Aspects of F-theory GUTs

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by

William Walters

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Abstract

We explicitly construct an $SO(10)$ GUT using a global description of the geometry and find two interesting properties: first, at the point of expected E_7 enhancement where the Yukawa interaction is generated, we see that there are not enough irreducible components in the resolved singular fiber to yield an E_7 structure; second, upon constructing the G-flux for this GUT, we see that it is not possible, in general, to satisfy the quantization condition. This leads to a set of extra conditions on the geometry to allow for the required quantization. We then look at an E_7 GUT using a similar construction.

Declaration

I hereby declare that all work described in this thesis is the result of my own research unless reference to others is given. None of this material has previously been submitted to this or any other university. All work was carried out in the Theoretical Physics Division of the Department of Mathematical Sciences, University of Liverpool, U.K. during the period of October 2009 until March 2013.

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Publication List

This thesis contains material that has appeared in the following publication by the author:

1. Radu Tatar and William Walters. "GUT theories from Calabi-Yau 4-folds with SO(10) Singularities". JHEP, 1212:092, 2012. arXiv:1206.5090 [hep-th].

There is another publication by the author, which is not presented in this thesis:

2. Chan-chi Chiou, Alon E. Faraggi, Radu Tatar and William Walters. "T-branes and Yukawa Couplings". JHEP, 1105:023, 2011. arXiv:1101.2455 [hep-th].

Contents

Chapter 1

Introduction

The existence of a Grand Unified Theory (GUT) of particle physics, where the known Standard Model interactions unify into a single force, is hinted at by the apparent unification of the gauge couplings of the strong and electroweak interactions, if one assumes supersymmetry. Unfortunately, there is not yet any experimental evidence for the existence of a GUT, and also no universally preferred GUT model. Thus our only requirement for a candidate GUT group is the consistent embedding of the Standard Model particles and interactions, although it is natural to choose a GUT group which gives rise to as few exotics as possible.

Unification of the three forces is not only elegant in its own right, but also predicts certain properties that are not explained by the Standard Model. For example, the quantization of electromagnetic charges and the relative strengths of the interactions of the Standard Model, which would otherwise be input parameters, are instead consequences of high energy unification. The fact that even the simplest GUT models accurately predict the relative $U(1)$ charges of the Standard Model representations, whilst not evidence of unification, certainly supports the assumption that the apparent unification of couplings is more than just a coincidence.

As we are going beyond the Standard Model to consider Grand Unification, it makes sense to do so in a framework which also allows for a description of gravity, which the Standard Model lacks. String theory is such a framework, whilst being far from complete, it is the most popular candidate description of quantum gravity. That string theory contains solutions in which one can embed the Standard Model is of course important, if we are going to use it to describe real world physics, however, the vast

number of possibilities is one of the major problems of string theory. In particular, the absence of top-down selection criteria makes it difficult to choose which string theory to focus on.

There are five string theories, which are interconnected via a web of dualities. The Type II string theories; Type IIA and Type IIB, have $\mathcal{N} = 2$ supersymmetry in 10 dimensions, and are dual to each other via T-duality. The difference between the two Type II theories is that Type IIA is non-chiral in 10 dimensions, because the left- and right-moving fermions in Type IIA have opposite chirality, whereas in Type IIB they have the same chirality. Type I string theory, which has $\mathcal{N} = 1$ supersymmetry in 10 dimensions, can be obtained by an orientifold of Type IIB, meaning that the strings in Type I are unoriented. Finally we have the two Heterotic string theories, where only the right-moving modes are supersymmetric, there are two different Heterotic string theories, one with an $SO(32)$ gauge group, and the other with $E_8 \times E_8$. Taking the strong coupling limit of Type IIA leads to an 11-dimensional theory known as M-theory. Note that whilst it is related to string theory, M-theory is not a theory of strings. Here the fundamental objects are believed to be 2-dimensional M2-branes. M-theory can be thought of as more fundamental than Type IIA, since Type IIA is obtained simply as a particular compactification of M-theory (on a circle). Another compactification of M-theory can yield $E_8 \times E_8$ Heterotic string theory. In this thesis we will be interested in a particular compactification of M-theory which can be shown to be dual to a strong coupling limit of Type IIB known as F-theory.

F-theory [1] is one of the most promising frameworks for GUT model building in string theory [2–5], combining the exceptional gauge groups, which are a natural feature of the $E_8 \times E_8$ Heterotic string theory, with the localization properties of Type IIB string theory. The presence of exceptional gauge groups is appealing as one can not only embed the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ easily, but also some of the most commonly studied simple candidate GUT groups: $SU(5)$, $SO(10)$ and E_6. The localization of gauge fields and matter to branes and their intersections means that we can decouple gravity from the theory and focus on GUT model building. The fact that we have decoupled gravity from the beginning means that our use of the

terms local and global do not refer to the exclusion or inclusion of gravitational effects but, instead, to whether we are only considering the restriction of fields to the GUT brane, or their extension to the whole compactification space.

Constructing a GUT in F-theory gives a very natural explanation as to why matter has the representation structure and interactions that we see in the Standard Model: it is the generic case once we have specified a unifying gauge group. This very pleasing origin for matter and its interactions, along with the knowledge that, even though we have chosen to decouple it, gravity could be included without the major problems one would encounter using point particles, are our main motivations for studying F-theory.

In Type II string theories, one can construct a gauge theory with matter using a stack of D-branes, upon which the open strings furnish an $SU(n)$ gauge symmetry, where n is the number of branes in the stack. In Type IIB, the low-energy effective action has an $SL(2,\mathbb{Z})$ symmetry, if this symmetry is taken as a symmetry of the entire Type IIB string theory, it interchanges F-strings and D-strings (D1-branes), allowing for a more general string state, consisting of a linear combination of F- and D-strings. D1-branes are dual to D7-branes, and so this symmetry allows one to also construct a more general type of 7-brane. Using different configurations of these 7-branes, it is possible to obtain more gauge groups than just $SU(n)$.

In F-theory, we introduce a torus whose complex modulus contains the field which couples to 7-branes. The presence of 7-branes is then encoded in singularities in this torus, which varies over spacetime due to the fact that its complex modulus is comprised of fields. We therefore compactify on an elliptically fibered 4-fold, with the understanding that we are taking the limit of vanishing fiber volume. By studying the structure of the singularities in the fiber, we learn about the exact configuration of 7-branes present at the singular locus and, in turn, the gauge group present on the brane. F-theory exploits the classification of singularities on an elliptic fibration in terms of Dynkin diagrams [6–8], so that the gauge group is encoded in the geometry of the fibration. Furthermore, because matter and Yukawa couplings correspond to higher codimension loci, where the singularity structure of the fiber changes, not only is the gauge group encoded in the geometry, but also the type of matter present and

the interactions possible between matter states.

F-theory is often looked at via the various dualities it has with the Heterotic and Type IIB string theories, and with M-theory. This is because, unlike these other theories, there is no fundamental description of F-theory; there is no 12-dimensional supergravity which F-theory reduces to at low energies. For this reason, it is best not to think of F-theory as being a 12-dimensional fundamental theory, but rather a 10-dimensional theory where we have geometrized the behaviour of some of the fields. The elliptic fiber we have added to the compactification is thus often described as being a fictional torus.

One possible way of constructing an F-theory GUT is to start from a higher rank gauge group and break it to the desired GUT group by giving a vev to a Higgs field lying in the adjoint of the original gauge group. By choosing this vev to depend on the coordinates of the GUT brane, it is possible to engineer a situation where, at certain loci, the vev vanishes and the gauge group is unhiggsed to a higher rank group. This leads to the intersecting 7-branes picture we look at in more detail in Section 2.5. Geometrically this corresponds to deforming the original singularity [9, 10].

In this thesis we focus on a method which is the opposite of deforming a higher gauge group. We instead start from the general form for the singularity corresponding to the GUT gauge group that we are interested in. We then look for generic enhancements as these can give matter and interaction terms. By enhancements we mean loci where the singularity worsens, i.e. the discriminant of our defining equation for the fibration vanishes to a higher order. Note that the word generic used here has a specific mathematical meaning [11], when dealing with a family of objects parametrised locally by a complex variety. Saying that a property is generic means that the subspace parametrising objects which do not have this property is contained in a lower dimensional subvariety.

We can see how our use of the word generic fits with its formal definition in the following example; for our $SO(10)$ case we focus our attention on elliptically fibered Calabi-Yau 4-folds of the form

$$y^2w + b_1zxyw + b_3z^2yw^2 = x^3 + b_2zx^2w + b_4z^3xw^2 + b_6z^5w^3\,. \tag{1.1}$$

With the b_i being unspecified functions on the three-dimensional base manifold B_3, with the condition that they do not vanish identically on the GUT surface S_2, given by $z = 0$. The codimension-3 enhancement which yields the Yukawa coupling is given by $b_2 = b_3 = 0$ at some point on the GUT surface S_2. At this point the other three b_i have specific values, which we can parametrise by three complex numbers, and hence by the space $\mathbb{C}^3$. We can say that, generically, none of these b_i vanish at this point, since that would be restricting us to a lower dimensional surface inside the parametrization space.

We focus on the example of an $SO(10)$ GUT group, which we were inspired to look at due to the interesting results found in [12] for SU(5), where the point of an expected E_6 enhancement was demonstrated not to give rise to an E_6-type fiber. The resolution of a D_5 singularity is explicitly constructed to see if there are any similar situations where an expected enhancement is not realised. We find that this occurs at the point of expected E_7 enhancement in a similar way to the $SU(5)$ case; there is one less irreducible component than would be necessary for an E_7 symmetry enhancement. Furthermore, we go on to show, using the formalism of [13], that this feature is not problematic. The generation of the Yukawa coupling that was thought to require this symmetry enhancement can be viewed as occuring due to one of the components of the fiber that supports matter, becoming homologous to the sum of two others, at the point of singularity enhancement. This method of showing the appearance of matter states makes their localization to matter curves very clear. The matter states correspond to M2-branes wrapping certain irreducible cycles in the fiber, but the particular irreducible cycles that give matter only arise on codimension-2 loci where the singularity enhances. So the matter is forced to live on the locus of enhancement.

The layout of this thesis is as follows, in Chapter 2, we give a brief introduction to F-theory, starting from its origins in Type IIB theory, and describing its various dualities with other theories. We also describe one of the usual methods for model building in F-theory, the intersecting 7-branes model, a terminology which we specifically use here to denote the formalism of starting from a larger group than the desired GUT group and breaking it to the GUT group using an adjoint Higgs vev. We also give a brief account of the ingredients required to construct the most studied GUTs at the most

basic level, i.e. the matter representations and couplings between them, along with a description of some common problems encountered in GUT construction.

In Chapter 3, we describe the general procedure for constructing a global GUT model in F-theory using the formalism of [13], starting from the Tate form for the singularity and focussing on the process of identifying the matter and couplings one would obtain in the generic case. These methods can be easily generalised to non-generic cases. We then give details on how to construct G-flux, firstly in the local case, and then the global one, which should match on the GUT divisor. Despite our focus being on global model building, we include the construction of the local flux, since using it as a starting point to construct a global flux is particularly useful in our example case of $SO(10)$. Here the G-flux only satisfies the required conditions after imposing certain constraints on the geometry, these constraints are much easier to derive from the local construction. It was the local flux construction that was used in [14] to demonstrate the requirement of extra constraints on the geometry for a consistently quantized G-flux in cases where the GUT gauge group is one whose commutant in E_8 is $SU(n)$, for even values of n.

Chapter 4 describes in detail the construction of an $SO(10)$ GUT as carried out in [15]. First we give an explicit resolution of the singularity starting from the general Tate form of a D_5 singularity. Then we look at the generic codimension-2 enhancements and show that these give matter states in the **16** and the **10**, and then the codimension-3 enhancements and show that these lead to the expected Yukawa couplings for an $SO(10)$ GUT. In studying the codimension-3 enhancements, we see the interesting result found in [15], which is that the expected point of E_7 enhancement does not give rise to this, since we have one less component than necessary.

In Chapter 5 we turn to the construction of G-flux to give chirality to the $SO(10)$ GUT model. By first constructing the local flux, we see that proper quantization sets extra conditions involving the geometry of the base manifold and the GUT surface. We then derive the set of possible conditions on the geometry, as carried out in [15]. When we construct the global G-flux, we see that the required quantization condition is only satisfied once we impose the conditions found using the local flux. Furthermore, we see

that the local and global fluxes match.

In Chapter 6 we look at constructing a model with an E_7 GUT group, again starting with the general Tate form for this group and explicitly detailing the process of resolving the singularity. Studying an E_7 GUT is motivated by the fact that this formalism does not require any actual gauge group enhancement to generate matter and interactions. The practice of associating these features to symmetry enhancements would usually be a barrier to E_7 GUTs being considered in F-theory, as E_8 is the highest possible symmetry without breaking the Calabi-Yau condition [8]. We find here however that there are no couplings for generic codimension-3 enhancements. For this reason we do not attempt to construct a G-flux for this model.

In Chapter 7 we summarise our findings and describe potential future directions for extending this research.

The appendices collect various useful results on intersection relations and representations. Appendix A contains intersection relations for the $SO(10)$ GUT manifold. The intersection relations in X_5 are useful for computing total intersections in chapters 4 and 5. The set of pairs of variables that cannot simultaneously vanish is used to see the irreducible components when the fiber splits at enhancement loci, and the intersection relations between classes that they imply are used to simplify the calculation of a global G-flux. Appendices B and C contain the weights of representations of $SO(10)$ and E_7, used in chapters 4 and 6 respectively to identify the matter states which appear at enhancement loci.

Chapter 2

F-theory

F-theory [1] is an interpretation of strongly coupled Type IIB theory, which in certain limits has dualities with specific compactifications of $E_8 \times E_8$ heterotic theory [1,16–18] and M-theory [19].

2.1 From Type IIB to F-theory

To describe F-theory (see [20,21] for introductory lecture notes, and [22–25] for reviews) we start with the low energy effective action of Type IIB (i.e. Type IIB supergravity in ten dimensions). Firstly we define

$$\tau = C_0 + \frac{i}{g_s}\,, \tag{2.1}$$

$$G_3 = F_3 - \tau H_3\,, \tag{2.2}$$

$$\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3\,. \tag{2.3}$$

C_p denote the R-R p-forms with $F_{p+1} = dC_p$, B_2 is the antisymmetric tensor, where $H_3 = dB_2$, and g_s is the string coupling. Then the bosonic part of the low energy effective action can be written as

$$\begin{aligned} S_{\text{IIB}} =& \frac{2\pi}{l_s^8}\left[\int d^{10}x\sqrt{-g}R - \frac{1}{2}\int \frac{1}{(\text{Im}\tau)^2} d\tau \wedge \star d\overline{\tau} + \frac{1}{\text{Im}\tau} G_3 \wedge \star\overline{G_3}\right. \\ &\left. + \frac{1}{2}\tilde{F}_5 \wedge \star\tilde{F}_5 + C_4 \wedge H_3 \wedge F_3\right]\,, \end{aligned} \tag{2.4}$$

which must be supplemented by the additional constraint

$$\star\tilde{F}_5 = \tilde{F}_5. \tag{2.5}$$

This action has an $SL(2,\mathbb{Z})$[1] symmetry:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} F_3 \\ H_3 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_3 \\ H_3 \end{pmatrix}. \tag{2.6}$$

The NS-NS 2-form B_2 couples to fundamental strings (F-strings), and the R-R 2-form C_2 couples to D1-branes (D-strings). Due to the $SL(2,\mathbb{Z})$ symmetry of F_3 and H_3, we combine F- and D-strings into an $SL(2,\mathbb{Z})$ doublet.

Denoting an F-string by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and a D-string by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we will therefore be considering $\begin{pmatrix} p \\ q \end{pmatrix}$ strings [26], which are bound states of F- and D-strings for p, q coprime (this ensures that it can be obtained from an F- or D-string using an $SL(2,\mathbb{Z})$ transformation). One can obtain a $\begin{pmatrix} p \\ q \end{pmatrix}$ string from an F-string with the $SL(2,\mathbb{Z})$ transformation

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad ps - rq = 1, \tag{2.7}$$

the matrix $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ is denoted by $g_{p,q}$, we note that abilily to choose integers r and s such that $ps - rq = 1$ required p and q to be coprime, as mentioned above.

In Type IIB, the R-R field C_0, which couples to D-strings, is dual to C_8, which couples to D7-branes. If we complexify the space in 10 dimensions, which is perpendicular to a D7-brane, i.e. set $z = x^8 + ix^9$, then for a D7-brane at $z = z_0$, we get a Poisson equation

$$d \star F_9 = \delta^{(2)}(z - z_0)\ , \tag{2.8}$$

which close to z_0 gives the solution for τ

$$\tau(z) = \tau_0 + \frac{1}{2\pi i} \log(z - z_0) + \dots\ . \tag{2.9}$$

In particular, we can see from this equation that encircling a D7-brane anticlockwise gives the monodromy

$$\tau \to \tau + 1\,, \tag{2.10}$$

which corresponds to the $SL(2,\mathbb{Z})$ transformation

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \equiv M_{1,0}. \tag{2.11}$$

[1] Actually the action by itself is invariant under $SL(2,\mathbb{R})$, but quantization of the associated charges allows only $SL(2,\mathbb{Z})$.

Therefore when a $\begin{pmatrix} p \\ q \end{pmatrix}$ string moves around a D7-brane, it transforms as

$$\begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p+q \\ q \end{pmatrix}. \tag{2.12}$$

A D7-brane is an object on which a $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ string can end, thus we define a (p,q) 7-brane to be a 7-brane on which a $\begin{pmatrix} p \\ q \end{pmatrix}$ string can end. When a string moves around a (p,q) 7-brane, the monodromy generalises to

$$M_{p,q} = g_{p,q} M_{1,0} g_{p,q}^{-1} = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}, \tag{2.13}$$

of which $\begin{pmatrix} p \\ q \end{pmatrix}$ is the only eigenvector. So when a $\begin{pmatrix} p \\ q \end{pmatrix}$ string encirles a (p,q) 7-brane, there is no monodromy.

The idea of F-theory is that because the $SL(2,\mathbb{Z})$ symmetry of τ is the same as the symmetry of a torus with complex modulus τ, we can geometrise its action. A torus with complex modulus τ is defined by

$$T^2 = \mathbb{C}/\left(\mathbb{Z} \oplus \tau\mathbb{Z}\right). \tag{2.14}$$

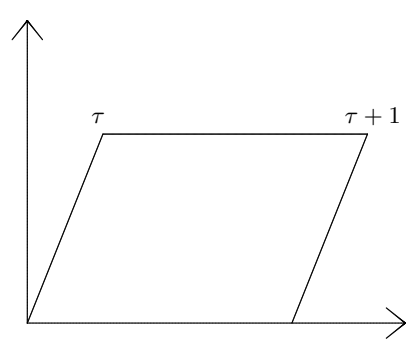

Figure 2.1: Torus identified with a parallelogram in the complex plane with opposite edges identified, the complex modulus τ describes the exact shape of the torus.

In F-theory this is interpreted as meaning that we can view the 10-dimensional action as the compactification of a 12-dimensional action with the extra two dimensions being an elliptically fibered torus with modulus τ. We take the torus to have a vanishing volume $V \to 0$, as otherwise the volume would appear as a parameter in the 10-dimensional action. In practice, we will do all calculations as if the torus' volume were finite, but it is understood that we will ultimately be taking this volume to zero,

so we still have 10 spacetime dimensions. In particular, the gauge symmetries that we associate with certain singularities are only present in the limit of vanishing fiber volume. We compactify on an elliptically fibered 4-fold Y_4 with a 3-fold base B_3. In the limit of vanishing fiber volume this corresponds to strongly coupled Type IIB compactified on the 3-fold B_3.

We can see from equation (2.9) that τ is singular in the presence of a 7-brane. So in F-theory, the locations of 7-branes are encoded in the geometry. Wherever the elliptic fiber degenerates (meaning that a one-cycle shrinks to zero size) we have a 7-brane. So 7-branes correspond to singularities in the elliptic fiber.

So far we have accounted for the action of the $SL(2,\mathbb{Z})$ on τ, the transformations of the R-R and NS-NS 2-form can be explained nicely via the M-theory duality, which we now turn to.

2.2 Duality with M-theory

One of the most useful ways to think about F-theory is via the duality with M-theory; in fact we will use this when we construct global F-theory GUTs, since it is from M-theory that some of the constraints on G-flux originate [2, 27].

We begin with M-theory on $\mathbb{R}^{1,3} \times Y_4$ where Y_4 is an elliptically fibered 4-fold over a base B_3. We focus on the fiber $T^2 = S^1_A \times S^1_B$, and take the limit of vanishing fiber volume in two stages. Firstly we take the radius R_A of the circle S^1_A to zero. This gives us Type IIA theory on a fibration with fiber S^1_B over the same base B_3. We can then T-dualize along S^1_B to give Type IIB theory on a circle with radius $\tilde{R}_B = \frac{l_s^2}{R_B}$ where R_B is the radius of the original S^1_B. Then taking the original circle to vanish in the Type IIA theory, means that it decompactifies in the Type IIB theory, giving Type IIB on $\mathbb{R}^{1,3} \times B_3$, which is exactly what we would have obtained if we had started with F-theory on the same Y_4 and taken the same limit. This duality is described in detail in [20].

It has been shown [28] that M-theory compactified on a Calabi-Yau 4-fold Y_4 gives a three dimensional effective theory with four supercharges. By using the duality with F-theory described above, we can dualize this to a four-dimensional effective theory, in

which case the four supercharges mean we have an $\mathcal{N} = 1$ supersymmetry. Therefore, we will always take the 4-fold Y_4 to be Calabi-Yau, since we wish to obtain $\mathcal{N} = 1$ supersymmetry in 4 dimensions.

Following the M-theory three-form C_3 through the duality yields an explanation for the action of the $SL(2,\mathbb{Z})$ action on the fields C_2 and B_2. We can decompose C_3 as

$$C_3 = \tilde{C}_3 + B_2 \wedge ds_A + C_2 \wedge ds_B + B_1 \wedge ds_A \wedge ds_B \,, \tag{2.15}$$

where s_A and s_B are the coordinates of the cycles S^1_A and S^1_B respectively.

From this we can see that after compactifying on S^1_A, T-dualizing along S^1_B and then taking the limits $R_A, R_B \to 0$, $\tilde{C}_3$ becomes $C_4 = \tilde{C}_3 \wedge ds_B$, B_2 and C_2 are the NS-NS and R-R 2-forms of Type IIB, and B_1 gives off-diagonal components of the metric in Type IIB, mixing the now decompactified circle with the other directions.

From this we can see how the $SL(2,\mathbb{Z})$ symmetry of the T^2 acts on B_2 and C_2. It acts on them in the same way that it acts on the two cycles of the torus, giving the required transformation.

2.3 Gauge Symmetry

In this work we will only consider matter coming from open strings ending on 7-branes, in particular we will only look at 7-branes which fill $\mathbb{R}^{1,3}$ and so wrap some 2-cycle, S_2, in the Calabi-Yau 4-fold Y_4. Since in F-theory, 7-branes correspond to singularities in the fiber, the gauge groups we deal with are determined by the geometry of Y_4, so this is what we study.

The easiest gauge group to construct is $SU(n)$: this corresponds to n parallel 7-branes, where the open strings stretching between them furnish the adjoint representation of $SU(n)$[2]. It is also possible to construct other gauge groups from configurations of 7-branes [29, 30]. However, exceptional gauge groups arise from multi-pronged open strings which can end on more than one 7-brane. These are impossible to realise in perturbative Type IIB, this is one of the reasons we turn to F-theory. Even if one tries to construct an $SU(5)$ GUT in Type IIB, it is impossible to generate $\mathbf{10} \cdot \mathbf{10} \cdot \mathbf{5}$

[2] Actually they furnish the adjoint representation of $U(n)$, however we decouple the $U(1)$ and instead deal with $SU(n)$

interaction term, since this corresponds to an E_6 enhancement[3].

The singularities we will deal with are those that have an ADE classification. This means that when one resolves the singularity by replacing the singular fibers by a network of $\mathbb{P}^1$s, in such a way as to preserve the Calabi-Yau condition of the 4-fold, the intersection structure of these $\mathbb{P}^1$s with each other then generates the corresponding ADE Dynkin diagram. This is demonstrated explicitly for D_5 and E_7 in chapters 4 and 6. It turns out that the corresponding ADE Lie algebra dictates the resulting gauge group on the 7-brane upon taking the limit of vanishing fiber volume. Considering non-ADE-type singularities is more complicated since one has to take monodromies into account, this is not done here since the groups we are interested in are of ADE-type.

The mapping from singularities to gauge groups is best seen via the duality with M-theory. In the M-theory picture, the gauge bosons are M2-branes wrapping 2-cycles [2], which become massless as these 2-cycles shrink to zero size. Above loci in the base manifold of the fibration where the fiber is singular, new 2-cycles appear, and hence we have gauge bosons that are specifically localised here.

Concretely, an elliptic fibration can be described by the Weierstrass equation:

$$y^2 = x^3 + fx + g\,, \tag{2.16}$$

where f and g are functions on the base B, and x and y parametrise the fiber. The fiber degenerates at loci on the base manifold where the discriminant,

$$\Delta = 4f^3 + 27g^2\,, \tag{2.17}$$

vanishes. This means that we have a point on the fiber where the derivative of the defining equation vanishes, since we have a double root.

To see why a singularity requires $\Delta = 0$, we first note that the derivative with respect to y only vanishes at $y = 0$, so we are interested in values of x which satisfy both

$$0 = x^3 + fx + g\,, \tag{2.18}$$

[3]Although it has been shown [13] that the required enhancement is not actually E_6, one still cannot generate this term in perturbative Type IIB.

and its derivative

$$3x^2 + f = 0\,. \tag{2.19}$$

If we plug $x^2 = -\frac{f}{3}$ into equation (2.18), we see that

$$\frac{2}{3}fx + g = 0\,, \tag{2.20}$$

thus we also have $x = -\frac{3g}{2f}$. Inserting this into equation (2.19), gives the required condition

$$4f^3 + 27g^2 = 0\,. \tag{2.21}$$

The different types of singularities one can obtain were classified by Kodaira [6], summarised in Table 2.1.

ord(f)	ord(g)	ord(Δ)	fiber type	singularity type
≥ 0	≥ 0	0	smooth	none
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	none
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

Table 2.1: Kodaira's Classification of Singularities.

The entries in the table give the vanishing order of the functions f and g, the discriminant Δ and the resulting singularity type.

The Kodaira classification however, only gives the full story when the base has complex dimension 1. For higher dimensional bases, we have the possibility of introducing monodromies, which break these gauge groups to non-simply laced subgroups; so instead of the expected A_n, D_n or E_n type group, we could end up with a B_n, C_n, F_4 or G_2 group. This can happen because in higher dimensions, the locus where the discriminant vanishes is not necessarily just a point and so the functions f and g need not be constant here. For this reason we will instead bring the Weierstrass equation into Tate form [7]:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\,. \tag{2.22}$$

This form can be achieved for most singularity types, with a few exceptions [31]. In particular, the most commonly studied GUT groups: $SU(5)$, $SO(10)$, and E_6 can all be brought into this form. One can recover the Weierstrass equation from the Tate form by completing the square in y and completing the cube in x, with the functions f and g given in terms of the a_i as

$$\begin{aligned} f &= -\tfrac{1}{48}(b_2^2 - 24b_4) \\ g &= -\tfrac{1}{864}(-b_2^3 + 36b_2b_4 - 216b_6)\,, \end{aligned} \tag{2.23}$$

where the b_i are

$$\begin{aligned} b_2 &= a_1^2 + 4a_2 \\ b_4 &= a_1a_3 + 2a_4 \\ b_6 &= a_3^2 + 4a_6\,. \end{aligned} \tag{2.24}$$

This, more general form contains more information about the structure of the singularity, and will in most cases account for the monodromies which can give non-simply laced groups. The results of the Tate algorithm are summarised in Table 2.2.

As with the Kodaira table, the entries give the vanishing multiplicities of the coefficients, which allow one to easily write down the general equation for the gauge group. In the table, some of the entries in the fiber column now have an additional superscript which was not present in the Kodaira classification. This refers to the monodromy acting on the irreducible components. In particular, the superscripts s / ns refer respectively to split / non-split. In the cases with these subscripts, this means that there is an irreducible component given by a quadratic equation (with the exception of I_0^*, which we discuss separately), and if this does not factorise (the non-split case) then the two solutions are exchanged under monodromy. In fact we must quotient the group by this monodromy giving a reduced non-simply laced group. If the quadratic factorises (the split case), then the two components are not exchanged and we get a simply laced group. In the case of I_0^* however, the relevant polynomial whose factorisation we are interested in is a cubic, hence there are three possibilities: The non-split case where all three roots are exchanged by monodromy and so we quotient D_4 by S_3 to obtain G_2; the semi-split case (denoted by the superscript ss) where the cubic factorises into a

type	group	a_1	a_2	a_3	a_4	a_6	Δ
I_0	—	0	0	0	0	0	0
I_1	—	0	0	1	1	1	1
I_2	$SU(2)$	0	0	1	1	2	2
I_3^{ns}	unconven.	0	0	2	2	3	3
I_3^{s}	unconven.	0	1	1	2	3	3
I_{2k}^{ns}	$Sp(k)$	0	0	k	k	$2k$	$2k$
I_{2k}^{s}	$SU(2k)$	0	1	k	k	$2k$	$2k$
I_{2k+1}^{ns}	unconven.	0	0	$k+1$	$k+1$	$2k+1$	$2k+1$
I_{2k+1}^{s}	$SU(2k+1)$	0	1	k	$k+1$	$2k+1$	$2k+1$
II	—	1	1	1	1	1	2
III	$SU(2)$	1	1	1	1	2	3
IV^{ns}	unconven.	1	1	1	2	2	4
IV^{s}	$SU(3)$	1	1	1	2	3	4
$I_0^{*\,ns}$	G_2	1	1	2	2	3	6
$I_0^{*\,ss}$	$SO(7)$	1	1	2	2	4	6
$I_0^{*\,s}$	$SO(8)^*$	1	1	2	2	4	6
$I_1^{*\,ns}$	$SO(9)$	1	1	2	3	4	7
$I_1^{*\,s}$	$SO(10)$	1	1	2	3	5	7
$I_2^{*\,ns}$	$SO(11)$	1	1	3	3	5	8
$I_2^{*\,s}$	$SO(12)^*$	1	1	3	3	5	8
$I_{2k-3}^{*\,ns}$	$SO(4k+1)$	1	1	k	$k+1$	$2k$	$2k+3$
$I_{2k-3}^{*\,s}$	$SO(4k+2)$	1	1	k	$k+1$	$2k+1$	$2k+3$
$I_{2k-2}^{*\,ns}$	$SO(4k+3)$	1	1	$k+1$	$k+1$	$2k+1$	$2k+4$
$I_{2k-2}^{*\,s}$	$SO(4k+4)^*$	1	1	$k+1$	$k+1$	$2k+1$	$2k+4$
$IV^{*\,ns}$	F_4	1	2	2	3	4	8
$IV^{*\,s}$	E_6	1	2	2	3	5	8
III^*	E_7	1	2	3	3	5	9
II^*	E_8	1	2	3	4	5	10
non-min	—	1	2	3	4	6	12

Table 2.2: F-theory Tate's algorithm.

linear factor and a quadratic factor, so two roots are exchanged leading to quotienting by Z_2 to obtain the group B_3; and the split case where we can factorise into three linear factors and so quotienting is not necessary, meaning that we have D_4.

2.4 Duality with Heterotic theory

It was originally argued [1] that F-theory compactified on an elliptically fibered $K3$, is dual to the $E_8 \times E_8$ Heterotic string theory compactified on a T^2. They are shown to be dual via a matching of the moduli spaces on each side of the duality. This duality can be extended by applying it fiberwise to cases where the elliptically fibered $K3$ is

itself fibered over a base B [32], and F-theory compactified on this geometry is then dual to the Heterotic string compactified on an elliptic fibration with the same base. For phenomenology we are interested in compactifications to 4 dimensions with $\mathcal{N} = 1$ and so we can use a two dimensional base to obtain a duality between F-theory on a Calabi-Yau 4-fold $Y_4 = K3 \to B_2$ and Heterotic on a Calabi-Yau 3-fold $Z_H = T^2 \to B_2$.

The Heterotic elliptic fibration Z_H can be obtained from the F-theory $K3$ fibration Y_4 by taking the stable degeneration limit [2, 16, 17, 33]. In this limit, the $K3$ fiber degenerates into two dP_9's, each contributing an E_8 factor, whose intersection is the elliptic fiber of the Heterotic compactification dual to the F-theory one. The Heterotic elliptic fibration is then given by this elliptic fiber, fibered over the base of the original $K3$ fibration.

We will not explicitly make use of this duality in this thesis, however, the local flux construction we use in Section 3.2 originated on the Heterotic side, and has been shown to still be valid in F-theory compactifications which do not possess a Heterotic dual [34].

We note that this requirement for the compactification to be a 4-fold which is $K3$ fibered means that not all F-theory models have Heterotic duals, whereas all F-theory models are dual to M-theory since all we require here is that the 4-fold was elliptically fibered (although it does not work the other way: not all M-theory compactifications will be dual to an F-theory one, since M-theory need not always be compactified on an elliptically fibered manifold).

2.5 Intersecting 7-branes

The easiest way to engineer a GUT in F-theory is using intersecting 7-branes. In order to get both matter and Yukawa couplings we need to have three stacks of 7-branes which intersect one another. Each pair of stacks will intersect along a 6-dimensional space, and it is here where matter localises. The matter is charged under the gauge group of each stack, and so is comprised of open strings starting on one stack and ending on another. Where the three stacks intersect, we also have an intersection of the three 6-dimensional spaces upon which we have localised matter. This gives the

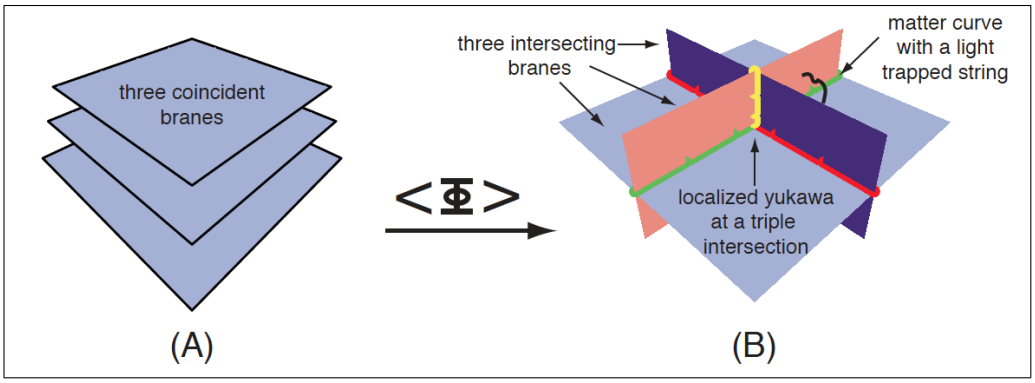

Figure 2.2: In this figure, taken from [35], we see how a stack of three branes can be deformed into an intersecting configuration, with three matter curves where a pair of branes intersect, and an interaction point where the three matter curves meet.

possibility of an interaction term involving three matter fields. This setup is illustrated in Figure 2.2.

One way of engineering this situation is to start with a 7-brane stack upon which we have an 8-dimensional gauge theory with a specified gauge group. Specifying the gauge group instantly constrains our geometry since we require a singularity corresponding to this group. This 8-dimensional gauge theory supports an adjoint-valued gauge field, Φ, whose expectation value is interpreted as parametrizing normal motion to the stack. Then, by specifying a vacuum expectation value $\langle\Phi\rangle$, we can deform the stack. The deformation involves rotating some of the 7-branes away from the stack so that they are no longer parallel to it, but intersect it, meaning our expectation value is spacetime dependent. We specifically take these rotations to happen in the compact 4-fold Y_4, so that all 7-branes fill $\mathbb{R}^{1,3}$.

It was first described in [10] how the act of deforming the singularity can result in localised matter. Suppose we take our original stack of 7-branes with a gauge group G, and deform by splitting it into two stacks with gauge groups G_1 and G_2, with the two stacks intersecting each other. Before rotating the gauge group, we have open string states living on the brane transforming in the adjoint representation of G. Then, by decomposing the adjoint of G under the breaking $G \to G_1 \times G_2$, we obtain something of the form

$$\mathrm{adj}(G) \to (\mathrm{adj}(G_1), \mathbf{1}) \oplus (\mathbf{1}, \mathrm{adj}(G_2)) \oplus \bigoplus_a (\mathbf{R}_a^1, \mathbf{R}_a^2)\,, \tag{2.25}$$

where $\mathbf{R}^i$ means a representation of the group G_i. Since the representations $(\mathbf{R}^1_a, \mathbf{R}^2_a)$ are charged under both groups, they must lie on both stacks and are hence localised to the intersection.

In practice we would choose either G_1 or G_2 to be the desired GUT group, then by choosing G appropriately, we can obtain the desired matter representations. For example, in an $SU(5)$ GUT model we require matter in the $\bar{\mathbf{5}}$ representation. Choosing $G = SU(6)$ and breaking it to $SU(5) \times U(1)$, we find via the branching rule:

$$\mathbf{35} \to \mathbf{24}_0 \oplus \mathbf{1}_0 \oplus \bar{\mathbf{5}}_1 \oplus \mathbf{5}_{-1}\,, \tag{2.26}$$

that we do indeed obtain the required $\bar{\mathbf{5}}$. This example would correspond physically to taking a stack of 6 D7-branes, and rotating one of them away from the stack. So we now have a stack of 5 D7-branes intersecting a single D7-brane, with matter transforming in the $\mathbf{5}$ and the $\overline{\mathbf{5}}$, localised at the intersection.

Yukawa couplings work in a similar way. Here we have three intersecting stacks of branes, which will generically all intersect at a point. This is obtained from further deformation of the original gauge group, G, into three gauge groups; G_1, G_2, G_3. As well as the branes all intersecting at this point, we will have three matter curves intersecting at this point, and it is here that the interaction between matter occurs, allowing the engineering of a Yukawa coupling. A Yukawa coupling requires that the cubic term be invariant under all three gauge groups.

The simplest situation to consider is one where we end up with one 7-brane stack upon which we have the GUT group, G_{GUT}, intersected with two single separate D7-branes. The overall gauge group is then given by

$$G_{\text{GUT}} \times U(1)_1 \times U(1)_2\,. \tag{2.27}$$

We describe how this set-up works for the most studied simple GUT groups in the next section.

2.6 GUTs in F-theory

Here we describe the basics involved in the construction of GUTs with gauge groups $SU(5)$, $SO(10)$, and E_6 using F-theory.

2.6.1 $SU(5)$

The most commonly studied GUT is $SU(5)$ [36], this is the minimal simple group containing the Standard Model group as a subgroup. Here quarks and leptons transform in the $\mathbf{10}$ and $\overline{\mathbf{5}}$ representations, with the possibility of a right handed neutrino as a singlet. We see that these give the required Standard Model representations when we break $SU(5) \to SU(3) \times SU(2) \times U(1)$:

$$\mathbf{10} \to (\mathbf{3},\mathbf{2})_{\frac{1}{6}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{1},\mathbf{1})_{1}\,. \tag{2.28}$$

from this breaking we can see that the $\mathbf{10}$ contains the quark doublets, anti-ups and the right-handed electron. The $\overline{\mathbf{5}}$ breaks as

$$\overline{\mathbf{5}} \to (\overline{\mathbf{3}},\mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{1},\mathbf{2})_{-\frac{1}{2}}\,, \tag{2.29}$$

from which we see that the $\overline{\mathbf{5}}$ contains the anti-downs and the lepton doublet. We also need Higgs fields transforming in a $\mathbf{5}$ and a $\overline{\mathbf{5}}$. The required Yukawa couplings are then $\mathbf{10}_M \cdot \mathbf{10}_M \cdot \mathbf{5}_H$ and $\mathbf{10}_M \cdot \overline{\mathbf{5}}_M \cdot \overline{\mathbf{5}}_H$. We can break to $SU(3) \times SU(2) \times U(1)$ to see that these reproduce the MSSM couplings:

$$\begin{aligned}
\mathbf{10}_M \cdot \mathbf{10}_M \cdot \mathbf{5}_H \to &((\mathbf{3},\mathbf{2})_{\frac{1}{6}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{1},\mathbf{1})_1) \cdot ((\mathbf{3},\mathbf{2})_{\frac{1}{6}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{1},\mathbf{1})_1) \\
&\cdot ((\mathbf{3},\mathbf{1})_{-\frac{1}{3}} \oplus (\mathbf{1},\mathbf{2})_{\frac{1}{2}}) \\
\mathbf{10}_M \cdot \overline{\mathbf{5}}_M \cdot \overline{\mathbf{5}}_H \to &((\mathbf{3},\mathbf{2})_{\frac{1}{6}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{1},\mathbf{1})_1) \cdot ((\overline{\mathbf{3}},\mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{1},\mathbf{2})_{-\frac{1}{2}}) \\
&\cdot ((\overline{\mathbf{3}},\mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{1},\mathbf{2})_{-\frac{1}{2}})\,.
\end{aligned} \tag{2.30}$$

By picking out the combinations which are still invariant under $SU(3) \times SU(2) \times U(1)$, we can see which couplings these would give in the MSSM. From the $\mathbf{10}_M \cdot \mathbf{10}_M \cdot \mathbf{5}_H$ we obtain $(\mathbf{3},\mathbf{2})_{\frac{1}{6}} \cdot (\overline{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}} \cdot (\mathbf{1},\mathbf{2})_{\frac{1}{2}}$ $(Q \cdot U \cdot H_u)$, and from $\mathbf{10}_M \cdot \overline{\mathbf{5}}_M \cdot \overline{\mathbf{5}}_H$ we get the $(\mathbf{3},\mathbf{2})_{\frac{1}{6}} \cdot (\overline{\mathbf{3}},\mathbf{1})_{\frac{1}{3}} \cdot (\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$ $(Q \cdot D \cdot H_d)$ and $(\mathbf{1},\mathbf{1})_1 \cdot (\mathbf{1},\mathbf{2})_{-\frac{1}{2}} \cdot (\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$ $(E \cdot L \cdot H_d)$ operators. However, we also get operators involving the Higgs triplets which lead to proton decay; we save this discussion for Section 2.8.

In F-theory, to construct an $SU(5)$ GUT, we first need to have a codimension-1 surface in the base B_3 of the elliptic fibration above which the fiber develops an $SU(5)$ singularity. This surface corresponds to the compact dimensions of a 7-brane configuration supporting an $SU(5)$ gauge theory, and so would be a stack of five coincident

D7-branes. The required matter is found at codimension-2 loci where the singularity type enhances to $SO(10)$ and $SU(6)$ singularities. This can easily be seen by decomposing the adjoints of those groups under $SU(5) \times U(1)$ (most of the results on group breaking used in this thesis can be found in [37]), for $SU(6)$:

$$\mathbf{35} \to \mathbf{24}_0 \oplus \mathbf{1}_0 \oplus \mathbf{5}_6 \oplus \overline{\mathbf{5}}_{-6}\,, \tag{2.31}$$

and for $SO(10)$:

$$\mathbf{45} \to \mathbf{24}_0 \oplus \mathbf{1}_0 \oplus \mathbf{10}_4 \oplus \overline{\mathbf{10}}_{-4}\,. \tag{2.32}$$

To generate the Yukawa couplings, further enhancements are needed. The further enhancements are $SO(12)$ for the $\mathbf{10}_M \cdot \overline{\mathbf{5}}_M \cdot \overline{\mathbf{5}}_H$ coupling and E_6 for the $\mathbf{10}_M \cdot \mathbf{10}_M \cdot \mathbf{5}_H$, or possibly larger enhancements containing these as subgroups. To see why these would be expected to give the required couplings, and why both are required, we first note that to break E_6 and $SO(12)$ to $SU(5)$, the Higgs must take a vev in a $U(1) \times U(1)$ subgroup. If we assume symmetry is restored at a point where we want a Yukawa coupling, then the coupling must be neutral under each of these $U(1)$s. So we look at the breaking of the adjoint of these two groups to $SU(5) \times U(1) \times U(1)$. For $SO(12)$ this is

$$\mathbf{66} \to \mathbf{24}_{(0,0)} \oplus 2 \times \mathbf{1}_{(0,0)} \oplus \mathbf{10}_{(4,0)} \oplus \overline{\mathbf{10}}_{(4,0)} \oplus \mathbf{5}_{(2,-2)} \oplus \overline{\mathbf{5}}_{(-2,-2)} \oplus \mathbf{5}_{(2,2)} \oplus \overline{\mathbf{5}}_{(-2,2)}\,, \tag{2.33}$$

while for E_6 this is

$$\begin{aligned} \mathbf{78} \to & \mathbf{24}_{(0,0)} \oplus 2 \times \mathbf{1}_{(0,0)} \oplus \mathbf{10}_{(4,0)} \oplus \overline{\mathbf{10}}_{(-4,0)} \oplus \mathbf{1}_{(-5,3)} \\ & \oplus \overline{\mathbf{5}}_{(3,-3)} \oplus \mathbf{10}_{(-1,-3)} \oplus \mathbf{1}_{(5,3)} \oplus \mathbf{5}_{(-3,3)} \oplus \overline{\mathbf{10}}_{(1,3)}\,. \end{aligned} \tag{2.34}$$

We can see from looking at the $U(1)$ charges that the $\mathbf{10}_M \cdot \overline{\mathbf{5}}_M \cdot \overline{\mathbf{5}}_H$ coupling should be at the point[4] of $SO(12)$ enhancement while the $\mathbf{10}_M \cdot \mathbf{10}_M \cdot \mathbf{5}_H$ should be at the E_6 enhancement. It is possible to combine both of these enhancements into a single point, corresponding to an E_7 singularity, or one can go further and obtain the $SU(5)$ from an original E_8 singularity [38–40], which can be of use for other phenomenological considerations such as right-handed neutrinos.

[4]When we say point here we mean a point in the base B_3 of the compactification space Y_4, which still fills $\mathbb{R}^{1,3}$.

2.6.2 $SO(10)$

For an $SO(10)$ GUT [41] the quarks and leptons are arranged in a $\mathbf{16}$[5] and we also have a Higgs which is given by the $\mathbf{10}$. It can be seen that this arrangement will give the desired representations when broken to the Standard Model by first breaking to $SU(5) \times U(1)$:

$$\mathbf{16} \to \mathbf{10}_{-1} \oplus \overline{\mathbf{5}}_3 \oplus \mathbf{1}_{-5}\,. \tag{2.35}$$

Here we can see from (2.28) and (2.29) that we will get the usual Standard Model matter content upon breaking further to $SU(3) \times SU(2) \times U(1)$.

However, unlike in the case of $SU(5)$, there are multiple ways to embed the Standard Model gauge group inside $SO(10)$. We could first break to $SU(5) \times U(1)_\chi$ similar to above [42,43], but interpret this as being flipped $SU(5)$ [44]. In flipped $SU(5)$ the anti-down and anti-up quarks are swapped between the $\mathbf{10}$ and the $\overline{\mathbf{5}}$, and hypercharge is then a linear combination of the $U(1)$ in $SU(5)$, whose commutant is $SU(3) \times SU(2) \times U(1)$, and $U(1)_\chi$.

One could also choose to break via a group which is left-right symmetric. For example the Pati-Salam [45] model $SU(4) \times SU(2)_L \times SU(2)_R$.

The required Yukawa coupling is of the form $\mathbf{16} \cdot \mathbf{16} \cdot \mathbf{10}$. We can see that this is as required by first breaking to $SU(5) \times U(1)$:

$$\mathbf{16} \cdot \mathbf{16} \cdot \mathbf{10} \to (\mathbf{10}_{-1} \oplus \overline{\mathbf{5}}_3 \oplus \mathbf{1}_{-5}) \cdot (\mathbf{10}_{-1} \oplus \overline{\mathbf{5}}_3 \oplus \mathbf{1}_{-5}) \cdot (\mathbf{5}_2 \oplus \overline{\mathbf{5}}_{-2})\,. \tag{2.36}$$

We can then pick out the couplings which are invariant under $SU(5) \times U(1)$ to see that this will give the $\mathbf{10} \cdot \mathbf{10} \cdot \mathbf{5}$ and $\mathbf{10} \cdot \overline{\mathbf{5}} \cdot \overline{\mathbf{5}}$ couplings required for $SU(5)$. Therefore when broken down to $SU(3) \times SU(2) \times U(1)$, the required MSSM interaction terms will be present. It is also possible to verify that the appearance of the desired MSSM Yukawa interaction terms when $SO(10)$ is broken in other ways, will require the $\mathbf{16} \cdot \mathbf{16} \cdot \mathbf{10}$ coupling in $SO(10)$.

As with $SU(5)$, in F-theory the first ingredient is the GUT surface itself, which must be a codimension-1 surface in the base, above which the fiber exhibits a D_5 singularity. We then look at enhancement loci for the matter and couplings. We expect to find the

[5] Note that the requirement of the spinorial $\mathbf{16}$ means that the GUT group is actually not $SO(10)$, but rather its double cover Spin(10), however we follow convention and refer to it as $SO(10)$.

10 matter at a curve of $SO(12)$ enhancement and the **16** at E_6 enhancement. We see this from breaking the adjoints of these to $SO(10)$. For $SO(12)$ we have

$$\mathbf{66} \to \mathbf{45}_0 \oplus \mathbf{1}_0 \oplus \mathbf{10}_{-2} \oplus \mathbf{10}_2\,, \tag{2.37}$$

and for E_6,

$$\mathbf{78} \to \mathbf{45}_0 \oplus \mathbf{1}_0 \oplus \mathbf{16}_{-3} \oplus \overline{\mathbf{16}}_3\,. \tag{2.38}$$

We would expect a Yukawa coupling again at a point of E_7 enhancement. To see why this is expected, we simply decompose the adjoint of E_7 to $SO(10) \times U(1) \times U(1)$:

$$\begin{aligned}\mathbf{133} \to &\mathbf{45}_{(0,0)} \oplus 2 \times \mathbf{1}_{(0,0)} \oplus \mathbf{16}_{(-3,0)} \oplus \overline{\mathbf{16}}_{(3,0)} \oplus \mathbf{1}_{(4,1)} \oplus \mathbf{10}_{(2,-1)} \\ &\oplus \mathbf{16}_{(1,1)} \oplus \mathbf{1}_{(-4,-1)} \oplus \mathbf{10}_{(2,-1)} \oplus \overline{\mathbf{16}}_{(-1,-1)}\,.\end{aligned} \tag{2.39}$$

From this we can have a $\mathbf{16} \cdot \mathbf{16} \cdot \mathbf{10}$ which is neutral under both $U(1)$ charges. As with the $SU(5)$ case, one can also go higher than an E_7 enhancement and use E_8 instead [46]. Examples of $SO(10)$ model building in F-theory can be found in [47, 48].

2.6.3 E_6

In an E_6 GUT both the matter and Higgs fields lie in the same **27** representation. We see that this gives the required representation content upon breaking $E_6 \to SO(10) \times U(1)$:

$$\mathbf{27} \to \mathbf{16}_1 \oplus \mathbf{10}_{-2} \oplus \mathbf{1}_4\,. \tag{2.40}$$

As in the $SO(10)$ case, there are multiple ways of embedding the Standard Model gauge group into E_6 [49].

For F-theory, requiring the **27** means we would look for an E_7 enhancement to support the matter curve, as decomposing the adjoint of E_7 under the breaking $E_7 \to E_6 \times U(1)$ gives

$$\mathbf{133} \to \mathbf{78}_0 \oplus \mathbf{1}_0 \oplus \mathbf{27}_1 \oplus \overline{\mathbf{27}}_{-1}\,. \tag{2.41}$$

For the Yukawa coupling of $\mathbf{27} \cdot \mathbf{27} \cdot \mathbf{27}$ we use an E_8 enhancement. This is because to break $E_8 \to E_6$ we would use an $SU(3)$ vev, and so we decompose the adjoint of E_8 to $SU(3) \times E_6$ to give

$$\mathbf{248} \to (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\overline{\mathbf{3}}, \overline{\mathbf{27}})\,, \tag{2.42}$$

which we can see would allow for the required Yukawa coupling term, as this combination is also invariant under the $SU(3)$.

We can see that this is the interaction term needed by decomposing it under the breaking $E_6 \to SO(10) \times U(1)$:

$$\mathbf{27} \cdot \mathbf{27} \cdot \mathbf{27} \to (\mathbf{16}_1 \oplus \mathbf{10}_{-2} \oplus \mathbf{1}_4) \cdot (\mathbf{16}_1 \oplus \mathbf{10}_{-2} \oplus \mathbf{1}_4) \cdot (\mathbf{16}_1 \oplus \mathbf{10}_{-2} \oplus \mathbf{1}_4) \,. \tag{2.43}$$

We can see that this coupling will give the required $SO(10)$ interaction term $\mathbf{16} \cdot \mathbf{16} \cdot \mathbf{10}$, since this is also invariant under the $U(1)$.

2.7 Breaking the GUT Group

In F-theory, there are a few methods of GUT breaking available, but if we use genericity as a guiding principle, then the preferred one is turning on a non-trivial gauge flux [5] which will break the gauge group to the commutant of generators that the flux corresponds to. Specifically, for an $SU(5)$ GUT this means that a $U(1)$ flux with a hypercharge component is used [5], since the commutant of the hypercharge, $U(1)_Y$, inside $SU(5)$ is the Standard Model gauge group. It is important to use a $U(1)$ flux with a component in the commutant of $SU(5)$ and not consisting solely of $U(1)_Y$ as otherwise a mass for the $U(1)_Y$ gauge boson is generated [4, 5]. The use of non-trivial flux as the preferred GUT breaking method is something specific to F-theory, and in particular, can only be used in F-theory compactifications that do not admit a Heterotic dual.

One could instead use discrete Wilson lines, but this puts constraints on the GUT surface S_2 as it must then have a non-trivial fundamental group. Thus, this is a non-generic GUT breaking method, for example in [50] only one such surface was constructed with the right properties to allow for this method.

2.8 Proton Decay

The prediction of rapid proton decay is a problem endemic to most GUT models, particularly $SU(5)$, which suffers from the doublet-triplet splitting problem as well as proton decay mediating operators such as $\overline{\mathbf{5}}_M \cdot \overline{\mathbf{5}}_M \cdot \mathbf{10}_M$. As such it has received a lot

of attention in GUT model building [51]. In F-theory, there are numerous ways to deal with the problem causing aspects of GUT models [52].

For $SU(5)$, the presence of the $\mathbf{10}_M \cdot \overline{\mathbf{5}}_M \cdot \overline{\mathbf{5}}_H$ means that to not have a proton decay inducing $\overline{\mathbf{5}}_M \cdot \overline{\mathbf{5}}_M \cdot \mathbf{10}_M$ requires that we have a method of distinguishing the $\overline{\mathbf{5}}_M$ and the $\overline{\mathbf{5}}_H$. This can be achieved by factorising the $\overline{\mathbf{5}}$ matter curve [5,53]. In $SO(10)$ one does not have this problem since the matter and Higgs fields live in different representations, and so are always able to be treated separately.

Dimension-5 proton decay operators are related to the presence of Higgs triplets at low energies. As stated in [5], by using the hypercharge flux method of GUT breaking, their zero modes come from different cohomology groups, allowing them to be treated separately. It is important to be able to treat the doublets separately from the triplets, since we specifically need the doublets to exist at low energies, whilst giving a triplets a large mass to avoid proton decay operators.

In $SO(10)$ GUT theories, there are also proton decay mediating operators, however, these are at least dimension-6, and therefore not as problematic, since they will naturally have a large suppression.

Chapter 3

Global GUT Construction with F-theory

Here we outline a procedure for constructing a global F-theory GUT to the level of the gauge group, matter representations, Yukawa couplings and G-flux. One first chooses the GUT gauge group, and then constructs an elliptically fibered Calabi-Yau 4-fold where the fiber develops a singularity of the appropriate type over a codimension-1 locus of the base. The easiest way of doing this is by using the Tate algorithm [8, 31]. We start with the general Tate form for an elliptically fibered Calabi-Yau manifold as described in Section 2.3:

$$wy^2 + a_1wxy + a_3w^2y = x^3 + a_2wx^2 + a_4w^2x + a_6w^3 \,. \tag{3.1}$$

Here we have expressed Y_4 as a surface inside an auxiliary 5-fold X_5, which we describe as a $\mathbb{P}^2$ bundle over the base B_3. w, x and y are coordinates on the elliptic fiber, and the a_i are functions depending on the coordinates of the base. Expressing the manifold in this way makes it easier to compute the Chern classes, which we will need to ensure proper quantization of the G-flux and to check that our manifold is Calabi-Yau.

We obtain the desired singularity type by requiring the a_i to have vanishing multiplicities in the codimension-1 locus given by Table 2.2. We will denote the GUT surface by S_2 and define it by $z = 0$, where z is a holomorphic coordinate on the base, and so the entries in the Tate table are the vanishing multiplicities in z. So our defining equation will be of the form

$$wy^2 + b_1wxyz^{n_1} + b_3w^2yz^{n_3} = x^3 + b_2wx^2z^{n_2} + b_4w^2xz^{n_4} + b_6w^3z^{n_6} \,, \tag{3.2}$$

where the b_i[1] are functions on the base, which do not vanish identically at $z = 0$.

For later use we note that w, x, y and z are sections of the following bundles:

Section	Bundle
w	$\mathcal{O}(\sigma)$
x	$\mathcal{O}(\sigma + 2c_1(B_3))$
y	$\mathcal{O}(\sigma + 3c_1(B_3))$
z	$\mathcal{O}(S_2)$

(3.3)

Sections of $\mathcal{O}(nA)$ are holomorphic functions that vanish to order n on the divisor A, and nowhere else.

3.1 Matter and Interactions

Having fixed the form of our Calabi-Yau 4-fold using the Table 2.2, we look for possible higher codimension enhancements. Codimension-2 enhancements can lead to matter, with the corresponding loci known as matter curves. Codimension-3 enhancements can lead to Yukawa interactions between matter. Note that one should not use Tate's algorithm to attempt to predict the singularity type of higher codimension enhancements, as it has been demonstrated that the singularity type will not always match the one expected from Table 2.2 [12, 54]. In all cases considered here however, the only difference occurs at codimension-3 enhancements.

To see where the singularity is enhanced, we look at the discriminant of our Tate form. This will be a polynomial in z, with coefficients depending on the b_i. Enhancements of the singularity occur when the b_i satisfy conditions which make the polynomial increase in order, i.e. the lowest order coefficients vanish.

In order to see what type of matter we have at the loci of codimension-2 enhancements, and what interaction terms can be generated in codimension-3, we must first resolve the singularity, which we do by performing a series of blow-ups. We check that the original singularity is resolved in all codimensions, not just codimension-1, before looking at enhancements. After fully resolving, the original singular locus becomes reducible, and the intersection matrix of the irreducible components, called Cartan divisors, should give the negative Cartan matrix of the extended Dynkin diagram of the singularity type. We therefore expect to obtain $n + 1$ irreducible components, where n

[1] Note that these b_i are not the same as those featured in equation (2.24)

is the rank of the GUT group.

We choose to perform the resolution in the same way as [13] as opposed to [12]. We illustrate how the two methods are related and the difference with an example. Suppose we had an elliptic fibration with an E_8 singularity. The general Tate form for this is

$$wy^2 + b_1 wxyz + b_3 w^2 yz^3 = x^3 + b_2 wx^2 z^2 + b_4 w^2 xz^4 + b_6 w^3 z^5 \,. \tag{3.4}$$

We can clearly see that it is singular at $x = y = z = 0$, so the first blow-up would be at this locus, and we replace the original coordinates by

$$x = x_1 \zeta, \qquad y = y_1 \zeta, \qquad z = z_1 \zeta \,, \tag{3.5}$$

This means that the original singular locus given by $x = y = z = 0$ is now located at $\zeta = 0$. Making this replacement in the Tate form yields what is known as the total transform of the surface, this is given by

$$wy_1^2 \zeta^2 + b_1 wx_1 y_1 z_1 \zeta^3 + b_3 w^2 y_1 z_1^3 \zeta^4 = x_1^3 \zeta^3 + b_2 wx_1^2 z_1^2 \zeta^4 + b_4 w^2 x_1 z_1^4 \zeta^5 + b_6 z_1^5 \zeta^5 \,. \tag{3.6}$$

Notice that the above equation can be factorised, as each term has a ζ^2 factor. If we remove this factor, we obtain what is known as the proper transform of the surface, and since this amounts to removing a $\zeta = 0$ component from the surface, which is where the original singular locus is located, we can think of the proper transform as somehow being less singular than what we started with.

The proper transform of the surface is now

$$wy_1^2 + b_1 wx_1 y_1 z_1 \zeta + b_3 w^2 y_1 z_1^3 \zeta^2 = x_1^3 \zeta + b_2 wx_1^2 z_1^2 \zeta^2 + b_4 w^2 x_1 z_1^4 \zeta^3 + b_6 z_1^5 \zeta^3 \,. \tag{3.7}$$

We have replaced the singular locus $x = y = z = 0$ with a $\mathbb{P}^2$ at $\zeta = 0$ with projective coordinates $[x_1, y_1, z_1]$. This new fibration is still singular at $x_1 = y_1 = \zeta = 0$ and so we would perform the next blow-up at this locus, repeating until we obtain a non-singular resolved manifold. The last few blow-ups will be small resolutions where instead of introducing a $\mathbb{P}^2$, we introduce a $\mathbb{P}^1$ with each blow-up, meaning that be only blow-up in two variables, instead of three. These last few blow-ups get rid of higher codimension singularities, where the manifold would become singular if we allow certain b_i or combinations thereof to vanish.

When following [12] however, we restrict to one of three patches of the resolved manifold, given by $x_1 \neq 0$, $y_1 \neq 0$ and $z_1 \neq 0$. In one of these patches, suppose we pick $x_1 \neq 0$, we can use the fact that $[x_1, y_1, z_1]$ are the projective coordinates of a $\mathbb{P}^2$ to set $x_1 = 1$, and instead of introducing the symbol ζ we just replace it by x_1 since this is no longer being used. Since we could have chosen any of the three patches, we can think of each as just one of three possible blow-ups:

$$\begin{aligned} 1: &\qquad [x, y, z] \to [x_1, x_1 y_1, x_1 z_1]\,, \\ 2: &\qquad [x, y, z] \to [x_2 y_2, y_2, y_2 z_2]\,, \\ 3: &\qquad [x, y, z] \to [x_3 z_3, y_3 z_3, z_3]\,. \end{aligned} \tag{3.8}$$

Each of these blow-ups will yield a different total transform equation for the manifold. In order to fully resolve the manifold and to obtain all of the reducible components of the fiber, one must perform each of these three blow-ups, and possibly a further three after each, and so on, until the manifold is fully resolved, this means checking the manifold resulting from each blow-up separately. Also, some irreducible components that the fiber splits into will only be present in certain patches, making their intersection properties more difficult to obtain. The benefit of this method is that the resulting equations are easier to work with, as one does not have to keep track of which variables are coordinates of projective spaces and so cannot simultaneously vanish. However, due to the need to perform more blow-ups, and to then obtain a set of equations for the resolved manifold valid in different patches, we instead choose not to work in patches. The price we pay for the reduced number of blow-ups and equations is more variables, making our equations less elegant. We refer to [55] for the resolution of the Tate form for an $SO(10)$ singularity using the methods of [12].

In the process of resolving, we introduce new divisors E_i (one for each blow-up), and in taking the proper transform after each blow-up, will factor out variables whose vanishing section corresponds to the divisor introduced. For this reason, the class of the resolved manifold $\tilde{Y}_4$ will be of the form

$$[\tilde{Y}_4] = 3\sigma + 6c_1(B_3) - \sum n_i E_i\,, \tag{3.9}$$

with the n_i being integers.

At the codimension-2 enhancement loci, some of the previously irreducible components of the fiber can become further reducible, and, by intersecting the new irreducible components with the original Cartan divisors (excluding the one corresponding to the extended node of the Dynkin diagram), one obtains a set of Cartan charges associated with the component. The set of Cartan charges associated to a particular component can then be matched up with a representation of the original gauge group, indicating the type of matter present.

In the GUTs considered here, it is also the case that if the new set of irreducible components at codimension-2 loci are intersected with each other, one obtains the negative Cartan matrix of a higher rank group, whose adjoint representation will yield the representation of the matter present when we break this to the GUT group. This matches the description presented in Section 2.5.

At codimension-3 enhancement loci, it is possible for the fiber to become further reducible, and, in particular, if one of the $\mathbb{P}^1$s corresponding to the generation of matter at codimension-2 loci becomes homologous to the sum of two others, we may generate a Yukawa coupling here.

3.2 Local Flux

G-flux is required in order to give chirality to the matter and is therefore a necessary ingredient in a prospective GUT. In Section 4 we will construct the G-flux in two different ways: firstly in the local limit, we do this because it is easier to see what conditions the base must satisfy in order to allow a properly quantized G-flux; secondly we construct a global G-flux, this uses a brute force approach, and again this will not be properly quantized for general geometries. However if we use the same conditions derived when constructing the local flux, we can obtain a properly quantized G-flux, which furthermore matches the locally constructed one.

To obtain a valid local flux, we first use the approach introduced in [56] to construct a spectral cover. Here when we say spectral cover, we mean a hypersurface in the fibration, given by a polynomial of order n, whose coefficients are functions on the base, and so for each point on the base, our polynomial defines n points, thus the

hypersurface defined by this polynomial will project to an n-fold cover of the base manifold [32]. This approach uses the Tate divisor, which is defined as the divisor inside Y_4 given by

$$wy^2 = x^3\,, \tag{3.10}$$

which restricts, in the local limit, to the spectral cover of the Higgs bundle[2], so called because it describes the behaviour of the Higgs field over the GUT surface, as described below. To take the local limit we first define the meromorphic section $t = \frac{y}{x}$ and take $t \to 0$, $z \to 0$ while holding $s = z/t$ fixed.

The spectral cover approach means that we are thinking of our GUT group as having been obtained from the higgsing of some underlying E_8 gauge theory, and the spectral cover equation that we obtain in the above limit can be thought of as $\det(sI - \langle\Phi\rangle)$, with $\langle\Phi\rangle$ the Higgs vev. Therefore for an $SU(n)$ spectral cover we would expect a polynomial of order n, with roots corresponding to the eigenvalues of $\langle\Phi\rangle$. For $\langle\Phi\rangle$ lying in an $SU(n)$ subgroup of E_8, these eigenvalues should sum to zero, and so our spectral equation should contain no term proportional to s^{n-1}. Using Table 3.3 we see that s is a section of the canonical bundle of S_2, K_{S_2} (The canonical bundle of an n-dimensional algebraic variety is defined as the nth exterior power of the cotangent bundle).

The above procedure gives the Higgs spectral cover in the singular space Y_4. In order to obtain it in the resolved space, $\tilde{Y}_4$, we take the total transform of the Tate divisor, and also write t and z in terms of the new variables introduced in the resolution procedure to see precisely how to take the required limits in a way that keeps $s = \frac{z}{t}$ fixed. For example, suppose that the resolution had transformed x, y and z in the following way:

$$\begin{aligned} x &\quad\to\quad x_2\alpha\beta^2\gamma^3\,, \\ y &\quad\to\quad y_3\alpha\beta^2\gamma^4\,, \\ z &\quad\to\quad z_1\alpha\beta\gamma\,. \end{aligned} \tag{3.11}$$

[2]Note that this only makes sense in cases where the GUT group G is given by the commutant of a group H in E_8, where H is either of $SU(n)$ or $Sp(n)$ type. The most commonly considered GUT groups, $SU(5)$, $SO(10)$, and E_6 correspond to $H = SU(5)$, $SU(4)$ and $SU(3)$ respectively.

This would give

$$t = \frac{y_3\gamma}{x_2}\,,$$
$$s = \frac{x_2 z_1 \alpha\beta}{y_3}\,, \tag{3.12}$$

and so the desired limit is accomplished by taking $\gamma \to 0$, since this leaves s invariant as required.

With the Higgs spectral cover equation obtained, and assuming that the Higgs field lies in an $SU(n)$ subgroup, the roots of the equation are the eigenvalues, λ_i, of the Higgs vev. This is a polynomial equation in s which defines a surface inside an auxiliary 3-fold consisting of the total space of the canonical bundle over the base S_2 [57]. For a generic point on the base S_2, the set of λ_i will denote n points on the fiber, and so the local sprectal cover is an n-sheeted cover of S_2.

It is convenient to compactify this total space into its projectivization $Z_3 = \mathbb{P}(\mathcal{O} \oplus K_{S_2})$. We then introduce a new divisor σ_0 which descends from the hyperplane of the $\mathbb{P}^1$ fiber. We denote the projective coordinates of the fiber by U and V which we define to be sections of the following bundles:

Section	Bundle
U	$\mathcal{O}(\sigma_0)$
V	$\mathcal{O}(\sigma_0 + c_1(S_2))$

(3.13)

We can then replace s by $\frac{U}{V}$ and rewrite the Higgs spectral cover equation as a homogeneous polynomial in U and V. This will give an equation of the form

$$b_0 U^n + b_2 U^{n-2} V^2 + \cdots + b_n V^n = 0\,, \tag{3.14}$$

where the coefficients b_i are functions on the base S_2.

We use this equation to define the class η in S_2 by $b_0 = 0$, so the class of the spectral divisor is given by

$$\mathcal{C}_{\text{Higgs,loc}} = n\sigma_0 + \pi^*\eta = 0\,. \tag{3.15}$$

Then the projection from the bundle to the base

$$\pi : Z_3 \to S_2\,, \tag{3.16}$$

will induce a projection from the spectral cover to the base,

$$p_{sc} : \mathcal{C}_{\text{Higgs,loc}} \to S_2\,. \tag{3.17}$$

We then specify a line bundle L on the spectral cover $\mathcal{C}_{\text{Higgs,loc}}$. For this line bundle to be an $SU(n)$ bundle requires the condition

$$c_1(p_{sc*}L) = 0\,. \tag{3.18}$$

We can then use the Grothendieck-Riemann-Roch theorem [32]:

$$p_{sc*}\left(e^{c_1(L)}\text{Td}\left(\mathcal{C}_{\text{Higgs,loc}}\right)\right) = \text{ch}\left(p_{sc*}L\right)\text{Td}\left(S_2\right)\,. \tag{3.19}$$

Where, Td is the Todd class, which for any complex manifold, X, has the expansion

$$\text{Td}\left(X\right) = 1 + \frac{c_1\left(X\right)}{2} + \frac{c_2\left(X\right) + c_1\left(X\right)^2}{12} + \cdots\,, \tag{3.20}$$

ch is the Chern character, defined for vector bundles V as

$$\text{ch}\left(V\right) = \dim\left(V\right) + c_1\left(V\right) + \frac{c_1\left(V\right)^2 - 2c_2\left(V\right)}{2} + \cdots\,. \tag{3.21}$$

Expanding out (3.19), and equating the first order terms gives

$$p_{sc*}c_1\left(L\right) + \frac{1}{2}p_{sc*}c_1\left(\mathcal{C}_{\text{Higgs,loc}}\right) = c_1\left(p_{sc*}L\right) + \frac{n}{2}c_1\left(S_2\right)\,. \tag{3.22}$$

From this we can see that condition (3.18) is then equivalent to

$$p_{sc*}c_1\left(L\right) + \frac{1}{2}\left(p_{sc*}c_1\left(\mathcal{C}_{\text{Higgs,loc}}\right) - nc_1\left(S_2\right)\right) = 0\,. \tag{3.23}$$

As the $\mathcal{C}_{\text{Higgs,loc}}$ is an n-sheeted cover of S_2, we can write $nc_1\left(S_2\right)$ as

$$nc_1\left(S_2\right) = p_{sc*}p_{sc}^*c_1\left(S_2\right)\,. \tag{3.24}$$

We also have

$$c\left(\mathcal{C}_{\text{Higgs,loc}}\right) = c\left(S_2\right)\frac{\left(1+\sigma_0\right)\left(1+\sigma_0+c_1\left(S_2\right)\right)}{1+\mathcal{C}_{\text{Higgs,loc}}}\,, \tag{3.25}$$

from which we can see that

$$c_1\left(\mathcal{C}_{\text{Higgs,loc}}\right) = 2c_1\left(S_2\right) + 2\sigma_0 - \mathcal{C}_{\text{Higgs,loc}}\,. \tag{3.26}$$

This allows us to write

$$p_{sc*}c_1\left(L\right) + \frac{1}{2}p_{sc*}\left(c_1\left(S_2\right) + 2\sigma_0 - \mathcal{C}_{\text{Higgs,loc}}\right) = 0\,, \tag{3.27}$$

which is often written as

$$p_{sc*}c_1(L) - \frac{1}{2}p_{sc*}r = 0, \tag{3.28}$$

where r is the ramification divisor of the covering, with class given by

$$r = [\mathcal{C}_{\text{Higgs,loc}} - \sigma_0 - \sigma_\infty]|_{\mathcal{C}_{\text{Higgs,loc}}} \,, \tag{3.29}$$

and

$$\sigma_\infty = \sigma_0 + c_1(S_2)\,. \tag{3.30}$$

Due to this requirement, we construct $c_1(L)$ by writing it as

$$c_1(L) = \frac{1}{2}r + \gamma\,, \tag{3.31}$$

where γ is a divisor which satisfies the condition

$$p_{sc*}\gamma = 0\,. \tag{3.32}$$

This γ will in general only have one possible form [32]:

$$\gamma = \alpha(n\sigma_0 - p_{sc}^*(\eta - nc_1(S_2))|_{\mathcal{C}_{\text{Higgs,loc}}} \,, \tag{3.33}$$

with α currently an arbitrary complex constant. We can see that this will satisfy $p_{sc*}\gamma = 0$ by computing $p_{sc}\sigma_0$. We first note that since U and V are the projective coordinates of a $\mathbb{P}^1$, they cannot vanish simultaneously, and results in the relation

$$\sigma_0 \cdot_{Z_3} (\sigma_0 + c_1(S_2)) = 0\,, \tag{3.34}$$

i.e. $\sigma_0^2 = -\sigma_0 \cdot c_1(S_2)$. Since $\mathcal{C}_{\text{Higgs,loc}}$ has the class $n\sigma_0 + \eta$ in Z_3, the pushdown of σ_0 from $\mathcal{C}_{\text{Higgs,loc}}$ to S_2 is the same as the pushdown of $\sigma_0 \cdot (n\sigma_0 + \eta)$ from Z_3 to S_2. Using the relation (3.34), we see that this is the same as the pushdown of $\sigma_0 \cdot (\eta - nc_1(S_2))$ from Z_3 to S_2 which is then just $\eta - nc_1(S_2)$. Overall we have

$$\begin{aligned} p_{sc*}\gamma &= p_{sc*}\alpha(n\sigma_0 - p_{sc}^*(\eta - Nc_1(S_2)) \\ &= \alpha\left(n(\eta - Nc_1(S_2)) - n(\eta - Nc_1(S_2))\right) = 0\,. \end{aligned} \tag{3.35}$$

The requirement that $c_1(L)$ be an integer class can then be used to constrain the possible values of α. The γ constructed has been shown to correspond to the local flux, and so we expect this to match the global one, since they should both give the same chirality structure. In particular, as it is simpler to construct the local flux than it is

the global G-flux (which we construct by brute force), it gives a much quicker way of deriving the necessary geometric conditions, which will allow for a properly quantized G-flux in the case of $SO(10)$ where this is not generically possible [14]. Working out all of the possible conditions from the global G-flux only would make it difficult to obtain an exhaustive list.

3.3 Global G-flux

As previously stated, the G-flux is useful due to the chirality formula [18]:

$$\chi(\mathbf{R}) = \int_{\mathcal{S}_{\mathbf{R}}} G_4 \,, \tag{3.36}$$

where $\mathcal{S}_{\mathbf{R}}$ is the surface associated to some component of the singular locus whose Cartan charges correspond to the weight vector of matter in the representation $\mathbf{R}$.

Using this formula, it is clear that to get a non-zero chirality, we need the G-flux to integrate non-trivially over matter surfaces in $\tilde{Y}_4$, therefore the G-flux will be a (2,2)-form. Identifying this with a holomorphic surface in $\tilde{Y}_4$ simplifies the chirality integral to a simple intersection calculation. Our convention is to use G_4 to mean the $(2,2)$-form flux and G to mean the class of the holomorphic surface it corresponds to. It is the class of this holomorphic surface that we construct.

We also require the G-flux to be orthogonal to the pullbacks of all vertical and horizontal surfaces in Y_4 to $\tilde{Y}_4$ [2]. This can be implemented as

$$G \cdot_{\tilde{Y}_4} \sigma \cdot_{\tilde{Y}_4} D_1 = G \cdot_{\tilde{Y}_4} D_1 \cdot_{\tilde{Y}_4} D_2 = 0 \,, \tag{3.37}$$

where D_1 and D_2 are the pullbacks of two divisors in B_3, and σ is the section of the elliptic fibration.

Also we require that the G-flux not break the GUT group, which means that it is not allowed to intersect any of the Cartan divisors. Concretely, this means that we need

$$G \cdot_{\tilde{Y}_4} \mathcal{D}_{-\alpha_i} \cdot_{\tilde{Y}_4} D = 0 \tag{3.38}$$

where D is any divisor in the base B_3, and $\mathcal{D}_{-\alpha_i}$ is any Cartan divisor.

Finally, we have the quantization condition [27][3]

$$G + \frac{1}{2}c_2(\tilde{Y}_4) \in H^4(\tilde{Y}_4, \mathbb{Z})\,. \tag{3.39}$$

This quantization condition is similar to the condition on the local flux that $c_1(L)$ be an integral class, i.e.

$$\gamma + \frac{1}{2}r \in H^2(\mathcal{C}_{\text{Higgs,loc}}, \mathbb{Z})\,, \tag{3.40}$$

So it should be the case that the odd part of the class of the surface in $\tilde{Y}_4$, which corresponds to the ramification divisor when restricted to the spectral cover, should match the odd part of the second Chern class of $\tilde{Y}_4$.

The conditions on G greatly restrict its form. Specifically, if we expand out the conditions (3.37) using the class of the resolved manifold $\tilde{Y}_4$, we get a total intersection in the auxiliary manifold $\tilde{X}_5$. The only non-zero total intersections are of the form $\sigma^2 \cdot D_1 \cdot D_2 \cdot D_3$, with the D_i being pullbacks of divisors in the base B_3. We see that G must satisfy

$$G \cdot (3\sigma + 6c_1 - \sum n_i E_i) \cdot \sigma \cdot D_1 = G \cdot (3\sigma + 6c_1 - \sum n_i E_i) \cdot D_1 \cdot D_2 = 0 \tag{3.41}$$

which rules out G containing any terms of the form σ^2, $\sigma \cdot D_1$, and $D_1 \cdot D_2$. This will become clearer after the discussion on explicitly computing intersections in Section 4.2. We are now left with two types of surfaces from which to construct G: firstly we have terms of the form

$$E_i \cdot_{\tilde{Y}_4} D\,, \tag{3.42}$$

where D is the pullback of some divisor in B_3 (note that terms of the form $E_i \cdot_{\tilde{Y}_4} \sigma$ will automatically vanish, due to the way in which we have resolved the singular manifold). These can be seen to automatically satisfy the orthogonality conditions using the fact that any total intersection terms containing exceptional divisors with less than cubic order will vanish (see Section 4.2 for a discussion on why this is the case).

We can also have terms of the form

$$E_i \cdot_{\tilde{Y}_4} E_j\,. \tag{3.43}$$

[3]Note that here the condition is given in terms of the first Pontryagin class $p_1(\tilde{Y}_4)$ as $\frac{G}{2\pi} - \frac{p_1(\tilde{Y}_4)}{4} \in H^4(\tilde{Y}_4, \mathbb{Z})$, however, since we are now dealing with complex geometry, we opt to instead write the quantization condition in terms of Chern classes using the formula $p_1 = c_1^2 - 2c_2$ and the fact that c_1 vanishes for a Calabi-Yau manifold. We also choose to define G with an additional factor of $\frac{1}{2\pi}$ as in [58].

Thus, our G-flux will be some linear combination of these two types of surfaces, upon which we impose the intersecting conditions (3.38). We can, however, be more specific in determining which surfaces to consider in the composition of the G-flux. We note that both types of surfaces will, in general, intersect the Cartan divisors, and so, the fact that our specific linear combination does not will be down to cancellations between the two types. The components of the form (3.43) when inserted into (3.38) yield terms proportional to

$$S_2 \cdot_{B_3} D \cdot_{B_3} \tilde{D}\,, \tag{3.44}$$

where D is the divisor in B_3 used in (3.38) and $\tilde{D}$ is either S_2 or c_1. This can be seen from the fact that the Cartan divisors are components of $z = 0$ which is the surface S_2, and the fact that our original variables used in defining the singular 4-fold were sections of bundles which were linear combinations of σ, S_2 and c_1 only. Therefore the relations used to eliminate the exceptional divisors from the intersection (see Section 4.2) cannot introduce any other classes in B_3, and any σ in the intersection will disappear upon reducing to an intersection in B_3.

In order to cancel off contributions of terms of the form (3.43) to the intersection with the Cartan divisors, the terms in the G-flux of the form (3.42) need only consist of those with D being c_1 or S_2.

Our general G-flux will then be of the form

$$G = \sum_i \Big(a_i c_1 \cdot_{\tilde{Y}_4} E_i + b_i S_2 \cdot_{\tilde{Y}_4} E_i \Big) + \sum_{i \leq j} c_{ij} E_i \cdot_{\tilde{Y}_4} E_j\,. \tag{3.45}$$

We then simply inpose the quantization constraint using the second Chern class, and the constraint that our G-flux does not break $SO(10)$, i.e. it does not intersect any of the Cartan divisors.

Chapter 4

Global $SO(10)$ GUT

Here we describe the first part of the work carried out in [15]. We obtain an $SO(10)$ GUT group by compactifying F-theory on a Calabi-Yau 4-fold with a D_5 singularity. As in the previous section, we construct this 4-fold in an auxiliary 5-fold X_5 defined as a $\mathbb{P}^2$ bundle over a base B_3:

$$X_5 = \mathbb{P}(\mathcal{O} \oplus K_{B_3}^{-2} \oplus K_{B_3}^{-3})\,. \tag{4.1}$$

We choose a divisor S_2 inside the base B_3 to support the singularities, which we define by the vanishing of a holomorphic section z. We can then use the Tate table to specify the Tate form and write down the surface Y_4 inside X_5 which will have a D_5 singularity at $z = 0$:

$$y^2 w + b_1 zxyw + b_3 z^2 yw^2 = x^3 + b_2 zx^2 w + b_4 z^3 xw^2 + b_6 z^5 w^3\,. \tag{4.2}$$

The objects in the above equation are sections of the following bundles within the auxiliary space X_5:

Section	Bundle
w	$\mathcal{O}(\sigma)$
x	$\mathcal{O}(\sigma + 2c_1)$
y	$\mathcal{O}(\sigma + 3c_1)$
z	$\mathcal{O}(S_2)$
b_1	$\mathcal{O}(c_1 - S_2)$
b_2	$\mathcal{O}(2c_1 - S_2)$
b_3	$\mathcal{O}(3c_1 - 2S_2)$
b_4	$\mathcal{O}(4c_1 - 3S_2)$
b_6	$\mathcal{O}(6c_1 - 5S_2)$

(4.3)

Where we use c_1 to mean $c_1(B_3)$. From its defining equation, we can then see that the class of Y_4 is

$$[Y_4] = 3\sigma + 6c_1\,. \tag{4.4}$$

Using this, we can work out its total Chern class, using

$$c(Y_4) = \frac{c(X_5)}{1+3\sigma+6c_1} = \frac{c(B_3)(1+\sigma)(1+\sigma+2c_1)(1+\sigma+3c_1)}{1+3\sigma+6c_1}\,. \tag{4.5}$$

Expanding this to first order, we see that the first Chern class of Y_4 vanishes, verifying that it is indeed a Calabi-Yau manifold, as required for the preservation of $\mathcal{N}=1$ supersymmetry in 4 dimensions.

4.1 Resolution of the $SO(10)$ Singularity

The locus

$$x = y = z = 0 \tag{4.6}$$

is singular. To blow-up along it, we introduce a $\mathbb{P}^2$ at the singular locus, by replacing the original coordinates with

$$x = \zeta x_1\,, \qquad y = \zeta y_1\,, \qquad z = \zeta z_1\,, \tag{4.7}$$

where $\zeta = 0$ gives rise to an exceptional divisor E_1, and $[x_1, y_1, z_1]$ are projective coordinates of a $\mathbb{P}^2$. The new classes of the sections are then

Section	Bundle
x_1	$\mathcal{O}(\sigma+2c_1-E_1)$
y_1	$\mathcal{O}(\sigma+3c_1-E_1)$
z_1	$\mathcal{O}(S_2-E_1)$
ζ	$\mathcal{O}(E_1)$

(4.8)

After a proper transform, the equation for Y_4 becomes

$$wy_1^2 + b_1 w x_1 y_1 z_1 \zeta + b_3 w^2 y_1 z_1^2 \zeta = x_1^3\zeta + b_2 w x_1^2 z_1 \zeta + b_4 w^2 x_1 z_1^3 \zeta^2 + b_6 w^3 z_1^5 \zeta^3 \tag{4.9}$$

The second blow-up is along $x_1 = y_1 = \zeta = 0$, which is obtained by setting

$$x_1 = x_2\alpha\,, \qquad y_1 = y_2\alpha\,, \qquad \zeta = \zeta_2\alpha\,. \tag{4.10}$$

The section $\alpha = 0$ gives rise to an exceptional divisor E_2, and the projective coordinates of the $\mathbb{P}^2$ introduced are $[y_2, x_2, \zeta_2]$. The new sections are

Section	Bundle
x_2	$\mathcal{O}(\sigma+2c_1-E_1-E_2)$
y_2	$\mathcal{O}(\sigma+3c_1-E_1-E_2)$
ζ_2	$\mathcal{O}(E_1-E_2)$
α	$\mathcal{O}(E_2)$

(4.11)

The proper transform of the equation defining the 4-fold is

$$wy_2^2 + b_1 w x_2 y_2 z_1 \zeta_2 \alpha + b_3 w^2 y_2 z_1^2 \zeta_2 = x_2^3 \zeta_2 \alpha^2 + b_2 w x_2^2 z_1 \zeta_2 \alpha + b_4 w^2 x_2 z_1^3 \zeta_2^2 \alpha + b_6 w^3 z_1^5 \zeta_2^3 \alpha \tag{4.12}$$

We can then blow-up along $y_2 = \zeta_2 = \alpha = 0$, which we do by setting

$$y_2 = y_3 \beta \,, \qquad \zeta_2 = \zeta_3 \beta \,, \qquad \alpha = \alpha_3 \beta \,. \tag{4.13}$$

The section $\beta = 0$ gives rise to a new exceptional divisor E_3, $[y_3, \zeta_3, \alpha_3]$ are projective coordinates of a $\mathbb{P}^2$. So the new sections are

Section	Bundle
y_3	$\mathcal{O}(\sigma + 3c_1 - E_1 - E_2 - E_3)$
ζ_3	$\mathcal{O}(E_1 - E_2 - E_3)$
α_3	$\mathcal{O}(E_2 - E_3)$
β	$\mathcal{O}(E_3)$

(4.14)

The proper transform of the equation for Y_4 is

$$\begin{aligned} wy_3^2 + b_1 w x_2 y_3 z_1 \zeta_3 \alpha_3 \beta + b_3 w^2 y_3 z_1^2 \zeta_3 = {} & x_2^3 \zeta_3 \alpha_3^2 \beta + b_2 w x_2^2 z_1 \zeta_3 \alpha_3 \\ & + b_4 w^2 x_2 z_1^3 \zeta_3^2 \alpha_3 \beta + b_6 w^3 z_1^5 \zeta_3^3 \alpha_3 \beta^2 \,. \end{aligned} \tag{4.15}$$

At this point, if we assume that all of the b_i are non-zero everywhere, we can see that there are no singularities, so this is smooth in codimension-1.

Up until this point we have had no choice along which loci we performed the blow-ups. However with the final two blow-ups there are multiple possibilities. As described in [12], these are expected to be related by flop transitions. We do not consider the other possibilities though, because as stated in the introduction to [59], in the F-theory limit of vanishing fiber volume, the resolved $\tilde{Y}_4$ and the singular Y_4 are indistinguishable, one can therefore see that the different possibilities for carrying out the resolution will lead to equivalent physics.

The last two resolutions are carried out as in [60]. We choose this method as opposed to a possibility more similar to [13] and [12], in order to end up with only one equation defining the resolved 4-fold $\tilde{Y}_4$, which then makes our calculations easier.

The first small resolution is along $y_3 = \zeta_3 = 0$ and we do this by setting

$$y_3 = y_4 \delta_4 \,, \qquad \zeta_3 = \zeta_4 \delta_4 \,. \tag{4.16}$$

The proper transform is

$$wy_4^2\delta_4 + b_1wx_2y_4z_1\zeta_4\alpha_3\beta\delta_4 + b_3w^2y_4z_1\zeta_4\delta_4 = x_2^3\zeta_4\alpha_3^2\beta + b_2wx_2^2z_1\zeta_4\alpha_3 + b_4w^2x_2z_1^3\zeta_4^2\alpha_3\beta\delta_4 + b_6w^3z_1^5\zeta_4^3\alpha_3\beta^2\delta_4^2\,. \tag{4.17}$$

The second small resolution is done at $y_4 = \alpha_3 = 0$ and is given by

$$y_4 = y_5\delta_5\,, \qquad \alpha_3 = \alpha_5\delta_5\,, \tag{4.18}$$

giving a proper transform

$$wy_5^2\delta_4\delta_5 + b_1wx_2y_5z_1\zeta_4\alpha_5\beta\delta_4\delta_5 + b_3w^2y_5z_1\zeta_4\delta_4 = x_2^3\zeta_4\alpha_5^2\beta\delta_5 + b_2wx_2^2z_1\zeta_4\alpha_5 + b_4w^2x_2z_1^3\zeta_4^2\alpha_5\beta\delta_4 + b_6w^3z_1^5\zeta_4^3\alpha_5\beta^2\delta_4^2\,. \tag{4.19}$$

The sections $\delta_4 = 0$ and $\delta_5 = 0$ give rise to new divisors E_4 and E_5 respectively, and we now have two $\mathbb{P}^1$s with projective coordinates $[y_4, \zeta_4] = [y_5\delta_5, \zeta_4]$ and $[y_5, \alpha_5]$. The sections are now

Section	Bundle
y_5	$\mathcal{O}(\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4 - E_5)$
ζ_4	$\mathcal{O}(E_1 - E_2 - E_3 - E_4)$
α_5	$\mathcal{O}(E_2 - E_3 - E_5)$
δ_4	$\mathcal{O}(E_4)$
δ_5	$\mathcal{O}(E_5)$
w	$\mathcal{O}(\sigma)$
x_2	$\mathcal{O}(\sigma + 2c_1 - E_1 - E_2)$
z_1	$\mathcal{O}(S_2 - E_1)$
β	$\mathcal{O}(E_3)$

(4.20)

The 4-fold is now completely resolved and has class

$$[\tilde{Y}_4] = 6c_1 + 3\sigma - 2E_1 - 2E_2 - 2E_3 - E_4 - E_5\,. \tag{4.21}$$

4.2 Cartan Divisors

The section $z = 0$, where the D_5 singularity is located, splits after the blow-ups as

$$z = z_1\zeta_4\alpha_5\beta^2\delta_4\delta_5 = 0\,. \tag{4.22}$$

Note that the component $\delta_4 = 0$ is reducible, with one component given by $\zeta_4 = 0$. To see this, we plug $\zeta_4 = 0$ the equation for the resolved $\tilde{Y}_4$ to obtain

$$wy_5^2\delta_4\delta_5 = 0\,. \tag{4.23}$$

Now, in our original $\mathbb{P}^2$ fiber, the projective coordinates are $[w, x, y]$, which after the five blow-ups become $[w, x_2\zeta_4\alpha_5^2\beta^3\delta_4\delta_5^2, y_5\zeta_4\alpha_5^2\beta^4\delta_4^2\delta_5^3]$. Since these cannot all vanish simultaneously, we can see that setting $\zeta_4 = 0$ requires that we have $w \neq 0$. Similarly, we can also see from the fourth blow-up that the set of coordinates $[y_4, \zeta_4]$ may also not simultaneously vanish. Performing the final blow-up makes this $[y_5\delta_5, \zeta_4]$, which means that because we have set $\zeta_4 = 0$, we must also have $y_5 \neq 0$ and $\delta_5 \neq 0$. Putting these conditions into (4.23) simply leaves $\delta_4 = 0$. So we have that $\zeta_4 = 0$ implies $\delta_4 = 0$ inside $\tilde{Y}_4$, but if we first set $\delta_4 = 0$, this does not imply that $\zeta_4 = 0$. From this we can conclude that $\zeta_4 = 0$ is a component of $\delta_4 = 0$.

The Cartan divisors are these six factors restricted to the resolved 4-fold $\tilde{Y}_4$, and are given by

Cartan Divisor	Component	Class in Y_4	multiplicity
$\mathcal{D}_{-\alpha_0}$	$(z_1 = 0)\vert_{Y_4}$	$S_2 - E_1$	1
$\mathcal{D}_{-\alpha_1}$	$(\delta_4 = 0)\vert_{Y_4,\zeta_4\neq 0}$	$-E_1 + E_2 + E_3 + 2E_4$	1
$\mathcal{D}_{-\alpha_2}$	$(\zeta_4 = 0)\vert_{Y_4}$	$E_1 - E_2 - E_3 - E_4$	2
$\mathcal{D}_{-\alpha_3}$	$(\beta = 0)\vert_{Y_4}$	E_3	2
$\mathcal{D}_{-\alpha_4}$	$(\delta_5 = 0)\vert_{Y_4}$	E_5	1
$\mathcal{D}_{-\alpha_5}$	$(\alpha_5 = 0)\vert_{Y_4}$	$(E_2 - E_3 - E_5)$	1

(4.24)

In order to demonstrate that we indeed have a resolved a D_5 singularity, we can perform the intersection of the set of Cartan divisors with the set of dual curves. As described in [13], the dual curves Σ_{α_i} are defined by intersecting the Cartan divisors with $D_1 \cdot D_2$ inside B_3 where D_1 and D_2 are any two divisors in B_3 such that

$$D_1 \cdot_{B_3} D_2 \cdot_{B_3} S_2 = 1\,. \tag{4.25}$$

To explicitly compute each intersection, we use the fact that in the original space X_5 the only non-zero complete intersections are of the form $\sigma^2 \cdot D_1 \cdot D_2 \cdot D_3$, where D_i are divisors in B_3. In blowing up the space X_5 we have introduced new exceptional divisors, if we blow-up along a singular locus of codimension-d, then this variety would not intersect any divisor with codimension greater than $5 - d$, and so the exceptional divisor will have zero intersection with the total transform of any variety of the original space with codimension greater that $5 - d$. This greatly simplifies our calculations.

Each blow-up performed also introduced a set of variables that cannot simultaneously vanish, giving a vanishing intersection relation involving the exceptional divisor.

In particular, if the blown up variety was of dimension d then the vanishing intersection relation can be expanded out to give a polynomial of order d in the exceptional divisor. This means that any power of the exceptional divisor greater than or equal to d can be written in terms of powers less than d. But we know that the complete intersection of any power of the exceptional divisor less than d will have to involve varieties with codimension greater than $5-d$, and so will vanish. So we can eliminate all exceptional divisors from our intersection computation by first writing any powers greater than or equal to d in terms of powers less than d, and then eliminating all remaining powers. It should be noted that this is only consistent if we eliminate the exceptional divisors in reverse order of where they appeared in the blow-up procedure. We illustrate this with an example of an intersection calculation.

To compute the intersection number of the Cartan divisor $\mathcal{D}_{-\alpha_5}$ with the dual curve Σ_{α_3}, we need to compute

$$(E_2 - E_3 - E_5) \cdot_{\tilde{Y}_4} E_3 \cdot_{\tilde{Y}_4} D_1 \cdot_{\tilde{Y}_4} D_2 \,. \tag{4.26}$$

We make this a complete intersection in $\tilde{X}_5$ by including the class of the resolved $\tilde{Y}_4$:

$$(3\sigma + 6c_1 - 2E_1 - 2E_2 - 2E_3 - E_4 - E_5) \cdot (E_2 - E_3 - E_5) \cdot E_3 \cdot D_1 \cdot D_2 \,. \tag{4.27}$$

We now wish to eliminate each of the exceptional divisors E_i in turn, starting with E_5 as we need to do it in reverse order. We can expand the above expression to obtain

$$\begin{aligned} &(3\sigma + 6c_1 - 2E_1 - 2E_2 - 2E_3 - E_4) \cdot (E_2 - E_3) \cdot E_3 \cdot D_1 \cdot D_2 \\ -E_5 \cdot &(3\sigma + 6c_1 - 2E_1 - E_2 - 3E_3 - E_4) \cdot E_3 \cdot D_1 \cdot D_2 + E_5^2 \cdot E_3 \cdot D_1 \cdot D_2 \,. \end{aligned} \tag{4.28}$$

We can see from the above reasoning that the term that is linear in E_5 will vanish, and we can use the last line of A.1, to rewrite E_5^2 in terms of lower powers. After again using the fact that any terms linear in E_5 will vanish, this gives

$$\begin{aligned} &(3\sigma + 6c_1 - 2E_1 - 2E_2 - 2E_3 - E_4) \cdot (E_2 - E_3) \cdot E_3 \cdot D_1 \cdot D_2 \\ &-(\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4) \cdot (E_2 - E_3) \cdot E_3 \cdot D_1 \cdot D_2 \,. \end{aligned} \tag{4.29}$$

We now move to E_4, which is easy as there are no quadradic terms, and so we can just use the fact that linear terms will vanish to remove them. After then expanding in E_3

and setting to zero all quadratic and linear terms, we are simply left with

$$E_3^3 \cdot D_1 \cdot D_2 \,. \tag{4.30}$$

Using again the relations in the appendix to eliminate successively E_3, E_2 and E_1, we obtain

$$(\sigma + 2c_1) \cdot (\sigma + 3c_1) \cdot S_2 \cdot D_1 \cdot D_2 \,, \tag{4.31}$$

but we know that all terms will vanish except

$$\sigma^2 \cdot S_2 \cdot D_1 \cdot D_2 = S_2 \cdot_{B_3} D_1 \cdot D_2 = 1 \,. \tag{4.32}$$

Repeating this process, the intersections of the Cartan divisors $\{\mathcal{D}_{-\alpha_0}, \mathcal{D}_{-\alpha_1}, \mathcal{D}_{-\alpha_2}, \mathcal{D}_{-\alpha_3}, \mathcal{D}_{-\alpha_4}, \mathcal{D}_{-\alpha_5}\}$ with the dual curves $\{\Sigma_{\alpha_0}, \Sigma_{\alpha_1}, \Sigma_{\alpha_2}, \Sigma_{\alpha_3}, \Sigma_{\alpha_4}, \Sigma_{\alpha_5}\}$ are

$$\begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \end{pmatrix} . \tag{4.33}$$

Which we recognise as the negative Cartan matrix of D_5. We can also translate this intersection matrix into the extended Dynkin diagram of D_5:

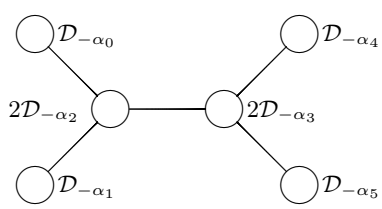

Figure 4.1: D_5 Dynkin diagram, we can see that the multiplicities of the Cartan divisors match what we would expect from a D_5 singularity.

4.3 Matter and Yukawas

The discriminant of the Tate form for the $SO(10)$ singularity has an expansion

$$\begin{aligned} \Delta = &- 16 b_2^3 b_3^2 z^7 \\ &+ \left(-27 b_3^4 - 8 b_1^2 b_2^2 b_3^2 + 72 b_2 b_4 b_3^2 + 4 b_1 b_2 \left(9 b_3^2 + 4 b_2 b_4\right) b_3 + 16 b_2^2 \left(b_4^2 - 4 b_2 b_6\right)\right) z^8 + O\left(z^9\right) \,. \end{aligned} \tag{4.34}$$

From this we expect the following enhancements.

Codimension-2:

$$b_3 = 0\,, \qquad b_2 = 0\,. \tag{4.35}$$

Codimension-3:

$$b_2 = b_3 = 0\,, \qquad b_3 = b_4^2 - 4b_2b_6 = 0\,. \tag{4.36}$$

4.3.1 16 Matter

We expect to get matter in the **16** of $SO(10)$ along $z = b_2 = 0$, as previously, this would naively have been expected to correspond to an E_6 enhancement, and as we saw in Section 2.6.2, the adjoint of E_6 yields the **16** when broken to $SO(10)$. To see why this would be expected to give an E_6 enhancement, we look Table 2.2 and see that the E_6 has the same vanishing multiplicities as $SO(10)$, except for the a_2 term, which for E_6 is one higher. Since here we have $a_2 = b_2 z$, one might expect that setting $b_2 = 0$ would increase the vanishing multiplicity of the a_2 term and hence yield an E_6 singularity.

We now look at the components of the fiber at $z = 0$ to see which will split at the enhancement locus $b_2 = 0$. Plugging each of the Cartan divisor equations into the equation for $\tilde{Y}_4$ gives

Cartan Divisor	Component	Equation in $\tilde{Y}_4$
$\mathcal{D}_{-\alpha_0}$	$(z_1 = 0)\|_{Y_4}$	$w\delta_4 - x_2^3\zeta_4 = 0$
$\mathcal{D}_{-\alpha_1}$	$(\delta_4 = 0)\|_{Y_4,\zeta_4\neq 0}$	$\zeta_4\alpha_5^2\beta + b_2 z_1 = 0$
$\mathcal{D}_{-\alpha_2}$	$(\zeta_4 = 0)\|_{Y_4}$	$\delta_4 = 0$
$\mathcal{D}_{-\alpha_3}$	$(\beta = 0)\|_{Y_4}$	$y_5^2\delta_4\delta_5 + b_3y_5\zeta_4\delta_4 - b_2\zeta_4\alpha_5 = 0$
$\mathcal{D}_{-\alpha_4}$	$(\delta_5 = 0)\|_{Y_4}$	$b_3y_5 - b_2x_2^2\alpha_5 = 0$
$\mathcal{D}_{-\alpha_5}$	$(\alpha_5 = 0)\|_{Y_4}$	$\delta_5 + b_3 = 0\,.$

(4.37)

Here in writing the equation in $\tilde{Y}_4$ that each Cartan divisor gives, we have set all variables equal to 1 that cannot vanish simultaneously with the one defining the Cartan divisor. The pairs of variables that cannot simultaneously vanish are all listed in the appendix, and can be obtained using the same techniques that were used to demonstrate that $\zeta_4 = 0$ implies $\delta_4 = 0$.

To see the relevant root splitting, we look at one specific component of $z = 0$, namely $\beta = 0$. We can see that $\beta = b_2 = 0$ gives

$$y_5\delta_4(y_5\delta_5 + b_3\zeta_4) = 0\,, \tag{4.38}$$

so it reduces to three components:

$$[\beta]\cdot[b_2] = [\beta]\cdot[y_5] + [\beta]\cdot([\delta_4]-[\zeta_4]) + [\beta]\cdot([b_2]-[y_5]-[\delta_4]+[\zeta_4]) \tag{4.39}$$

The second component is specifically $[\beta]\cdot([\delta_4]-[\zeta_4])$ since the first Cartan divisor restricted to $\beta = 0$ gives $b_2 = 0$, whereas if we just take $\beta = \delta_4 = 0$, this would not automatically imply that $b_2 = 0$, and so is not a component of $\beta = b_2 = 0$. The other components of $z = 0$ do not undergo any splitting. So $z = 0$ splits into 7 irreducible components along $b_2 = 0$, these are

Component of $(z = b_2 = 0)\|_{\tilde{Y}_4}$	Equations in $\tilde{Y}_4$	Cartan charges	Multiplicity
$(S_2 - E_1)\cdot(2c_1 - S_2)$	$z_1 = 0$ $b_2 = 0$	$(0,1,0,0,0)$	1
$(-E_1 + E_2 + E_3 + 2E_4)\cdot(2c_1 - S_2)$	$\delta_4 = 0\|_{\zeta_4\neq 0}$ $b_2 = 0$	$(-2,1,0,0,0)$	3
$(E_1 - E_2 - E_3 - E_4)\cdot(2c_1 - S_2)$	$\zeta_4 = 0$ $b_2 = 0$	$(1,-2,1,0,0)$	2
$(E_3)\cdot(\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4 - E_5)$	$\beta = 0$ $y_5 = 0$	$(1,0,-1,1,0)$	2
$(E_3)\cdot(-\sigma - c_1 - S_2 + 2E_1 - E_4 + E_5)$	$\beta = 0$ $b_2 = 0\|_{y_5,\delta_4\neq 0}$	$(1,0,-1,0,1)$	2
$(E_5)\cdot(2c_1 - S_2)$	$\delta_5 = 0$ $b_2 = 0$	$(0,0,1,-2,0)$	1
$(E_2 - E_3 - E_5)\cdot(2c_1 - S_2)$	$\alpha_5 = 0$ $b_2 = 0$	$(0,0,1,0,-2)$	1

(4.40)

The Cartan charges listed above are computed by intersecting the component with some divisor on the base B_3 which intersects the matter curve exactly once, we then intersect this with each of the Cartan divisors inside $\tilde{Y}_4$. Explicitly, for some component C of $z = b_2 = 0$, its ith Cartan charge is given by

$$C \cdot_{\tilde{Y}_4} \mathcal{D}_{-\alpha_i} \cdot_{\tilde{Y}_4} D\,, \tag{4.41}$$

where D is some divisor in B_3 such that

$$D \cdot_{B_3} [b_2] \cdot_{B_3} S_2 = 1\,. \tag{4.42}$$

As an example we show the calculation of the first Cartan charge of the component given by $\beta = y_5 = 0$, whose class is $(E_3)\cdot(\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4 - E_5)$. We use the same techniques as in the previous section; we systematically eliminate the

exceptional divisors in reverse order with the aim of bringing the total intersection to a form where we can use equation (4.42) to get a numerical answer. We start by intersecting with the first Cartan divisor, the class of $\tilde{Y}_4$ and some surface D satisfying (4.42), this is now a total intersection in X_5, given by

$$\begin{aligned}(E_3)\cdot(\sigma+3c_1-E_1-E_2-E_3-E_4-E_5)\cdot(-E1+E_2+E_3+2E_4)\\ \cdot(3\sigma+6c_1-2E_1-2E_2-2E_3-E_4-E_5)\cdot D\,.\end{aligned} \tag{4.43}$$

We firstly expand in E_5, dropping the linear term, to give

$$\begin{aligned}&(E_3)\cdot(\sigma+3c_1-E_1-E_2-E_3-E_4)\cdot(-E_1+E_2+E_3+2E_4)\\ &\cdot(3\sigma+6c_1-2E_1-2E_2-2E_3-E_4)\cdot D\\ &+E_5^2\cdot E_3\cdot(-E1+E_2+E_3+2E_4)\cdot D\,.\end{aligned} \tag{4.44}$$

We can then use the relation in Appendix A to write E_5^2 in terms of other divisors, to give

$$\begin{aligned}(E_3)\cdot(\sigma+3c_1-E_1-E_2-E_3-E_4)\cdot(-E_1+E_2+E_3+2E_4)\\ \cdot(3\sigma+6c_1-2E_1-3E_2-E_3-E_4)\cdot D\,.\end{aligned} \tag{4.45}$$

Expanding in E_4, and again dropping the linear terms, we get

$$\begin{aligned}&2E_3\cdot E_4^3\cdot D+E_3\cdot E_4^2\cdot(-8\sigma-18c_1+5E_1+9E_2+5E_3)\cdot D\\ &+E_3\cdot(\sigma+3c_1-E_1-E_2-E_3)\cdot(-E_1+E_2+E_3)\cdot(3\sigma+6c_1-2E_1-3E_2-E_3)\cdot D\,.\end{aligned} \tag{4.46}$$

To get rid of the E_4^3 term, we first write it as $E_4\cdot E_4^2$, and then use the relations in the appendix to rewrite the E_4^2 term, and then only keep the order 2 part of the result, which we can then combine with the next term in the expansion before using the same relation to eliminate E_4:

$$2E_3\cdot E_4^3\cdot D=2E_3\cdot E_4^2\cdot(\sigma+3c_1-2E_2-2E_3)\cdot D \tag{4.47}$$

combining this with the other terms gives

$$\begin{aligned}&E_3\cdot E_4^2\cdot(-6\sigma-12c_1+5E_1+5E_2+E_3)\cdot D\\ &+E_3\cdot(\sigma+3c_1-E_1-E_2-E_3)\cdot(-E_1+E_2+E_3)\cdot(3\sigma+6c_1-2E_1-3E_2-E_3)\cdot D\,.\end{aligned} \tag{4.48}$$

After using the relation again to eliminate E_4 and expanding in E_3 we obtain

$$E_3^3 \cdot (3\sigma + 6c_1 - 3E_1 - 2E_2) \cdot D\,, \tag{4.49}$$

and it is simple to use again the relations from the appendix to eliminate E_3 to give

$$\begin{aligned} &(\sigma + 3c_1 - E_1 - E_2) \cdot (E_1 - E_2) \cdot E_2 \cdot (3\sigma + 6c_1 - 3E_1 - 2E_2) \cdot D \\ =& -2E_2^4 \cdot D + E_2^3 \cdot (5\sigma + 12c_1 - 3E_1) \cdot D\,. \end{aligned} \tag{4.50}$$

The E_2^4 term is treated similarly to the E_4^3 term above, we write it as $E_2 \cdot E_2^3$ and use the relations from the appendix on the E_2^3 part:

$$E_2^4 \cdot D = E_2^3 \cdot (2\sigma + 5c_1 - E_1) \cdot D\,, \tag{4.51}$$

we plug this into the intersection expression above, which leads to

$$\begin{aligned} &E_2^3 \cdot (\sigma + 2c_1 - E_1) \cdot D \\ =&(\sigma + 2c_1 - E_1) \cdot (\sigma + 3c_1 - E_1) \cdot E_1 \cdot (\sigma + 2c_1 - E_1) \cdot D \\ =& -E_1^4 \cdot D + E_1^3 \cdot (3\sigma + 7c_1) \cdot D\,. \end{aligned} \tag{4.52}$$

Repeating the same procedure as was done for the E_2^4 and E_2^3 terms to the E_1^4 and E_1^3 terms, and then removing the terms linear and quadratic in E_1, we are finally left with an intersection expression free of exceptional divisors:

$$(\sigma + 2c_1) \cdot (\sigma + 3c_1) \cdot S_2 \cdot (\sigma + 2c_1 - S_2) \cdot D\,, \tag{4.53}$$

we can then use the intersection relation coming from the fact that $[w, x, y]$ are the projective coordinates of a $\mathbb{P}^2$ to rewrite the σ^3 term in terms of σ^2, and all other terms with smaller powers of σ will vanish as they involve the intersection of more than 3 divisors in the 3-dimensional base B_3. This leaves us with

$$\sigma^2 \cdot S_2 \cdot (2c_1 - S_2) \cdot D = S_2 \cdot_{B_3} (2c_1 - S_2) \cdot_{B_3} D = S_2 \cdot_{B_3} [b_2] \cdot_{B_3} D = 1\,. \tag{4.54}$$

All other Cartan charges can be calculated similarly, although it can be easily seen that those of the Cartan divisors which do not split further will be unchanged.

The splitting of the weight associated to the third root is

$$-\alpha_3 = (0, 1, -2, 1, 1) \to (-2, 1, 0, 0, 0) + (1, 0, -1, 1, 0) + (1, 0, -1, 0, 1)\,. \tag{4.55}$$

we can use the tables in Appendix B to see that the latter two components correspond to $-(\mu_{\mathbf{16}} - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5)$ and $\mu_{\mathbf{16}} - \alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$, which confirms the matter in the **16** representation.

4.3.2 10 Matter

We expect to get matter in the **10** along $z = b_3 = 0$, as this is where one would normally see a D_6 enhancement.

To see the relevant root splitting for the matter in the vector representation, we look at another specific component of $z = 0$, namely $\delta_5 = 0$. We can see that the Cartan divisor $\delta_5 = 0$ splits here to

$$\alpha_5(b_2 x_2^2 + b_4 x_2 \beta + b_6 \beta^2) \tag{4.56}$$

where we have set equal to 1 any variables which cannot vanish simultaneously with δ_5. So this has split into three components, the first is just another Cartan divisor restricted to $b_3 = 0$. The expression in brackets is factorised if we assume $b_2 \neq 0$. We do this as $b_2 = b_3 = 0$ corresponds to a Yukawa coupling which we consider in the next section. We call the 2 factors γ_+ and γ_- and they have the same homology class. Overall, $\delta_5 = 0$ reduces as

$$[\delta_5] \cdot [b_3] = [\delta_5] \cdot [\alpha_5] + 2 \times \frac{[\delta_5] \cdot ([b_2] - [\alpha_5])}{2} \,. \tag{4.57}$$

So we see that $z = b_3 = 0$ splits into 7 components altogether, as one would expect from a "D_6" enhancement. The seven components are

Component of $(z = b_3 = 0)\|_{\tilde{Y}_4}$	Equations in $\tilde{Y}_4$	Cartan charges	Multiplicity
$(S_2 - E_1) \cdot (3c_1 - 2S_2)$	$z_1 = 0$ $b_2 = 0$	$(0, 1, 0, 0, 0)$	1
$(-E_1 + E_2 + E_3 + 2E_4) \cdot (3c_1 - 2S_2)$	$\delta_4 = 0\|_{\zeta_4 \neq 0}$ $b_2 = 0$	$(-2, 1, 0, 0, 0)$	1
$(E_1 - E_2 - E_3 - E_4) \cdot (3c_1 - 2S_2)$	$\zeta_4 = 0$ $b_2 = 0$	$(1, -2, 1, 0, 0)$	2
$(E_3) \cdot (3c_1 - 2S_2)$	$\beta = 0$ $b_3 = 0$	$(0, 1, -2, 1, 1)$	2
$\frac{1}{2}(E_5) \cdot (3c_1 - 2S_2 - E_2 + E_3 + E_5)$	$\delta_5 = 0$ $\gamma_+ = 0$	$(0, 0, 0, -1, 1)$	1
$\frac{1}{2}(E_5) \cdot (3c_1 - 2S_2 - E_2 + E_3 + E_5)$	$\delta_5 = 0$ $\gamma_- = 0$	$(0, 0, 0, -1, 1)$	1
$(E_2 - E_3 - E_5) \cdot (3c_1 - 2S_2)$	$\alpha_5 = 0$ $b_2 = 0$	$(0, 0, 1, 0, -2)$	2

(4.58)

The $\delta_5 = 0$ root splits as

$$(0, 0, 1, -2, 0) \to (0, 0, 1, 0, -2) + (0, 0, 0, -1, 1) + (0, 0, 0, -1, 1)\ . \tag{4.59}$$

The first component is a Cartan divisor, but using Appendix B, we see that the other two are both given by

$$\mu_{\mathbf{10}} - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4\,. \tag{4.60}$$

So indeed this corresponds to matter in the **10**.

4.3.3 Yukawa Coupling

We expect to get a Yukawa interaction at the point which corresponds to an "E_7" enhancement which is given by $b_2 = b_3 = 0$. We could think of this as a further enhancement of the "E_6" curve, therefore instead of looking at how the six components of $z = 0$ split, we study how the seven components of $z = b_2 = 0$ split.

Firstly we can see from above that the third component of $\beta = b_2 = 0$ will split further to

$$y_5\delta_5 = 0\,. \tag{4.61}$$

Also, we have $\delta_5 = b_2 = b_3 = 0$, which gives

$$\alpha_5\beta(b_4x_2 + b_6\beta) = 0\,. \tag{4.62}$$

We already had the first two components, but the last is new. So altogether we see that $z = b_2 = b_3 = 0$ has 7 components, given by

Component of $(z = b_2 = b_3 = 0)\|_{\tilde{Y}_4}$	Equations in $\tilde{Y}_4$	Cartan charges
$(S_2 - E_1) \cdot (2c_1 - S_2) \cdot (3c_1 - 2S_2)$	$z_1 = 0$ $b_2 = 0$ $b_3 = 0$	$(0, 1, 0, 0, 0)$
$(-E_1 + E_2 + E_3 + 2E_4) \cdot (2c_1 - S_2) \cdot (3c_1 - 2S_2)$	$\delta_4 = 0\|_{\zeta_4 \neq 0}$ $b_2 = 0$ $b_3 = 0$	$(-2, 1, 0, 0, 0)$
$(E_1 - E_2 - E_3 - E_4) \cdot (2c_1 - S_2) \cdot (3c_1 - 2S_2)$	$\zeta_4 = 0$ $b_2 = 0$ $b_3 = 0$	$(1, -2, 1, 0, 0)$
$(E_3) \cdot (\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4 - E_5) \cdot$ $(3c_1 - 2S_2)$	$\beta = 0$ $y_5 = 0$ $b_3 = 0$	$(1, 0, -1, 1, 0)$
$(E_3) \cdot (-\sigma - c_1 - S_2 + 2E_1 - E_4 + E_5) \cdot$ $(3c_1 - 2S_2)$	$\beta = 0$ $b_2 = 0\|_{y_5, \delta_4 \neq 0}$ $\delta_5 = 0$	$(0, 0, 0, -1, 1)$
$(E_5) \cdot (2c_1 - S_2) \cdot (3c_1 - 2S_2)$	$\delta_5 = 0$ $b_2 = 0$ $b_3 = 0\|_{\alpha_5, \beta \neq 0}$	$(0, 0, 0, -1, 1)$
$(E_2 - E_3 - E_5) \cdot (2c_1 - S_2)$	$\alpha_5 = 0$ $b_2 = 0$ $b_3 = 0$	$(0, 0, 1, 0, -2)$

(4.63)

The associated multiplicities of the components are

Cartan Charges	Multiplicity
$(0, 1, 0, 0, 0)$	1
$(-2, 1, 0, 0, 0)$	3
$(1, -2, 1, 0, 0)$	2
$(1, 0, -1, 1, 0)$	4
$(0, 0, 0, -1, 1)_1$	3
$(0, 0, 0, -1, 1)_2$	1
$(0, 0, 1, 0, -2)$	2

(4.64)

We can see that this corresponds to the Yukawa coupling $\mathbf{16} \cdot \mathbf{16} \cdot \mathbf{10}$ by approaching this point along the $\mathbf{16}$ matter curve, where we can see the splitting:

$$\begin{aligned} (1, 0, -1, 0, 1) \rightarrow & (1, 0, -1, 1, 0) + (0, 0, 0, -1, 1) \\ (\mu_{16} - \alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5) \rightarrow & -(\mu_{16} - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) \\ & + (\mu_{10} - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)\,. \end{aligned} \tag{4.65}$$

This gives the desired Yukawa coupling. It is interesting to note here that our "E_7"

enhancement has only 7 components instead of the expected 8, this is similar to what was shown to happen with E_6 in [12] (see also [54]), where one node was missing.

4.3.4 D7 Enhancement

We expect to get a "D7" enhancement at $b_3 = b_4^2 - 4b_2b_6 = 0$ [47]. Starting from the "D6" enhancement, i.e. with $b_3 = 0$, the effect of setting $b_4^2 - 4b_2b_6 = 0$ is to make $\gamma_+ = \gamma_- \equiv \gamma$, since we can factorise equation (4.56) to

$$\alpha_5 \left(2b_2x_2 + b_4\beta\right)^2 = 0\,. \tag{4.66}$$

So the two previously separate components, γ_+ and γ_-, are now the same. So despite it being a "D7" enhancement, we actually only get six components instead of the expected eight. These are:

Component of $\left(z = b_3 = b_4^2 - 4b_2b_6 = 0\right)\big\vert_{\tilde{Y}_4}$	Equations in $\tilde{Y}_4$	Cartan charges
$(S_2 - E_1)\cdot(3c_1 - 2S_2)\cdot(8c_1 - 6S_2)$	$z_1 = 0$ $b_2 = 0$ $b_4^2 - 4b_2b_6 = 0$	$(0,1,0,0,0)$
$(-E_1 + E_2 + E_3 + 2E_4)\cdot(3c_1 - 2S_2)\cdot(8c_1 - 6S_2)$	$\delta_4 = 0\vert_{\zeta_4\neq 0}$ $b_2 = 0$ $b_4^2 - 4b_2b_6 = 0$	$(-2,1,0,0,0)$
$(E_1 - E_2 - E_3 - E_4)\cdot(3c_1 - 2S_2)\cdot(8c_1 - 6S_2)$	$\zeta_4 = 0$ $b_2 = 0$ $b_4^2 - 4b_2b_6 = 0$	$(1,-2,1,0,0)$
$(E_3)\cdot(3c_1 - 2S_2)\cdot(8c_1 - 6S_2)$	$\beta = 0$ $b_3 = 0$ $b_4^2 - 4b_2b_6 = 0$	$(0,1,-2,1,1)$
$\frac{1}{2}(E_5)\cdot(3c_1 - 2S_2 - E_2 + E_3 + E_5)\cdot(8c_1 - 6S_2)$	$\delta_5 = 0$ $\gamma = 0$ $b_4^2 - 4b_2b_6 = 0$	$(0,0,0,-1,1)$
$(E_2 - E_3 - E_5)\cdot(3c_1 - 2S_2)\cdot(8c_1 - 6S_2)$	$\alpha_5 = 0$ $b_2 = 0$ $b_4^2 - 4b_2b_6 = 0$	$(0,0,1,0,-2)$

(4.67)

With the multiplicities given by

Cartan Charges	Multiplicity
$(0,1,0,0,0)$	1
$(-2,1,0,0,0)$	1
$(1,-2,1,0,0)$	2
$(0,1,-2,1,1)$	2
$(0,0,0,-1,1)$	2
$(0,0,1,0,-2)$	2

(4.68)

Here, at the point of enhancement, we see that the two previously separate $\mathbf{10}$ matter curves become one, we believe that this corresponds to a $\mathbf{10} \cdot \mathbf{10} \cdot \mathbf{1}$ coupling, we do not see a curve for the singlet as its trivial transformation under the gauge group means that it is not part of the GUT divisor.

Chapter 5

$SO(10)$ G-flux

One of the most interesting results of [15], is that having a consistent G-flux puts restrictions on the geometry of S_2 and B_3, which we demonstrate explicitly in this chapter. We construct the local flux first, as the required restrictions are most easily deduced in the local case, we then construct the global flux and demonstrate both that it matches the global extension of the local flux, and also that the same conditions are required.

5.1 Tate divisor in Y_4

Our original 4-fold Y_4 is given by

$$y^2w + b_1zxyw + b_3z^2yw^2 = x^3 + b_2zx^2w + b_4z^3xw^2 + b_6z^5w^3, \tag{5.1}$$

and we are interested in the Tate divisor

$$wz\left(b_2x^2 + b_4z^2xw + b_6z^4w^2 - b_1xy - b_3zyw\right) . \tag{5.2}$$

Along the Tate divisor we have $wy^2 = x^3$. Since taking the local limit amounts to zooming in on $x = y = 0$, we can assume that $w \neq 0$, and therefore set it equal to 1, we can then rewrite the Tate divisor in terms of the $t = y/x$. This is done by using $y^2 = x^3$ to rewrite powers of t as holomorphic sections instead of meromorphic sections. For example we have

$$t^2 = \frac{y^2}{x^2} = \frac{x^3}{x^2} = x\,, \tag{5.3}$$

and similarly for other powers of t. By doing similar calculations up to t^5, we see that the Tate divisor can be written in terms of t and z as

$$z\left(b_2t^4 + b_4z^2t^2 + b_6z^4 - b_1t^5 - b_3zt^3\right) . \tag{5.4}$$

By setting $s = z/t$ and holding s fixed in the limit $t \to 0$, $z \to 0$, we obtain

$$st^5 \left(b_2 + b_4 s^2 + b_6 s^4 - b_3 s\right) . \tag{5.5}$$

5.1.1 Tate divisor in resolved $\tilde{Y}_4$

Now we look at the resolved Calabi-Yau $\tilde{Y}_4$, setting

$$\begin{aligned} x &= x_2 \zeta_4 \alpha_5^2 \beta^3 \delta_4 \delta_5^2 \,, \\ y &= y_5 \zeta_4 \alpha_5^2 \beta^4 \delta_4^2 \delta_5^3 \,, \\ z &= z_1 \zeta_4 \alpha_5 \beta^2 \delta_4 \delta_5 \,. \end{aligned} \tag{5.6}$$

This makes

$$\begin{aligned} t &= \frac{y}{x} = \frac{y_5 \beta \delta_4 \delta_5}{x_2} \,, \\ s &= \frac{z}{t} = \frac{z_1 x_2 \zeta_4 \alpha_5 \beta}{y_5} \,. \end{aligned} \tag{5.7}$$

The limit $t \to 0$ with s held fixed can be achieved by taking the limit $\delta_4 \to 0$ or $\delta_5 \to 0$.

Now we take the total transform of the Tate divisor,

$$wy^2 - x^3 = 0 \tag{5.8}$$

which gives

$$\zeta_4^2 \alpha_5^4 \beta^8 \delta_4^3 \delta_5^6 \left(w y_5^2 \delta_4 - x_2^3 \zeta_4 \alpha_5^2 \beta\right) = 0 \,. \tag{5.9}$$

The term

$$w y_5^2 \delta_4 - x_2^3 \zeta_4 \alpha_5^2 \beta = 0 \tag{5.10}$$

is then the proper transform of the Tate divisor, which we then restrict to the resolved $\tilde{Y}_4$.

With this restriction, the proper transform of the Tate divisor is reducible, with components given by $\zeta_4 = 0$, $z_1 = 0$, and the remainder.

To see this, set $\zeta_4 = 0$. By using (A.4), we see that we cannot have $w = 0$, $y_5 = 0$, $\alpha_5 = 0$ or $\delta_5 = 0$, so we set these equal to one. The Tate divisor equation is now

$$\delta_4 = 0 \,, \tag{5.11}$$

and the equation for the resolved $\tilde{Y}_4$ also becomes

$$\delta_4 = 0\,. \tag{5.12}$$

Which means that the Tate divisor equation is automatically satisfied, meaning that $\zeta_4 = 0$ must be a component of it.

For the component $z_1 = 0$, we cannot have α_5, β or δ_5 equal to zero, so again we set these equal to one. The Tate divisor equation takes the form

$$wy_5^2\delta_4 - x_2^3\zeta_4 = 0\,, \tag{5.13}$$

and the equation for $\tilde{Y}_4$ becomes

$$wy_5^2\delta_4 = x_2^3\zeta_4\,. \tag{5.14}$$

This again means that the Tate divisor equation is satisfied automatically in $\tilde{Y}_4$. So we define the Tate divisor to be the remaining component after removing $\zeta_4 = 0$ and $z_1 = 0$:

$$\mathcal{C}_{\text{Tate}} = [wy_5^2\delta_4 - x_2^3\zeta_4\alpha_5^2\beta]\cdot[\tilde{Y}_4] - [\zeta_4 = 0]\cdot[\tilde{Y}_4] - [z_1 = 0]\cdot[\tilde{Y}_4]\,. \tag{5.15}$$

Which is in the class

$$3\sigma + 6c_1 - S_2 - 2E_1 - E_2 - E_3 - 2E_5\,. \tag{5.16}$$

Its intersection with the Cartan divisors takes the form:

$$\mathcal{C}_{\text{Tate}}\cdot_{\tilde{Y}_4}\Sigma_{\alpha_i} = (0,0,0,1,0)\times 4\,. \tag{5.17}$$

This fits with the branching rule for the adjoint of E_8 under the breaking to $SU(4)\times SO(10)$:

$$\mathbf{248} \to (\mathbf{15},\mathbf{1}) \oplus (\mathbf{1},\mathbf{45}) \oplus (\mathbf{4},\overline{\mathbf{16}}) \oplus (\mathbf{6},\mathbf{10}) \oplus (\overline{\mathbf{4}},\mathbf{16})\,, \tag{5.18}$$

as the roots of the spectral equation correspond to the fundamental of the $SU(4)$ Higgs field, and $(0,0,0,1,0)$ is the highest weight of the $\overline{\mathbf{16}}$ of $SO(10)$.

5.2 Local limit

As discussed before, the local limit may be either $\delta_4 \to 0$ or $\delta_5 \to 0$. However, if we set δ_4 equal to zero in the Tate divisor equation, and take $x_2 = \alpha_5 = 1$ as they cannot vanish together with δ_4, then we obtain

$$\zeta_4 \beta = 0\,. \tag{5.19}$$

But since $\zeta_4 = 0$ is one of the reducible components we remove from the Tate divisor, this leaves $\beta = 0$, which means that $s = z/t$ becomes zero, instead of being held fixed. So the local limit here must be given by $\delta_5 \to 0$, so we intersect with E_5:

$$\begin{aligned} \mathcal{C}_{\text{Tate}} \cdot_{\tilde{Y}_4} E_5 &= [wy_4^2 - x_2^3\alpha_4^2\beta\zeta_3] \cdot_{\tilde{Y}_4} E_5 - [\zeta_3 = 0] \cdot_{\tilde{Y}_4} E_5 - [z_1 = 0] \cdot_{\tilde{Y}_4} E_5 \\ &= [wy_4^2 - x_2^3\alpha_4^2\beta\zeta_3] \cdot_{\tilde{Y}_4} E_5\,, \end{aligned} \tag{5.20}$$

where the last terms vanish because we cannot have $\delta_5 = z_1 = 0$ or $\delta_5 = \zeta_4 = 0$. Setting $\delta_5 = 0$, and $w = z_1 = \zeta_4 = 1$, the equation for the Tate divisor becomes

$$y_5^2 = x_2^3\alpha_5^2\beta\,. \tag{5.21}$$

Notice that if $\alpha_5 = 0$, then $y_5 = 0$, but these two conditions are not allowed to hold simultaneously, so we set $\alpha_5 = 1$. This implies that the equation for $\tilde{Y}_4$ becomes

$$b_3 y_5 = b_2 x_2^2 \zeta_4 \alpha_+ b_4 x_2 \beta + b_6 \beta^2\,. \tag{5.22}$$

If we now set $x_2 = 0$, the Tate divisor equation gives $y_5 = 0$ and so the equation for $\tilde{Y}_4$ becomes

$$0 = b_6 \beta^2\,. \tag{5.23}$$

For generic b_6 this sets $\beta = 0$. Under the identification $\delta_5 = 0$, $\zeta_4 = \delta_4 = 1$, the coordinates of the $\mathbb{P}^2$ from the second blow-up become

$$[x_2, 0, \beta]\,, \tag{5.24}$$

$x_2 = 0$ would imply $\beta = 0$ so $x_2 = 0$ is not allowed and we can set $x_2 = 1$. The Tate divisor equation is

$$y_5^2 = \beta\,, \tag{5.25}$$

which, after substituting it into $\tilde{Y}_4$, gives the required spectral equation

$$b_6 y_5^4 + b_4 y_5^2 - b_3 y_5 + b_2 = 0\,. \tag{5.26}$$

5.3 Local flux

In Section 3.2 we derived the form of the local flux to be

$$\gamma = \alpha \left(n\sigma_0 - p_{sc}^*\left(\eta - nc_1(S_2)\right)\right)\big|_{\mathcal{C}_{\text{Higgs,loc}}} \,. \tag{5.27}$$

This is the flux on the GUT divisor S_2, we wish to construct it as a surface in $\tilde{Y}_4$ which will have the same properties, but should then be equivalent to the global G-flux.

To construct the local flux, as in [13], we construct surfaces in $\tilde{Y}_4$, $\mathcal{S}_C$ which correspond to curves inside $\mathcal{C}_{\text{Higgs,loc}}$. We firstly construct the surface $\mathcal{S}_{p^*D}$ by

$$\begin{aligned} \mathcal{S}_{p^*D} &= \mathcal{C}_{\text{Tate}} \cdot D - (3\sigma + 6c_1) \cdot D \\ &= (-S_2 - 2E_1 - E_2 - E_3 - 2E_5) \cdot D \,, \end{aligned} \tag{5.28}$$

which only intersects the Cartan root $\mathcal{D}_{-\alpha_4}$, and we have made a subtraction to get the required orthogonality properties.

We now construct the surface corresponding to $\sigma_0|_{\mathcal{C}_{\text{Higgs,loc}}}$, which we denote by $\mathcal{S}_{\sigma\cdot\mathcal{C}}$. We recall that σ_0 is the section of the local $\mathbb{P}^1$ bundle which we identified with $U = 0$, which we then restricted to $\mathcal{C}_{\text{Higgs,loc}}$ with the spectral cover equation. For $SO(10)$ this local spectral cover equation in Z_3 would be

$$b_6U^4 + b_4U^2V^2 - b_3UV^3 + b_2 = 0 \,. \tag{5.29}$$

Our spectral cover equation inside $\tilde{Y}_4$ is given above by

$$b_6y_5^4 + b_4y_5^2 - b_3y_5 + b_2 = 0 \,, \tag{5.30}$$

which means that we want a surface such that when we take the local limit, i.e. intersect with $\delta_5 = 0$, implies that $y_5 = b_2 = 0$.

On the GUT surface S_2, $b_2 = 0$ is where the singularity enhances and we obtain matter in the **16**. This is therefore our starting point. Looking at the components of the fiber along the **16** matter curve, it is obvious that the component $\beta = y_5 = 0$.

In constructing the local flux, we will use equation (5.27) and therefore the local flux in $\tilde{Y}_4$ will be given by

$$G_{\text{local}} = \alpha(4\mathcal{S}_{\sigma\cdot\mathcal{C}} - \mathcal{S}_{p_{sc}^*D}) \,, \tag{5.31}$$

where D is a divisor in B_3 that we will determine shortly. The flux must not break $SO(10)$ and therefore cannot intersect any Cartan divisors. The intersection numbers of $\mathcal{S}_{p_{sc}^* D}$ with the Cartan divisors are given by

$$\mathcal{S}_{p_{sc}^* D} \cdot_{\tilde{Y}_4} \mathcal{D}_{-\alpha_i} \cdot_{\tilde{Y}_4} D_1 = (0,0,0,4,0)\,, \tag{5.32}$$

where D_1 is some divisor in B_3 such that $S_2 \cdot_{B_3} D \cdot_{B_3} D_1 = 1$.

However, we know that the component of the **16** matter curve $\beta = y_5 = 0$ has Cartan charges $(1,0,-1,1,0)$ and therefore if our surface $\mathcal{S}_{\sigma\cdot\mathcal{C}}$ were just this component, then the flux would break $SO(10)$. What we require is for the surface $\mathcal{S}_{\sigma\cdot\mathcal{C}}$ to have Cartan charges $(0,0,0,1,0)$, which we can achieve by making subtractions from $\beta = y_5 = 0$. By making subtractions of the form $[b_2]\cdot\mathcal{D}_{-\alpha_i}$, we can easily construct a surface with the desired properties. We end up with

$$\begin{aligned}\mathcal{S}_{\sigma\cdot\mathcal{C}} &= [\beta] \cdot_{\tilde{Y}_4} [y_5] - [b_2] \cdot_{\tilde{Y}_4} (\mathcal{D}_{-\alpha_2} + 2\mathcal{D}_{-\alpha_3} + \mathcal{D}_{-\alpha_4} + \mathcal{D}_{-\alpha_5}) \\ &= E_3 \cdot_{\tilde{Y}_4} (\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4 - E_5) - (2c_1 - S_2) \cdot_{\tilde{Y}_4} (E_1 - E_4)\,,\end{aligned} \tag{5.33}$$

which we can easily check has the correct Cartan charges since we already know those of the $\mathcal{D}_{-\alpha_i}$.

Note that in making these subtractions, we have not changed the charge under $\mathcal{D}_{-\alpha_4}$, which is the Cartan divisor corresponding to the exceptional divisor E_5. Since intersection with E_5 is what corresponds to the local limit, we therefore see that making these subtractions has not changed the necessary property of $\mathcal{S}_{\sigma\cdot\mathcal{C}}$ corresponds to σ_0 in the local limit.

Using equation (5.27) and noting that for $SO(10)$, we take $n = 4$, we see that the D that appeared in equation (5.31) is $\eta - 4c_1(S_2)$. Using equation (5.26) and the definition of η from Section 3.2, we have

$$\eta = [b_6] = 6c_1 - 5S_2\,. \tag{5.34}$$

Because S_2 is a divisor in B_3, we have

$$c\,(S_2) = \frac{c(B_3)}{1 + S_2}\,, \tag{5.35}$$

which gives

$$c_1(S_2) = c_1 - S_2\,, \tag{5.36}$$

recalling that when we say c_1 we mean specifically $c_1(B_3)$.

Putting these together means that in equation (5.31) we have

$$D = 2c_1 - S_2 \,. \tag{5.37}$$

We can construct a local G-flux with the traceless combination

$$\begin{aligned} G_{\text{local}} &= \alpha(4\mathcal{S}_{\sigma\cdot\mathcal{C}} - \mathcal{S}_{p^*(2c_1-S_2)}) \\ &= \alpha(c_1 \cdot_{\tilde{Y}_4} (-4E_1 + 2E_2 + 14E_3 + 8E_4 + 4E_5) + S_2 \cdot_{\tilde{Y}_4} (2E_1 - E_2 - E_3 - 4E_4 - 2E_5) \\ &\quad - 4E_3 \cdot_{\tilde{Y}_4} (E_1 + E_2 + E_3 + E_4 + E_5)) \,. \end{aligned} \tag{5.38}$$

This can be further simplified by using the relations in the appendix to

$$\begin{aligned} G_{\text{local}} =& \alpha(c_1 \cdot_{\tilde{Y}_4} (4E_1 - 6E_2 + 6E_3 - 8E_4 + 4E_5) + S_2 \cdot_{\tilde{Y}_4} (-2E_1 + 3E_2 - 5E_3 + 4E_4 - 2E_5) \\ &+ 4E_3 \cdot_{\tilde{Y}_4} E_4 - 4E_3 \cdot_{\tilde{Y}_4} E_5) \,. \end{aligned} \tag{5.39}$$

We now use the quantization condition to constrain the parameter α. This first requires constructing $\mathcal{S}_r$, where r is the ramification divisor of the local spectral cover. The ramification divisor can be computed using

$$\begin{aligned} r &= (\mathcal{C}_{\text{Higgs,loc}} - \sigma_0 - \sigma_\infty)|_{\mathcal{C}_{\text{Higgs,loc}}} \\ &= (\eta + 4\sigma_0 - \sigma_0 - \sigma_0 - c_1(S_2))|_{\mathcal{C}_{\text{Higgs,loc}}} \\ &= (5c_1 - 4S_2 - 2\sigma_0)|_{\mathcal{C}_{\text{Higgs,loc}}} \,. \end{aligned} \tag{5.40}$$

So the surface in $\tilde{Y}_4$ corresponding to r is given by

$$\mathcal{S}_r = \mathcal{S}_{p^*_{sc}(5c_1-4S_2)} - 2S_{\sigma\cdot\mathcal{C}} \,, \tag{5.41}$$

whose odd part is given by

$$\mathcal{S}_r^{(\text{odd})} = c_1 \cdot E_2 + c_1 \cdot E_3 \,. \tag{5.42}$$

We have the quantization condition that

$$G_{\text{local}} + \frac{1}{2}\mathcal{S}_r^{(\text{odd})} \tag{5.43}$$

be integrally quantized[1], for some choice of $\alpha \in \mathbb{C}$. Looking at G_{local} and $\mathcal{S}_r^{(\text{odd})}$, [14] has argued that that this can not be done generically. We can satisfy this requirement

[1] It is actually the combination $G_{\text{local}} + \frac{1}{2}\mathcal{S}_r$ that is required to be integrally quantized, but the even part of $\mathcal{S}_r$ will trivially satisfy this condition, so we are free to ignore it.

by imposing extra conditions on B_3 and S_2. Now if we look at the last two terms of G, we see that they do not depend on B_3 or S_2, and so to satisfy the quantization condition we require $\alpha \in \frac{\mathbb{Z}}{4}$. Since our condition only concerns the non-integer part, we then have four cases to consider: $\alpha \in \mathbb{Z}$, $\alpha \in \mathbb{Z} + \frac{1}{4}$, $\alpha \in \mathbb{Z} + \frac{1}{2}$ and $\alpha \in \mathbb{Z} + \frac{3}{4}$.

Firstly, for $\alpha \in \mathbb{Z}$, we simply require that the odd part of the ramification divisor vanishes, which can be done by choosing the base such that $c_1(B_3)$ is even.

For $\alpha \in \mathbb{Z} + \frac{1}{2}$, we require $c_1(S_2)$ even, for later use we note that this means that the odd parts of $c_1(B_3)$ and S_2 now match.

The last two possibilities, $\alpha \in \mathbb{Z} \pm \frac{1}{4}$ both give the same condition, which is that the class S_2 be a multiple of 4.

5.4 Global Flux

As discussed in Section 3.3, we require that the G-flux be quantized according to

$$G + \frac{1}{2} c_2(\tilde{Y}_4) \in H^4(\tilde{Y}_4, \mathbb{Z}) \,. \tag{5.44}$$

Hence we must compute the second Chern class of the resolved 4-fold $\tilde{Y}_4$. In particular, we are only interested in the odd part of the second Chern class, since the even part satisfies this equation trivially and so does not constrain the G-flux. We compute the total Chern class using the following formula, taken from [61][2]:

$$\begin{aligned} c(\tilde{X}_5) =& c(X_5) \frac{(1+E_1)(1+\sigma+2c_1-E_1)(1+\sigma+3c_1-E_1)(1+S_2-E_1)}{(1+\sigma+2c_1)(1+\sigma+3c_1)(1+S_2)} \\ &\times \frac{(1+E_2)(1+\sigma+2c_1-E_1-E_2)(1+\sigma+3c_1-E_1-E_2)(1+E_1-E_2)}{(1+\sigma+2c_1-E_1)(1+\sigma+3c_1-E_1)(1+E_1)} \\ &\times \frac{(1+E_3)(1+\sigma+3c_1-E_1-E_2-E_3)(1+E_1-E_2-E_3)(1+E_2-E_3)}{(1+\sigma+3c_1-E_1-E_2)(1+E_1-E_2)(1+E_2)} \\ &\times \frac{(1+E_4)(1+\sigma+3c_1-E_1-E_2-E_3-E_4)(1+E_1-E_2-E_3-E_4)}{(1+\sigma+3c_1-E_1-E_2-E_3)(1+E_1-E_2-E_3)} \\ &\times \frac{(1+E_5)(1+\sigma+3c_1-E_1-E_2-E_3-E_4-E_5)(1+E_2-E_3-E_5)}{(1+\sigma+3c_1-E_1-E_2-E_3-E_4)(1+E_2-E_3)} \,. \end{aligned} \tag{5.45}$$

[2]Note that in [61], it is specified that the formula is only valid when there are no singularities present, so it may seem that it is not applicable here, however, we use the formula here in the non-singular X_5 where it is valid, and only afterwards do we then restrict to $\tilde{Y}_4$.

We can then restrict this to $\tilde{Y}_4$ by

$$c(\tilde{Y}_4) = \frac{c(\tilde{X}_5)}{1 + 3\sigma + 6c_1 - 2E_1 - 2E_2 - 2E_3 - E_4 - E_5}\,. \tag{5.46}$$

After expanding this out, we can read off the individual Chern classes, in particular, the first chern class vanishes, confirming that our resolved 4-fold is indeed Calabi-Yau. We could use the same equation to see that at all stages in the resolution procedure the Calabi-Yau condition was kept intact. Using the fact that $c_2(Y_4)$ is even [13, 62], after expanding, we see that the odd part of $c_2(\tilde{Y}_4)$ is given by

$$\begin{aligned} c_2^{(odd)}(\tilde{Y}_4) = c_1 \cdot (E_1 + E_2 + E_3 + E_4 + E_5) + S_2 \cdot E_1 + E_1 \cdot E_2 \\ + E_1 \cdot E_5 + E_2 \cdot E_3 + E_4 \cdot E_4 + E_5 \cdot E_5\,. \end{aligned} \tag{5.47}$$

Simplifying using the relations in the appendix gives

$$c_2^{(odd)}(\tilde{Y}_4) = c_1 \cdot E_2 + c_1 \cdot E_3\,. \tag{5.48}$$

As well as the quantization condition, we also require that the G-flux be orthogonal to horizontal and vertical surfaces in Y_4. Also, to preserve the $SO(10)$ symmetry we require that the intersection between G and the Cartan divisors vanish. As in [13], this restricts G to be a linear combination of $c_1 \cdot_{\tilde{Y}_4} E_i$, $S_2 \cdot_{\tilde{Y}_4} E_i$ and $E_i \cdot_{\tilde{Y}_4} E_j$. Using the relations between exceptional divisors in the appendix, we can eliminate all combinations of $E_i \cdot_{\tilde{Y}_4} E_j$ except for two of them, which we choose to be $E_3 \cdot_{\tilde{Y}_4} E_4$ and $E_3 \cdot_{\tilde{Y}_4} E_5$. So overall G is then given by

$$G = \frac{1}{2}c_1 \cdot (E_2 + E_3) + \sum_{i=1}^{5}(a_i c_1 \cdot E_i + b_i S_2 \cdot E_i) + pE_3 \cdot E_4 + qE_3 \cdot E_5\,. \tag{5.49}$$

Here a_i, b_i, p and q are integers, and the first two terms are present to enforce the quantization condition.

However, using this expression for G and requiring it to vanish when intersected with the Cartan divisors does not give integer answers for all of a_i, b_i, p and q. This was to be expected, since in the previous section we saw that G cannot be quantized without imposing extra conditions. We now solve for the G-flux for two of these conditions.

1) For $c_1(B_3)$ even, the odd part of c_2 vanishes and our general form for G is

$$G = \sum_{i=1}^{5}(a_i c_1 \cdot E_i + b_i S_2 \cdot E_i) + pE_3 \cdot E_4 + qE_3 \cdot E_5\,. \tag{5.50}$$

The requirement that this vanishes when intersected with the Cartan divisors gives the one parameter solution

$$\begin{aligned}
a_1 &= 4n\,, \\
a_2 &= -6n\,, \\
a_3 &= 6n\,, \\
a_4 &= -8n\,, \\
a_5 &= 4n\,, \\
b_1 &= -2n\,, \\
b_2 &= 3n\,, \\
b_3 &= -5n\,, \\
b_4 &= 4n\,, \\
b_5 &= -2n\,, \\
p &= 4n\,, \\
q &= -4n\,,
\end{aligned} \tag{5.51}$$

where n is an integer. So the flux is

$$\begin{aligned}
G =& n\,(c_1 \cdot (4E_1 - 6E_2 + 6E_3 - 8E_4 + 4E_5) + S_2 \cdot (-2E_1 + 3E_2 - 5E_3 + 4E_4 - 2E_5) \\
&+4E_3 \cdot E_4 - 4E_3 \cdot E_5)\ .
\end{aligned} \tag{5.52}$$

2) For $c_1(S_2)$ to be even, the odd part of c_2 can now be written as $S_2 \cdot E_2 + S_2 \cdot E_3$, and so we take G to be

$$G = \frac{1}{2} S_2 \cdot (E_2 + E_3) + \sum_{i=1}^{5} (a_i c_1 \cdot E_i + b_i S_2 \cdot E_i) + pE_3 \cdot E_4 + qE_3 \cdot E_5\,. \tag{5.53}$$

As before, we impose that G does not intersect any of the Cartan divisors, and obtain the one parameter solution

$$
\begin{aligned}
a_1 &= 2 + 4n\,, \\
a_2 &= -3 - 6n\,, \\
a_3 &= 3 + 6n\,, \\
a_4 &= -4 - 8n\,, \\
a_5 &= 2 + 4n\,, \\
b_1 &= -1 - 2n\,, \\
b_2 &= 1 + 3n\,, \\
b_3 &= -3 - 5n\,, \\
b_4 &= 2 + 4n\,, \\
b_5 &= -1 - 2n\,, \\
p &= 2 + 4n\,, \\
q &= -2 - 4n\,,
\end{aligned}
\tag{5.54}
$$

again with n integral. This gives

$$
\begin{aligned}
G = \left(n + \frac{1}{2}\right) &\left(c_1 \cdot (4E_1 - 6E_2 + 6E_3 - 8E_4 + 4E_5) + S_2\,(-2E_1 + 3E_2 - 5E_3 + 4E_4 - 2E_5)\right. \\
&\left. +4E_3 \cdot E_4 - 4E_3 \cdot E_5\right)\,.
\end{aligned}
\tag{5.55}
$$

3) When the class of S_2 is a multiple of 4, one also gets the same answer as in the local case. So altogether we see that with each of the three possible conditions, the global and local fluxes match.

We can now intersect the flux with the matter surfaces. We take our matter surfaces as

$$
\begin{aligned}
\mathcal{S}_{\mathbf{16}} &= E_3 \cdot (\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4 - E_5)\,, \\
\mathcal{S}_{\mathbf{10}} &= \frac{1}{2} E_5 \cdot (3c_1 - 2S_2 - E_2 + E_3 + E_5)\,.
\end{aligned}
\tag{5.56}
$$

We then obtain

$$
\begin{aligned}
G \cdot_{\tilde{Y}_4} \mathcal{S}_{\mathbf{16}} &= \alpha(6c_1 - 5S_2) \cdot_{S_2} (2c_1 - S_2)\,, \\
G \cdot_{\tilde{Y}_4} \mathcal{S}_{\mathbf{10}} &= 0\,.
\end{aligned}
\tag{5.57}
$$

This is problematic since phenomenology requires a non-zero chirality for the **10**, however both of these are in agreement with [46]. In [46] in order to obtain a non-zero

chirality for the **10**, a split spectral cover was used, this means that the spectral cover is reducible, which corresponds to a factorising of the spectral cover equation. This situation has been looked at in the context of global model building in [63], and is dependent on the gluing data between the components [64, 65]. This adds more complexity since it requires more specification of the G-flux other than just its homology class as has been used here, we leave this extension for future work.

Chapter 6

Global E_7 GUT

The formalism used in the previous sections is different to what has previously been done with F-theory, as there is no actual group enhancement going on. In the $SO(10)$ example, the occurence of matter was not due to enhancements to $SO(12)$ and E_6, but due to the Cartan roots splitting into new components and these components being identified with weights of the required represention. Similarly, the codimension-3 enhancements just led to more splitting, and no group enhancement. It is interesting then to see what happens if we start with an E_7 GUT, since the codimension-3 enhancement here would previously have required going to an exceptional group beyond E_8, whereas now we can carry out the process just using the weight lattice of E_7 and so do not encounter these problems. We proceed as with the $SO(10)$ GUT, and start with the appropriate Tate form, which can simply be read off from Table 2.2 as

$$y^2w + b_1zxyw + b_3z^3yw^2 = x^3 + b_2z^2x^2w + b_4z^3xw^2 + b_6z^5w^3\,. \tag{6.1}$$

This is embedded in the same auxiliary space X_5 as the $SO(10)$ example, and the objects in the defining equation are sections of the same bundles of X_5, except that the different powers of z mean that some of the b_is are changed. We list everything for

convenience:

Section	Bundle
w	$\mathcal{O}(\sigma)$
x	$\mathcal{O}(\sigma + 2c_1)$
y	$\mathcal{O}(\sigma + 3c_1)$
z	$\mathcal{O}(S_2)$
b_1	$\mathcal{O}(c_1 - S_2)$
b_2	$\mathcal{O}(2c_1 - 2S_2)$
b_3	$\mathcal{O}(3c_1 - 3S_2)$
b_4	$\mathcal{O}(4c_1 - 3S_2)$
b_6	$\mathcal{O}(6c_1 - 5S_2)$

(6.2)

6.1 Resolution of the E_7 Singularity

As with the $SO(10)$ case, the locus $x = y = z = 0$ is singular, so we perform blow-ups to fully resolve the manifold. We must perform five blow-ups to remove the codimension-1 singularity, these are:

$$\begin{array}{llll}
\text{Blow-up 1:} & x = x_1\alpha\,, & y = y_1\alpha\,, & z = z_1\alpha\,. \\
\text{Blow-up 2:} & x_1 = x_2\beta\,, & y_1 = y_2\beta\,, & \alpha = \alpha_2\beta\,. \\
\text{Blow-up 3:} & y_2 = y_3\gamma\,, & \alpha_2 = \alpha_3\gamma\,, & \beta = \beta_3\gamma\,. \\
\text{Blow-up 4:} & y_3 = y_4\delta\,, & \alpha_3 = \alpha_4\delta\,, & \gamma = \gamma_4\delta\ . \\
\text{Blow-up 5:} & y_4 = y_5\epsilon\,, & \beta_3 = \beta_5\epsilon\,, & \gamma_4 = \gamma_5\epsilon\ .
\end{array}$$

The loci $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ and $\epsilon = 0$ give rise to the exceptional divisors E_1, E_2, E_3, E_4 and E_5 respectively. After these five blow-ups, the proper transform of the E_7 Tate form is

$$\begin{aligned}
&wy_5^2 + b_1 w x_2 y_5 z_1 \alpha_4 \beta_5 \gamma_5 \delta\epsilon + b_3 w^2 y_5 z_1^3 \alpha_4^2 \beta_5 \gamma_5^2 \delta^3 \epsilon^2 = x_2^3 \alpha_4 \beta_5^2 \gamma_5 \epsilon \\
&+ b_2 w x_2^2 z_1^2 \alpha_4^2 \beta_5^2 \gamma_5^2 \delta^2 \epsilon^2 + b_4 w^2 x_2 z_1^3 \alpha_4^2 \beta_5 \gamma_5 \delta + b_6 w^3 z_1^5 \alpha_4^3 \beta_5 \gamma_5^2 \delta^3 \epsilon\,.
\end{aligned} \tag{6.3}$$

Each blow-up performed introduces a $\mathbb{P}^2$, after these first five blow-ups we have the projectivity relations

$$\begin{aligned}
[x_2\beta_5\gamma_5\delta\epsilon^2, y_5\beta_5\gamma_5^2\delta^3\epsilon^4, z_1] \\
[x_2, y_5\gamma_5\delta^2\epsilon^2, \alpha_4\gamma_5\delta^2\epsilon] \\
[y_5\delta\epsilon, \alpha_4\delta\beta_5\epsilon] \\
[y_5\epsilon, \alpha_4, \gamma_5\epsilon] \\
[y_5, \beta_5, \gamma_5]\,.
\end{aligned} \tag{6.4}$$

The first of these relations comes from the fact that our auxiliary space X_5 is a $\mathbb{P}^2$ bundle with projective coordinates $[w, x, y]$. Using these projectivity relations we can see that the blown up space of equation 6.3 is smooth in codimension-1. As with the $SO(10)$ case, the small resolutions are carried out as in [60], and are required to remove higher codimension singularities. These are given by

$$\begin{aligned}
&\text{Blow-up 6:} \qquad y_5 = y_6\zeta_6\,, \qquad \alpha_4 = \alpha_6\zeta_6\,. \\
&\text{Blow-up 7:} \qquad y_6 = y_7\zeta_7\,, \qquad \beta_5 = \beta_7\zeta_7\,. \\
&\text{Blow-up 8:} \qquad y_7 = y_8\zeta_8\,, \qquad \gamma_5 = \gamma_8\zeta_8\,.
\end{aligned}$$

The loci $\zeta_i = 0$ give rise to new exceptional divisors E_i for $i = 6\,,7\,,8$. Note that we actually only needed two further blow-ups to fully resolve the singularity in higher codimension, but we choose to do an extra one, since otherwise, one of the Cartan divisors will have a non-integral class. In doing one more blow up than is necessary, we end up with a linear relation between some of the exceptional divisors, so we do in fact have only seven linearly independent ones, consistent with E_7 being a rank 7 group. This is similar to the situation for E_6 in [14]. The final resolved manifold is given by

$$\begin{aligned}
wy_8^2\zeta_6\zeta_7\zeta_8 + b_1wx_2y_8z_1\alpha_6\beta_7\gamma_8\delta\epsilon\zeta_6\zeta_7\zeta_8 + b_3w^2y_8z_1^3\alpha_6^2\beta_7\gamma_8^2\delta^3\epsilon^2\zeta_6^2\zeta_7\zeta_8^2 = \\
x_2^3\alpha_6\beta_7^2\gamma_8\epsilon\zeta_7 + b_2wx_2^2z_1^2\alpha_6^2\beta_7^2\gamma_8^2\delta^2\epsilon^2\zeta_6\zeta_7\zeta_8 + b_4w^2x_2z_1^3\alpha_6^2\beta_7\gamma_8\delta\zeta_6 + b_6w^3z_1^5\alpha_6^3\beta_7\gamma_8^2\delta^3\epsilon\zeta_6^2\zeta_8\,.
\end{aligned} \tag{6.5}$$

The small resolutions also give rise to more projectivity relations:

$$\begin{aligned}
[y_8\zeta_7\zeta_8, \alpha_6] \\
[y_8\zeta_8, \beta_7] \\
[y_8, \zeta_8]
\end{aligned} \tag{6.6}$$

The variables lie in the following sections

Section	Bundle
w	$\mathcal{O}(\sigma)$
x_2	$\mathcal{O}(\sigma + 2c_1 - E_1 - E_2)$
y_8	$\mathcal{O}(\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8)$
z_1	$\mathcal{O}(S_2 - E_1)$
α_6	$\mathcal{O}(E_1 - E_2 - E - 3 - E_4 - E_6)$
β_7	$\mathcal{O}(E_2 - E_3 - E_5 - E_7)$
γ_8	$\mathcal{O}(E_3 - E_4 - E_5 - E_8)$
δ	$\mathcal{O}(E_4)$
ϵ	$\mathcal{O}(E_5)$
ζ_6	$\mathcal{O}(E_6)$
ζ_7	$\mathcal{O}(E_7)$
ζ_8	$\mathcal{O}(E_8)$

(6.7)

6.2 Cartan Divisors

The original E_7 singularity was located at $z = 0$, and after resolving, this becomes

$$z_1\alpha_6\beta_7\gamma_8^2\delta^3\epsilon^3\zeta_6\zeta_7\zeta_8^2 = 0\,, \tag{6.8}$$

and so is now reducible. Note that whilst it may appear that $z = 0$ has split into 9 components (one more than is expected for an E_7 singularity), the components $\alpha_6 = 0$ and $\zeta_6 = 0$ are actually equivalent, and so we have eight unique components, as expected. As with the $SO(10)$ case, the Cartan divisors are given by these components restricted to the resolved manifold $\tilde{Y}_4$, and have the following homology classes and multiplicities:

Cartan Divisor	Component	Class in $\tilde{Y}_4$	multiplicity
$\mathcal{D}_{-\alpha_0}$	$(z_1 = 0)\vert_{\tilde{Y}_4}$	$S_2 - E_1$	1
$\mathcal{D}_{-\alpha_1}$	$(\alpha_6 = 0)\vert_{\tilde{Y}_4}$	$E_1 - E_2 - E_3 - E_4 - E_6$	2
$\mathcal{D}_{-\alpha_2}$	$(\delta = 0)\vert_{\tilde{Y}_4}$	E_4	3
$\mathcal{D}_{-\alpha_3}$	$(\gamma_8 = 0)\vert_{\tilde{Y}_4}$	$E_3 - E_4 - E_5 - E_8$	4
$\mathcal{D}_{-\alpha_4}$	$(\epsilon = 0)\vert_{\tilde{Y}_4}$	E_5	3
$\mathcal{D}_{-\alpha_5}$	$(\beta_7 = 0)\vert_{\tilde{Y}_4}$	$(E_2 - E_3 - E_5 - E_7)$	2
$\mathcal{D}_{-\alpha_6}$	$(\zeta_7 = 0)\vert_{\tilde{Y}_4,\beta_7\neq 0}$	$(-E_2 + E_3 + E_5 + 2E_7)$	1
$\mathcal{D}_{-\alpha_7}$	$(\zeta_8 = 0)\vert_{\tilde{Y}_4,\gamma_8\neq 0}$	$(-E_3 + E_4 + E_5 + 2E_8)$	2

(6.9)

We can use the same techniques as with the $SO(10)$ case to derive an intersection matrix

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \end{pmatrix} . \tag{6.10}$$

This confirms that we do indeed have an E_7-type singularity.

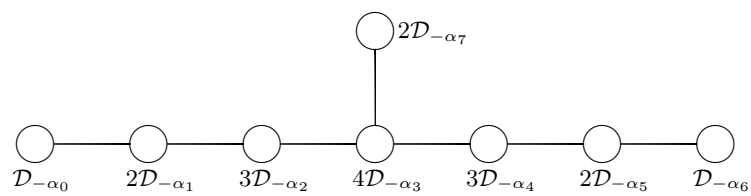

Figure 6.1: E_7 Dynkin diagram, as with the D_5 case the multiplicities of the Cartan divisors are those expected.

6.3 Matter and Yukawas

We now look for higher codimension enhancements of the singularity by computing the discriminant of the Tate form, this is given by

$$\begin{aligned} \Delta = &- 1024 b_4^3 w^{10} z^9 + 16 \Big(\Big(\left(b_1^2 + 4b_2\right)^2 - 96 b_1 b_3 \Big) b_4^2 \\ &+ 72 \left(b_1^2 + 4b_2\right) b_4 b_6 - 432 b_6^2 \Big) w^{10} z^{10} + O\left(z^{11}\right) . \end{aligned} \tag{6.11}$$

From here we can clearly see that a codimension-2 enhancement occurs at $b_4 = 0$ and we have a codimension-3 enhancement at $b_4 = b_6 = 0$. We thus expect to see matter states arising from splittings of the Cartan divisors at $b_4 = 0$ and Yukawa interactions at $b_4 = b_6 = 0$.

6.3.1 56 Matter

$b_4 = 0$ would usually correspond to an E_8 enhancement and so we expect to see matter in the **56** here. We can see this by decomposing the adjoint of E_8 under the breaking $E_8 \to E_7 \times SU(2)$:

$$\mathbf{248} \to (\mathbf{133}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{56}, \mathbf{2}) . \tag{6.12}$$

If we look at the component $\epsilon = 0$ restricted to the resolved 4-fold $\tilde{Y}_4$, setting equal to 1, anything which cannot vanish simulateously with ϵ, we have

$$y_8^2 \zeta_7 \zeta_8 - b_4 \beta_7 \gamma_8 = 0\,, \tag{6.13}$$

and can see that this Cartan divisor will become reducible at $b_4 = 0$, splitting into three components:

$$[\epsilon] \cdot [b_4] = 2[\epsilon] \cdot [y_8] + [\epsilon] \cdot ([\zeta_7] - [\beta_7]) + [\epsilon] \cdot ([\zeta_8] - [\gamma_8])\,. \tag{6.14}$$

The latter two components are specifically involve $([\zeta_7] - [\beta_7])$ and $([\zeta_8] - [\gamma_8])$ (as opposed to just $[\zeta_7]$ and $[\zeta_8]$) as it is these two Cartan divisors with give $\epsilon = 0$ when intersected with $b_4 = 0$. None of the other Cartan divisors split at $b_4 = 0$, so we can see that $z = 0$ has eight irreductible components at $b_4 = 0$, these are

Component of $(z = b_4 = 0)\|_{\tilde{Y}_4}$	Equations in $\tilde{Y}_4$	Cartan charges	Multiplicity
$(S_2 - E_1) \cdot (4c_1 - 3S_2)$	$z_1 = 0$ $b_4 = 0$	$(1, 0, 0, 0, 0, 0, 0)$	1
$(E_1 - E_2 - E_3 - E_4 - E_6) \cdot (4c_1 - 3S_2)$	$\alpha_6 = 0$ $b_4 = 0$	$(-2, 1, 0, 0, 0, 0, 0)$	2
$E_4 \cdot (4c_1 - 3S_2)$	$\delta = 0$ $b_4 = 0$	$(1, -2, 1, 0, 0, 0, 0)$	3
$(E_3 - E_4 - E_5 - E_8) \cdot (4c_1 - 3S_2)$	$\gamma_8 = 0$ $b_4 = 0$	$(0, 1, -2, 1, 0, 0, 1)$	4
$E_5 \cdot (\sigma + 3c_1 - E_1$ $-E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8)$	$\epsilon = 0$ $y_8 = 0$	$(0, 0, 0, -1, 0, 1, 1)$	6
$(E_2 - E_3 - E_5 - E_7) \cdot (4c_1 - 3S_2)$	$\beta_7 = 0$ $b_4 = 0$	$(0, 0, 0, 1, -2, 1, 0)$	2
$(-E_2 + E_3 + E_5 + 2E_7) \cdot (4c_1 - 3S_2)$	$\zeta_7 = 0, \beta_7 \neq 0$ $b_4 = 0$	$(0, 0, 0, 0, 1, -2, 0)$	4
$(-E_3 + E_4 + E_5 + 2E_8) \cdot (4c_1 - 3S_2)$	$\zeta_8 = 0, \gamma_8 \neq 0$ $b_4 = 0$	$(0, 0, 1, 0, 0, 0, -2)$	5

(6.15)

The splitting of the weight associated to the fourth root is

$$(0, 0, 1, -2, 1, 0, 0) \to 2 \times (0, 0, 0, -1, 0, 1, 1) + (0, 0, 0, 0, 1, -2, 0) + (0, 0, 1, 0, 0, 0, -2)\,. \tag{6.16}$$

The latter two components here are just Cartan divisors, but we can see from the table in Appendix C that the repeated component corresponds to $\mu_{\mathbf{56}} - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6 - \alpha_7$ and so along this curve we have matter in the **56**, which fits with

the new irreducible components along this curve intersecting to give an E_8 structure, due to decomposing the adjoint of E_8 under the breaking $E_8 \to E_7 \times SU(2)$:

$$\mathbf{248} \to (\mathbf{133}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{56}, \mathbf{2}) \,. \tag{6.17}$$

6.3.2 Yukawa Coupling

We expect to get an interaction structure at the codimension-3 enhancement point $z = b_4 = b_6 = 0$, however, none of the Cartan divisors undergo further splitting here. The only one which features b_6 in its equation vanishes entirely. Hence we have no interaction terms, and so conclude that an E_7 GUT is not possible in this formalism if we are only interested in generic enhancements.

Chapter 7

Summary and Outlook

In this thesis we have focussed on constructing GUTs in F-theory, starting from a general Tate form for singularity type corresponding to the desired GUT group, we have seen explicitly for SO(10) how this will generically lead to the required matter and Yukawa couplings. This is also the case for $SU(5)$ [13] and E_6 [14] GUTs in F-theory. The fact that matter with the desired interactions and structure appears without needing any more specific criteria satisfied is particularly appealing, since after assuming a GUT group, one does not need to ask the question of why the matter arranges itself in this way, as it is the generic case. Said another way, whilst we do need the extra ingredients of enhancement loci, in addition to the GUT surface, in order to obtain matter and Yukawa couplings, these will be present in most cases without putting any severe constraints on the geometry.

Interestingly we have seen that we do not get the naively expected enhancements at codimension-3. This is in line with the assertions of [13] that we should not be looking for symmetry enhancement to give matter and Yukawa couplings, but instead be looking at the Cartan charges of the irreducible components of the fiber and how they change at the loci of enhancement. One thing to note is that whilst we do not get the expected symmetry enhancements, the resulting interaction terms still match those that were assumed to come from these expected enhancements. This means that the resulting physics (at the level of the interaction terms) is unchanged from previous assumptions.

One feature of $SO(10)$ which was not seen in the $SU(5)$ case, is that the constraints on the G-flux also put constraints on the geometry of the base B_3 or the surface S_2,

which before the construction of the G-flux we had said nothing about the structure of, other than them being holomorphic. Whilst this particular construction turned out not to be phenomenologically promising due to the vanishing chirality of the **10**, this is still an interesting feature.

Motivated by the lack of actual symmetry enhancement occuring, instead simply obtaining new states within the current gauge group, we then looked at E_7, which would have been difficult to consider if we had tried to use the intersecting 7-branes picture of starting from a gauge group of at least 2 ranks higher than E_7 yet still containing it as a subgroup. Due to the fact that an E_8 singularity is the maximal allowed if we wish to preserve the Calabi-Yau condition [57]. This construction turned out not to contain any interaction terms, and so is not viable as a GUT model, however it is still interesting, since it was not obvious beforehand that we would not obtain a Yukawa term.

$SO(10)$ GUTs are less studied than $SU(5)$, since $SU(5)$ is considered the simplest, as it has both the smallest rank and adjoint, leading to fewer exotics upon breaking to the Standard Model. However, there is no experimental reason for choosing $SU(5)$ over $SO(10)$. Here, we have seen that in $SO(10)$ we will generically have a problem with the chirality of the **10**, but this is expected to be rectified if one uses a split spectral cover as in [46]. A split spectral cover is a non-generic situation, so this instantly makes the model less appealing. Split spectral covers have recently been studied with the formalism used in this thesis [63]. $SO(10)$ does have some advantages over $SU(5)$ however, for example one has the benefit of the right-handed neutrinos being contained in the **16**, whereas in $SU(5)$ they are instead a singlet. This means that in $SO(10)$ we can localise them on the brane and so we can generate their interaction terms in the same way as the other matter. In $SU(5)$ however, right-handed neutrinos are more difficult to accommodate since they are not localised to the GUT surface, thus must be treated differently to the other matter [66].

GUT breaking is more complicated for $SO(10)$ than it would be for the case of $SU(5)$, since there are multiple ways to embed the Standard Model gauge group inside $SO(10)$, as discussed in Section 2.6.2. However we expect that we could break the GUT

by turning on $U(1)$ fluxes on the GUT-brane worldvolume.

The $SO(10)$ model described in this thesis does not yet have all the required features of a realistic GUT, but is a promising start, which has so far yielded some previously unexpected results.

Appendix A

Intersection Relations for $SO(10)$

A.1 Intersection Relations in X_5

Here we list relations that hold in X_5 coming from the constraints we get at each blow-up concerning the non-vanishing of sets of homogeneous coordinates.

$$\begin{aligned}
0 &= \sigma\,(\sigma + 2c_1)\,(\sigma + 3c_1) \\
0 &= (\sigma + 2c_1 - E_1)\,(\sigma + 3c_1 - E_1)\,(S_2 - E_1) \\
0 &= (\sigma + 2c_1 - E_1 - E_2)\,(\sigma + 3c_1 - E_1 - E_2)\,(E_1 - E_2) \\
0 &= (\sigma + 3c_1 - E_1 - E_2 - E_3)\,(E_1 - E_2 - E_3)\,(E_2 - E_3) \\
0 &= (\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4)\,(E_1 - E_2 - E_3 - E_4) \\
0 &= (\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4 - E_5)\,(E_2 - E_3 - E_5)
\end{aligned} \tag{A.1}$$

As all blow-ups are done in the $w = 1$ patch, we have that

$$0 = \sigma \cdot E_i \tag{A.2}$$

since the class σ given by $w = 0$ cannot then intersect any of the exceptional divisors.

A.2 Intersection Relations in $\tilde{Y}_4$

Consider the sets of coordinates associated to each blow-up and see which ones cannot simultaneously vanish:

$$\begin{aligned}
[w, x, y] &= [w, x_2\zeta_4\alpha_5^2\beta^3\delta_4\delta_5^2, y_5\zeta_4\alpha_5^2\beta^4\delta_4^2\delta_5^3] \\
[x_1, y_1, z_1] &= [x_2\alpha_5\beta\delta_5, y_5\alpha_5\beta^2\delta_4\delta_5^2, z_1] \\
[x_2, y_2, \zeta_2] &= [x_2, y_5\beta\delta_4\delta_5, \zeta_4\beta\delta_4] \\
[y_3, \zeta_3, \alpha_3] &= [y_5\delta_4\delta_5, \zeta_4\delta_4, \alpha_5\delta_5] \\
[y_4, \zeta_4] &= [y_5\delta_5, \zeta_4] \\
[y_5, \alpha_5]
\end{aligned} \tag{A.3}$$

With these relations and the equation for $\tilde{Y}_4$, we see that we cannot have solutions to any of the following equations:

$$\begin{aligned}
x_2 = \zeta_4 &= 0 \\
x_2 = \beta &= 0 \\
x_2 = \delta_4 &= 0 \\
y_5 = z_1 &= 0 \\
y_5 = \zeta_4 &= 0 \\
y_5 = \alpha_5 &= 0 \\
z_1 = \alpha_5 &= 0 \\
z_1 = \beta &= 0 \\
z_1 = \delta_5 &= 0 \\
\zeta_4 = \alpha_5 &= 0 \\
\zeta_4 = \delta_5 &= 0 \\
\alpha_5 = \delta_4 &= 0 \\
\delta_4 = \delta_5 &= 0\,.
\end{aligned} \tag{A.4}$$

These imply the following intersection relations in $\tilde{Y}_4$:

$$\begin{aligned}
(\sigma + 2c_1 - E_1 - E_2) \cdot_{\tilde{Y}_4} (E_1 - E_2 - E_3 - E_4) &= 0 \\
(\sigma + 2c_1 - E_1 - E_2) \cdot_{\tilde{Y}_4} (E_3) &= 0 \\
(\sigma + 2c_1 - E_1 - E_2) \cdot_{\tilde{Y}_4} (E_4) &= 0 \\
(\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4 - E_5) \cdot_{\tilde{Y}_4} (S_2 - E_1) &= 0 \\
(\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4 - E_5) \cdot_{\tilde{Y}_4} (E_1 - E_2 - E_3 - E_4) &= 0 \\
(\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4 - E_5) \cdot_{\tilde{Y}_4} (E_2 - E_3 - E_5) &= 0 \\
(S_2 - E_1) \cdot_{\tilde{Y}_4} (E_2 - E_3 - E_5) &= 0 \\
(S_2 - E_1) \cdot_{\tilde{Y}_4} (E_3) &= 0 \\
(S_2 - E_1) \cdot_{\tilde{Y}_4} (E_5) &= 0 \\
(E_1 - E_2 - E_3 - E_4) \cdot_{\tilde{Y}_4} (E_2 - E_3 - E_5) &= 0 \\
(E_1 - E_2 - E_3 - E_4) \cdot_{\tilde{Y}_4} (E_5) &= 0 \\
(E_2 - E_3 - E_5) \cdot_{\tilde{Y}_4} (E_4) &= 0 \\
(E_4) \cdot_{\tilde{Y}_4} (E_5) &= 0\,.
\end{aligned} \tag{A.5}$$

There are two more relations we can get by considering the surface $z_1 = 0$ in $\tilde{Y}_4$. Since $z_1 = 0$ means that we cannot have y_5 α_5, β or δ_5 vanishing, we set these equal to 1, this leaves

$$w\delta_4 = x_2^3\zeta_4\,. \tag{A.6}$$

By setting $\zeta_4 = 0$ implies $\delta_4 = 0$, and vice versa, so these are equivalent, giving

$$(S_2 - E_1) \cdot_{\tilde{Y}_4} (E_1 - E_2 - E_3 - E_4) = (S_2 - E_1) \cdot_{\tilde{Y}_4} (E_4)\,, \tag{A.7}$$

or

$$(S_2 - E_1) \cdot_{\tilde{Y}_4} (E_1 - E_2 - E_3 - 2E_4) = 0\,. \tag{A.8}$$

Also we see that setting $w = 0$ implies $x_2 = 0$ and vice versa, so these are also equivalent, which gives the relation

$$(S_2 - E_1) \cdot_{\tilde{Y}_4} (2c_1 - E_1 - E_2) = 0\,. \tag{A.9}$$

Appendix B

$SO(10)$ weights and roots

Cartan charges of **10**	Root
$(1,0,0,0,0)$	μ_{10}
$(-1,1,0,0,0)$	$\mu_{10}-\alpha_1$
$(0,-1,1,0,0)$	$\mu_{10}-\alpha_1-\alpha_2$
$(0,0,-1,1,1)$	$\mu_{10}-\alpha_1-\alpha_2-\alpha_3$
$(0,0,0,-1,1)$	$\mu_{10}-\alpha_1-\alpha_2-\alpha_3-\alpha_4$
$(0,0,0,1,-1)$	$\mu_{10}-\alpha_1-\alpha_2-\alpha_3-\alpha_5$
$(0,0,1,-1,-1)$	$\mu_{10}-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5$
$(0,1,-1,0,0)$	$\mu_{10}-\alpha_1-\alpha_2-2\alpha_3-\alpha_4-\alpha_5$
$(1,-1,0,0,0)$	$\mu_{10}-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4-\alpha_5$
$(-1,0,0,0,0)$	$\mu_{10}-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4-\alpha_5$
Cartan charges of **16**	Root
$(0,0,0,0,1)$	μ_{16}
$(0,0,1,0,-1)$	$\mu_{16}-\alpha_5$
$(0,1,-1,1,0)$	$\mu_{16}-\alpha_3-\alpha_5$
$(1,-1,0,1,0)$	$\mu_{16}-\alpha_2-\alpha_3-\alpha_5$
$(0,1,0,-1,0)$	$\mu_{16}-\alpha_3-\alpha_4-\alpha_5$
$(-1,0,0,1,0)$	$\mu_{16}-\alpha_1-\alpha_2-\alpha_3-\alpha_4$
$(1,-1,1,-1,0)$	$\mu_{16}-\alpha_2-\alpha_3-\alpha_4-\alpha_5$
$(-1,0,1,-1,0)$	$\mu_{16}-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5$
$(1,0,-1,0,1)$	$\mu_{16}-\alpha_2-2\alpha_3-\alpha_4-\alpha_5$
$(-1,1,-1,0,1)$	$\mu_{16}-\alpha_1-\alpha_2-2\alpha_3-\alpha_4-\alpha_5$
$(1,0,0,0,-1)$	$\mu_{16}-\alpha_2-2\alpha_3-\alpha_4-2\alpha_5$
$(0,-1,0,0,1)$	$\mu_{16}-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4-\alpha_5$
$(-1,1,0,0,-1)$	$\mu_{16}-\alpha_1-\alpha_2-2\alpha_3-\alpha_4-2\alpha_5$
$(0,-1,1,0,-1)$	$\mu_{16}-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4-2\alpha_5$
$(0,0,-1,1,0)$	$\mu_{16}-\alpha_1-2\alpha_2-3\alpha_3-\alpha_4-2\alpha_5$
$(0,0,0,-1,0)$	$\mu_{16}-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4-2\alpha_5$

(B.1)

Appendix C

E_7 weights and roots

Cartan charges of **56**	Root
$(0,0,0,0,0,1,0)$	μ_{56}
$(0,0,0,0,1,-1,0)$	$\mu_{56}-\alpha_6$
$(0,0,0,1,-1,0,0)$	$\mu_{56}-\alpha_5-\alpha_6$
$(0,0,1,-1,0,0,0)$	$\mu_{56}-\alpha_4-\alpha_5-\alpha_6$
$(0,1,-1,0,0,0,1)$	$\mu_{56}-\alpha_3-\alpha_4-\alpha_5-\alpha_6$
$(1,-1,0,0,0,0,1)$	$\mu_{56}-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6$
$(0,1,0,0,0,0,-1)$	$\mu_{56}-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7$
$(-1,0,0,0,0,0,1)$	$\mu_{56}-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6$
$(1,-1,1,0,0,0,-1)$	$\mu_{56}-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7$
$(-1,0,1,0,0,0,-1)$	$\mu_{56}-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7$
$(1,0,-1,1,0,0,0)$	$\mu_{56}-\alpha_2-2\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7$
$(-1,1,-1,1,0,0,0)$	$\mu_{56}-\alpha_1-\alpha_2-2\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7$
$(1,0,0,-1,1,0,0)$	$\mu_{56}-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6-\alpha_7$
$(0,-1,0,1,0,0,0)$	$\mu_{56}-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7$
$(-1,1,0,-1,1,0,0)$	$\mu_{56}-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6-\alpha_7$
$(1,0,0,0,-1,1,0)$	$\mu_{56}-\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-\alpha_7$
$(0,-1,1,-1,1,0,0)$	$\mu_{56}-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6-\alpha_7$
$(-1,1,0,0,-1,1,0)$	$\mu_{56}-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-\alpha_7$
$(1,0,0,0,0,-1,0)$	$\mu_{56}-\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-2\alpha_6-\alpha_7$
$(0,0,-1,0,1,0,1)$	$\mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4-\alpha_5-\alpha_6-\alpha_7$
$(0,-1,1,0,-1,1,0)$	$\mu_{56}-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-\alpha_7$
$(-1,1,0,0,0,-1,0)$	$\mu_{56}-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-2\alpha_6-\alpha_7$
$(0,0,-1,1,-1,1,1)$	$\mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-\alpha_7$
$(0,0,0,0,1,0,-1)$	$\mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4-\alpha_5-\alpha_6-2\alpha_7$
$(0,-1,1,0,0,-1,0)$	$\mu_{56}-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-2\alpha_6-\alpha_7$
$(0,0,0,-1,0,1,1)$	$\mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-3\alpha_4-2\alpha_5-\alpha_6-\alpha_7$
$(0,0,-1,1,0,-1,1)$	$\mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4-2\alpha_5-2\alpha_6-\alpha_7$
$(0,0,0,1,-1,1,-1)$	$\mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-2\alpha_7$
$(0,0,0,-1,1,-1,1)$	$\mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-3\alpha_4-2\alpha_5-2\alpha_6-\alpha_7$
$(0,0,1,-1,0,1,-1)$	$\mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-3\alpha_4-2\alpha_5-\alpha_6-2\alpha_7$
$(0,0,0,1,0,-1,-1)$	$\mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4-2\alpha_5-2\alpha_6-2\alpha_7$
$(0,0,0,0,-1,0,1)$	$\mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-3\alpha_4-3\alpha_5-2\alpha_6-\alpha_7$

(C.1)

$$
\begin{array}{c|l}
(0,0,1,-1,1,-1,-1) & \mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-3\alpha_4-2\alpha_5-2\alpha_6-2\alpha_7 \\
(0,1,-1,0,0,1,0) & \mu_{56}-\alpha_1-2\alpha_2-4\alpha_3-3\alpha_4-2\alpha_5-\alpha_6-2\alpha_7 \\
(0,0,1,0,-1,0,-1) & \mu_{56}-\alpha_1-2\alpha_2-3\alpha_3-3\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7 \\
(0,1,-1,0,1,-1,0) & \mu_{56}-\alpha_1-2\alpha_2-4\alpha_3-3\alpha_4-2\alpha_5-2\alpha_6-2\alpha_7 \\
(1,-1,0,0,0,1,0) & \mu_{56}-\alpha_1-3\alpha_2-4\alpha_3-3\alpha_4-2\alpha_5-\alpha_6-2\alpha_7 \\
(0,1,-1,1,-1,0,0) & \mu_{56}-\alpha_1-2\alpha_2-4\alpha_3-3\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7 \\
(1,-1,0,0,1,-1,0) & \mu_{56}-\alpha_1-3\alpha_2-4\alpha_3-3\alpha_4-2\alpha_5-2\alpha_6-2\alpha_7 \\
(-1,0,0,0,0,1,0) & \mu_{56}-2\alpha_1-3\alpha_2-4\alpha_3-3\alpha_4-2\alpha_5-\alpha_6-2\alpha_7 \\
(1,-1,0,1,-1,0,0) & \mu_{56}-\alpha_1-3\alpha_2-4\alpha_3-3\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7 \\
(0,1,0,-1,0,0,0) & \mu_{56}-\alpha_1-2\alpha_2-4\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7 \\
(-1,0,0,0,1,-1,0) & \mu_{56}-2\alpha_1-3\alpha_2-4\alpha_3-3\alpha_4-2\alpha_5-2\alpha_6-2\alpha_7 \\
(-1,0,0,1,-1,0,0) & \mu_{56}-2\alpha_1-3\alpha_2-4\alpha_3-3\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7 \\
(1,-1,1,-1,0,0,0) & \mu_{56}-\alpha_1-3\alpha_2-4\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7 \\
(-1,0,1,-1,0,0,0) & \mu_{56}-2\alpha_1-3\alpha_2-4\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7 \\
(1,0,-1,0,0,0,1) & \mu_{56}-\alpha_1-3\alpha_2-5\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7 \\
(-1,1,-1,0,0,0,1) & \mu_{56}-2\alpha_1-3\alpha_2-5\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7 \\
(-1,1,0,0,0,0,-1) & \mu_{56}-\alpha_1-3\alpha_2-5\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-3\alpha_7 \\
(0,-1,0,0,0,0,1) & \mu_{56}-2\alpha_1-4\alpha_2-5\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7 \\
(-1,1,0,0,0,0,-1) & \mu_{56}-2\alpha_1-3\alpha_2-5\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-3\alpha_7 \\
(0,-1,1,0,0,0,-1) & \mu_{56}-2\alpha_1-4\alpha_2-5\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-3\alpha_7 \\
(0,0,-1,1,0,0,0) & \mu_{56}-2\alpha_1-4\alpha_2-6\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-3\alpha_7 \\
(0,0,0,-1,1,0,0) & \mu_{56}-2\alpha_1-4\alpha_2-6\alpha_3-5\alpha_4-3\alpha_5-2\alpha_6-3\alpha_7 \\
(0,0,0,0,-1,1,0) & \mu_{56}-2\alpha_1-4\alpha_2-6\alpha_3-5\alpha_4-4\alpha_5-2\alpha_6-3\alpha_7 \\
(0,0,0,0,0,-1,0) & \mu_{56}-2\alpha_1-4\alpha_2-6\alpha_3-5\alpha_4-4\alpha_5-3\alpha_6-3\alpha_7
\end{array}
\tag{C.2}
$$

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