \in \mathcal{D}(A)$ for $\tau \in [-r,0]$, $\theta \in [\tau,0]$, $\mathcal{R}(A_i) \in \mathcal{D}(A)$,

(b) $G(t-s)B(s)k \in \mathcal{D}(A)$ *for almost all* $s \le t \in [0,T]$ *and* $AG(t-\cdot)B(\cdot)k \in L^2([0,T];H)$ *for any* $t \in [0,T]$,

(c) $AG(t+\cdot)D(\cdot) \in L^2([0,T];H)$ *and* $AG(t+\cdot)\phi_1(\cdot) \in L^2([0,T];H)$.

Then the mild solution of Equation (5.2.1) is also a strong one.

Proof. Suppose that $y(t) = y(t,\phi)$ is a mild solution of Equation (5.2.1), that is, it satisfies the following variation of constants formula

$$\begin{aligned} y(t) &= G(t)\phi_0 - V(t,0)\phi_1(0) + \int_{-r}^{0} U(t,\theta)\phi_1(\theta)d\theta + \int_{-r}^{0} V(t,\theta)\phi_1'(\theta)d\theta \\ &\quad + \int_0^t G(t-s)B(s)dW_Q(s), \quad T \ge 0, \end{aligned}$$

almost surely where for any $t \in [0,T]$, the kernels

$$U(t,\theta) = \int_{-r}^{\theta} G(t-\theta+\tau)d\eta(\tau) \in L^2([-r,0];\mathcal{L}(H)),\ \theta \in [-r,0],$$

$$V(t,\theta) = \int_{-r}^{\theta} G(t-\theta+\tau)D(\tau)d\tau \in L^2([-r,0];\mathcal{L}(H)),\ \theta \in [-r,0].$$

We can deduce that $\int_{-r}^{\theta} G(t-\theta+\tau)d\eta(\tau)\phi_1(\theta) \in \mathcal{D}(A)$ by condition (a). Since $\mathcal{R}(D(\theta)) \subset \mathcal{D}(A)$, we have $\int_{-r}^{0} G(t+\tau)D(\tau)d\tau\phi_1(0) \in \mathcal{D}(A)$ and $\int_{-r}^{\theta} G(t-\theta+\tau)D(\tau)\phi_1'(\theta)d\tau \in \mathcal{D}(A)$. Therefore, it can be shown that $y(t) \in \mathcal{D}(A)$ for any $t \ge 0$ by the conditions (a) - (c) in the theorem. Note that $G(t) = \mathbf{O}$ for $t < 0$ and A is a closed operator. By using Lemma 5.3, we can deduce that for the initial datum $\phi_0 \in \mathcal{D}(A)$, $\phi_1 \in W^{1,2}([-r,0];H)$,

$$\begin{aligned} A\int_0^t G(s)\phi_0 ds &= G(t)\phi_0 - \phi_0 - \int_0^t \int_{-r}^{0} d\eta(v)G(s+v)\phi_0 ds \\ &\quad - \int_{-r}^{0} D(v)G(t+v)\phi_0 dv. \end{aligned} \tag{5.3.14}$$

Also, by using Fubini's theorem and Lemma 5.3, we can deduce the following Equations (5.3.15)-(5.3.17). Firstly,

$$
\begin{aligned}
& -A\int_0^t\int_{-r}^0 G(s+\tau)D(\tau)d\tau\phi_1(0)ds \\
= & -\int_{-r}^0 A\int_0^t G(s+\tau)D(\tau)ds\phi_1(0)d\tau \\
= & -\int_{-r}^0\bigg(G(t+\tau)D(\tau)-G(\tau)D(\tau)-\int_0^t\int_{-r}^0 d\eta(v)G(s+\tau+v)D(\tau)ds \\
& -\int_{-r}^0 D(v)G(t+\tau+v)D(\tau)dv\bigg)\phi_1(0)d\tau \\
= & -\int_{-r}^0 G(t+\tau)D(\tau)\phi_1(0)d\tau+\int_{-r}^0\int_0^t\int_{-r}^0 d\eta(v)G(s+\tau+v)D(\tau)\phi_1(0)dsd\tau \\
& +\int_{-r}^0\int_{-r}^0 D(v)G(t+\tau+v)D(\tau)\phi_1(0)dvd\tau. \qquad (5.3.15)
\end{aligned}
$$

Secondly,

$$
\begin{aligned}
& A\int_0^t\int_{-r}^0\int_{-r}^\theta G(s-\theta+\tau)d\eta(\tau)\phi_1(\theta)d\theta ds \\
= & A\int_{-r}^0\int_0^t\int_{-r}^\theta G(s-\theta+\tau)d\eta(\tau)\phi_1(\theta)dsd\theta \\
= & \int_{-r}^0\int_{-r}^\theta A\int_0^t G(s-\theta+\tau)\phi_1(\theta)dsd\eta(\tau)d\theta \\
= & \int_{-r}^0\int_{-r}^\theta\bigg(G(t-\theta+\tau)\phi_1(\theta)-G(-\theta+\tau)\phi_1(\theta) \\
& -\int_0^t\int_{-r}^0 d\eta(v)G(s-\theta+\tau+v)\phi_1(\theta)ds \\
& -\int_{-r}^0 D(v)G(t-\theta+\tau+v)\phi_1(\theta)dv \\
& +\int_{-r}^0 D(v)G(-\theta+\tau+v)\phi_1(\theta)dv\bigg)d\eta(\tau)d\theta \\
= & \int_{-r}^0\int_{-r}^\theta G(t-\theta+\tau)\phi_1(\theta)d\eta(\tau)d\theta \\
& -\int_{-r}^0\int_{-r}^\theta\int_0^t\int_{-r}^0 d\eta(v)G(s-\theta+\tau+v)\phi_1(\theta)dsd\eta(\tau)d\theta \\
& -\int_{-r}^0\int_{-r}^\theta\int_{-r}^0 D(v)G(t-\theta+\tau+v)\phi_1(\theta)dvd\eta(\tau)d\theta \\
& +\int_{-r}^0\int_{-r}^\theta\int_{-r}^0 D(v)G(-\theta+\tau+v)\phi_1(\theta)dvd\eta(\tau)d\theta, \qquad (5.3.16)
\end{aligned}
$$

and thirdly,

$$
\begin{aligned}
& A\int_0^t \int_{-r}^0 \int_{-r}^\theta G(s-\theta+\tau)D(\tau)d\tau\phi_1'(\theta)d\theta ds \\
= \;& A\int_{-r}^0 \int_0^t \int_{-r}^\theta G(s-\theta+\tau)D(\tau)d\tau\phi_1'(\theta)dsd\theta \\
= \;& \int_{-r}^0 \int_{-r}^\theta A\int_0^t G(s-\theta+\tau)D(\tau)ds\phi_1'(\theta)d\tau d\theta \\
= \;& \int_{-r}^0 \int_{-r}^\theta \Big(G(t-\theta+\tau)D(\tau) - G(-\theta+\tau)D(\tau) \\
& - \int_0^t \int_{-r}^0 d\eta(v)G(s-\theta+\tau+v)D(\tau)ds \\
& - \int_{-r}^0 D(v)G(t-\theta+\tau+v)D(\tau)dv \\
& + \int_{-r}^0 D(v)G(-\theta+\tau+v)D(\tau)dv\Big)\phi_1'(\theta)d\tau d\theta \\
= \;& \int_{-r}^0 \int_{-r}^\theta G(t-\theta+\tau)D(\tau)\phi_1'(\theta)d\tau d\theta \\
& - \int_{-r}^0 \int_{-r}^\theta \int_0^t \int_{-r}^0 d\eta(v)G(s-\theta+\tau+v)D(\tau)\phi_1'(\theta)dsd\tau d\theta \\
& - \int_{-r}^0 \int_{-r}^\theta \int_{-r}^0 D(v)G(t-\theta+\tau+v)D(\tau)\phi_1'(\theta)dvd\tau d\theta \\
& + \int_{-r}^0 \int_{-r}^\theta \int_{-r}^0 D(v)G(-\theta+\tau+v)D(\tau)\phi_1'(\theta)dvd\tau d\theta. \qquad (5.3.17)
\end{aligned}
$$

Using the standard stochastic Fubini's theorem (c.f. Da Prato and Zabczyk (1992), Theorem 4.18), ones can deduce that

$$
\begin{aligned}
& A\int_0^t \int_0^s G(s-u)B(u)dW_Q(u)ds \\
= \;& \int_0^t A\int_u^t G(s-u)B(u)dsdW_Q(u) \\
= \;& \int_0^t \Big(G(t-u)B(u) - B(u) - \int_u^t \int_{-r}^0 d\eta(v)G(s-u+v)B(u)ds \\
& - \int_{-r}^0 D(v)G(t-u+v)B(u)dv\Big)dW_Q(u)
\end{aligned}
$$

$$
\begin{aligned}
= & \int_0^t G(t-u)B(u)dW_Q(u) - \int_0^t B(u)dW_Q(u) \\
& - \int_0^t \int_u^t \int_{-r}^0 d\eta(v)G(s-u+v)B(u)dsdW_Q(u) \\
& - \int_0^t \int_{-r}^0 D(v)G(t-u+v)B(u)dvdW_Q(u). \qquad (5.3.18)
\end{aligned}
$$

On the other hand, since $y(t)$ is a mild solution of Equation (5.2.1), thus

$$
\begin{aligned}
& \int_0^t \int_{-r}^0 d\eta(v)y(s+v)ds \\
= & \int_0^t \int_{-r}^0 d\eta(v)G(s+v)\phi_0 ds - \int_0^t \int_{-r}^0 d\eta(v) \int_{-r}^0 G(s+v+\tau)D(\tau)d\tau\phi_1(0)ds \\
& + \int_0^t \int_{-r}^0 d\eta(v) \int_{-r}^0 \int_{-r}^\theta G(s-\theta+\tau+u)d\eta(\tau)\phi_1(\theta)d\theta ds \\
& + \int_0^t \int_{-r}^0 d\eta(v) \int_{-r}^0 \int_{-r}^\theta G(s-\theta+\tau+v)D(\tau)d\tau\phi_1'(\theta)d\theta ds \\
& + \int_0^t \int_{-r}^0 d\eta(v) \int_0^s G(s+v-u)B(u)dW_Q(u)ds. \qquad (5.3.19)
\end{aligned}
$$

Therefore, by the closeness of A and Equation (5.2.8), it follows that

$$
\begin{aligned}
& \int_0^t \left(Ay(s) + \int_{-r}^0 d\eta(v)y(s+v) \right) ds \\
= & \ A\int_0^t y(s)ds + \int_0^t \int_{-r}^0 d\eta(v)y(s+v)ds \\
= & \ A\int_0^t \Bigg(G(s)\phi_0 - \int_{-r}^0 G(s+\tau)D(\tau)d\tau\phi_1(0) \\
& + \int_{-r}^0 \int_{-r}^\theta G(s-\theta+\tau)d\eta(\tau)\phi_1(\theta)d\theta \\
& + \int_{-r}^0 \int_{-r}^\theta G(s-\theta+\tau)D(\tau)d\tau\phi_1'(\theta)d\theta + \int_0^s G(s-u)B(u)dW_Q(u) \Bigg) ds \\
& + \int_0^t \int_{-r}^0 d\eta(v)y(s+v)ds
\end{aligned}
$$

$$
\begin{aligned}
&= A\int_0^t G(s)\phi_0 ds - A\int_0^t\int_{-r}^0 G(s+\tau)D(\tau)d\tau\phi_1(0)ds \\
&\quad +A\int_0^t\int_{-r}^0\int_{-r}^\theta G(s-\theta+\tau)d\eta(\tau)\phi_1(\theta)d\theta ds \\
&\quad +A\int_0^t\int_{-r}^0\int_{-r}^\theta G(s-\theta+\tau)D(\tau)d\tau\phi_1'(\theta)d\theta ds \\
&\quad +A\int_0^t\int_0^s G(s-u)B(u)dW_Q(u)ds + \int_0^t\int_{-r}^0 d\eta(v)y(s+v)ds.
\end{aligned}
$$

which, together with Equations (5.3.14)-(5.3.19), yields that

$$
\begin{aligned}
&\int_0^t\left(Ay(s)+\int_{-r}^0 d\eta(v)y(s+v)\right)ds \\
= \ & G(t)\phi_0 - \phi_0 - \int_0^t\int_{-r}^0 d\eta(v)G(s+v)\phi_0 ds - \int_{-r}^0 D(v)G(t+v)\phi_0 dv \\
& -\int_{-r}^0 G(t+\tau)D(\tau)\phi_1(0)d\tau + \int_{-r}^0\int_0^t\int_{-r}^0 d\eta(v)G(s+\tau+v)D(\tau)\phi_1(0)dsd\tau \\
& +\int_{-r}^0\int_{-r}^0 D(v)G(t+\tau+v)D(\tau)\phi_1(0)dvd\tau \\
& +\int_{-r}^0\int_{-r}^\theta G(t-\theta+\tau)\phi_1(\theta)d\eta(\tau)d\theta \\
& -\int_{-r}^0\int_{-r}^\theta\int_0^t\int_{-r}^0 d\eta(v)G(s-\theta+\tau+v)\phi_1(\theta)dsd\eta(\tau)d\theta \\
& -\int_{-r}^0\int_{-r}^\theta\int_{-r}^0 D(v)G(t-\theta+\tau+v)\phi_1(\theta)dvd\eta(\tau)d\theta \\
& +\int_{-r}^0\int_{-r}^\theta\int_{-r}^0 D(v)G(-\theta+\tau+v)\phi_1(\theta)dvd\eta(\tau)d\theta \\
& +\int_{-r}^0\int_{-r}^\theta G(t-\theta+\tau)D(\tau)\phi_1'(\theta)d\tau d\theta \\
& -\int_{-r}^0\int_{-r}^\theta\int_0^t\int_{-r}^0 d\eta(v)G(s-\theta+\tau+v)D(\tau)\phi_1'(\theta)dsd\tau d\theta \\
& -\int_{-r}^0\int_{-r}^\theta\int_{-r}^0 D(v)G(t-\theta+\tau+v)D(\tau)\phi_1'(\theta)dvd\tau d\theta \\
& +\int_{-r}^0\int_{-r}^\theta\int_{-r}^0 D(v)G(-\theta+\tau+v)D(\tau)\phi_1'(\theta)dvd\tau d\theta \\
& +\int_0^t G(t-u)B(u)dW_Q(u) - \int_0^t B(u)dW_Q(u) \\
& -\int_0^t\int_u^t\int_{-r}^0 d\eta(v)G(s-u+v)B(u)dsdW_Q(u)
\end{aligned}
$$

$$
\begin{aligned}
& -\int_0^t \int_{-r}^0 D(v)G(t-u+v)B(u)dvdW_Q(u) \\
& +\int_0^t \int_{-r}^0 d\eta(v)G(s+v)\phi_0 ds - \int_0^t \int_{-r}^0 d\eta(v) \int_{-r}^0 G(s+v+\tau)D(\tau)d\tau\phi_1(0)ds \\
& +\int_0^t \int_{-r}^0 d\eta(v) \int_{-r}^0 \int_{-r}^\theta G(s-\theta+\tau+u)d\eta(\tau)\phi_1(\theta)d\theta ds \\
& +\int_0^t \int_{-r}^0 d\eta(v) \int_{-r}^0 \int_{-r}^\theta G(s-\theta+\tau+v)D(\tau)d\tau\phi_1'(\theta)d\theta ds \\
& +\int_0^t \int_{-r}^0 d\eta(v) \int_0^s G(s+v-u)B(u)dW_Q(u)ds \\
= \; & y(t) - \phi_0 - \Big(\int_{-r}^0 D(v)G(t+v)\phi_0 dv \\
& \quad - \int_{-r}^0 \int_{-r}^0 D(v)G(t+\tau+v)D(\tau)\phi_1(0)d\tau dv \\
& \quad + \int_{-r}^0 \int_{-r}^0 \int_{-r}^\theta D(v)G(t-\theta+\tau+v)\phi_1(\theta)d\eta(\tau)d\theta dv \\
& \quad + \int_{-r}^0 \int_{-r}^0 \int_{-r}^\theta D(v)G(t-\theta+\tau+v)D(\tau)\phi_1'(\theta)d\tau d\theta dv \\
& \quad + \int_{-r}^0 \int_0^t D(v)G(t-u+v)B(u)dW_Q(u)dv \Big) \\
& + \Big(\int_{-r}^0 \int_{-r}^0 \int_{-r}^\theta D(v)G(-\theta+\tau+v)\phi_1(\theta)d\eta(\tau)d\theta dv \\
& \quad + \int_{-r}^0 \int_{-r}^0 \int_{-r}^\theta D(v)G(-\theta+\tau+v)D(\tau)\phi_1'(\theta)d\tau d\theta dv \Big) - \int_0^t B(u)dW_Q(u) \\
= \; & y(t) - \phi_0 - \int_{-r}^0 D(v)y(t+v)dv + \int_{-r}^0 D(v)\phi_1(v)dv - \int_0^t B(u)dW_Q(u).
\end{aligned}
$$

That is,

$$
\begin{aligned}
\int_0^t \Big(Ay(s) + \int_{-r}^0 d\eta(\theta)y(s+\theta) \Big) ds &= y(t) - \phi_0 - \int_{-r}^0 D(\theta)y(t+\theta)d\theta \\
&\quad + \int_{-r}^0 D(\theta)\phi_1(\theta)d\theta - \int_0^t B(s)dW_Q(s).
\end{aligned}
$$

Thus, the mild solution $y(t)$ of Equation (5.2.1) is also a strong solution. The proof is complete.

■

5.4 Conclusion

In conclusion, we have studied the stochastic neutral functional differential equations in a separable Hilbert space. The theory of fundamental solutions (Green's operator) from Liu (2008, 2009) has been used to establish the representation of the mild solutions. Then we investigated relations among strong, weak and mild solutions of our SNFDEs and suitable conditions has been founded under which mild solutions become strong ones. By suitably applying the above theorem of fundamental solutions (Green's operator), one can show that the results of Theorem 5.1, Proposition 5.1 and Theorem 5.2 can be extended to the following semi-linear neutral stochastic functional differential equations:

for any $T > 0$, $t \in (0, T]$

$$d\left[y(t) - \int_{-r}^{0} D(\theta)y(t+\theta)d\theta\right] = Ay(t)dt + \int_{-r}^{0} d\eta(\theta)y(t+\theta)dt + G(y_t)dt + B(y_t)dW_Q(t),$$

$$y(0) = \phi_0,\ y_0 = \phi_1,\ \phi = (\phi_0, \phi_1) \in \mathcal{H},$$

Chapter 6

Fixed point theory and asymptotic stability

6.1 Asymptotic stability of impulsive stochastic partial delay differential equations with Poisson jumps

6.1.1 Introduction

Many evolution processes are characterized by the fact that they experience a change of state abruptly at certain moments of time. It is natural to assume that these perturbations act instantaneously, that is, in the form of impulsive. It is known that many models in sciences and economics exhibit impulse effects. For example, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics. Impulsive differential equations (IDEs) are often used to describe such systems. For instance, in Lakshmikantham et al. (1989), a mathematical model of a simple impulsive differential system in which impulses occur at fixed times may be described by

$$\begin{cases} \frac{dx}{dt} = f(t,x), \quad t \neq t_k, \quad k = 1, 2, \cdots \\ \\ \triangle x = I_k(x), \quad t = t_k \end{cases}$$

where $\{t_k\}$ is a sequence of times such that $t_k \to \infty$ as $k \to \infty$, for $t = t_k$, $\triangle x(t_k) = x(t_k^+) - x(t_k)$ and $x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$, $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is a open set, $I_k : \Omega \to \Omega$.

In recent years, significant progress has been made in the theory of impulsive differential equations, please refer to Lakshmikantham et al. (1989) and references therein. For the abrupt change, there may emerge jumps in the evolution, which lead to the non-smooth effects of the system. Thus, the qualitative properties of impulsive differential systems are very important and many results have been obtained in analysis of systems with impulse effect or design of control systems via impulsive control laws. For example, in Lakshmikantham et al. (1989), some basic theories including the theory of stability of an impulsive control scheme is equivalent to the stability of trivial solution of an impulsive differential equation are given; Yang (2001) studied the problem of impulsive control and impulsive synchronization by using the comparison method of impulsive differential equations to judge whether the system under consideration is stable or not. In Shen (1999) and Zhang and Sun (2008), Lyapunov-Razumikhin stability theorems for impulsive functional differential equations (IFDEs) are presented. Meanwhile, the studies of impulsive differential equations with delays (IDDEs) have received significant attention. For example, the stability of zero solution of IDDEs has been investigated by Liu and Ballinger (2001), Xu and Yang (2005) and Zhang and Sun (2008).

However, besides delay and impulsive effects, stochastic effects likewise exist in real systems. A lot of dynamical systems have variable structures subject to abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc. So it is necessary to study the impulsive system with stochastic factor i.e. impulsive stochastic differential equations with delays (ISDDEs). For example, Yang et al. (2006) studied the p-moment stability of ISDDEs and established stability criteria of the system. Wu et al. (2004) investigated the p-moment stability of stochastic differential equations with jumps and a theory of the p-moment stability was constructed. By applying fixed point theory, a few researchers, for instance, J. Luo, T. Taniguchi and R. Sakthivel (Luo (2007, 2008), Luo and Taniguchi (2009), and Sakthivel and Luo (2009a,b)) have obtained the asymptotic behavior of solutions of stochastic differential equations.

In particular, Sakthivel and Luo (2009b) studied the existence and asymptotic in p-th moment of mild solutions of nonlinear impulsive stochastic differential equations in a real separable Hilbert space H,

$$\begin{cases} dx(t) = [Ax(t) + f(t, x(t))]dt + g(t, x(t))dw(t), \quad t \geq 0, \quad t \neq t_k, \\ \\ \triangle x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad t = t_k, \quad 1, 2, \cdots, m, \\ \\ x(0) = x_0, \end{cases}$$

where $f : \mathbb{R}_+ \times H \to H$, $g : \mathbb{R}_+ \times H \to \mathcal{L}(K, H)$ are all Borel measurable; $I_k : H \to H$; A is the infinitesimal generator of a semigroup of bounded linear operator $T(t)$, $t \geq 0$ in H; Furthermore the fixed moments of time t_k satisfies $0 < t_1 < \cdots < t_m < \lim_{k\to\infty} = \infty$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively; $\triangle x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump in the state x at time t_k with I_k determining the size of jump; w is a K-valued Wiener process.

Also, Sakthivel and Luo (2009a) studied the asymptotic stability in p-th moment of mild solutions to nonlinear impulsive stochastic partial differential equations with infinite delay on H,

$$\begin{cases} dx(t) = [Ax(t) + f(t, x(t - \tau(t)))]dt + g(t, x(t - \delta(t)))dw(t), \quad t \geq 0, \quad t \neq t_k, \\ \\ \triangle x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad t = t_k, \quad 1, 2, \cdots, m, \\ \\ x_0(\cdot) = \varphi \in D^b_{\mathcal{F}_0}([\tilde{m}(0), 0], H), \end{cases}$$

where $\tau(t)$, $\delta(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy $t - \tau(t) \to \infty$, $t - \delta(t) \to \infty$ as $t \to \infty$ and $\tilde{m}(0) = \max\{\inf(s - \tau(s), s \geq 0), \inf(s - \delta(s), s \geq 0)\}$.

To the best of our knowledge, there are a few work about the asymptotic stability for mild solutions to impulsive stochastic partial delay differential equations with Poisson jumps. In this section, we shall apply fixed point theorem to investigate the asymptotic stability in mean square of mild solution to the following equations,

$$\begin{cases} dX(t) = \Big[AX(t) + F\big(t, X(t), X(t - \delta(t))\big)\Big]dt + G\big(t, X(t), X(t - \rho(t))\big)dW_Q(t) \\ \\ \qquad\qquad + \int_{\mathbb{Z}} L\big(t, X(t - \theta(t)), u\big)\widetilde{N}(dt, du), \quad t \geq 0, \quad t \neq t_k, \\ \\ \triangle X(t_k) = X(t_k^+) - X(t_k^-) = I_k(X(t_k^-)), \quad t = t_k, \quad k = 1, 2, \cdots, m, \\ \\ X_0(\cdot) = \varphi \in D^b_{\mathcal{F}_0}([-r, 0]; H), \quad -r \leq t < 0, \end{cases}$$

where $\delta(t)$, $\rho(t)$ and $\theta(t) : [0, \infty) \to [0, r]$, $r > 0$ are continuous functions. Details of this equations are explained in Section 6.1.2.

The rest of this section is organized as follows. In section 6.1.2, we briefly present some basic notations and preliminaries for impulsive stochastic partial delay differential equations with Poisson jumps. Section 6.1.3 is devoted to the study of asymptotic stability in mean square of mild solutions to our ISPDDEs with jumps. By employing a fixed point approach, sufficient conditions are derived for achieving the required result. These conditions do not require the monotone decreasing behavior of the delays.

6.1.2 Impulsive stochastic partial delay differential equations with Poisson jumps

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with probability measure $\mathbb{P}$ on Ω and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is the filtration is right-continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets.

Let H, K be two real separable Hilbert spaces with their norms and inner products denoted by $\|\cdot\|_H$, $\|\cdot\|_K$ and $\langle\cdot,\cdot\rangle_H$, $\langle\cdot,\cdot\rangle_K$ respectively. We denote by $\mathcal{L}(K, H)$ the set of all linear bounded operators from K into H, equipped with the usual operator norm $\|\cdot\|$. In this work, we always use the same symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion may arise. Let $r > 0$ and $D := D([-r, 0]; H)$ denote the family of all right-continuous functions with left-hand limits φ from $[-r, 0]$ to H. The space $D([-r, 0]; H)$ is assumed to be equipped with the supremum norm $\|\varphi\|_D = \sup_{-r\leq\theta\leq 0}\|\varphi(\theta)\|_H$. We also use $D_{\mathcal{F}_0^b}([-r, 0]; H)$ to denote the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $D([-r, 0]; H)$-valued random variables.

Let $W_Q(t)$ be a Wiener process and $\int_0^t \Phi(s)dW_Q(s)$ be the stochastic integral with respect to $W_Q(t)$, which is a continuous square-integrable martingale (c.f. Section 2.2). Let $\widetilde{N}(dt, du) := N(dt, du) - dt\lambda(du)$ be the compensated Poisson random measures and $\int_0^T \int_{\mathbb{Z}} L(t, u)\widetilde{N}(dt, du)$ the stochastic integral with respect to $\widetilde{N}(dt, du)$ which is a centered square-integrable martingale, where $\mathbb{Z} \in \mathcal{B}(K - \{0\})$ with $0 \notin \bar{\mathbb{Z}}$, the closure of $\mathbb{Z}$ in K and $\mathcal{B}(K - \{0\})$ denotes the Borel σ-filed of $K - \{0\}$ (c.f. Section 2.3). We always assume in this chapter that W_Q and $\widetilde{N}$ are independent of the $\mathcal{F}_0$ and of each other.

In this section, we consider a mathematical model given by the following impulsive stochastic delay differential equations with Poisson jumps,

$$\begin{cases} dX(t) = \big[AX(t) + F\big(t, X(t-\delta(t))\big)\big]dt + G\big(t, X(t-\rho(t))\big)dW_Q(t) \\ \qquad\qquad + \int_{\mathbb{Z}} L\big(t, X(t-\theta(t)), u\big)\widetilde{N}(dt, du), \quad t \geq 0, \quad t \neq t_k, \\ \triangle X(t_k) = X(t_k^+) - X(t_k^-) = I_k(X(t_k^-)), \quad t = t_k, \quad k = 1, 2, \cdots, m, \\ X_0(\cdot) = \varphi \in D^b_{\mathcal{F}_0}([-r, 0]; H), \quad -r \leq t < 0. \end{cases} \tag{6.1.1}$$

where $F : \mathbb{R}_+ \times H \to H$, $G : \mathbb{R}_+ \times H \to \mathcal{L}(K, H)$ and $L : \mathbb{R}_+ \times H \times \mathbb{Z} \to H$ are all Borel measurable; A is the infinitesimal generator of a semigroup of bounded linear operators $T(t)$, $t \geq 0$, in H; $I_k : H \to H$. Furthermore the fixed moments of time t_k satisfies $0 < t_1 < \cdots < t_m < \lim_{k\to\infty} t_k = \infty$, $X(t_k^+)$ and $X(t_k^-)$ represent the right and left limits of $X(t)$ at $t = t_k$, respectively. Also $\triangle X(t_k) = X(t_k^+) - X(t_k^-) = I_k(X(t_k^-))$ represents the jump in the state X at time t_k with I_k determining the size of the jump. Moreover, for $r > 0$, let $\delta(t)$, $\rho(t)$ and $\theta(t)$ be continuous functions from $[0, \infty)$ to $[0, r]$.

Let us recall the definition of mild solution of impulsive stochastic delay differential equations (6.1.1) and the definitions of mean square stability.

Definition 6.1 (Mild solution) *A stochastic process* $\{x(t), t \in [0, T]\}$, $0 \leq T < \infty$, *is called a mild solution of Equation (6.1.1) if*

(i) $X(t)$ *is adapted to* $\mathcal{F}_t$ *and has càdlàg path on* $t \geq 0$ *almost surely and*

(ii) for arbitrary $t \in [0,T]$, $\mathbb{P}\{\omega : \int_0^t \|X(t)\|_H^2 ds < \infty\} = 1$ and almost surely

$$\begin{aligned}
X(t) &= T(t)\varphi(0) + \int_0^t T(t-s)F\big(s, X(s-\delta(s))\big)ds \\
&\quad + \int_0^t T(t-s)G\big(s, X(s-\rho(s))\big)dW_Q(s) \\
&\quad + \int_0^t \int_{\mathbb{Z}} T(t-s)L\big(s, X(s-\theta(s)), u\big)\widetilde{N}(ds, du) \\
&\quad + \sum_{0<t_k<t} T(t-s)I_k(X(t_k^-)) \qquad (6.1.2)
\end{aligned}$$

and

$$X_0(\cdot) = \varphi \in D^b_{\mathcal{F}_0}([-r,0];H), \quad -r \le t < 0.$$

Definition 6.2 (Mean square stability) *The Equation (6.1.1) is said to be stable in mean square if for arbitrarily given $\epsilon > 0$, there exists $\delta > 0$ such that $\|\varphi\|_D < \delta$ guarantees that*

$$\mathbb{E}\Big\{\sup_{t\ge 0} \|X(t)\|_H^2\Big\} < \epsilon.$$

Definition 6.3 (Asymptotic mean square stability) *The Equation (6.1.1) is said to be mean square asymptotically stable if it is stable in mean square and for any $\varphi \in D^b_{\mathcal{F}_0}([-r,0];H)$,*

$$\lim_{T\to\infty} \mathbb{E}\Big\{\sup_{t\ge T} \|X(t)\|_H^2\Big\} = 0.$$

6.1.3 Asymptotic stability

In this section, we shall formulate and prove the conditions for the asymptotic stability in mean square of mild solution to Equation (6.1.1) by using fixed point theorem. For the purpose of stability, we shall assume that $F(t,0) = 0$, $G(t,0) = 0$, $L(t,0,u) = 0$ and $I_k(0) = 0\,(k = 1, 2, \cdots, m)$. Then Equation (6.1.1) has a trivial solution when $\varphi = 0$.

In order to obtain our main result, we impose the following assumptions:

Assumption 6.1 *A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $T(t)$, $t \geq 0$, in H satisfying*

$$\|T(t)\| \leq Me^{-\alpha t},\ t \geq 0 \tag{6.1.3}$$

for some constants $M \geq 1$ and $0 < \alpha \in \mathbb{R}_+$.

Assumption 6.2 *For all x, $y \in H$ and $t \geq 0$, the functions F, G and L satisfy the following Lipschitz conditions:*

$$\|F(t,x) - F(t,y)\|_H^2 \leq C_1\|x-y\|_H^2, \quad C_1 > 0; \tag{6.1.4}$$

$$\|G(t,x) - G(t,y)\|_{\mathcal{L}_2^0}^2 \leq C_2\|x-y\|_H^2, \quad C_2 > 0; \tag{6.1.5}$$

$$\int_{\mathbb{Z}} \|L(t,x,u) - G(t,y,u)\|_H^2 \lambda(du) \leq C_3\|x-y\|_H^2, \quad C_3 > 0. \tag{6.1.6}$$

Assumption 6.3 *$I_k \in C(H,H)$ and there exists a constant $q_k > 0$ such that for each x, $y \in H$ and $k = 1, 2, \cdots, m$,*

$$\|I_k(x) - I_k(y)\| \leq q_k\|x-y\|. \tag{6.1.7}$$

Theorem 6.1 *Suppose that Assumptions (6.1) - (6.3) hold. If the following inequality is satisfied:*

$$5M^2\Big(C_1^2\alpha^{-2} + C_2^2(2\alpha)^{-1} + C_3^2(2\alpha)^{-1} + C_4\Big) < 1, \tag{6.1.8}$$

where $C_4 = e^{-\alpha T}\mathbb{E}\Big(\sum_{k=1}^m \|q_k\|_H^2\Big)$, then the mild solution to Equation (6.1.1) is mean square asymptotically stable.

Proof. Let $\mathscr{B}$ denote the Banach space of all bounded and mean square continuous (c.f. definition of mean square continuity on Page 17) $\mathcal{F}_0$-adapted process $\phi(t,\omega) : [-r,\infty) \times \Omega \to H$ equipped with the supremum norm

$$\|\phi\|_{\mathscr{B}} := \sup_{t\geq 0} \mathbb{E}\|\phi(t)\|_H^2 \qquad \text{for } \phi \in \mathscr{B}.$$

Denoted by S the complete metric space with the supremum metric consisting of function $\phi \in \mathscr{B}$ such that $\phi(t) = \varphi(t)$ for $t \in [-r, 0]$ and $\mathbb{E}\|\phi(t, \omega)\|_H^2 \to 0$ as $t \to \infty$.

Let the operator $\Phi : S \to S$ be the operator defined by $\Phi(X)(t) = \varphi(t)$ for $t \in [-r, 0]$ and for $t \geq 0$ defined as follows:

$$
\begin{aligned}
\Phi(X)(t) \quad := \quad & T(t)\varphi(0) + \int_0^t T(t-s)F\big(s, X(s-\delta(s))\big)ds \\
& + \int_0^t T(t-s)G\big(s, X(s), X(s-\rho(s))\big)dW_Q(s) \\
& + \int_0^t \int_{\mathbb{Z}} T(t-s)\big(L(s, X(s-\theta(s))), u\big)\widetilde{N}(ds, du) \\
& + \sum_{0<t_k<t} T(t-s)I_k(X(t_k^-)) \\
:= \quad & \sum_{i=1}^{5} J_i(t).
\end{aligned}
\tag{6.1.9}
$$

As mentioned in Luo (2008), in order to obtain asymptotic stability result, it is enough to show that the operator Φ has a fixed point in S. To prove this result, we use the contraction mapping principle. We first verify the mean square continuity of Φ on $[0, \infty)$.

Let $X \in S$, $u_1 \geq 0$ and $|r|$ be sufficiently small, then

$$
\mathbb{E}\|\Phi(X)(u_1 + r) - \Phi(X)(u_1)\|_H^2 \leq 5 \sum_{i=1}^{5} \mathbb{E}\|J_i(u_1 + r) - J_i(u_1)\|_H^2.
$$

By the strongly continuous property of $T(t)$ and Lebesgue's dominated convergence theorem, it can be easily obtained that $\mathbb{E}\|J_i(u_1 + r) - J_i(u_1)\|_H^2 \to 0$, $i = 1, 2, 3, 4, 5$. Let us firstly verify the result for $i = 1$, we have

$$
\begin{aligned}
& \mathbb{E}\|J_1(u_1 + r) - J_1(u_1)\|_H^2 \\
= \quad & \mathbb{E}\|T(u_1 + r)\varphi(0) - T(u_1)\varphi(0)\|_H^2 \\
= \quad & \mathbb{E}\|T(u_1)[T(r) - I]\varphi(0)\|_H^2 \\
\to \quad & 0 \quad \text{as} \quad r \to 0.
\end{aligned}
$$

For $i = 2$, it can be shown that

$$
\begin{aligned}
& \mathbb{E}\|J_2(u_1+r) - J_2(u_1)\|_H^2 \\
= & \ \mathbb{E}\Big\| \int_0^{u_1+r} T(u_1+r-s)F\big(s, X(s-\delta(s))\big)ds \\
& \quad - \int_0^{u_1} T(u_1-s)F\big(s, X(s-\delta(s))\big)ds \Big\|_H^2 \\
= & \ \mathbb{E}\Big\| \int_0^{u_1} T(u_1+r-s)F(s, X\big(s-\delta(s))\big)ds \\
& \quad + \int_{u_1}^{u_1+r} T(u_1+r-s)F\big(s, X(s-\delta(s))\big)ds \\
& \quad - \int_0^{u_1} T(u_1-s)F\big(s, X(s-\delta(s))\big)ds \Big\|_H^2 \\
= & \ \mathbb{E}\Big\| \int_0^{u_1} [T(u_1+r-s) - T(u_1-s)]F\big(s, X(s-\delta(s))\big)ds \\
& \quad + \int_{u_1}^{u_1+r} T(u_1+r-s)F(s, X\big(s-\delta(s))\big)ds \Big\|_H^2 \\
\leq & \ 2\mathbb{E}\int_0^{u_1} \|[T(u_1+r-s) - T(u_1-s)]F\big(s, X(s-\delta(s))\big)\|_H^2 ds \\
& \quad +2\mathbb{E}\int_{u_1}^{u_1+r} \|T(u_1+r-s)F\big(s, X(s-\delta(s))\big)\|_H^2 ds \\
\to & \ 0 \quad \text{as} \quad r \to 0.
\end{aligned}
$$

Moreover, for $i = 3$, by the Hölder inequality and the Burkholder-Davis-Gundy inequality we have

$$
\begin{aligned}
& \mathbb{E}\|J_3(u_1+r) - J_3(u_1)\|_H^2 \\
= & \ \mathbb{E}\Big\| \int_0^{u_1+r} T(u_1+r-s)G\big(s, X(s-\rho(s))\big)dW_Q(s) \\
& \quad - \int_0^{u_1} T(u_1-s)G\big(s, X(s-\rho(s))\big)dW_Q(s) \Big\|_H^2 \\
\leq & \ 2\mathbb{E}\Big\| \int_0^{u_1} [T(u_1+r-s) - T(u_1-s)]G\big(s, X(s-\rho(s))\big)dW_Q(s) \Big\|_H^2 \\
& \quad + 2\mathbb{E}\Big\| \int_{u_1}^{u_1+r} T(u_1+r-s)G\big(s, X(s-\rho(s))\big)dW_Q(s) \Big\|_H^2
\end{aligned}
$$

$$
\begin{aligned}
&\le \quad 2\int_0^{u_1} \mathbb{E}\|T[(u_1+r-s)-T(u_1-s)]G\big(s,X(s-\rho(s))\big)\|^2_{\mathcal{L}_2^0}ds \\
&\qquad +2\int_{u_1}^{u_1+r} \mathbb{E}\|T(u_1+r-s)G\big(s,X(s-\rho(s))\big)\|^2_{\mathcal{L}_2^0}ds \\
&\to \quad 0 \quad \text{as} \quad r\to 0.
\end{aligned}
$$

Similarly, for $i=4$, we have

$$
\begin{aligned}
&\mathbb{E}\|J_4(u_1+r)-J_4(u_1)\|^2_H \\
= \quad & \mathbb{E}\Big\|\int_0^{u_1+r} T(u_1+r-s)L\big(s,X(s-\theta(s),u)\big)\widetilde{N}(ds,du) \\
&\quad -\int_0^{u_1} T(u_1-s)L\big(s,X(s-\theta(s),u)\big)\widetilde{N}(ds,du)\Big\|^2_H \\
\le \quad & 2\mathbb{E}\Big\|\int_0^{u_1}\int_{\mathbb{Z}}[T(u_1+r-s)-T(u_1-s)]L\big(s,X(s-\theta(s),u)\big)\widetilde{N}(ds,du)\Big\|^2_H \\
&\quad +2\mathbb{E}\Big\|\int_0^{u_1+r}\int_{\mathbb{Z}} T(u_1+r-s)L\big(s,X(s-\theta(s),u)\big)\widetilde{N}(ds,du)\Big\|^2_H \\
\to \quad & 0 \quad \text{as} \quad r\to 0.
\end{aligned}
$$

Finally, for $i=5$ we deduce that

$$
\begin{aligned}
&\mathbb{E}\|J_5(u_1+r)-J_5(u_1)\|^2_H \\
= \quad & \Big\|\sum_{0<t_k<t} T(u_1+r-t_k)I_k(X(t_k^-)) - \sum_{0<t_k<t} T(u_1-t_k)I_k(X(t_k^-))\Big\|^2_H \\
= \quad & \Big\|\sum_{0<t_k<t} T(u_1-t_k)[T(r)-I]I_k(X(t_k^-))\Big\|^2_H \\
\to \quad & 0 \quad \text{as} \quad r\to 0.
\end{aligned}
$$

Therefore,

$$
\mathbb{E}\|J_i(u_1+r)-J_i(u_1)\|^2_H \to 0, \quad i=1,\,2,\,3,\,4,\,5, \quad \text{as} \quad r\to 0,
$$

which means Φ is mean square continuous on $[0,\infty)$.

Next, we show that $\Phi(S)\subset S$, i.e. Φ maps S into S. Let $X\in S$, from (6.1.9) we have that

$$\begin{aligned}
\mathbb{E}\|(\Phi X)(t)\|_H^2 \ \le\ & 5\mathbb{E}\|T(t)\varphi(0)\|_H^2 + 5\mathbb{E}\bigg\|\int_0^t T(t-s)F\big(s, X(s-\delta(s))\big)ds\bigg\|_H^2 \\
& + 5\mathbb{E}\bigg\|\int_0^t T(t-s)G\big(s, X(s-\rho(s))\big)dW_Q(s)\bigg\|_H^2 \\
& + 5\mathbb{E}\bigg\|\int_0^t\int_{\mathbb{Z}} T(t-s)L\big(s, X(s-\delta(s), u)\big)\widetilde{N}(ds, du)\bigg\|_H^2 \\
& + 5\sum_{0<t_k<t}\mathbb{E}\|T(t-t_k)I_k(X(t_k^-))\|_H^2. \qquad (6.1.10)
\end{aligned}$$

Now we estimate the terms on the right-hand side of (6.1.10). First using (6.1.3) and (6.1.7) we get

$$5\mathbb{E}\|T(t)\varphi(0)\|_H^2 \le 5M^2e^{-2\alpha t}\|\varphi\|_D^2 \to 0 \quad \text{as} \quad t\to\infty, \qquad (6.1.11)$$

and

$$\begin{aligned}
& 5\sum_{0<t_k<t}\mathbb{E}\|T(t-t_k)I_k(X(t_k^-))\|_H^2 \\
\le\ & 5M^2e^{-2\alpha t}\|I_k(X(t_k^-))\|_H^2 \to 0 \quad \text{as} \quad t\to\infty. \qquad (6.1.12)
\end{aligned}$$

Second, using Hölder inequality, Assumption 6.1 and 6.2 yield

$$\begin{aligned}
& 5\mathbb{E}\bigg\|\int_0^t T(t-s)F\big(s, X(s-\delta(s))\big)ds\bigg\|_H^2 \\
\le\ & \mathbb{E}\bigg[\int_0^t \|T(t-s)F\big(s, X(s-\delta(s))\big)\|_H ds\bigg]^2 \\
\le\ & \mathbb{E}\bigg[\int_0^t Me^{-\alpha(t-s)}\|F\big(s, X(s-\delta(s))\big)\|_H ds\bigg]^2 \\
\le\ & M^2C_1^2\mathbb{E}\bigg[\int_0^t e^{-\alpha(t-s)}\|X(s)-\delta(s)\|_H ds\bigg]^2 \\
=\ & M^2C_1^2\mathbb{E}\bigg[\int_0^t e^{-\frac{1}{2}\alpha(t-s)}e^{-\frac{1}{2}\alpha(t-s)}\|X(s)-\delta(s)\|_H ds\bigg]^2 \\
\le\ & 5M^2C_1^2\bigg[\int_0^t e^{-\alpha(t-s)}ds\bigg]\bigg[\int_0^t e^{-\alpha(t-s)}\mathbb{E}\|X(s)-\delta(s)\|_H^2 ds\bigg] \\
\le\ & 5M^2C_1^2\alpha^{-1}\int_0^t e^{-\alpha(t-s)}\mathbb{E}\|X(s)-\delta(s)\|_H^2 ds.
\end{aligned}$$

Since $X(t)\in S$, for any $\epsilon>0$ there exists a $u_1>0$ such that $\mathbb{E}\|X(s-\delta(s))\|_H^2<\epsilon$

for $t \geq u_1$. Thus we obtain

$$
\begin{aligned}
& 5\mathbb{E}\Big\| \int_0^t T(t-s)F\big(s, X(s-\delta(s))\big)ds \Big\|_H^2 \\
\leq\ & 5M^2C_1^2\alpha^{-1}\int_0^{u_1} e^{-\alpha(t-s)}\mathbb{E}\|X(s)-\delta(s)\|_H^2 ds \\
& + 5M^2C_1^2\alpha^{-1}\int_{u_1}^{t} e^{-\alpha(t-s)}\mathbb{E}\|X(s)-\delta(s)\|_H^2 ds \\
\leq\ & 5M^2C_1^2\alpha^{-1}\Big(\int_0^{u_1} e^{\alpha s}\mathbb{E}\|X(s)-\delta(s)\|_H^2 ds\Big)e^{-\alpha t} \\
& + 5M^2C_1^2\alpha^{-1}\int_{u_1}^{t} \epsilon\, e^{-\alpha(t-s)} ds \\
\leq\ & 5M^2C_1^2\alpha^{-1}e^{-\alpha t}\int_0^{u_1} e^{\alpha s}\mathbb{E}\|X(s-\delta(s))\|_H^2 ds + 5M^2C_1^2\alpha^{-2}\epsilon.
\end{aligned}
$$

As $e^{-\alpha t} \to 0$ as $t \to \infty$ and by condition (6.1.8) on Theorem 6.1, there exists $u_2 \geq u_1$ such that for any $t \geq u_2$ we have that

$$
5M^2C_1^2\alpha^{-1}e^{-\alpha t}\int_0^{u_1} e^{\alpha s}\mathbb{E}\|X(s-\delta(s))\|_H^2 ds \leq \epsilon - 5M^2C_1^2\alpha^{-2}\epsilon.
$$

Thus, we obtain for any $t \geq u_2$,

$$
5\mathbb{E}\Big\| \int_0^t T(t-s)F\big(s, X(s-\delta(s))\big)ds \Big\|_H^2 \leq \epsilon.
$$

That is to say,

$$
5\mathbb{E}\Big\| \int_0^t T(t-s)F\big(s, X(s-\delta(s))\big)ds \Big\|_H^2 \to 0 \quad \text{as} \quad t \to \infty. \tag{6.1.13}
$$

Third, using Hölder inequality, Assumption 6.1 and 6.2 yield

$$
\begin{aligned}
& 5\mathbb{E}\Big\| T(t-s)G\big(s, X(s-\rho(s))\big)dW(s) \Big\|_H^2 \\
\leq\ & 5M^2\int_0^t e^{-2\alpha(t-s)}\mathbb{E}\|G\big(s, X(s-\rho(s))\big)\|_{\mathcal{L}_2^0}^2 ds \\
\leq\ & 5M^2C_2^2\int_0^t e^{-2\alpha(t-s)}\mathbb{E}\|X(s-\rho(s))\|_H^2 ds.
\end{aligned}
$$

Similarly, it follows that

$$\begin{aligned}
& 5\mathbb{E}\Big\| \int_0^t \int_{\mathbb{Z}} T(t-s)L\big(s, X(s-\delta(s),u)\big)\widetilde{N}(ds,du)\Big\|_H^2 \\
\leq\ & 5M^2 \int_0^t e^{-2\alpha(t-s)}\mathbb{E}\int_{\mathbb{Z}} \|L\big(s, X(s-\delta(s),u)\big)\|_H^2 \lambda(du)ds \\
\leq\ & 5M^2 C_3^2 \int_0^t e^{-2\alpha(t-s)}\mathbb{E}\|X(s-\delta(s))\|_H^2 ds.
\end{aligned}$$

Thus, similar to the proof of (6.1.13), we have

$$5\mathbb{E}\Big\| \int_0^t T(t-s)G\big(s, X(s-\rho(s))\big)dW_Q(s)\Big\|_H^2 \to 0 \quad \text{as} \quad t\to\infty. \quad (6.1.14)$$

and

$$5\mathbb{E}\Big\| \int_0^t \int_{\mathbb{Z}} T(t-s)L\big(s, X(s-\theta,u)\big)\widetilde{N}(ds,du)\Big\|_H^2 \to 0 \quad \text{as} \quad t\to\infty. \quad (6.1.15)$$

Thus, from (6.1.11) through (6.1.15), we have that

$\mathbb{E}\|\Phi(X)(t)\|_H^2 \to 0$ as $t\to\infty$. So, we conclude that $\Phi(S)\subset S$.

Finally, we will show that the mapping $\Phi: S\to S$ is contractive. For $X, Y\in S$, by (6.1.4), (6.1.5) and (6.1.6) we get

$$\begin{aligned}
& \sup_{t\in[0,T]} \mathbb{E}\|(\Phi X)(t)-(\Phi Y)(t)\|_H^2 \\
\leq\ & 4\sup_{t\in[0,T]} \mathbb{E}\Big\| \int_0^t T(t-s)\Big[F\big(s, X(s-\delta(s))\big)-F\big(s, Y(s-\delta(s))\big)\Big]ds\Big\|_H^2 \\
& +4\sup_{t\in[0,T]} \mathbb{E}\Big\| \int_0^t T(t-s)\Big[G\big(s, X(s-\rho(s))\big)-G\big(s, Y(s-\rho(s))\big)\Big]dW_Q(s)\Big\|_H^2 \\
& +4\sup_{t\in[0,T]} \mathbb{E}\Big\| \int_0^t \int_{\mathbb{Z}} T(t-s)\Big[L\big(s, X(s-\delta(s),u)\big) \\
& \quad -L\big(s, Y(s-\delta(s),u)\big)\Big]\widetilde{N}(ds,du)\Big\|_H^2 \\
& +4\sup_{t\in[0,T]} \mathbb{E}\sum_{0<t_k<t} \mathbb{E}\|T(t-t_k)\Big(I_k(X(t_k^-))-I_k(Y(t_k^-))\Big)\|_H^2
\end{aligned}$$

$$
\begin{aligned}
&\le 4M^2C_1^2\alpha^{-2}\Big(\sup_{t\in[0,T]}\mathbb{E}\|X(t)-Y(t)\|_H^2\Big)\\
&\quad +4M^2C_2^2(2\alpha)^{-1}\Big(\sup_{t\in[0,T]}\mathbb{E}\|X(t)-Y(t)\|_H^2\Big)\\
&\quad +4M^2C_3^2(2\alpha)^{-1}\Big(\sup_{t\in[0,T]}\mathbb{E}\|X(t)-Y(t)\|_H^2\Big)\\
&\quad +4M^2C_4\Big(\sup_{t\in[0,T]}\mathbb{E}\|X(t)-Y(t)\|_H^2\Big)\\
&\le 4M^2\Big[C_1^2\alpha^{-2}+(C_2^2+C_3^2)(2\alpha)^{-1}+C_4\Big]\Big(\sup_{t\in[0,T]}\mathbb{E}\|X(t)-Y(t)\|_H^2\Big)
\end{aligned}
\tag{6.1.16}
$$

where

$$C_4=e^{-\alpha T}\mathbb{E}\Big(\sum_{k=1}^{m}\|q_k\|_H^2\Big).$$

Thus, by condition (6.1.8) it follows that the mapping Φ is contractive. Hence, by the Banach fixed point theorem we have a fixed point $X(t)$ of Φ which is a unique solution to Equation (6.1.1) with $X(s)=\varphi(s)$ on $[-r,0]$ such that $\mathbb{E}\|X(t)\|_H^2\to 0$ as $t\to\infty$.

Next, we show that the solution $X(t)$ is stable in mean square. For any fixed positive real number ϵ, we can choose a $\delta_\epsilon\in(0,\epsilon)$ satisfying

$$5M^2(C_1^2\alpha^{-2}+C_2^2(2\alpha)^{-1}+C_3^2(2\alpha)^{-1}+C_4)\epsilon<\epsilon-5M^2\delta_\epsilon.$$

Let $X(t)=X(t,0;\varphi)$ is a mild solution to Equation (6.1.1) with $\|\varphi\|_D^2<\delta_\epsilon$. We claim that $\mathbb{E}\|X(t)\|_H^2<\epsilon$ for all $t\ge 0$. Notice that $\mathbb{E}\|X(t)\|_H^2<\epsilon$ on $t\in[-r,0]$.

If there exists a time $t^*>0$ such that $\mathbb{E}\|X(t^*)\|_H^2=\epsilon$ and $\mathbb{E}\|X(t)\|_H^2<\epsilon$ for $0\le t<t^*$, then it follows from (6.1.10) that

$$\mathbb{E}\|X(t^*)\|_H^2<5M^2e^{-2\alpha t^*}\delta_\epsilon+5M^2(C_1^2\alpha^{-2}+C_2^2(2\alpha)^{-1}+C_3^2(2\alpha)^{-1}+C_4)\epsilon<\epsilon,$$

which contradicts the definition of t^*. Thus, since the mild solution is stable in mean square, the mild solution of Equation (6.1.1) is mean square asymptotically stable if assumption in Theorem 6.1 hold. The proof is complete.

■

6.2 Asymptotic stability of stochastic retarded evolution equations with jumps

6.2.1 Introduction

There exists a wide literature devoted to various problems of distributed parameter systems with time delays in infinite dimensional spaces. Particularly, in control and approximation theory, it was found very convenient to choose the state space in an appropriate product Hilbert space and then use semigroup theory or variational method, which usually give a unified treatment of a variety of parabolic, hyperbolic and functional differential equations.

For any fixed constant $r > 0$ and the Hilbert space H, we denote by $L_r^2 = L^2([-r, 0]; H)$ the usual Hilbert space of all H-valued equivalence classes of measurable functions which are square integrable on $[-r, 0]$. Let $\mathcal{H}$ denote the Hilbert space $H \times L_r^2$, with the norm

$$\|\phi\|_{\mathcal{H}} = \sqrt{\|\phi_0\|_H^2 + \|\phi_1\|_{L_r^2}^2}, \text{ for all } \phi = (\phi_0, \phi_1) \in \mathcal{H}.$$

Liu (2008) studied the fundamental problems of the following class of stochastic retarded differential equations in Hilbert spaces. For any $T > 0$,

$$\begin{aligned} dy(t) &= Ay(t)dt + Fy_t dt + B(t)dW_Q(t), \quad t \in (0, T], \\ y(0) &= \phi_0, \quad y_0 = \phi_1, \quad \phi = (\phi_0, \phi_1) \in \mathcal{H}. \end{aligned}$$

where A is the infinitesimal generator of a strongly continuous semigroup $T(t)$,

$t \geq 0$, on H. $B(t) \in \mathcal{W}^([0,T]; \mathcal{L}_2^0)$, $y_t(\theta) = y(t+\theta)$, $\theta \in [-r, 0]$ and $F : L^2([-r,0]; H) \to H$ is a linear, generally unbounded operator having the property that $F : C([-r,0]); H) \to H$ is bounded. Liu (2008) has constructed the fundamental solutions and established a stochastic version of variation of constants formula of mild solutions to the above equations. Furthermore, Liu (2008) also investigated the relations among strong, weak and mild solutions for infinite dimensional stochastic retarded systems and studied strong solution approximation of mild solutions which was used to establish the Burkholder's type of inequalities of stochastic convolutions for linear stochastic retarded systems.

In recent years, stochastic differential equations driven by Poisson jumps is an emerging field drawing attention from both theoretical and applied disciplines, which has been successfully applied to problems in mechanics, economics, physics and several fields in engineering (c.f. Bertoin (1996), Protter (2004), Applebaum (2004) and references therein). Taking into account the Poisson jumps effect, we intend to study the following class of stochastic retarded evolution equations with Poisson jumps,

$$\begin{aligned} dx(t) &= Ax(t)dt + Fx_t(t) + B(t, x(t))dW_Q(t) + \int_{\mathbb{Z}} L(t, x(t), u)\widetilde{N}(dt, du), \\ & \qquad t \in (0, T], \\ x(0) &= \phi_0, \quad x_0 = \phi_1, \quad \phi = (\phi_0, \phi_1) \in \mathcal{H} \end{aligned}$$

The details of this equation are explained in Section 6.2.2.

We wish to adopt the theory of fundamental solutions which established in Liu (2008), then construct a variation of constants formula of mild solution to our equation. We shall investigate the mean square asymptotic stability of the equation which we are interested by using the method of fixed point theorem.

The rest of this section is organized as follows. In Section 6.2.2, we shall give the statement of problem formulation and introduce some basic notations and preliminaries. Section 6.2.3 is devoted to the topic of approximations of strong solutions. In Section 6.2.4, we shall establish a basic tool in stochastic analysis, that is the Burkholder-Davis-Gundy's inequality of stochastic convolutions involving the Green's operator. In Section 6.2.5, we shall study the mean square asymptotic stability of stochastic retarded evolution equations with Poisson jumps by fixed point theorem.

6.2.2 Stochastic retarded evolution equations with jumps

The stochastic integral with respect to K-valued Q-Wiener process $\{W_Q(t), t \geq 0\}$ and stochastic integral with respect to compensated Poisson random measure $\widetilde{N}(dt, du)$ are defined as in Section 2.2 and 2.3.

For any fixed constant $r > 0$ and the Hilbert space H, we denote by $L_r^2 = L^2([-r, 0]; H)$ the usual Hilbert space of all H-valued equivalence classes of measurable functions which are square integrable on $[-r, 0]$. Let $\mathcal{H}$ denote the Hilbert space $H \times L_r^2$, with the norm

$$\|\phi\|_{\mathcal{H}} = \sqrt{\|\phi_0\|_H^2 + \|\phi_1\|_{L_r^2}^2}, \text{ for all } \phi = (\phi_0, \phi_1) \in \mathcal{H}.$$

In this section, we consider the following stochastic retarded evolution equations with Poisson jumps: for any $T > 0$,

$$\begin{aligned} dx(t) &= Ax(t)dt + Fx_t(t) + B(t, x(t))dW_Q(t) + \int_{\mathbb{Z}} L(t, x(t), u)\widetilde{N}(dt, du), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0, T], \quad (6.2.1) \\ x(0) &= \phi_0, \quad x_0 = \phi_1, \quad \phi = (\phi_0, \phi_1) \in \mathcal{H} \end{aligned}$$

where A is the infinitesimal generator of a strongly continuous semigroup $T(t)$ or e^{tA}, $t \geq 0$ on H. $B : [0, \infty] \times H \to \mathcal{L}_2^0(K, H)$, $L : [0, \infty] \times H \times \mathbb{Z} \to H$ and $F : L^2([-r, 0] : H) \to H$ is a bounded linear operator such that the map F allows for a bounded linear extension $F : L^2([-r, T]; H) \to L^2([0, T; H])$ which is defined by $(Fx)(t) = Fx_t$, $x \in L^2([0, T]; H)$ with $x_t(\theta) := x(t + \theta)$. That is there exists a real number $K > 0$ such that

$$\int_0^T \|(Fx)(t)\|_H^2 dt \leq K \int_{-r}^T \|x(t)\|_H^2 dt. \tag{6.2.2}$$

for any $x(t) \in L^2([-r, T]; H)$.

The existence and uniqueness of mild solution to the corresponding deterministic system of Equation (6.2.1) have been studied in Liu (2008). Furthermore, a strongly continuous one-parameter family of bounded linear operators which will completely describe the corresponding deterministic systematical dynamics with time delays was introduced. This family, which constitutes the fundamental solutions (also called the Green's operator $G(t) \in \mathcal{L}(H)$) is applied subsequently to defining mild solutions of the stochastic retarded differential equation (c.f. Liu (2008) and Chapter 5, Section 5.2).

Let us firstly introduce the Green's operator $G(t)$. Let $x(t, \phi)$ be the mild solution of the corresponding deterministic system of Equation (6.2.1). For any $h \in H$, let $\phi_0 = h$, $\phi_1(\theta) = 0$ for $\theta \in [-r, 0]$ and $\phi = (h, 0)$, we define the fundamental solution $G(t)$ of the corresponding deterministic system of Equation (6.2.1) by

$$G(t)h = \begin{cases} x(t, \phi), & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{6.2.3}$$

This relation implies that $G(t)$ is a unique solution of

$$G(t) = \begin{cases} T(t) + \int_0^t T(t-s)FG(s+\cdot)ds, & \text{if } t \geq 0, \\ \mathbf{O}, & \text{if } t < 0. \end{cases} \tag{6.2.4}$$

where $G_t(\theta) = G(t+\theta)$, $\theta \in [-r, 0]$, and $\mathbf{O}$ denotes the null operator on H.

For simplicity, we denote $x(t, \phi)$ and $x_t(\cdot, \phi)$ by $x(t)$ and $x_t(\cdot)$ respectively, in the sequel.

The following theorem from Liu (2008), Theorem 3.2 gives the variation of constants formula which describes the representation of mild solution to the corresponding deterministic system of Equation (6.2.1) by the fundamental solutions $G(t)$.

Theorem 6.2 *Let $\phi = (\phi_0, \phi_1) \in \mathcal{H}$, then the solution of the corresponding deterministic system of Equation (6.2.1) may be represented by*

$$\begin{aligned} x(t) &= G(t)\phi_0 + \int_0^t G(t-s)F\vec{\phi}_s ds, \quad t \geq 0, \\ x(t) &= \phi_1(t), \quad t \in [-r, 0), \end{aligned} \tag{6.2.5}$$

where $\vec{\phi} : [-r, \infty) \to H$ is defined by

$$\vec{\phi}(t) = \begin{cases} \phi_1(t) & \text{if} \quad t \in [-r, 0], \\ 0 & \text{if} \quad t \in (0, \infty). \end{cases} \tag{6.2.6}$$

and $\vec{\phi}_s = \vec{\phi}(s + \theta)$, $\theta \in [-r, 0]$.

Definition 6.4 (Strong solution) *A stochastic process $x(t)$, $t \in [-r, T]$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is called a strong solution of Equation (6.2.1) if*

(i) $x(t) \in \mathcal{D}(A)$, $0 \leq t \leq T$, almost surely and is adapted to $\mathcal{F}_t$, $t \in [0, T]$;

(ii) $x(t) \in H$ has càdlàg paths on $t \in [0, T]$ and satisfies

$$\begin{aligned} x(t) &= \phi_0 + \int_0^t [Ax(s) + Fx_s]ds + \int_0^t B(s, x(s))dW_Q(s) \\ &\quad + \int_0^t \int_{\mathbb{Z}} L(s, x(s), u)\widetilde{N}(ds, du) \\ x(0) &= \phi_0, \quad x_0 = \phi_1, \end{aligned} \tag{6.2.7}$$

for arbitrary $\phi = (\phi_0, \phi_1) \in \mathcal{H}$ and $0 \leq t \leq T$ almost surely.

Generally speaking, the above solution concept is too strong to include important examples. The weaker one described below is more appropriate for practical purposes. Adopting the solution concepts from Liu (2008), we give the following definition of mild solution of our stochastic retarded evolution equations with jumps.

Definition 6.5 (Mild solution) *A stochastic process $x(t)$, $t \in [-t, T]$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is called a mild solution of Equation (6.2.1) if*

(i) $x(t)$ is adapted to $\mathcal{F}_t$, $t \in [0,T]$ and for arbitrary $0 \leq t \leq T$,

$$P\Big\{\omega : \int_0^t \|x(t,\omega)\|_H^2 ds < \infty\Big\} = 1;$$

(ii) $x(t) \in H$ has càdlàg paths on $t \in [0,T]$ almost surely, for arbitrary $0 \leq t \leq T$ and $\phi \in \mathcal{H}$,

$$\begin{aligned} x(t) \quad &= \quad G(t)\phi_0 + \int_0^t G(t-s)\vec{\phi_s} ds + \int_0^t G(t-s)B(s,x(s))dW_Q(s) \\ &\quad + \int_0^t \int_{\mathbb{Z}} G(t-s)L(s,x(s),u)\widetilde{N}(ds,du) \end{aligned} \tag{6.2.8}$$

where $\vec{\phi}$ is defined as in (6.2.6).

Proposition 6.1 *Suppose that the following conditions hold:*

(a) $\phi_0 \in \mathcal{D}(A)$ and $\phi_1(\theta) \in L^2([-r,0];H)$ for any $\theta \in [-r,0]$;

(b) $G(t-s)F\vec{\phi_s} \in \mathcal{D}(A)$, $G(t-s)B(x(s))k \in \mathcal{D}(A)$, $G(t-s)L(x(s),u) \in \mathcal{D}(A)$ for any $k \in K$, $u \in K$ and almost all $s \leq t \in [0,T]$;

(c) $\|AG(t-s)\vec{\phi_s}\|_H \in L^2([0,T];H)$;

(d) $\|AG(t-s)B(s,x)\|_{\mathcal{L}_2^0}^2 \leq z_1(t-s)\|x\|_D^2, \quad z_1(\cdot) \in L^2(0,T;\mathbb{R}_+)$;

(e) $\int_{\mathbb{Z}} \|AG(t-s)L(s,x,u)\|_H^2 \lambda(du) \leq z_2(t-s)\|x\|_D^2, \quad z_2(\cdot) \in L^2(0,T;\mathbb{R}_+)$.

Then a mild solution $x(t)$ of Equation (6.2.1) is also a strong solution almost surely.

Proof. Suppose that $x(t) = x(t, \phi)$ is a mild solution of Equation (6.2.1), that is, it satisfies the following variation of constants formula

$$\begin{aligned} x(t) &= G(t)\phi_0 + \int_0^t G(t-s)F\vec{\phi_s}ds + \int_0^t G(t-s)B(s, x(s))dW_Q(s) \\ &\quad + \int_0^t \int_{\mathbb{Z}} G(t-s)L(s, x(s), u)\widetilde{N}(ds, du). \end{aligned}$$

It is not difficult to see that $x(t) \in \mathcal{D}(A)$ for any $t \geq 0$ by the above conditions in the theorem. By using the following proposition, Proposition 3.2 from Liu (2008) which states that for $h \in H$ and $\int_0^t G(s)hds \in \mathcal{D}(A)$ for almost all $t \in \mathbb{R}_+$, the following holds:

$$A \int_0^t G(s)hds = G(s)h - h - \int_0^t FG_shds, \quad t \in \mathbb{R}_+, \quad h \in H,$$

we can deduce that for the initial datum $\phi_0 \in \mathcal{D}(A)$ and $\phi_1 \in L^2([-r, 0]; H)$,

$$A \int_0^t G(s)\phi_0 ds = G(t)\phi_0 - \phi_0 - \int_0^t FG_s\phi_0 ds, \tag{6.2.9}$$

and

$$\begin{aligned} & A \int_0^t \int_0^s G(s-v)F\vec{\phi_v}dvds \\ = \;& A \int_0^t \int_u^t G(s-v)F\vec{\phi_v}dsdv \\ = \;& \int_0^t G(t-v)F\vec{\phi_v}dv - \int_0^t F\vec{\phi_v}dv - \int_0^t \int_0^s FG_{s-v}F\vec{\phi_v}dvds. \end{aligned} \tag{6.2.10}$$

Note that $G(t) = \mathbf{O}$ for $t < 0$ and A is a closed operator. Using the standard stochastic Fubini's theorem (c.f. Theorem 4.18 in Da Prato and Zabczyk (1992) and Applebaum (2006)), ones can deduce that

$$
\begin{aligned}
& A\int_0^t \int_0^s G(s-v)B(s,x(s))dW_Q(v)ds \\
= \;& \int_0^t G(t-v)B(v,x(v))dW_Q(v) - \int_0^t B(v,x(v))dW_Q(v) \\
& \quad - \int_0^t \int_v^t FG_{s-v}B(v,x(v))dsdW_Q(v) \\
= \;& \int_0^t G(t-v)B(v,x(v))dW_Q(v) - \int_0^t B(v,x(v))dW_Q(v) \\
& \quad - \int_0^t \int_0^s FG_{s-v}B(v,x(v))dW_Q(v)ds,
\end{aligned}
\tag{6.2.11}
$$

and

$$
\begin{aligned}
& A\int_0^t \int_0^s \int_{\mathbb{Z}} G(s-v)L(v,x(v),u)\widetilde{N}(dv,du)ds \\
= \;& A\int_0^t \int_v^t \int_{\mathbb{Z}} G(s-v)L(v,x(v),u)ds\widetilde{N}(dv,du) \\
= \;& \int_0^t \int_{\mathbb{Z}} G(t-v)L(v,x(v),u)\widetilde{N}(dv,du) - \int_0^t \int_{\mathbb{Z}} L(v,x(v),u)\widetilde{N}(dv,du) \\
& \quad - \int_0^t \int_0^s \int_{\mathbb{Z}} FG_{s-v}L(v,x(v),u)\widetilde{N}(dv,du)ds.
\end{aligned}
\tag{6.2.12}
$$

On the other hand, since $x(t)$ is a mild solution of Equation (6.2.1), thus

$$
\begin{aligned}
\int_0^t Fx_s ds \;=\; & \int_0^t FG_s\phi_0 ds + \int_0^t F\vec{\phi}_s ds + \int_0^t \int_0^s FG_{s-v}F\vec{\phi}_v dvds \\
& + \int_0^t \int_0^s FG_{s-v}B(v,x(v))dW_Q(v)ds \\
& + \int_0^t \int_0^s \int_{\mathbb{Z}} FG_{s-v}L(v,x(v),u)\widetilde{N}(dv,du)ds.
\end{aligned}
$$

Therefore, by the closeness of A and (6.2.8), it follows that

$$
\int_0^t [Ax(s) + Fx_s]ds \;=\; A\int_0^t x(s)ds + \int_0^t Fx_s ds
$$

$$
\begin{aligned}
&= A\int_0^t G(s)\phi_0 ds + A\int_0^t \int_0^s G(s-v)F\vec{\phi_v} dv ds \\
&\quad +A\int_0^t \int_0^s G(s-v)B(v,x(v))dW_Q(v)ds \\
&\quad +A\int_0^t \int_0^s \int_{\mathbb{Z}} G(s-v)L(v,x(v),u)\widetilde{N}(dv,du)ds \\
&\quad +\int_0^t Fx_s ds
\end{aligned} \tag{6.2.13}
$$

which, together with (6.2.9)-(6.2.13), yields that

$$
\begin{aligned}
&\int_0^t [Ax(s)+Fx_s]ds \\
&= G(t)\phi_0 - \phi_0 - \int_0^t FG_s\phi_0 ds \\
&\quad +\int_0^t G(t-v)F\vec{\phi_v} dv - \int_0^t F\vec{\phi_v} dv \\
&\quad -\int_0^t \int_0^s FG_{t-s}F\vec{\phi_v} dv ds \\
&\quad +\int_0^t G(t-v)B(v,x(v))dW_Q(v) - \int_0^t B(v,x(v))dW_Q(v) \\
&\quad -\int_0^t \int_0^s FG_{s-v}B(v,x(v))dW_Q(v)ds \\
&\quad +\int_0^t \int_{\mathbb{Z}} G(t-v)L(v,x(v),u)\widetilde{N}(dv,du) - \int_0^t \int_{\mathbb{Z}} L(v,x(v),u)\widetilde{N}(dv,du) \\
&\quad -\int_0^t \int_0^s \int_{\mathbb{Z}} FG_{s-v}L(v,x(v),u)\widetilde{N}(dv,du)ds \\
&\quad +\int_0^t FG_s\phi_0 ds + \int_0^t F\vec{\phi_s} ds + \int_0^t \int_0^s FG_{s-v}F\vec{\phi_v} dv ds \\
&\quad +\int_0^t \int_0^s FG_{s-v}B(v,x(v))dW_Q(v)ds \\
&\quad +\int_0^t \int_0^s \int_{\mathbb{Z}} FG_{s-v}L(v,x(v),u)\widetilde{N}(dv,du)ds \\
&= x(t) - \phi_0 - \int_0^t B(s,x(s))dW_Q(s) - \int_0^t \int_{\mathbb{Z}} L(s,x(s),u)\widetilde{N}(ds,du).
\end{aligned}
$$

Thus, the mild solution $x(t)$ of Equation (6.2.1) is also a strong solution. The proof is complete.

■

6.2.3 Approximation of strong solutions

In association with the Equation (6.2.1), we shall investigate in this section a family of stochastic retarded evolution equations which have strong solutions converging to the mild solutions of Equation (6.2.1). Firstly, we introduce a closed linear operator $\triangle(n, A, F)$ on H from Liu (2008), which is defined by

$$\triangle(n, A, F) = nI - A - F(e^{n\cdot}),$$

for each $n \in \mathbb{C}$. Then the retarded resolvent set $\rho(A, F)$ is defined as the set of all values n in $\mathbb{C}$ for which the operator $\triangle(n, A, F)$ has a bounded inverse, denoted by $R(n, A, F)$, on H. Moreover, if $\mathcal{R}(n) > \gamma_0$, $\gamma_0 \in \mathbb{R}$, then $n \in \rho(A, F)$ and the retarded resolvent $R(n, A, F)$ is given by the Laplace transform of $G(t)$:

$$R(n, A, F) = \int_0^\infty e^{-nt} G(t) dt. \tag{6.2.14}$$

To this end, we introduce an approximation system of Equation (6.2.1) as follows:

$$\begin{aligned} x(t) &= \int_0^t [Ax(s) + Fx_s] ds + \int_0^t R(n) B(s, x(s)) dW_Q(s) \\ &\quad + \int_0^t \int_{\mathbb{Z}} R(n) L(s, x(s), u) \widetilde{N}(ds, du), \quad t \in [0, T], \\ x(0) &= R(n)\phi_0, \quad x(t) = R(n)\phi_1(t), \quad t \in [-r, 0), \quad \phi \in \mathcal{H}, \end{aligned} \tag{6.2.15}$$

where $n \in \rho(A, F)$, the resolvent set of the pair (A, F) and $R(n) := nR(n, A, F)$, $R(n, A, F)$ is the retarded resolvent of A. Recall that $\|e^{tA}\| \leq Me^{\mu t}$, $\mu \in \mathbb{R}$, for all $t \geq 0$. Then it can be shown by (6.2.14) that $\|R(n)\| \leq 2M$ when n is large enough and

$$R(n) \xrightarrow{s} I \quad \text{as} \quad n \to \infty. \tag{6.2.16}$$

Denote by $M^2_\lambda([0, T] \times \mathbb{Z}; H)$ the space of H-valued mappings $L(t, u)$, progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ such that

$$\mathbb{E}\left\{\int_0^T \int_{\mathbb{Z}} \|L(t,u)\|_H^2 \lambda(du)dt\right\} < \infty. \tag{6.2.17}$$

Proposition 6.2 *Let $\phi = (\phi_0, \phi_1) \in \mathcal{H}$ and $L(t,u) \in M_\lambda^2([0,T] \times \mathbb{Z}; H)$. Then, for each $n \in \rho(A,F)$, the stochastic retarded differential equation (6.2.15) has a unique strong solution $x^n(t) \in \mathcal{D}(A)$, which has almost surely continuous paths, such that for arbitrary $t \in [0,T]$, $x^n(t) \to x(t)$, the mild solution of (6.2.1), in mean square sense as $n \to \infty$.*

Proof. Recall Proposition 3.2 from Liu (2008), $G(t)$ acquires the following strongly differentiable properties:

$$\frac{d}{dt}G(t)h = AG(t)h + FG_t h = G(t)Ah + G_t Fh, \tag{6.2.18}$$

for $h \in \mathcal{D}(A)$ and $G(t)h \in \mathcal{D}(A)$ for almost all $t \in \mathbb{R}_+$. The existence of a unique strong solution $x^n(t)$ of Equation (6.2.15) is an immediate consequence of Proposition 6.1, (6.2.18) and the fact that

$$AR(n) = n^2R(n,A,F) - nI - F(e^{n\cdot})R(n) \in \mathcal{L}(H), \quad \forall\, n \in \rho(A,F).$$

To prove the remainder of the proposition, note that for any $t \in [0,T]$,

$$\begin{aligned}
& x(t) - x^n(t) \\
= \; & G(t)[\phi_0 - R(n)\phi_0] + \int_0^t G(t-s)[F\vec{\phi}_s - F\vec{\phi}_s^n]ds \\
& \quad + \int_0^t G(t-s)[B(s,x(s)) - R(n)B(s,x(s))]dW_Q(s) \\
& \quad + \int_0^t \int_{\mathbb{Z}} G(t-s)[L(s,x(s),u) - R(n)L(s,x(s),u)]\widetilde{N}(ds,du)
\end{aligned} \tag{6.2.19}$$

where $\vec{\phi}_s^n = R(n)\vec{\phi}_s$. This immediately yields that for any $t \in [0,T]$,

$$
\begin{aligned}
& \mathbb{E}\|x(t) - x^n(t)\|_H^2 \\
\le\ & 4^2\Big\{\mathbb{E}\|G(t)[\phi_0 - R(n)\phi_0]\|_H^2 + \mathbb{E}\Big\|\int_0^t G(t-s)[F\vec{\phi}_s - F\vec{\phi}_s^n]ds\Big\|_H^2 \\
& + \mathbb{E}\Big\|\int_0^t G(t-s)[I - R(n)]B(s, x(s))dW_Q(s)\Big\|_H^2 \\
& + \mathbb{E}\Big\|\int_0^t \int_{\mathbb{Z}} G(t-s)[I - R(n)]L(s, x(s), u)\widetilde{N}(ds, du)\Big\|_H^2\Big\} \\
:=\ & 16[I_1 + I_2 + I_3 + I_4]. \qquad (6.2.20)
\end{aligned}
$$

Note that $G(t) \le C\, e^{\gamma t}$, $\gamma \in \mathbb{R}$ and (6.2.16), we can deduce that for any $t \in [0, T]$,

$$
\begin{aligned}
I_1 &= \mathbb{E}\|G(t)[\phi_0 - R(n)\phi_0]\|_H^2 \\
&\le C^2 e^{2\gamma T}\|[I - R(n)]\phi_0\|_H^2 \to 0 \quad \text{as} \quad n \to \infty. \qquad (6.2.21)
\end{aligned}
$$

By using the Hölder inequality, dominated convergence theorem and (6.2.2), we can obtain that for any $t \in [0, T]$,

$$
\begin{aligned}
I_2 &= \mathbb{E}\Big\|\int_0^t G(t-s)[F\vec{\phi}_s - F\vec{\phi}_s^n]ds\Big\|_H^2 \\
&\le C^2 e^{2\gamma T} K \int_{-r}^0 \|[I - R(n)]\phi_1(\theta)\|_H^2 d\theta \to 0 \quad \text{as} \quad n \to \infty. \qquad (6.2.22)
\end{aligned}
$$

for some constant $K > 0$. On the other hand, by virtue of the definition of stochastic integrals and dominated convergence theorem, we have that for any $t \in [0, T]$,

$$
\begin{aligned}
I_3 &= \mathbb{E}\Big\|\int_0^t G(t-s)[I - R(n)]B(s, x(s))dW_Q(s)\Big\|_H^2 \\
&\le C^2 e^{2\gamma T} \int_0^T \mathbb{E}\|[I - R(n)]B(s, x(s))\|_{\mathcal{L}_2^0}^2 ds \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}
\qquad (6.2.23)
$$

Similarly,

$$
\begin{aligned}
I_4 &= \mathbb{E}\left\| \int_0^t \int_{\mathbb{Z}} G(t-s)[I-R(n)L(s,x(s),u)]\widetilde{N}(ds,du) \right\|_H^2 \\
&\leq \mathbb{E} \int_0^t \int_{\mathbb{Z}} \|G(t-s)[I-R(n)]L(s,x(s),u)\widetilde{N}(ds,du)\|_H^2 \\
&\leq C^2 e^{2\gamma T}\, \mathbb{E} \int_0^T \int_{\mathbb{Z}} \|[I-R(n)]L(s,x(s),u)\|_H^2 \lambda(du)ds \to 0, \ \text{ as } \ n \to \infty.
\end{aligned}
\tag{6.2.24}
$$

Combining (6.2.21), (6.2.22), (6.2.23) and (6.2.24), it is to deduce that $\mathbb{E}\|x(t) - x^n(t)\|_H^2 \to 0$ as $n \to \infty$. The proof is now complete.

■

6.2.4 Burkholder-Davis-Gundy's inequality

We shall derive a retarded version of the classical Burkholder-Davis-Gundy's inequality for the stochastic convolution

$$
W_G^{\widetilde{N}} = \int_0^t \int_{\mathbb{Z}} G(t-s)L(s,u)\widetilde{N}(ds,du).
$$

We impose the following assumptions:

Assumption 6.4 *$A : \mathcal{D}(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, on H such that*

$$
\langle Ah, h \rangle_H \leq \alpha \|h\|_H^2, \quad \forall\, h \in \mathcal{D}(A),
$$

for some constant $\alpha \in \mathbb{R}_+$.

Assumption 6.5 *For any $x(t) \in L^2([-r,T];H)$, there exists some constant $K > 0$ such that*

$$
\int_0^T \|Fx_t\|_H^2 \, dt \leq K \int_{-r}^T \|x(t)\|_H^2.
$$

Theorem 6.3 *Suppose that Assumption 6.4 and 6.5 hold, and there exists a number $\delta > 0$ such that*

$$\int_{\mathbb{Z}} \|L(\xi, u) - L(\eta, u)\|_H^2 \, \lambda(du) \le \delta \|\xi - \eta\|_H^2, \tag{6.2.25}$$

then for any $L(\cdot,\cdot) \in M_\lambda^2([0,T] \times \mathbb{Z}; H)$, there exists some positive constant $C_{\alpha,T,K}$, dependent on α, T and K, such that

$$\begin{aligned} & \mathbb{E}\Big[\sup_{0\le t\le T}\Big\|\int_0^t \int_{\mathbb{Z}} G(t-s)L(s,u)\widetilde{N}(ds,du)\Big\|_H^2\Big] \\ \le\ & C_{\alpha,T,K}\, \mathbb{E}\int_0^T \int_{\mathbb{Z}} \|L(s,u)\|_H^2 \, ds\lambda(du). \end{aligned} \tag{6.2.26}$$

Proof. **Step 1**: We first suppose that $L(t,u) \in M_\lambda^2([0,T] \times \mathbb{Z}; \mathcal{D}(A))$, and $x(t)$ is a strong solution of the following equation

$$x(t) = \int_0^t [Ay(s) + Fx_s]\, ds + \int_0^t \int_{\mathbb{Z}} L(s,u)\widetilde{N}(ds,du), \quad t > 0, \tag{6.2.27}$$

with $x(t) = 0$ for $t \in [-r, 0]$. By applying Itô's formula to the function $\|h\|_H^2$, $h \in H$, and the strong solution $x(t)$ of Equation (6.2.27), we obtain that

$$\begin{aligned} \|x(t)\|_H^2 &= 2\int_0^t \langle Ay(s) + Fx_s, x(s)\rangle_H \, ds + \int_0^t \int_{\mathbb{Z}} \|L(s,u)\|_H^2 \, \lambda(du)ds \\ &\quad + \int_0^t \int_{\mathbb{Z}} [2\langle x(s-), L(s,u)\rangle_H + \|L(s,u)\|_H^2]\widetilde{N}(ds,du), \end{aligned} \tag{6.2.28}$$

by using Assumption 6.4 and 6.5 yields that

$$\begin{aligned} \|x(t)\|_H^2 &\le 2\alpha \int_0^t \|x(s)\|_H^2 ds + \int_0^t \|x(s)\|_H^2 ds + \int_0^t \|x_s\|_H^2 ds \\ &\quad + \int_0^t \int_{\mathbb{Z}} \|L(s,u)\|_H^2 \, \lambda(du)ds \\ &\quad + \int_0^t \int_{\mathbb{Z}} [2\langle x(s-), L(s,u)\rangle_H + \|L(s,u)\|_H^2]\widetilde{N}(ds,du) \\ &\le (2\alpha + 1 + K)\int_0^t \|x(s)\|_H^2 ds + \int_0^t \int_{\mathbb{Z}} \|L(s,u)\|_H^2 \, \lambda(dy)ds \\ &\quad + \int_0^t \int_{\mathbb{Z}} [2\langle x(s-), L(s,u)\rangle_H + \|L(s,u)\|_H^2]\widetilde{N}(ds,du). \end{aligned} \tag{6.2.29}$$

Let us denote

$$M(t) := \int_0^t \int_{\mathbb{Z}} [2\langle x(s-), L(s,u)\rangle_H + \|L(s,u)\|_H^2] \widetilde{N}(ds, du)$$

and $[M(t)]$ the corresponding quadratic variation.

By Burkholder-Davis-Gundy inequality (c.f. Protter (2004), Theorem 48), it follows that there is a constant $\bar{C} > 0$ such that

$$\mathbb{E}\left(\sup_{0 \le s \le t} \|M(s)\|_H^2 \right) \le \bar{C}\, \mathbb{E}[M(t)]^{\frac{1}{2}}. \tag{6.2.30}$$

In what follows, we compute

$$\begin{aligned}
[M(t)]^{\frac{1}{2}} &= \left\{ \sum_{s \in D_p, 0 \le s \le t} (2\langle x(s), L(s, p(s))\rangle_H + \|L(s, p(s))\|_H^2)^2 \right\}^{\frac{1}{2}} \\
&\le \sqrt{2} \left\{ \sum_{s \in D_p, 0 \le s \le t} \|L(s, p(s))\|_H^4 \right\}^{\frac{1}{2}} \\
&\quad + 2\sqrt{2} \left\{ \sum_{s \in D_p, 0 \le s \le t} \|x(s)\|_H^2 \|L(s, p(s))\|_H^2 \right\}^{\frac{1}{2}} \\
&\le \sqrt{2} \sum_{s \in D_p, 0 \le s \le t} \|L(s, p(s))\|_H^2 \\
&\quad + 2\sqrt{2} \sup_{0 \le s \le t} \|x(s)\|_H \left\{ \sum_{s \in D_p, 0 \le s \le t} \|L(s, p(s))\|_H^2 \right\}^{\frac{1}{2}} \\
&\le \frac{1}{2\bar{C}} \sup_{0 \le s \le t} \|x(s)\|_H^2 + (\sqrt{2} + 4\bar{C}) \sum_{s \in D_p, 0 \le s \le t} \|L(s, p(s))\|_H^2,
\end{aligned} \tag{6.2.31}$$

where $\bar{C}$ is the positive constant appearing in the right-hand side of (6.2.30).

Consequently,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}\|x(t)\|_H^2\Big)\\
\le\;& 2(2\alpha+1+K)\,\mathbb{E}\int_0^T \|x(t)\|_H^2 dt\\
&+2[1+(\sqrt{2}+4\bar{C})\bar{C}]\,\mathbb{E}\int_0^T\int_{\mathbb{Z}}\|L(t,u)\|_H^2\,\lambda(du)dt\\
\le\;& 2(2\alpha+1+K)\int_0^T \mathbb{E}\Big(\sup_{0\le s\le t}\|x(s)\|_H^2\Big)dt\\
&+2[1+(\sqrt{2}+4\bar{C})\bar{C}]\,\mathbb{E}\int_0^T\int_{\mathbb{Z}}\|L(t,u)\|_H^2\,\lambda(du)dt \qquad (6.2.32)
\end{aligned}
$$

which, combining with Gronwall's inequality, immediately implies that

$$\mathbb{E}\Big(\sup_{0\le t\le T}\|x(t)\|_H^2\Big)\le C_{\alpha,T,K}\,\mathbb{E}\int_0^T\int_{\mathbb{Z}}\|L(t,u)\|_H^2\,\lambda(du)dt,$$

where

$$C_{\alpha,T,K}=2[1+(\sqrt{2}+4\bar{C})\bar{C}]e^{2(2\alpha+1+K)T}>0.$$

Step 2: In the general case that $x(t)$ is a mild solution of the Equation (6.2.27), let $L_n(t,u)=nR(n,A,F)$ where $R(n,A,F)$, $n\in\rho(A,F)$, is the retarded resolvent of A. Define

$$x^n(t)=\int_0^t\int_{\mathbb{Z}}G(t-s)L_n(s,u)\widetilde{N}(ds,du),\quad t\in(0,T],$$

it is obvious that $x^n(t)$, together with $x^n(t)=0$ for $t\in[-r,0]$, is a strong solution of

$$x(t)=\int[Ax(s)+Fx_s]ds+\int_0^t\int_{\mathbb{Z}}L_n(s,u)\widetilde{N}(ds,du),\quad t\in(0,T],$$

with $x(t)=0$ for $t\in[-r,0]$. Moreover, by virtue of Proposition 6.2 we know that $x^n(t)\to x(t)$ in $L^2(\Omega,\mathcal{F},\mathbb{P};H)$ for any $t\in[0,T]$ as $n\to\infty$. Now we can apply (6.2.26) to the difference $x^n(t)-x^m(t)$ with $L(\cdot,\cdot)$ replaced by the difference

$L_n - L_m$ from which we deduce that

$$\mathbb{E}\Big(\sup_{0\le t\le T}\|x(t)-x^n(t)\|_H^2\Big)\to 0, \quad \text{as} \quad n\to\infty,$$

hence (6.2.26) is true for any such a $L \in M_\lambda^2([0,T]\times\mathbb{Z},\mathcal{D}(A))$ in the theorem. The proof is now complete.

■

6.2.5 Asymptotic stability

In this section, we shall formulate and prove the conditions for the asymptotic stability in mean square of mild solutions to Equation (6.2.1) by using a fixed point approach. To prove the result, on the functions B and L we impose some Lipschitz and linear growth conditions. Also for the purpose of stability, we shall assume that $B(t,0)=0$ and $\int_{\mathbb{Z}} L(t,0,u)\widetilde{N}(ds,du)=0$. Then Equation (6.2.1) has a trivial solution when $x_0=\phi_1=0$ and $x(0)=\phi_0=0$.

We impose the following assumptions:

Assumption 6.6 *$G(t)$, $t\ge 0$, the strongly continuous one-parameter family of bounded linear operator satisfies that $\|G(t)\|_H \le Me^{-\alpha t}$, $t\ge 0$ for some constants $M\ge 1$ and $0<\alpha\in\mathbb{R}_+$.*

Assumption 6.7 *The functions B and L satisfy the Lipschitz condition and there exists constants C_1 and C_2 for every $t\ge 0$ and x, $y\in H$ such that*

$$\begin{aligned}\|B(t,x)-B(t,y)\|_H^2 &\le C_1\|x-y\|_H^2,\\ \int_{\mathbb{Z}}\|L(t,x,u)-L(t,y,u)\|_H^2\lambda(du) &\le C_2\|x-y\|_H^2\end{aligned}$$

Theorem 6.4 *Suppose that Assumption 6.6 and 6.7 hold. If the following inequality is satisfied:*

$$4M^2(C_1^2(2a)^{-1}+C_2^2(2a)^{-1})<1, \tag{6.2.33}$$

then the mild solution to Equation (6.2.1) is mean square asymptotically stable.

Proof. Let $\mathscr{B}$ denote the Banach space of all bounded and mean square continuous $\mathcal{F}_0$-adapted process $\varphi(t,\omega) : [-r,T) \times \Omega \to H$ equipped with the supremum norm

$$\|\varphi\|_{\mathscr{B}} := \sup_{t\geq 0} \mathbb{E}\|\varphi(t)\|_H^2 \quad \text{for } \varphi \in \mathscr{B}.$$

Denoted by S the complete metric space with the supremum metric consisting of function $\varphi \in \mathscr{B}$ such that $\varphi(t) = \phi_1(t)$ for $t \in [-r,0]$ and $\mathbb{E}\|\varphi(t,\omega)\|_H^2 \to 0$ as $t \to \infty$.

Let the operator $\Phi : S \to S$ be the operator defined by $\Phi(x)(t) = \phi_1(t)$ for $t \in [-r,0]$ and for $t \geq 0$ defined as follows:

$$\begin{aligned} \Phi(x)(t) \quad &:= \quad G(t)\phi_0 + \int_0^t G(t-s)F\vec{\phi}_s ds + \int_0^t G(t-s)B(s,x(s))dW_Q(s) \\ &\qquad + \int_0^t \int_{\mathbb{Z}} G(t-s)L(s,x(s),u)\widetilde{N}(ds,du) \\ &:= \quad \sum_{i=1}^4 J_i(t). \end{aligned} \tag{6.2.34}$$

We first verify the mean square continuity of Φ on $[0,\infty)$. Let $x \in S$, $t_1 \geq 0$ and $|r|$ be sufficiently small, then

$$\mathbb{E}\|\Phi(x)(t_1+r) - \Phi(x)(t_1)\|_H^2 \leq 4\sum_{i=1}^4 \mathbb{E}\|J_i(t_1+r) - J_i(t_1)\|_H^2.$$

By the strongly continuous property of $G(t)$ and Lebesgue's dominated convergence theorem, it can be obtained that $\mathbb{E}\|J_i(t+r) - J_i(t_1)\|_H^2 \to 0$, $i = 1,2,3,4$. The case for $i = 1$ is obvious.

For $i = 2$, it can be shown that

$$\begin{aligned} &\mathbb{E}\|J_2(t_1+r) - J_2(t_1)\|_H^2 \\ = \quad &\mathbb{E}\left\| \int_0^{t_1+r} G(t_1+r-s)F\vec{\phi}_s ds - \int_0^{t_1} G(t_1-s)F\vec{\phi}_s ds \right\|_H^2 \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\Big\| \int_0^{t_1} G(t_1+r-s)F\vec{\phi}_s ds + \int_{t_1}^{t_1+r} G(t_1+r-s)F\vec{\phi}_s ds \\
&\qquad - \int_0^{t_1} G(t_1-s)F\vec{\phi}_s ds\Big\|_H^2 \\
&= \mathbb{E}\Big\| \int_0^{t_1} [G(t_1+r-s)-G(t_1-s)]F\vec{\phi}_s ds \\
&\qquad + \int_{t_1}^{t_1+r} G(t_1+r-s)F\vec{\phi}_s ds\Big\|_H^2 \\
&\leq 2\mathbb{E}\int_0^{t_1} \|[G(t_1+r-s)-G(t_1-s)]F\vec{\phi}_s\|_H^2 ds \\
&\qquad +2\mathbb{E}\int_{t_1}^{t_1+r} \|G(t_1+r-s)F\vec{\phi}_s\|_H^2 ds \\
&\to 0 \quad \text{as} \quad r \to 0.
\end{aligned}
$$

Furthermore, for $i = 3$, by the Hölder inequality an the Burkholder-Davis-Gundy inequality we have we deduce that

$$
\begin{aligned}
&\mathbb{E}\|J_3(t_1+r)-J_3(t_1)\|_H^2 \\
&= \mathbb{E}\Big\| \int_0^{t_1+r} G(t_1+r-s)B(s,x(s))dW_Q(s) - \int_0^{t_1} G(t_1-s)B(s,x(s))dW_Q(s)\Big)\Big\|_H^2 \\
&= \mathbb{E}\Big\| \int_0^{t_1} G(t_1+r-s)B(s,x(s))dW_Q(s) + \int_{t_1}^{t_1+r} G(t_1+r-s)B(s,x(s))dW_Q(s) \\
&\qquad - \int_0^{t_1} G(t_1-s)B(s,x(s))dW_Q(s)\Big\|_H^2 \\
&\leq 2\,\mathbb{E}\Big\| \int_0^{t_1} [G(t_1+r-s)-G(t_1-s)]B(s,x(s))dW_Q(s)\Big\|_H^2 \\
&\qquad + 2\,\mathbb{E}\Big\| \int_{t_1}^{t_1+r} G(t_1+r-s)B(s,x(s))dW_Q(s)\Big\|_H^2 \\
&\leq 2\int_0^{t_1} \mathbb{E}\|[G(t_1+r-s)-G(t_1-s)B(s,x(s))]\|_{\mathcal{L}_2^0}^2 ds \\
&\qquad + 2\int_{t_1}^{t_1+r} \mathbb{E}\|G(t_1+r-s)B(s,x(s))\|_{\mathcal{L}_2^0}^2 ds \\
&\to 0 \quad \text{as} \quad r \to 0.
\end{aligned}
$$

Similarly, for $i = 4$, we get

$$\begin{aligned}
& \mathbb{E}\|J_4(t_1+r) - J_4(t_1)\|_H^2 \\
= \ & \mathbb{E}\Big\| \int_0^{t_1+r} G(t_1+r-s)L(s,x(s),u)\widetilde{N}(ds,du) \\
& \quad - \int_0^{t_1} G(t_1-s)L(s,x(s),u)\widetilde{N}(ds,du)\Big\|_H^2 \\
\leq \ & 2\mathbb{E}\Big\| \int_0^{t_1}\int_{\mathbb{Z}} [G(t_1+r-s) - G(t_1-s)]L(s,x(s),u)\widetilde{N}(ds,du)\Big\|_H^2 \\
& \quad + 2\mathbb{E}\Big\| \int_0^{t_1+r}\int_{\mathbb{Z}} G(t_1+r-s)L(s,x(s),u)\widetilde{N}(ds,du)\Big\|_H^2 \\
\leq \ & 2\int_0^{t_1}\int_{\mathbb{Z}} \mathbb{E}\|[G(t_1+r-s) - G(t_1-s)]L(s,x(s),u)\|_H^2 \lambda(du)ds \\
& \quad + 2\int_{t_1}^{t_1+r}\int_{\mathbb{Z}} \mathbb{E}\|G(t_1+r-s)L(s,x(s),u)\|_H^2 \lambda(du)ds \\
\to \ & 0 \quad \text{as} \quad r \to 0.
\end{aligned}$$

Therefore,

$$\mathbb{E}\|J_i(t_1+r) - J_i(t_1)\|_H^2 \to 0, \quad i = 1,\, 2,\, 3,\, 4, \quad \text{as} \quad r \to 0,$$

which means Φ is mean square continuous on $[0,\infty)$.

Next, we show that $\Phi(S) \subset S$. Let $x \in S$, from (6.2.34) we have that

$$\begin{aligned}
\mathbb{E}\|\Phi(x)(t)\|_H^2 \ \leq \ & 4\mathbb{E}\|G(t)\phi_0\|_H^2 + 4\mathbb{E}\int_0^t \|G(t-s)F\vec{\phi_s}\|_H^2 ds \\
& + 4\mathbb{E}\Big\| \int_0^t G(t-s)B(s,x(s))dW_Q(s)\Big\|_H^2 \\
& + 4\mathbb{E}\Big\| \int_0^t \int_{\mathbb{Z}} G(t-s)L(s,x(s),u)\widetilde{N}(ds,du)\Big\|_H^2.
\end{aligned} \tag{6.2.35}$$

Now we estimate the terms on the right-hand side of (6.2.35). First, using Assumption 6.6 and 6.7 we get

$$4\mathbb{E}\|G(t)\phi_0\|_H^2 \leq 4M^2 e^{-2\alpha t}\|\phi_0\|_H^2 \to 0 \quad \text{as} \quad t \to \infty, \tag{6.2.36}$$

Second, using Hölder inequality yields that

$$\begin{aligned}
& 4\mathbb{E}\left\|\int_0^t G(t-s)F\vec{\phi_s}\right\|_H^2 \\
\leq \;& 4M^2K^2\int_0^t e^{-\alpha(t-s)}ds\int_0^t e^{-\alpha(t-s)}\mathbb{E}\|\vec{\phi_s}\|_H^2ds \\
\leq \;& 4M^2K^2\alpha^{-1}\int_0^t e^{-\alpha(t-s)}\mathbb{E}\|\vec{\phi_s}\|_H^2ds. \qquad (6.2.37)
\end{aligned}$$

For any $x(t)\in S$ and any $\epsilon>0$ there exists a $t_1>0$ such that $\mathbb{E}\|\vec{\phi_s}\|_H^2<\epsilon$ for $t\geq t_1$. Thus from (6.2.37) we obtain

$$\begin{aligned}
& 4\mathbb{E}\left\|\int_0^t G(t-s)F\vec{\phi_s}ds\right\|_H^2 \\
\leq \;& 4M^2K^2\alpha^{-1}e^{-\alpha t}\int_0^{t_1} e^{\alpha s}\mathbb{E}\|\vec{\phi_s}\|_H^2ds+4M^2L^2\alpha^{-1}\epsilon. \qquad (6.2.38)
\end{aligned}$$

As $e^{-\alpha t}\to 0$ as $t\to\infty$ and by assumption (6.2.33) in Theorem 6.4, there exists $t_2\geq t_1$ such that for any $t\geq t_2$ we obtain

$$4M^2L^2\alpha^{-1}e^{-\alpha t}\int_0^{t_1} e^{\alpha s}\mathbb{E}\|\vec{\phi_s}\|_H^2ds\leq\epsilon-4M^2L^2\alpha^{-1}\epsilon. \qquad (6.2.39)$$

Thus, from (6.2.38) and (6.2.39) we obtain for any $t\geq t_2$,

$$4\mathbb{E}\left\|\int_0^t G(t-s)F\vec{\phi_s}ds\right\|_H^2\leq\epsilon.$$

That is to say,

$$4\mathbb{E}\left\|\int_0^t G(t-s)F\vec{\phi_s}ds\right\|_H^2\to 0 \quad \text{as} \quad t\to\infty. \qquad (6.2.40)$$

Third, using Hölder inequality and Assumption 6.6 and 6.7 yield

$$\begin{aligned}
& 4\mathbb{E}\left\|G(t-s)B(s,x(s))dW_Q(s)\right\|_H^2 \\
\leq \;& 4M^2C_1^2\int_0^t e^{-2\alpha(t-s)}\mathbb{E}\|B(s,x(s))\|_{\mathcal{L}_2^0}^2ds. \qquad (6.2.41)
\end{aligned}$$

Similarly, it follows that

$$
\begin{aligned}
&4\mathbb{E}\left\| \int_0^t \int_{\mathbb{Z}} G(t-s)L(s,x(s),u)\widetilde{N}(ds,du) \right\|_H^2 \\
\leq\ & 4M^2C_2^2 \int_0^t e^{-2\alpha(t-s)}\mathbb{E}\|L(s,x(s),u)\|_H^2 \lambda(du)ds
\end{aligned}
$$

Further, similar to the proof of (6.2.40), from (6.2.41) we get

$$
4\mathbb{E}\left\| \int_0^t G(t-s)B(s,x(s))dW_Q(s) \right\|_H^2 \to 0 \quad \text{as} \quad t \to \infty. \tag{6.2.42}
$$

and

$$
\mathbb{E}\left\| \int_0^t \int_{\mathbb{Z}} G(t-s)L(s,x(s),u)\widetilde{N}(ds,du) \right\|_H^2 \to 0 \quad \text{as} \quad t \to \infty. \tag{6.2.43}
$$

Thus, using (6.2.36), (6.2.40), (6.2.42) and (6.2.43), we have $\mathbb{E}\|\Phi(x)(t)\|_H^2 \to 0$ as $t \to \infty$. So we conclude that $\Phi(S) \subset S$.

Finally, we will show that the mapping $\Phi : S \to S$ is contractive. For $x, y \in S$, by Assumption 6.7 we get

$$
\begin{aligned}
&\sup_{s\in[0,T]} \mathbb{E}\|(\Phi x)(t) - (\Phi y)(t)\|_H^2 \\
\leq\ & 2\sup_{s\in[0,T]} \left\| \int_0^t G(t-s)[B(s,x(s)) - B(s,y(s))]dW_Q(s) \right\|_H^2 \\
&\quad +2\sup_{s\in[0,T]} \left\| \int_0^t \int_{\mathbb{Z}} G(t-s)[L(s,x(s),u) - L(s,y(s),u)]\widetilde{N}(ds,du) \right\|_H^2 \\
\leq\ & [2M(C_1^2(2a)^{-1} + C_2^2(2a)^{-1})]\Big(\sup_{s\in[0,T]} \mathbb{E}\|x(t) - y(t)\|_H^2 \Big).
\end{aligned} \tag{6.2.44}
$$

Next, we show that the solution $x(t)$ is stable in mean square. For any fixed positive real number ϵ, we can choose a $\delta_\epsilon \in (0,\epsilon)$ satisfying

$$
4M^2(C_1^2(2a)^{-1} + C_2^2(2a)^{-1})\epsilon < \epsilon - 4M^2\delta_\epsilon.
$$

Let $x(t) = x(t,0;\varphi)$ is a mild solution to Equation (6.2.1) with $\|\varphi\|_{\mathcal{H}}^2 < \delta_\epsilon$. We claim that $\mathbb{E}\|X(t)\|_H^2 < \epsilon$ for all $t \geq 0$. Notice that $\mathbb{E}\|X(t)\|_H^2 < \epsilon$ on $t \in [-r,0]$.

If there exists a time $t^* > 0$ such that $\mathbb{E}\|X(t^*)\|_H^2 = \epsilon$ and $\mathbb{E}\|X(t)\|_H^2 < \epsilon$ for

$0 \leq t < t^*$, then it follows from (6.2.35) that

$$\mathbb{E}\|X(t^*)\|_H^2 < 4M^2 e^{-2\alpha t^*}\delta_\epsilon + 4M^2C_1^2(2a)^{-1} + 4M^2C_2^2(2a)^{-1})\epsilon < \epsilon,$$

which contradicts the definition of t^*. Thus, since the mild solution is stable in mean square, the mild solution of Equation (6.2.1) is mean square asymptotically stable if assumption in Theorem 6.4 hold. The proof is complete.

■

6.3 Conclusion

To sum up, in this chapter, we have focussed on the mean square asymptotic stability of impulsive stochastic partial delay differential equations with Poisson jumps and stochastic retarded evolution equations with Poisson jumps. Since the Lyapunov direct method has some difficulties with the theory and application to specific problems when discussing the asymptotic behavior of solutions in stochastic differential equations, we adopt the fixed point theory which recently has been successfully applied in Sakthivel and Luo (2009a,b), in order to obtain the desire asymptotic stability results under some suitable conditions. It should be pointed out that for the impulsive stochastic partial delay differential equations with Poisson jumps, we do not require the monotone decreasing behavior of the delays (i.e. $\delta'(t) \leq 0$, $\rho'(t) \leq 0$ and $\theta'(t) \leq 0$, $\forall\, t \geq 0$) when obtaining the asymptotic stability. Furthermore, in Section 6.2.4, we have derived a retarded version of the classical Burkholder-Davis-Gundy's inequality for the stochastic convolution involving the Green's operator, which plays an important role in the study of asymptotic stability of stochastic retarded evolution equations with jumps.

Chapter 7

General Conclusion

We have studied and analyzed five different stochastic models: stochastic delay differential equations with Poisson jumps, stochastic partial differential equations with Markovian switching and Poisson jumps, stochastic neutral functional differential equations, impulsive stochastic partial delay differential equations with Poisson jumps and stochastic retarded evolution equations with Poisson jumps.

In Chapter 3, we have studied the stability of mild solutions to stochastic delay differential equations with Poisson jumps. We have showed by using the strong solution approximation and by constructing a metric between transition probability of mild solutions, the stability in distribution of mild solutions can be obtained under some suitable conditions. The results of stability in distribution have been obtained under some different criteria compares to the study of some other types of equations in Basak et al. (1996), Yuan and Mao (2003) and Bao et al. (2010, 2009b). In contrast with Basak et al. (1996) and Yuan and Mao (2003), we have studied this stability result in the case of infinite dimensions, which is more complicated. Comparing to Bao et al. (2010, 2009b), we have studied a more general type of equations which includes Poisson jumps with a slightly different form of delay term. Furthermore, we have improved the sufficient conditions (Assumption 3.2) for the stability in distribution of mild solutions by improving the estimations of Burkholder type of inequality for stochastic convolution driven by compensated Poisson random measures in Lemma 3.1.

In Chapter 4, we have studied the exponential stability and almost sure exponential stability of energy solutions to stochastic partial differential equations with Markovian switching and Poisson jumps. Taniguchi (2007) firstly studied the exponential stability and almost sure exponential stability of energy solutions to stochastic delay differential equations with finite delays. Defining the energy solution (c.f. Theorem 4.1) in the same way as in Taniguchi (2007), we have showed the existence and uniqueness of energy solutions together with the stochastic energy equality in Theorem 4.1. It is worth noting that the stochastic energy equality (4.3.3) has played an key role in the study of exponential stability of our equations. By estimating the coefficients in the stochastic energy equality, we have derived the desired stability results. At last, comparing to Taniguchi (2007), Wan and Duan (2008) and Hou et al. (2010), we have improved the existing result to cover a more general class of stochastic partial delay differentia equations in Theorem 4.2 and 4.3. Moreover, unlike in Taniguchi (2007), we do not require the functions in the delay terms to be differentiable.

In Chapter 5, we have discussed the relationship among strong, weak and mild solutions to a stochastic functional differential equations of neutral type. Both fundamental solutions and variation of constants formula of mild solutions are introduced in the same way as in Liu (2008, 2009). We have extended the deterministic neutral systems in Liu (2009) to the stochastic systems driven by Wiener processes. By using the fundamental solutions (Green's operator), the mild solutions can be represented by the variation of constants formula (5.3.5). Hence we have showed the strong solution actually is a mild one in Theorem 5.1; the weak solution can be represented by the variation of constants formula of mild solutions in Proposition 5.1; and the mild solution is a strong one under some conditions in Theorem 5.2.

Finally, in Chapter 6, we have obtained the mean square asymptotic stability of two types of equations, impulsive stochastic delay differential equations with Poisson jumps and stochastic retarded evolution equations with Poisson jumps by using fixed point theorem under some suitable criteria. It is worth pointing out that the use of fixed point theorem method is quite concise (c.f. Luo (2007, 2008) and Luo and Taniguchi (2009)). We have generalized the results in Luo (2007) to a class of impulsive stochastic delay differential equations with Poisson jumps. In many applications, due to the complex random nature of situation, the stochastic problem should be considered in a stochastic integro-differential framework. Such stochastic integro-differential equations allow some long-range dependence of the noise, hence they are more general type of equations. It is important to note that one can easily prove that by adopting and employing the method used in Theorem 6.1, impulsive stochastic integro-differential equations are mean squared asymptotically stable. In addition, in the study of stochastic retarded evolution equations with Poisson jumps, an approximation of strong solutions has been established in Section 6.2.3 and consequently, a retarded Burkholder-Davis-Gundy's inequality for the stochastic convolution involving the Green's operator $W_G^{\widetilde{N}} = \int_0^t \int_{\mathbb{Z}} G(t-s)L(s,u)\widetilde{N}(ds,du)$ has been derived (c.f. Section 6.2.4).

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