On the Theory of Truthful and Fair Pricing for Banner Advertisements

Thesis submitted in accordance with the requirements of
the University of Liverpool for the degree of Doctor in Philosophy by

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September 2013

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Notations

The following notations and abbreviations are found throughout this thesis:

n	The number of advertisers or buyers
m	The number of slots or items
q_j	The quality or desirability of slot j
d_i	The demand of buyer i
v_i	The valuation of buyer i
F_i	Cumulative distribution function of buyer i's value
f_i	Probability density function of buyer i's value
$u_i(\mathbf{X}, \mathbf{p})$	The utility of buyer i under outcome $(\mathbf{X}, \mathbf{p})$
$\phi_i(v_i)$	Buyer i's virtual value defined by $v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$
B_i	Buyer i's budget
$t_i(\mathbf{v})$	The (expected) total quality of items that buyer i is assigned
$T_i(v_i)$	The expectation of $t_i(\mathbf{v})$ over all other buyers' bids except buyer i
$P_i(v_i)$	The expectation of $p_i(\mathbf{v})$ over all other buyers' bids except buyer i
v_{-i}	The joint valuations of all bidders other than i
$\mathbf{v}$	The valuation profile of all the buyers
$\mathbf{q}$	The quality profile of all the slots
K	The number of distinct values in the set $\{v_1, \ldots, v_n\}$
A_k	The set of buyers that have the kth largest value
$\mathbf{x}$	The indicator variable profile, where $x_{ij} = 1$ denotes item j is assigned to buyer i
$[s]$	The set {1,2,. . . ,s}
$\mathrm{E}_{\mathbf{v}}$	Short for $\mathrm{E}_{\mathbf{v} \in \mathbf{V}}$
$\mathrm{E}_{v_{-i}}$	Short for $\mathrm{E}_{v_{-i} \in V_{-i}}$
M	An auction or a mechanism
$\mathbf{X}$	The allocation set of items
$\mathbf{p}$	The buyers' payments or the prices of the slots or items
$Rev(M)$	The expected revenue of the mechanism M
BIC	Bayesian incentive compatible
IR	Individual rationality
α	The approximate ratio
CE	Competitive equilibrium
EF	Envy-free

LP	Linear programming
DP	Dynamic programming
PTAS	A polynomial time approximate schemes
Δ	The upper bound on the demand of any single buyer
X3C	The exact cover by 3-sets problem
GSP	Generalized Second Prize
V_k	The kth sample vector $\{v_k^1, v_k^2, \cdots, v_k^n\}$, where each v_k^i is sampled from $U[20, 80]$.

Definitions

The following definitions and concepts are found in this thesis:

Bayesian Incentive Compatible	Definition 2.1.1
Individual Rationality	Definition 2.1
Relaxed Envy-free Pricing	Definition 2.2.1
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Preface

The sources of other materials are identified. This work has not been submitted for any other degree or professional qualification except as specified. We illustrate that which chapters in this thesis have appeared in which paper as below. Chapter 3.1, 3.2 3.3, 3.4 and 3.7 has been published in the paper [23]; Chapter 3.5, 3.6, 4.5, 5.5 and 6 has been published in the paper [22]; Chapter 4.1, 4.2, 4.3, 4.4, 5.1, 5.2, 5.3 and 5.4 appeared in the paper [17].

Abstract

We consider revenue maximization problem in banner advertisements under two fundamental concepts: Envy-freeness and truthfulness. Envy-freeness captures fairness requirement among buyers while truthfulness gives buyers the incentive to announce truthful private bids. A extension of envy-freeness named competitive equilibrium, which requires both envy-freeness and market clearance conditions, is also investigated. For truthfulness also called incentive compatible, we adapt Bayesian settings, where each buyer's private value is drawn independently from publicly known distributions. Therefore, the truthfulness we adopt is Bayesian incentive compatible mechanisms.

Most of our results are positive. We study various settings of revenue maximizing problem e.g. competitive equilibrium and envy-free solution in relaxed demand, sharp demand and consecutive demand case; Bayesian incentive compatible mechanism in relaxed demand, sharp demand, budget constraints and consecutive demand cases. Our approach allows us to argue that these simple mechanisms give optimal or approximate-optimal revenue guarantee in a very robust manner.

Acknowledgements

Pursuing a PhD degree is a wonderful intellectual journey. This journey would not be successful or as fun without the help and support from the various people I have had the fortune of knowing both during and before grad school. Above all, I'd like to thank my PhD primal advisor Xiaotie Deng. Xiaotie taught me a lot on almost every important aspect of research. Xiaotie taught me how to judge the importance of a research problem by examining its interpretations carefully, helped me to learn how to discover a problem and explore its properties and overcome it, and showed me what is the right way of presenting things in writing papers. I've got tremendous benefit from learning these things from Xiaotie. I would like to thank my second supervisor Paul Goldberg, whose intuition of attacking a problem impresses me a lot and who is also a very nice collaborator. My thanks goes to Ning Chen, one of my collaborators, for always being encouraging to me even if I went the wrong way, who also shapes my mind that there is always a simple explanation of apparently complex issues. I would like to thank Bo Tang, a very brilliant guy and very smart collaborator for considerate discussion with me to tackle problems, resulting in many new ideas. I also thank Bozidar and Milan for the wonderful summer internship experiences, which were eye-openers to me. I thank my defense examiners Mingyu (internal) and Kousha (external) for giving me various feedback on the thesis.

I thank University of Liverpool for providing such a great academic environment, and for surrounding me with an amazing group of smart people: Piotr, Leslie, Mingyu, Rahul, Giorgos, Martin among others. Pursuing a doctorate degree and conducting this research would not have been possible without financial support. For this I thank the University of Liverpool for providing me a scholarship for the last three years. I thank Clare and Helen for being a great help with my PhD studies, who assistant to supervise PhD requirement issues. My interest in theory was ignited by seminars organized by Prof. Fengshan Bai at Tsinghua University, who later became my master thesis advisor. I thank Prof. Bai for guiding me into theory, and for being extremely kind while being stern on mathematical rigor, providing a stress-free environment of research. Finally, last but not least, I thank my parents for the unconditional support throughout.

Chapter 1

Introduction

1.1 Introduction

Arguably, online advertising has been the most successful new business enabled through the World Wide Web. At the same time as it has grown to a modern media realm and as it has evidenced the feasibility of Internet monetization, it has also been relied on as a major, and sometimes the primary, source of financing to fuel the creations of thousands of new Internet services. According to the Internet Advertising Bureau [47], the online ad annual revenues hit $31 Billion in 2011. Moreover, the internet advertising revenues of the first half year of 2012 continued to reach a new height of $17 billion, representing a 14 percent increase year-on-year [48].

How to price the online advertisement has been a central problem in the important industry. The search engine based advertising model, the sponsored search auction, has been extensively studied in the literatures since the pioneer work of Edelman, Ostrovsky, and Schwarz [28], as well as Varian [57], in the context of position auction, especially on the generalized second price auction (GSP). Here each position for placing an ad is associated with a quality value representing the prominence of the position for search engine users to be attracted to click on it. On the other hand, each advertiser is associated with a value related to its potential gains from the attracted users because of its profitability and possibly the attractiveness of its ad design. The product of the two factors is used to decide on the placement and to price the ads by the GSP protocol and created a success for the future designers of the Internet advertisement model to emulate.

It is difficult to fully carry over the success of the sponsored search model to many other settings of advertisement. One problem is caused by the use of banner ads which may require more than one slot which was used by a single ad in the text mode. How to allocate and price ads of different sizes efficiently to match up to the success of the GSP protocol in placing the text ads in the sponsored search model is an important challenge in the new form of advertisement. In banner advertisement other display-related advertisement mode, a banner is often priced under the CPM (cost per thousand impression) scheme based on bringing up the ads to webpage users impressions. Their market share has been steadily increasing. In 2011, 35 percent online advertisement income came from the display-related advertising sector [47].

Such Internet banner advertising can involve various elements of mixed media, now ranging from texts and graphics to streaming audios and videos. Hence, demand is a practical consideration. Advertisers are granted options to choose the proper way to display their own ads, which may occupy more than one slot for the traditional text ads, which can be formulated as sharp/consecutive models. Subsequently, the Web designer has to address the problem of how to display ads with distinct sizes and as a market maker, to devise an allocation and pricing rule to increase her/his own revenue. There may be different values (per unit slot) associated with those different types and sizes of ads.

Another motivation of our study of banner ads on matching market comes from TV advertising where inventories of a commercial break are usually divided into slots of several seconds each, and slots have various qualities measuring their expected number of viewers and corresponding attraction. We may have a longer ad of 3 slots or a short one with simply one slot. Advancing into multiple choices of multi-slot ads from every advertiser, it has made obsolete the nice structures developed in the GSP auction protocol. In banner (or newspaper) advertising, advertisers may request different sizes or areas for their displayed ads, which may be decomposed into a number of base units. Some advertisers may only care about how many areas of ads displayed without concerning any other things, which can be formulated as sharp demand models. The relaxed demand model can be directly used to classic text ads in sponsored search advertisement.

In this thesis we develop new algorithmic insight to design a methodology for efficient pricing mechanisms to achieve social optimality and to extract optimum revenue.

1.1.1 Our Modeling Approach

We have a set of *buyers* (advertisers) and a set of *items* to be sold (the ad slots on a web page). We address the challenge of computing prices that satisfy certain desirable properties. Next we describe the elements of the model in more details.

- **Items.** Our model considers the geometric organization of the ad slots, which commonly has the slots arranged in some sequence (typically, from top to bottom in the right-hand side of a web page). The slots are of variable quality. In the study of sponsored search auctions, a standard assumption is that the quality (corresponding to click-through rate) is highest at the beginning of the sequence and then monotonically decreases. Here we consider a generalization where the quality may go down and up, subject to a limit on the total number of local maxima (which we call *peaks*), corresponding to focal points on the web page. As we will show later in some pricing model, without this limit the revenue maximization problem is NP-hard.

- **Buyers.** A buyer (advertiser) may want to purchase multiple slots, so as to display a larger ad. We will consider various case of demand models. Thus, consider each buyer i has a fixed *demand* d_i, which is the number of slots she needs for her ad. Three important aspects of this are
  - *relaxed* multi-unit demand, where buyer i can buy the number of slots up to d_i
  - *sharp* multi-unit demand, referring to the fact that buyer i should be allocated d_i items, or none at all; there is no point in allocating any fewer
  - *consecutive* demand for the allocated items, where buyer i is allocated d_i consecutive slots or nothing.

  These constraints give rise to new and interesting combinatorial pricing problems.

- **Valuations.** We assume that each buyer i has a parameter v_i representing the value she assigns to a slot of unit quality. Valuations for multiple slots are additive, so that a buyer with demand d_i would value a block or discrete sequence of d_i slots to be their total quality, multiplied by v_i. This valuation model has been considered by Edelman et al. [28] and Varian [57] in their seminal work for keywords advertising.

This scenario of sharp and consecutive demand captures some similarity but is still quite different from single-minded buyers (i.e., each one desires a fixed combination of items).

Pricing mechanisms. Given the valuations and demands from the buyers, the market maker decides on a price vector for all slots and an allocation of slots to buyers, as an output of the market. The question is one of which output the market maker should choose to achieve certain objectives. We consider three approaches:

- **Truthful mechanism** whereby the buyers report their demands and values to the market maker; then prices are set in such a way as to ensure that the buyers have the incentive to report their true valuations. We give a revenue-maximizing approach (i.e., maximizing the total price paid), within this framework.

- **Envy-free solution** whereby we prescribe certain constraints on the prices so as to guarantee each buyer is envy-free (fairness), as explained below.

- **Competitive equilibrium** whereby we prescribe certain constraints on the prices so as to guarantee certain well-known notions of fairness and efficiency, as explained in Chapter 2.

The mechanisms we exhibit are computationally efficient.

Regarding the design of truthful mechanisms, the point here is that the value v_i of each buyer i is initially private information of that buyer. Truthful market design relies on the general revelation principle [50] to simplify the search for mechanisms with desirable properties, such as one that brings in the maximum revenue. Therefore, it is natural for us to consider market mechanisms that bring in the optimum revenue, while ensuring the participants' incentives to reveal the truth about their private values. Some of the work is in a Bayesian setting, which assumes some prior knowledge of a buyer's value, represented by a probability distribution over her possible values. We exhibit a Bayesian Incentive Compatible mechanism that always extracts the maximal revenue in expectation.

Note that the private information about the value for each advertiser creates an asymmetry among the participants and the market maker. Truthful market design relies on the general revelation principle [50] to simplify the search for mechanisms with desirable properties, such as one that brings in the maximum revenue. Therefore, assuming some prior knowledge about advertisers' valuations, in most of the cases, we propose a Bayesian Incentive Compatible mechanism that always extracts the maximal revenue in expectation.

While truthful mechanisms rule out the possibility of buyers' deviations, it might produce the price discrimination phenomenon that causes discontent among users and lures the arbitrage behavior. If we insist on truthful markets, the unfair auction is the only possibility [20, 33]. Market equilibrium (*Competitive equilibrium*) offers an alternative which offers a sense of fairness to all customers in terms of that no one would prefer to shift to another allocation under the current price vector. Further, all goods are sold, and otherwise priced at zero (market clearance) which none would improve their utility taking those in. In fact, all market participants have maximized their utilities with their allocation under the current price vector.

As one of the central solution concepts in economics, competitive equilibrium has been studied and applied in a variety of domains [49]. In particular, we show that, when pricing with sharp/consecutive demand buyers, competitive equilibrium may not exist; even if an equilibrium is guaranteed to exist, a maximum equilibrium (in which each price is as high as it can be in any solution) may not exist. Thus, we design an algorithm that determines the existence of an equilibrium, and computes a revenue maximizing one if it does.

While (revenue maximizing) competitive equilibrium has a number of nice economic properties and has been recognized as an elegant tool for the analysis of competitive markets, its possible non-existence largely ruins its applicability. Such non-existence is a result of the market clearance condition required in the equilibrium (i.e., unallocated items have to be priced at zero). In most applications, however, especially in advertising markets, market makers are able to manage the amount of supplies. For instance, in TV advertising, publishers can 'freely' adjust the length of a commercial break. Therefore, the market clearance condition becomes arguably unnecessary in those applications. This motivates the study of *envy-free pricing* (here envy-free pricing we mean relaxed or sharp or consecutive or bundle envy-free pricing [32]) which only requires the *fairness* condition in the competitive equilibrium, where no buyer can get a larger utility from any other allocation for the given prices. In contrast with competitive equilibrium, an envy-free solution always exists (a trivial one is obtained by setting all prices to ∞). Once again, taking the interests of both sides of the market into account, revenue maximizing envy-free pricing is a natural solution concept that can be applied in those marketplaces.

The study of algorithmic computation of revenue maximizing envy-free pricing was initiated by Guruswami et al. [39], where the authors considered two special settings with unit demand buyers and single-minded buyers and showed that a revenue maximizing envy-free pricing is NP-hard to compute. Because envy-free pricing has applications in various settings and efficient computation is a critical condition for its applicability, there is a surge of studies on its computational issues since the pioneering work of [39], mainly focusing on approximation solutions and special cases that admit polynomial time algorithms, e.g., [4, 5, 8, 9, 14, 18, 29, 34, 37, 40].

The NP-hardness result of [39] for unit demand buyers implies that we cannot hope for a polynomial time algorithm for general v_{ij} valuations in the multi-unit demand setting, even for the very special case when one has positive values for at most three items [14]. However, it does not rule out positive computational results for special, but important, cases of multi-unit demand. For $v_i q_j$ valuations with multi-unit demand, where the hardness reductions of [14, 39] does not apply.

Despite the recent surge in the studies of algorithmic pricing, multi-unit demand models have not received much attention. Most previous work has focused on two simple special settings: unit demand and single-minded buyers, but arguably multi-unit demand has much more applicability. While the relaxed demand model shares similar properties to unit demand (e.g., existence, solution structure, and computation), the sharp/consecutive demand model has a number of features that unit demand does not possess.

- Existence of equilibrium. In unit or relaxed demand ($v_i q_j$) case, the competitive equilibrium always exists, moreover, the maximum and minimum equilibrium always exists. As discussed above, a competitive equilibrium may not exist in the sharp/consecutive demand model. Further, even if a solutions exist, the solution space may not form a distributive lattice.

- Over-priced items. In unit-demand, the price p_j of any item j is always at most the value v_{ij} of the corresponding winner i. This no longer holds for sharp multi-unit demand. Specifically, even if $p_j > v_{ij}$, buyer i may still want to get j since his net profit from other items may compensate his loss from item j (see Example 2.2.4)[1]. This property enlarges the solution space and adds an extra challenge to finding a revenue maximizing solution.

[1]This phenomenon does occur in our real life. For example, in most travel packages offered by travel agencies, they could lose money for some specific programs; but their overall net profit could always be positive.

Our main results in this thesis are summarized in the following table (Table 1.1), where RM denotes revenue maximization.

RM	Relaxed Demand	Sharp Demand	Consecutive Demand	With Budget
Bayesian Auction	P solved	P solved	NP-hard (arbitrary peak)	2-Approx
			P solved (constant peak)	
Competitive Equilibrium	P solved	P solved	NP-hard (arbitrary peak)	Unkown
			P solved (constant peak)	
Envy-free Solution	P solved	NP-hard (arbitrary demand)	NP-hard (same qualities)	Unkown
		P solved (constant demand)	P solved (same demand)	

Table 1.1: Summary of main results of this thesis

1.2 Organization

This thesis is organized as follows. We begin in Chapter 2 with a detailed description of our banner ads model and the related solution concepts. In Chapter 3, we study the problem in Bayesian model and provide a Bayesian Incentive Compatible Auction with optimal expected revenue for the special case of the single peak in quality values of advertisement positions. Then we extend it to the general case of multiple peaks/valleys in the same section, for multiple peaks the problem is shown to be NP-hard. Next, in Chapter 4, we turn to the prior-free model and propose an algorithm to compute the competitive equilibrium with maximum revenue. Finally, we present results related to various envy-free concepts in Chapter 5. The simulation is presented in Section 6.

1.3 Related Works

This thesis merges three work [17, 23**?**]. A study on search based text ads and display based multi-media ads was conducted by Li and Li [46] to explore their profitabilities. Hunter discussed experimental results identifying factors that affect the prices of banner ads [56], for three types of ad size: 1 slot (mini), 2 slots (standard) and 6 lots (hi-rise).

The theoretical study of position auction (of 1 slot) under the generalized second price auction was initiated in [28, 57]. There has been a series of studies of position auctions in deterministic settings [45]. Our consideration of position auctions in the Bayesian setting fits in the general one dimensional auction design framework. Our study considers continuous distributions on buyers' values. For discrete distribution, [10] presents an optimal mechanism for budget constrained buyers without demand constraints in multi-parameter settings and very recently they also give a general reduction from revenue to welfare maximization in [11]; for buyers with both budget constraints and demand constraints, 2-approximate mechanisms [1] and 4-approximate mechanisms [6] exist in the literature.

There are extensive studies on multi-unit demand in economics, see, e.g., [3, 12, 30]. While our study for relaxed demand model shares the similar property of unit demand model where it is well known that the set of competitive equilibrium prices is non-empty and forms a distributive lattice [38, 55]. This immediately implies the existence of an equilibrium with maximum possible prices; hence, revenue is maximized. Demange, Gale, and Sotomayor [21] proposed a combinatorial dynamics which always converges to a revenue maximizing (or minimizing) equilibrium for unit demand. However, Our study on sharp/consecutive demand buyers exhibit different structure property as unit demand model.

From an algorithmic point of view, the problem of revenue maximization in envy-free pricing was initiated by Guruswami et al. [39], who showed that computing an optimal envy-free pricing is APX-hard for unit-demand bidders and gave an $O(\log n)$ approximation algorithm. Briest [8] showed that given appropriate complexity assumptions, the unit-demand envy-free pricing problem in general cannot be approximated within $O(\log^{\epsilon} n)$ for some $\epsilon > 0$. Hartline and Yan [41] characterized optimal envy-free pricing for unit-demand and showed its connection to mechanism design. Recently, Devanur, Ha and Hartline generalize and characterize the envy-free benchmark from [41] to settings with budgets and characterize the optimal envy-free outcomes for both welfare and revenue, and give prior-free mechanisms that approximate these benchmarks [27]. For the multi-unit demand setting, Chen et al. [18] gave an $O(\log D)$ approximation algorithm when there is a metric space behind all items, where D is the maximum demand, and Briest [8] showed that the problem is hard to approximate within a ratio of $O(n^{\epsilon})$ for some ϵ, unless $NP \subseteq \bigcap_{\epsilon>0} BPTIME(2^{n^{\epsilon}})$. It should be noticed that recent work by M, Feldman et al studies envy-free revenue maximization problem with budget but without demand constraints and present a 2-approximate mechanism for envy-free pricing problem [32]. Another stream of research is on single-minded bidders, including, for example, [4, 5, 9, 19, 29, 39]. To the best of our knowledge, we are the first to study algorithmic computation of multi-unit demand.

Several works in the literature also made an effort to model online advertising [7, 31, 53]. However their focus on the design of expressive auctions and clearing algorithms is substantially different from this work. In their work, the advertisers' consecutive demand are not taken into consideration. Deng et al., had a study on the problem for the VCG protocol, and various GSP type protocols, together with a simulated study [25]. However, it only works in the special case of the sponsored search model where the slots are usually ranked from top to bottom in a decreasing order of their quality scores.

It should be noticed that, besides the research on banner advertisements, we also have some work [24] on Bayesian double auctions, which are motivated from mechanisms of groupon. In [24], optimal mechanisms or constant approximate mechanisms are presented in various settings, e.g. single dimension versus multi dimension, continuous distribution versus discrete distribution, supply limit versus demand limit. Double auction paradigm can be viewed as an extension of our Bayesian setting work in Chapter 3 from single side bidding market to two sides bidding market. The double auction design problem becomes more complicated compared to single side auction since the market maker acts as the middle man to bring buyers and sellers together. A guide to the literature in micro-economics on this topic can be found in [35]. The profit maximization problem for the single buyer/single seller setting has been studied by Myerson and Satterthwaite [52]. Our optimal double auction is a direct extension of their work and, to our best knowledge, fills a clear gap in the economic theory of double auctions. Deshmukh et al. [26], studied the revenue maximization problem for double auctions when the auctioneer has no prior knowledge about bids. Their prior-free model is essentially different from ours. More auction mechanism design problems were studied by many researchers in recent years, but as far as we know, not in the context of optimal double auction design in the Bayesian setting. The most related one is by Jain and Wilkens [43], where they studied the market intermediation problem in a setting with a single unit-demand buyer and a group of sellers. They gave several constant approximate mechanism with various buyer behaviour assumptions. While our setting assumes the existence of a monopoly platform, Rochet and Tirole [54] and Armstrong [2] introduced several different models for two-sided markets and studied platform competition.

Chapter 2

Preliminaries

In our model, a banner advertisement instance consists of n advertisers and m advertising slots (with same size) that are lined up in a line. Each slot is characterized by a number q_j which can be viewed as the quality or desirability of the slot. Each advertiser (or buyer) i wants to display his own ad on the webpage, which may occupy slots in three different demand types:

- *relaxed demand* where buyers can occupy the number of slots up to d_i;
- *sharp demand* where buyers occupy d_i not necessarily consecutive slots or nothing;
- *consecutive demand* where buyers occupy consecutive d_i slots or nothing.

In addition, each buyer has a private number v_i representing her valuation and thus, the ith buyer's value for item j is $v_{ij} = v_i q_j$. In other words, the valuation matrix for n buyers and m items is the outer product of the vectors $\mathbf{v} = (v_i)_i$ and $\mathbf{q} = (q_j)_j$. We use slot and item interchangeably if there is no confusion. Let K be the number of distinct values in the set $\{v_1, \ldots, v_n\}$. Let $A_1, \ldots, A_K$ be a partition of all buyers where each A_k, $k = 1, 2, \ldots, K$, contains the set of buyers that have the kth largest value.

The vector of all the buyers' values is denoted by $\mathbf{v}$ or sometimes $(v_i; v_{-i})$ where v_{-i} is the joint bids of all bidders other than i. Let $\mathbf{V}$ denote the state space or the distribution of v if there is no confusion. We represent a feasible assignment by a vector $\mathbf{x} = (x_{ij})_{i,j}$, where $x_{ij} \in \{0, 1\}$ is simply the indicator variable where $x_{ij} = 1$ denotes item j is assigned to buyer i. Thus we have $\sum_i x_{ij} \leq 1$ for every item j. Given a fixed assignment $\mathbf{x}$, we use t_i to denote the (expected) total quality of items that buyer i is assigned, precisely, $t_i = \sum_j q_j x_{ij}$. In general, when $\mathbf{x}$ is a function of buyers' bids $\mathbf{v}$, we define t_i to be a function of $\mathbf{v}$ such that $t_i(\mathbf{v}) = \sum_j q_j x_{ij}(\mathbf{v})$.

Throughout this thesis, we will often say that slot j is assigned to a buyer set B to denote that j is assigned to some buyer in B. We will call the set of all slots assigned to B the allocation to B. In addition, a buyer will be called a winner if he succeeds in displaying his ad and a loser otherwise. We use the standard notation $[s]$ to denote the set of integers from 1 to s, i.e. $[s] = \{1, 2, \ldots, s\}$. We sometimes use $\sum_i$ instead of $\sum_{i\in[n]}$ to denote the summation over all buyers and $\sum_j$ instead of $\sum_{j\in[m]}$ for items, and the terms $\mathrm{E}_{\mathbf{v}}$ and $\mathrm{E}_{v_{-i}}$ are short for $\mathrm{E}_{\mathbf{v}\in\mathbf{V}}$ and $\mathrm{E}_{v_{-i}\in V_{-i}}$.

We also introduce a special case of the revenue maximizing problem of banner advertisement. In that case, the qualities of items are single peaked structured illustrated in Figure 2.1. That is, there exists a peak slot k such that for any slot $j < k$ on the left side of k, $q_j \geq q_{j-1}$ and for any slot $j > k$ on the right side of k, $q_j \geq q_{j+1}$. In Bayesian settings, we will study this single peak case in Section 3.5 and show how to handle the general case in Section 3.6. Similarly, we say the qualities of items with constant peak if the number of peaks formed by qualities of items is bounded by a constant.

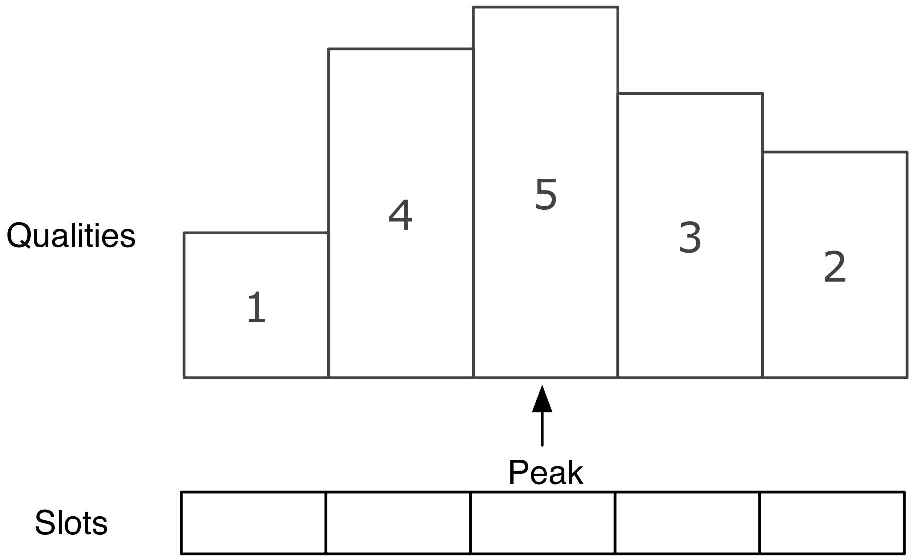

FIGURE 2.1: A case with qualities {1,4,5,3,2}

2.1 Bayesian Mechanism Design

In Chapter 3, we consider the revenue maximization problem of the above banner advertisement instances in the context of mechanism design. Mechanism design studies algorithmic procedures where the input data is not always objective but reported from selfish agents. Following the work of [51], we consider this problem in a Bayesian setting where the seller has a prior knowledge about the buyers' distribution of valuations. This has been shown to be a standard assumption if one wants to optimize the auctioneer's revenue. The auctioneer holds the set of items that can be sold, but does not know the (true) valuations of these items for different buyers. Each buyer is a selfish entity, that privately knows her own valuation for each item (which constitutes its type). Obviously, a strategic buyer may choose to misreport her valuations, which are private, in order to increase her utility, i.e. the valuation of assigned item minus her payment. Since we only consider the auction setting, we may use auction and mechanism interchangeably when there is no ambiguity.

We consider direct-revelation mechanisms: each buyer reports her valuation, and the reported valuation needs to be the same as the buyer's true valuation. The mechanism then computes a feasible assignment and charges the players (i.e., buyers) the payment for the items they have been assigned. An auction M thus consists of a tuple $(\mathbf{X}, \mathbf{p})$, where $\mathbf{X}$ specifies the allocation of items and $\mathbf{p} = (p_i)_i$ specifies the buyers' payments where both $\mathbf{X}$ and $\mathbf{p}$ are functions of the reported valuations $\mathbf{v}$. Thus, the expected revenue of the mechanism is $Rev(M) = \mathrm{E}_{\mathbf{v}}\left[\sum_i p_i(\mathbf{v})\right]$ where $\mathrm{E}_{\mathbf{v}}$ denotes the expectation with respect to components of $\mathbf{v}$ sampled from their respective distributions. From the viewpoint of a single buyer i with private value v_i, her expected utility is given by $\mathrm{E}_{v_{-i}}\left[v_i t_i(\mathbf{v}) - p_i(\mathbf{v})\right]$. The goal of an auctioneer is to maximize her expected revenue; a buyer i is however only interested in maximizing her own expected utility, and may declare a false value if this could increase her utility. The mechanism therefore needs to incentivize the buyers/players to truthfully reveal their values. This is made precise using the notion of Bayesian Incentive Compatibility.

Definition 2.1.1 (Bayesian Incentive Compatible)**.** A mechanism M is called *Bayesian Incentive Compatible* (BIC) iff the following inequalities hold for all i, v_i, v_i'.

$$\mathrm{E}_{v_{-i}}[v_i t_i(\mathbf{v}) - p_i(\mathbf{v})] \geq \mathrm{E}_{v_{-i}}\left[v_i t_i(v_i'; v_{-i}) - p_i(v_i'; v_{-i})\right] \tag{2.1}$$

Besides, we say M is *Incentive Compatible* if M satisfies a stronger condition that $v_i t_i(\mathbf{v}) - p_i(\mathbf{v}) \geq v_i t_i(v_i'; v_{-i}) - p_i(v_i'; v_{-i})$, for all $\mathbf{v}, i, v_i'$,

To put it in words, in a BIC mechanism, no player can improve her *expected* utility (expectation taken over other players' bids) by misreporting her value. An IC mechanism satisfies the stronger requirement that no matter what the other players declare, no player has incentives to deviate.

In the Bayesian setting, we also assume all buyers' values are distributed independently according to publicly known bounded distributions and also the valuations have known upper and lower bounds, i.e. $v_i \in [\underline{v}_i, \overline{v}_i]$ and $\mathbf{v} \in \mathbf{V} = \prod_i [\underline{v}_i, \overline{v}_i]$. The distribution of each buyer is represented by a Cumulative Distribution Function (CDF) F_i and a Probability Density Function (PDF) f_i. In addition, we assume that the concave closure or convex closure or integration of those functions can be computed efficiently.

We assume the buyers are self-interested and rational, thus, each buyer's expected utility is non negative.

Definition 2.1 (Individual Rationality)**.** A mechanism M is called *ex-interim Individual Rational* (IR) iff the following inequalities hold for all i, v_i.

$$\mathrm{E}_{v_{-i}}[v_i t_i(\mathbf{v}) - p_i(\mathbf{v})] \geq 0 \tag{2.2}$$

If $v_i t_i(\mathbf{v}) - p_i(\mathbf{v}) \geq 0$ for all $\mathbf{v}, i$, we say M is *ex-post Individual Rational.*

Obviously, an ex-post Individual Rational mechanism must be ex-interim Individual Rational. The term "ex-interim" here indicates the non-negativity of each agent's utility holds for every possible valuation of this agent, averaged over the possible valuations of the other agents. Ex-post IR holds if and only if the utility of each player cannot be negative for any bidding profile $\mathbf{v}$.

2.2 Competitive Equilibrium and Envy-freeness

In Chapters 4 and 5, we study the revenue maximizing competitive equilibrium and envy-free solution in the full information setting and Bayesian setting respectively. Various concepts of envy-free solution and competitive equilibrium will be investigated. When studying competitive equilibrium, we can make item-wise pricing, that is setting a price for each advertising slot. When studying envy-freeness, we can do both item pricing and bundle pricing (only the bundle of the items is priced). More precisely, an outcome of the market is a tuple $(\mathbf{X}, \mathbf{p})$, where

- $\mathbf{X} = (X_1, \ldots, X_n)$ is an *allocation* vector, where X_i is the set of items that i wins. If $X_i \neq \emptyset$, we say i is a winner and have X_i is a set with no more than d_i items or exactly d_i items, which depends on the concept we'll investigate. Precisely, if the winner is relaxed demand, X_i is a set of items containing no more than d_i items; if the winner is sharp demand, then X_i is a set of items containing exactly d_i items; otherwise, if the winner is consecutive demand, X_i is a set of items containing consecutive d_i items. If $X_i = \emptyset$, i does not win any items and we say i is a loser. Further, since every item has unit supply, we require $X_i \cap X_{i'} = \emptyset$ for any $i \neq i'$.
- $\mathbf{p} = (p_1, \ldots, p_m) \geq 0$ is a *price* vector, where p_j is the price charged for item j, we also use p_i denote the payment of buyer i if there is no confusion;

Given an output $(\mathbf{X}, \mathbf{p})$, recall $v_{ij} = v_i q_j$, let $u_i(\mathbf{X}, \mathbf{p})$ denote the *utility* of i. That is, if $X_i \neq \emptyset$, then $u_i(\mathbf{X}, \mathbf{p}) = \sum_{j \in X_i} (v_{ij} - p_j)$; if $X_i = \emptyset$, then $u_i(\mathbf{X}, \mathbf{p}) = 0$.

As mentioned above, in addition to the efficiency (market clearance) condition where every unsold item is priced at zero, the competitive equilibrium also provide a "free market" where buyers always pick their favorite bundles. This property is also called envy-freeness or fairness in the literature of Economics and Computer Science. We present various concepts of envy-freeness as follows. First, we consider relaxed envy-freeness concept, which means the buyer would not envy the bundle of items up to his demand.

Definition 2.2.1 (Relaxed Envy-free Pricing). We say a tuple $(\mathbf{X}, \mathbf{p})$ is an *relaxed envy-free pricing* solution if every buyer is relaxed envy-free, where a buyer i is relaxed envy-free if the following conditions are satisfied:

- if $X_i \neq \emptyset$, then (i) $|X_i| \leq d_i$ $u_i(\mathbf{X}, \mathbf{p}) = \sum_{j \in X_i} (v_{ij} - p_j) \geq 0$, and (ii) for any other subset of items T with $|T| \leq d_i$, $u_i(\mathbf{X}, \mathbf{p}) = \sum_{j \in X_i} (v_{ij} - p_j) \geq \sum_{j \in T} (v_{ij} - p_j)$;

- if $X_i = \emptyset$ (i.e., i wins nothing), then, for any subset of items T with $|T| \leq d_i$, $\sum\limits_{j \in T} (v_{ij} - p_j) \leq 0$.

The sharp envy-free concept are presented in the following.

Definition 2.2.2 (Sharp Envy-free Pricing)**.** We say a tuple $(\mathbf{X}, \mathbf{p})$ is an *sharp envy-free pricing* solution if every buyer is sharp envy-free, where a buyer i is sharp envy-free if the following conditions are satisfied:

- if $X_i \neq \emptyset$, then (i) X_i is a set of exactly d_i items. $u_i(\mathbf{X}, \mathbf{p}) = \sum\limits_{j \in X_i} (v_{ij} - p_j) \geq 0$, and (ii) for any other subset of items T with $|T| = d_i$, $u_i(\mathbf{X}, \mathbf{p}) = \sum\limits_{j \in X_i} (v_{ij} - p_j) \geq \sum\limits_{j \in T} (v_{ij} - p_j)$;
- if $X_i = \emptyset$ (i.e., i wins nothing), then, for any subset of items T with $|T| = d_i$, $\sum\limits_{j \in T} (v_{ij} - p_j) \leq 0$.

And consecutive envy-free is defined as follows.

Definition 2.2.3 (Consecutive Envy-free Pricing)**.** We say a tuple $(\mathbf{X}, \mathbf{p})$ is an *consecutive envy-free pricing* solution if every buyer is consecutive envy-free, where a buyer i is consecutive envy-free if the following conditions are satisfied:

- if $X_i \neq \emptyset$, then (i) X_i is d_i consecutive items w.r.t. a given total order on the items. $u_i(\mathbf{X}, \mathbf{p}) = \sum\limits_{j \in X_i} (v_{ij} - p_j) \geq 0$, and (ii) for any other subset of consecutive items T with $|T| = d_i$, $u_i(\mathbf{X}, \mathbf{p}) = \sum\limits_{j \in X_i} (v_{ij} - p_j) \geq \sum\limits_{j \in T} (v_{ij} - p_j)$;
- if $X_i = \emptyset$ (i.e., i wins nothing), then, for any subset of consecutive items T with $|T| = d_i$, $\sum\limits_{j \in T} (v_{ij} - p_j) \leq 0$.

Before defining (relaxed) bundle envy-free pricing, we will use the notation $v_i(T)$ to denote the valuation of buyer i for a bundle T, which is given by $v_i(T) = \sum_{j \in T} v_{ij}$ if $|T| \leq d_i$ and $v_i(T) = \max_{T'}\{\sum_{j \in T'} v_{ij} | T' \subset T, |T'| \leq d_i\}$, otherwise. We make clarification that

Definition 2.2.4 (Bundle Envy-free Pricing)**.** We say a tuple $(\mathbf{p}, \mathbf{X})$ is an *bundle envy-free pricing* solution if every buyer is bundle envy-free, where a buyer i is bundle envy-free if the following conditions are satisfied:

- if $X_i \neq \emptyset$, then (i) $u_i(\mathbf{p}, \mathbf{X}) = \sum\limits_{j \in X_i} (v_{ij} - p_j) \geq 0$, and (ii) for any other bundle X_j received by buyer j, $u_i(\mathbf{p}, \mathbf{X}) = \sum\limits_{j \in X_i} (v_{ij} - p_j) \geq v_i(X_j) - \sum_{k \in X_j} p_k$;
- if $X_i = \emptyset$ (i.e., i wins nothing), then, for any bundle X_j obtained by buyer j, $v_i(X_j) - \sum_{k \in X_j} p_k \leq 0$.

To be precise, the above definition of bundle envy-freeness should be called relaxed bundle envy-freeness. If we change the definition of $v_i(T)$ to be the maximum value among all the subsets of T with size exactly d_i or among all consecutive d_i subsets, we can similarly get sharp bundle envy-free and consecutive bundle envy-free concept. We use notations bundle envy-free pricing to denote one of relaxed/sharp/consecutive bundle envy-free concepts if the definition is clear in the corresponding settings. It is not difficult to see that differences among these envy-free concepts are that the set of items the buyer envy is different, e.g. relaxed envy-freeness indicates that the buyer would not envy any set with the number of items no more than the buyer's demand. Envy-freeness captures fairness in the market e.g. for consecutive envy-free pricing, the utility of everyone is maximized at the corresponding allocation for the given prices. That is, if i wins a consecutive subset X_i, then i cannot obtain a higher utility from any other consecutive subset of the same size; if i does not win anything, then i cannot obtain a positive utility from any consecutive subset with size d_i. It is easy to see that envy-free solutions always exists (e.g., set all prices to be ∞ and allocate nothing to every buyer).

Given the definition of above various envy-free concepts, it is interesting to see the inclusion relationship among them. By their definitions we have the following inclusion relationships:
relaxed envy-free $\Rightarrow$ (relaxed) bundle envy-free,
sharp envy-free $\Rightarrow$ (sharp) bundle envy-free,
consecutive envy-free $\Rightarrow$ (consecutive) bundle envy-free.

Example 2.2.1 (Four types of envy-freeness)**.** Suppose there are two buyers i_1 and i_2 with values per unit of quality $v_{i_1} = 10$, $v_{i_2} = 8$ and $d_{i_1} = 1$, $d_{i_2} = 2$. The item j_1, j_2, j_3 with quality as $q_{j_1} = q_{j_3} = 1$ and $q_{j_2} = 3$. By fundamental calculations, the optimal solutions of the three types of envy-freeness are as follows:

- Optimal relaxed envy-free solution, $X_{i_1} = \{j_2\}$, $X_{i_2} = \{j_1, j_3\}$ and $p_{j_1} = p_{j_3} = 8$ and $p_{j_2} = 28$ with total revenue 44;
- Optimal sharp envy-free solution, $X_{i_1} = \{j_2\}$, $X_{i_2} = \{j_1, j_3\}$ and $p_{j_1} = p_{j_3} = 8$ and $p_{j_2} = 28$ with total revenue 44;
- Optimal consecutive envy-free solution, $X_{i_1} = \{j_3\}$, $X_{i_2} = \{j_1, j_2\}$ and $p_{j_1} = p_{j_3} = 6$ and $p_{j_2} = 26$ with total revenue 38;
- Optimal (relaxed) bundle envy-free solution, $X_{i_1} = \{j_2\}$, $X_{i_2} = \{j_1, j_3\}$ and $p_{j_1} = p_{j_3} = 8$ and $p_{j_2} = 30$ with total revenue 46;

The other concept we will consider is competitive equilibrium, which requires that, besides envy-freeness, every unsold item must be priced at zero (or at any given reserve price). Such market clearance condition captures efficiency of the whole market.

Definition 2.2.5 (Competitive Equilibrium)**.** We say a market mechanism $(\mathbf{X}, \mathbf{p})$ is a relaxed/sharp/consecutive competitive equilibrium if it satisfies two conditions.

- $(\mathbf{X}, \mathbf{p})$ must be a relaxed/sharp/consecutive envy-free pricing.
- the unsold items must be priced at zero.

For brevity, we can call a solution envy-free instead of relaxed/sharp/consecutive envy-free, if there is no confusion in the following sections. For a given output $(\mathbf{X}, \mathbf{p})$, the *revenue* collected by the market maker is defined as $\sum_{j=1}^{m} p_j$ (equivalently, $\sum_{i=1}^{n} \sum_{j \in X_i} p_j$). We are interested in revenue maximizing solutions, specifically, revenue maximizing competitive equilibrium.

It is well known that a competitive equilibrium always exists for unit demand buyers (even for general v_{ij} valuations) [55]; for our sharp/consecutive multi-unit demand model, however, a competitive equilibrium may not exist, as the following example shows (however, the relaxed competitive equilibrium always exists)

Example 2.2.2 (Sharp/consecutive competitive equilibrium need not exist)**.** There are two buyers i_1, i_2 with values $v_{i_1} = 10$ and $v_{i_2} = 9$, and demands $d_{i_1} = 1$ and $d_{i_2} = 2$, respectively, and two items j_1, j_2 with unit quality $q_{j_1} = q_{j_2} = 1$. If i_1 wins an item, without loss of generality, say j_1, then j_2 is unsold and $p_{j_2} = 0$; by envy-freeness of i_1, we have $p_{j_1} = 0$. Thus, i_2 envies the bundle $\{j_1, j_2\}$. If i_2 wins both items, then $p_{j_1} + p_{j_2} \leq v_{i_2 j_1} + v_{i_2 j_2} = 18$, implying that $p_{j_1} \leq 9$ or $p_{j_2} \leq 9$; thus, i_1 is not envy-free. Hence, there is no competitive equilibrium in the given instance.

In the unit demand case, it is well-known that the set of equilibrium prices forms a distributive lattice; hence, there exist extremes which correspond to the maximum and the minimum equilibrium price vectors. In our sharp/consecutive demand model, however, even if a competitive equilibrium exists, maximum equilibrium prices may not exist (however, the maximum relaxed competitive equilibrium always exists).

Example 2.2.3 (Maximum equilibrium need not exist for sharp/consecutive buyers)**.** There are two buyers i_1, i_2 with values $v_{i_1} = 10, v_{i_2} = 1$ and demands $d_{i_1} = 2, d_{i_2} = 1$, and two items j_1, j_2 with unit quality $q_{j_1} = q_{j_2} = 1$. It can be seen that allocating the two items to i_1 at prices $(19, 1)$ or $(1, 19)$ are both revenue maximizing equilibria; but there is no equilibrium price vector which is at least both $(19, 1)$ and $(1, 19)$.

In the context of the sharp/consecutive multi-unit demand, an interesting and important property is that it is possible that some items are 'over-priced'; this is a significant difference between sharp/consecutive multi-unit and unit demand models. Formally, in a solution $(\mathbf{X}, \mathbf{p})$, we say an item j is *over-priced* if there is a buyer i such that $j \in X_i$ and $p_j > v_i q_j$. That is, the price charged for item j is larger than its contribution to the utility of its winner.

Example 2.2.4 (Over-priced items in sharp/consecutive envy-freeness)**.** There are two buyers i_1, i_2 with values $v_{i_1} = 20, v_{i_2} = 10$ and demands $d_{i_1} = 1$ and $d_{i_2} = 2$, and three items j_1, j_2, j_3 with qualities $q_{j_1} = 3, q_{j_2} = 2, q_{j_3} = 1$. We can see that the allocations $X_{i_1} = \{j_1\}, X_{i_2} = \{j_2, j_3\}$ and prices $(45, 25, 5)$ constitute a revenue maximizing envy-free solution with total revenue 75, where item j_2 is over-priced. If no items are over-priced, the maximum possible prices are $(40, 20, 10)$ with total revenue 70.

In all, the relaxed envy-free solution and relaxed competitive equilibrium share the similar property as unit demand model (One can replace each multi-unit demand bidder by multiple unit-demand bidders, and the problem is then reduced to the unit-demand case) while the sharp/consecutive model exhibits different structure property, which is important in banner advertisement. While relaxed demand or sharp demand model serves as general banner advertisements, the consecutive demand model fits properly advertisement of rich media ads and TV ads, where the media (pic,video, flash) may require a fix number of consecutive number of slots. Thus, when we refer to consecutive buyers, we mean rich media advertisement buyers.

Chapter 3

Multi-unit Bayesian Auction with Demand or Budget Constraints

In this chapter, we will study revenue maximization problem under Bayesian settings for various demand or budget constraints. Our main theorem for this chapter is the following theorem.

Theorem 3.1. *For the relaxed demand, the sharp demand and consecutive demand (when qualities have constant peak) case without budget constraints, an optimal mechanism can be constructed efficiently. The problem for the consecutive demand model with arbitrary peak is shown to be NP-hard; for the case with the budget constraint but without demand constraint, a* 2*-approximate mechanism can be constructed efficiently.*

The road map of this chapter are as follows. In Section 3.1, elementary settings are introduced. We will review the classical characterization of Bayesian Incentive Compatibility in Section 3.2 and show how the payments can be discarded in the objective by incorporating Myerson's virtual value functions. We will solve the pure optimization problems for relaxed demand and sharp demand in Section 3.3 and Section 3.4 respectively. Section 3.5 and 3.6 contribute for consecutive demand buyers. At the end, in Section 3.7, a 2-approximate mechanism is proposed for budget constrained buyers.

3.1 Preliminaries

3.1.1 Demand Constraints

In our auction design problem, recall we want to sell m items to n buyers. Each buyer has a private number v_i representing her valuation and each item is characterized by a number q_j which can be viewed as the quality or desirability of the item. Thus, the ith buyer's value for item j is $v_i q_j$. In other words, the valuation matrix for n buyers and m items is the outer product of $\mathbf{v}$ and $\mathbf{q}$. Buyers are also assumed to abide by additional constraints as follows. We consider four specific constraints of this problem.

1 *Relaxed Demand Constraint:* Buyer i's demand is relaxedly constrained by d_i if i may buy any number of items up to a maximum d_i in this auction.

2 *Sharp Demand Constraint:* Buyer i's demand is sharply constrained by d_i if i must buy exactly d_i items in this auction or alternatively buys nothing.

3 *Consecutive Demand Constraint:* Buyer i's demand is consecutively constrained by d_i (w.r.t. a given total order on the items) if i must buy exactly d_i consecutive items in this auction or alternatively buys nothing.

4 *Budget Constraint:* Buyer i's budget is constrained by a publicly known number B_i if i cannot pay more than B_i.

3.1.2 Goal and Objectives

Given the buyers' value distributions, our goal is to design BIC and ex-interim IR mechanisms to allocate items to buyers so as to maximize the auctioneer's expected revenue. As is common in Computer Science, the optimal solution may be hard to compute efficiently, so we also consider the mechanisms which implement this objective *approximately.* More precisely, our aim is to devise a mechanism that for any distributions of buyers' values, the mechanism guarantees at least $1/\alpha$ times the optimum, where α is a constant. We call such mechanisms α-approximate mechanisms.

Definition 3.2 (α-approximate Mechanism)**.** We say a BIC and ex-interim IR mechanism M is an α-approximate mechanism if and only if for any BIC and ex-interim IR mechanism M', $Rev(M) \geq 1/\alpha \cdot Rev(M')$. We say a mechanism is optimal if it is a 1-approximate mechanism.

We are also interested in obtaining *computationally efficient* mechanisms, which is made precise by requiring that they should be computable in polynomial time. That is of course a standard requirement in the context of algorithmic mechanism design.

3.2 Characterization of Optimal Mechanism

In this section, we show that, in these auction domains, the optimal randomized, BIC and ex-interim IR auction can be represented by a simple deterministic, IC and ex-post IR auction. Furthermore, this optimal auction can be constructed efficiently.

Our constructions and proofs are simple and based on a basic idea of converting the optimization problem with allocation rules and payment rules to a problem only involving allocations. This can be done in two steps. First, due to the fact that our mechanism design problems fall within the single parameter domain where each player can be represented by a single parameter (i.e., his value per unit of quality), we can replace the complicated BIC conditions with a much simpler requirement of monotonicity on allocation rules. After that, all of the constraints are related with allocation functions instead of payments. Second, although the objective of our auction is to maximize the revenue, we can show that to maximize the auctioneer's revenue in a BIC auction is equivalent to maximizing a specific function of allocations, more precisely, the virtual surplus which is developed in [51]. Thus, we can get rid of the payments in our optimizing goal as well.

After this transformation, the original revenue optimization problems can be viewed as simple combinatorial optimization problems. As we will show later, our problems can be solved efficiently and even in a deterministic way.

3.2.1 Monotonicity

Although the Incentive Compatibility is defined in the terms of payments, it can be boiled down to a simple condition of monotonicity in single parameter settings. The proof can be sketched as follows. Fix a player i and all other players' bids v_{-i}. Recall that we use t_i, a function of $\mathbf{v}$ to denote the total quality of items assigned to i. Consider two possible values v_i and v_i' player i may hold. By the definition of IC, we have $v_i t_i(v_i; v_{-i}) - p_i(v_i; v_{-i}) \geq v_i t_i(v_i'; v_{-i}) - p_i(v_i'; v_{-i})$ and similarly $v_i' t_i(v_i'; v_{-i}) - p_i(v_i'; v_{-i}) \geq v_i' t_i(v_i; v_{-i}) - p_i(v_i; v_{-i})$. Summing up these two inequalities, we got $(v_i - v_i')(t_i(v_i; v_{-i}) - t_i(v_i'; v_{-i})) \geq 0$. It follows that, $t_i(x; v_{-i})$ must be a monotone non-decreasing function of x for any given v_{-i}. Regarding the Bayesian setting, the BIC condition can be similarly characterized in the following Lemma 3.3 adapted from [51]. For convenience in the Bayesian model, let $T_i(v_i)$ be the expectation of $t_i(\mathbf{v})$ over all other players' bids, more precisely, $T_i(v_i) = \mathrm{E}_{v_{-i}}[t_i(\mathbf{v})]$. Similarly, we define an expected version of payment rules, thus $P_i(v_i) = \mathrm{E}_{v_{-i}}[p_i(\mathbf{v})]$.

Lemma 3.3 (From [51])**.** *A mechanism $M = (x, p)$ is Bayesian Incentive Compatible if and only if:*

a) $T_i(x)$ is monotone non-decreasing for any buyer i.

b) $P_i(v_i) = v_i T_i(v_i) - \int_{\underline{v}_i}^{v_i} T_i(z)dz$

Proof. If $M = (x, p)$ is Bayesian Incentive Compatible, then for any v_i and v_i',

$$v_i T_i(v_i) - P_i(v_i) \geq v_i T_i(v_i') - P_i(v_i')$$

$$v_i' T_i(v_i') - P_i(v_i') \geq v_i' T_i(v_i) - P_i(v_i)$$

Plus above two inequality, we could get $(v_i - v_i')(T_i(v_i) - T_i(v_i')) \geq 0$, hence, $T_i(x)$ is monotone non-decreasing for any buyer i. in addition, let $U_i(v_i, T, P) = v_i T_i(v_i) - P_i(v_i)$, $v_i T_i(v_i) - P_i(v_i) \geq v_i T_i(v_i') - P_i(v_i')$ is equivalent to $U_i(v_i, T, P) \geq (v_i - v_i')T_i(v_i') + U_i(v_i', T, P)$, for any i, v_i, v_i'. Therefore, $U_i(v_i, T, P) = U_i(\underline{v}_i, T, P) + \int_{\underline{v}_i}^{v_i} T_i(z)dz$. Since, w.l.o.g. $U_i(\underline{v}_i, T, P) = 0$, thus,

$$P_i(v_i) = v_i T_i(v_i) - \int_{\underline{v}_i}^{v_i} T_i(z) dz.$$

Similarly, if a) and b) hold, then $v_i T_i(v_i) - P_i(v_i) \geq v_i T_i(v_i') - P_i(v_i')$ is equivalent to (by b)) $\int_{v_i'}^{v_i} T_i(z) dz \geq (v_i - v_i') T_i(v_i')$, which is true by monotonicity of $T_i(x)$. □

3.2.2 Virtual Surplus

For single item settings where the auctioneer has only one item to be sold, [51] showed that to maximize the seller's revenue is equivalent to maximizing the social welfare when each buyer's bid is his virtual value defined as $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$, where recall that $F_i(x)$ and $f_i(x)$ are respectively the Cumulative Distribution Function and Probability Density Function of the buyer i's value distribution. That is, the virtual value of an buyer is her true value minus the Hazard Rate of her value and distribution. Then given buyers' distributions, we define the virtual surplus as the expectation of the summation of every buyer's virtual value times her allocation, more precisely, $\mathrm{E}_{\mathbf{v}}[\sum_i \phi_i(v_i) t_i(\mathbf{v})]$. Then we can show that in both of our auction domains in this section, expected revenue is equal to expected virtual surplus.

Lemma 3.4 (From [51])**.** *For any BIC mechanism $M = (x, p)$, the expected revenue $\mathrm{E}_{\mathbf{v}}[\sum_i P_i(v_i)]$ is equal to the virtual surplus $\mathrm{E}_{\mathbf{v}}[\sum_i \phi_i(v_i) t_i(\mathbf{v})]$.*

Proof.

$$\begin{aligned}
\mathrm{E}_{\mathbf{v}}[P_i(v_i)] &= \mathrm{E}_{\mathbf{v}}[-U_i(v_i, T, P) + v_i T_i(v_i)] \\
&= \mathrm{E}_{\mathbf{v}}[-U_i(v_i, T, P)] + \mathrm{E}_{\mathbf{v}}[v_i T_i(v_i)] \\
&= -\mathrm{E}_{\mathbf{v}}[\int_{\underline{v}_i}^{v_i} T_i(z) dz] + \mathrm{E}_{\mathbf{v}}[v_i t_i(\mathbf{v})] \\
&= -\int_{\underline{v}_i}^{\overline{v}_i} dv_i \int_{\underline{v}_i}^{v_i} f_i(v_i) T_i(z) dz + \mathrm{E}_{\mathbf{v}}[v_i t_i(\mathbf{v})] \\
&= -\int_{\underline{v}_i}^{\overline{v}_i} T_i(z) dz \int_{z}^{\overline{v}_i} f_i(v_i) dv_i + \mathrm{E}_{\mathbf{v}}[v_i t_i(\mathbf{v})] \\
&= -\int_{\underline{v}_i}^{\overline{v}_i} T_i(z)(1 - F_i(z)) dz + \mathrm{E}_{\mathbf{v}}[v_i t_i(\mathbf{v})] \\
&= -\int_{\underline{v}_i}^{\overline{v}_i} T_i(v_i) \frac{1 - F_i(v_i)}{f_i(v_i)} f_i(v_i) dz + \mathrm{E}_{\mathbf{v}}[v_i t_i(\mathbf{v})] \\
&= -\mathrm{E}_{\mathbf{v}}[T_i(v_i) \frac{1 - F_i(v_i)}{f_i(v_i)}] + \mathrm{E}_{\mathbf{v}}[v_i t_i(\mathbf{v})] \\
&= \mathrm{E}_{\mathbf{v}}[\phi_i(v_i) t_i(\mathbf{v})]
\end{aligned}$$

□

We assume $\phi_i(t)$ is monotone increasing, i.e. the distribution is regular. Otherwise, Myerson's ironing technique can be utilized to make $\phi_i(t)$ monotone — it is here that we invoke our assumption that we can efficiently compute the convex closure of a continuous function and integration. More precisely, let $\bar{\phi}_i(t)$ be the ironing virtual value of buyer i (we refer to [51] for concrete definitions), where $\bar{\phi}_i(t)$ is regular. Myerson's classic results give the following lemmas.

Lemma 3.5 (From [51]). *Let x be an allocation that maximizes the ironing virtual surplus $\mathrm{E}_{\mathbf{v}}[\sum_i \bar{\phi}_i(v_i)t_i(\mathbf{v})]$, satisfying (1) monotone property (e.g. $T_i(v_i)$ is monotone non-decreasing for any buyer i) and supply and demand constraints (2) if $\bar{\phi}_i(v_i) = \bar{\phi}_i(v_i')$ then $T_i(v_i) = T_i(v_i')$, $\forall v_i \neq v_i'$, and p be the payment such that $p_i(\mathbf{v}) = v_i t_i(\mathbf{v}) - \int_{\underline{v}_i}^{v_i} t_i(v_{-i}, s_i) ds_i$ $\forall i$, then (x, p) is an optimal BIC mechanism.*

Since the allocation in the following sections is computed by deterministic algorithms, the property (2) in Lemma 3.5 is satisfied naturally. Hence, by Lemma 3.5, w.l.o.g., we always suppose virtual value $\phi_i(t)$ is regular (e.g. monotone increasing).

By Lemma 3.3 and 3.4, in order to maximize the expected revenue, one needs only to design algorithms to find the allocation to maximize the virtual surplus $\sum_i \phi_i(v_i)t_i(\mathbf{v})$ for each given v and simultaneously make the algorithm satisfy monotonicity (e.g. part (a) in Lemma 3.3). Fortunately, we will prove in the following powerful lemma that any deterministic algorithm maximizing the virtual surplus must be monotone. Hence, our main task will be seeking deterministic algorithm to maximize virtual surplus in various settings.

Lemma 3.6. *Any deterministic algorithm that achieves the maximum virtual surplus $\sum_i \phi_i(v_i)t_i(\mathbf{v})$ for any given v, must be monotone, that is, $T_i(v_i)$ is monotone non-decreasing for any buyer i.*

Proof. We will prove a stronger fact, that $t_i(v_i, v_{-i})$ is non-decreasing as v_i increases. Given other buyers' bids v_{-i}, the monotonicity of t_i is equivalent to $t_i(v_i, v_{-i}) \leq t_i(v_i', v_{-i})$ if $v_i' > v_i$. Assuming that $v_i' > v_i$, the regularity of ϕ_i implies that $\phi_i(v_i) \leq \phi_i(v_i')$. If $\phi_i(v_i) = \phi_i(v_i')$, then $t_i(v_i, v_{-i}) = t_i(v_i', v_{-i})$ and we are done.

Consider the case that $\phi_i(v_i) < \phi_i(v_i')$. Let Q and Q' denote the total quantities obtained by all the other buyers except buyer i in the mechanism when buyer i bids v_i and v_i' respectively.

$$\phi_i(v_i')t_i(v_i', v_{-i}) + Q' \geq \phi_i(v_i')t_i(v_i, v_{-i}) + Q$$
$$\phi_i(v_i)t_i(v_i, v_{-i}) + Q \geq \phi_i(v_i)t_i(v_i', v_{-i}) + Q'.$$

Above inequalities are due to the optimality of allocations when i bids v_i and v_i' respectively. It follows that

$$\phi_i(v_i')(t_i(v_i, v_{-i}) - t_i(v_i', v_{-i})) \leq Q' - Q$$
$$\phi_i(v_i)(t_i(v_i, v_{-i}) - t_i(v_i', v_{-i})) \geq Q' - Q$$

By the fact that $\phi_i(v_i) < \phi_i(v_i')$, it must be $t_i(v_i, v_{-i}) \le t_i(v_i', v_{-i})$. □

3.3 Relaxed Demand Case

Recall that a mechanism $M = (x, p)$ satisfies the relaxed demand constraint d_i for buyer i if for any realization of the mechanism, any buyer i cannot be assigned more than d_i items. Note that our mechanism only considers the allocation probability x, not the realized allocations. To convert the randomized mechanism to a realized allocation, we need a randomized rounding procedure satisfying the demand constraints. Fortunately, such a procedure is explicit in the the Birkhoff-Von Neumann theorem [42]. Thus, the relaxed demand constraint can be rewritten as $\sum_j(x_{ij}) \le d_i$ for each buyer i. By using the characterization of BIC and virtual surplus, we can transform the revenue optimization problem to an essentially simpler combinatorial optimization problem. The following lemma follows from Lemma 3.3 and 3.4.

Lemma 3.7. *Suppose that x is the allocation function that maximizes $\mathrm{E}_{\mathbf{v}}[\sum_i \phi_i(v_i)t_i(\mathbf{v})]$ subject to the constraints that $T_i(v_i)$ is monotone non-decreasing and inequalities*

$$\sum_j x_{ij}(\mathbf{v}) \le d_i, \quad \sum_i x_{ij}(\mathbf{v}) \le 1, \quad x_{ij}(\mathbf{v}) \ge 0. \tag{3.1}$$

Suppose also that

$$p_i(\mathbf{v}) = v_i t_i(\mathbf{v}) - \int_{\underline{v}_i}^{v_i} t_i(v_{-i}, s_i) ds_i \tag{3.2}$$

Then (x, p) represents an optimal mechanism for the relaxed demand case.

A main observation on Lemma 3.7 is that all inequalities in Eq. (3.1) only constrain $\mathbf{v}$ independently, not correlatively with different $\mathbf{v}$s. This allows us to consider the optimization problem for each $\mathbf{v}$ separately. After that, we will prove T_i is still monotone increasing in the resulting mechanism. In other words, we consider the problem of maximizing $\sum_i \phi_i(v_i)t_i(\mathbf{v})$ for each $\mathbf{v}$ separately instead of maximizing its expectation overall. This problem can be solved by a simple greedy algorithm in the spirit of assigning items with good quality to buyers with higher virtual value. For completeness, we describe our mechanism for the relaxed demand case in Algorithm 1.

Ultimately, we prove that the T_i deduced from our mechanism is monotone non-decreasing in the following theorem — our summary statement.

Theorem 3.8. *The mechanism that applies the allocation rule according to Algorithm 1 and payment rule according to Equation (3.2) is an optimal mechanism for the multi unit auction design problem with relaxed demand constrained buyers.*

Proof. It suffices to prove that $T_i(v_i)$ is monotone non-decreasing, which directly comes from Lemma 3.6. □

Algorithm 1: RELAXED

Input: Demands d_i, CDFs F_i, PDFs f_i, qualities q_j and bids $\mathbf{v}$
Output: Allocation x_{ij}

$\phi_i \leftarrow v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$;
Sort buyers in decreasing order of ϕ_i;
Sort items in decreasing order of q_j;
$x_{ij} \leftarrow 0$;
for *each buyer* i **do**
    for *each item* j **do**
        if $\phi_i > 0$ *and* $\sum_i x_{ij} < 1$ *and* $\sum_j x_{ij} < d_i$ **then**
            $x_{ij} \leftarrow 1$;
        end
    end
end
return x;

3.4 Sharp Demand Case

We now describe how to design an optimal mechanism for sharp demand cases. Recall that if a buyer is sharply constrained by d_i, he only wants to buy exactly d_i items or nothing. Thus the only difference between this problem with the one with relaxed constraints is that the inequalities (3.1) in Lemma 3.7 should be replaced by the following inequalities[1].

$$\sum_j x_{ij}(\mathbf{v}) = d_i \text{ or } 0, \quad \sum_i x_{ij}(\mathbf{v}) \le 1, \quad x_{ij}(\mathbf{v}) \ge 0 \quad \forall i, j, \mathbf{v} \tag{3.3}$$

By Lemma 3.3 and 3.4, similar to the relaxed demand case, we can convert the revenue optimization problem in the sharp demand case to a simple combinatorial optimization problem.

Lemma 3.9. *Suppose that x is the allocation function maximizing $\mathrm{E}_{\mathbf{v}}[\sum_i \phi_i(v_i)t_i(\mathbf{v})]$ subject to the constraints that $T_i(v_i)$ is non-decreasing monotone and inequalities (3.3). Suppose also that*

$$p_i(v) = v_i t_i(\mathbf{v}) - \int_{\underline{v}_i}^{v_i} t_i(s_i, v_{-i}) ds_i.$$

Then (x, p) represents an optimal mechanism for the sharp demand case.

Considering each bidding profile v_{-i} separately, we observe that the optimal mechanism always maximizes $\sum_i \phi_i(v_i)t_i(\mathbf{v})$ for all $\mathbf{v}$ subject to sharp demand constraints. By incorporating the definition of t_i, our goal is to maximize $\sum_i \sum_j \phi_i(v_i) q_j x_{ij}(\mathbf{v})$ subject to $\sum_j x_{ij}(\mathbf{v}) = d_i$ or 0 and $\sum_i x_{ij}(\mathbf{v}) \le 1$. It is not hard to see this problem is

[1] The formula $\sum_j x_{ij}(\mathbf{v}) = d_i$ or 0 here is not precise since in the random mechanism $\sum_j x_{ij}(\mathbf{v})$ may be arbitrary number between 0 and d_i. More precise definition may be complex e.g. distribution over deterministic mechanism. However, we didn't explicitly use the randomized value of $\sum_j x_{ij}(\mathbf{v})$ in our algorithm, and our mechanism is deterministic implying $x_{ij} \in \{0, 1\}$, and, $\sum_j x_{ij}(\mathbf{v}) = d_i$ or 0 is correct if $x_{ij} \in \{0, 1\}$, hence we still use this formula here.

equivalent to a maximum weighted matching problem on a bipartite graph with n left nodes and m right nodes. For any pair of nodes $(i, j) \in [n] \times [m]$, there exists an edge with weight $\phi_i(v_i)q_j$. Besides, the matching should satisfy an additional constraint that each left node must be matched with exact d_i right nodes or nothing. We call this problem maximum weighted matching with sharp constraints. An essential observation our algorithm relies on is a property of the optimal solution as we will show in Lemma 3.10. For convenience, we sort all left nodes in decreasing order of their $\phi_i(v_i)$ and all right nodes in decreasing order of their q_j.

Lemma 3.10. *There must exist an optimal solution for the maximum weighted matching problem with sharp constraints such that each left node is matched with consecutive d_i right nodes or nothing.*

Proof. Assume by contradiction, there exists a left node that the optimal match it with a set of non-consecutive right nodes. Let i be the first left node (w.r.t. the decreasing order of $\phi_i(v_i)$) with this property and U_i be the set of right nodes assigned to i. By our assumption, U_i is not consecutive. Thus, there exists a right node j not in U_i such that $\min_{k \in U_i}\{q_k\} \leq q_j \leq \max_{k \in U_i}\{q_k\}$. It is easy to see that j must be assigned to a left node with smaller ϕ than i otherwise i is not the first left node with non-consecutive matching set. Let r be the last node of U_i, i.e. with the largest index in U_i. Thus $q_j \geq q_r$. After that, we can refine the optimal solution by exchanging the assignment of node j and node r. The resulting assignment is still feasible and has larger weight. Keep doing this, we can get the desired optimal solution. □

By using this property, the problem can be solved by dynamic programming precisely. Let $w[i, j]$ denote the weight of the maximum weighted matchings with first i left nodes and all the first j right nodes being assigned. Initially, $w[0, 0] = 0$ and $w[0, j] = -\infty$, $\forall j \neq 0$. Then we have the transition function,

$$w[i, j] = \max\left\{w[i-1, j], w[i-1, j-d_i] + \sum_{k=j-d_i+1}^{j} \phi_i(v_i)c_k\right\}$$

Finding the maximum $w[i, j]$ over $i \in [n]$ and $j \in [m]$ gives the maximum weighted matchings and optimal solutions.

Theorem 3.11. *The mechanism which applies the allocation rule w.r.t. the above Dynamic Programming and payment rule w.r.t equation (3.2) is an optimal mechanism for multi unit auction design problem with sharp demand constrained buyers.*

Proof. To complete the proof of Theorem 3.11, it is sufficient to show $T_i(v_i)$ is non-decreasing, which follows directly from Lemma 3.6. □

3.5 Optimal Auction of Consecutive Demand for the Single Peak Case

The goal of this section is to present our optimal auction for the single peak case that serves as an elementary component in general case later. En route, several principal techniques are examined exhaustively to the extent that they can be applied directly in next section. By employing these techniques, we show that the optimal Bayesian Incentive Compatible auction can be represented by a simple Incentive Compatible one. Furthermore, this optimal auction can be implemented efficiently.

As mentioned above, we attempt to attain the BIC auction that maximizes the auctioneer's expected revenue. The same as above, we will transform the problem of BIC revenue maximization problem to a optimization problem only involved with allocation in objective function and where BIC is replaced by monotonicity of total expected qualities. After this transformation, the original revenue optimization problems can be viewed as simple combinatorial optimization problems. Fortunately, we observe delicate structures in the optimal solution that allow us to solve the problem entirely.

As pointed out in Section 3.1 and 3.2, a banner (maybe rich media) advertisement auction meets a buyer i's demand denoted by a number d_i if for any realization of the auction, the buyer i must be assigned either d_i consecutive slots or nothing. By incorporating the characterization of BIC and virtual surplus, we can transform the revenue optimization problem to an essentially simpler combinatorial optimization problem. The following lemma that follows from Lemma 3.3 and 3.4, formalizes the optimization problem.

Lemma 3.12. *Suppose that x is the allocation function that maximizes $\mathrm{E}_{\mathbf{v}}[\sum_i \phi_i(v_i)t_i(\mathbf{v})]$ subject to the constraints that $T_i(v_i)$ is monotone non-decreasing and for any bidders' profile $\mathbf{v}$, any buyer i is assigned either d_i consecutive slots or nothing. Suppose also that*

$$p_i(\mathbf{v}) = v_i t_i(\mathbf{v}) - \int_{\underline{v}_i}^{v_i} t_i(v_{-i}, s_i) ds_i \tag{3.4}$$

Then (x, p) represents an optimal mechanism for the rich media advertisement problem in single-peak case.

A main observation on Lemma 3.12 is that except the monotonicity all requirements on the mechanism only constrain $\mathbf{v}$ independently, not correlatively with different $\mathbf{v}$s. This allows us to consider the optimization problem for each $\mathbf{v}$ separately. In other words, we consider the problem of maximizing $\sum_i \phi_i(v_i)t_i(\mathbf{v})$ for each $\mathbf{v}$ separately instead of maximizing its expectation overall. After that, we will prove T_i is still monotone increasing in the resulting mechanism.

Given above discussions, a very important component of the optimal banner advertisement auction is to assign ad slots to advertisers consecutively and simultaneously to maximize the summation of virtual values, i.e. $\phi_i(v_i)t_i(\mathbf{v})$. When exerting ourselves on this specific problem, we make several preliminary observations that allow us to derive a dynamic programming for the single-peak case. Without loss of generality, we assume all buyers are sorted in the decreasing order of their virtual values. First, we show that the optimal assignment must be consecutive, i.e. there is no unassigned slots between any bidders' allocated slots (see Figure 3.1).

Lemma 3.13. *There exists an optimal allocation x that maximizes $\sum_i \phi_i(v_i)t_i(\mathbf{v})$ in single peak case, satisfies the following condition. For any unassigned slot j, it must be either $\forall j' > j$, slot j' is unassigned or $\forall j' < j$, slot j' is unassigned.*

Proof. We pick an arbitrary optimal allocation x that maximizes the summation of virtual values. If x satisfies the property, it is the desired allocation and we are done. Otherwise, we do the following modification on x. Let slot j $(1 < j < m)$ be the unassigned slot between buyers' allocated slots. Since the quality function are single peaked, we have $q_j \geq q_{j+1}$ or $q_j \geq q_{j-1}$. We only prove the lemma for the case $q_j \geq q_{j+1}$ and the proof for the other case is symmetric. Let slot $j' > j$ be the leftmost assigned slot on the right side of j. We modify x by assigning the buyer i who got the slot j' the d_i consecutive slots from j. It is easy to check the resulting allocation is still feasible and optimal. Moreover, the slot j becomes assigned now. By keep doing this, we can eliminate all unassigned slots between buyers' allocations. Thus, the resulting allocation must be consecutive. □

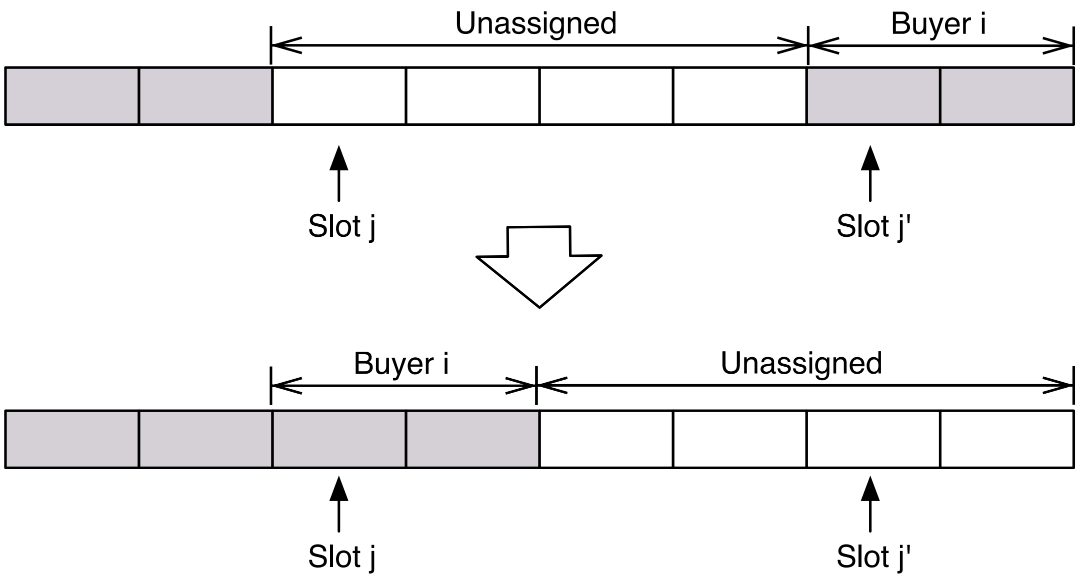

FIGURE 3.1: By re-assigning i the slots from j, we make the set of assigned slots consecutive.

Next, we prove that this consecutiveness even holds for all set $[s] \subseteq [n]$. That is, there exists an optimal allocation that always assigns the first s buyers consecutively for all $s \in [n]$. For convenience, we call a slot is out of a set of buyers if the slot is not assigned to any buyers in that set. Then the consecutiveness can be formalized in the following lemma.

Lemma 3.14. *There exists an optimal allocation x in single peak case, satisfies the following condition. For any slot j out of $[s]$, it must be either $\forall j' > j$, slot j' is out of $[s]$ or $\forall j' < j$, slot j' is out of $[s]$.*

Proof. The idea is to pick an arbitrary optimal allocation x and modify it to the desired one. Suppose x does not satisfy the property on a subset $[s]$. By Lemma 3.13, there is no unassigned slots in the middle of allocations to set $[s]$. Then there must be a slot assigned to a buyer i out of the set $[s]$ that separates the allocations to $[s]$ We use W_i to denote the allocated slots of buyer i. Suppose slot k is the peak. There are two cases to be considered:

Case 1. $k \notin W_i$.

Let j and j' be the leftmost and rightmost slot in W_i respectively. We consider two cases $q_j \geq q_{j'}$ and $q_j < q_{j'}$. We only prove for the first case and the proof for the other case is symmetric. If $q_j \geq q_{j'}$, we find the leftmost slot $j_1 > j'$ assigned to $[s]$ and the rightmost slot $j_2 < j_1$ not assigned to $[s]$. In addition, let $i_1 \in [s]$ be the buyer that j_1 is assigned to and $i_2 > s$ be the buyer that j_2 is assigned to. In single peak case, it is easy to check $q_j \geq q_{j'}$ implies that all the slots assigned to i_2 have higher quality than i_1's. Thus swapping the positions of i_1 and i_2, as illustrated in Figure 3.2, will always increase the virtual surplus, $\sum_i \phi_i(v_i)t_i(\mathbf{v})$. By keep doing this, we can eliminate all slots out of $[s]$ in the middle of allocation to $[s]$ and attain the desired optimal solution.

Case 2. $k \in W_i$

Suppose $W_i = \{j_1^i, j_2^i, \cdots, j_{u_i}^i\}$ with $j_1^i < j_2^i < \cdots < j_{u_i}^i$ and there exists $1 \leq e \leq u_i$ such that $k = j_e^i$. Let a and b be the left and right neighbour buyers of i winning slots next to W_i. As we know $a, b \in [s]$, hence, $\phi_a(v_a) \geq \phi_i(v_i)$ and $\phi_b(v_b) \geq \phi_i(v_i)$. Let $W_a = \{j_1^a, j_2^a, \cdots, j_{u_a}^a\}$ and $W_b = \{j_1^b, j_2^b, \cdots, j_{u_i}^b\}$ denote the allocated slots of buyer a and b respectively, where $j_1^a < j_2^a < \cdots < j_{u_a}^a$ and $j_1^b < j_2^b < \cdots < j_{u_b}^b$. As $k \in W_i$, then $q_{j_1^i} \geq q_{j_{u_a}^a}$ and $q_{j_{u_i}^i} \geq q_{j_1^b}$ (noting that $j_{u_a}^a$ and j_1^b are the indices of slots with the largest qualities in W_a and W_b respectively). We will show that either swapping winning slots of i with a or with b will increase the virtual surplus. To prove this, there four cases needed to be considered: (1). $u_i \geq u_a$ and $u_i \geq u_b$; (2). $u_i \geq u_a$ and $u_i < u_b$; (3). $u_i < u_a$ and $u_i \geq u_b$; (4). $u_i < u_a$ and $u_i < u_b$. We only prove the case (1) since the other cases can be proved similarly. Now, suppose $u_i \geq u_a$ and $u_i \geq u_b$, then we must have either (i). $\sum_{k=1}^{u_b} q_{j_k^i} \geq \sum_{k=1}^{u_b} q_{j_k^b}$ or (ii). $\sum_{k=1}^{u_a} q_{j_{u_i-k+1}^i} \geq \sum_{k=1}^{u_a} q_{j_k^a}$. Suppose (i) is not true, that is $\sum_{k=1}^{u_b} q_{j_k^i} < \sum_{k=1}^{u_b} q_{j_k^b}$, if $u_b \leq e$, then we have $q_{j_1^i} \leq q_{j_{u_b}^i}$, as a result,

$$u_b q_{j_1^i} \leq \sum_{k=1}^{u_b} q_{j_k^i} < \sum_{k=1}^{u_b} q_{j_k^b} \leq u_b q_{j_1^b} \leq u_b q_{j_{u_i}^i},$$

thus, $q_{j_1^i} < q_{j_{u_i}^i}$; otherwise $u_b > e$, then it must also hold that $q_{j_1^i} \leq q_{j_{u_b}^i}$ (otherwise, for any $1 \leq \ell \leq u_b$, $q_{j_\ell^i} \geq q_{j_{u_b}^i} \geq q_{j_1^b}$ implying that $\sum_{k=1}^{u_b} q_{j_k^i} \geq u_b q_{j_1^b} \geq \sum_{k=1}^{u_b} q_{j_k^b}$, contradiction), hence, for any $1 \leq \ell \leq u_b$, $q_{j_\ell^i} \geq q_{j_1^i}$, it follows,

$$u_b q_{j_1^i} \leq \sum_{k=1}^{u_b} q_{j_k^i} < \sum_{k=1}^{u_b} q_{j_k^b} \leq u_b q_{j_1^b} \leq u_b q_{j_{u_i}^i},$$

in both cases, it is obtained that $q_{j_1^i} < q_{j_{u_i}^i}$, therefore,

$$\sum_{k=1}^{u_a} q_{j_{u_i-k+1}^i} > u_a q_{j_1^i} \geq \sum_{k=1}^{u_a} q_{j_k^a}$$

implying (ii) is true. Thus, if (i) is true, by simple calculations, swapping winning slots of i with b will increase the virtual value (since $\phi_b(v_b) \geq \phi_i(v_i)$), otherwise swapping winning slots of i with a will increase the virtual surplus (since $\phi_a(v_a) \geq \phi_i(v_i)$). Then keep doing it by the method of Case 1 until eliminating all slots out of $[s]$ in the middle of allocation to $[s]$ and attaining the desired optimal solution.

□

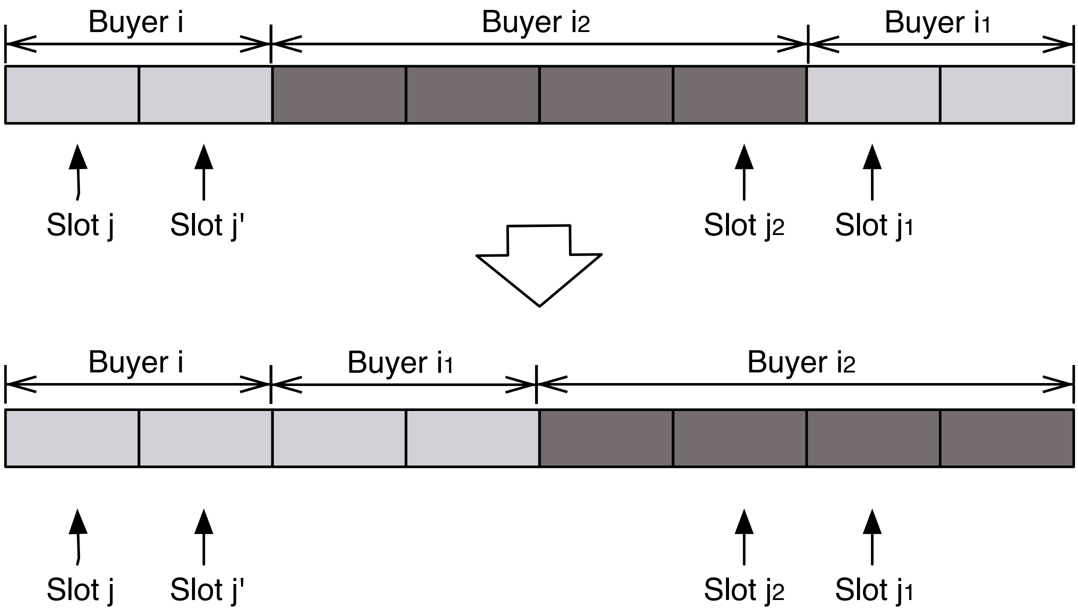

FIGURE 3.2: Slots with light color are assigned to $[s]$. By swapping the positions of i_1 and i_2, we make the allocations to $[s]$ consecutive.

Lemma 3.14 reveals the optimal substructure that allows us to solve the problem by dynamic programming. Since the optimal solution always assigns to $[s]$ consecutively, we can boil the allocations to $[s]$ down to an interval denoted by $[l, r]$. Let $g[s, l, r]$ denote the maximized value of our objective function $\sum_i \phi_i(v_i)t_i(\mathbf{v})$ when we only consider first s buyers and the allocation of s is exactly the interval $[l, r]$. Initially, we have $g[0, l, l-1] = 0$, $1 \leq l \leq m+1$ and $g[0, l, r] = -\infty$, otherwise. Then we have the following transition function.

$$g[s,l,r] = \max \begin{cases} g[s-1,l,r] \\ \\ g[s-1,l,r-d_s] + \phi_s(v_s)\sum_{j=r-d_s+1}^{r} q_j \\ \\ g[s-1,l+d_s,r] + \phi_s(v_s)\sum_{j=l}^{l+d_s-1} q_j \end{cases} \tag{3.5}$$

The optimality of the allocation obtained from the Dynamic Programming just follows from Lemma 3.14. More precisely, the optimal solution always assign the sth buyer nothing or slots next to (on the left or right) the interval allocated to first $s-1$ buyers.

Ultimately, by Lemma 3.12, we prove that the T_i deduced from our mechanism is monotone non-decreasing in the following theorem — our summary statement.

Theorem 3.15. *The mechanism that applies the allocation rule according to Dynamic Programming (3.5) and payment rule according to Equation (3.4) is an optimal mechanism for the banner advertisement problem with single peak qualities.*

Proof. The proof comes straightforwardly from Lemma 3.6. □

3.6 The General Case of Consecutive Demand Buyers

We now move to the general case where the qualities of item may have several peaks. We assume the number of peaks h is constant. This is a reasonable assumption when we consider the rich media advertisement and TV advertisement. For arbitrary peak, NP-hardness will be shown. It should be emphasized again that we study the revenue maximization problem in the Bayesian setting and our goal is to find out the optimal auction (with maximum revenue) among all Bayesian Incentive Compatible auction.

As we have shown in Section 3.5, the Bayesian Incentive Compatibility can be replaced with a simple requirement for monotonicity of allocation functions. Moreover, as Myerson showed in [51], to maximize the revenue in Bayesian setting is equivalent to maximizing the virtual surplus. Similar with Lemma 3.12, Lemma 3.16 following from Lemma 3.3 and Lemma 3.4 show the essential part of our optimal auction construction.

Lemma 3.16. *Suppose that x is the allocation function that maximizes $\mathrm{E}_{\mathbf{v}}[\sum_i \phi_i(v_i)t_i(\mathbf{v})]$ subject to the constraints that $T_i(v_i)$ is monotone non-decreasing and for any bidders' profile $\mathbf{v}$, any buyer i is assigned either d_i consecutive slots or nothing. Suppose also that*

$$p_i(\mathbf{v}) = v_i t_i(\mathbf{v}) - \int_{\underline{v}_i}^{v_i} t_i(v_{-i}, s_i)ds_i \tag{3.6}$$

Then (x,p) represents an optimal mechanism for the banner advertisement problem in general case.

Consider the problem of maximizing $\sum_i \phi_i(v_i)t_i(\mathbf{v})$ for each $\mathbf{v}$. Recall that there are only h peaks (local maximum) in the qualities. Thus, there are at most $h-1$ valleys (local minimum). Since h is a constant, we can enumerate all the buyers occupying the valleys. After this enumeration, we can divide the qualities into at most h consecutive pieces and each of them forms a single-peak. Then using similar properties as those in Lemma 3.13 and 3.14, we can obtain a larger size dynamic programming (still runs in polynomial time) similar to dynamic programming (3.5) to solve the problem.

Theorem 3.17. *There is a polynomial algorithm to compute revenue maximization problem in Bayesian settings where the qualities of slots have constant number of peaks.*

Proof. Our proof is based on the single peak algorithm. Assume there are h peaks, then there must be $h-1$ valleys. Suppose these valleys are indexed $j_1, j_2, \cdots, j_{h-1}$. In optimal allocation, for any j_k, $k = 1, 2, \cdots, h-1$, j_k must be allocated to a buyer or unassigned to any buyer. If j_k is assigned to a buyer, say, buyer i, since i would buy d_i consecutive slots, j_k may appear in ℓth position of this d_i consecutive slots. Hence, by this brute force, each j_k will at most have $\sum_i d_i + 1 \leq mn + 1$ possible positions to be allocated. In all, all the valleys have $(mn+1)^h$ possible allocated positions. For each of this allocation, the slots is broken into h single peak slots. We can obtain similar properties as those in Lemma 3.13 and 3.14. Without loss of generality, suppose the rest buyers are still the set $[n]$, with non-increasing virtual value. Since the optimal solution always assigns to $[s]$ concentrating in h intervals, we can boil the allocations to $[s]$ down to intervals denoted by $[l_i, r_i]$, $i = 1, 2, \cdots, h$, where $[l_i, r_i]$ lies in the i-th single peak slot. Let $g[s, l_1, r_1, \cdots, l_h, r_h]$ denote the maximized value of our objective function $\sum_i \phi_i(v_i) t_i(\vec{v})$ when we only consider first s buyers and the allocations of $[s]$ are exactly intervals $[l_i, r_i]$, $i = 1, 2, \cdots, h$. Then we have the following transition function.

$$g[s, l_1, r_1, \cdots, l_h, r_h] = \max_{i \in [d]} \begin{cases} g[s-1, l_1, r_1, \cdots, l_h, r_h] \\ \\ g[s-1, l_1, r_1, \cdots, l_i, r_i - d_s, \cdots, l_h, r_h] + \phi_s(v_s) \sum_{j=r_i-d_s+1}^{r_i} q_j \\ \\ g[s-1, l_1, r_1, \cdots, l_i + d_s, r_i, \cdots, l_h, r_h] + \phi_s(v_s) \sum_{j=l_i}^{l_i+d_s-1} q_j \end{cases}$$

□

Theorem 3.18. *If the qualities of slots have arbitrary peaks, the revenue maximization problem of Bayesian settings is NP-hard*

Proof. We prove the NP-hardness by reducing the 3 partition problem that is to decide whether a given multi-set of integers can be partitioned into certain number of subsets that all have the same sum. More precisely, given a multi-set S of $3n$ positive integers, can S be partitioned into n subsets $S_1, \ldots, S_n$ such that the sum of the numbers in each subset is equal? The 3 partition problem has been proven to be NP-complete in a strong sense in [36], meaning that it remains NP-complete even when the integers in S are bounded above by a polynomial in n.

Given a instance of 3 partition $(a_1, a_2, \ldots, a_{3n})$, we construct a instance for advertising problem with $3n$ advertisers and $m = n + \sum_i a_i$ slots. It should be mentioned that

m is polynomial of n due to the fact that all a_i are bouned by a polynomial of n. In the advertising instance, the valuation v_i for each advertiser i is 1 and his demand d_i is defined as a_i. Moreover, for any advertiser, his valuation distribution is that $v_i = 1$ with probability 1. Then everyone's virtual value is exactly 1. By Lemma 3.16, to maximize revenue is equivalent to maximize the simplified function $\sum_i \sum_j x_{ij} q_j$.

Let $B = \sum_i a_i/n$. We define the quality of slot j is 0 if j is times of $B+1$, otherwise $q_j = 1$. That can be illustrated as follows.

$$\underbrace{1\,1\cdots 1}_{B}\ 0\ \underbrace{1\,1\cdots 1}_{B}\ 0 \ldots \underbrace{1\,1\cdots 1}_{B}\ 0$$

It is not hard to see that the optimal revenue is $\sum_i a_i$ iff there is a solution to this 3 partition instance. □

3.7 Approximate Mechanism for Budget Constraints

In this section, we will present a 2-approximate mechanism for the Multi-item auction with budget constrained buyers. It should be noted that there is no demand constraints for all the buyers considered in this section. Recall that a mechanism $M = (x, p)$ satisfies the buyer i's budget constraint iff $p_i(\mathbf{v}) \le B_i$ for all buyer profiles v. If $m = 1$, i.e. the auctioneer only has one slot, a 2- approximate mechanism has been suggested in [1] and [6]. Thus, our approach is to reduce the Multi-item Auction to Single-item Auction, i.e. the case for $m = 1$. Recall that B_i denotes bidder i's budget, $x_{ij}(\mathbf{v})$ denote the probability of allocating item j to buyer i when the buyers' bids revealed type is $\mathbf{v}$ and recall we use $t_i(\mathbf{v}) = \sum_j q_j x_{ij}(\mathbf{v})$, a function of $\mathbf{v}$ to denote the total quality of items assigned to i. Then the multi-item auction problem can be formalized as the following optimization problem.

$$\begin{aligned}
\text{Max:}\quad & \mathrm{E}_v\left[\sum_i p_i(\mathbf{v})\right] & & \\
\text{s.t.}\quad & \mathrm{E}_{v_{-i}}[v_i t_i(\mathbf{v}) - p_i(\mathbf{v})] \ge \mathrm{E}_{v_{-i}}[v_i t_i(v_i', v_{-i}) - p_i(v_i', v_{-i})], & \forall \mathbf{v}, i, v_i' & \\
& \mathrm{E}_{v_{-i}}[v_i t_i(\mathbf{v}) - p_i(\mathbf{v})] \ge 0, & \forall \mathbf{v}, i & \quad (\textsc{Multi-item}) \\
& p_i(\mathbf{v}) \le B_i, & \forall \mathbf{v}, i & \\
& x_{ij}(\mathbf{v}) \ge 0 & \forall \mathbf{v}, i, j & \\
& \sum_i x_{ij}(\mathbf{v}) \le 1 & \forall \mathbf{v}, j &
\end{aligned}$$

Now consider the following single-item problem. Denote $B_i' = \frac{B_i}{\sum_j q_j}$, and let $y_i(\mathbf{v})$ be the allocation function for bidder i and $s_i(\mathbf{v})$ be the payment function for bidder i. The single-item auction with budget constraints can be formalized as following optimization problem.

$$\begin{aligned}
\text{Max:}\quad & \mathrm{E}_v\left[\sum_i s_i(\mathbf{v})\right] && \\
\text{s.t.}\quad & \mathrm{E}_{v_{-i}}\left[v_i y_i(\mathbf{v}) - s_i(\mathbf{v})\right] \geq \mathrm{E}_{v_{-i}}\left[v_i y_i(v_i', v_{-i}) - s_i(v_i'; v_{-i})\right], && \forall \mathbf{v}, i, v_i' \\
& \mathrm{E}_{v_{-i}}\left[v_i y_i(\mathbf{v}) - s_i(\mathbf{v})\right] \geq 0, && \forall \mathbf{v}, i \\
& s_i(\mathbf{v}) \leq B_i', && \forall \mathbf{v}, i \\
& y_i(\mathbf{v}) \geq 0 && \forall \mathbf{v}, i \\
& \sum_i y_i(\mathbf{v}) \leq 1 && \forall \mathbf{v}
\end{aligned} \tag{Single}$$

Our main observation for the above optimization problems is the following proposition.

Proposition 3.19. *The problems* MULTI-ITEM *and* SINGLE *are equivalent:*

- *for any feasible mechanism $M(\mathbf{v}) = (x(\mathbf{v}), p(\mathbf{v}))$ of problem* MULTI-ITEM*, the following mechanism $\hat{M}(\mathbf{v}) = (y(\mathbf{v}), s(\mathbf{v}))$ is a feasible mechanism for problem* SINGLE *where $y_i(\mathbf{v}) = \frac{t_i(\mathbf{v})}{\sum_j q_j}$, $s_i(\mathbf{v}) = \frac{p_i(\mathbf{v})}{\sum_j q_j}$, $\forall i \in [n]$.*
- *for any feasible mechanism $\hat{M}(\mathbf{v}) = (y(\mathbf{v}), s(\mathbf{v}))$ of problem* SINGLE*, the following mechanism $M(v) = (x(\mathbf{v}), p(\mathbf{v}))$, where $x_{ij}(\mathbf{v}) = y_i(\mathbf{v})\ \forall i \quad j$ and $p_i(\mathbf{v}) = s_i(\mathbf{v})(\sum_j q_j)\ \forall i$, is a feasible mechanism for problem* MULTI-ITEM.

Proof. The proof is based on direct calculations. First, for any feasible mechanism $M(\mathbf{v}) = (x(\mathbf{v}), p(\mathbf{v}))$ of problem MULTI-ITEM, let $y_i(\mathbf{v}) = \frac{t_i(\mathbf{v})}{\sum_j q_j}$, $s_i(\mathbf{v}) = \frac{p_i(\mathbf{v})}{\sum_j q_j}$, $\forall i \in [n]$, it is not difficult to check that $\hat{M}(\mathbf{v}) = (y(\mathbf{v}), s(\mathbf{v}))$ is a feasible mechanism to SINGLE. On the other hand side, for any feasible mechanism $\hat{M}(\mathbf{v}) = (y(\mathbf{v}), s(\mathbf{v}))$ of problem SINGLE, let $x_{ij}(\mathbf{v}) = y_i(\mathbf{v})\ \forall i \quad j$, $p_i(\mathbf{v}) = s_i(\mathbf{v})(\sum_j q_j)\ \forall i$, it is easy to show that $M(v) = (x(\mathbf{v}), p(\mathbf{v}))$ is a feasible mechanism for problem MULTI-ITEM. □

Ultimately, we reduce the multi-item auction design problem to the single-item auction design problem. By the results of [1] and [6], there exists a 2-approximate mechanism for problem SINGLE. Thus, we have a 2-approximate mechanism for problem MULTI-ITEM.

Remark 3.20. For the discrete distribution case, [10] presents an optimal mechanism, for multi-buyers with multi-items. Their algorithm can be extended to the case where buyers are budget constrained but not demand constrained. Given buyers' discrete distribution and bid profiles, a revised version of their mechanism is an optimal mechanism and runs in time polynomial in $\sum_i |T_i|$, where $|T_i|$ is the number of types of buyer i for all the items. Hence, restricting their results to MULTI-ITEM auction, that optimal mechanism is indeed an optimal mechanism for each buyer having a budget constraint but no demand constraint, with values independently drawn from discrete distribution, running in time polynomial in the input.

Chapter 4

Revenue Maximization of Competitive Equilibrium

We study the revenue maximization problem of various competitive equilibriums in this chapter. In the simple relaxed demand case, an algorithm to compute an optimal relaxed competitive equilibrium is presented for the revenue maximization problem. The situation become complicated for the sharp demand and consecutive demand case. Indeed, the equilibrium may not exist in these two cases, however, we find a polynomial algorithm to decide whether an equilibrium exists or not and compute a revenue maximizing one if one does in both settings. Hence, the problems are solved completely.

The structure of this chapter is as follows. In Section 4.1, the main results are introduced. We solve the revenue maximization problem for relaxed demand buyers in Section 4.2. A hardness result for deciding whether a sharp competitive equilibrium exists or not for general valuation is proposed in Section 4.3. For the sharp demand and consecutive demand cases, we present polynomial algorithms to decide whether an equilibrium exists or not and find a revenue maximizing one if it does in Section 4.4 and 4.5 respectively.

4.1 Introduction

As discussed in Chapter 2, *competitive equilibrium* provides a solution concept that captures both market efficiency and fairness for the buyers. In a competitive equilibrium, every buyer obtains a best possible allocation that maximizes his own utility and every unallocated item is priced at zero (i.e., market clearance). Competitive equilibrium is one of the central solution concepts in economics and has been studied and applied in a variety of domains [49]. Combining the considerations from the two sides of the market, an ideal solution concept therefore would be revenue maximizing competitive equilibrium. Strongly polynomial time is defined in the arithmetic model of computation. In this model of computation the basic arithmetic operations (addition, subtraction, multiplication, division, and comparison) take a unit time step to perform, regardless of the sizes of the operands.

Definition 4.1.1 (Strongly Polynomial Time)**.** The algorithm runs in strongly polynomial time if

- the number of operations in the arithmetic model of computation is bounded by a polynomial in the number of integers in the input instance;
- the space used by the algorithm is bounded by a polynomial in the size of the input

For relaxed demand buyers, the relaxed competitive equilibrium exhibits almost the same property of the unit demand model. Since the maximum relaxed competitive equilibrium always exists, simultaneously the maximum revenue is attained. The result for relaxed competitive equilibrium are summarized as follows.

Theorem 4.1. *There always exists a maximum equilibrium with relaxed demand constraint. Further, there is a strongly polynomial time algorithm to compute a maximum equilibrium (i.e., a revenue maximizing equilibrium).*

For sharp multi-unit demand buyers, when the valuations v_{ij} are arbitrary, even determining the existence of a competitive equilibrium is NP-complete (see Section 4.3).

Theorem 4.2. *It is NP-hard to determine the existence of a competitive equilibrium for general valuations in the sharp demand model (even when all demands are 3, and valuations are 0/1).*

For our correlated valuation $v_i q_j$ model, the sharp and consecutive demand buyers exhibit similar structure properties. As mentioned above, the equilibrium may not exist and maximum equilibrium may not exist even if the competitive equilibrium exists. Further , there may exist overpriced items (see Example 2.2.4) in sharp/consecutive competitive equilibriums.

Theorem 4.3. *For sharp multi-unit demand, a sharp competitive equilibrium may not exist; even if an equilibrium is guaranteed to exist, a maximum sharp equilibrium (in which each price is as high as it can be in any solution) may not exist. Further, there is a polynomial time algorithm that determines the existence of a sharp equilibrium, and computes a revenue maximizing one if it does.*

The results for consecutive demand buyers are summarized as follows.

Theorem 4.4. *For consecutive multi-unit demand, a consecutive competitive equilibrium may not exist; even if an equilibrium is guaranteed to exist, a maximum consecutive equilibrium (in which each price is as high as it can be in any solution) may not exist. Further, if the number of peaks is bounded by a constant number, there is a polynomial time algorithm that determines the existence of a consecutive equilibrium, and computes a revenue maximizing one if it does. If the number of peaks is arbitrary, the complexity of determining the existence of a consecutive equilibrium is NP-hard, and computing a revenue maximizing consecutive equilibrium is also NP-hard even if the equilibrium exists.*

Recall K is the number of distinct values in the set $\{v_1, \ldots, v_n\}$, and let $A_1, \ldots, A_K$ be a partition of all buyers where each A_k, $k = 1, 2, \ldots, K$, contains the set of buyers that have the kth largest value.

4.2 Relaxed Competitive Equilibrium

In this section, we study relaxed competitive equilibrium and assume that buyer i can win any number of items, upper bounded by d_i. That is, in a feasible output $(\mathbf{p}, \mathbf{X})$, we only require that $|X_i| \le d_i$ for each buyer i.

The difference between Definition 2.2.1 and 2.2.2 is that in Definition 2.2.1, for relaxed demand, any buyer i does not envy any other subset S of size $|S| \le d_i$, whereas in Definition 2.2.2, for sharp demand, i does not envy any subset S of size $|S| = d_i$. Hence, in the relaxed multi-unit demand model, as buyers are interested in a larger set of candidate items, the definition of envy-freeness is stronger.

Given the new notion of relaxed envy-freeness, we have the following new definition for relaxed competitive equilibrium.

Definition 4.2.1 (Relaxed Maximum Equilibrium)**.** A price vector $\mathbf{p}$ is called a relaxed maximum equilibrium price vector if for any other relaxed equilibrium price vector $\mathbf{q}$, $p_j \ge q_j$ for every item j. An relaxed equilibrium $(\mathbf{p}, \mathbf{X})$ is called a relaxed maximum equilibrium if $\mathbf{p}$ is a relaxed maximum price vector.

Still we are interested in computing revenue maximizing relaxed competitive equilibrium. Actually, we compute the relaxed maximum equilibrium, which simultaneously is a revenue maximizing equilibrium.

Lemma 4.5. *For any relaxed envy-free solution* $(\mathbf{p}, \mathbf{X})$*, there is no overpriced item in* $(\mathbf{p}, \mathbf{X})$*.*

Proof. Suppose there is an item j which is allocated to buyer i and j is overpriced (i.e., $p_j > v_i q_j$), then i would get more utility from the bundle $X_i \backslash \{j\}$ than X_i, which is a contradiction. □

Lemma 4.6. *For any relaxed envy-free solution* $(\mathbf{p}, \mathbf{X})$*, suppose i is a winner, then for any buyer i' with $v_{i'} > v_i$, i' is a winner as well. In addition, the number of items that i' wins equals $d_{i'}$, i.e., $|X_{i'}| = d_{i'}$.*

Proof. Since i is a winner, by the above Lemma 4.5, for any $j \in X_i$, $v_{i'} q_j > v_i q_j \ge p_j$. That is, i' is able to obtain a positive utility from j. This implies that i' must be a winner. Further, if $|X_{i'}| < d_{i'}$, then i' would envy bundle $X_{i'} \cup \{j\}$, which is a contradiction. □

Another significant difference between the relaxed demand models and sharp/consecutive demand is that a maximum competitive equilibrium may not exist in the sharp/-consecutive demand model (as shown by Example 2.2.2), whereas as we will see in the next subsection, a relaxed maximum equilibrium always exists in the relaxed demand model.

4.2.1 Maximum Equilibrium

Theorem 4.7. *There always exists a relaxed maximum equilibrium with relaxed demand constraint. Further, there is a strongly polynomial time algorithm to compute a relaxed maximum equilibrium (i.e., a revenue maximizing relaxed equilibrium).*

Now we present the algorithm to compute a relaxed maximum equilibrium. Such an algorithm denoted by ALG-MEQ have two steps. The first step is to select the winner set and the second step to allocate items and settle prices.

ALG-MAX-EQ

1. Assume buyers are ordered by $i_1, \ldots, i_n$ where $v_{i_1} \geq \cdots \geq v_{i_n}$.
2. Let $i_{\max} = \arg\min_k \sum_{j=1,\ldots,k} d_{i_j} \geq m$ and $S^* = \{i_1, \ldots, i_{\max}\}$.
3. Allocation $\mathbf{X}^*$
   - Let $X_i^* = \emptyset$ for each buyer $i \notin S^*$.
   - Allocate items to buyers in S^* according to the following rules: Buyers with larger values obtain items with larger qualities. (Note that $i_{\max}$ gets $\min\left\{d_{i_{\max}},\ m - \sum_{i \in S^* \setminus \{i_{\max}\}} d_i\right\}$ items.)
4. Price $\mathbf{p}^*$
   - Let $p_j^* = 0$ for each unallocated item j.
   - Let $p_j^* = v_{i_{\max}} q_j$ for each item $j \in X_{i_{\max}}^*$.
   - For each remaining item j in the reverse order.
     - let i_u be the buyer that wins item j.
     - let k be the item with the smallest index that i_{u+1} wins.
     - let $p_j^* = v_{i_u}(q_j - q_k) + p_k^*$.
5. Output the tuple $(\mathbf{p}^*, \mathbf{X}^*)$.

Proof of Theorem 4.7. It is easy to see that ALG-MAX-EQ takes strongly polynomial time. Consider any relaxed competitive equilibrium $(\mathbf{p}, \mathbf{X})$ and let $S = \{i \mid X_i \neq \emptyset\}$ be its set of winners. By the rule of defining S^* and Lemma 4.6, we know that $S \subseteq S^*$. If $S \subset S^*$, then $i_{\max} \notin S$ (recall that $i_{\max} = \max(S)$) and there is an item j which is not allocated to any buyer in $(\mathbf{p}, \mathbf{X})$; thus, $p_j = 0$. This would imply that $i_{\max}$ is not envy-free in $(\mathbf{p}, \mathbf{X})$, a contradiction. Hence, we have $S = S^*$. By the allocation and pricing rules of ALG-MAX-EQ, we know that the price vector $\mathbf{p}^*$ computed by ALG-MAX-EQ is a maximum relaxed price vector and the tuple $(\mathbf{p}^*, \mathbf{X}^*)$ is a relaxed maximum equilibrium. □

4.3 Hardness of General Valuations of Sharp Competitive Equilibrium

Theorem 4.8. *It is NP-hard to determine the existence of a sharp competitive equilibrium for general valuations in the sharp demand model (even when all demands are 3, and valuations are 0/1).*

Proof. We reduce from exact cover by 3-sets (X3C): Given a ground set $A = \{a_1, \ldots, a_{3n}\}$ and a collection of subsets $S_1, \ldots, S_m \subset A$ where $|S_i| = 3$ for each i, we are asked whether there are n subsets that cover all elements in A. Given an instance of X3C, we construct a market with $3n + 3$ items and $9n + m + 1$ buyers as follows. Every element in A corresponds to an item; further, we introduce another three items $B = \{b_1, b_2, b_3\}$. We use index j to denote one item. For each subset S_i, there is a buyer with value $v_{ij} = 1$ if $j \in S_i$ and $v_{ij} = 0$ otherwise; further, for every possible subset $\{x, y, z\}$ where $x \in A$ and $y, z \in B$, there is a buyer with value $v_{ij} = 1$ if $j \in \{x, y, z\}$ and $v_{ij} = 0$ otherwise; finally, there is a buyer with value $v_{ij} = 1$ if $j \in B$ and $v_{ij} = 0$ otherwise. The demand of every buyer is 3.

We claim that there is a positive answer to the X3C instance if and only if there is a sharp competitive equilibrium in the constructed market. Assume that there is $T \in \{S_1, \ldots, S_m\}$ with $|T| = n$ that covers all elements in A. Then we allocate items in A to the buyers in T and allocate B to the buyer who desires B, and set all prices to be 1. It can be seen that this defines a sharp competitive equilibrium.

On the other hand, assume that there is a sharp competitive equilibrium $(\mathbf{p}, \mathbf{X})$. We first claim that all items are allocated out in the sharp equilibrium. Otherwise, there must exist an item $a_j \in A$ that is not allocated to any buyer. (If all unallocated items just belonged to B, then all 3 items in B would be unallocated, contradicting envy-freeness of the buyer who values B.) Then we have $p_{a_j} = 0$. Consider the buyers who desire subsets $\{a_j, b_1, b_2\}, \{a_j, b_1, b_3\}, \{a_j, b_2, b_3\}$. They do not win since a_j is not sold. Due to envy-freeness, we have

$$
\begin{aligned}
p_{b_1} + p_{b_2} &\geq 3 \\
p_{b_1} + p_{b_3} &\geq 3 \\
p_{b_2} + p_{b_3} &\geq 3
\end{aligned}
$$

This implies that $p_{b_1} + p_{b_2} + p_{b_3} \geq 4.5$. Hence, the buyer who desires B cannot afford the price of B and at least one item in B, say b_1, is not allocated out. Thus $p_{b_1} = 0$ and $p_{b_2} + p_{b_3} \geq 4.5$. This contradicts envy-freeness of the buyer who gets b_2 and b_3.

Now since all items in A are allocated out, because of the construction of the market, we have to allocate all items in A to n buyers and allocate B to one buyer; the former gives a solution to the X3C instance. □

4.4 Computation of Sharp Competitive Equilibrium

It is well known that a sharp competitive equilibrium always exists for unit demand buyers (even for general v_{ij} valuations) [55]; for our sharp multi-unit demand model, however, a sharp competitive equilibrium may not exist. In the unit (relaxed) demand case, it is well-known that the set of equilibrium prices forms a distributive lattice; hence, there exist extremes which correspond to the maximum and the minimum equilibrium price vectors. In our multi-unit demand model, however, even if a sharp competitive equilibrium exists, sharp maximum equilibrium prices may not exist. Because of the sharp multi-unit demand, an interesting and important property is that it is possible that some items are 'over-priced'; this is a significant difference between sharp multi-unit and unit (relaxed) demand models (see Example 2.2.4).

We have the following characterization for over-priced items in an equilibrium solution.

Lemma 4.9. *For any given sharp competitive equilibrium* $(\mathbf{p}, \mathbf{X})$*, the following claims hold:*

- *If there is any unallocated item, then there are no over-priced items.*
- *At most one winner can have over-priced items; further, that winner, say* i*, must be the one with the smallest value among all winners in the sharp equilibrium* $(\mathbf{p}, \mathbf{X})$*. That is, for any other winner* $i' \neq i$*, we have* $v_{i'} > v_i$*.*

Proof. The first claim is obvious since any unallocated item j' is priced at 0; thus if there is a winner i and item $j \in X_i$ such that $p_j > v_i q_j$, then i would envy the subset $X_i \cup \{j'\} \setminus \{j\}$.

To prove the second claim, suppose there are two winners i, i' where $v_i \geq v_{i'}$, and suppose that i has over-priced item j. Since i' is envy-free, his own utility must be non-negative; we know there is an item $j' \in X_{i'}$ such that $v_{i'} q_{j'} \geq p_{j'}$. This implies that $v_i q_{j'} \geq p_{j'}$; thus, i would envy the subset $X_i \cup \{j'\} \setminus \{j\}$, a contradiction. □

4.4.1 Properties

We present some observations regarding sharp envy-freeness and sharp competitive equilibrium. Our first observation implies that a winner is sharp envy-free if and only if he prefers each of his allocated items to any other item.

Lemma 4.10. *Given any solution* $(\mathbf{p}, \mathbf{X})$ *and any winner* i*, if* i *is sharp envy-free then* $v_{ij} - p_j \geq v_{ij'} - p_{j'}$ *for any items* $j \in X_i$ *and* $j' \notin X_i$*. On the other hand, if* i *is not sharp envy-free, then there is* $j \in X_i$ *and* $j' \notin X_i$ *such that* $v_{ij} - p_j < v_{ij'} - p_{j'}$*.*

Proof. If i is sharp envy-free but (for $j \in X_i$ and $j' \notin X_i$) $v_{ij} - p_j < v_{ij'} - p_{j'}$, it is easy to see that i would envy subset $X_i \cup \{j'\} \setminus \{j\}$, a contradiction. If i is not sharp envy-free, then there is a subset T of items with $|T| = d_i$ such that $\sum_{j \in X_i}(v_{ij} - p_j) < \sum_{j' \in T}(v_{ij'} - p_{j'})$. Since $|X_i| = |T|$, the inequality holds for at least one item, i.e., there is $j \in X_i$ and $j' \notin X_i$ such that $v_{ij} - p_j < v_{ij'} - p_{j'}$. □

Lemma 4.11. *For any sharp envy-free solution* $(\mathbf{p}, \mathbf{X})$*, suppose there are two buyers* i, i' *with values* $v_i > v_{i'}$ *and two items* j *and* j' *that are allocated to* i *and* i' *respectively, i.e.,* $j \in X_i$ *and* $j' \in X_{i'}$*. Then* $q_j \geq q_{j'}$*.*

Proof. By the above Lemma 4.10, we have

$$\begin{aligned} v_i q_j - p_j &\geq v_i q_{j'} - p_{j'} \\ v_{i'} q_{j'} - p_{j'} &\geq v_{i'} q_j - p_j \end{aligned}$$

Adding the two inequalities together, we get $(v_i - v_{i'})(q_j - q_{j'}) \geq 0$, yielding the desired result. □

Lemma 4.11 implies that in any sharp envy-free solution, the allocation of items is monotone in terms of their amount of qualities and the values of the winners, i.e., winners with larger values win items with larger qualities. However, it does not imply that the value of every winner is larger than or equal to the value of any loser. For instance, consider three buyers i_1, i_2, i_3 and two items j_1, j_2 with $q_{j_1} = 2$ and $q_{j_2} = 1$. The values and demands are $v_{i_1} = 1.3, v_{i_2} = 1, v_{i_3} = 0.9$ and $d_{i_1} = 1, d_{i_2} = 2, d_{i_3} = 1$. Then prices $p_{j_1} = 2.2, p_{j_2} = 0.9$ and allocations $X_{i_1} = \{j_1\}, X_{i_2} = \emptyset, X_{i_3} = \{j_2\}$ constitute a revenue maximizing sharp envy-free solution. In this solution, $v_{i_2} > v_{i_3}$, but i_2 does not win any item (because of the sharp demand constraint) whereas i_3 wins item j_2.

Lemma 4.12. *If there is a sharp competitive equilibrium* $(\mathbf{p}, \mathbf{X})$*, then for any winner* i*, item* $j \in X_i$ *and unallocated item* j'*, we have* $q_j \geq q_{j'}$*.*

Proof. Since item j' is not allocated to any buyer, its price $p_{j'} = 0$. By sharp envy-freeness, we have $v_i q_j \geq v_i q_j - p_j \geq v_i q_{j'} - p_{j'} = v_i q_{j'}$, which implies that $q_j \geq q_{j'}$. □

By the above characterization, in any sharp competitive equilibrium, all allocated items have larger qualities. Hence, by Lemmas 4.11 and 4.12, we know that if the set of winners is fixed in a sharp competitive equilibrium, the allocation is determined implicitly as well. On the other hand, we observe that Lemma 4.12 does not hold if $(\mathbf{p}, \mathbf{X})$ is a (revenue maximizing) sharp envy-free solution. For instance, consider two buyers i_1, i_2 with values $v_{i_1} = 10, v_{i_2} = 1$ and demand $d_{i_1} = 1, d_{i_2} = 10$, and twelve items $j_1, j_2, \ldots, j_{12}$ with qualities $q_{j_1} = 10, q_{j_2} = 5, q_{j_3} = \cdots = q_{j_{12}} = 1$. It can be seen that in the optimal sharp envy-free solution, we set prices $p_{j_1} = 91, p_{j_2} = \infty, p_{j_3} = \cdots = p_{j_{12}} = 1$, and allocate $X_{i_1} = \{j_1\}, X_{i_2} = \{j_3, \ldots, j_{12}\}$, which generates total revenue $91 + 10 = 101$. In this solution, $q_{j_2} > q_{j_3} = \cdots = q_{j_{12}}$, but item j_2 is not allocated to any buyer.

Lemma 4.13. *Given a sharp envy-free solution* $(\mathbf{p}, \mathbf{X})$*, a loser* ℓ *and any subset* T *of* d_ℓ *items, the following property cannot hold:*
A non-empty subset of items in T *are either allocated to winners with values smaller than* v_ℓ *or priced at 0; any other elements of* T *are allocated to winners having the same value* v_ℓ *as* ℓ*.*

Note that Lemma 4.13 is not only for sharp equilibrium but also available for sharp envy-free solutions.

Proof. Let $(\mathbf{p}, \mathbf{X})$ be a sharp envy-free pair of price and allocation vectors. Given the loser ℓ and T satisfying the conditions of the statement of the lemma, we show how to construct a set T' of items that ℓ envies.

Let $T = T_0 \cup T_1 \cup \cdots \cup T_s$ be a partition of T where T_0 consists of items priced at 0 in $(\mathbf{p}, \mathbf{X})$ and for $i > 0$, $T_i = T \cap X_i$. Note that any non-empty T_i satisfies $v_i \leq v_\ell$, and if $T_0 = \emptyset$ then $T_i \neq \emptyset$ for some $i > 0$ with $v_i < v_\ell$.

Note that T_0 satisfies $\sum_{j \in T_0} v_i q_j - p_j \geq 0$, where the inequality is strict if T_0 is non-empty. Let $T'_0 = T_0$.

Consider any non-empty T_i (with $i > 0$). Let T'_i be the $|T_i|$ items $j \in X_i$ that maximize $v_i q_j - p_j$. We have $\sum_{j \in T'_i} v_i q_j - p_j \geq 0$. Hence $\sum_{j \in T'_i} v_\ell q_j - p_j \geq 0$, with strict inequality if $v_i < v_\ell$.

Summing these inequalities, we have $\sum_{i=0}^{s} \sum_{j \in T'_i} v_\ell q_j - p_j \geq 0$, and in fact the inequality is strict since at least one of the $s+1$ inequalities is strict. Let $T' = T'_0 \cup T'_1 \cup \cdots \cup T'_s$; $|T'| = |T| = d_\ell$ and we have shown that ℓ envies T'. $\square$

4.4.2 Algorithm

Our main result of this section is the following.

Theorem 4.14. *There is a polynomial algorithm to determine the existence of a sharp competitive equilibrium; and if one exists, it computes a revenue maximizing sharp equilibrium.*

Thus, both the existence problem and the maximization problem become tractable, as a result of the correlated valuations $v_{ij} = v_i q_j$.

The algorithm, called Max-CE, is divided into two steps. The first step is to compute a set of 'candidate' winners if an equilibrium exists. The second step is to calculate a 'candidate' equilibrium and verify if it is indeed a (revenue maximizing) equilibrium. Let A_k, $1 \leq k \leq K$ denotes all the buyers with the kth largest value in $\{v_1, v_2, \cdots, v_n\}$.

MAX-CE STAGE 1.

1. `Let` $S^* \leftarrow \emptyset$ `be the set of candidate winners`
2. `Let` $k \leftarrow 1$ `and let` $D \leftarrow m$ `be the number of ``available items''`
3. `While` $k \leq K$
   - `If` $d_i > D$ `for every` $i \in A_k$`, let` $k \leftarrow k + 1$
   - `Else`
     - `Let` $S = \{i \mid i \in A_k,\ d_i \leq D\}$
     - `If` $\sum_{i \in S} d_i > D$
       (a) `If there is` $S' \subseteq S$ `s.t.` $\sum_{i \in S'} d_i = D$
       `let` $S^* \leftarrow S^* \cup S'$`, and goto` MAX-CE STAGE 2
       (b) `Else, a sharp competitive equilibrium does not exist, and return`
     - `Else` $\sum_{i \in S} d_i \leq D$
       (c) `Let` $S^* \leftarrow S^* \cup S$, $D \leftarrow D - \sum_{i \in S} d_i$, $k \leftarrow k + 1$
4. `Goto` MAX-CE STAGE 2

Note that in the above step 3(a) we check whether there is $S' \subseteq S$ such that $\sum_{i \in S'} d_i = D$; this is equivalent to solving a subset sum problem. However, in our instance, each demand satisfies $d_i \leq m$. Hence, a dynamic programming approach can solve the problem in time $O(n^2 m)$. Hence, STAGE 1 runs in strongly polynomial time.

An input to MAX-CE is all the n buyers with valuation v_i and demand d_i and all the m items with qualities q_j.

Lemma 4.15. *If an input to* MAX-CE *has a sharp competitive equilibrium* $(\mathbf{p}, \mathbf{X})$*, then* STAGE 1 *will not return that a sharp equilibrium does not exist at step 3(b).*

Proof. Let $(\mathbf{p}, \mathbf{X})$ be a sharp competitive equilibrium of an input to MAX-CE. In this proof, when we refer to winning/losing buyers, or allocated/unallocated items, we mean with respect to $(\mathbf{p}, \mathbf{X})$. In particular, let W be the set of winners of $(\mathbf{p}, \mathbf{X})$.

Suppose that MAX-CE STAGE 1 exits on the k-th iteration of the loop. We claim that during the first $k-1$ iterations, all buyers added to S^* must be winners. To see this, suppose alternatively that at iteration $k' < k$, buyer ℓ is the first loser to be added to S^*. In that case, ℓ has d_ℓ items that satisfy the conditions of Lemma 4.13, contradicting sharp envy-freeness. (Suppose that the winners found by the algorithm during the first $k' - 1$ iterations are given their allocation in $(\mathbf{p}, \mathbf{X})$. At iteration k', the algorithm has more than d_ℓ available items, some of which are allocated to buyers with value less than ℓ, or are unallocated.)

At the final iteration k we must have $S \neq \emptyset$ (otherwise the algorithm will begin a new iteration). Since $\sum_{i \in S} d_i > D$, we have $S \backslash W \neq \emptyset$ (members of S have too much demand for them all to be able to win). Since there is no subset $S' \subseteq S$ such that $\sum_{i \in S'} d_i = D$, we have $\sum_{i \in S \cap W} d_i < D$. Hence, there are items that are not allocated to buyers in

$S^* \cup (S \cap W)$. Let $i' \in S \setminus W$; we can find $d_{i'}$ items that satisfy the condition of Lemma 4.13, implying that a buyer in $S \backslash W$ is not sharp envy-free, a contradiction. □

Lemma 4.16. *A revenue maximizing sharp competitive equilibrium* $(\mathbf{p}, \mathbf{X})$ *can be converted to one with equal revenue whose winning set is* S^*.

Proof. Assume that the given instance has a sharp competitive equilibrium $(\mathbf{p}, \mathbf{X})$ and that MAX-CE enters MAX-CE STAGE 2 at the kth iteration with the set of candidates S^*. Let W be the set of winners of $(\mathbf{p}, \mathbf{X})$, and let $W' = W \cap (A_1 \cup \cdots \cup A_{k-1})$ and $W'' = W \setminus W'$. Let $S^1 = S^* \cap (A_1 \cup \cdots \cup A_{k-1})$ and $S^2 = S^* \setminus S^1$ (note that $S^2 \subseteq A_k$). From the analysis of the above lemma and Lemma 4.13, we know that (i) $W' = S^1$, (ii) $W'' \subseteq A_k$, and (iii) $\sum_{i \in W''} d_i = \sum_{i \in S^2} d_i$. Thus, the only difference between S^* and W lies in the selection of buyers in A_k (this is due to possibly multiple choices in step 3(a) in MAX-CE STAGE 1). Due to envy-freeness, we have

$$\sum_{i \in W'' \setminus S^2} u_i(\mathbf{p}, \mathbf{X}) = \sum_{i \in W'' \setminus S^2} \sum_{j \in X_i} (v_i q_j - p_j) \geq 0 \geq \sum_{i \in S^2 \setminus W''} u_i(\mathbf{p}, \mathbf{X})$$

Since all buyers in $W'' \setminus S^2$ and $S^2 \setminus W''$ have the same value, we know that the above inequalities are tight. Thus, if we reassign the items in $\cup_{i \in W''} X_i$ to the buyers in S^2 and keep the same prices, the resulting output will still be a sharp equilibrium. □

Given the above characterization, the second step of the algorithm MAX-CE is described as follows.

MAX-CE STAGE 2.

5. `Allocation` $\mathbf{X}^*$ `is constructed as follows:`
   - `Let` $X_i^* \leftarrow \emptyset$, `for each buyer` $i \notin S^*$
   - `For each` $i \in S^*$ `in non-increasing order of` v_i
     - `allocate` d_i `of the remaining items to` i `in non-increasing order of` q_j
6. `Price` $\mathbf{p}^*$ `is computed according to the following linear program:`

$$\begin{array}{lll} \max & \sum_{i \in S^*} \sum_{j \in X_i^*} p_j^* & \\ s.t. & p_j^* \geq 0 & \forall\ j \\ & p_j^* = 0 & \forall\ j \notin \cup_{i \in S^*} X_i^* \\ & v_i q_j - p_j^* \geq v_i q_{j'} - p_{j'}^* & \forall\ i \in S^*, j \in X_i^*, j' \notin X_i^* \\ & \sum_{j \in T} (v_i q_j - p_j^*) \leq 0 & \forall\ i \notin S^*,\ T \text{ with } |T| = d_i \end{array}$$

7. `If the above linear program has a feasible solution, output the tuple` $(\mathbf{p}^*, \mathbf{X}^*)$
8. `Else, return that a sharp competitive equilibrium does not exist`

In the above LP, there are m variables where each item j has a variable p_j^*. The first two constraints ensure that the price vector is a set of feasible market clearing prices. The third condition guarantees that all winners are envy-free. The last condition says that for each loser i and any subset of items T with $T = |d_i|$, i cannot obtain a positive utility from T. Notice that it is possible that there are exponentially many combinations of T; thus the LP has an exponential number of constraints. However, observe that for any given solution $\mathbf{p}^*$, it is easy to verify if $\mathbf{p}^*$ is a feasible solution of the LP or find a violated constraint. In particular, for every loser i, we can order all items j in decreasing order of $v_i q_j - p_j^*$ and verify the subset T composed of the first d_i items; if i cannot obtain a positive utility from such T, then i is envy-free. Therefore, there is a separation oracle to the LP, and thus, the ellipsoid method can solve the LP in polynomial time. Hence, the total running time of MAX-CE is in polynomial time.

If the algorithm returns a tuple $(\mathbf{p}^*, \mathbf{X}^*)$, certainly it is a sharp competitive equilibrium; further, it is a revenue maximizing sharp equilibrium because of the objective function in the LP. It is therefore sufficient to show the following claim to complete the proof of Theorem 4.14.

Lemma 4.17. *If there exists a sharp competitive equilibrium, then* STAGE 2 *will not claim that an equilibrium does not exist at step 8.*

Proof. If there is a sharp competitive equilibrium $(\mathbf{p}, \mathbf{X})$, let W be the set of winners of the equilibrium. By Lemma 4.15, we know that MAX-CE will enter MAX-CE STAGE 2. By the above discussions, we know that W and S^* only differ in the last kth iteration of the main loop of MAX-CE STAGE 1 and replacing all winners in $W \cap A_k$ with $S^* \cap A_k$ gives a sharp equilibrium as well. Further, by Lemma 4.11 and 4.12, the allocation of items to the winners in W is fixed. Hence, the sharp equilibrium price vector $\mathbf{p}$ gives a feasible solution to the LP in the STAGE 2, which implies that the algorithm will not claim that a sharp equilibrium does not exist. □

4.5 Consecutive Competitive Equilibrium

In this section, we study the revenue maximizing consecutive competitive equilibrium in the full information setting. To simplify the following discussions, we sort all buyers and items in non-increasing order of their values, i.e., $v_1 \geq v_2 \geq \cdots \geq v_n$.

We say an allocation $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$ is efficient if $\mathbf{Y}$ maximizes the total social welfare e.g. $\sum_i \sum_{j \in Y_i} v_{ij}$ is maximized over all the possible allocations. We call $\mathbf{p} = (p_1, p_2, \cdots, p_m)$ an equilibrium price if there exists consecutive an allocation $\mathbf{X}$ such that $(\mathbf{X}, \mathbf{p})$ is a competitive equilibrium. The following lemma is implicitly stated in [38], for completeness, we'll prove it.

Lemma 4.18. *Let allocation* $\mathbf{Y}$ *be efficient, then for any equilibrium price* $\mathbf{p}$*,* $(\mathbf{Y}, \mathbf{p})$ *is a consecutive competitive equilibrium.*

Proof. Since $\mathbf{p}$ is an equilibrium price, there exists an allocation $\mathbf{X}$ such that $(\mathbf{X}, \mathbf{p})$ is a consecutive competitive equilibrium. As a result, by consecutive envy-freeness, $u_i(\mathbf{X}, \mathbf{p}) \geq u_i(\mathbf{Y}, \mathbf{p})$ for any $i \in [n]$. Let $T = [m] \setminus \cup_i Y_i$, then we have

$$\begin{aligned}
&\sum_i \sum_{j \in Y_i} v_{ij} - \sum_{j=1}^{m} p_j \geq \sum_i \sum_{j \in X_i} v_{ij} - \sum_{j=1}^{m} p_j \\
&= \sum_i \sum_{j \in X_i} v_{ij} - \sum_i \sum_{j \in X_i} p_j = \sum_i u_i(\mathbf{X}, \mathbf{p}) \\
&\geq \sum_i u_i(\mathbf{Y}, \mathbf{p}) = \sum_i \sum_{j \in Y_i} v_{ij} - \sum_i \sum_{j \in Y_i} p_j \\
&= \sum_i \sum_{j \in Y_i} v_{ij} - \sum_{j=1}^{m} p_j + \sum_{j \in T} p_j
\end{aligned}$$

where the first inequality is due to $\mathbf{Y}$ being efficient and first equality due to $u_i(\mathbf{X}, \mathbf{p})$ being consecutive competitive equilibrium (unallocated item priced at 0). Therefore, $\sum_{j \in T} p_j = 0$ and the above inequalities are all equalities. $\forall i : u_i(\mathbf{X}, \mathbf{p}) = u_i(\mathbf{Y}, \mathbf{p})$. Further, because the price is the same,

$\forall i$ a loser $\forall Z$ consecutive items and $|Z| = d_i$, we have $u_i(Z) \leq 0$.

$\forall i$ a winner $\forall Z$ consecutive items and $|Z| = d_i$, we have

$$u_i(Y_i) = u_i(X_i) \geq u_i(Z).$$

Therefore, $(\mathbf{Y}, \mathbf{p})$ is a consecutive competitive equilibrium. □

By Lemma 4.18, to find a revenue maximizing consecutive competitive equilibrium, we can first find an efficient allocation and then use linear programming to settle the prices. We develop the following dynamic programming to find an efficient allocation. We first only consider there is one peak in the quality order of items. The case with constant peaks is similar to the above approaches, for general peak case, as shown in above Theorem 3.18, finding one consecutive competitive equilibrium is NP-hard if the competitive equilibrium exists, and determining existence of consecutive competitive equilibrium is also NP-hard.

Recall that all the values are sorted in non-increasing order e.g. $v_1 \geq v_2 \geq \cdots \geq v_n$. $g[s, l, r]$ denotes the maximized value of social welfare when we only consider first s buyers and the allocation of s is exactly the interval $[l, r]$. Then we have the following transition function.

$$g[s, l, r] = \max \begin{cases} g[s-1, l, r] \\ \\ g[s-1, l, r-d_s] + v_s \sum_{j=r-d_s+1}^{r} q_j \\ \\ g[s-1, l+d_s, r] + v_s \sum_{j=l}^{l+d_s-1} q_j \end{cases} \tag{4.1}$$

By tracking procedure 4.1, an efficient allocation denoted by $\mathbf{X}^* = (X_1^*, X_2^*, \cdots, X_n^*)$ can be found. The price $\mathbf{p}^*$ such that $(\mathbf{X}^*, \mathbf{p}^*)$ is a revenue maximization consecutive competitive equilibrium can be determined from the following linear programming. Let T_i be any consecutive number of d_i slots, for all $i \in [n]$.

$$\begin{aligned}
\max \quad & \sum_{i\in[n]} \sum_{j\in X_i^*} p_j & \\
s.t. \quad & p_j \geq 0 & \forall\, j \in [m] \\
& p_j = 0 & \forall\, j \notin \cup_{i\in[n]} X_i^* \\
& \sum_{j\in X_i^*} (v_i q_j - p_j) \geq \sum_{j'\in T_i} (v_i q_{j'} - p_{j'}) & \forall\, i \in [n] \\
& \sum_{j\in X_i^*} (v_i q_j - p_j) \geq 0 & \forall i \in [n]
\end{aligned}$$

Clearly there is only a polynomial number of constraints. The constraints in the first line represent that all the prices are non negative (no positive transfers). The constraint in the second lines means unallocated items must be priced at zero (market clearance condition). And the constraint in third line contains two aspects of information. First for all the losers e.g. loser k with $X_k = \emptyset$, the utility k gets from any consecutive number of d_k is no more than zero, which makes all the losers envy-free. The second aspect is that the winners e.g. winner i with $X_i \neq \emptyset$ must receive a bundle with d_i consecutive slots maximizing its utility over all d_i consecutive slots, which together with the constraint in fourth line (winners' utilities are non negative) guarantees that all winners are consecutive envy-free.

Theorem 4.19. *If the number of peaks is upper bounded by a constant. Then there is a polynomial time algorithm to decide whether there exists a consecutive competitive equilibrium or not and to compute a revenue maximizing consecutive market equilibrium if one does exist. If the number of peaks is arbitrary, both problems (deciding existence or computing the maximum one if one does exist) is NP-hard.*

Proof. The efficient allocation can be found efficiently if the number of peaks is bounded by a constant, see Theorem 3.17. Clearly the above linear programming and procedure (4.1) run in polynomial time. If the linear programming output a price $\mathbf{p}^*$, then by its constraint conditions, $(\mathbf{X}^*, \mathbf{p}^*)$ must be a consecutive competitive equilibrium. On the other hand, if there exists a consecutive competitive equilibrium $(\mathbf{X}, \mathbf{p})$ then by Lemma 4.18, $(\mathbf{X}^*, \mathbf{p})$ is a consecutive competitive equilibrium, providing a feasible solution of the above linear programming. By the objective of the linear programming, we know it must be a revenue maximizing one. For the general peak case, as shown in the above Theorem 3.18, finding one consecutive competitive equilibrium is NP-hard if the competitive equilibrium exists since finding the efficient allocation is NP-hard, and determining existence of competitive equilibrium is also NP-hard. This is because that considering the instance in the proof of Theorem 3.18, it is not difficult to see the constructed instance has an equilibrium if and only if 3 partition has a solution. □

Chapter 5

Revenue Maximization of Envy-free Multi-unit Models

We study revenue maximization problems of various envy-free buyers in this chapter. The main work of this chapter focuses on relaxed/sharp/consecutive demand buyers. For the relaxed demand case, as before, the problem is simple, which can be solved completely by dynamic programming as shown in Section 5.2. The circumstances become difficult for sharp demand cases. Indeed, if the demand of the buyers is arbitrary, the problem is NP-hard, however, for the very important case where buyers' demands are bounded by a constant, a polynomial algorithm is presented to compute an optimal sharp envy-free solution. The cases for consecutive demand buyers are still very hard. If the demand is arbitrary, the problem is NP-hard even if all the qualities are the same, yet, for uniform demand buyers, we present a polynomial algorithm solving the problems if the qualities are ordered from top to down. Some partial results on bundle envy-free buyers and budgets constraint buyers are also given.

This chapter is organized as follows. We begin in Section 5.1 to present the main results in this chapter. Next, the revenue maximization problem for relaxed demand buyers is totally solved in Section 5.2. For sharp envy-freeness, the general demand case is NP-hard, see Section 5.4, however, a polynomial algorithm is presented to find an optimal envy-free solution when the demand is bounded by a constant in Section 5.3. The problem becomes more difficult in consecutive demand cases. Indeed, the problem is NP-hard even if all the qualities are all the same, yet, a polynomial algorithm is presented when the buyers have common demand in Section 5.5. Some results on bundle envy-freeness are presented in Section 5.6. Studies on identical items with budgets for relaxed demand buyers are shown in Section 5.7.

5.1 Introduction

First, we present the positive result for relaxed envy-free solutions as follows.

Theorem 5.1. *There is a strongly polynomial time algorithm to compute a revenue maximizing (maximum) relaxed envy-free solution with relaxed demand constraint.*

The NP-hardness result of [39] for unit demand buyers implies that we cannot hope for a polynomial time algorithm for general v_{ij} valuations in the multi-unit demand setting, even for the very special case when one has positive values for at most three items [14]. However, it does not rule out positive computational results for special, but important, cases of multi-unit demand. For $v_i q_j$ valuations with multi-unit demand, where the hardness reductions of [14, 39] does not apply, we have the following results.

Theorem 5.2. *There is a polynomial time algorithm that computes a revenue maximizing sharp envy-free solution in the sharp multi-unit demand model with $v_i q_j$ valuations if the demand of every buyer is bounded by a constant. On the other hand, the problem is NP-hard if the sharp demand is arbitrary, even if there are only three buyers.*

Here, we have a complete overview of sharp/consecutive envy-free pricing with multi-unit demand buyers. Most of our results are positive, suggesting that sharp/consecutive envy-free pricing are candidate solution concepts to be applicable in the domains where the valuations are correlated with respect to the quality of the items.

We prove that it is NP-hard to compute an optimal sharp envy-free solution even if there are only three buyers. Hence, our efforts focus on the special, yet very important bounded-demand case. To compute an optimal sharp envy-free solution for bounded demand, certain candidate winner sets (the number of such sets is polynomial) are defined and found; and crucially, there is at least one optimal winner set in our selected candidate winner sets. For each candidate winner set, if the demand is bounded by a constant, we present a linear programming to characterize its optimal solution when the allocation is known. Finally, a dynamic programming algorithm is provided to find the allocation sets when a candidate winner set is fixed. Both the linear programming and the dynamic programming run in polynomial time. For consecutive demand buyers, the problem of revenue maximizing consecutive envy-free solution is NP-hard. However, for the very important case that the qualities are order from top to bottom, and all the buyers' demands are the same, a polynomial algorithm would be presented to compute maximum revenue of consecutive envy-free solutions.

Theorem 5.3. *The revenue maximization problem of consecutive envy-free buyers is NP-hard even if all the qualities are the same.*

Definition 5.1.1 (Uniform Demand)**.** We say buyers have uniform demands if all the buyers have the same demand in the model.

Definition 5.1.2 (PTAS)**.** An algorithm is called a polynomial time approximation scheme (PTAS) for a given problem with input I if for any $\epsilon > 0$, the algorithm runs in time polynomial in the size of input I to output a solution with value $M(I)$ such that $M(I) \geq (1-\epsilon)OPT(I)$, where $OPT(I)$ is the optimal value of the problem.

For bundle envy-free case, the situation become much complicated, yet we still have some positive results. Noting that taking budget into consideration, each winner i's payment should be no more than b_i e.g. $\sum_{j \in X_i} p_j \leq b_i$.

Theorem 5.4. *For bundle envy-free pricing, each buyer only having uniform demand constraints, a PTAS is presented for identical items and an exponential time algorithm is presented for distinct items with valuation $v_i q_j$; for relaxed envy-free pricing, each buyer only have budget constraints, an optimal algorithm is proposed for identical items.*

See Section 5.6 and 5.7 for the proof of Theorem 5.4.

Recall K is the number of distinct values in the set $A = \{v_1, \ldots, v_n\}$, and $A_1, \ldots, A_K$ is a partition of all buyers where each A_k, $k = 1, 2, \ldots, K$, contains the set of buyers that have the kth largest value.

5.2 Relaxed Envy-Free Pricing

Theorem 5.5. *There is a strongly polynomial time algorithm to compute a revenue maximizing relaxed envy-free solution with relaxed demand constraint.*

Next we present an algorithm denoted by ALG-RLE-EF to solve the relaxed envy-free pricing problem. Suppose $S = \{i_1, i_2, \ldots, i_t\}$ to be a candidate winner set and $T = \{j_1, j_2, \ldots, j_\ell\}$ a subset of items, where $i_1 < i_2 < \cdots < i_t$, $j_1 < j_2 < \cdots < j_\ell$ and $d(S \backslash \{i_t\}) < \ell \leq d(S)$. Let $\mathbf{X}$ be the allocation produced by the following procedure denoted by RLE-ALLOCATION.

RLE-ALLOCATION

`Allocation` $\mathbf{X}$`:`

- `Let` $X_i \leftarrow \emptyset$`, for each buyer` $i \notin S$
- `For each` $i \in S \backslash \{i_t\}$ `with the decreasing order of` v_i
    - `buers with smaller indices obtain items with smaller indices`
- `allocate all the remaining items in` T `to` i_t

It is easy to check RLE-ALLOCATION takes strongly polynomial time.

ALG-RLE-EF

1. If $d(A) \leq m$, let $k_{\max} \leftarrow K$
2. Else let $k_{\max} = \arg\min_k d\big(A_1 \cup \cdots \cup A_k\big) \geq m$
3. For $k = 1$ to $k_{\max}$
   - Let $L \leftarrow d\big(A_1 \cup \cdots \cup A_{k-1}\big)$ and $i_{\min} = |\bigcup_{i=1}^{k-1} A_i|$
   - If $k < k_{\max}$ let $\ell \leftarrow d(A_k)$; else let $\ell \leftarrow m - L$
   - For $r = 1$ to ℓ
     - let $i_{\max} = \arg\min_j \sum_{i=i_{\min}+1}^{j} d_i \geq \ell$
     - let $S = \{1, 2, \dots, i_{\max}\}$
     - let $T = \{z_1, z_2, \dots, z_{L+\ell}\} \subseteq \{1, 2, \dots, m\}$ with $z_1 < z_2 < \cdots < z_{L+\ell}$ be items to be defined
     - run RLE-ALLOCATION on S and T getting $\mathbf{X}$, let $w_j \leftarrow v_i$ if $j \in X_i$
     - Pricing getting price vector $\mathbf{p}$ as follows
       let $p_j = \infty$ for $j \notin T$;
       let $p_{z_{L+\ell}} = w_{L+\ell} q_{z_{L+\ell}}$ and $p_{z_j} = w_j(q_{z_j} - q_{z_{j+1}}) + p_{z_{j+1}}$, for $j \in T \backslash \{z_{L+\ell}\}$
     - getting $T = \{z_1, z_2, \dots, z_{L+\ell}\}$ from the optimal solution of the following optimization
       problem and denote the optimal value as $R^{k,w}$

$$\text{Maximize} \quad R = \sum_{j=1}^{L+\ell} (j w_j - (j-1) w_{j-1}) q_{z_j}$$

$$\text{Subject to} \quad z_1 < z_2 < \cdots < z_{L+\ell}, \{z_1, z_2, \dots, z_{L+\ell}\} \subseteq \{1, 2, \dots, m\}$$

4. Let $R^{k^*,r^*} = \max\{R^{k,r}\}$ and $(\mathbf{p}, \mathbf{X})$ be the corresponding tuple to R^{k^*,r^*}; Output R^{k^*,r^*} and the tuple $(\mathbf{p}, \mathbf{X})$

Since the optimization problem of ALG-RLE-EF can be solved by dynamic programming method as in SOLVE-DLP (see subsection 5.3.5) in strongly polynomial time, the time taken in ALG-RLE-EF is strongly polynomial time.

Proof of Theorem 5.5. We need to prove ALG-RLE-EF actually outputs an optimal solution. First, it is easy to check our output is a relaxed envy-free solution. Second, there must be an optimal winner set (a set is called an optimal winner set if there exists a optimal relaxed envy-free solution such that the set is the winner set for this optimal relaxed envy-free solution) in our select possible winner sets like S. For any optimal winner set S', suppose there is an optimal solution $(\mathbf{p}', \mathbf{X}')$ corresponding to S'. Suppose $i_{\max} = \max\{i \in S'\}$ and $i_{\max} \in A_k$, suppose $w = |S' \cap A_k|$, then it is easy to check that $R^{k,r} \geq \sum_{i=1}^{n} \sum_{j \in X_i'} p_j'$. Hence, the output is an optimal relaxed envy-free solution. $\square$

5.3 Algorithm for Constant Sharp Demands

We noted earlier that a sharp envy-free solution always exists. We may use envy-free instead of sharp envy-free alternatively in this section for convenience if there is no confusion. Our main results are the following.

Theorem 5.6. *If the demand of each buyer is bounded by a constant, then the revenue-maximizing sharp envy-free pricing problem can be solved in polynomial time.*

We note that the correlated $v_i q_j$ valuations are crucial to derive an efficient computation when the demands are bounded by a constant; in contrast, for unit-demand, the sharp envy-free pricing is NP-hard for general valuations v_{ij} even if every buyer is interested in at most three items [14].

Throughout this section, let Δ be a constant where $d_i \leq \Delta$ for any buyer i, and again, buyers and items are sorted according to their values and qualities. For any tuple $(\mathbf{p}, \mathbf{X})$, we assume that all unsold items are priced at ∞. This assumption is without loss of generality for sharp envy-freeness. We will first explore some important properties of an (optimal) sharp envy-free solution, then at the end of the section present our algorithm.

5.3.1 Candidate Winner Sets

For a given set S of buyers, let $\max(S)$ and $\min(S)$ denote the buyer in S that has the largest and smallest index (buyers are indexed in increasing order with their values decreasing), respectively.

Definition 5.3.1 (Candidate Winner Set)**.** Given a subset of buyers $S \neq \emptyset$, let $k = \max\{r | A_r \cap S \neq \emptyset\}$. We say S is a *candidate winner set* if the total demand of buyers in S is at most m, i.e., $d(S) \leq m$, and for any $i \in A_1 \cup \cdots \cup A_{k-1} \setminus S$, $d_i > \sum_{i' \in S:\ i' > i} d_{i'}$.

The definition of candidate winner set is closely related to sharp envy-freeness. Indeed, due to Lemma 4.13, the above definition defines a slightly larger set (than all possible sets of winners in sharp envy-free solutions) as the inequality does not consider all the buyers completely in the same value category v_j. However, this definition makes it easier for us to describe and analyze the algorithm.

Proposition 5.7. *For any sharp envy-free solution* $(\mathbf{p}, \mathbf{X})$*, let* $S = \{i \mid X_i \neq \emptyset\}$ *be the set of winners. Then* S *is either a candidate winner set or* $S = \emptyset$*.*

Proof. The claim follows directly from Lemma 4.13. $\square$

FindWinners(S): `Input a set of buyers` S

- `Let` $i_{\max} = \max(S)$ `and assume` $i_{\max} \in A_k$
- `Initially let` $W_S = S$
- `For each buyer` $j \in A_1 \cup \cdots \cup A_{k-1}$ `in reverse order`
  - `If` $j \notin S$ `and` $d_j \leq \sum_{i \in W_S:\ i>j} d_i$, `let` $W_S \leftarrow W_S \cup \{j\}$
- `Return` W_S

Proposition 5.8. *For any subset of buyers S, let $W_S =$*FindWinners*(S).*

- *If $d(W_S) \leq m$, then W_S is a candidate winner set and for any candidate winner set $S' \supseteq S$, $W_S \subseteq S'$.*
- *If $d(W_S) > m$, then there is no candidate winner set containing S.*

Proof. Obviously, if $d(W_S) \leq m$, then from the definition of candidate winner set, we know W_S is a candidate winner set. Still, by the definition of candidate winner set, for any j in $W_S \backslash S$, any candidate winner set $S' \supseteq S$, since $d_j \leq \sum_{i \in W_S:\ i>j} d_i$, then $d_j \leq \sum_{i \in S':\ i>j} d_i$ (since $S' \supseteq S$), thus, $j \in S'$, hence, $W_S \subseteq S'$. Therefore, the second statement follows. □

FindLoser(S): `Input a candidate winner set` S

- `Let` $i_{\min} = \min(S)$ `and assume` $i_{\min} \in A_j$
- `Initially let` $L_S = \emptyset$, `and` $\alpha = \infty$
- `For each` $k = j, j+1, \ldots, K$
  - `Let` $i_0 = \arg\min\{d_i \mid i \in A_k \backslash\ S\}$
  - `If` $d_{i_0} < \alpha$, `let` $L_S \leftarrow L_S \cup \{i_0\}$ `and` $\alpha \leftarrow d_{i_0}$
- `Return` L_S

Proposition 5.9. *For any given tuple $(\mathbf{p}, \mathbf{X})$ with winner set S, suppose that S is a candidate winner set and let $L_S =$*FindLoser*(S). If all losers in L_S are sharp envy-free with respect to $(\mathbf{p}, \mathbf{X})$, then all other losers are sharp envy-free as well.*

Proof. For any $i \in L_S$, if i is sharp envy-free, then for any subset T of items with $|T| = d_i$, $\sum_{j \in T}(v_i q_j - p_j) \leq 0$. Hence, for any $v \leq v_i$ and T' with $|T'| \geq d_i$ (Consider sum of $vq_j - p_j$ over all the elements j of subset T with d_i of T', then each value $vq_j - p_j$ is counted by $\binom{|T'|-1}{d_i-1}$ times), we have

$$\sum_{j \in T'}(vq_j - p_j) = \frac{1}{\binom{|T'|-1}{d_i-1}} \sum_{T \subseteq T', |T|=d_i} \sum_{j \in T}(vq_j - p_j) \leq \frac{1}{\binom{|T'|-1}{d_i-1}} \sum_{T \subseteq T', |T|=d_i} \sum_{j \in T}(v_i q_j - p_j) \leq 0.$$

Hence, by the rules of FINDLOSER, we know that if all the losers in L_S are sharp envy-free, all other losers in $A_j \cup \cdots \cup A_K$ are sharp envy-free as well. On the other hand, for any loser $j \in A_1 \cup \cdots \cup A_{j-1}$, since S is a candidate winner set, we know that $d_j > \sum_{i \in S:\ i>j} d_i = \sum_{i \in S} d_i$. Since all unsold items are priced at ∞, we know that j is sharp envy-free. Hence, all losers are sharp envy-free. □

5.3.2 Bounding the Number of Candidate Winner Sets

We have the following bound on the number of candidate winner sets.

Lemma 5.10. *For any $k \in \{2, \ldots, K\}$ and $S \subseteq A_k$, suppose $d(S) \leq m$. Let*

$$\mathcal{C} = \left\{ S \cup S' \mid S' \subseteq A_1 \cup \cdots \cup A_{k-1} \text{ and } S \cup S' \text{ is a candidate winner set} \right\}$$

Then $|\mathcal{C}| \leq \left\lfloor \frac{m}{d(S)} \right\rfloor$.

Proof. Let $a = d(S)$ and ℓ be the index of the buyer $\max(A_{k-1})$. We add buyers $\ell, \ell-1, \ell-2, \ldots, 1$ into S in sequence and maintain all the possible candidate winner sets. Let $\mathcal{C}_0 = \{S\}$. In general, we have constructed $\mathcal{C}_t$ containing all the candidate winner sets of $\{\ell, \ell-1, \ell-2, \ldots, \ell-t+1\} \cup S$. We order $\mathcal{C}_t = \{S_{t,1}, S_{t,2}, \ldots, S_{t,n_t}\}$ such that $d(S_{t,1}) \leq d(S_{t,2}) \leq \cdots \leq d(S_{t,n_t}) \leq m$. We should prove that $d(S_{t,i}) \geq i * d(S)$.

We now add $\ell - t$ into $\mathcal{C}_t$ to construct $\mathcal{C}_{t+1}$. Let $t_s = \max\{i : d(S_{t,i}) < d_{\ell-t}\}$ if $\{i : d(S_{t,i}) < d_{\ell-t}\} \neq \emptyset$, otherwise $t_s = 0$. Let $S_{t+1,j} = S_{t,j}$ for $j = 1, 2, \cdot, t_s$, $S_{t+1,j+t_s} = S_{t,j} \cup \{\ell - t\}$ for $j = 1, 2, \ldots, n_t$. Clearly that $d(S_{t+1,i}) \geq i * d(S)$ for $i \leq t_s$ by inductive hypothesis. And

$$d(S_{t+1,j+t_s}) = d(S_{t,j}) + d_{\ell-t} \geq j * d(S) + d(S_{t,t_s}) \geq (j + t_s) * d(S).$$

Let $n_{t+1} = \max\{i : d(S_{t+1,i}) \leq m\}$. Clearly the claim follows for $\ell - t$ and $\mathcal{C}_t$.

The lemma follows by the condition $m \geq d(S_{\ell,n_\ell}) \geq n_\ell * d(S)$. □

5.3.3 Optimal Winner Sets

Definition 5.3.2 (Optimal Winner Set)**.** A subset of buyers S is called an *optimal winner set* if there is a revenue maximizing sharp envy-free solution $(\mathbf{p}, \mathbf{X})$ such that S is its set of winners.

Proposition 5.11. *Let S be an optimal winner set and let $k = \max\{r \mid A_r \cap S \neq \emptyset\}$. For any $S' \subseteq A_k$, if $d(S') = d(S \cap A_k)$, then $S' \cup (S \backslash A_k)$ is an optimal winner set as well.*

Before proving the proposition, we first establish the following claim.

Claim 5.3.1. *Suppose there exists a revenue-maximizing sharp envy-free solution $(\mathbf{p}, \mathbf{X})$, and let S be the winning set in $(\mathbf{p}, \mathbf{X})$, and let $k = \max\{r \mid A_r \cap S \neq \emptyset\}$. Then every buyer in A_k has utility zero.*

Proof. Of course, every loser in A_k has utility zero. To show that every winner in A_k has utility zero, we show that if such a winner has positive utility, then prices can be raised to the point where his utility becomes zero, while maintaining sharp envy-freeness (contradicting the assumption that $(\mathbf{p}, \mathbf{X})$ maximizes revenue).

Let $i_{\max} = \max(S)$. Let

$$\delta = \frac{u_{i_{\max}}(\mathbf{p}, \mathbf{X})}{d_{i_{\max}}}.$$

We claim that $(\mathbf{p} + \delta, \mathbf{X})$ is an sharp envy-free solution as well, where the price of each item is increased by δ.

Obviously we have $\delta \geq 0$, and the conclusion holds trivially if $\delta = 0$. Suppose $\delta > 0$. For the tuple $(\mathbf{p} + \delta, \mathbf{X})$, since all items have their prices increased by the same amount, all losers are still sharp envy-free and all winners would not envy the items they don't get. Hence, we need only to check that each winner still gets a non-negative utility. For $i_{\max}$, we have $u_{i_{\max}}(\mathbf{p} + \delta, \mathbf{X}) = 0$. For any other winner $i \neq i_{\max}$, it holds that $v_i \geq v_{i_{\max}}$. Since i does not envy any item in $(\mathbf{p}, \mathbf{X})$, for any item $j' \in X_i$ and $j \in X_{i_{\max}}$, it holds that $v_i q_{j'} - p_{j'} \geq v_i q_j - p_j$, hence, $p_{j'} \leq v_i(q_{j'} - q_j) + p_j$. Then, we get

$$p_{j'} \leq \frac{\sum_{j \in X_{i_{\max}}} (v_i(q_{j'} - q_j) + p_j)}{d_{i_{\max}}} = v_i q_{j'} - \frac{\sum_{j \in X_{i_{\max}}} (v_i q_j - p_j)}{d_{i_{\max}}}.$$

This implies that

$$\begin{aligned} p_{j'} + \delta &\leq v_i q_{j'} - \frac{\sum_{j \in X_{i_{\max}}} \big((v_i q_j - p_j) - (v_{i_{\max}} q_j - p_j)\big)}{d_{i_{\max}}} \\ &= v_i q_{j'} - \frac{\sum_{j \in X_{i_{\max}}} (v_i - v_{i_{\max}}) q_j}{d_{i_{\max}}} \leq v_i q_{j'}. \end{aligned}$$

Hence, $u_i(\mathbf{p} + \delta, \mathbf{X}) = \sum_{j' \in X_i} (v_i q_{j'} - p_{j'} - \delta) \geq 0$. Therefore, $(\mathbf{p} + \delta, \mathbf{X})$ is an envy-free solution. ☐

We are now ready for the proof of Proposition 5.11.

Proof of Proposition 5.11. Since S is an optimal winner set, there is an optimal sharp envy-free solution $(\mathbf{p}, \mathbf{X})$ such that $S = \{i \mid X_i \neq \emptyset\}$. We construct a new allocation $\mathbf{X}'$ with winner set $S' \cup (S \backslash A_k)$ as follows:

- For any $i \notin A_k$, $X'_i = X_i$.
- For any $i \in A_k \setminus S'$, $X'_i = \emptyset$.
- For all the buyers in S', allocate items in $\bigcup_{i \in S \cap A_k} X_i$ to them arbitrarily. (The allocation is feasible as $d(S') = d(S \cap A_k)$.)

Obviously, $(\mathbf{p}, \mathbf{X}')$ generates the same revenue as $(\mathbf{p}, \mathbf{X})$. We claim that $(\mathbf{p}, \mathbf{X}')$ is an envy-free solution (this implies our desired result that $S' \cup (S \backslash A_k)$ is an optimal winner set). For any buyer $i \notin A_k$, since prices are not changed, i is still sharp envy-free.

Next we prove that all buyers $i \in A_k$ are sharp envy-free in $(\mathbf{p}, \mathbf{X}')$. Let $J = \cup_{i \in S \cap A_k} X_i$ be the set of items allocated to buyers in A_k; we also have $J = \cup_{i \in S'} X'_i$. Suppose first that $|S \cap A_k| = |S'| = 1$; in this case $(\mathbf{p}, \mathbf{X})$ differs trivially from $(\mathbf{p}, \mathbf{X}')$, so $(\mathbf{p}, \mathbf{X}')$ is sharp envy-free.

Alternatively, there is some buyer $\bar{i} \in A_k$ with $d_{\bar{i}} < d(S \cap A_k)$. We show that any item $j \in J$ allocated to $i \in A_k$ in $(\mathbf{p}, \mathbf{X}')$, affords zero utility to i, i.e. j satisfies $v_i q_j = p_j$. Let v be the value shared by all $i \in A_k$, i.e. $v = v_i$ for any $i \in A_k$. Since $(\mathbf{p}, \mathbf{X})$ is sharp envy-free, we have using Claim 5.3.1 that $u_i(\mathbf{p}, \mathbf{X}) = 0$ for all $i \in A_k$, hence $\sum_{j \in J} vq_j - p_j = 0$. Suppose some $j \in J$ does not satisfy $vq_j - p_j = 0$. Arrange all $j \in J$ in descending order of $vq_j - p_j$. Any proper prefix P of this sequence satisfies $\sum_{j \in P} vq_j - p_j > 0$. Then buyer $\bar{i}$ envies this prefix. □

5.3.4 Maximizing Revenue for a Given Set of Winners and Allocated Items

Suppose that S is a candidate winner set and T is a subset of items, where $|T| = d(S)$. We want to know if there is an envy-free solution such that S is the set of winners and S wins items in T; if yes, we want to find one that maximizes revenue. This problem can be solved easily by a linear program with an exponential number of constraints for each $i \in S$. The following combinatorial algorithm does the same thing; the idea inside is critical to our main algorithm.

We will use the following notations: $S = \{i_1, i_2, \ldots, i_t\}$ with $i_1 < i_2 < \cdots < i_t$ and $T = \{j_1, j_2, \ldots, j_\ell\}$ with $j_1 < j_2 < \cdots < j_\ell$. Let $i_{b(s)}$ be the winner of j_s, $s = 1, 2, \ldots, \ell$.

MaxRevenue(S, T): Input a candidate winner set S and a subset of items T where $|T| = d(S)$

- Let $L_S = \text{FindLoser}(S)$.
- Allocation $\mathbf{X}$
  - Let $X_i \leftarrow \emptyset$, for each buyer $i \notin S$.
  - Allocate items in T to buyers in S according to the following rule (by Lemma 4.11):
    Buyers with smaller indices obtain items with smaller indices.
- Price $\mathbf{p}$
  - Let $Y = \emptyset$
  - For each item $j \notin T$, let $p_j = \infty$.
  - For each item $k \in X_{i_t}$, do the following

    (a) Let J be the last 2Δ items with the largest indices in T. Run the following linear program (denoted by LP$^{(k)}$), which computes prices for items in $X_{i_{t-1}} \cup X_{i_t}$

$$\begin{aligned}
\min \quad & v_{i_{t-1}} q_k - p_k & & \\
s.t. \quad & v_{i_{t-1}} q_k - p_k \geq v_{i_{t-1}} q_j - p_j & \forall\ j \in X_{i_t} & \quad (1)\\
& \sum_{j \in X_{i_t}} (v_{i_t} q_j - p_j) = 0 & & \quad (2)\\
& v_{i_{t-1}} q_j - p_j = v_{i_{t-1}} q_k - p_k & \forall\ j \in X_{i_{t-1}} & \quad (3)\\
& v_{i_t} q_j - p_j \leq v_{i_t} q_{j'} - p_{j'} & \forall\ j \in X_{i_{t-1}}, j' \in X_{i_t} & \quad (4)\\
& \textstyle\sum_{j \in J'} (v_i q_j - p_j) \leq 0 & \forall\ i \in L_S,\ J' \subseteq J \text{ with } |J'| = d_i & \quad (5)\\
& p_{j_s} = v_{b(s)}(q_{j_s} - q_{j_{s+1}}) + p_{j_{s+1}} & \forall j_s \in J - X_{i_t} - X_{i_{t-1}} & \quad (6)
\end{aligned}$$

    (b) If the LP$^{(k)}$ in (a) has a feasible solution, let $Y \leftarrow Y \cup \{k\}$.

    (c) For each item $j_s \in X_{i_1} \cup \cdots \cup X_{i_{t-2}}$ in the reverse order

    * let $p_{j_s} = v_{i_{b(s)}}(q_{j_s} - q_{j_{s+1}}) + p_{j_{s+1}}$

    (d) Denote the price vector computed above by $\mathbf{p}^{(k)}$.
- If $Y = \emptyset$, return that there is no price vector $\mathbf{p}$ such that $(\mathbf{p}, \mathbf{X})$ is sharp envy-free.
- Otherwise,
  - Let $k^* \in Y$ have the largest total revenue for which $(\mathbf{p}^{(\mathbf{k}^*)}, \mathbf{X})$ is an envy-free solution.
  - Output the tuple $(\mathbf{p}^{(\mathbf{k}^*)}, \mathbf{X})$.

Remark 5.12. It should be noting that in LP^k, the objective function is equivalent to maximize p_k. By the pricing rule (2), (3), (6) and (c) of MaxRevenue(S, T), the total revenue $\sum_{j \in T} p_j$ obtained is a linear increasing function of p_k, hence maximizing p_k is equivalent to maximizing the total revenue.

We establish the following properties:

Proposition 5.13. *Let* $(\mathbf{p}, \mathbf{X})$ *is computed in terms of* $LP^{(k^*)}$ *where* $k^* \in X_{i_t}$ *in* MaxRevenue(S, T)*. Let* $i_{b(u)}$ *be the winner of* j_u*. Use the convention* $j_{\ell-d_{i_t}+1} = k^*$*. For* $i = 1, 2, \ldots, l - d_{i_t}$*, we have*

1. $v_{i_{b(i)}} q_{j_{i+1}} - p_{j_{i+1}} \geq 0$;
2. $\frac{p_{j_i}}{q_{j_i}} \geq \frac{p_{j_{i+1}}}{q_{j_{i+1}}}$;
3. $p_{j_i} \geq p_{j_{i+1}}$.

Proof. For the first inequality, consider the last case, $v_{i_{t-1}} q_{k^*} - p_{k^*} \geq 0$. Assume it does not hold. By Formula (1) in Algorithm MaxRevenue, $\sum_{j \in X_{i_t}} (v_{i_{t-1}} q_j - p_j) < 0$. Therefore, $\sum_{j \in X_{i_t}} (v_{i_t} q_j - p_j) < 0$, which contradicts Formula (2). Further, $v_{i_u} q_{k^*} - p_{k^*} \geq 0$ for all $u : 1 \leq u \leq t-1$. That is, all other buyers have nonnegative utility on item k^*. Now consider $s = 1, 2, \ldots, \ell - d_{i_t}$. By (6) and (c) in the algorithm, using the convention $j_{\ell-d_{i_t}+1} = k^*$, item 1 holds as following

$$v_{b(s)} q_{j_{s+1}} - p_{j_{s+1}} \geq v_{b(s+1)} q_{j_{s+1}} - p_{j_{s+1}} = v_{b(s+1)} q_{j_{s+2}} - p_{j_{s+2}} \geq \cdots \geq v_{i_{t-1}} q_{k^*} - p_{k^*} \geq 0.$$

For the second inequality, by pricing rule (c), we know that

$$\frac{p_{j_i}}{q_{j_i}} \geq \frac{p_{j_{i+1}}}{q_{j_{i+1}}}$$

holds if and only if

$$\frac{v_{i_{b(i)}}(q_{j_i} - q_{j_{i+1}}) + p_{j_{i+1}}}{q_{j_i}} \geq \frac{p_{j_{i+1}}}{q_{j_{i+1}}}$$

which holds if and only if

$$(v_{i_{b(i)}} q_{j_{i+1}} - p_{j_{i+1}})(q_{j_i} - q_{j_{i+1}}) \geq 0,$$

which follows from the first inequality.

The third inequality follows immediately from the second one and the non-increasing ordering of q's. □

Lemma 5.14. *Suppose that* S *is a candidate winner set and* T *is a subset of items, where* $|T| = d(S)$*. Let* $\mathbf{X}$ *be the allocation computed in the procedure* MaxRevenue(S, T)*. Then* MaxRevenue(S, T) *determines whether there exists a price vector* $\mathbf{p}$ *such that* $(\mathbf{p}, \mathbf{X})$ *is a sharp envy-free solution, and if the answer is 'yes', it outputs one that maximizes total revenue given the allocation* $\mathbf{X}$.

Proof. Assume that there is a price vector $\mathbf{p}'$ such that $(\mathbf{p}', \mathbf{X})$ is a revenue maximizing sharp envy-free solution, with the winner set S and the sold item set T. In one direction, we prove that the algorithm given the input sets S and T returns a solution with at least the same total revenue. On another direction, we prove that the solution found by the Algorithm is an envy-free solution for the fixed sets S and T. By Remark 5.12, this sharp envy-free solution must be an optimal one. The two parts together complete the proof.

For the first direction, let $S = \{i_1, i_2, \ldots, i_t\}$ with $v_{i_1} \geq v_{i_2} \geq \cdots \geq v_{i_t}$ and $T = \{j_1, j_2, \ldots, j_l\}$ with $q_{j_1} \geq q_{j_2} \geq \cdots \geq q_{j_l}$. By Claim 5.3.1, $\sum_{j \in X_{i_t}} (v_{i_t} q_j - p'_j) = 0$. Consider an item $k' = \arg\max_{k \in X_{i_t}} (v_{i_{t-1}} q_k - p'_k)$. Define a new price vector $\mathbf{p}$ as follows:

- For $j \in X_{i_t}$, $p_j = p'_j$.
- For $j \in X_{i_{t-1}}$, $p_j = v_{i_{t-1}}(q_j - q_{k'}) + p'_{k'}$.
- For $j \in X_{i_1} \cup \cdots \cup X_{i_{t-2}}$, p_j is defined according to Step (c) of the procedure MaxRevenue.

It is easy to see that the formulas (1), (2) and (3) of $\mathrm{LP}^{(k')}$ are satisfied for price vector $\mathbf{p}$. By induction on the reverse order of items, we can show that $\mathbf{p'} \leq \mathbf{p}$. This implies that formula (4) of $\mathrm{LP}_{k'}$ is satisfied as well. Further, since prices are monotonically increasing, all losers (in particular, those in L_S) are still sharp envy-free, which implies formula (5) is satisfied. Formula (6) is automatically satisfied. Hence, $\mathbf{p}$ is a feasible solution of $\mathrm{LP}^{(k')}$. Hence, there is a feasible solution in the above procedure MaxRevenue(S, T) for item k'; this implies that $Y \neq \emptyset$ in the course of the procedure.

In addition, again because of $\mathbf{p'} \leq \mathbf{p}$, the total revenue generated by $(\mathbf{p}, \mathbf{X})$ is at least that by $(\mathbf{p'}, \mathbf{X})$. By the objective of the linear program, we know that the revenue generated by the solution at $\mathrm{LP}^{(k')}$ is at least that by $(\mathbf{p}, \mathbf{X})$ Therefore, MaxRevenue(S, T) computes a revenue no less than that of $(\mathbf{p}, \mathbf{X})$.

For the second direction, let $(\mathbf{p}, \mathbf{X})$ be the output of the procedure MaxRevenue(S, T). We need to show that $(\mathbf{p}, \mathbf{X})$ is an envy-free solution. Suppose $(\mathbf{p}, \mathbf{X})$ is computed in terms of $\mathrm{LP}^{(k^*)}$, where $k^* \in X_{i_t}$.

We first claim that all losers are sharp envy-free. By Proposition 5.9, we need only to check if all the losers in L_S are sharp envy-free for $(\mathbf{p}, \mathbf{X})$. Since $p_j = \infty$, $\forall j \notin T$, we only need to check that all the losers in L_S would not envy the items in T.

According to (5) in Step (a) of MaxRevenue(S, T), for any $i \in L_S$, we know that, for any buyer i, $\sum_{j \in T'} (v_i q_j - p_j) \leq 0$ for any $T' \subseteq J$ with $|T'| = d_i$. Choose $T' = \{j_{\ell - d_{i_t} - d_i + 1}, j_{\ell - d_{i_t} - d_i + 2}, \cdots, j_{\ell - d_{i_t}}\} \subseteq J$ (as $d_i \leq \Delta$). Let $j_{\max}$ be the largest index in T' such that $v_i q_{j_{\max}} - p_{j_{\max}} \leq 0$. Then, by monotonicity of price-per-unit-quality in Proposition 5.13, we have

$$q_{j_1}\left(v_i - \frac{p_{j_1}}{q_{j_1}}\right) \leq q_{j_2}\left(v_i - \frac{p_{j_2}}{q_{j_2}}\right) \leq \cdots \leq q_{j_{\max}}\left(v_i - \frac{p_{j_{\max}}}{q_{j_{\max}}}\right) \leq 0,$$

and $v_i q_j - p_j > 0$, $\forall j \in \{j_{\max+1}, j_{\max+2}, \ldots, j_{\ell - d_{i_t}}\}$.

Hence, for every loser i in L_S, its largest d_i values in the set $\{v_i q_j - p_j \mid j \in T\}$ are contained in $\left\{v_i q_j - p_j \mid j \in \{j_{\ell - d_{i_t} - d_i + 1}, j_{\ell - d_{i_t} - d_i + 2}, \ldots, j_\ell\} \subset J\right\}$. Therefore, the requirement (5) in Step (a) of MaxRevenue(S, T) would imply that for any $T' \subset T$

with $|T'| = d_i$, we have $\sum_{j\in T'}(v_i q_j - p_j) \leq 0$, which means that i is sharp envy-free. Hence, all the losers are sharp envy-free for the tuple.

It remains to show that all winners are sharp envy-free as well. Before doing this, by the pricing rule in subroutine (c), we can easily see that for any i_u and $j \in X_{i_u}$ with $u < t$, there exists item $j' \in X_{i_{u+1}}$ such that $p_j = v_{i_u}(q_j - q_{j'}) + p_{j'}$. We will use this particular property to show all winners are sharp envy-free. Since $p_j = \infty$ for any $j \notin T$, it suffices to show that any winner would not envy the items of other winners. The claim follows from the following arguments.

- All winners get non-negative utility. Formula (2) guarantee that i_t gets nonnegative utility for X_{i_t}. For any winner $i_u < i_t$, none has over-priced item. It follows by the fact that, $\forall s \in J - X_{i_t}$, $p_{j_s} = v_{i_{b(s)}}(q_{j_s} - q_{j_{s+1}}) + p_{j_{s+1}}$ in the algorithm and $v_{i_{b(s)}} q_{j_{s+1}} - p_{j_{s+1}} \geq 0$ in Proposition 5.13.

- Buyer i_t would not envy items won by any other winner i_u, where $i_u < i_t$. We show this by induction. Formula (4) shows the base case hold (i.e., i_t would not envy items won by i_{t-1}). Then, for any item $j' \in X_{i_t}$ and any item $j \in X_{i_u}$, (notice that by the pricing rule, there exists $k \in X_{i_{u+1}}$ such that $p_j = v_{i_u}(q_j - q_k) + p_k$), we have

$$\begin{aligned} v_{i_t} q_j - p_j &= v_{i_t} q_j - (v_{i_u}(q_j - q_k) + p_k) = (v_{i_t} - v_{i_u})(q_j - q_k) + v_{i_t} q_k - p_k \\ &\leq v_{i_t} q_k - p_k \leq v_{i_t} q_{j'} - p_{j'}, \end{aligned}$$

  where the first inequality follows from $v_{i_t} - v_{i_u} \leq 0$ and $q_j - q_k \geq 0$, and the second inequality follows from the induction hypothesis.

- For any i_u, $i_u < i_t$, i_u would not envy items won by i_t. Again, the proof is by induction. The base case $i_u = i_{t-1}$, for any item $j \in X_{i_{t-1}}$ and item $j' \in X_{i_t}$, it holds that

$$v_{i_{t-1}} q_j - p_j = v_{i_{t-1}} q_j - (v_{i_{t-1}}(q_j - q_{k^*}) + p_{k^*}) = v_{i_{t-1}} q_{k^*} - p_{k^*} \geq v_{i_{t-1}} q_{j'} - p_{j'},$$

  where the first equality follows from formula (3) and the inequality follows from formula (1). Hence, the base case holds. Next for any $j \in X_{i_u}$ and item $j' \in X_{i_t}$, (notice by pricing rule, there exists $k \in X_{i_{u+1}}$ such that $p_j = v_{i_u}(q_j - q_k) + p_k$), we have

$$\begin{aligned} v_{i_u} q_j - p_j &= v_{i_u} q_j - (v_{i_u}(q_j - q_k) + p_k) = v_{i_u} q_k - p_k \\ &= (v_{i_u} - v_{i_{u+1}})(q_k - q_{j'}) + v_{i_u} q_{j'} + (v_{i_{u+1}}(q_k - q_{j'}) - p_k). \end{aligned}$$

  Since $v_{i_u} - v_{i_{u+1}} \geq 0$ and $q_k - q_{j'} \geq 0$, and by the induction hypothesis, $v_{i_{u+1}} q_k - p_k \geq v_{i_{u+1}} q_{j'} - p_{j'}$, it holds that $v_{i_u} q_j - p_j \geq v_{i_u} q_{j'} - p_{j'}$.

- Every winner in $S\backslash\{i_t\}$ would not envy the items won by other winner in $S\backslash\{i_t\}$. Use the convention $j_{\ell - d_{i_t} + 1} = k^*$, recall $\forall u, 1 \leq u \leq \ell - d_{i_t}$, $p_{j_u} = v_{i_{b(u)}}(q_{j_u} -$

$q_{j_{u+1}}) + p_{j_{u+1}}$, then for $1 \le s < s' \le \ell - d_{i_t}$,

$$\begin{aligned} p_{j_s} - p_{j_{s'}} &= \sum_{u=s}^{s'-1}(p_{j_u} - p_{j_{u+1}}) = \sum_{u=s}^{s'-1} v_{i_{b(u)}}(q_{j_u} - q_{j_{u+1}}) \\ &\le v_{i_{b(s)}} \sum_{u=s}^{s'-1}(q_{j_u} - q_{j_{u+1}}) = v_{i_{b(s)}}(q_{j_s} - q_{j_{s'}}). \end{aligned}$$

Rewrite $p_{j_s} - p_{j_{s'}} \le v_{i_{b(s)}}(q_{j_s} - q_{j_{s'}})$ as $v_{i_{b(s)}} q_{j_s} - p_{j_s} \ge v_{i_{b(s)}} q_{j_{s'}} - p_{j_{s'}}$, which means buyer with smaller index would not envy items won by buyer with larger index. Similarly, noting that

$$p_{j_s} - p_{j_{s'}} = \sum_{u=s}^{s'-1} v_{i_{b(u)}}(q_{j_u} - q_{j_{u+1}}) \ge v_{i_{b(s')}} \sum_{u=s}^{s'-1}(q_{j_u} - q_{j_{u+1}}) = v_{i_{b(s')}}(q_{j_s} - q_{j_{s'}}).$$

Rewrite $p_{j_s} - p_{j_{s'}} \ge v_{i_{b(s')}}(q_{j_s} - q_{j_{s'}})$ as $v_{i_{b(s')}} q_{j_s} - p_{j_s} \le v_{i_{b(s')}} q_{j_{s'}} - p_{j_{s'}}$, which means buyer with larger index would not envy items won by buyer with smaller index. In all, every winner in $S \backslash \{i_t\}$ would not envy the items won by other winner in $S \backslash \{i_t\}$.

Therefore, we know that the tuple $(\mathbf{p}, \mathbf{X})$ is an envy-free solution. □

Observe that the computation of Step (a) of MaxRevenue does not depend on the whole set T. In fact, we only need to know the last 2Δ items with largest indices in T to check whether Y is empty or not. Therefore, whether MaxRevenue(S, T) will output a tuple only depends on the last 2Δ items in T. The prices for those 2Δ items are determined in one of the linear programs there. Suppose that the last 2Δ items in T are J and let $j_{\min} = \min\{j \in J\}$, then if MaxRevenue$(S, T)$ output a tuple $(\mathbf{p}, \mathbf{X})$, we can re-choose any other set $Z \subseteq \{1, 2, 3, \dots, j_{\min} - 1\}$ with $|Z| = \ell - 2\Delta$ and run MaxRevenue$(S, Z \cup J)$, which would always output an envy-free tuple $(\mathbf{p}', \mathbf{X}')$ as well. Similarly, if MaxRevenue(S, T) claims that there is no tuple $(\mathbf{p}, \mathbf{X})$ which is an envy-free solution, then MaxRevenue$(S, Z \cup J)$ also claims that no tuple exists. These observations are critical in our main algorithm Max-EF.

5.3.5 Only the Winner Set is Known

Suppose that we are given a candidate winner set $S = \{i_1, i_2, \dots, i_t\}$ and a set of items $J = \{j_1, \dots, j_{2\Delta}\}$ with $i_1 < i_2 < \cdots < i_t$ and $j_1 < \cdots < j_{2\Delta}$. Assume that $\ell = d(S) > 2\Delta$. Let $Y = \{1, 2, \dots, j_1 - 1\}$ denote the set of items that have indices smaller than j_1. Our objective is to pick a subset $Z \subseteq Y$ with $|Z| = \ell - 2\Delta$ such that the revenue given by MaxRevenue$(S, Z \cup J)$ is as large as possible. By Steps (a) and (c) of MaxRevenue, for the given set of winners S, the prices of the items in J are already fixed (no matter which Z is chosen). Hence, to maximize revenue from MaxRevenue$(S, Z \cup J)$, it suffices to maximize revenue (or equivalently, prices) from the items in Z. To this end, we use the approach of dynamic programming to find an optimal solution.

Consider any subset $Z = \{z_1, z_2, \ldots, z_{\ell-2\Delta}\} \subseteq Y$ with $z_1 < z_2 < \cdots < z_{\ell-2\Delta}$; denote $z_{\ell-2\Delta+1} = j_1$. Suppose MAXREVENUE$(S, Z \cup J)$ will output a tuple $(\mathbf{p}, \mathbf{X})$. As we already know that each z_j will be allocated to which winner by MAXREVENUE$(S, Z \cup J)$, let $w_j = v_i$ if $z_j \in X_i$, for $j = 1, 2, \ldots, \ell - 2\Delta$; further, let $w_0 = 0$. An important observation is that the values of all w_j's are independent to the selection of Z. By the pricing rule in MAXREVENUE$(S, Z \cup J)$, it holds that $p_{z_j} = w_j(q_{z_j} - q_{z_{j+1}}) + p_{z_{j+1}}$, for $j = 1, 2, \ldots, \ell - 2\Delta$. Hence, we have

$$\begin{aligned}
\sum_{j=1}^{\ell-2\Delta} p_{z_j} &= \sum_{j=1}^{\ell-2\Delta} \left(\sum_{u=j}^{\ell-2\Delta} (p_{z_u} - p_{z_{u+1}}) + p_{j_1} \right) \\
&= \sum_{j=1}^{\ell-2\Delta} \sum_{u=j}^{\ell-2\Delta} \big((q_{z_u} - q_{z_{u+1}}) w_u \big) + (\ell - 2\Delta) p_{j_1} \\
&= \sum_{j=1}^{\ell-2\Delta} (j \cdot q_{z_j} w_j - j \cdot q_{z_{j+1}} w_j) + (\ell - 2\Delta) p_{j_1} \\
&= \left[\sum_{j=1}^{\ell-2\Delta} \big(j \cdot w_j - (j-1) \cdot w_{j-1} \big) q_{z_j} \right] - \Big[(\ell - 2\Delta)(q_{j_1} w_{\ell-2\Delta} - p_{j_1}) \Big] \\
&\triangleq R_1 - R_2,
\end{aligned}$$

where R_1 and R_2 are the first and second term of the difference, respectively. By the rule of MAXREVENUE, the allocation of $z_{\ell-2\Delta}$ (thus, the value $w_{\ell-2\Delta}$) and the price p_{j_1} are fixed. Hence, to maximize $\sum_{j=1}^{\ell-2\Delta} p_{z_j}$, it suffices to maximize R_1. For any α, β with $1 \leq \alpha \leq \beta \leq j_1 - 1$, let $opt(\alpha, \beta)$ denote the optimal value of the following problem, denoted by $DLP(\alpha, \beta)$, which picks α items from the first β items to maximize a given objective (recall that w_j is defined above for $j = 1, \ldots, \ell - 2\Delta$).

$$\begin{aligned}
\max \quad & \sum_{j=1}^{\alpha} \big(j \cdot w_j - (j-1) \cdot w_{j-1} \big) q_{z_j} \\
\text{s.t.} \quad & z_1 < z_2 < \cdots < z_\alpha, \{z_1, z_2, \ldots, z_\alpha\} \subseteq \{1, 2, \ldots, \beta\}.
\end{aligned}$$

The problem that maximizes R_1 is exactly $DLP(\ell - 2\Delta, j_1 - 1)$, which can be solved by the following dynamic programming.

SOLVE-DLP

1. `Compute` $opt(1,1), opt(1,2), \ldots, opt(1, j_1 - 1)$.

2. `Compute`

$$opt(\alpha, \beta{+}1) = \begin{cases} \max\left\{opt(\alpha,\beta), opt(\alpha-1,\beta) + (\alpha \cdot w_\alpha - (\alpha-1)w_{\alpha-1})q_{\beta+1}\right\} & \texttt{if } \beta+1 \geq \alpha \\ 0 & \texttt{Otherwise} \end{cases}$$

3. `Find a subset` Z^* `that maximizes` $opt(\ell - 2\Delta, j_1 - 1)$.

4. `Return the output of` MAXREVENUE$(S, Z^* \cup J)$.

The following claim is straightforward from the definition of $DLP(\alpha, \beta)$ and the above dynamic programming.

Proposition 5.15. *Given a candidate winner set S and a subset J of 2Δ items, the above* SOLVE-DLP *picks in polynomial time a subset $Z \subseteq Y$ with $|Z| = \ell - 2\Delta$ such that the revenue given by* MAXREVENUE$(S, Z \cup J)$ *is the maximum if we guessed S and J correctly.*

5.3.6 Algorithm

In this subsection, we will present our main algorithm MAX-EF. The algorithm has two stages: STAGE 1 is to select the set of possible winners (candidate winners) who will be allocated items, and STAGE 2 is designed to calculate all the 'candidate' maximum revenue and presents an optimal sharp envy-free solution and maximum revenue.

The algorithm is described as follows.

MAX-EF STAGE 1.

1. Initialize $D = \emptyset$ (denote the collection of candidate winner sets).
2. Find $S \subseteq A_1$ such that $d(S) = \max\{d(S') \mid d(S') \leq m, S' \subseteq A_1\}$, let $D \leftarrow \{S\}$.
3. For $k = 2, \ldots, K$
   - Sort $A_1 \cup A_2 \cup \cdots \cup A_k$ in the decreasing order of their values.
   - For each d such that $1 \leq d \leq m$
     - Let $S = \text{argmax}_S\{d(S)|d(S) \leq d, S \subset A_k\}$.
     - Let $S_{0,1} = S$, $n_0 = 1$ and $\mathcal{C}_0 = \{S_{0,1}\}$.
     - Let $\ell = |A_1 \cup A_2 \cup \cdots \cup A_{k-1}|$.
     - For $t = 1, 2, \ldots, \ell$ do:
       * In general, we have constructed $\mathcal{C}_t$ containing all the candidate winner sets of $\{\ell - t + 1, \ell - t + 2, \ldots, \ell\} \cup S$.
       * We order $\mathcal{C}_t = \{S_{t,1}, S_{t,2}, \ldots, S_{t,n_t}\}$ such that $d(S_{t,1}) \leq d(S_{t,2}) \leq \cdots \leq d(S_{t,n_t}) \leq m$.
       * We now add $\ell - t$ into $\mathcal{C}_t$ to construct $\mathcal{C}_{t+1}$.
         · Let $t_s = \max\{i : d(S_{t,i}) < d_{\ell-t}\}$ if $\{i : d(S_{t,i}) < d_{\ell-t}\} \neq \emptyset$, otherwise $ts = 0$.
         · Let $S_{t+1,j} = S_{t,j}$ for $j = 1, 2, \cdots, t_s$.
         · Let $S_{t+1,j+t_s} = S_{t,j} \cup \{\ell - t\}$ for $j = 1, 2, \ldots, n_t$.
         · Let $n_{t+1} = \max\{i \leq t_s + n_t : d(S_{t+1,i}) \leq m\}$.
         · Let $\mathcal{C}_{t+1} = \{S_{t+1,i} : i \leq n_{t+1}\}$.
     - $D \leftarrow D \cup \mathcal{C}_\ell$.
4. return D

STAGE 1 of MAX-EF is designed to select candidate winner sets one of which contains exactly the winners in an optimal sharp envy-free solution. For each $1 \leq k \leq K \leq n$ and $1 \leq d \leq m$ the problem is of one discussed in Lemma 5.10. It constructs $\mathcal{C}$, consisting of up to $\frac{m}{d}$ subsets of total size $O(\frac{m*n}{d})$ in time $O(\frac{m*n^2}{d})$. The total time complexity then adds up to $O(m * n^3 \log m)$. Hence, MAX-EF runs in strongly polynomial time.

Proposition 5.16. *There is an optimal winner set contained in the set D.*

Proof. Now suppose there is an optimal winner set W, if $W \subseteq A_1$, then by Proposition 5.11, the set S selected in above algorithm is an optimal winner set and we are done. Otherwise, let $i_{\max} = \max(W)$; suppose $i_{\max} \in A_{k^*}$, where $k^* \geq 2$, and let $w^* = d(W \cap A_{k^*})$. Now consider the k^*th and w^*th round of the for loop. There exists $T \subseteq A_{k^*}$ such that $d(T) = w^*$. By Proposition 5.11, we know that $(W \backslash (W \cap A_k)) \cup T$ is an optimal winner set. By the procedure of the algorithm and Proposition 5.8, the algorithm would find all the candidate winner sets with the form $C \cup T$ where $C \subseteq A_1 \cup \cdots \cup A_{k-1}$. Hence, $(W \backslash (W \cap A_k)) \cup T \in D$. □

MAX-EF STAGE 2.

5. `For each candidate winner set` $S \in D$
   - `Let` $\ell = d(S)$
   - `If` $\ell \leq 2\Delta$
     - `For any set` $J \subseteq \{1, 2, \ldots, m\}$ `with` $|J| = \ell$
       - `Run` MAXREVENUE(S, J).
       - `If it outputs a tuple` $(\mathbf{p}, \mathbf{X})$, `let` $R^{S,J} \leftarrow \sum_{i=1}^{n} \sum_{j \in X_i} p_j$
       - `Else, let` $R^{S,J} \leftarrow 0$.
   - `Else` $\ell > 2\Delta$
     - `For any set` $J \subseteq \{\ell - 2\Delta + 1, \ell - 2\Delta + 2, \ldots, m\}$ `with` $|J| = 2\Delta$
       - `Let` $j_{\min} \leftarrow \min\{j \in J\}$
       - `Choose any` $Z \leftarrow \{z_1, \ldots, z_{\ell-2\Delta}\} \subseteq \{1, 2, \ldots, j_{\min} - 1\}$, `where` $z_1 > z_2 > \cdots > z_{\ell-2\Delta}$.
       - `Run` MAXREVENUE$(S, J \cup Z)$
       - `If it outputs a tuple`
         - `run` SOLVE-DLP `on` S `and` J `to get a tuple` $(\mathbf{p}, \mathbf{X})$
         - `let` $R^{S,J} \leftarrow \sum_{i=1}^{n} \sum_{j \in X_i} p_j$
       - `Else, let` $R^{S,J} \leftarrow 0$

6. `Output a tuple` $(\mathbf{p}, \mathbf{X})$ `which gives the maximum` $R^{S,J}$.

Since MAXREVENUE and SOLVE-DLP takes polynomial time, and $|D| \leq nm \log m$, we know STAGE 2 of MAX-EF runs in polynomial time.

Proof of Theorem 5.6. Since MAX-EF takes polynomial time, we only need to check that MAX-EF will output an optimal sharp envy-free solution. By the above analysis, we know that MAX-EF will output an envy-free solution. Since there is an optimal winner $S \in D$, there exists an optimal sharp envy-free solution $(\mathbf{p}, \mathbf{X})$ such that $S = \{i | X_i \neq \emptyset\}$. W.l.o.g. suppose that the items in $T = \bigcup_{i=1}^{n} X_i$ are allocated to S by the rules of allocation of MAXREVENUE(S, T) (otherwise, there exists $i > i'$ and $j < j'$ such that $j \in X_i$ and $j' \in X_{i'}$, if $v_i = v_{i'}$, then $v_i q_j - p_j \geq v_i q_{j'} - p_{j'}$ and $v_{i'} q_j - p_j \leq v_{i'} q_{j'} - p_{j'}$, hence $v_i q_j - p_j = v_i q_{j'} - p_{j'}$, then exchanging the allocation j and j' without changing their prices would still make everyone sharp envy-free. If $v_i < v_{i'}$, then by Lemma 4.11, we have $q_j = q_{j'}$, then exchanging allocation j and j' and their prices would still make everyone sharp envy-free). If $d(S) \leq 2\Delta$, then by the argument of Lemma 5.14, we know $R^{S,T} \geq \sum_{i=1}^{n} \sum_{j \in X_i} p_j$. Similarly if $d(S) > 2\Delta$, let J be the 2Δ largest values in T, by the argument of Lemma 5.14 and Proposition 5.15, we know $R^{S,J} \geq \sum_{i=1}^{n} \sum_{j \in X_i} p_j$. Therefore, the output $(\mathbf{p}, \mathbf{X})$ of MAX-EF is an optimal sharp envy-free solution. □

5.4 Hardness of Arbitrary Sharp Demand

Theorem 5.17. *For the sharp multi-unit demand with $v_i q_j$ valuations, it is NP-hard to solve the revenue-maximizing sharp envy-free pricing problem, even if there are only three buyers.*

We next prove the NP-hardness result that is Theorem 5.17.

We reduce from the exact cover by 3-sets problem (X3C): Given a ground set $A = \{a_1, a_2, \ldots, a_{3n}\}$ and collection $T = \{S_1, S_2, \ldots, S_m\}$ where each $S_i \subset A$ and $|S_i| = 3$, we are asked if there are n elements of T that cover all elements in A. We assume that $n \le m \le 2n-1$; it is easy to see that the problem still remains NP-complete (as we can add dummy elements x, y, z to A and subset $\{x, y, z\}$ to T to balance the sizes of A and T).

Given an instance of X3C, we construct a market with 3 buyers and $n+m$ items as follows. Let $M = 3nm + 1$, $L = \sum_{i=1}^{3n} M^i$. Note that $L < 3nM^{3n}$, whose binary representation is of size polynomial in m and n. Consider m values $R_i = \sum_{a_j \in S_i} M^j$, for $i = 1, 2, \ldots, m$, and rearranging if necessary, let $R_1 \ge R_2 \ge \cdots \ge R_m$ be a non-increasing order of these values. The valuations and demands of buyers are

$$
\begin{aligned}
d_1 &= n, \quad v_1 = 3 \\
d_2 &= 2n, \quad v_2 = \frac{3n+1}{n+1} \\
d_3 &= n, \quad v_3 = 2
\end{aligned}
$$

The qualities of items are defined as follows: Let $q_j = L$, for $j = 1, 2, \ldots, n$, and $q_{n+j} = R_j$, for $j = 1, 2, \ldots, m$. Obviously, the unit values and qualities are in non-increasing order, and the construction is polynomial.

Consider the winner set in an optimal envy-free solution $(\mathbf{p}, \mathbf{X})$. Since $n \le m \le 2n-1$, the possible winner sets are $\{1\}$, $\{2\}$, $\{3\}$, and $\{1, 3\}$. There is no sharp envy-free solution where $\{2\}$ or $\{3\}$ is the winner set, since buyer 1 would be envious. It remains to consider $\{1\}$ and $\{1, 3\}$. If the winner set is $\{1\}$, then the optimal revenue is $v_1 \cdot \left(\sum_{i=1}^{n} q_i\right) = 3nL$ where buyer 1 gets the first n items. If the winner set is $\{1, 3\}$, it is not difficult to see that in the optimal sharp envy-free solution $(\mathbf{p}, \mathbf{X})$, it holds that $X_1 = \{1, 2, \ldots, n\}$. Suppose that $X_3 = \{j_1, j_2, \ldots, j_n\} \subset \{n+1, n+2, \ldots, n+m\}$ where $j_1 > j_2 > \cdots > j_n$. Applying the characterizations of optimal sharp envy-freeness MaxRevenue(S, T) and Lemma 5.14 in Section 5.3.4, in the optimal solution $(\mathbf{p}, \mathbf{X})$ with $X_1 = \{1, 2, \ldots, n\}$ and $X_3 = \{j_1, j_2, \ldots, j_n\}$, we should prove the following claim

Claim 5.4.1.

$$v_1 q_k - p_k = v_1 q_j - p_j \quad \forall k, j \in X_3$$

proof of Claim 5.4.1. According to MaxRevenue(S, T), there exists $k^* : n+1 \le k^* \le m+n$ such that $(\mathbf{p}, \mathbf{X})$ is the optimal solution of the following linear program (denoted by $LP^{(k^*)}$).

$$
\begin{array}{llll}
\min & v_1 q_{k^*} - p_{k^*} & & \\
s.t. & v_1 q_{k^*} - p_{k^*} \geq v_1 q_j - p_j & \forall\ j \in X_3 & (1^*) \\
& \sum\limits_{j \in X_3} (v_3 q_j - p_j) = 0 & & (2^*) \\
& v_1 q_j - p_j = v_1 q_{k^*} - p_{k^*} & \forall\ j \in X_1 & (3^*) \\
& v_3 q_j - p_j \leq v_3 q_{j'} - p_{j'} & \forall\ j \in X_1, j' \in X_3 & (4^*) \\
& \sum_{j \in X_1 \cup X_3} (v_2 q_j - p_j) \leq 0 & & (5^*)
\end{array}
$$

Please note that the last set of equations (6^*) in the original LP are not needed since they are empty under the current restriction of three buyers. We first prove all the inequalities in (1^*) must be equalities. Suppose it is not true. Then there exists $\ell \in X_3$ such that

$$v_1 q_{k^*} - p_{k^*} > v_1 q_\ell - p_\ell.$$

Set $b_j = v_1 q_j - p_j$, $j \in X_3$. From (2^*), it follows that $\sum\limits_{j \in X_3} b_j = (v_1 - v_3) \sum\limits_{j \in X_3} q_j$. Take the average

$$\bar{b} = \frac{\sum\limits_{j \in X_3} a_j}{|X_3|} = \frac{(v_1 - v_3) \sum\limits_{j \in X_3} q_j}{|X_3|}$$

We introduce the price vector $\mathbf{p}' = (p'_1, p'_2, \cdots, p'_n, p'_{j_1}, p'_{j_2}, \cdots, p'_{j_n})$ such that $\forall j \in X_3$: $p'_j = v_1 q_j - \bar{b}$ and $\forall j \in X_1$: $p'_j = v_1(q_j - q_{k^*}) + p'_{k^*}$. If we can prove that $(\mathbf{p}', \mathbf{X})$ is still a feasible solution for LP^{k^*}, then $p'_{k^*} > p_{k^*}$ (due to $b_{k^*} > \bar{b}$ by (1^*)). It results in a smaller objective value than $v_1 q_{k^*} - p_{k^*}$, a contradiction to the optimality of $(\mathbf{p}, \mathbf{X})$.

First, (1^*) (2^*) (3^*) follows directly from definition of $\mathbf{p}'$. We need only to check (4^*) and (5^*). From $p'_{k^*} > p_{k^*}$, $\forall\ j \in X_1$ $p'_j = v_1(q_j - q_{k^*}) + p'_{k^*} > v_1(q_j - q_{k^*}) + p_{k^*} = p_j$. We have $\forall j \in X_1$: $p'_j > p_j$. Hence, the inequality (5^*) holds. To see inequality (4^*), notice

$$
\begin{aligned}
v_3 q_j - p'_j &= v_3 q_j - v_1(q_j - q_{k^*}) - p'_{k^*} \\
&= v_3 q_j - v_1(q_j - q_{j'}) - p'_{j'} \\
&= (v_3 - v_1)(q_j - q_{j'}) + v_3 q_{j'} - p'_{j'} \\
&\leq v_3 q_{j'} - p'_{j'}, \quad \forall j \in X_1, j' \in X_3.
\end{aligned}
$$

Claim 5.4.1 is proven. □

By Claim 5.4.1 and the above condition (3^*), we have

$$v_1 q_i - p_i = v_1 q_j - p_j,\ \forall\ i \in X_1, j \in X_3 \tag{5.1}$$

By the above condition (2^*),

$$\sum_{j \in X_3} p_j = v_3 \cdot \sum_{k=1}^{n} q_{j_k}. \tag{5.2}$$

Combining (5.1) and (5.2), the total revenue is

$$R = \sum_{i=1}^{n} p_i + \sum_{j \in X_3} p_j = v_1 \cdot \sum_{i=1}^{n} q_i + (2v_3 - v_1) \cdot \sum_{k=1}^{n} q_{j_k}.$$

Since buyer 2 is sharp envy-free, we have

$$v_2 \cdot \Big(\sum_{i=1}^{n} q_i + \sum_{k=1}^{n} q_{j_k} \Big) - R = (v_2 - v_1) \cdot \sum_{i=1}^{n} q_i + (v_1 + v_2 - 2v_3) \cdot \sum_{k=1}^{n} q_{j_k} \le 0.$$

Therefore, computing the maximum revenue when the winner set is $\{1, 3\}$ is equivalent to solving the following program:

$$\begin{aligned} \max \quad & R = v_1 \cdot \sum_{i=1}^{n} q_i + (2v_3 - v_1) \cdot \sum_{k=1}^{n} q_{j_k} \\ \text{s.t.} \quad & (v_2 - v_1) \cdot \sum_{i=1}^{n} q_i + (v_1 + v_2 - 2v_3) \cdot \sum_{k=1}^{n} q_{j_k} \le 0 \\ & j_1 > j_2 > \cdots > j_n, \; j_k \in \{n+1, n+2, \ldots, n+m\}, k = 1, 2, \ldots, n. \end{aligned} \tag{5.3}$$

Considering $v_1 = 3$, $v_2 = \frac{3n+1}{n+1}$, $v_3 = 2$, and $q_i = L$, $i = 1, 2, \ldots, n$, the program (5.3) is equivalent to

$$\begin{aligned} \max \quad & R = 3nL + \sum_{k=1}^{n} q_{j_k} \\ \text{s.t.} \quad & \sum_{k=1}^{n} q_{j_k} \le L \\ & j_1 > j_2 > \cdots > j_n, \; j_k \in \{n+1, n+2, \ldots, n+m\}, k = 1, 2, \ldots, n. \end{aligned} \tag{5.4}$$

It is not difficult to see that the maximum revenue (i.e., the optimal value of the above program) is $(3n+1)L$ if and only if there is a positive answer to the instance of X3C. This completes the proof.

5.5 Consecutive Envy-free Solutions

We first prove a negative result on computing the revenue maximization problem in general demand case. We show it is NP-hard if all the qualities are the same.

Theorem 5.18. *The revenue maximization problem of consecutive envy-free buyers is NP-hard even if all the qualities are the same.*

Proof. We prove the NP-hardness by reducing the 3 partition problem that is to decide whether a given multi-set of integers can be partitioned into triples that all have the same sum. More precisely, given a multi-set S of $n = 3m$ positive integers, can S be partitioned into m subsets $S_1, \ldots, S_m$ such that the sum of the numbers in each subset is equal? The 3 partition problem has been proven to be NP-complete in a strong sense in [36], meaning that it remains NP-complete even when the integers in S are bounded above by a polynomial in n.

Given a instance of 3 partition $(a_1, a_2, \ldots, a_{3n})$. Let $B = \sum_i a_i/n$. we construct a instance for advertising problem with $3n+1$ advertisers and $m = B+1+n+\sum_i a_i$ slots. It should be mentioned that m is polynomial of n due to the fact that all a_i are bounded by a polynomial of n. In the advertising instance, the valuation v_i for each advertiser i is 1 and his demand d_i is defined as a_i and there is another buyer with valuation 2 for each slot and with demand $B+1$. The quality of each slot j is 1. It is not hard to see that the optimal revenue is $nB + 2(B+1)$ if and only if there is a solution to this 3 partition instance, the optimal solution is illustrated as follows.

$$\underbrace{1\,1\cdots 1}_{B+1}\ \underbrace{1}_{\text{unassigned}}\ \underbrace{1\,1\cdots 1}_{B}\ \underbrace{1}_{\text{unassigned}}\ \underbrace{1\,1\cdots 1}_{B}\ \underbrace{1}_{\text{unassigned}}\ \cdots\underbrace{1\,1\cdots 1}_{B}$$

□

Although the hardness in Theorem 5.18 indicates that finding the optimal revenue for general demand in polynomial time is impossible , however, it doesn't rule out the very important case where the demand is uniform, e.g. $d_i = d$. We assume slots are in a decreasing order from top to bottom, that is, $q_1 \geq q_2 \geq \cdots \geq q_m$. The result is summarized as follows.

Theorem 5.19. *There is a polynomial time algorithm to compute the consecutive envy-free solution when all the buyers have the same demand and slots are ordered from top to bottom.*

The proof of Theorem 5.19 is based on (consecutive) bundle envy-free solutions, in fact we will prove the (consecutive) bundle envy-free solution is also a consecutive envy-free solution by defining price of items properly. Thus, we need first give the result on (consecutive) bundle envy-free solutions. Suppose d is the uniform demand for all the buyers. Let T_i be the slot set allocated to buyer i, $i = 1, 2, \cdots, n$. Let P_i be the total payment of buyer i and p_j be the price of slot j. Let t_i denote the total qualities obtained by buyer i, e.g. $t_i = \sum_{j \in T_i} q_j$ and $\alpha_i = iv_i - (i-1)v_{i-1}$, $\forall i \in [n]$.

Theorem 5.20. *The revenue maximization problem of (consecutive) bundle envy-freeness is equivalent to solving the following LP.*

$$\begin{aligned} \textit{Maximize:} \quad & \sum_{i=1}^{n} \alpha_i t_i \\ \textit{s.t.} \quad & t_1 \geq t_2 \geq \cdots \geq t_n \\ & T_i \subset [m], \quad T_i \cap T_k = \emptyset \quad \forall i, k \in [n] \end{aligned} \tag{5.5}$$

Proof. Recall P_i denote the payment of buyer i, we next prove that the linear programming (5.5) actually gives optimal solution of (consecutive) bundle envy-free. By the definition of (consecutive) bundle envy-free, where buyer i would not envy buyer $i+1$ and versus, we have

$$v_i t_i - P_i \geq v_i t_{i+1} - P_{i+1} \tag{5.6}$$

$$v_{i+1} t_{i+1} - P_{i+1} \geq v_{i+1} t_i - P_i \tag{5.7}$$

Plus above two inequalities gives us that $(v_i - v_{i+1})(t_i - t_{i+1}) \geq 0$. Hence, if $v_i > v_{i+1}$, then $t_i \geq t_{i+1}$. From (5.6), we could get $P_i \leq v_i(t_i - t_{i+1}) + P_{i+1}$. The maximum payment of buyer i is

$$P_i = v_i(t_i - t_{i+1}) + P_{i+1}, \tag{5.8}$$

together with $t_i \geq t_{i+1}$, implying (5.6) and (5.7). Besides the maximum payment of n is $P_n = t_n v_n$. (5.8) together with $t_i \geq t_{i+1}$ and $P_n = t_n v_n$ would make everyone bundle envy-free, the arguments are as follows.

- All the buyers must be (consecutive) bundle envy free. By (5.8), we have $P_i - P_{i+1} = v_i(t_i - t_{i+1})$, hence $P_i = \sum_{k=i}^{n-1} v_k(t_k - t_{k+1}) + P_n$. Noticing that if $t_i = 0$, then $P_i = 0$, which means i is loser. For any buyer $j < i$, we have $P_j - P_i = \sum_{k=j}^{i-1} v_k(t_k - t_{k+1}) \leq \sum_{k=j}^{i-1} v_j(t_k - t_{k+1}) = v_j(t_j - t_i)$. rewrite $P_j - P_i \leq v_j(t_j - t_i)$ as $v_j t_i - P_i \leq v_j t_j - P_j$, which means buyer j would not envy buyer i. Similarly, $P_j - P_i = \sum_{k=j}^{i-1} v_k(t_k - t_{k+1}) \geq \sum_{k=j}^{i-1} v_i(t_k - t_{k+1}) = v_i(t_j - t_i)$, rewrite $P_j - P_i \geq v_i(t_j - t_i)$ as $v_i t_i - P_i \geq v_i t_j - P_j$, which means i would not envy buyer j.

Now let's calculate $\sum_{i=1}^{n} P_i$ based on (5.8) using notation $t_{n+1} = 0$, one has

$$\begin{aligned}
\sum_{i=1}^{n} P_i &= \sum_{i=1}^{n} \left[\sum_{k=i}^{n-1} v_k(t_k - t_{k+1}) + P_n\right] = \sum_{i=1}^{n}\sum_{k=i}^{n} v_k(t_k - t_{k+1}) \\
&= \sum_{k=1}^{n}\sum_{i=1}^{k} v_k(t_k - t_{k+1}) = \sum_{k=1}^{n} k v_k(t_k - t_{k+1}) \\
&= \sum_{k=1}^{n} k v_k t_k - \sum_{k=1}^{n}(k-1)v_{k-1}t_k = \sum_{i=1}^{n} \alpha_i t_i
\end{aligned}$$

We know the revenue maximizing problem of bundle envy-freeness can be formalized as (5.5). □

Since consecutive envy-free solutions are a subset of (consecutive) bundle envy-free solutions, hence the optimal value of optimization (5.5) gives an upper bound of optimal objective value of consecutive envy-free solutions. Noting optimization LP (5.5) can be solved by dynamic programming. Let $g[s, j]$ denote the optimal objective value of the following LP with some set in $[j]$ allocated to all the buyers in $[s]$:

$$\begin{aligned}
\text{Maximize:} \quad & \sum_{i=1}^{s} \alpha_i t_i \\
\text{s.t.} \quad & t_1 \geq t_2 \geq \cdots \geq t_s \\
& T_i \subset [j], \quad T_i \cap T_k = \emptyset \;\; \forall i, k \in [s]
\end{aligned} \tag{5.9}$$

Then

$$g[s,j] = \max \begin{cases} g[s, j-1] \\ \\ g[s-1, j-d] + \alpha_s \sum_{u=j-d+1}^{j} q_u \end{cases}$$

Next, we show how to modify the (consecutive) bundle envy-free solution to consecutive envy-free solutions by properly defining the slot price of T_i, for all $i \in [n]$. Suppose the optimal winner set of bundle envy-free solution is $[L]$. Assume the optimal allocation and price of bundle envy-free solution are $T_i = \{j_1^i, j_2^i, \cdots, j_d^i\}$ with $j_1^i \geq j_2^i \geq \cdots \geq j_d^i$ and P_i respectively, for all $i \in [L]$.

Proof of Theorem 5.19. Define the price of T_i iteratively as follows:
$p_{j_k^L} = v_L q_{j_k^L}$, for all $k \in [d]$;
$p_{j_k^i} = v_i(q_{j_k^i} - q_{j_k^{i+1}}) + p_{j_k^{i+1}}$ for $k \in [d]$ and $i \in [n]$
Now we could see that the price defined by above procedure is still a bundle envy-free solution. This is because by induction, we have $P_i = \sum_{k=1}^{d} p_{j_k^i}$. Hence, we need only to check the prices defined as above and allocations T_i constitute a consecutive envy-free solution. In fact, we prove a strong version, suppose T_is are consecutive from top to down in a line S, we will show each buyer i would not envy any consecutive sub line of S comprising d slots. For any i,
Case 1, buyer i would not envy the slots below his slots.
for any consecutive line T bellow i with size d, suppose T comprises of slots won by buyer k (denoted such slot set by U_k) and $k+1$ (denoted such slot set by U_{k+1} and let $\ell = |U_{k+1}|$) where $k \geq i$. Recall that $t_i = \sum_{j \in T_i} q_j$, then

$$\begin{aligned}
&\sum_{j \in T_i} p_j - \sum_{j \in T} p_j \\
&= v_i(t_i - t_{i+1}) + P_{i+1} - \sum_{j \in U_k \cup U_{k+1}} p_j \\
&= v_i(t_i - t_{i+1}) + v_{i+1}(t_{i+1} - t_{i+2}) + \cdots + P_k - \sum_{j \in U_k \cup U_{k+1}} p_j \\
&= v_i(t_i - t_{i+1}) + v_{i+1}(t_{i+1} - t_{i+2}) + \cdots + \sum_{j \in T_k \setminus U_k} p_j - \sum_{j \in U_{k+1}} p_j \\
&= v_i(t_i - t_{i+1}) + v_{i+1}(t_{i+1} - t_{i+2}) + \cdots + \sum_{u=1}^{\ell} v_k(q_{j_u^k} - q_{j_u^{k+1}}) \\
&\leq v_i(t_i - t_{i+1}) + v_i(t_{i+1} - t_{i+2}) + \cdots + \sum_{u=1}^{\ell} v_i(q_{j_u^k} - q_{j_u^{k+1}}) \\
&= v_i t_i - v_i \sum_{j \in T} q_j
\end{aligned}$$

Rewrite

$$\sum_{j \in T_i} p_j - \sum_{j \in T} p_j \leq v_i t_i - v_i \sum_{j \in T} q_j$$

as

$$v_i t_i - \sum_{j\in T_i} p_j \geq v_i \sum_{j\in T} q_j - \sum_{j\in T} p_j,$$

we get the desired result.

Case 2, buyer i would not envy the slots above his slots.

for any consecutive line T above i with size d, suppose T comprises of slots won by buyer k (denoted such slot set by U_k) and $k-1$ (denoted such slot set by U_{k-1} and let $\ell = |U_{k-1}|$) where $k \leq i$. Recall that $t_i = \sum_{j\in T_i} q_j$, then

$$\begin{aligned}
&\sum_{j\in T} p_j - \sum_{j\in T_i} p_j \\
&= \sum_{j\in U_{k-1}\cup U_k} p_j - \sum_{j\in T_i} p_j \\
&= \sum_{u=d-\ell+1}^{d} v_{k-1}(q_{j_u^{k-1}} - q_{j_u^k}) + \sum_{j\in T_k} p_j - \sum_{j\in T_i} p_j \\
&= \sum_{u=d-\ell+1}^{d} v_{k-1}(q_{j_u^{k-1}} - q_{j_u^k}) + v_k(t_k - t_{k+1}) + \cdots + v_{i-1}(t_{i-1} - t_i) \\
&\geq \sum_{u=d-\ell+1}^{d} v_i(q_{j_u^{k-1}} - q_{j_u^k}) + v_i(t_k - t_{k+1}) + \cdots + v_i(t_{i-1} - t_i) \\
&= v_i \sum_{j\in T} q_j - v_i t_i
\end{aligned}$$

Rewrite

$$\sum_{j\in T} p_j - \sum_{j\in T_i} p_j \geq v_i \sum_{j\in T} q_j - v_i t_i$$

as

$$v_i t_i - \sum_{j\in T_i} p_j \geq v_i \sum_{j\in T} q_j - \sum_{j\in T} p_j,$$

we get the desired result.

□

5.6 (Relaxed) Bundle Envy-free Pricing with Uniform Demand

The concept of bundle envy-free we investigate in this section refers to relaxed bundle envy-free. To simplify the following discussions, we sort all buyers and items in non-increasing order of their unit values and qualities, respectively, i.e., $v_1 \geq v_2 \geq \cdots \geq v_n$ and $q_1 \geq q_2 \geq \cdots \geq q_m$. Recall K is the number of distinct values in the set $\{v_1, \ldots, v_n\}$, and $A_1, \ldots, A_K$ is a partition of all buyers where each A_k, $k = 1, 2, \ldots, K$, contains the set of buyers that have the kth largest value.

5.6.1 Identical Items

In this subsection, we consider the model with $q_1 = q_2 = \cdots = q_m = 1$ and each buyer with the same demand d. Let d_i denote the number of items received by buyer i. Let $\alpha_i = iv_i - (i-1)v_{i-1}$, $\forall i \in [n]$. Then we have the following characterization of the problem by ILP:

Theorem 5.21. *The revenue maximizing problem with identical item and uniform demand of bundle envy-freeness is equivalent to solve the following integer linear programming:*

$$\begin{aligned} \textit{Maximize:} \quad & \sum_{i=1}^{n} \alpha_i d_i \\ s.t. \quad & d \geq d_1 \geq d_2 \geq \cdots \geq d_n \\ & \sum_{i=1}^{n} d_i \leq m, \\ & d_i \in Z^+, \forall i \in [n] \end{aligned} \tag{5.10}$$

Proof. Denote by p_i the payment of buyer i, by the definition of envy-free bundle pricing, we have

$$v_i d_i - p_i \geq v_i d_{i+1} - p_{i+1} \tag{5.11}$$

$$v_{i+1} d_{i+1} - p_{i+1} \geq v_{i+1} d_i - p_i \tag{5.12}$$

Plus above two inequalities gives us that $(v_i - v_{i+1})(d_i - d_{i+1}) \geq 0$. Hence, if $v_i > v_{i+1}$, then $d_i \geq d_{i+1}$. From (5.11), we could get $p_i \leq v_i(d_i - d_{i+1}) + p_{i+1}$. The maximum payment of buyer i is

$$p_i = v_i(d_i - d_{i+1}) + p_{i+1}, \tag{5.13}$$

together with $d_i \geq d_{i+1}$, implying (5.11) and (5.12). Besides the maximum payment of n is $p_n = d_n v_n$. (5.8) together with $d_i \geq d_{i+1}$ and $p_n = d_n v_n$ would make everyone envy-free, the arguments are as follows.

- All the buyers must be envy free. By (5.13), we have $p_i - p_{i+1} = v_i(d_i - d_{i+1})$, hence $p_i = \sum_{k=i}^{n-1} v_k(d_k - d_{k+1}) + p_n$. Noticing that if $d_i = 0$, then $p_i = 0$, which means i is loser. For any buyer $j < i$, we have $p_j - p_i = \sum_{k=j}^{i-1} v_k(d_k - d_{k+1}) \leq \sum_{k=j}^{i-1} v_j(d_k - d_{k+1}) = v_j(d_j - d_i)$. rewrite $p_j - p_i \leq v_j(d_j - d_i)$ as $v_j d_i - p_i \leq v_j d_j - p_j$, which means buyer j would not envy buyer i. Similarly, $p_j - p_i = \sum_{k=j}^{i-1} v_k(d_k - d_{k+1}) \geq \sum_{k=j}^{i-1} v_i(d_k - d_{k+1}) = v_i(d_j - d_i)$, rewrite $p_j - p_i \geq v_i(d_j - d_i)$ as $v_i d_i - p_i \geq v_i d_j - p_j$, which means i would not envy buyer j.

Now let's calculate $\sum_{i=1}^{L} p_i$ based on (5.13) using notation $d_{n+1} = 0$, one has

$$\begin{aligned}\sum_{i=1}^{n} p_i &= \sum_{i=1}^{n}\left[\sum_{k=i}^{n-1} v_k(d_k - d_{k+1}) + p_n\right] = \sum_{i=1}^{n}\sum_{k=i}^{n} v_k(d_k - d_{k+1})\\ &= \sum_{k=1}^{n}\sum_{i=1}^{k} v_k(d_k - d_{k+1}) = \sum_{k=1}^{n} k v_k(d_k - d_{k+1})\\ &= \sum_{k=1}^{n} k v_k d_k - \sum_{k=1}^{n}(k-1)v_{k-1}d_k = \sum_{i=1}^{n}\alpha_i d_i\end{aligned}$$

Since $\sum_{i=1}^{n} d_i \leq m$, we know the revenue maximizing problem can be formalized as (5.10). □

Let $y_i = d_i - d_{i+1}$, for $i \leq n-1$, and $y_n = d_n$ then $d_i = \sum_{j=i}^{n} y_j$. the programming (5.10) can be further simplified as follows:

$$\begin{aligned}\text{Maximize:} \quad & \sum_{i=1}^{n} i v_i y_i\\ \text{s.t.} \quad & \sum_{i=1}^{n} i y_i \leq m,\\ & \sum_{i=1}^{n} y_i \leq d,\\ & y_i \in Z^+, \forall i \in [n]\end{aligned} \tag{5.14}$$

Since (5.14) is a special two dimensional knapsack, then there is a PTAS for (5.14)[13]. If d is a constant, a brute force method takes n^d time to valuate all the y_i's and gives the optimal solution.

Remark 5.22. If the demand is sharp demand (which means buyer i would buy exactly d_i items or buy nothing), the problem is trivial and the optimal solution can be reached easily.

5.6.2 Distinct Items with Different Qualities q_j

In this subsection, we generalize the model for identical item to different items, where each item j is associated with number q_j representing the quality of the item. Each buyer i has a per unit valuation v_i, hence i's valuation for item j is $v_i q_j$. We still consider uniform demand case where each buyer's demand is bounded by same d. Let x_{ij} denote variables whether item j is received by buyer i. Denote by $\alpha_i = i v_i - (i-1)v_{i-1}$. Similarly as uniform demand with identical items, we have the following theorem.

Theorem 5.23. *The revenue maximizing problem with uniform demand and distinct qualities is equivalent to solve the following integer linear programming:*

$$\begin{aligned}
\textit{Maximize:} \quad & \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i x_{ij} q_j \\
\textit{s.t.} \quad & \sum_{j=1}^{m} x_{ij} q_j \geq \sum_{j=1}^{m} x_{i+1j} q_j, && \forall i \in [n-1] \\
& \sum_{j=1}^{m} x_{ij} \leq d, && \forall i \in [n] \\
& \sum_{i=1}^{n} x_{ij} \leq 1, && \forall j \in [m] \\
& x_{ij} \in \{0,1\}, && \forall i \in [n], j \in [m]
\end{aligned} \tag{5.15}$$

Proof. The proof is identical to identical item case. □

Theorem 5.24. *The programming (5.15) is NP-hard even if there are only three buyers.*

Proof. We reduce the partition problem to (5.15). Given an instance of partition problem, $A = \{a_1, a_2, \cdots, a_m\}$,where a_i, $i = 1, 2, \cdots, m$ are positive integers, the partition problem asks whether there is a subset B of A such that the sum of elements in B equal half of the sum of elements of A. We construct an instance of (5.15) with $2m$ items, three buyers with relaxed demand m. The values of buyers are $v_1 = 10$, $v_2 = 6$, $v_3 = 5$ and the qualities are $q_j = a_j$ for $j \in [m]$ and $q_j = \sum_{k=1}^{m} a_k$ for $j = m+1, m+2, \cdots, 2m$. Noting $\alpha_1 = 10$, $\alpha_2 = 2$, $\alpha_3 = 3$. Clearly, in optimal solution of such an instance, the item $m+1, m+2, \cdots, 2m$ must be allocated to buyer 1. Therefore, optimization of such an instance is equivalent to the following integer linear programming:

$$\begin{aligned}
\text{Maximize:} \quad & (10m \sum_{j=1}^{m} a_j) + \sum_{i=2}^{3}\sum_{j=1}^{m} \alpha_i x_{ij} q_j \\
\text{s.t.} \quad & \sum_{j=1}^{m} x_{ij} q_j \geq \sum_{j=1}^{m} x_{i+1j} q_j, && i = 2 \\
& \sum_{j=1}^{m} x_{ij} \leq m, && i = 2, 3 \\
& \sum_{i=2}^{3} x_{ij} \leq 1, && \forall j \in [m] \\
& x_{ij} \in \{0,1\}, && \forall i = 2, 3, j \in [m]
\end{aligned} \tag{5.16}$$

In the optimal solution of (5.16), we must have $\sum_{i=2}^{3} x_{ij} = 1$, hence, (5.16) can be simplified as

$$\begin{aligned}
\text{Maximize:} \quad & [(10m+2)\sum_{j=1}^{m} a_j] + \sum_{j=1}^{m} x_{3j}q_j \\
\text{s.t.} \quad & \sum_{j=1}^{m} x_{2j}q_j \geq \sum_{j=1}^{m} x_{3j}q_j, \\
& \sum_{j=1}^{m} x_{ij} \leq m, & i = 2,3 \\
& \sum_{i=2}^{3} x_{ij} \leq 1, & \forall j \in [m] \\
& x_{ij} \in \{0,1\}, & \forall i = 2,3, j \in [m]
\end{aligned} \tag{5.17}$$

It is not difficult to see that (5.17) has an optimal solution $(10m + 2.5)\sum_{j=1}^{m} a_j$ if and only if the partition problem has a positive answer. □

It should be noticed that the optimization problem (5.15) can be solved by the following dynamic programming. Let $W[i, j, T]$ denote the optimal value when there are i buyers, the total qualities of buyer i is larger than or equals j only items indexed in T being sold. Noting that the optimal solution of (5.15) is given by $W[L, 0, [m]]$. Let $M = \sum_{j=1}^{d} q_j$ and $D = \{S | \sum_{k\in S} q_k \geq j, |S| \leq d, S \subset T\}$

SOLVE-IP(II)

1. `Compute` $W[1, j, T]$, $j = 0, 1, 2, \cdots M$, $T = 2^{[m]}$.

2. `Compute`

$$W[i,j,T] = \begin{cases} \max_{S_i \in D}\{\alpha_i \sum_{k\in S_i} q_k + W[i-1, \sum_{k\in S_i} q_k, T\backslash S_i]\} & \texttt{if } D \neq \emptyset \\ 0 & \texttt{Otherwise} \end{cases}$$

3. `Find` S_i, $i = 1, 2, \cdots, L$ `that maximize` $W[L, 0, [m]]$.

5.7 Identical Items with Budget (Relaxed Envy-free)

In this section, we adapt relaxed envy-free pricing schemes, which mean the items received by each buyer maximizes his total utility. In our model, there are n buyers facing m identical items. The valuation of buyer i for each item is v_i. Besides, buyer i has a budget b_i. We present an optimal solution for this problem, which improves the previous 2-approximate solution [32].

Lemma 5.25. *The price of each sold item must be the same.*

Proof. Suppose there are more than two buyers and Lemma 5.25 is not true. The buyer who receives item with higher price would envy the buyer who receives the item with low price. □

With Lemma 5.25, we first calculate revenues when $p = v_1, v_2, \cdots, v_n$. Now we need only to consider $p \neq v_i \ \forall i$ case, which is given bellow.

$$\begin{aligned} \text{Maximize:} \quad & \sum_{i=1}^{n} \min\{m, \left\lfloor \frac{b_i}{p} \right\rfloor\} \mathbf{1}_{v_i > p} p \\ \text{s.t.} \quad & \sum_{i=1}^{n} \min\{m, \left\lfloor \frac{b_i}{p} \right\rfloor\} \mathbf{1}_{v_i > p} \leq m, \\ & p > 0. \end{aligned} \tag{5.18}$$

Observation 5.7.1. $\left\lfloor \frac{b_i}{p} \right\rfloor \leq m, \forall i \in [n], \quad \textit{where} \quad v_i > p.$

The optimization problem (5.18) can be further simplified as follows.

$$\begin{aligned} \text{Maximize:} \quad & \sum_{i=1}^{n} \left\lfloor \frac{b_i}{p} \right\rfloor \mathbf{1}_{v_i > p} p \\ \text{s.t.} \quad & \sum_{i=1}^{n} \left\lfloor \frac{b_i}{p} \right\rfloor \mathbf{1}_{v_i > p} \leq m, \\ & p > 0. \end{aligned} \tag{5.19}$$

Lemma 5.26. *In any optimal solution p^* for optimization problem (5.19), there exists i such that $\left\lfloor \frac{b_i}{p^*} \right\rfloor$ is integer.*

Proof. Otherwise, we can increase the price by a small amount achieving high objective value, contradicting that the price p^* is optimal. □

By Lemma 5.26, to get optimal solution of (5.19), we need only to calculate the revenue when $p \in \{\frac{b_i}{j} | i \in [n], j \in [m]\}$.

Chapter 6

Simulation

6.1 Simulation

Since the consecutive model has a direct application for rich media advertisement, the simulation for comparing the schemes e.g. Bayesian optimal mechanism (Bayesian for simplicity in this chapter), consecutive CE (CE for simplicity in this chapter), consecutive EF (EF for simplicity in this chapter), generalized GSP, will be presented in this chapter. Our simulation shows a convincing result for these schemes. We did a simulation to compare the expected revenue among those pricing schemes. The sampling method is applied to the competitive equilibrium, envy-free solution, Bayesian truthful mechanism, as well as the generalized GSP, which is the widely used pricing scheme for text ads in most advertisement platforms nowadays.

The value samples v come from the same uniform distribution $U[20, 80]$. With a random number generator, we produced 200 group samples $\{V_1, V_2, \cdots, V_{200}\}$, they will be used as the input of our simulation. Each group contains n samples, e.g. $V_k = \{v_k^1, v_k^2, \cdots, v_k^n\}$, where each v_k^i is sampled from uniform distribution $U[20, 80]$. For the parameters of slots, we assume there are 6 slots to be sold, and fix their position qualities:

$$\begin{aligned} Q &= \{q_1,\ \ q_2,\ \ q_3,\ \ q_4,\ \ q_5,\ \ q_6\} \\ &= \{0.8,\ 0.7,\ 0.6,\ 0.5,\ 0.4,\ 0.3\} \end{aligned} \tag{6.1}$$

The actual ads auction is complicated, but we simplified it in our simulation, we do not consider richer conditions, such as set all bidders' budgets unlimited, and there is no reserve prices in all mechanisms. We vary the group size n from 5 to 12, and observe their expected revenue variation. From $j = 1$ to $j = 200$, at each j, invoke the function EF (V_j, D, Q), GSP (V_j, D, Q), CE (V_j, D, Q) and Bayesian (V_j, D, Q) respectively. Thus, those mechanisms use the same data from the same distribution as inputs and compare their expected revenue fairly. Finally, we average those results from different mechanisms respectively, and compare their expected revenue at sample size n.

The generalized GSP mechanism for rich ads in the simulation was not introduced in the previous sections. Here, in our simulation, it is a simple generalization of the standard GSP which is used in keywords auction. In our generalization of GSP, the allocations of bidders are given by maximizing the total social welfare, which is compatible with GSP in keywords auction, and each winner's price per quality is given by the next highest bidder's bid per quality. Since the real generalization of GSP for rich ads is unknown and the generalization form may be various, our generalization of GSP for rich ads may not be a revenue maximizing one, however, it is a natural one. The pseudo-codes are listed in Appendix A.

Incentive analysis is also considered in our simulation, except Bayesian mechanism (it is truthful bidding, $b_i = v_i$). Since the bidding strategies in other mechanisms (GSP, CE, EF) are unclear, we present a simple bidding strategy for bidders to converge to an equilibrium. We try to find the equilibrium bids by searching the bidder's possible bids ($b_i < v_i$) one by one, from top rank bidders to lower rank bidders iteratively, until reaching an equilibrium where no one would like to change his bid. If any equilibrium exists, we count the expected revenue for this sample; if not, we ignore this sample. All the pseudo-codes are listed in Appendix A.

Since the Envy-Free solution in our paper only works for the condition that all the bidders have the same demand, thus, we did the simulation in 2 separate ways:

1. Simulation I is for bidders with a fixed demands, we set $d_i = 2$, for all i and compares expected revenues obtained by GSP, CE, EF, Bayesian.

2. Simulation II is for bidders with different demands and compares expected revenues obtained by GSP, CE, Bayesian.

Figure 6.1 shows I's results when all bidders' demand fixed at 2. Obviously, the expected revenue is increasing when more bidders involved. When the bidders' number rises, the rank of expected revenue of different mechanisms remains the same in the order Bayesian > EF > CE > GSP.

Simulation II is for bidders with various demands. With loss of generality, we assume that bidder's demand $D = \{d_1, d_2, \cdots, d_i\}, d_i \in \{1, 2, 3\}$, we assign those bidders' demand randomly, with equal probability.

Figure 6.2 shows our simulation results for II when bidders' demand varies in {1,2,3}, the rank of expected revenue of different mechanisms remains the same as simulation I, From this chart, we can see that Bayesian truthful mechanism and competitive equilibrium get more revenues than generalized GSP.

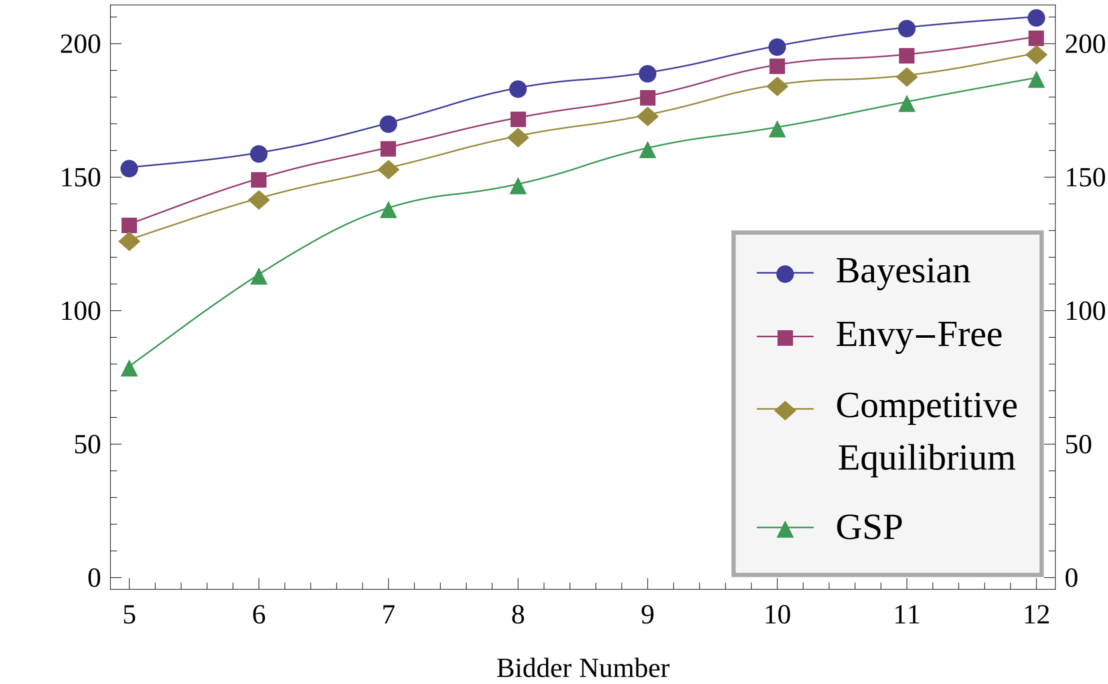

FIGURE 6.1: Simulation results from different mechanisms, all bidders' demand fixed at $d_i = 2$

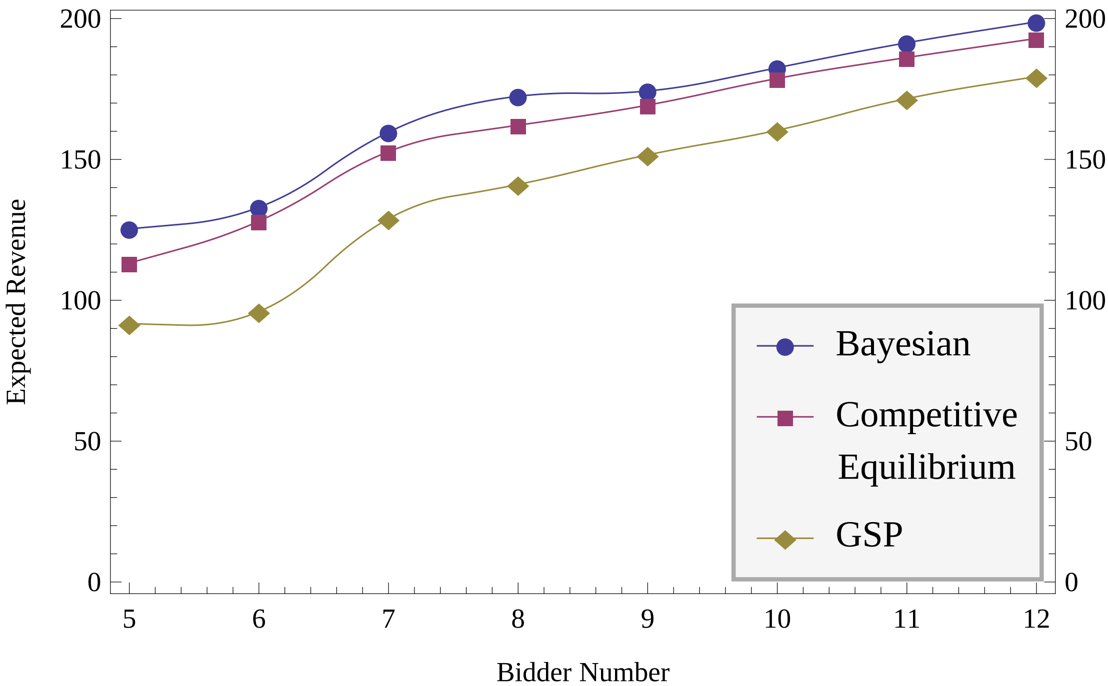

FIGURE 6.2: Simulation results from different mechanisms, bidders' demand varies in {1,2,3}

Chapter 7

Conclusion and Open Problems

7.1 Conclusion

We study the optimal Bayesian truthful mechanism design issues for the multi-item auction problem with correlated valuations $v_{ij} = v_i q_j$. We focus on three demand models, the relaxed demand, the sharp demand and the consecutive demand model. We develop optimal (revenue) mechanisms for the seller. In addition, for the budget constrained model (without demand constraints), we develop a 2-approximate truthful mechanism. We prove that the solution is polynomial time solvable.

Question 7.1.1. A major open problem is to find a constant approximation scheme when the demand constraints and the budget constraints are used simultaneously.

For discrete distribution, [1] and [6] suggested a constant approximate mechanism for multi unit auction with budget and relaxed demand constrained buyers. However, their approach which is based on solving an associated linear program cannot be extended to the continuous distribution case. Of course, another direction is to improve the approximation ratio for budget constrained cases.

Our models have potential applications to various settings. E.g. TV ads can also be modeled under our consecutive demand adverts where inventories of a commercial break are usually divided into slots of fixed sizes, and slots have various qualities measuring their expected number of viewers and corresponding attractiveness (see Figure 7.1). With an extra effort to explore the periodicity of TV ads, we can extend our multiple peak model to one involved with cyclic multiple peaks.

Besides single consecutive demand where each buyer only have one demand choice, the buyer may have more options to display his ads. e.g. select a large picture or a small one to display his adverts. Our dynamic programming algorithm A.1 can also be applied to this case (Transition function in each step selects maximum value from $2k+1$ possible values, where k is the number of choices of the buyer).

Another remarkable extension of our model is to add budget constraints for sharp or consecutive buyers, e.g., each buyer can't afford the payment more than his budget. By relaxing the requirement of Bayesian incentive compatible (BIC) to one of approximate

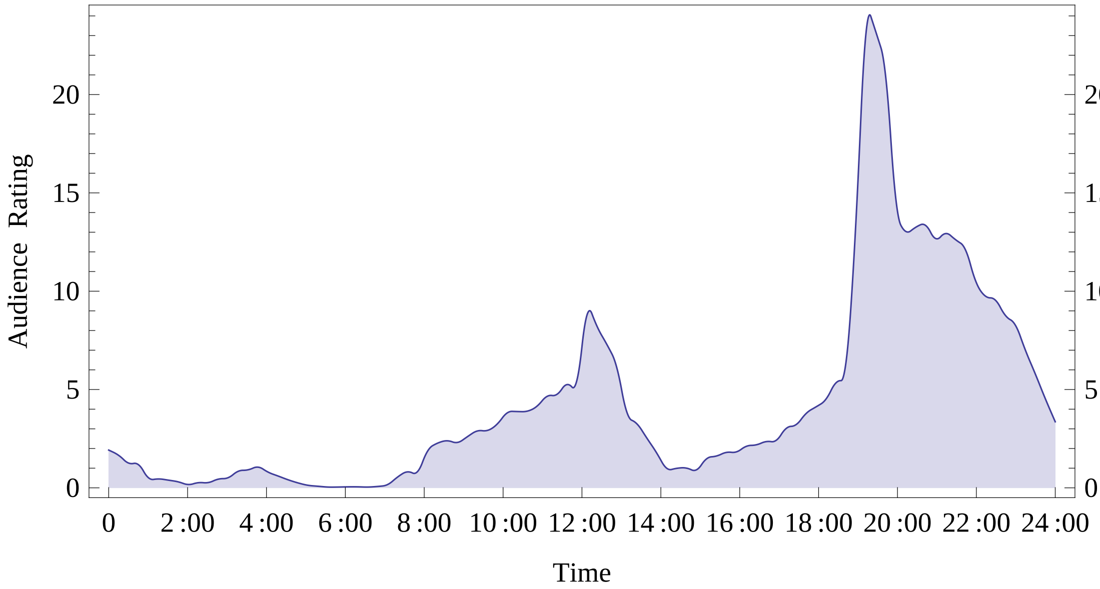

FIGURE 7.1: Audience Rating Curve of One TV Channel of China with Several Peaks in One Day

BIC, this extension can be obtained by the recent milestone work of Cai et al. [11]. It remains an open problem how to do it under the exact BIC requirement. Further, it is also interesting to handle it under the market equilibrium paradigm for our model.

We study revenue maximization problem under two concepts: envy-free bundle pricing and envy-free item pricing, which is complementary compared with the recent work [55]. For envy-free bundle pricing, we suppose the buyer are uniform demand constrains meaning each buyer's demand is up to the same number. A PTAS is presented for identical items and a exponential time algorithm is given for v_iq_j valuations. Besides we show the problem is NP-hardness for identical items model. For envy-free item pricing, we present an optimal algorithm for budgets constraint buyer with no demand constraints for identical items. Our work inspires the following problem, which leaves for future work.

Question 7.1.2. v_iq_j model. Revenue Maximization with budget no demand constraints (or unit demand), consider relaxed envy-free pricing case first, then consider bundle envy-free pricing case.

Question 7.1.3. How to handle revenue maximization of the case $q_1 = q_2 = \cdots = q_m = 1$, with budget no demand (or unit demand) constraints? PTAS?

Question 7.1.4. How to calculate consecutive envy-free solution when the number of distinct demands is bounded by a constant number, where the NP-hardness result in Section 5.5 does not apply to this very important case.

Through out the thesis, our main focus is for correlated valuations $v_{ij} = v_iq_j$ except in Section 4.3 of Chapter 4, we present a hardness result for finding sharp competitive equilibria of revenue maximization problem for general valuation v_{ij}. The valuation we will discuss in the following paragraph refers to general valuations v_{ij} unless specified.

In fact, for general valuations, as introduced in Chapter 1, for the unit-demand setting, Guruswami et al. [39] initialized the study of envy-freeness in computer science perspective and gave an $O(\log n)$ approximation algorithm, and Briest [8] showed that the problem is hard to approximate within a ratio of $O(\log^{\epsilon} n)$ for some ϵ, under some complexity assumptions. Chen et al. [18] provided a polynomial time algorithm to compute a revenue maximization envy-free pricing when there is a metric space behind all items. One of major open problems is the following:

Question 7.1.5. How to generalize the results on envy-freeness for unit demand model to multi-unit demand model (including generalization of hardness result)?

The work of Chen et al. [18] can be viewed as one direction of generalization to answer open problem 7.1.5. Our results of relaxed/sharp/consecutive demand concepts can also be viewed as other generalization of open problem 7.1.5. Another accompanied open problem should be the following:

Question 7.1.6. How to generalize the results on envy-freeness for unit demand model to the same setting with budget constraints (including generalization of hardness result)?

The study of Kempe et al. [44] and Devanur et al. [27] can be viewed as a generalization to solve problem 7.1.6.

Another streams of general valuations are competitive equilibria. Chen et al. [16] studies competitive equilibrium on unit demand matching market with budget constraints (e.g. utility function has one discontinuity point) and consistent conditions for the utility function and propose a strongly polynomial time algorithm to determine whether the equilibrium exists or not and output the minimum one if one does. One open problem should be the following:

Question 7.1.7. How to generalize the results of Chen et al. [16] to multi-unit case with budget constraints?

Chen et al. [15] study a Nash dynamic process for unit demand auction model in matching market using maximum competitive equilibrium mechanism. They proved that the aligned best response always converges and converges to a minimum competitive equilibrium of truthful bidding of each buyer. Hence, we would like to investigate Nash dynamic process for our envy-free case under envy-free mechanisms.

Question 7.1.8. How to do Nash dynamics under envy-free mechanism framework?

For the problem 7.1.8, as far as we know, there is no literature studying this interesting problem. Convergence or uniqueness or converging results of Nash dynamics under envy-free framework would be very interesting.

Appendix A

Pseudo-code

A.1 Pseudo-code of Simulation

We present the pseudo codes of simulation in Chapter 6.

A.1.1 Expected Revenue for Bayesian Truthful Mechanism (See Section 3.5 of Chapter 3)

Suppose with loss of generality, $b_1 > b_2 > \ldots > b_n > 10$, and $q_1 > q_2 > \ldots > q_n$, let $\phi_i(v_i) = 2v_i - b_i - 10$.

Algorithm 2: Bayesian Expected Revenue

Input: Demands d_i, qualities (CTR) q_j and bids b_i, number of samples K
Output: Expected Revenue R
Generate uniform distribution for b_i as $\mathbf{I}_i$ uniformly distributed on $I_i = [b_i - 10, b_i + 10]$;
Repeat ;
for $r = 1, 2, \cdots, K$ **do**
    Generate v_i^r from $\mathbf{I}_i$ independently, $i = 1, 2, \cdots, n$;
    Calculate $\phi_i(v_i^r)$ and sort it decreasing order as $\phi_i'(v_i^r) > \phi_{i+1}'(v_i^r)$, $i = 1, 2, \cdots, n$;
    Use dynamic programming

$$g[s, r] = \max \begin{cases} g[s-1, r] \\ g[s-1, r-d_s] + \phi_s'(v_s^r) \sum_{j=r-d_s+1}^{r} q_j \end{cases} \tag{A.1}$$

    By tracking dynamic programming find allocation X_i;
    Calculate $R^r = \sum_i \phi_i(v_i^r) \sum_{j \in X_i} q_j$
end
return $R = \frac{1}{K} \sum_{r=1}^{K} R^r$;

The following the sub algorithm for finding the allocations X_i when ϕ_i, $i = 1, 2, \cdots, n$ are known.

Algorithm 3: sharp

Input: virtual surplus ϕ_i qualities q_j
Output: Allocation x_{ij}

Sort buyers i in decreasing order of ϕ_i;
$g[i, j] \leftarrow -\infty$; $g[0, 0] \leftarrow 0$;
$u[i, j] \leftarrow 0$; $x_{ij} \leftarrow 0$;
for *each buyer i with positive ϕ_i* **do**
  for *each item j* **do**
    $tmp \leftarrow g[i-1, j-d_i] + \sum_{k=j-d_i+1}^{j} \phi_i q_k$;
    $g[i, j] \leftarrow g[i-1, j]$;
    if $g[i, j] < tmp$ **then**
      $u[i, j] \leftarrow 1$;
      $g[i, j] \leftarrow tmp$;
    end
  end
end
$g[i^*, j^*] = \max_{i,j}\{g[i, j]\}$;
while $i^* > 0$ **do**
  if $u[i^*, j^*] = 1$ **then**
    for *each item k from $j^* - d_{i^*} + 1$ to j^** **do**
      $x_{i^*,k} \leftarrow 1$;
    end
    $j^* \leftarrow j^* - d_{i^*}$;
  end
  $i^* \leftarrow i^* - 1$;
end
return x;

A.1.2 Revenue from Competitive Equilibrium (See Section 4.5 of Chapter 4)

Suppose $q_1 \geq q_2 \geq q_3 \geq \cdots \geq q_n$

Algorithm 4: Sub-algorithm for CE denoted by CE(d,q,b)

Input: Demands d_i, qualities (CTR) q_j and bids b_i
Output: Equilibrium ($\mathbf{X}$,$\mathbf{p}$)
Sort the bids b_i in decreasing order e.g. $b_1 > b_2 > \cdots > b_n$;
Use dynamic programming

$$g[s,r] = \max \begin{cases} g[s-1,r] \\ \\ g[s-1,r-d_s] + b_s \sum_{j=r-d_s+1}^{r} q_j \end{cases} \tag{A.2}$$

By tracking dynamic programming find allocation $\mathbf{X}$;
Using following LP to settle price $\mathbf{p}$;
Let T_i be any consecutive number of d_i slots, for all $i \in [n]$;

$$\begin{aligned} \max \quad & \sum_{i \in [n]} \sum_{j \in X_i} p_j \\ s.t. \quad & p_j \geq 0 && \forall\ j \in [m] \\ & p_j = 0 && \forall\ j \notin \cup_{i \in [n]} X_i \\ & \sum_{j \in X_i} (v_i q_j - p_j) \geq \sum_{j' \in T_i} (v_i q_{j'} - p_{j'}) && \forall\ i \in [n] \\ & \sum_{j \in X_i} (v_i q_j - p_j) \geq 0 && \forall i \in [n] \end{aligned}$$

if *LP has a feasible solution* **then**
  return ($\mathbf{X}$,$\mathbf{p}$)
end
else
  return null;
end

Algorithm 5: Main Algorithm for CE

Input: Demands d_i, qualities (CTR) q_j and bids b_i, Accuracy ϵ, biding times K
Output: R revenue

$b_i^1 = b_i$, $v_i = b_i$ $i = 1, 2, \cdots, n$.
invoke Sub-algorithm for CE on (d, q, b^1),
if *output is not null* **then**
    Suppose the output is $(\mathbf{X}, \mathbf{p})$
    calculate the utility for all i. e.g. $u_i = v_i \sum_{j \in X_i} q_j - \sum_{j \in X_i} p_j$
end
for $r = 1, 2, \cdots, K$ **do**
    for $i = 1, 2, \cdots, n$ **do**
        let $M_i^r = \lfloor b_i^r / \epsilon \rfloor$;
        for $t_i^r = \epsilon, 2\epsilon, \cdots, M_i^r * \epsilon$ **do**
            invoke Sub-algorithm for CE on input $(d, q, (t_i^r, b_{-i}^r))$
            if *the output is not null* **then**
                Suppose the output is $(\mathbf{X}, \mathbf{p})$
                Calculate the current utility $u = v_i \sum_{j \in X_i} q_j - \sum_{j \in X_i} p_j$
                if $u > u_i$ **then**
                    let $u_i = u$ and $b_i^{r+1} = t_i^r$, $b_i^r = t_i^r$.
                end
                else
                    $b_i^{r+1} = b_i^r$;
                end
            end
        end
    end
    $R^r = \sum_j p_j$
end

A.1.3 Revenue from generalized GSP (See more details in Chapter 6)

Algorithm 6: Algorithm GSP

Input: Demands d_i, qualities (CTR) q_j and bids b_i, Accuracy ϵ, biding times K
Output: R revenue

$b_i^1 = b_i$, $v_i = b_i$ $i = 1, 2, \cdots, n$.
Suppose the allocation of GSP is $\mathbf{X} = sharp(b, q)$;
calculate the utility for all i. e.g. $u_i = v_i \sum_{j \in X_i} q_j - \sum_{j \in X_i} p_j$
for $r = 1, 2, \cdots, K$ **do**
    for $i = 1, 2, \cdots, n$ **do**
        let $M_i^r = \lfloor b_i^r / \epsilon \rfloor$;
        for $t_i^r = \epsilon, 2\epsilon, \cdots, M_i^r * \epsilon$ **do**
            Suppose the output of GSP on $(d, q, (t_i^r, b_{-i}^r))$ is $(\mathbf{X}, \mathbf{p})$
            Calculate the current utility $u = v_i \sum_{j \in X_i} q_j - \sum_{j \in X_i} p_j$ of bidder i
            if $u > u_i$ **then**
                let $u_i = u$ and $b_i^{r+1} = t_i^r$ $b_i^r = t_i^r$.
            end
            else
                $b_i^{r+1} = b_i^r$;
            end
        end
    end
    return $R^r = \sum_j p_j$
end

A.1.4 Revenue from Envy-free Solution (See Section 5.5 of Chapter 5)

Suppose $q_1 \geq q_2 \geq q_3 \geq \cdots \geq q_n$

Algorithm 7: Sub-algorithm for EF denoted by EF(d,q,b)

Input: Demands d, qualities (CTR) q_j and bids b_i

Output: Equilibrium ($\mathbf{X}$,$\mathbf{p}$)

1 Sort the bids b_i in decreasing order e.g. $b_1 > b_2 > \cdots > b_n$;

2 Use dynamic programming (similar as sharp) (initial values $g[0,0] = 0$, $g[1,r] = -\infty$, $r \leq d$)

$$g[s,r] = \max \begin{cases} g[s, r-1] \\ \\ g[s-1, r-d] + b_s \sum_{j=r-d+1}^{r} q_j \end{cases} \tag{A.3}$$

By tracking dynamic programming find allocation $\mathbf{X}$;

3 The payment of buyers are $\mathbf{P}$, where P_i is the payment of buyer i ;

4 $P_n = b_n \sum_{j \in X_n} q_j$, and $P_i = b_i(\sum_{j \in X_i} q_j - \sum_{j \in X_{i+1}} q_j) + P_{i+1}$ for $i = 1, 2, \cdots, n-1$

Algorithm 8: Main Algorithm for EF

Input: Demands d, qualities (CTR) q_j and bids b_i, Accuracy ϵ, true value v_i, biding times K

Output: R revenue

$b_i^1 = b_i$, $i = 1, 2, \cdots, n$.
invoke Sub-algorithm for EF on (d, q, b^1),
if *output is not null* **then**
    Suppose the output is $(\mathbf{X},\mathbf{P})$
    calculate the utility for all i. e.g. $u_i = v_i \sum_{j \in X_i} q_j - P_i$
end
for $r = 1, 2, \cdots, K$ **do**
    for $i = 1, 2, \cdots, n$ **do**
        let $M_i^r = \lfloor b_i^r / \epsilon \rfloor$;
        for $t_i^r = \epsilon, 2\epsilon, \cdots, M_i^r * \epsilon$ **do**
            invoke Sub-algorithm for EF on input $(d, q, (t_i^r, b_{-i}^r))$
            if *the output is not null* **then**
                Suppose the output is $(\mathbf{X},\mathbf{P})$
                Calculate the current utility $u = v_i \sum_{j \in X_i} q_j - P_i$
                if $u > u_i$ **then**
                    let $u_i = u$ and $b_i^{r+1} = t_i^r$, $b_i^r = t_i^r$.
                end
                else
                    $b_i^{r+1} = b_i^r$;
                end
            end
            else
                $b_i^{r+1} = b_i^r$;
            end
        end
    end
    $R^r = \sum_i P_i$
end

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