Renormalisation of general $\mathcal{N} = 2$ Supersymmetric Chern-Simons Theories in Three Dimensions

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy by

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Declaration

I hereby declare that all the work described in this thesis is the result of my own research activities unless reference is given to others. None of this material has been previously submitted to this or any other University. All work was carried out in the Theoretical Physics Division during the period of September 2009 to October 2013.

Contributions from this work have been published, or are awaiting publication, elsewhere in the following references:

1. I. Jack, C.Luckhurst, JHEP **1103** (2011), 076, "Exact β-functions in softly-broken $\mathcal{N} = 2$ Chern-Simons matter theories".

2. I. Jack, C.Luckhurst, arXiv:1304.3344, "Results for the four-loop anomalous dimension for a general $\mathcal{N} = 2$ supersymmetric Chern-Simons theory in three dimensions".

3. I. Jack and C. Luckhurst, JHEP **1312** (2013), 078, "Superconformal Chern-Simons theory in three dimensions beyond leading order".

Abstract

We present exact results for the β-functions for the soft-breaking parameters in softly-broken $\mathcal{N} = 2$ Chern-Simons matter theories in terms of the anomalous dimension in the unbroken theory. We check our results explicitly up to the two loop level. We then go on to present results for the planar contribution to the four-loop anomalous dimension for a general $\mathcal{N} = 2$ supersymmetric Chern-Simons theory in three dimensions. These results should facilitate higher-order superconformality checks for theories relevant for the AdS/CFT correspondence. We then go on to discuss possible higher-loop corrections to superconformal invariance for a class of $\mathcal{N} = 2$ supersymmetric Chern-Simons theories including the ABJM model. We argue that corrections are inevitable even for simple generalisations of the ABJM model; but that it is likely that any corrections are of a particular "maximally transcendental" form.

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Contents

1 Introduction

This thesis begins by looking at the Standard model of particle physics, one of great achievements of science, giving a brief overview of the basics of theoretical physics such as the various particle types such as the ones composing matter and those governing the forces by which the matter particles interact, eg electromagnetism and the strong and weak nuclear forces. Following on from this we look at the weaknesses of the standard model and the requirement of new physics to expand it in order to correct things that are wrong (eg massless neutrinos) or to add new particles and interactions for phenomena currently considered beyond the standard model (eg gravity). After this there is an overview of some of the most popular theories attempting to solve these problems including string theory and supersymmetry, before a more in depth look at supersymmetry starting with the extension of the Poincaré algebra and how this leads to a symmetry between the fermions and bosons. Following on from this we look at renormalisation discussing its motivation and development, looking at dimensional regularization and dimensional reduction, before moving on to the specific supersymmetric (SUSY) theories (softly broken $\mathcal{N} = 2$ Chern-Simons matter theories in three dimensions) which is the focus of this thesis, starting out with the two-loop component calculation of the anomalous dimension and β-functions of the soft-breaking parameters. After this we present the results of our planar four-loop calculation of the anomalous dimension for a general $\mathcal{N} = 2$ Chern-Simons theory where we describe the diagrams using a novel labelling system. In the final chapter we use the four-loop planar result to discuss the possible higher-loop corrections to superconformal invariance for a range of $\mathcal{N} = 2$ supersymmetric Chern-Simons theories. We show that there is a strong case to be made for the view that the majority of superconformal theories will require a coupling redefinition beyond leading order in order to preserve superconformality.

1.1 The Standard Model

One of the biggest events of twentieth century physics was the creation of the Standard Model. It was a great collaboration between experimentalists and theorists which made several predictions (eg the top and bottom quarks) which were later confirmed. It is a gauge theory with the gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ which combines all of the leptons (electrons etc.) and quarks (up, down etc.) as well as the force mediating particles (gauge bosons) into a framework that contained all of their interactions as well as including particles that at the time were yet to be discovered such as the Higgs boson (2012). While it has been one of the great successes of science it is not the final "theory of everything" as it still leaves many questions unanswered. A comprehensive guide can be found here [1].

1.1.1 Problems with the Standard Model

One of the main problems with the standard model is that it makes several assumptions or makes no mention at all of certain phenomena.

- Parameters: There are 19 free parameters in the standard model which needed to be determined by experiments as these values cannot be calculated from the model. This is the main reason that the standard model is considered an effective theory, valid to a certain energy scale, but not a fundamental theory.

- Generations of Leptons and Quarks: The standard model cannot explain why there are three generations of quarks and leptons.

- Massless neutrinos: the standard model assumes that neutrinos are massless, which was the assumption when they were first postulated, however experiments have shown that not to be the case and the more recent discovery of neutrino oscillation (where neutrinos spontaneously change from one flavour to another) would not be possible with massless neutrinos.

- Gravity: one of the standard model's problems is the exclusion of gravity. It does not include the graviton or any mechanism for the inclusion of gravity.

- The Hierarchy Problem: The Hierarchy problem relates to the mass of the Higgs boson, when the calculation to predict its mass is performed quadratic divergences are produced which make very large contributions to its mass which give a very large mass. This was originally a problem due to restrictions placed on the Higgs mass by electroweak theory, although now the problem is that it has been discovered with a mass of 125GeV [2] which is well below the mass predicted by the model.

- Unification of the gauge couplings: The three forces which are accounted for in the standard are thought to unify at some very high energy scale and in the running of the couplings of the standard model they do not meet at the same place.

- Dark Matter: Due to observations of our galaxy it has been theorized that a large portion of matter that is responsible for the distribution of gravity throughout the galaxy is invisible to us. It doesn't seem to interact or at least its interactions are so rare or weak that we cannot observe them. This has led to the proposal of the existence of so called Dark Matter [3] (also called WIMPs, Weakly Interacting Massive Particles) and there are no candidate particles in the standard model.

1.1.2 Extensions to the Standard Model

Just as there are problems with the standard model there have been many attempts to extend it to address its problems, some examples are:

- Grand Unified Theories (GUTs): These theories attempt to unify all the forces into a single unified framework one of the most common is SU(5) which breaks to give $SU(3)_C \otimes\ SU(2)_L \otimes\ U(1)_Y$.

- Supersymmetry: A symmetry between fermionic and bosonic degrees of freedom so that every particle has a superpartner distinguished from itself by the new particle having a different spin. One of the strengths of supersymmetry is that it can solve the Hierarchy problem as well as provide candidate particles for dark matter.

- String Theory: This theory is an attempt at a truly fundamental theory built from the starting assumption that all particles are tiny one dimensional strings, as opposed to the point particles of QFT, that vibrate in different ways to form the different particles. It also includes higher dimensional objects called D-branes and M-branes Many models derived from string theory involve supersymmetry, although there are theories such as bosonic string theory which do not require it. A notable feature of string theory is its use of extra spatial dimensions, string theories are typically 10 dimensional (11 for M-theory) although bosonic string theory requires 26 dimensions [4].

- Technicolour: A theory that replaces the Higgs mechanism for electro-weak symmetry breaking with symmetry breaking by a composite of massless fermions which are introduced into the Lagrangian [5].

Of the theories currently being looked at to extend the standard model the one being tested most at CERN is supersymmetry. Although so far there is no evidence for it there are many different models which will need much higher energies to test than the Large Hadron Collider can generate.

2 Supersymmetry (SUSY)

2.1 Motivation for Supersymmetry

Supersymmetry was first introduced by Gol'fund and Likhtman in 1971 as an extension to the Poincaré algebra [6] followed in 1973 by Volkov and Akulov [7] and then in 1974 Wess and Zumino formulated a basic field theory that possessed "Supergauge invariance", the Wess-Zumino model [8]. Although some of the main arguments for SUSY today are naturalness and providing a solution to the Hierarchy problem (the quadratic divergences are cancelled off by the additional terms generated by the superpartners) these were not the original motivations and it is a testament to the power of the theory that despite its origins as an attempt to extend the Poincaré algebra it has found uses in many branches of theoretical physics.

2.2 The SUSY algebra

The starting point for any field theory is the Poincaré group. In 1967 [9] Coleman and Mandula proved that any quantum field theory which has non-trivial interactions must be the direct product of a symmetry Lie algebra with the Poincaré algebra if there is a mass gap (the mass gap is the difference in energy between the vacuum and the next lowest energy state). The Poincaré group is defined by its group algebra which describe translations and Lorentz transformations (boosts and rotations). The generators of the group are the four translation generators P_μ and the six generators of the Lorentz transformations $M_{\mu\nu}$. The standard Poincaré algebra is as follows:

$$[P_\mu, P_\nu] = 0, \tag{2.1}$$

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \tag{2.2}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}). \tag{2.3}$$

Supersymmetry gets around the restriction of the Coleman-Mandula theorem by relaxing one condition, that a Lie algebra can only consist of commutators. This method is known as the Haag-Lopuszanski-Söhnius theorem [10] which involves generalizing the definition of a Lie algebra to include algebras that are defined by relations between anticommutators as well as commutators. These algebras are called graded Lie algebras or superalgebras. In four dimensions the superalgebra (the $\mathcal{N} = 1$ superalgebra) adds one pair of spinorial generators to the Poincaré algebra with the following anticommutation relation:

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu, \tag{2.4}$$

$$\{Q_\alpha, Q_\beta\} = 0, \tag{2.5}$$

$$\{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0, \tag{2.6}$$

with $\sigma^\mu = (1, \sigma^1, \sigma^2, \sigma^3)$ where σ^i are the usual Pauli matrices and $\alpha, \dot{\alpha} = 1, 2$.

These spinorial generators commute with the Poincaré algebra to produce:

$$[Q_\alpha, P_\mu] = 0, \tag{2.7}$$

$$[\overline{Q}_{\dot{\alpha}}, P_\mu] = 0, \tag{2.8}$$

$$[Q_\alpha, M_{\mu\nu}] = \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, \tag{2.9}$$

$$[\overline{Q}_{\dot{\alpha}}, M_{\mu\nu}] = -\tfrac{1}{2}(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}}\overline{Q}_{\dot{\beta}}, \tag{2.10}$$

with

$$\sigma^{\mu\nu} = \tfrac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu), \tag{2.11}$$

$$\bar{\sigma}^{\mu\nu} = \tfrac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu). \tag{2.12}$$

Further pairs of spinorial generators can be added to create extended supersymmetries. Therefore the most general supersymmetry algera is:

$$[P_\mu, P_\nu] = 0, \tag{2.13}$$

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \tag{2.14}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}), \tag{2.15}$$

$$[Q^i_\alpha, P_\mu] = [\overline{Q}^i_{\dot{\alpha}}, P_\mu] = 0, \tag{2.16}$$

$$[Q^i_\alpha, M_{\mu\nu}] = \tfrac{1}{2}(\sigma^{\mu\nu})_\alpha{}^\beta Q^i_\beta, \tag{2.17}$$

$$[\overline{Q}^i_{\dot{\alpha}}, M_{\mu\nu}] = -\tfrac{1}{2}\overline{Q}^i_{\dot{\beta}}(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}, \tag{2.18}$$

$$\{Q_\alpha{}^i, \overline{Q}_{\dot{\alpha}}{}^j\} = 2\delta^{ij}(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu, \tag{2.19}$$

$$\{Q^i_\alpha, Q^j_\beta\} = 2\epsilon_{\alpha\beta}Z^{ij}, \tag{2.20}$$

$$\{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 2\epsilon_{\dot{\alpha}\dot{\beta}}Z^{ij}, \tag{2.21}$$

$$[Z^{ij}, \text{anything}] = 0, \tag{2.22}$$

where $\alpha, \dot{\alpha} = 1, 2$ and $i, j = 1, 2, ..., N$ and Z^{ij} are the central charges. There is a constraint on the number of SUSY generators [11] which arises from the requirement for consistency with the corresponding QFT. The maximum number of supersymmetries is connected to the maximal spin of the particle in the multiplet such that

$$N \leq 4S, \tag{2.23}$$

with S being the maximal spin so for theories whose maximal spin is 1 such as Super Yang-Mills [12] the maximum value for $\mathcal{N}$ is 4 whereas for SUSY theories incorporating gravity (SUGRA [13]), which has the spin-2 graviton, the maximally supersymmetric theory is $\mathcal{N} = 8$.

2.3 The Wess-Zumino model

The first (and simplest) supersymmetric model is the Wess-Zumino model [8] which combines a massless complex scalar field ϕ with a massless spinor field ψ as well as an auxiliary (non-propagating) scalar field F:

$$L = (\partial_\mu \overline{\phi})(\partial^\mu \phi) + i\overline{\psi}\partial\!\!\!/\psi + F^* F. \tag{2.24}$$

The Lagrangian is now invariant under the following transforms

$$\delta\phi = i[\xi^\alpha Q_\alpha, \phi] = \sqrt{2}\xi\psi, \tag{2.25}$$

$$\delta\psi = i[\xi^\alpha Q_\alpha, \psi] = \sqrt{2}F\xi - i\sqrt{2}\sigma^\mu\overline{\xi}\partial_\mu\phi, \tag{2.26}$$

$$\delta F = i[\xi^\alpha Q_\alpha, F] = -i\sqrt{2\xi}\bar{\sigma}^\mu\partial_\mu\psi, \tag{2.27}$$

where ξ and $\overline{\xi}$ are both anticommuting parameters. The reason for the addition of the auxiliary field is that the degrees of freedom for the scalar and spinor fields are not equal. On-shell a Majorana fermion has two degrees of freedom and four states. So on-shell we need the propagating complex scalar field to match this. However we also need this property to hold off-shell, where the spinor has four degrees of freedom, which means we need to include the two non-propagating fields F and F^*.

2.4 The Superfield Formulation of SUSY

While it is possible to formulate Supersymmetry in term of component fields (scalars, spinors, etc.) it is often desirable to formulate it in a more compact formalism. The starting point for what is called the Superspace formalism is Grassmann algebra.

2.4.1 Grassmann Algebra

The primary feature of Grassmann numbers is that they anticommute. So for any two such numbers θ and η,

$$\theta\eta = -\eta\theta. \tag{2.28}$$

The obvious result of this is that

$$\theta^2 = 0, \tag{2.29}$$

and this makes Taylor expansions much simpler. The most important thing needed is to define integration over the Grassmann numbers. Since the product of two Grassmann numbers $(\chi\eta)$ will commute with any other Grassmann number, it seems reasonable for the product of two Grassmann variables to be an ordinary number. Therefore the integral $\int d\theta\theta$ is just an ordinary number which is 1. Grassmann integration is defined as:

$$\int d\theta = 0, \int d\theta\theta = 1. \tag{2.30}$$

The general integration of a function of an anticommuting variable is:

$$\int d\theta f(\theta) = \int d\theta(A + B\theta) = B. \tag{2.31}$$

Since $\theta^2 = 0$ the Taylor expansion for $f(\theta)$ is simply $f(\theta) = A + B\theta$.

2.4.2 Superspace

In constructing supersymmetric models it is very useful to have a formalism where supersymmetry is inherently manifest. To achieve this the superfield formalism was introduced by Salam and Strathdee [14] which extends Minkowski space to superspace which consists of the usual Minkowski space-time coordinates x^μ with $\mu = 0, ..., 3$ as well as four constant (space-time independent), anticommuting Grassmann numbers $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$ ($\alpha, \dot{\alpha} = 1, 2$), which can be formulated in terms of the 2-component Weyl spinor formalism and are considered to be independent of each other. The anticommutation relations of θ and $\bar{\theta}$ are

$$\{\theta^\alpha, \theta^\beta\} = 0, \tag{2.32}$$

$$\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0, \tag{2.33}$$

$$\{\theta^\alpha, \bar{\theta}^{\dot{\alpha}}\} = 0, \tag{2.34}$$

and a coordinate in superspace is given by $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. Note these are the rules for $\mathcal{N} = 1$. For $\mathcal{N} > 1$ the θ's get an additional index running from 1,...,N as each pair of spinorial generators gets its own pair of superspace coordinates.

2.4.3 Superfields

Using the superspace formalism we can now define fields in terms of these new supercoordinates. Just as a translational element of the Poincaré group may be written as

$$e^{-ix.P}, \tag{2.35}$$

a general superspace translation can be written

$$e^{-i(x.p-\theta Q-\bar{\theta}\overline{Q})}. \tag{2.36}$$

Using Q and $\overline{Q}$ and the Baker-Campbell-Hausdorff identity

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots} \tag{2.37}$$

which if the higher order terms vanish reduces to

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}. \tag{2.38}$$

When the higher order terms are zero it can be shown that the coordinates transform as

$$x'^\mu = x^\mu + i\epsilon\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\epsilon}, \tag{2.39}$$

$$\theta' = \theta + \epsilon, \tag{2.40}$$

$$\bar{\theta}' = \bar{\theta} + \bar{\epsilon}. \tag{2.41}$$

These transformations show that the SUSY generators can be expressed in terms of superspace derivatives. So

$$Q_\alpha = i(\tfrac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu), \tag{2.42}$$

$$\overline{Q}_{\dot{\alpha}} = -i(\tfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu). \tag{2.43}$$

The most general expression for a superfield expanded in terms of the superspace coordinates and general component fields which are only dependent on x^μ is:

$$F(x,\theta,\bar{\theta}) = f(x)+\theta\phi(x)+\bar{\theta}\overline{\chi}(x)+\theta\theta m(x)+\bar{\theta}\bar{\theta}n(x)+\theta\sigma^\mu\bar{\theta}v(x)+\theta\theta\bar{\theta}\overline{\lambda}(x)+\bar{\theta}\bar{\theta}\theta\psi(x)+\theta\theta\bar{\theta}\bar{\theta}d(x), \tag{2.44}$$

since all higher powers of θ will disappear. In general superfield representations are highly reducible. By applying constraints to the superfield we can extra component fields by imposing covariant constraints.

2.4.4 The Chiral Superfield

From the earlier definitions of Q and $\overline{Q}$ we can also define a pair of covariant derivatives D and $\overline{D}$. By starting with $A(x,\theta,\bar{\theta})A(y,\xi,\bar{\xi})$ rather than $A(y,\xi,\bar{\xi})A(x,\theta,\bar{\theta})$ we get

$$D_\alpha = (\tfrac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu), \tag{2.45}$$

$$\overline{D}_{\dot{\alpha}} = -(\tfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu), \tag{2.46}$$

which implies that D and $\overline{D}$ obey the anticommutation relations

$$\{D_\alpha, \overline{D}_{\dot{\alpha}}\} = 2i(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu, \tag{2.47}$$

$$\{D_\alpha, D_\beta\} = 0, \tag{2.48}$$

$$\{\overline{D}_{\dot{\alpha}}, \overline{D}_{\dot{\beta}}\} = 0, \tag{2.49}$$

and

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \overline{Q}_{\dot{\alpha}}\} = \{\overline{D}_{\dot{\alpha}}, Q_\alpha\} = \{\overline{D}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0. \tag{2.50}$$

Using the covariant derivatives to impose a constraint on the field Φ we can define an irreducible representation as

$$\overline{D}^{\dot{\alpha}}\Phi = 0, \tag{2.51}$$

which is the chiral superfield. And similarly an antichiral field is one that satisfies

$$D^\alpha\overline{\Phi} = 0. \tag{2.52}$$

After applying this constraint the superfield reduces to

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + i\partial_\mu\phi(x)\theta\sigma^\mu\bar{\theta} + \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x), \tag{2.53}$$

These constraints are easier to solve in terms of a new coordinates system

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}, \tag{2.54}$$

(these are known as chiral coordinates). When expressed in terms of these coordinates the chiral superfield becomes

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x). \tag{2.55}$$

The same thing can be done for the antichiral superfield in terms of the coordinates

$$\overline{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}. \tag{2.56}$$

Under SUSY transformations the fields all transform into each other

$$\delta\phi = \sqrt{2}\epsilon\psi, \tag{2.57}$$

$$\delta\psi = i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu\phi + \sqrt{2}\epsilon F, \tag{2.58}$$

$$\delta F = i\sqrt{2}\bar{\epsilon}\sigma^\mu\partial_\mu\psi. \tag{2.59}$$

Here we can see that the F field transforms as a total derivative, i.e. δF vanishes when integrated over the spacetime. As can easily be seen from these definitions of Φ and $\overline{\Phi}$ the products of chiral (antichiral) superfields, Φ^2, Φ^3 etc. are also chiral (antichiral) superfields as they still only depend on θ ($\bar{\theta}$). However the product $\Phi^\dagger\Phi$ is a general superfield.

2.4.5 The Vector Superfield

As well as the chiral superfield whose components contain scalar and spinor fields which are used to represent the fermionic fields and their superpartners, we also need a superfield which will allow us to construct gauge invariant interactions. The way to do this is to define a real vector superfield i.e. a superfield defined by the constraint

$$V = V^{\dagger}. \tag{2.60}$$

It is a general superfield (not chiral) and has the following expansion:

$$\begin{aligned} V(x,\theta,\bar{\theta}) = {} & C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{2}\theta\theta[M(x) + iN(x)] \\ & - \frac{i}{2}\bar{\theta}\bar{\theta}[M(x) - iN(x)] - \theta\sigma^{\mu}\bar{\theta}A_{\mu}(x) \\ & + i\theta\theta\bar{\theta}[\lambda(x) + \frac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\chi(x)] - i\bar{\theta}\bar{\theta}\theta[\lambda(x) + \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\chi}(x)] \\ & + i\theta\theta\bar{\theta}\bar{\theta}[D(x) + \frac{1}{2}\Box C(x)]. \end{aligned} \tag{2.61}$$

Here χ, λ are spinors A_{μ} is a real vector and C, M, N and D are real scalars. Under the abelian gauge transformation the V transforms in the following way:

$$V \to V + \Phi + \Phi^{\dagger}. \tag{2.62}$$

Here Φ and $\Phi^{\dagger}$ are both chiral superfields. In components

$$C \to C + \phi + \phi^{*}, \tag{2.63}$$

$$\chi \to \chi - i\sqrt{2}\psi, \tag{2.64}$$

$$M + iN \to M + iN - 2iF, \tag{2.65}$$

$$A_{\mu} \to A_{\mu} - i\partial_{\mu}(\phi - \phi^{*}), \tag{2.66}$$

$$\lambda \to \lambda, \tag{2.67}$$

$$D \to D. \tag{2.68}$$

The physical degrees of freedom for V are the gauge field A_μ and the Majorana spinor field λ (commonly referred to as the gaugino). All of the other fields are unphysical and so this representation is still reducible. This can be changed by using additional constraints such as by using gauge-fixing where the unphysical fields can have their values set to zero. A common gauge-fixing condition is the Wess-Zumino gauge [12], where $C = \chi = M = N = 0$ which leaves the following expression for V

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu(x) + i\theta\theta\bar{\theta}\lambda(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + i\theta\theta\bar{\theta}\bar{\theta}D(x). \tag{2.69}$$

This makes it very easy to calculate powers of V

$$V^2 = -\theta\theta\bar{\theta}\bar{\theta}A_\mu(x)A^\mu(x), \tag{2.70}$$

$$V^n = 0, \tag{2.71}$$

for $n \geq 3$.

2.5 The Construction of SUSY Lagrangians

Using the superfield formalism we can construct supersymmetric Lagrangians out of chiral and antichiral superfields which are invariant under SUSY transformations. A general form of Lagrangian which only contains chiral and antichiral superfields is, when written in superspace,

$$L = \int d^4\theta(\sum_i \Phi_i^\dagger \Phi_i) + (\int d^2\theta W(\Phi) + \text{h.c.}), \tag{2.72}$$

with

$$\int d^2\theta = -\frac{1}{4}\int d\theta^\alpha d\theta_\alpha, \tag{2.73}$$

$$\int d^2\bar{\theta} = -\frac{1}{4}\int d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}^{\dot{\alpha}}, \tag{2.74}$$

$$\int d^4\theta = -\frac{1}{4}\int d^2\theta d^2\bar{\theta}, \tag{2.75}$$

where $W(\Phi)$ is the superpotential and h.c. stands for hermitian conjugate which contain the antichiral fields. The first term in the Lagrangian is the kinetic term and since the product of a chiral superfield with an antichiral superfield is a general superfield such products are not allowed in the superpotential because of the need for the superpotential to be holomorphic.

A typical superpotential is

$$W(\Phi) = g\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3. \tag{2.76}$$

When entered into the Lagrangian it is preceded by $\int d^2\theta$ which projects out the highest order components of the superpotential as these components always transform as a total derivative and so makes the action manifestly supersymmetric. If we ignore the superpotential and we expand in terms of the component fields we find

$$L = (\partial_\mu\bar{\phi})(\partial^\mu\phi) + i\bar{\psi}\partial\!\!\!/\psi + F^*F, \tag{2.77}$$

which is the Wess-Zumino model from before. The basic scalar Lagrangian is

$$L = \int d^4\theta\Phi_i^\dagger\Phi_i + \int d^2\theta(g_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}y_{ijk}\Phi_i\Phi_j\Phi_k) + \text{h.c.} \tag{2.78}$$

To obtain the Lagrangian in terms of on-shell component fields we need to expand the superfields in terms of their components and then eliminate the auxiliary fields using their equations of motion.

For a more realistic theory we must also include terms involving gauge fields and their superpartners. It needs gauge invariant interactions of the matter fields with the gauge fields as well as the kinetic and self-interaction terms for the gauge fields. To do this first we need a supersymmetric analog for the field strength tensor. For a general gauge group the supersymmetric field strengths are defined in terms of the vector superfield in the following way

$$W_\alpha = -\frac{1}{4}\overline{D}^2(e^V D_\alpha e^{-V}), \tag{2.79}$$

$$\overline{W}_{\dot\alpha} = -\frac{1}{4}D^2(e^V \overline{D}_{\dot\alpha} e^{-V}). \tag{2.80}$$

Here D and $\overline{D}$ are the covariant derivatives and from the Grassmann algebra we can see that the field strength tensors W_α and $\overline{W}_{\dot\alpha}$ are chiral and antichiral superfields respectively. The field strength W_α transforms in the following way

$$W_\alpha \rightarrow W'_\alpha = e^{-i\Lambda} W_\alpha e^{i\Lambda}, \tag{2.81}$$

where Λ is a chiral superfield. Just as before where the product $\Phi^\dagger\Phi$ is used to generate the kinetic terms for the scalar Lagrangian the products $W^\alpha W_\alpha$ and $\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}$ are used to generate the kinetic terms for the gauge fields. In the Wess-Zumino gauge we get

$$W^\alpha W_\alpha|_{\theta\theta} = -2i\lambda\sigma^\mu D_\mu\bar{\lambda} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^2 + i\frac{1}{4}F^{\mu\nu}F^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma}. \tag{2.82}$$

Here $D_\mu = \partial_\mu + ig[A_\mu,]$ is the usual Lie colour group covariant derivative. Using these terms along with their hermitian conjugates we get

$$\begin{aligned} L &= \frac{1}{4}\int d^2\theta W^\alpha W_\alpha + \frac{1}{4}\int d^2\bar{\theta}\overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}} \\ &= \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda}. \end{aligned} \tag{2.83}$$

To obtain a gauge invariant coupling with the chiral superfields we alter the chiral antichiral product to include the gauge fields

$$\Phi_i^\dagger\Phi_i|_{\theta\theta\bar{\theta}\bar{\theta}} \rightarrow \Phi_i^\dagger e^{gV}\Phi_i|_{\theta\theta\bar{\theta}\bar{\theta}}. \tag{2.84}$$

So the full gauge invariant supersymmetric Lagrangian has the form

$$\begin{aligned} L = \frac{1}{4}\int d^2\theta W^\alpha W_\alpha + \frac{1}{4}\int d^2\bar{\theta}\overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}} + \int d^4\theta\Phi_i^\dagger e^{gV}\Phi_i \\ + \int d^2\theta(\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}y_{ijk}\Phi_i\Phi_j\Phi_k) + \text{h.c.} \end{aligned} \tag{2.85}$$

The linear Φ term is not included as it is not gauge invariant. As we can see the form of a supersymmetric Lagrangian is heavily restricted by all of the symmetry requirements. The only real freedom is the field content, the values of the couplings and the masses. From the way the superfields are defined all the components of a supermultiplet should have the same masses. Therefore the superpartners should have the same masses as the standard particles, however since this has not been observed supersymmetry must be a broken symmetry. The three dimensional $\mathcal{N} = 2$ theory is obtained from the four dimensional $\mathcal{N} = 1$ theory using dimensional reduction [15].

3 Renormalisation

3.1 Motivation for Renormalisation

When a process is being calculated in Quantum Field Theory (QFT) there is a problem: when trying to calculate an amplitude for a specific process there is often no way to find an exact solution and so perturbation theory must be used. Unfortunately it was soon discovered that when using perturbation theory in QFT one encounters divergent (infinite) results which are nonsensical. The root of this problem is that when we calculate the amplitude for a particular process we must sum all the possible ways in which the process can occur and integrate over all momenta for the intermediate (unobservable) particles involved. Over a period of many years a procedure has been developed to systematically remove these divergent results which is called renormalisation. Not all QFTs can be renormalised however QFTs that have tried to incorporate gravity for example are non-renormalisable which is one of the main reasons that no definitive fully experimentally tested quantum theory of gravity has ever been formulated. One of the criteria for renormalisability is the mass dimension of the couplings of the theory, if the lowest dimension coupling has a mass dimension of 0 or higher then the theory may be renormalisable, if it has a mass dimension which is lower than zero then the theory will be non-renormalisable.

3.2 Renormalisation Procedure

There are two ways that divergent results can be found when renormalising a theory: Infrared divergences which occur when the integral results in a momentum term of highest power appearing in the denominator which, as the momentum tends to zero, sends the integrand to infinity and ultraviolet divergences which result from having a higher momentum term in the numerator which tends to infinity as the momentum tends to infinity. We have only considered ultraviolet divergences in this work.

Before renormalisation is performed we must first determine what the divergences are. There are several different methods for determining divergences the most common of which is dimensional regularisation.

3.2.1 Dimensional Regularisation

While the integrals may be divergent in four (or whatever number of spacetime dimensions is being looked at) the integral will not be divergent in an arbitrary number of spacetime dimensions and so this procedure looks at the integral in d dimensions and then once the potentially divergent terms have been removed allows the number of spacetime dimensions to tend toward the desired number of spacetime dimensions and so allows the result to continue analytically back to the desired case.

Starting from the Lagrangian of the most basic interacting QFT, the scalar ϕ^4 theory we have

$$L = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4, \tag{3.1}$$

where the ϕ^4 term shows the scalar field interacting with λ the coupling constant. From this equation we can calculate a set of rules for constructing Feynman diagrams. Feynman diagrams are used to represent the terms that we get from the perturbative expansion of the Lagrangian where each diagram represents an integral which can be determined by the Feynman rules. The diagrams have three components, external lines denoting physical, observable particles, internal lines denoting unobservable 'virtual particles' whose contribution to the amplitude is dictated by propagators, one for each internal line, and vertices, which are determined by the interaction terms in the Lagrangian. As can easily be seen for the ϕ^4 theory we have only the one quartic self-interaction term and so all interacting diagrams can only contain four point vertices for example.

Initially the terms in the Lagrangian are assumed to be 'bare' (i.e divergent) terms i.e

$$L = \frac{1}{2}\partial^\mu \phi_0 \partial_\mu \phi_0 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{1}{4!}\lambda_0 \phi_0^4, \tag{3.2}$$

and after renormalisation they are thought of as physical quantities. The way that the two are related is by the renormalisation constants (Z's) which are chosen so that they rescale the bare quantities to remove the divergences. So for the ϕ^4 theory

$$\phi_0 = (Z_\phi)^{\frac{1}{2}}\phi, \tag{3.3}$$

$$m_0 = Z_m m, \tag{3.4}$$

$$\lambda_0 = Z_\lambda \lambda. \tag{3.5}$$

There are several ways to do this (known as schemes) one of which is to choose the Zs so that only the divergent parts of the calculation are removed by leaving the finite parts alone. This is called the "minimal subtraction scheme" (denoted by MS) [16]. Normally the pole terms generated in perturbation theory are also accompanied by constant terms involving γ, the Euler-Mascheroni constant, and $\log(4\pi)$. Another scheme, called the "modified minimal subtraction scheme", (denoted by $\overline{\text{MS}}$) eliminates these constant terms as well. Another way to renormalise a theory is to calculate the counterterms as one is performing a calculation diagram by diagram, which involves picking out divergent subdiagrams and replacing those parts in the diagram with a counterterm. This is a faster process for performing specific calculations as only counterterms specific to the calculation being carried out are necessary.

As well as renormalising a theory so that it gives finite results certain quantities which can describe various properties of the theory can also be calculated in terms of Feynman diagrams. Two such quantities are the anomalous dimension, γ, and the β-function which show the dependence on the renormalisation scale μ of the field normalisation and the shift in the coupling constant (every coupling constant has its own β-function) and are defined by

$$\gamma = \frac{1}{2}\frac{\mu}{Z}\frac{\partial Z}{\partial \mu} = \frac{1}{2}\mu\frac{\partial lnZ}{\partial \mu} = \frac{1}{2}\frac{\partial lnZ}{\partial ln\mu},$$
$$\beta_\lambda = \mu\frac{\partial \lambda}{\partial \mu}. \tag{3.6}$$

The β-function is very useful as it describes the behaviour of a running coupling constant with respect to changes in the energy scale. If the β-function has a positive sign this indicates that the strength of the coupling constant is greater at higher energies. If it has a negative sign this indicates that the coupling constant gets weaker at higher energies. This latter property has been observed in QCD making the theory “asymptotically free” which consequently allows the use of perturbation theory at high energies [17], [18].

3.3 Superfield Perturbation Theory

A disadvantage of Dimension Regularisation (DREG) with respect to supersymmetric theories is that it does not preserve the symmetry between the fermions and bosons. This is due to the number of gauge fields being equal to the dimension of the integral, when the integral is taken to have an arbitrary dimension so is the number of gauge fields. This changes the bosonic degrees of freedom but not the fermionic degrees of freedom and thus breaks the fermion boson symmetry. However another method of renormalisation exists which is commonly used for supersymmetric theories which is called Dimensional Reduction (DRED) [19] which does preserve supersymmetry via the use of supergraphs. DRED was developed specifically by attempting to modify DREG so that it would be compatible with supersymmetry and therefore preserve the boson/fermion symmetry. The essential difference between the two methods is that in DRED the continuation from 4 to d dimensions is made by dimensional reduction so that while the momentum integrals are d-dimensional, just like in DREG, the number of field components is unchanged and so supersymmetry is preserved.

3.3.1 Supergraphs

Supergraphs [20] are very similar to standard Feynman diagrams, the main difference being that in a Feynman diagram each line represents one particular field; in a supergraph each line represents a superfield and so represents several component fields at the same time thus often dramatically reducing the number of diagrams that need to be calculated. However as superfield propagators typically involve the covariant derivatives and the superspace Lagrangian also includes superspace integrals there is a certain amount of algebra that needs to be solved before the normal spacetime integral can be performed. So firstly all the θ integrals are performed. This is achieved by moving the Ds and $\overline{D}$s around the diagram (typically by using integration by parts to move them from one vertex to another) and eliminating them using their anticommutation relations until there are only two D's and two $\overline{D}$'s on each loop. They can also be moved from one end of the propagator to the other using the formula

$$D_\alpha(p,\theta)\delta^4(\theta-\theta') = -\delta^4(\theta-\theta')D_\alpha(-p,\theta). \tag{3.7}$$

Once the required number of D's has been obtained we can use the identities

$$\delta^4(\theta-\theta')X\delta^4(\theta-\theta') = 0, \tag{3.8}$$

for

$$X = 1, D_\alpha, \overline{D}_{\dot\alpha}, D^2, \overline{D}^2, D^2\overline{D}_{\dot\alpha}, \overline{D}^2 D_\alpha \tag{3.9}$$

and

$$\delta^4(\theta-\theta')X\delta^4(\theta-\theta')=\delta^4(\theta-\theta'), \tag{3.10}$$

if

$$X=D^2\overline{D}^2,\overline{D}^2D^2,D^\alpha\overline{D}^2D_\alpha,\overline{D}_{\dot\alpha}D^2\overline{D}^{\dot\alpha}, \tag{3.11}$$

to solve all but one of the θ integrals, which leaves only a "normal" Feynman integral. If there are fewer than two D or $\overline{D}$ terms then the diagram is zero. In four dimensions the standard relations are:

$$[D_\alpha,\overline{D}^2]=p_{\alpha\dot\alpha}\overline{D}^{\dot\alpha}, \tag{3.12}$$

$$[\overline{D}_{\dot\alpha},D^2]=-p_{\alpha\dot\alpha}D^\alpha, \tag{3.13}$$

where $p_{\alpha\dot\alpha}=i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$. From these relations it is straightforward to derive the following relations:

$$D^2\overline{D}^2D^2=-\partial_\mu\partial^\mu D^2, \tag{3.14}$$

$$D^\alpha\overline{D}^2D_\alpha=\overline{D}_{\dot\alpha}D^2\overline{D}^{\dot\alpha}, \tag{3.15}$$

$$D^2\overline{D}^2+\overline{D}^2D^2-2D^\alpha\overline{D}^2D_\alpha=-\partial_\mu\partial^\mu. \tag{3.16}$$

3.4 Conventions in $\mathcal{N}=2$ Supersymmetric Chern-Simons

In $\mathcal{N}=2$ Supersymmetric Chern-Simons in three dimensions the following superspace and supersymmetry conventions apply. We use a metric signature $(+,-,-)$ so that a possible choice of γ matrices is $\gamma^0=\sigma_2$, $\gamma^1=i\sigma_3$, $\gamma^2=i\sigma_1$ with, for instance

$$(\gamma^0)_\alpha{}^\beta=(\sigma_2)_\alpha{}^\beta. \tag{3.17}$$

We then have

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} - i\epsilon^{\mu\nu\rho}\gamma_\rho, \tag{3.18}$$

where $\epsilon^{\mu\nu\rho}$ is the antisymmetric tensor with $\epsilon^{012} = 1$. We have [21] two complex two-spinors θ^α and $\bar{\theta}^\alpha$ with indices raised and lowered according to

$$\theta^\alpha = C^{\alpha\beta}\theta_\beta, \quad \theta_\alpha = \theta^\beta C_{\beta\alpha}, \tag{3.19}$$

with $C^{12} = -C_{12} = i$. We then have

$$\theta_\alpha \theta_\beta = C_{\beta\alpha}\theta^2, \quad \theta^\alpha \theta^\beta = C^{\beta\alpha}\theta^2, \tag{3.20}$$

where

$$\theta^2 = \frac{1}{2}\theta^\alpha \theta_\alpha. \tag{3.21}$$

The supercovariant derivatives are defined by

$$D_\alpha = \partial_\alpha + \tfrac{i}{2}\bar{\theta}^\beta \partial_{\alpha\beta}, \tag{3.22}$$

$$\overline{D}_\alpha = \overline{\partial}_\alpha + \tfrac{i}{2}\theta^\beta \partial_{\alpha\beta}, \tag{3.23}$$

where

$$\partial_{\alpha\beta} = \partial_\mu (\gamma^\mu)_{\alpha\beta}, \tag{3.24}$$

satisfying

$$\{D_\alpha, \overline{D}_\beta\} = i\partial_{\alpha\beta}. \tag{3.25}$$

We also define

$$d^2\theta = \frac{1}{2}d\theta^\alpha d\theta_\alpha, \quad d^2\bar{\theta} = \frac{1}{2}d\bar{\theta}^\alpha d\bar{\theta}_\alpha, \quad d^4\theta = d^2\theta d^2\bar{\theta}, \tag{3.26}$$

so that

$$\int d^2\theta\theta^2 = \int d^2\bar{\theta}\bar{\theta}^2 = -1. \tag{3.27}$$

The vector superfield $V(x, \theta, \bar{\theta})$ is expanded in Wess-Zumino gauge as

$$V = i\theta^\alpha\bar{\theta}_\alpha\sigma + \theta^\alpha\bar{\theta}^\beta A_{\alpha\beta} - \theta^2\bar{\theta}^\alpha\bar{\lambda}_\alpha - \bar{\theta}^2\theta^\alpha\lambda_\alpha + \theta^2\bar{\theta}^2 D, \tag{3.28}$$

and the chiral field is expanded as

$$\Phi = \phi(y) + \theta^\alpha\psi_\alpha(y) - \theta^2 F(y), \tag{3.29}$$

where

$$y^\mu = x^\mu + i\theta\gamma^\mu\bar{\theta}. \tag{3.30}$$

Using the conventions from the $\mathcal{N} = 2$ theory in three dimensions that we will be looking at these relations are modified due to the chosen conventions eliminating the need for dotted indices and so now

$$[D_\alpha, \overline{D}^2] = p_{\alpha\beta}\overline{D}^\beta, \tag{3.31}$$

$$[\overline{D}_\alpha, D^2] = -p_{\alpha\beta}D^\beta, \tag{3.32}$$

$$\{D_\alpha, \overline{D}_\beta\} = 2ip_{\alpha\beta}, \tag{3.33}$$

which if D and $\overline{D}$ have the same index means that $\{D^\alpha, \overline{D}_\alpha\} = 2ip^\alpha{}_\alpha = 0$ since the Pauli matrices are traceless. This property simplifies the D-algebra significantly.

4 Two-Loop Renormalisation of $\mathcal{N} = 2$ Softly Broken Chern-Simons Matter Theories

4.1 Chern-Simons Theory

Chern-Simons gauge theories have attracted attention for a considerable time due to their topological nature [22–24] (in the pure gauge case) and their possible relation to the quantum Hall effect, fractional quantum Hall effect [25] and high-T_c superconductivity. Another area of interest has been the "Anyon models" where instead of composite particles which follow either Bose-Einstein or Fermi-Dirac statistics there are composite particles which obey any intermediate statistics or "q-statistics". In these models the gauge field from which the Chern-Simons term is composed may be regarded as the q-statistics inducing field. Another area where there has been substantial interest is in $\mathcal{N} = 2$ supersymmetric Chern-Simons matter theories in the context of the AdS/CFT correspondence (see Refs. [15, 21, 26] for details and a comprehensive list of references). Most of the AdS/CFT correspondence related theories are motivated from string theory where a 4-dimensional $\mathcal{N} = 1$ theory undergoes a process known as dimensional reduction which converts it from a 4-dimensional $\mathcal{N} = 1$ theory to a 3-dimensional $\mathcal{N} = 2$ theory. The main two models are the BLG (Bagger, Lambert, Gustavsson) model [27] and the ABJ/ABJM (Aharony, Bergman, Jafferis, Maldacena) [28, 29] models. The ABJ and ABJM models will be looked at in more depth in chapter 6.

4.2 Two-Loop Renormalisation

It is already well-known that the β-functions for the soft-breaking parameters in softly-broken $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions may be expressed exactly in terms of the anomalous dimensions and gauge β-function for the unbroken theory. (See

Ref. [30] for a complete description of the most general case.) Moreover this leads [31] to exact renormalisation group invariant solutions for the soft-breaking parameters–the “anomaly-mediated supersymmetry-breaking” (AMSB) solutions [32,33]. Renormalisation group invariant in this context meaning a solution for the soft couplings in terms of the regular couplings and constant parameters which do not change under variations of the scale μ. Here we show that similar results hold for $\mathcal{N}=2$ Chern-Simons matter theories in three dimensions; indeed the results are simpler due to the absence of a gauge coupling (which reflects the topological nature of the gauge part of the theory).

Our results are based on a set of rules devised by Yamada [34] for obtaining the β-functions for the scalar soft-breaking couplings (in four dimensions) starting from the anomalous dimension for the chiral superfields. We shall present here an abridged derivation based on Ref. [35]; see Ref. [30] for the complete version. Yamada's rules are based on the spurion formalism [36], which enables one to write the softly broken $N=2$ theory in terms of superfields. The Lagrangian for the theory can be written

$$L = L_{SUSY} + L_{SB} + L_{GF} + L_{FP}, \tag{4.1}$$

where L_{SUSY} is the usual $N=2$ supersymmetric Lagrangian [37],

$$\begin{aligned} L_{SUSY} = &\int d^4\theta \left(2k\int_0^1 dt\,\mathrm{Tr}[\overline{D}^\alpha(e^{-tV}D_\alpha e^{tV})] + \Phi^j(e^{V_A R_A})^i{}_j\Phi_i\right) \\ &+ \left(\int d^2\theta W(\Phi) + \text{h.c.}\right), \end{aligned} \tag{4.2}$$

where V is the vector superfield, Φ the chiral matter superfield and where the superpotential $W(\Phi)$ is given by

$$W(\Phi) = \frac{1}{4!}Y^{ijkl}\Phi_i\Phi_j\Phi_k\Phi_l + \frac{1}{3!}Z^{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{2!}\mu^{ij}\Phi_i\Phi_j. \tag{4.3}$$

(We use the convention that $\Phi^i = (\Phi_i)^*$.) We assume a simple gauge group; a gauge group with a $U(1)$ factor could also include a linear term in the superpotential. Gauge invariance requires the gauge coupling k to be quantised, so that $2\pi k$ is an integer. The vector superfield V is in the adjoint representation, $V = V_A T_A$ where T_A are the generators of the fundamental representation, satisfying

$$[T_A, T_B] = \quad i f_{ABC} T_C, \tag{4.4}$$

$$\mathrm{Tr}(T_A T_B) = \quad \delta_{AB}. \tag{4.5}$$

Note that this choice of convention differs from those used in Ref. [38] to make the conventions consistent throughout this thesis. The chiral superfield can be in a general representation, with gauge matrices denoted R_A satisfying

$$[R_A, R_B] = \quad i f_{ABC} R_C, \tag{4.6}$$

$$\mathrm{Tr}(R_A R_B) = \quad T(R)\delta_{AB}. \tag{4.7}$$

In three dimensions the Yukawa couplings Y^{ijkl} are dimensionless and the theory is renormalisable. The soft breaking part L_{SB} may be written [39]

$$\begin{aligned} L_{SB} = & \int d^2\theta\eta \left(\frac{1}{4!} h^{ijkl}\Phi_i\Phi_j\Phi_k\Phi_l + \frac{1}{3!} g^{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{2!} b^{ij}\Phi_i\Phi_j + \text{h.c.} \right) \\ & - \int d^4\theta\eta^*\eta\Phi^j (m^2)^i{}_j (e^{V_A R_A})_i{}^k \Phi_k, \end{aligned} \tag{4.8}$$

where $\eta = \theta^2$ is the spurion external field. For convenience we set b^{ij} and g^{ijk} to zero. Note that in three dimensions there is no soft term corresponding to the four-dimensional gaugino mass term. The gauge-fixing and Fadeev-Popov terms are contained in L_{GF} and L_{FP} respectively. It is convenient to introduce a generalised form γ_η of the anomalous dimension γ of the chiral supermultiplet, given by:

$$\gamma_\eta = \gamma + \gamma_1\eta + \gamma_1^\dagger\eta^* + \gamma_2\eta^*\eta. \tag{4.9}$$

It was shown by Yamada [34] that $(\gamma_\eta)^i{}_j$ could be obtained from $(\gamma)^i{}_j$ by the following rules (simpler in three than in four dimensions due to the absence of a running gauge coupling):

1. Replace Y^{lmno} by $Y^{lmno} - h^{lmno}\eta$. (Additional terms for b and g are omitted as we set them to zero).

2. Insert $\delta^{l'}{}_l + (m^2)^{l'}{}_l\eta^*\eta$ between contracted indices l and l' in Y and Y^*, respectively: $Y^{lmno}Y_{lm'n'o'} \to Y^{lmno}Y_{lm'n'o'} + Y^{lmno}(m^2)^{l'}{}_l Y_{l'm'n'o'}\eta^*\eta$ (where, here and subsequently, $Y_{lmno} = (Y^{lmno})^*$).

3. Replace a term $T^i{}_j$ in $\gamma^i{}_j$ with no Yukawa couplings by $T^i{}_j - (m^2)^i{}_k T^k{}_j\eta^*\eta$.

γ_1 and γ_2 may then be obtained by extracting the coefficients of η and $\eta^*\eta$ respectively. In the case of γ_1, the above rules can be subsumed by the simple relation

$$(\gamma_1)^i{}_j = \mathcal{O}\gamma^i{}_j, \tag{4.10}$$

where

$$\mathcal{O} = -h^{lmno}\frac{\partial}{\partial Y^{lmno}}. \tag{4.11}$$

It is straightforward to show that

$$\beta_h^{ijkl} = \gamma^{(i}{}_m h^{jkl)m} - 2\gamma_1^{(i}{}_m Y^{jkl)m}. \tag{4.12}$$

This result is similar in form to the standard result for β_Y which follows from the non-renormalisation theorem [20] (which is valid for $\mathcal{N} = 2$ supersymmetric theories in three dimensions [21]), namely

$$\beta_Y^{ijkl} = \gamma^{(i}{}_m Y^{jkl)m}. \tag{4.13}$$

It also follows from Eqs. (4.8) and (4.9) that

$$(\beta_{m^2})^i{}_j = \frac{1}{2}\gamma^i{}_k (m^2)^k{}_j + \frac{1}{2}(m^2)^i{}_k \gamma^k{}_j + \gamma^i_{2j}, \tag{4.14}$$

which we may write using Yamada's rules as

$$(\beta_{m^2})^i{}_j = \left[2\mathcal{O}\mathcal{O}^* + \tilde{Y}_{lmn}\frac{\partial}{\partial Y_{lmn}} + \tilde{Y}^{lmn}\frac{\partial}{\partial Y^{lmn}}\right]\gamma^i{}_j, \tag{4.15}$$

where

$$\tilde{Y}^{ijkl} = (m^2)^i{}_m Y^{mjkl} + (m^2)^j{}_m Y^{imkl} + (m^2)^k{}_m Y^{ijml} + (m^2)^l{}_m Y^{ijkm}. \tag{4.16}$$

The exact results Eqs. (4.12) and (4.15) for the β-functions lead to exact renormalisation group invariant solutions for the soft-breaking couplings, namely

$$h^{ijkl} = -M_0 \beta_Y^{ijkl}, \tag{4.17}$$

$$(m^2)^i{}_j = \frac{1}{2}|M_0|^2 \mu \frac{d\gamma^i{}_j}{d\mu}, \tag{4.18}$$

where M_0 is a constant mass. These results can be proved following the four-dimensional discussion in Ref. [31] (though the terms with $\kappa_{1,2}$ were given for the first time in Ref. [40]); but once more the details are simpler due to the non-running of the gauge coupling. We note that in the case of a gauge group with a $U(1)$ factor and a linear term in the superpotential, additional terms are expected [30] in the expressions for β_g and β_b (which for us are zero), and thence corresponding extra terms in Eqs. (4.18); there should also be an exact expression for the β-function corresponding to the linear soft coupling, and an exact RGinvariant solution for this coupling. There is also potentially an additional term [41] in the solution for m^2 corresponding to the possible Fayet-Iliopoulos term.

We now turn to our check of the results Eqs. (4.12) and (4.15) up to two loops using the component formulation of the theory (there are no divergences at odd loop orders for a theory in odd dimensions, so this is the simplest non-trivial check). The first ingredient is the anomalous dimension of the chiral superfield, which is given at two loops by

$$64\pi^2\gamma^{(2)} = \frac{1}{3}Y_2 - 2k^{-2}C_2(R)C_2(R) - k^{-2}T(R)C_2(R) + k^{-2}C_2(G)C_2(R), \tag{4.19}$$

where

$$(Y_2)^i{}_j = Y^{iklm}Y_{jklm}, \tag{4.20}$$

$$C_2(R) = R_A R_A, \tag{4.21}$$

$$C_2(G)\delta_{AB} = f_{ACD}f_{BCD}, \tag{4.22}$$

and $T(R)$ is defined in Eq. (4.7). This result may readily be obtained by $\mathcal{N} = 2$ superfield methods [21, 26, 39, 42]; see the for the $\mathcal{N} = 2$ superfield conventions.

An expression for the two-loop anomalous dimension for an $\mathcal{N} = 1$ theory in three dimensions (with no Yukawa coupling) is given in Ref. [43]. This does not agree with the k^{-2} terms in Eq. (5.19) when specialised to the $\mathcal{N} = 2$ case. Presumably this is because the result is in general gauge-dependent and the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ Feynman gauges are not equivalent. Since $\mathcal{N} = 2$ supersymmetry is not manifest in the $\mathcal{N} = 1$ formalism, one would not expect Eq. (4.13) to be valid using the anomalous dimension computed using the $\mathcal{N} = 1$ formalism. We have however checked explicitly via a component calculation that the β function for the Yukawa coupling is indeed given by Eq. (4.13) with the anomalous dimension of Eq. (5.19).

We then find from Eq. (4.10) that

$$64\pi^2(\gamma_1^{(2)})^i{}_j = -\frac{1}{3}h^{ilmn}Y_{jlmn} \tag{4.23}$$

and that therefore (using Eq. (4.12))

$$\begin{aligned} 64\pi^2\beta_h^{ijkl(2)} &= [\frac{1}{3}Y_2 - 2k^{-2}C_2(R)C_2(R) - k^{-2}T(R)C_2(R) + k^{-2}C_2(G)C_2(R)]^i{}_m h^{mjkl} \\ &+ \frac{2}{3}h^{ilmn}Y_{plmn}Y^{pjkl} + \text{cyclic perms.} \end{aligned} \tag{4.24}$$

We also find from Eq. (4.15) that

$$64\pi^2(\beta_{m^2})^i{}_j = \frac{2}{3}h^{iklm}h_{jklm} + \frac{1}{3}(m^2)^i{}_k(Y_2)^k{}_j + \frac{1}{3}(Y_2)^i{}_k(m^2)^k{}_j + 2Y^{iklm}(m^2)^{k'}{}_kY_{jk'lm}.$$

It is straightforward to verify these results by a component calculation. The supersymmetric Lagrangian is given in components by [44]

$$L_{SUSY} = L_{CS} + L_m, \tag{4.25}$$

$$L_{CS} = 2k\,\mathrm{Tr}[\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho) - \bar{\lambda}\lambda + 2D\sigma], \tag{4.26}$$

$$\begin{aligned} L_m &= D_\mu\phi^i D^\mu\phi_i + i\overline{\psi}^i\gamma^\mu D_\mu\psi_i + F^iF_i \\ &- \phi^i\sigma^2\phi_i + \phi^iD\phi_i + i\phi^\dagger\bar{\lambda}\psi - i\overline{\psi}\lambda\phi \\ &+ \left(\frac{1}{3!}Y^{ijkl}\phi_i\phi_j\phi_kF_l + \frac{1}{4}Y^{ijkl}\phi_i\phi_j\overline{\psi}_k\psi_l + \text{h.c.}\right), \end{aligned} \tag{4.27}$$

where λ and ψ are two-component Dirac spinors, $\bar{\lambda} = \lambda^\dagger\gamma_0$, $D_\mu = \partial_\mu + iA_\mu$ and we have set $\mu^{ij} = Z^{ijk} = 0$ for simplicity, in order to focus on the dimensionless couplings. After eliminating the auxiliary fields D, σ we obtain

$$L_{CS} = 2k\,\mathrm{Tr}[\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho)], \tag{4.28}$$

$$\begin{aligned} L_m = {} & D_\mu\phi^i D^\mu\phi_i + i\overline{\psi}^i\gamma^\mu D_\mu\psi_i - (\phi^\dagger R_A\phi)(\phi^* R_B\phi)(\phi^* R_A R_B\phi) \\ & + (\phi^* R_A\phi)(\overline{\psi}^* R_A\psi) + 2(\overline{\psi}^* R_A\phi)(\phi^* R_A\psi) \\ & - \frac{1}{(3!)^2}Y^{ijkn}Y_{i'j'k'n}\phi_i\phi_j\phi_k\phi^{i'}\phi^{j'}\phi^{k'} + (\frac{1}{4}Y^{ijkl}\phi_i\phi_j\overline{\psi}_k\psi_l + \text{h.c.}). \end{aligned} \tag{4.29}$$

The soft-breaking Lagrangian is given by

$$L_{SB} = -\left(\frac{1}{4!}h^{ijkl}\phi_i\phi_j\phi_k\phi_l + \text{h.c.}\right) - (m^2)^i{}_j\phi_i\phi^j, \tag{4.30}$$

where we have set $b^{ij} = g^{ijk} = 0$.

The diagrams contributing to the anomalous dimension of the scalar component field ϕ at two loops are depicted in Fig. 1, with scalar, fermion, gauge and ghost propagators denoted by dashed, unbroken, wavy and dotted lines respectively. We work in a standard Feynman gauge in components which gives us the following scalar, fermion and gauge propagators

$$\Delta_S = \frac{1}{k^2}, \qquad \Delta_F = \frac{k_\mu\gamma^\mu}{k^2}, \qquad \Delta_V = \frac{i\epsilon^{\mu\nu\rho}k_\rho}{k^2}. \tag{4.31}$$

The divergent contributions from the diagrams in Fig. 1 to $\partial_\mu\phi^*\partial^\mu\phi$ are given by (using dimensional regularisation and working in $d = 3-\epsilon$ dimensions)

$$L\Gamma^{(2)}_{\phi(a)} = \frac{1}{3}Y_2 + \frac{1}{6}k^{-2}[4C_2(R) - 2C_2(G) + 5T(R)]C_2(R), \tag{4.32}$$

$$L\Gamma^{(2)}_{\phi(b)} = \frac{1}{12}k^{-2}[-4C_2(R) + C_2(G)]C_2(R), \tag{4.33}$$

$$L\Gamma^{(2)}_{\phi(c)} = -\frac{2}{3}k^{-2}T(R)C_2(R), \tag{4.34}$$

$$L\Gamma^{(2)}_{\phi(d)} = -\frac{2}{3}k^{-2}T(R)C_2(R), \tag{4.35}$$

$$L\Gamma^{(2)}_{\phi(e)} = \frac{1}{6}k^{-2}C_2(G)C_2(R), \tag{4.36}$$

$$L\Gamma^{(2)}_{\phi(f)} = -\frac{1}{6}k^{-2}C_2(G)C_2(R), \tag{4.37}$$

$$L\Gamma^{(2)}_{\phi(g)} = \frac{2}{3}k^{-2}[-2C_2(R) + C_2(G)]C_2(R), \tag{4.38}$$

$$L\Gamma^{(2)}_{\phi(h)} = \frac{1}{3}k^{-2}C_2(G)C_2(R), \tag{4.39}$$

where $L = 64\pi^2\epsilon$, leading to

$$\gamma^{(2)}_{\phi} = \frac{1}{3}Y_2 - k^{-2}C_2(R)C_2(R) - \frac{1}{2}k^{-2}T(R)C_2(R) + \frac{3}{4}k^{-2}C_2(G)C_2(R), \tag{4.40}$$

which agrees (up to an overall factor of 4, whose origin we have not been able to identify) with the component-field calculation in Ref. [43], when the relevant result is specialised to the case of $\mathcal{N} = 2$ supersymmetry. Note that since there are no simple poles at one loop, there are no double poles at two loops and no need to consider diagrams with counterterm insertions at this order. The list of diagrams contributing to β_h and β_{m^2} can be shortened by noting that any logarithmically divergent diagram where an external scalar emerges from a $\phi^* A\phi$ vertex is zero by symmetry, due to the form of the gauge propagator (see Eq. (4.31)). The diagrams contributing to the two-loop β functions for m^2 and h are shown in Figs. 2 and 3 respectively. They yield divergent contributions to the effective action given by

$$\begin{aligned} L\Gamma^{(2)}_{m^2(a)} =& \{Y^{iklm}(m^2)^{k'}{}_k Y_{jk'lm} + \frac{1}{2}k^{-2}[4C_2(R) - 2C_2(G) + T(R)]C_2(R)(m^2)^i{}_j\}\phi_i\phi^j \\ &+ 2k^{-2}\mathrm{tr}[R_A R_B m^2]\phi^* R_A R_B\phi, \end{aligned} \tag{4.41}$$

$$L\Gamma^{(2)}_{m^2(b)} = -\frac{1}{4}k^{-2}[4C_2(R) - C_2(G)]C_2(R)\phi^* m^2\phi, \tag{4.42}$$

$$L\Gamma^{(2)}_{m^2(c)} = -2k^{-2}\mathrm{tr}[R_A R_B m^2]\phi^* R_A R_B\phi, \tag{4.43}$$

$$L\Gamma^{(2)}_{m^2(d)} = \frac{1}{3}h^{iklm}h_{jklm}\phi_i\phi^j, \tag{4.44}$$

and

$$L\Gamma^{(2)}_{h(a)} = \frac{1}{4}k^{-2}[h^{ijmn}(R_AR_B)^k{}_m(R_AR_B)^l{}_n - \frac{1}{12}h^{ijkm}[C_2(G)C_2(R)]^l{}_m]\phi_i\phi_j\phi_k\phi_l, \tag{4.45}$$

$$L\Gamma^{(2)}_{h(b)} = \frac{1}{4}k^{-2}[-2h^{ijmn}(R_AR_B)^k{}_m(R_AR_B)^l{}_n + \frac{1}{6}h^{ijkm}\{4C_2(R)C_2(R) + T(R)C_2(R)\}^l{}_m]\phi_i\phi_j\phi_k\phi_l, \tag{4.46}$$

$$L\Gamma^{(2)}_{h(c)} = \{\frac{1}{3}h^{ilmn}Y_{plmn}Y^{pjkl} + \frac{1}{4}k^{-2}h^{ijmn}(R_AR_B)^k{}_m(R_AR_B)^l{}_n - \frac{1}{12}k^{-2}h^{ijkm}[C_2(R)C_2(R)]^l{}_m\}\phi_i\phi_j\phi_k\phi_l. \tag{4.47}$$

These add to

$$L\Gamma^{(2)}_{m^2} = \{Y^{iklm}(m^2)^{k'}{}_kY_{jk'lm} + \frac{1}{3}h^{iklm}h_{jklm} + \frac{1}{4}k^{-2}[4C_2(R) - 3C_2(G) + 2T(R)]C_2(R)(m^2)^i{}_j\}\phi_i\phi^j \tag{4.48}$$

and

$$L\Gamma^{(2)}_h = \frac{1}{6}\{\frac{1}{3}h^{iqmn}Y_{pqmn}Y^{pjkl} - \frac{1}{8}k^{-2}h^{ijkm}[2T(R)C_2(R) + 4C_2(R)C_2(R) - C_2(G)C_2(R)]^l{}_m\}\phi_i\phi_j\phi_k\phi_l. \tag{4.49}$$

We expect from elementary renormalisation theory that the soft-breaking β-functions will satisfy

$$2L\Gamma^{(2)}_h = \frac{1}{4!}\left(\beta_h^{ijkl(2)} - 4(\gamma_\phi^{(2)})^l{}_mh^{ijkm}\right)\phi_i\phi_j\phi_k\phi_l, \tag{4.50}$$

$$2L\Gamma^{(2)}_{m^2} = (\beta^{(2)}_{m^2})^i{}_j\phi^j\phi_i - (\gamma^{(2)}_\phi m^2)^i{}_j\phi^j\phi_i - (m^2\gamma^{(2)}_\phi)^i{}_j\phi^j\phi_i, \tag{4.51}$$

writing the results in this form to avoid cumbersome symmetrisations. We easily verify these identities using Eqs. (4.25), (4.24), (4.48), (4.49), (4.40).

4.3 Summary

We have presented the results for the anomalous dimension and β-functions in the component formalism, for a softly broken version of $\mathcal{N} = 2$ supersymmetric Chern-Simons matter theory, to leading order (two-loops) and we have shown that the results obtained from renormalisation in the component formalism are equivalent to those of the superfield formalism. As we have seen the number of diagrams in the component formalism was more than twice the number of superfield diagrams (although there is no fixed correspondence between the number of diagrams in each formalism). This led us to determining the next to leading order terms for the theory in terms of superfields so as to reduce the number of diagrams we needed to evaluate.

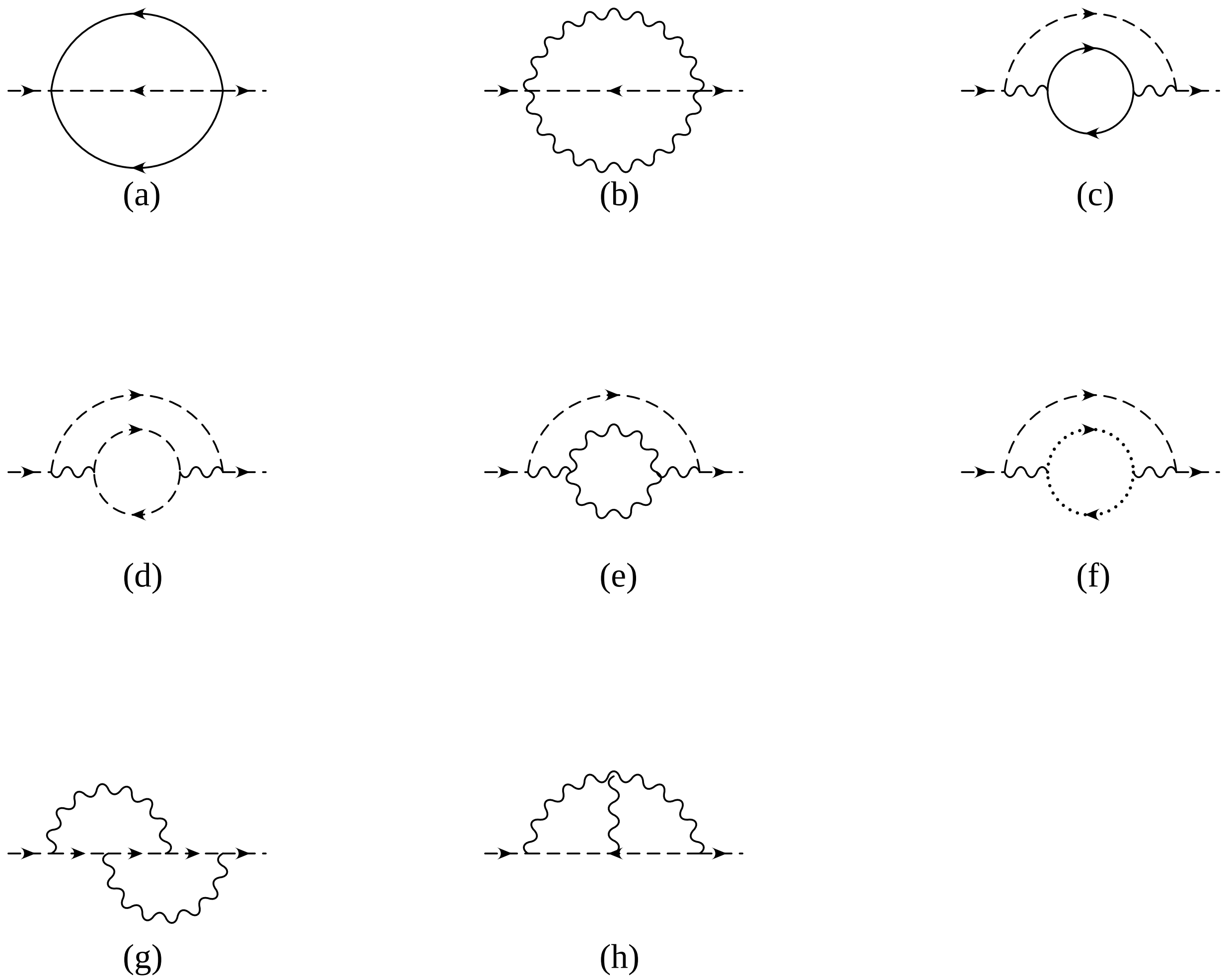

Figure 1: Diagrams contributing to $\gamma_\phi^{(2)}$.

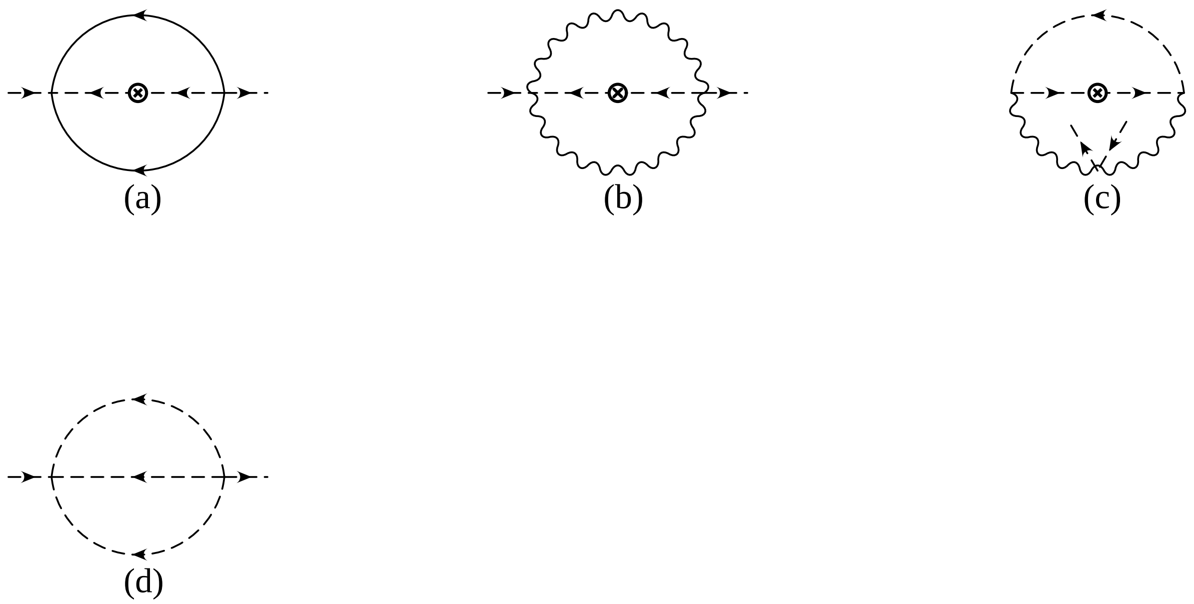

Figure 2: Diagrams contributing to $\beta_{m^2}^{(2)}$.
(The $\otimes$ symbol represents the $(m^2)^i{}_j$ vertex).

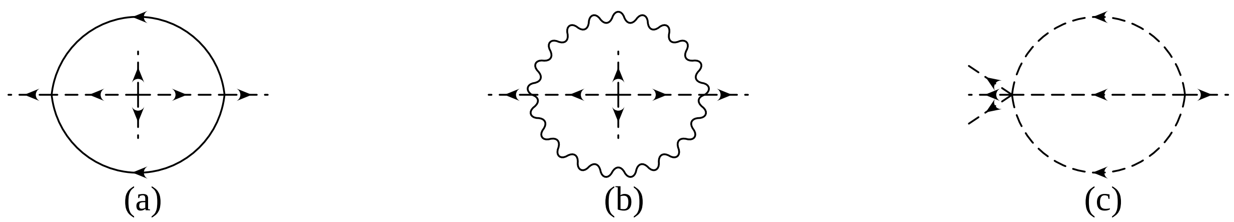

Figure 3: Diagrams contributing to $\beta_h^{(2)}$.

5 4-Loop Renormalisation of a General $\mathcal{N} = 2$ Supersymmetric Chern-Simons Theory

We present results for the planar contribution to the four-loop anomalous dimension for a general $N = 2$ supersymmetric Chern-Simons theory in three dimensions. These results should facilitate higher-order superconformality checks for theories relevant for the AdS/CFT correspondence.

5.1 Introduction

There has been substantial interest in $\mathcal{N} = 2$ supersymmetric Chern-Simons matter theories in the context of the AdS/CFT correspondence and in particular, a wide range of superconformal theories has been discovered [44, 45], starting with the BLG [27, 46] and ABJ/ABJM [28,29] models. Although a more familiar formulation is in terms of "quiver"-type gauge theories, many of them may be understood in terms of an underlying "3-algebra" structure [27, 47, 48]. Explicit perturbative computations to corroborate the superconformal property have been carried out in Refs. [15, 21, 26] at lowest order (two loops for a theory in three dimensions). Since the gauge coupling is unrenormalised for any Chern-Simons theory due to the topological nature of the theory, it is only necessary to compute the anomalous dimensions of the chiral fields in order to check for superconformality (in view of the non-renormalisation theorem). Our purpose here is to provide results to enable the extension of this check to the next (four-loop) order. As may readily be imagined, this is a highly non-trivial undertaking. Consequently we envisage an abridged version of the full task. Firstly, we have calculated only the planar diagrams, corresponding to the leading N contributions. Even then, and even after discarding large classes of diagrams which can be seen in advance not to contribute to the anomalous dimension, one is faced with the order of a hundred distinct diagrams. The process of automation which has made it feasible to perform high-loop calculations in non-supersymmetric theories using packages such as MINCER [49] is not available to us here; we are not aware of any available package for performing superspace calculations. Secondly, therefore, we have confined ourselves to computing the coefficients of only a subset (albeit a large one) of the invariants contributing to the anomalous dimension. Initially we suspected that it might be possible to derive the remaining coefficients by *assuming* the superconformality of a small number of the known examples of such theories. However as we shall see later this turns out not to be the case. In any case, we have tried to facilitate an extension of the check to include further invariants in the anomalous dimension, in the following sense: for the subset of invariants on which we have focussed our attention, we have (of course) computed all the diagrams which can contribute. Many of these diagrams also contribute to other invariants, and in these cases we have listed the contributions to these other invariants so that they can readily be combined with the contributions from the remaining diagrams at a later date.

5.2 $\mathcal{N} = 2$ Chern-Simons theory in three dimensions

The action for the theory can be written

$$S = S_{SUSY} + S_{GF}, \tag{5.1}$$

where S_{SUSY} is the usual supersymmetric action [37]

$$\begin{aligned} S_{SUSY} = &\int d^3x \int d^4\theta \left(k \int_0^1 dt\, \mathrm{Tr}[\overline{D}^{\alpha}(e^{-tV} D_{\alpha} e^{tV})] + \Phi^j (e^{V_A R_A})^i{}_j \Phi_i \right) \\ &+ \left(\int d^3x \int d^2\theta W(\Phi) + \text{h.c.} \right). \end{aligned} \tag{5.2}$$

Here V is the vector superfield, Φ the chiral matter superfield and the superpotential (quartic for renormalisability in three dimensions) $W(\Phi)$ is given by

$$W(\Phi) = \frac{1}{4!} Y^{ijkl} \Phi_i \Phi_j \Phi_k \Phi_l. \tag{5.3}$$

(We use the convention that $\Phi^i = (\Phi_i)^*$.) We assume a simple gauge group, though we comment later on the extension to non-simple groups. Gauge invariance requires the gauge coupling k to be quantised, so that $2\pi k$ is an integer. The vector superfield V is in the adjoint representation, $V = V_A T_A$ where T_A are the generators of the fundamental representation, satisfying Eq. (4.5) and the chiral superfield can be in a general representation, with gauge matrices denoted R_A satisfying Eq.(4.7).

In three dimensions the Yukawa couplings Y^{ijkl} are dimensionless and (as mentioned earlier) the theory is renormalisable. In Eq. (5.1) the gauge-fixing term S_{GF} is given by [21]

$$S_{GF} = \frac{k}{2\alpha} \int d^3x d^2\theta \mathrm{tr}[ff] - \frac{k}{2\alpha} \int d^3x d^2\bar{\theta} \mathrm{tr}[\overline{f}\overline{f}] \tag{5.4}$$

and we introduce into the functional integral a corresponding ghost term

$$\int \mathcal{D}f \mathcal{D}\overline{f} \Delta(V) \Delta^{-1}(V), \tag{5.5}$$

with

$$\Delta(V) = \int d\Lambda d\overline{\Lambda} \delta(F(V, \Lambda, \overline{\Lambda}) - f) \delta(\overline{F}(V, \Lambda, \overline{\Lambda}) - \overline{f}), \tag{5.6}$$

with $\overline{F} = D^2 V$, $F = \overline{D}^2 V$. With $\alpha = 0$ this results in a gauge propagator

$$\langle V^A(1) V^B(2) \rangle = -\frac{1}{K} \frac{1}{\partial^2} \overline{D}^{\alpha} D_{\alpha} \delta^4(\theta_1 - \theta_2) \delta^{AB}. \tag{5.7}$$

The gauge vertices are obtained by expanding $S_{SUSY}+S_{GF}$ as given by Eqs. (5.2), (5.4):

$$\begin{aligned} S_{SUSY}+S_{GF}\to &-\frac{i}{6}f^{ABC}\int d^3xd^4\theta\overline{D}^{\alpha}V^AD_{\alpha}V^BV^C \\ &-\frac{1}{24}f^{ABE}f^{CDE}\int d^3xd^4\theta\overline{D}^{\alpha}V^AV^BD_{\alpha}V^CV^D+\ldots. \end{aligned} \tag{5.8}$$

The ghost action resulting from Eq. (5.6) has the same form as in the four-dimensional $\mathcal{N}=1$ case [50, 51]

$$S_{gh}=\int d^3xd^4\theta\mathrm{tr}\{\overline{c}'c-c'\overline{c}+\frac{1}{2}(c+\overline{c}')[V,c+\overline{c}]+\frac{1}{12}(c+\overline{c}')[V,[V,c-\overline{c}]]+\ldots\}, \tag{5.9}$$

leading to ghost propagators

$$\langle\overline{c}'(1)c(2)\rangle=-\langle c'(1)\overline{c}(2)\rangle=-\frac{1}{\partial^2}\delta^4(\theta_1-\theta_2) \tag{5.10}$$

and cubic, quartic vertices which may easily be read off from Eq. (5.9). Finally the chiral propagator and chiral-gauge vertices are readily obtained by expanding Eq.(5.2); the chiral propagator is given by:

$$\langle\Phi^i(1)\Phi_j(2)\rangle=-\frac{1}{\partial^2}\delta^4(\theta_1-\theta_2)\delta^i{}_j. \tag{5.11}$$

The regularisation of the theory is effected by replacing V, Φ, Y by the corresponding bare quantities V_B, Φ_B, Y_B, with the bare and renormalised fields related by

$$V_B=Z_VV,\quad \Phi_B=Z_\Phi\Phi. \tag{5.12}$$

Since the Chern-Simons level k is expected to be unrenormalised for a generic Chern-Simons theory due to the topological nature of the theory (so that $k_B=k$), superconformality will be determined purely by the vanishing of the β-functions for the superpotential coupling.

These will be given according to the non-renormalisation theorem by

$$\beta_Y^{ijkl} = \gamma_{\Phi m}^{(i} Y^{jkl)m}, \tag{5.13}$$

where the anomalous dimension γ_Φ is defined by

$$\gamma_\Phi = \mu \frac{d}{d\mu} \ln Z_\Phi. \tag{5.14}$$

Writing

$$Z_\Phi = \sum_{L\,\text{even},\text{m}=1\ldots\frac{\text{L}}{2}} \frac{Z_\Phi^{(L,m)}}{\epsilon^m}, \tag{5.15}$$

where L is the number of loops. γ_Φ is determined by the simple poles in Z_Φ according to $\overline{\text{MS}}$ with DRED as

$$\gamma_\Phi^{(L)} = L Z_\Phi^{(L,1)} \tag{5.16}$$

and the higher order poles in Z_Φ are determined by consistency conditions, the one relevant for our purposes being

$$Z_\Phi^{(4,2)} = \beta_Y^{(2)} . \frac{\partial}{\partial Y} \gamma_\Phi^{(2)} - 2 \left(\gamma_\Phi^{(2)} \right)^2, \tag{5.17}$$

where β_Y is given by Eq. (6.11) and

$$\beta_Y . \frac{\partial}{\partial Y} \equiv \beta_Y^{klmn} . \frac{\partial}{\partial Y^{klmn}}. \tag{5.18}$$

At lowest order (two loops) it was found that superconformality (i.e. the vanishing of β_Y) was equivalent to the vanishing of γ_Φ in all the cases considered [21, 26] and it appears likely that this will remain true at higher orders.

5.3 Perturbative Calculations

In this section we review the two-loop calculation and describe in detail our four-loop results.

The anomalous dimension of the chiral superfield is given at two loops by [21, 26]

$$(8\pi)^2\gamma_\Phi^{(2)} = \frac{1}{3}Y_2 - 2k^{-2}C_2(R)C_2(R) - k^{-2}T(R)C_2(R) + k^{-2}C_2(G)C_2(R), \tag{5.19}$$

where $(Y_2)^i{}_j$, $C_2(R)$ and $C_2(G)$ are defined in Eqs. (4.21), (4.22) and (4.22) and $T(R)$ is defined in Eq. (4.7). This result may readily be obtained by $\mathcal{N}=2$ superfield methods [21, 26, 39, 42]; see section 3.4 for our $\mathcal{N}=2$ superfield conventions. Henceforth we set $k=1$ for simplicity; it may easily be restored if desired. Two-loop results for general Chern-Simons theories have also been obtained in Ref. [43] but are not directly comparable since they were computed in the $\mathcal{N}=1$ framework.

As explained earlier, in this paper we confine ourselves to the contributions to the four-loop anomalous dimension from planar diagrams. From a consideration of possible group invariants, the four-loop anomalous dimension is expected to take the form

$$\begin{aligned}(8\pi)^4\gamma_\Phi^{(4)} &= \alpha_1 Z_1 + \alpha_2 Z_2 + \alpha_3 W_1 + \alpha_4 W_2 + \alpha_5 W_3 + \alpha_6 W_4 + (\alpha_7 X + \alpha_8 C_2(G))U_1 \\ &+ (\alpha_9 X + \alpha_{10} C_2(G))U_2 + \alpha_{11}C_{40} + \alpha_{12}C_{31} + \alpha_{13}C_{22} + \alpha_{14}C_{13} + \alpha_{15}F_4 \\ &+ X\left(\alpha_{16}C_{30} + \alpha_{17}C_{21} + \alpha_{18}C_{12}\right) + X^2(\alpha_{19}C_{20} + \alpha_{20}C_{11}) + \alpha_{21}X^3C_2(R) \\ &+ \alpha_{22}X_2 + \alpha_{23}X_4 + (\alpha_{24}X + \alpha_{25}C_2(R) + \alpha_{26}C_2(G))X_1 \\ &+ \alpha_{27}X_5C_2(R) + (\alpha_{28}X + \alpha_{29}C_2(G))X_5 + \alpha_{30}X_3 \\ &+ \alpha_{31}\mathrm{tr}(C_2(R)\{R_A, R_B\}R_C)R_AR_BR_C + \alpha_{32}X_6 + \alpha_{33}d_{CDA}d_{CDB}R_AR_B,\end{aligned} \tag{5.20}$$

where the invariants involving Yukawa couplings are given by

$$
\begin{aligned}
(Y_3)^i{}_j &= Y^{ikmn}(Y_2)^l{}_k Y_{jlmn}, \\
(Y_4)^i{}_j &= Y^{iklr}Y_{klmn}Y^{mnpq}Y_{pqrj}, \\
Z_1 &= Y_2 C_2(R) C_2(R), \\
(Z_2)^i{}_j &= Y^{iklm}Y_{jkln}(C_2(R)C_2(R))^n{}_m, \\
(W_1)^i{}_j &= Y^{iklm}Y_{rknp}(R_A)^n{}_l(R_B)^p{}_m(R_A R_B)^r{}_j, \\
(W_2)^i{}_j &= Y^{iklm}Y_{pkln}(R_A R_B)^n{}_m(R_B R_A)^p{}_j, \\
(W_3)^i{}_j &= Y^{ikmp}Y_{jkln}(R_A R_B)^l{}_m(R_A R_B)^n{}_p, \\
(W_4)^i{}_j &= Y^{iklm}Y_{pkln}(R_A C_2(R))^n{}_m(R_A)^p{}_j, \\
U_1 &= Y_2 C_2(R), \\
(U_2)^i{}_j &= Y^{iklm}Y_{jkln}(C_2(R))^m{}_n,
\end{aligned}
\tag{5.21}
$$

and the remaining ones are

$$
\begin{aligned}
C_{mn} &= C_2(R)^m C_2(G)^n, \\
F_4 &= f_{EAB}f_{ECD}f_{HAF}f_{HCG}R^B R^D R^F R^G, \\
X &= T(R) - \frac{1}{2}C_2(G), \\
X_1 &= \mathrm{tr}(C_2(R)R_A R_B)R_A R_B, \\
X_2 &= \mathrm{tr}(R_A R_B R_C R_D)R_A R_B R_C R_D, \\
X_3 &= \mathrm{tr}(C_2(R)C_2(R)R_A R_B)R_A R_B, \\
X_4 &= \mathrm{tr}(Y_2 R_A R_B)R_A R_B, \\
X_5 &= D_{ABC}R_A R_B R_C, \\
X_6 &= f_{EAB}D_{ECD}R_A R_C R_B R_D,
\end{aligned}
\tag{5.22}
$$

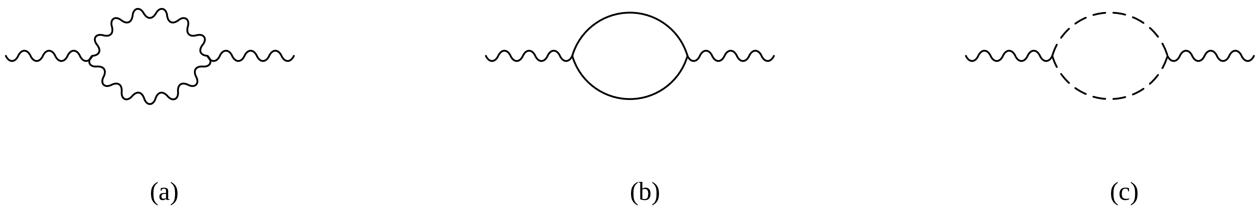

Figure 4: The one-loop insertions contributing to X.

with

$$D_{ABC} = \frac{1}{2}\mathrm{tr}(\{R_A, R_B\}R_C). \tag{5.23}$$

The quantity X in Eq. (5.22) is produced by one-loop vector two-point insertions as depicted in Fig. 4. One can show using results from Ref. [52] that the structure X_6 vanishes for the case of the fundamental representation; but we have not been able to prove this in general. We have decided to omit the computation of the coefficients α_{14}, α_{15}, α_{18}, α_{20}, α_{26}, α_{29} and $\alpha_{30} - \alpha_{33}$, and therefore we shall leave out those diagrams which can only contribute to these coefficients. Our rationale broadly speaking has been to avoid coefficients which derive contributions from large numbers of diagrams. This typically entails avoiding invariants with factors of $C_2(G)$, since it is clear for instance from Table 1 that invariants with more factors of $C_2(G)$ can arise from a larger number of diagrams. The coefficients α_{12}, α_{13}, α_{17} are exceptions to this. We computed these since the corresponding invariants C_{31}, C_{22} and XC_{21} have non-zero double poles (see Eq. (5.27)), which we wished to compute as a consistency check.

We are therefore concerned with the calculation of two-point diagrams. We have used the package *FeynArts* [53] to assist in generating the full set of diagrams. This package requires as an input the basic four-loop planar vacuum topologies, since only the topologies up to three loops are contained in the standard package. The topologies which we have used in *FeynArts* are depicted in Fig. 5 and also in Fig. 11; the remaining topologies consist of insertions of loops on simple three-loop topologies and we have enumerated diagrams in

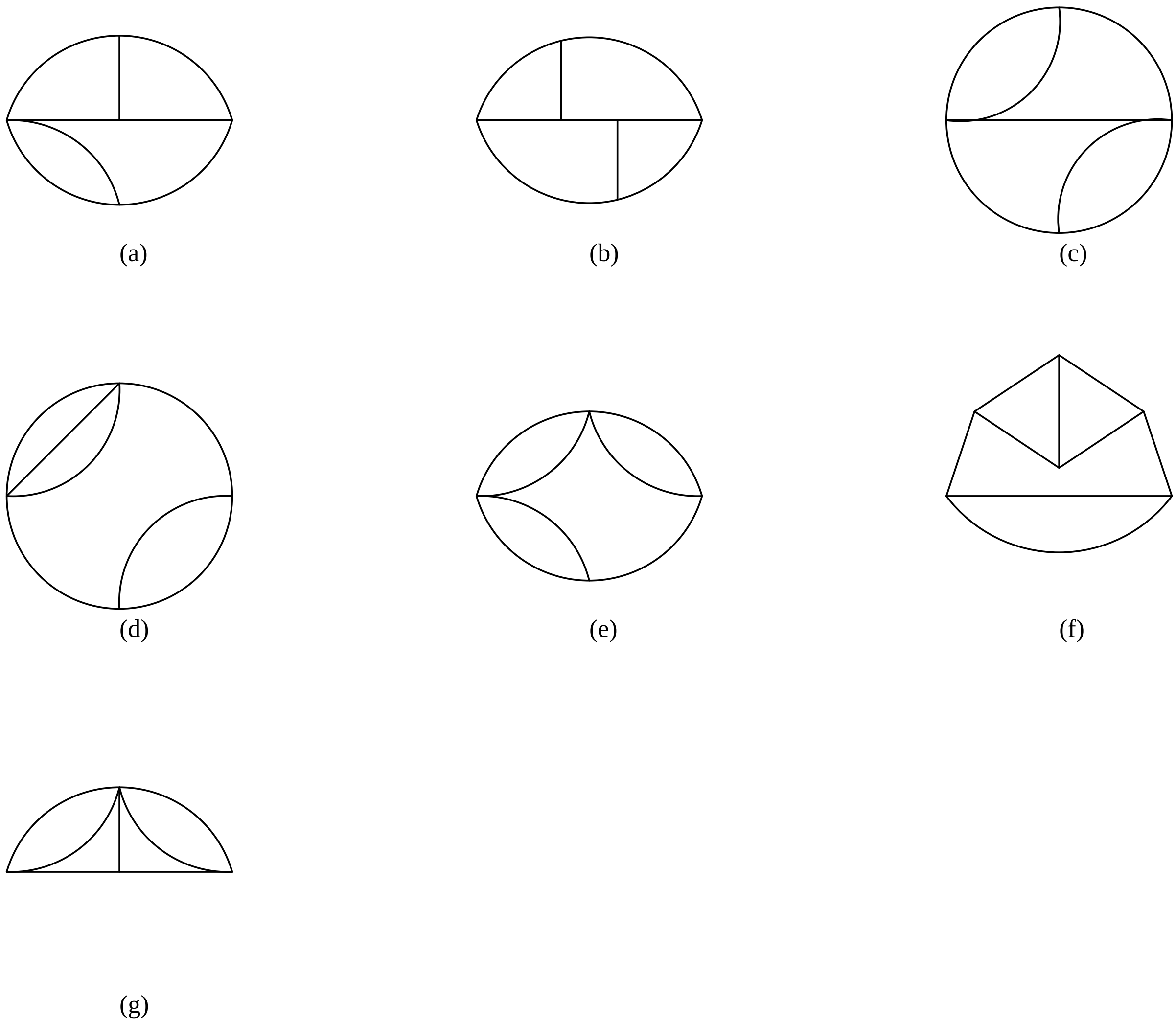

Figure 5: Four-loop topologies.

these classes “by hand”.

Two large classes of diagrams may be immediately discarded as having no logarithmic divergences and therefore no contribution to the anomalous dimension [39]. The first consists of those diagrams in which the first (last) vertex encountered along the incoming (outgoing) chiral line has a single gauge line. These are shown schematically in Fig. 6(a). The second class consists of those diagrams which contain a one-loop subdiagram with one gauge and one chiral line; depicted in Fig. 6(b). We were able to use the features of *FeynArts* to discard such diagrams of the type in Fig. 6(a) automatically.

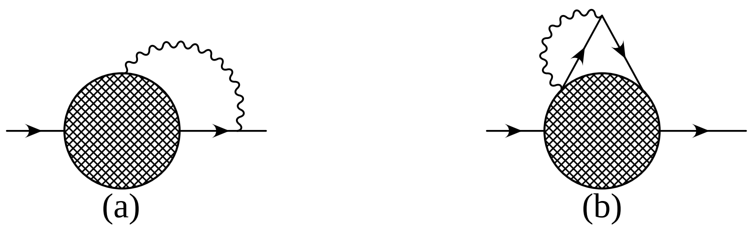

Figure 6: Classes of diagram which do not contribute.

Instead of displaying each of the divergent graphs pictorially, which would be very laborious, we introduce a notation for various classes of diagram and illustrate it by means of representative examples, depicted in Fig. 7. The main exception is the graphs with Yukawa vertices, which we shall describe shortly. The majority of our graphs have no Yukawa vertices and most can be described using a fairly uniform notation. We start with graphs which have only a single chiral line, and only gauge/chiral vertices. The gauge-chiral vertices along the chiral line are noted in order from left to right with the numbers representing which vertices the vertex in question connects to. In Fig. 7(a) the bracket containing 355 represents the first vertex connecting to the third vertex and then to the fifth vertex twice. The 4 following the bracket represents the second vertex connecting to the fourth vertex and then the 1 and the 2 following the represent the third and fourth vertices connecting to the first and second vertices respectively. Lastly the (11) represents the fifth vertex connecting to the first vertex twice.

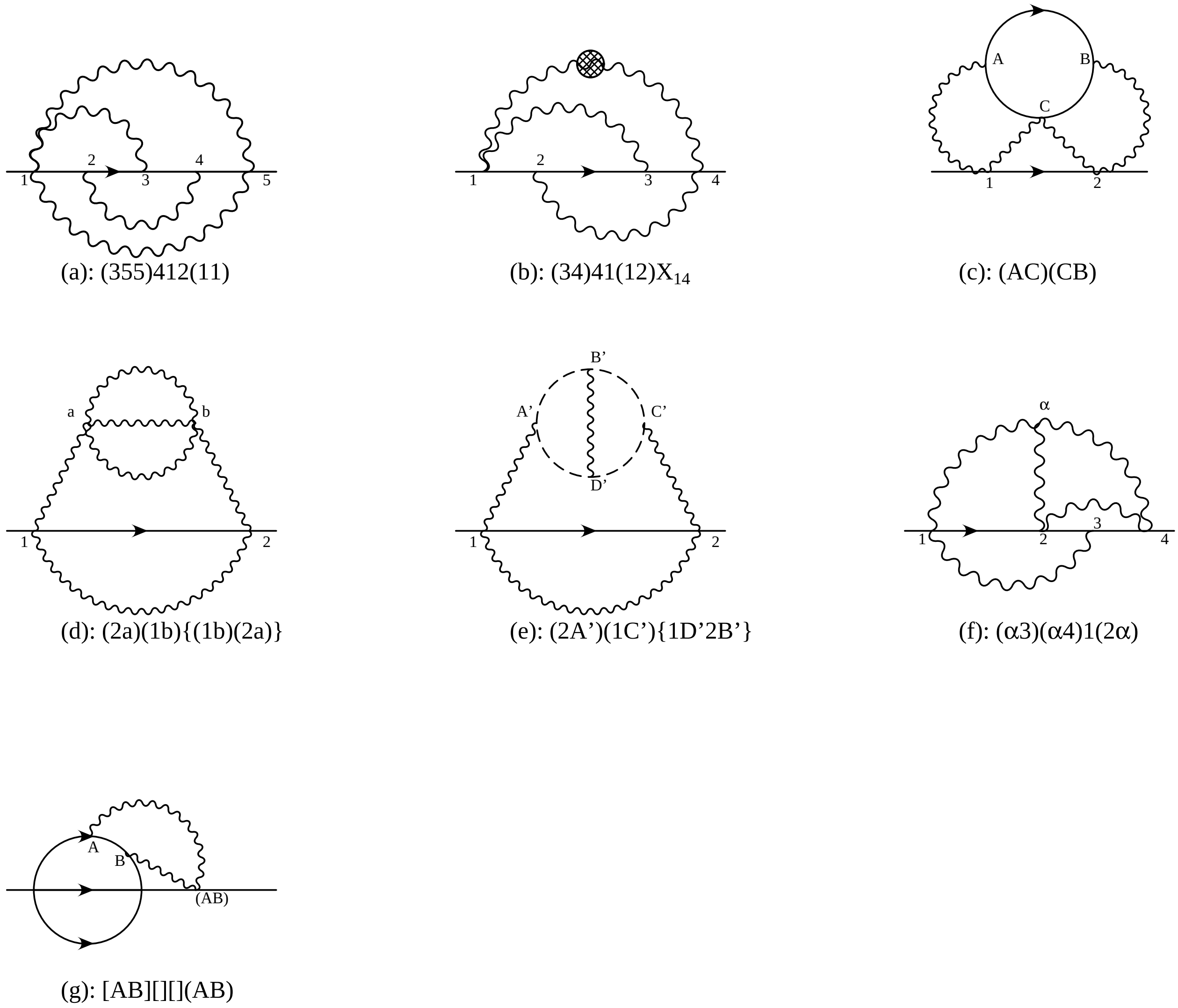

Figure 7: Typical diagrams with their notation.

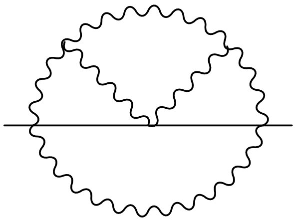

Figure 8: The additional graph from Table 5.

The insertion of a one-loop gauge 2-point function on the propagator joining vertices i and j is denoted by the addition of X_{ij}; see Fig. 7(b). More complex chiral loops (but without internal lines) are described by labelling the vertices on the loop by A, B, alphabetically, following the direction of the chiral arrows, and listing their connections to the vertices on the "main" chiral line as in the previous examples; in Fig. 7(c) the first bracket containing AC represents the first vertex on the main chiral line connecting to the A and C vertices on the chiral loop, the second bracket denotes the second vertex on the main chiral line connecting to the C and B vertices on the chiral loop. Gauge loops are described similarly, but with lower-case letters. If there are internal lines within the loop, these are denoted by listing the connections alphabetically in the same way as the main chiral line, but enclosed within brackets {}, as in Fig. 7(d) after the vertices on the main chiral line have been denoted. Ghost loops are denoted by labelling their vertices with primed letters, as in Fig. 7(e). Gauge vertices which do not lie within loops are denoted by Greek letters and their connections with the main line or with loops denoted as usual, as in Fig. 7(f). A single graph which does not fall into any of these categories is that shown in Fig. 8 (the result for this graph will be given in Table 5).

Now we come to the graphs with Yukawa vertices. There are two graphs with four Yukawa vertices (and no gauge lines). Their structure can easily be derived from the corresponding group invariant (one for each graph). They will therefore simply be labelled by their group structure (Y_3 and Y_4, as defined in Eq. (5.21)). The graphs with two Yukawa vertices (connected to an external line) and two gauge lines are described using a

Figure 9: The graphs of Table 9.

somewhat different notation to the above. The gauge matrices on chiral lines (represented by square brackets) are labelled A, B, etc and the matrices on each chiral line in the chiral loop are enclosed within square brackets [ ]. Two matrices labelled with the same letter are connected by a gauge propagator. This is exemplified in Fig. 7(g) where the three square brackets represent the three chiral lines connecting the two Yukawa vertices, the AB in the first square bracket represents two gauge matrices connected to the chiral line on two separate vertices and the regular bracket containing AB representing the two gauge matrices connecting at a single vertex outside of the two Yukawa vertices. Finally, we have found it simplest to depict the diagram explicitly for a small class of diagrams with two Yukawa vertices, in Fig. 9.

Several diagrams clearly give no contribution by virtue of group theoretic considerations. For each remaining diagram the D-algebra is performed using the conventions and useful identities listed in section 3.4. A large number (almost all, in fact) of diagrams containing 3-point gauge vertices yield vanishing contributions when the results of all possible arrangements of the Ds and $\overline{D}$s are added together. Unfortunately we have not succeeded in establishing a criterion to predict in advance which diagrams give non-vanishing results. Our results for the non-vanishing divergences are listed diagram-by-diagram in Tables 3-13. Note that the graph $(3\alpha)\alpha\alpha(1\alpha)$ in Table 5 yields two distinct group structures which have been listed separately, the second occurrence distinguished by a prime.

Let us now explain how the Tables have been constructed. The results have been expressed in terms of a relatively small basis of momentum integrals [54, 55] which are depicted in Fig. 11 and whose divergences are also listed in Appendix C. Figs. 11(a)-(g) depict I_4, I_{4bbb}, I_{22}, I_{42bbc}, $I_{422qAbBd}$, J_4 and J_5, respectively. The results given later, and also most of these conventions for labelling the diagrams, are taken from Refs. [54,55]. In Fig. 11 the arrows denote momenta in the numerator contracted as indicated and the small vertical line denotes two propagators, one before and one after the line. These momentum integrals multiply a variety of group structures, which appear in the final columns of Tables 3-13. In Table 1 we give the decompositions of some of these group structures into the basis of group invariants. The definitions of these group structures are not given explicitly as they may easily be read off from the structure of the diagrams where they appear. Two examples should suffice: for instance, to take the diagrams Fig. 7(a), (b) respectively

$$\begin{aligned} S_4 &= R_{(A}R_BR_{C)}R_DR_AR_DR_{(B}R_{C)}, \\ S_{X4} &= XR_{(A}R_{B)}R_CR_AR_{(B}R_{C)}. \end{aligned} \tag{5.24}$$

Finally the first columns of Tables 3-13 contain an overall symmetry factor. The resulting contribution to the two-point function for each diagram is therefore obtained by adding the momentum integrals with the coefficients listed in the appropriate row and multiplying the resulting sum by the corresponding symmetry factor and group structure. For instance, the fourth row of Table 3 denotes a contribution

$$(-1)(-2I_4 + I_{4bbb})\left(W_2 - \frac{1}{12}C_2(G)U\right). \tag{5.25}$$

The combination of momentum integrals

$$\frac{1}{4}I_4 - \frac{5}{8}I_{22} - I_{4bbb} + I_{42bbc} - 2I_{422qAbBd}, \tag{5.26}$$

which one frequently observes in the tables, results from a momentum integral correspond-

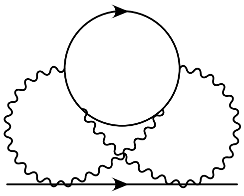

Figure 10: The non-planar graph.

ing to the topology Fig. 11(e), but with a trace over a product of "$p_{\alpha\beta}$" around the perimeter (constructed as in Eq. (3.24), where p is the momentum on one of the perimeter lines).

The first check on our results is provided by the consistency conditions Eq. (5.17) for the double poles. These, with the aid of Eq. (5.19), give

$$\begin{aligned}(8\pi)^4 Z_\Phi^{(4,2)} &= \frac{1}{6}Y_3 - \frac{1}{12}Z_1 + \frac{1}{4}Z_2 + \frac{1}{8}\left(U_2 - \frac{1}{3}U_1\right)\left(X - \frac{1}{2}C_2(G)\right) \\ &+ \frac{1}{2}C_{40} - \frac{1}{4}C_{31} + \frac{1}{32}C_{22} + \frac{1}{2}XC_{30} - \frac{1}{8}XC_{21} + \frac{1}{8}X^2C_{20}. \end{aligned} \tag{5.27}$$

We have checked all these non-zero coefficients, and moreover we have verified that the double poles for the remaining invariants whose coefficients we are computing vanish as they should. Of course the double pole contributions can in principle come from non-planar as well as planar diagrams. However, one can check that the only double-pole contribution from a non-planar diagram to one of the group structures whose divergent contribution we have computed is that from the diagram Fig. 10 (which contributes to α_{22}). Indeed, diagrams with three-point gauge vertices only have simple poles and the majority of non-planar diagrams are of this type. Including this double-pole contribution along with those from the planar diagrams in Table. 13, we find that the double pole proportional to X_2 in Eq. (5.22) is indeed cancelled. Of course the double poles corresponding to the invariants with coefficients α_{14}, α_{15}, α_{18}, α_{20}, α_{26}, α_{29} and $\alpha_{30} - \alpha_{33}$ should also cancel, but this we have not checked.

We note that the diagrams listed in Table 12 consist of insertions of a two-loop contribution to the gauge two-point function. These would be relevant to a superspace calculation of the corrections to the Chern-Simons level k; also required would be the similar contributions to the ghost two-point function; and the two-loop corrections to the $V\overline{\Phi}\Phi$ vertex. Such calculations have been performed in components [56], but there may be some interest in corroborating them in the superspace context.

Our final result for the four-loop anomalous dimension is

$$\begin{aligned}
(8\pi)^4\gamma_\Phi^{(4)} &= \frac{2}{3}Y_3 + \frac{\pi^2}{4}Y_4 - \frac{4}{3}Z_1 + \left(4 - \frac{2}{3}\pi^2\right)(4W_1 - Z_2) \\
&+ \left(8 - \frac{5}{3}\pi^2\right)W_2 - \frac{1}{3}\pi^2 W_3 + \frac{2}{3}\pi^2 W_4 \\
&+ \left[2\left(1 - \frac{1}{8}\pi^2\right)X - \left(1 - \frac{1}{4}\pi^2\right)C_2(G)\right]\left(\frac{1}{3}U_1 - U_2\right) \\
&- 4\left(6 + \pi^2\right)C_{40} + \left(32 + \frac{17}{6}\pi^2\right)C_{31} - \frac{1}{2}\left(25 + \frac{23}{24}\pi^2\right)C_{22} \\
&+ \alpha_{14}C_{13} + \alpha_{15}F_4 + X\left[-(8 + 3\pi^2)C_{30} + \left(2 + \frac{19}{6}\pi^2\right)C_{21} + \alpha_{18}C_{12}\right] \\
&+ X^2[-(2 + \pi^2)C_{20} + \alpha_{20}C_{11}] - \frac{1}{8}\pi^2 X^3 C_2(R) + \left(16 - \frac{7}{3}\pi^2\right)X_2 \\
&- \frac{2}{3}X_4 + \left[(8 - 3\pi^2)X - 2\pi^2 C_2(R) + \alpha_{26}C_2(G)\right]X_1 + \frac{16\pi^2}{3}X_5 C_2(R) \\
&+ (-\pi^2 X + \alpha_{29}C_2(G))X_5 + \alpha_{30}X_3 + \alpha_{31}\mathrm{tr}(C_2(R)\{R_A, R_B\}R_C)R_A R_B R_C \\
&+ \alpha_{32}X_6 + \alpha_{33}d_{CDA}d_{CDB}R_A R_B. \qquad (5.28)
\end{aligned}$$

As we explained in the introduction to this section, it is possible that the remaining undetermined coefficients may be determined by comparison with a small number of the known superconformal theories. We have therefore paused at this point in the explicit calculation and in the next Section we shall see what we can deduce already about higher-order superconformality and what are the prospects of using superconformality to efficiently determine remaining coefficients.

5.4 Summary

Of course further weight would be given to any superconformality checks by continuing the computation of the remaining unknown coefficients in Eq. (5.28). This would be hugely simplified if we could understand in advance which diagrams with 3-point gauge vertices will yield a vanishing contribution. In any case the remainder of the computation is certainly not insuperable, merely somewhat laborious. The extension to the non-planar diagrams is also in principle feasible, though we do not at present have available a convenient basis of momentum integrals already tabulated for this case. On the other hand, many of the non-planar diagrams may not actually contribute since most of them will contain 3-point gauge vertices.

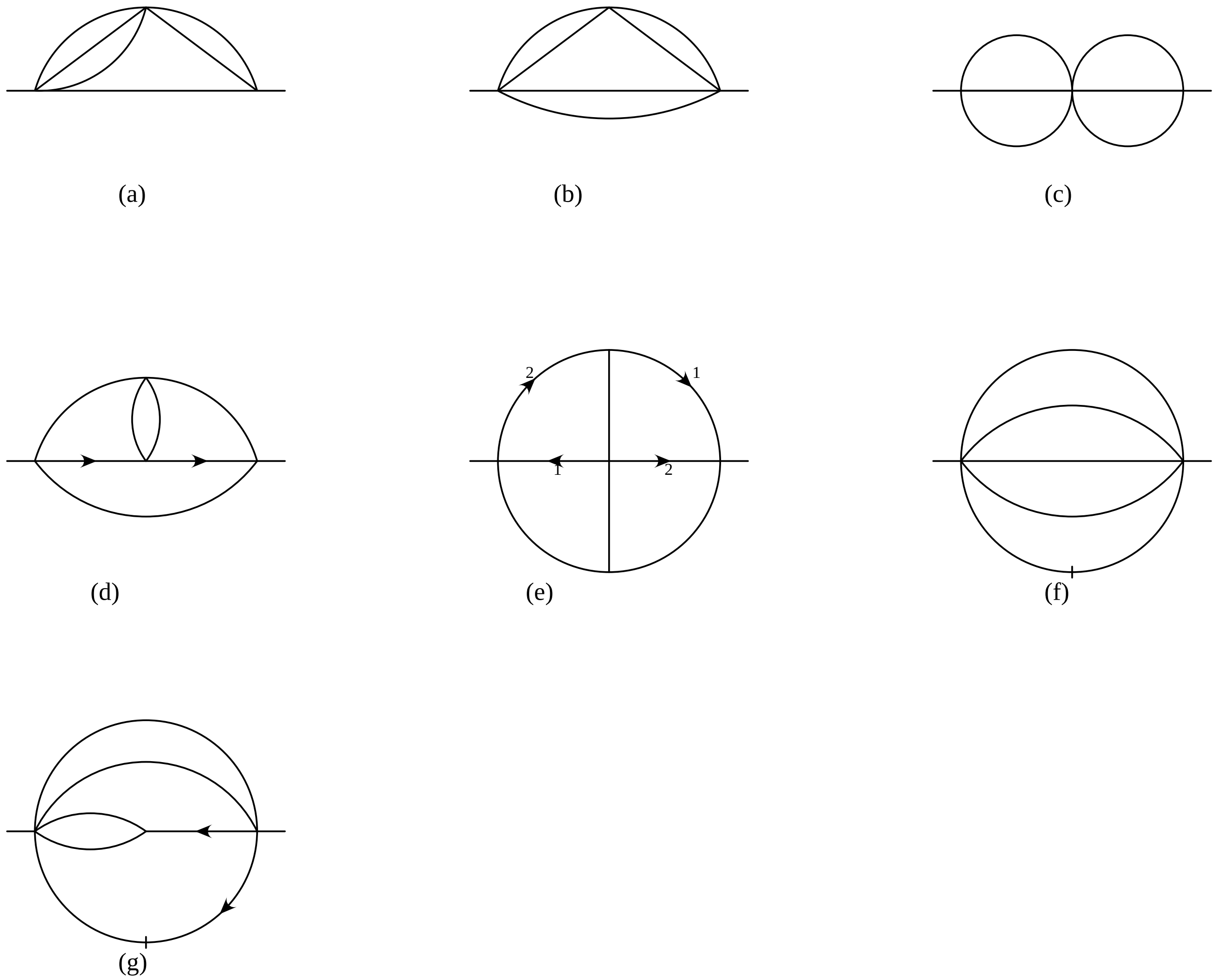

Figure 11: The basis of momentum integrals.

	C_{40}	C_{31}	C_{22}	C_{13}	F_4
S_1	1	$-\frac{1}{2}$	$\frac{1}{16}$	0	0
S_2	1	$-\frac{3}{2}$	$\frac{41}{48}$	$-\frac{33}{192}$	$\frac{5}{24}$
S_3	1	$-\frac{3}{4}$	$\frac{1}{8}$	0	0
S_4	1	$-\frac{5}{4}$	$\frac{13}{24}$	$-\frac{1}{12}$	0
S_5	1	$-\frac{3}{2}$	$\frac{5}{6}$	$-\frac{47}{288}$	$\frac{7}{36}$
S_6	1	-2	$\frac{65}{48}$	$-\frac{29}{96}$	$\frac{5}{12}$
S_7	1	$-\frac{5}{2}$	$\frac{33}{16}$	$-\frac{17}{32}$	1
S_8	1	-1	$\frac{3}{8}$	$-\frac{7}{128}$	$\frac{1}{16}$
S_9	1	$-\frac{3}{2}$	$\frac{13}{16}$	$-\frac{5}{32}$	$\frac{1}{4}$
S_{10}	1	$-\frac{5}{4}$	$\frac{9}{16}$	$-\frac{3}{32}$	$\frac{1}{8}$
S_{11}	1	$-\frac{7}{4}$	$\frac{33}{32}$	$-\frac{13}{64}$	$\frac{1}{4}$
S_{12}	1	$-\frac{3}{2}$	$\frac{3}{4}$	$-\frac{1}{8}$	0
S_{13}	1	$-\frac{3}{2}$	$\frac{3}{4}$	$-\frac{1}{8}$	$\frac{1}{16}$
S_{14}	1	$-\frac{7}{4}$	1	$-\frac{3}{16}$	$\frac{1}{8}$
S_{15}	1	$-\frac{21}{12}$	$\frac{103}{96}$	$-\frac{43}{192}$	$\frac{7}{24}$
S_{16}	1	-2	$\frac{21}{16}$	$-\frac{9}{32}$	$\frac{1}{4}$
S_{17}	1	$-\frac{9}{4}$	$\frac{27}{16}$	$-\frac{13}{32}$	$\frac{5}{8}$
S_{18}	0	1	$-\frac{11}{8}$	$\frac{7}{16}$	-1
S_{19}	0	1	$-\frac{7}{8}$	$\frac{3}{16}$	0
S_{20}	0	0	1	$-\frac{3}{8}$	2
S_{21}	0	0	$\frac{3}{2}$	$-\frac{5}{8}$	1
S_{22}	0	0	2	-1	2
S_{23}	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	2
S_{24}	0	0	1	$-\frac{7}{18}$	$\frac{4}{3}$

Table 1: Decompositions into group invariants for diagrams of type Fig. 7(a).

	C_{30}	C_{21}	C_{12}
S_{X1}	1	$-\frac{3}{4}$	$\frac{1}{6}$
S_{X2}	1	$-\frac{7}{12}$	$\frac{5}{48}$
S_{X3}	1	$-\frac{3}{4}$	$\frac{5}{32}$
S_{X4}	1	-1	$\frac{1}{4}$
S_{X5}	1	$-\frac{1}{4}$	0
S_{X6}	0	1	$-\frac{3}{8}$

Table 2: Decompositions into group invariants for diagrams of type Fig. 7(b).

		I_4	I_{22}	I_{4bbb}	I_{42bbc}	$I_{422qAbBd}$	
Y_3	$\frac{1}{12}$	1	0	0	0	0	Y_3
Y_4	$\frac{1}{8}$	0	0	1	0	0	Y_4
$[A][B][](AB)$	-1	$-\frac{7}{4}$	$-\frac{5}{8}$	1	1	-2	W_1
$[AB][](AB)$	-1	-2	0	1	0	0	$W_2 - \frac{1}{12}C_2(G)U$
$[(AB)][](AB)$	$\frac{1}{2}$	-2	0	0	0	0	$W_2 - \frac{1}{12}C_2(G)U_1$
$[(AB)(AB)][][]$	$\frac{1}{4}$	-2	0	0	0	0	$Z_2 - \frac{1}{4}C_2(G)U_2$
$[(AB)][(AB)][]$	$\frac{1}{4}$	0	0	-2	0	0	$W_3 - \frac{1}{4}C_2(G)U_2 + \frac{1}{12}C_2(G)U_1$
$[A][(AB)][B]$	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{5}{8}$	-1	1	-2	$\frac{1}{2}W_2 - W_3 + W_4 + \frac{1}{2}Z_2$
$[(AB)][AB][]$	-1	$\frac{1}{4}$	$-\frac{5}{8}$	-1	1	-2	$W_3 - \frac{1}{4}C_2(G)U_2 + \frac{1}{12}C_2(G)U_1$
$[ABAB][][]$	$\frac{1}{2}$	0	0	1	0	0	$Z_2 - \frac{1}{2}C_2(G)U_2$
$[ABA][B][]$	1	$-\frac{1}{4}$	$\frac{5}{8}$	1	-1	2	$-\frac{1}{2}Z_2 + \frac{1}{4}C_2(G)U_2 - \frac{1}{12}C_2(G)U_1$
$(AB)[][][](AB)$	$\frac{1}{12}$	-2	0	0	0	0	$Z_1 - \frac{1}{4}C_2(G)U_1$
$[AA][][]X_{AA}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$X\left(U_1 - \frac{1}{2}U_2\right)$
$[A][A]X_{AA}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	XU_2
$[(AA)][][]X_{AA}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	XU_2

Table 3: Results for diagrams of type Fig. 7(g).

		I_4	I_{22}	I_{4bbb}	I_{42bbc}	$I_{422qAbBd}$	
(44)(33)(22)(11)	$\frac{1}{4}$	4	0	0	0	0	S_1
(233)(13)(112)	$-\frac{1}{2}$	0	0	4	0	0	S_5
(66)4523(11)	$\frac{1}{2}$	0	0	-2	0	0	S_3
(355)412(11)	-1	0	0	-2	0	0	S_4
(24)(13)(24)(13)	1	0	0	-2	0	0	S_8
(36)516(14)	1	0	0	0	-2	0	S_9
(25)(14)52(13)	-2	0	0	-1	0	0	S_{10}
(35)4(15)2(13)	-1	0	0	0	-2	0	S_{11}
(46)6513(12)	2	$-\frac{1}{4}$	$\frac{5}{8}$	0	-1	2	S_{12}
(34)(34)(12)(12)	1	0	0	-1	2	0	S_{13}
(35)(45)12(12)	-2	0	0	-1	2	0	S_{14}
(223)(113)(12)	-1	4	0	0	0	0	S_5
(234)(14)1(12)	2	0	0	1	0	0	S_{15}
(334)4(11)(12)	1	4	0	-2	0	0	S_6
(345)511(12)	-2	$\frac{1}{4}$	$-\frac{5}{8}$	0	1	-2	S_6
(2233)(11)(11)	$-\frac{1}{2}$	4	0	0	0	0	S_2
(33)(44)(11)(22)	$\frac{1}{4}$	0	4	0	0	0	S_7
(22)(1133)(22)	$-\frac{1}{4}$	0	4	0	0	0	S_2
(33)4(114)(23)	1	0	2	0	0	0	S_6
(34)51(15)(24)	-2	-2	1	1	0	0	S_{14}
(23)(14)(14)(23)	1	-2	1	0	0	0	S_{13}
(35)6161(24)	1	$-\frac{5}{2}$	$\frac{9}{4}$	2	-2	4	S_{16}
(34)5(15)1(23)	-1	-2	1	2	0	0	S_{17}
(34)(55)11(22)	-1	0	2	0	0	0	S_7
(45)6611(23)	1	-2	1	2	0	0	S_7
$(3\alpha)(4\alpha)1(2\alpha)$	1	$-\frac{3}{4}$	$-\frac{1}{8}$	0	1	-2	S_{18}
$(5\alpha)4\alpha2(1\alpha)$	-1	$-\frac{1}{8}$	$\frac{5}{16}$	1	-1	1	S_{19}

Table 4: Results for diagrams of type Fig. 7(a),(f).

		I_4	I_{22}	I_{4bbb}	I_{42bbc}	$I_{422qAbBd}$	
$(\alpha\alpha)(3\alpha)(2\alpha)$	$\frac{1}{24}$	1	$\frac{1}{2}$	0	0	0	S_{20}
$(3\alpha\alpha)\alpha(1\alpha)$	$\frac{1}{72}$	0	0	$-\frac{1}{2}$	0	0	S_{21}
$(3\alpha)(\alpha\alpha)(1\alpha)$	$-\frac{1}{192}$	0	0	-4	-4	0	S_{20}
$(3\alpha)\alpha\alpha(1\alpha)$	$\frac{1}{24}$	$\frac{1}{2}$	$-\frac{5}{4}$	-1	3	-4	S_{20}
$(3\alpha)\alpha\alpha(1\alpha)'$	$\frac{1}{8}$	$-\frac{1}{4}$	$\frac{5}{8}$	0	-1	2	F_4
$(ab)(ab)(ab)$	$-\frac{1}{4}$	$\frac{5}{8}$	$-\frac{9}{16}$	-1	$\frac{1}{2}$	-1	S_{22}
$(\alpha\alpha)4\alpha(2\alpha)$	$-\frac{1}{12}$	0	0	$\frac{1}{2}$	0	0	S_{23}
$(3\alpha\alpha)(1\alpha\alpha)$	$\frac{1}{48}$	0	0	1	0	0	S_{24}
Fig. 8	$-\frac{1}{8}$	-1	$\frac{1}{2}$	0	0	0	S_{20}

Table 5: Results for graphs with 4-point gauge vertex and graphs of type Fig. 7(c),(d).

		I_4	I_{22}	I_{4bbb}	I_{42bbc}	$I_{422qAbBd}$	
$(3A)C(1B)\{132\}$	1	0	0	0	-1	0	$X_5C_2(R) + \frac{1}{4}T(R)(C_{21} - \frac{3}{8}C_{12})$
$(3A)B(1C)\{123\}$	1	$\frac{1}{2}$	$-\frac{5}{4}$	-1	2	-4	$X_5C_2(R) - \frac{1}{4}T(R)(C_{21} - \frac{3}{8}C_{12})$
$(3A')C'(1B')\{132\}$	$\frac{1}{16}$	0	0	0	-1	0	$C_{22} - \frac{3}{8}C_{13}$
$(3A')B'(1C')\{123\}$	$-\frac{1}{16}$	$\frac{1}{2}$	$-\frac{5}{4}$	-1	2	-4	$C_{22} - \frac{3}{8}C_{13}$
$(A3)A(B1)\{(12)3\}$	-2	0	0	$-\frac{1}{2}$	0	0	$X_5C_2(R)$
$(AB2)(C1)\{112\}$	-2	-2	0	0	0	0	$X_5\left(C_2(R) - \frac{1}{3}C_2(G)\right)$
$(AB2)(B1)\{1(12)\}$	2	-2	0	0	0	0	$X_5\left(C_2(R) - \frac{1}{3}C_2(G)\right)$
$(AA2)(B1)\{(11)2\}$	1	0	0	-2	0	0	$X_5\left(C_2(R) - \frac{1}{3}C_2(G)\right)$
$(3A)B(1A)\{(13)2\}$	-1	0	0	0	-1	0	$X_5(C_2(R) - \frac{3}{8}C_2(G))$

Table 6: Results for graphs contributing to $X_5C_2(R)$, and similar topologies.

		I_4	I_{22}	I_{4bbb}	I_{42bbc}	
$(222)(111)X_{12}$	-1	1	0	0	0	XS_{X1}
$(1122)(11)X_{11}$	-1	0	1	0	0	XS_{X2}
$(33)(22)(11)X_{22}$	$\frac{1}{2}$	0	1	0	0	XS_{X5}
$(233)1(11)X_{12}$	2	0	$\frac{1}{2}$	0	0	XS_{X1}
$(44)32(11)X_{23}$	-1	0	$\frac{1}{2}$	0	0	XS_{X5}
$(2a)(1a)X_{aa}$	$\frac{1}{6}$	0	1	0	0	XC_{21}
$(23)(13)(12)X_{12}$	-4	$\frac{1}{2}$	$-\frac{1}{4}$	0	0	XS_{X3}
$(23)(13)(12)X_{13}$	-1	0	0	1	0	XS_{X3}
$(34)41(12)X_{14}$	1	0	0	$\frac{1}{2}$	0	XS_{X4}
$(34)41(12)X_{13}$	2	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	XS_{X4}
$(\alpha3)\alpha(\alpha1)X_{\alpha2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	XS_{X6}

Table 7: Results for diagrams of type Fig. 7(b).

		I_4	I_{22}	I_{4bbb}	I_{42bbc}	J_4	J_5	
$(AB)\{(1B)(1A)\}X_{AB}$	$-\frac{1}{2}$	2	0	0	0	2	2	$X(X_1-\frac{1}{4}C_2(G)T(R)C_2(R))$
$(AB)\{(1B)(1A)\}X_{1A}$	-1	2	0	0	0	0	0	$X(X_1-\frac{1}{4}C_2(G)T(R)C_2(R))$
$(AC)\{(1B)A1\}X_{AB}$	1	0	1	0	0	1	1	$X(X_1-\frac{1}{4}C_2(G)T(R)C_2(R))$
$(AB)\{(1C)1A\}X_{AC}$	1	0	1	0	0	1	1	$X(X_1-\frac{1}{4}C_2(G)T(R)C_2(R))$
$(AD)\{1CB1\}X_{BC}$	-1	0	1	0	0	1	1	$X(X_1-\frac{1}{2}C_2(G)T(R)C_2(R))$
$(AC)\{1D1B\}X_{BD}$	$-\frac{1}{2}$	-2	2	1	-2	0	0	$X(X_1-\frac{1}{2}C_2(G)T(R)C_2(R))$
$(AC)\{1D1B\}X_{1A}$	-1	-2	0	1	0	0	0	$X(X_1-\frac{1}{4}C_2(G)T(R)C_2(R))$
$(AC)\{1(BB)1\}X_{BB}$	1	0	2	0	0	0	0	XX_1
$(AB)\{(1AA)1\}X_{AA}$	1	0	-2	0	0	0	0	$X(X_1-\frac{1}{2}C_2(G)T(R)C_2(R))$

Table 8: Results for diagrams contributing to XX_1.

		I_4	J_4	I_{42bbc}	
Fig. 9(a)	$\frac{1}{12}$	2	-2	-2	$X_4C_2(R)$
Fig. 9(b)	$-\frac{1}{6}$	1	-1	0	$X_4C_2(R)$

Table 9: Results for diagrams contributing to $X_4C_2(R)$.

		I_4	I_{22}	I_{4bbb}	
$(11)(11)X_{11}X_{11}$	$-\frac{1}{8}$	0	1	0	$X^2(C_{20}-\frac{1}{6}C_{11})$
$(22)(11)X_{12}X_{12}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$	0	$X^2(C_{20}-\frac{1}{4}C_{11})$
$(22)(11)(X^2)_{12}$	1	0	0	$-\frac{1}{2}$	$X^2(C_{20}-\frac{1}{4}C_{11})$
$(11)(X^3)_{11}$	$-\frac{1}{2}$	0	0	$\frac{1}{8}$	$X^3C_2(R)$

Table 10: Results for diagrams contributing to $X^2C_2(R)^2$ and $X^3C_2(R)$.

		I_4	I_{22}	I_{4bbb}	
$(ABC)X_{1A}$	1	1	0	0	XX_5
$(ABB)X_{1B}$	-1	1	0	0	XX_5
$(ABB)X_{1A}$	$-\frac{1}{2}$	0	0	1	XX_5

Table 11: Results for diagrams contributing to XX_5.

		I_4	I_{4bbb}	
$(2a)(1b)\{(1b)(2a)\}$	$\frac{1}{24}$	1	0	$C_{22}-\frac{1}{4}C_{13}$
$(2A')(1B')\{(1B')(2A')\}$	$\frac{1}{6}$	1	0	$C_{22}-\frac{1}{4}C_{13}$
$(2A')(1C')\{1D'2B'\}$	$-\frac{1}{8}$	1	$-\frac{1}{2}$	$C_{22}-\frac{1}{4}C_{13}$
$(2A')(1\alpha)\{1\alpha\alpha\}$	$-\frac{1}{8}$	0	1	$C_{22}-\frac{1}{4}C_{13}$
$(2A)(1B)\{(1B)(2A)\}$	-2	1	0	$(X_1-\frac{1}{4}C_2(G)T(R)C_2(R))(C_2(R)-\frac{1}{4}C_2(G))$
$(2A)(1C)\{1D2B\}$	2	1	$-\frac{1}{2}$	$(X_1-\frac{1}{4}C_2(G)T(R)C_2(R))(C_2(R)-\frac{1}{4}C_2(G))$
$(2A)(1\alpha)\{1\alpha\alpha\}$	1	0	2	$\frac{1}{4}C_2(G)T(R)$

Table 12: Results for two-loop vector two-point insertion diagrams.

		I_4	I_{22}	I_{4bbb}	I_{42bbc}	$I_{422qAbBd}$	
$(AB)(CD)$	1	-4	0	1	1	0	$X_2-\frac{1}{16}T(R)C_{12}+\frac{1}{2}C_2(G)X_5$
$(AC)(BC)$	-1	-4	0	1	1	0	$X_2-\frac{3}{32}T(R)C_{12}+\frac{5}{8}C_2(G)X_5-\frac{1}{4}iX_6$
$(AB)(BC)$	-1	$-\frac{15}{4}$	$-\frac{5}{8}$	1	2	-2	$X_2-\frac{3}{32}T(R)C_{12}+\frac{5}{8}C_2(G)X_5-\frac{1}{4}iX_6$
$(AB)(AB)$	$\frac{1}{2}$	-4	0	1	2	0	$X_2-\frac{3}{32}T(R)C_{12}+\frac{5}{8}C_2(G)X_5-\frac{1}{4}iX_6$
$(AA)(BC)$	-1	-2	0	0	0	0	$X_2-\frac{1}{16}T(R)C_{12}+\frac{1}{2}C_2(G)X_5$
$(AA)(AB)$	1	-2	0	0	0	0	$X_2-\frac{3}{32}T(R)C_{12}+\frac{5}{8}C_2(G)X_5-\frac{1}{4}iX_6$
$(ABCD)$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{5}{8}$	-1	1	-2	$X_2-\frac{3}{32}T(R)C_{12}+\frac{5}{8}C_2(G)X_5-\frac{1}{4}iX_6$
$(AABB)$	$-\frac{1}{8}$	0	0	4	0	0	$X_2-\frac{3}{32}T(R)C_{12}+\frac{5}{8}C_2(G)X_5-\frac{1}{4}iX_6$

Table 13: Results for diagrams contributing to X_2.

6 Superconformal Chern-Simons theory in three dimensions beyond leading order

We discuss possible higher-loop corrections to superconformal invariance for a class of $\mathcal{N} = 2$ supersymmetric Chern-Simons theories including the ABJM model. We argue that corrections are inevitable even for simple generalisations of the ABJM model; but that it is likely that any corrections are of a particular "maximally transcendental" form.

6.1 Introduction

Since the gauge coupling is unrenormalised for any Chern-Simons theory due to the topological nature of the theory (and indeed is quantised at certain values–the Chern-Simons "level") it is only necessary to compute the anomalous dimensions of the chiral fields in order to check for superconformality (in view of the non-renormalisation theorem). Our purpose here is to attempt to extend the explicit check of superconformality beyond lowest order. Many of the superconformal theories involve a simple choice of the superpotential couplings in terms of the Chern-Simons level, and the simplest expectation would be that this choice renders the theory finite at higher orders too. This would be analogous to the case of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetric theories in four dimensions, where the finiteness properties are manifest to all orders in the $\mathcal{N} = 1$ superfield description once the field content and superpotential have been specified (assuming a supersymmetric regulator such as DRED). However, an alternative possibility is that one might have to adjust the couplings order by order so as to achieve finiteness [57,58]. This would be more analogous to the case of finite $\mathcal{N} = 1$ theories in four dimensions, where the finiteness is obtained through an order-by-order adjustment of the couplings. We might certainly expect the theories where superconformality is achieved by solving a somewhat non-trivial condition at lowest order to behave like this.

In odd spacetime dimensions, divergences only occur at even loop order, so to go beyond leading order we are driven to consider a four-loop calculation. The total number of diagrams is colossal; so here we report on what can be learnt from the consideration of a subset of the full set of diagrams, namely those which have at least one (in fact at least two) Yukawa vertices. We were able to compute all the relevant diagrams with the exception of a single non-planar diagram. Our conclusions are as follows: firstly, we note that the contributions to the anomalous dimension at this order fall into two classes, proportional respectively to F^4 and $\pi^2 F^4$, where F is the usual factor associated with loops in dimensional regularisation, in 3 dimensions $F = \frac{1}{8\pi}$. The latter class has been called "maximally transcendental" [54], and we shall call the former "rational". We then show that the maximally-transcendental contributions to the four-loop anomalous dimension inevitably require a coupling redefinition to restore superconformality as soon as "multi-trace deformations" are included; and in fact we shall argue that there is evidence that maximally-transcendental redefinitions may be needed even in simpler cases such as the ABJM and ABJ models. On the other hand, there is no such evidence that redefinitions are needed for the rational contributions; and we shall show that (at least to leading order in N, M, and probably to all orders) the "rational" contributions to the four-loop anomalous dimension adopt a universal form for a wide class of theories once the lowest-order superconformality conditions are imposed; and thus, if a "rational" coupling redefinition is unnecessary for the ABJ and ABJM models, it would also would be unnecessary for this class of related models.

6.2 $\mathcal{N} = 2$ Chern-Simons theory in three dimensions

We consider an $\mathcal{N} = 2$ supersymmetric $U(N) \times U(M)$ Chern-Simons theory with vector multiplets V, $\hat{V}$ in the adjoint representations of $U(N)$ and $U(M)$ respectively, and we write

$$V^b{}_a = V^A (T_A)^b{}_a, \quad \hat{V}^{\hat{b}}{}_{\hat{a}} = \hat{V}^A (\hat{T}_A)^{\hat{b}}{}_{\hat{a}}, \tag{6.1}$$

where $T_A,\ A = 1, \dots N^2$ and $\hat{T}_A,\ A = 1, \dots M^2$ are the generators for the fundamental representations of $U(N)$, $U(M)$ respectively.

The vector multiplets are coupled to chiral multiplets $(A^i)^a{}_{\hat{a}}$ and $(B_i)^{\hat{a}}{}_a$, $i = 1, 2$ in the $(N, \overline{M})$ and $(\overline{N}, M)$ representations of the gauge group, respectively. The gauge matrices T_A satisfy Eq. (4.5). The action for the theory can be written

$$S = S_{SUSY} + S_{GF}, \tag{6.2}$$

where S_{SUSY} is the usual supersymmetric action [37]

$$\begin{aligned} S_{SUSY} = & \int d^3x \int d^4\theta \int_0^1 dt \left\{ K_1 \operatorname{Tr}[\overline{D}^\alpha (e^{-tV} D_\alpha e^{tV})] + K_2 \operatorname{Tr}[\overline{D}^\alpha (e^{-t\hat{V}} D_\alpha e^{t\hat{V}})] \right\} \\ & + \int d^3x \int d^4\theta \operatorname{Tr} \left(\overline{A}_i e^V A^i e^{-\hat{V}} + \overline{B}^i e^{\hat{V}} B_i e^{-V} \right) \\ & + \left(\int d^3x \int d^2\theta W(A^i, B_i) + \text{h.c.} \right). \end{aligned} \tag{6.3}$$

Here the superpotential (quartic for renormalisability in three dimensions) $W(A^i, B_i)$ is given by

$$\begin{aligned} W(A^i, B_i) \;=\; & \operatorname{Tr}[h_1 (A^1 B_1)^2 + h_2 (A^2 B_2)^2 + h_3 A^1 B_1 A^2 B_2 + h_4 A^2 B_1 A^1 B_2] \\ + \; & \frac{1}{2} H_1 [\operatorname{Tr}(A^1 B_1)]^2 + H_{12} \operatorname{Tr}(A^1 B_1) \operatorname{Tr}(A^2 B_2) + \frac{1}{2} H_2 [\operatorname{Tr}(A^2 B_2)]^2. \end{aligned} \tag{6.4}$$

Gauge invariance requires $2\pi K_1$ and $2\pi K_2$ to be integers.

A variety of interesting theories may be obtained by specialising the superpotential in Eq. (6.4) and the gauge group and associated Chern-Simons levels in various ways.

Setting $h_1 = h_2 = 0,\ h_3 = -h_4 = h,\ H_1 = H_2 = H_{12} = 0,\ K_1 = -K_2$, we obtain the

$\mathcal{N}=2$ ABJM/ABJ-like theories studied in Ref. [15]. In particular, for $h=\frac{1}{K}$ one obtains the $\mathcal{N}=6$ superconformal ABJ theory and for $N=M$ the ABJM theory.

On the other hand, for

$$h_1=h_2=\frac{1}{2}\left(\frac{1}{K_1}+\frac{1}{K_2}\right),\quad h_3=\frac{1}{K_1},\quad h_4=\frac{1}{K_2}, \tag{6.5}$$

we obtain the $\mathcal{N}=3$ superconformal theory described in Ref. [59].

Additional, more general, superconformal theories may be found by solving the lowest order finiteness conditions (see next section). Further superconformal theories may also be obtained by adding flavour matter [21].

The details of gauge fixing and quantisation for our Chern-Simons theory are the same as those used in Eqs. (5.4), (5.5) and (5.6) with the same procedure for each gauge sector. With $\alpha=0$ this results in a gauge propagator for V of the form

$$\langle V^A(1)V^B(2)\rangle=-\frac{1}{K_1}\frac{1}{\partial^2}\overline{D}^\alpha D_\alpha\delta^4(\theta_1-\theta_2)\delta^{AB}, \tag{6.6}$$

with a similar propagator for $\hat{V}$. The gauge vertices are obtained by expanding $S_{SUSY}+S_{GF}$ as given by Eqs. (6.3), (5.4):

$$\begin{aligned} S_{SUSY}+S_{GF}\to &-\frac{i}{6}K_1 f^{ABC}\int d^3x d^4\theta\overline{D}^\alpha V^A D_\alpha V^B V^C \\ &-\frac{i}{6}K_2 f^{ABC}\int d^3x d^4\theta\overline{D}^\alpha \hat{V}^A D_\alpha \hat{V}^B \hat{V}^C+\dots. \end{aligned} \tag{6.7}$$

Again we obtain the same ghost action, Eq. (5.9), resulting from Eq. (5.6) leading to ghost propagators

$$\langle\overline{c}'(1)c(2)\rangle=-\langle c'(1)\overline{c}(2)\rangle=-\frac{1}{\partial^2}\delta^4(\theta_1-\theta_2), \tag{6.8}$$

(together with similar expressions involving $\hat{V}$ and its own ghosts), and cubic and higher-order vertices which may easily be read off from Eq. (5.9). Finally the chiral propagator and chiral-gauge vertices are readily obtained by expanding Eq.(5.2); the chiral propagators are given by:

$$\langle \overline{A}_{i}^{\hat{a}}{}_{a} A^{jb}{}_{\hat{b}} \rangle = -\frac{1}{\partial^2}\delta^4(\theta_1-\theta_2)\delta_a{}^b\delta^{\hat{a}}{}_{\hat{b}}\delta^j{}_i, \tag{6.9}$$

with a similar expression for the B-propagator.

The regularisation of the theory is effected by replacing V, $\hat{V}$, A, B, h_i, H_i (and the various ghost fields) by corresponding bare quantities V_B, $\hat{V}_B$, A_B, B_B, h_{Bi}, H_{Bi} (and similarly for the ghost fields) with the bare and renormalised fields related by

$$V_B = Z_V V, \tag{6.10}$$

etc. Since the Chern-Simons levels K_1 and K_2 are expected to be unrenormalised for a generic Chern-Simons theory due to the topological nature of the theory (so that $K_{B1} = K_1$ and $K_{B2} = K_2$), superconformality will be determined purely by the vanishing of the β-functions for the superpotential couplings. These will be given according to the non-renormalisation theorem in terms of the anomalous dimensions of the fields associated with each coupling; for instance

$$\beta_{h_3} = (\gamma_{A^1}+\gamma_{B_1}+\gamma_{A^2}+\gamma_{B_2})h_3, \tag{6.11}$$

where anomalous dimensions such as γ_{A^1} are defined in the same way as γ_Φ in Eq. (5.14). Similarly Z_{A^1} is defined in the same way as Z_Φ in Eq. (5.15) and γ_{A^1} is determined from Z_{A^1} according to Eq. (5.16). The higher order poles in Z_{A^1} are determined by consistency conditions, the one relevant for our purposes being

$$Z_{A^1}^{(4,2)} = \sum_r \beta_{\lambda_r}^{(2)}.\frac{\partial}{\partial\lambda_r}\gamma_{A^1}^{(2)} - 2\left(\gamma_{A^1}^{(2)}\right)^2, \tag{6.12}$$

where $\{\lambda_r, r = 1, \ldots, 14\} = \{h_i, \overline{h}_i, H_i, \overline{H}_i, H_{12}, \overline{H}_{12}\}$ and β_{h_3} is given by Eq. (6.11) with similar expressions for the other superpotential couplings (and the β-function for a coupling is the same as that for its conjugate). At lowest order (two loops) it was found that superconformality (i.e. the vanishing of β_{λ_r}) was equivalent to the vanishing of all the corresponding anomalous dimensions (for the fields involved in the λ_r coupling) in all the cases considered [21, 26].

Finally we discuss what may be inferred about possible higher-order corrections from previous work, in particular Ref. [54]. In Ref. [54], four-loop corrections were discussed to the magnon dispersion relation for the ABJM model. These corrections were computed in the planar limit, which corresponds for us to the $O(N^4, M^4)$ terms at four loops; they were found to be maximally transcendent, with no rational contribution. Now implicit in Ref. [54] is the assumption that the ABJ model has the simple form discussed earlier. The corrections required for the dispersion relation may therefore be assumed to include the corrections needed to restore conformal invariance, in addition to corrections peculiar to the dispersion relation itself. Barring any accidental cancellations, the absence of rational corrections to the dispersion relation may be taken to imply that there is no need for rational corrections to the couplings to maintain four-loop superconformal invariance for the ABJ model. Conversely, given that transcendental corrections are required for the dispersion relation, there is no *a priori* reason not to expect transcendental corrections to the superconformality conditions.

6.3 Perturbative Calculations

In this section we review the two-loop calculation and describe in detail our four-loop results.

The renormalisation constants of the chiral superfields A_1, B_1 are given at two loops

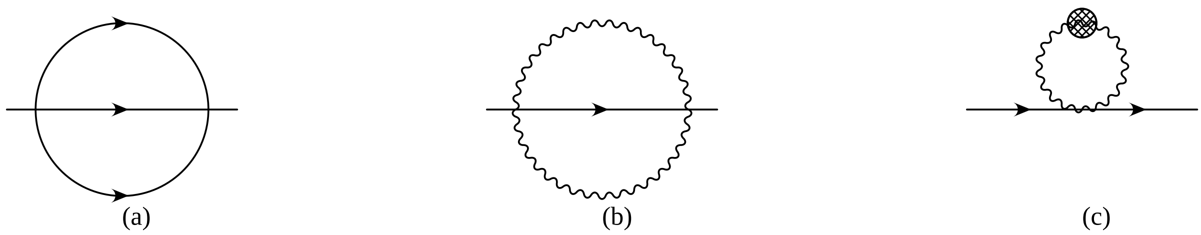

Figure 12: Two-loop diagrams.

by [21, 26]

$$F^{-2}\gamma^{(2)}_{A^1} = \rho_{A^1} - \rho_k, \tag{6.13}$$

(with similar expressions for A^2, B_1 and B_2) where $F = \frac{1}{8\pi}$ as defined before and

$$\begin{aligned}
\rho_{A^1} = \rho_{B_1} &= 4|h_1|^2(MN+1) + (|h_3|^2 + |h_4|^2)MN + (h_3\overline{h}_4 + h_4\overline{h}_3) \\
&+ MN(|H_1|^2 + |H_{12}|^2) + |H_1|^2, \\
\rho_{A^2} = \rho_{B_2} &= 4|h_2|^2(MN+1) + (|h_3|^2 + |h_4|^2)MN + (h_3\overline{h}_4 + h_4\overline{h}_3) \\
&+ MN(|H_2|^2 + |H_{12}|^2) + |H_2|^2, \\
\rho_k &= (k_1^2 + k_2^2)(2MN+1) + 2(MN+2)k_1k_2,
\end{aligned} \tag{6.14}$$

with

$$k_1 = \frac{1}{K_1}, \quad k_2 = \frac{1}{K_2}. \tag{6.15}$$

This result may readily be obtained by $\mathcal{N}=2$ superfield methods [21,26,39,42] from the two-loop two-point diagrams depicted in Fig. 12; see section 3.4 for our $\mathcal{N}=2$ superfield conventions. Here and later we do not distinguish in the diagrams between the different chiral or gauge fields, so that each diagram in Fig. 12 is a schematic representation of several distinct Feynman diagrams. ρ_{A^1} etc correspond to Fig. 12(a) while it may easily be checked that

$$\rho_k = \rho_b + \rho_c \tag{6.16}$$

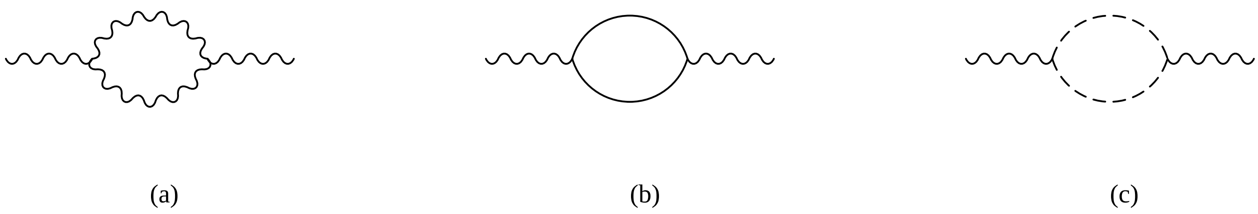

Figure 13: One-loop insertion diagrams.

where the contributions $\rho_{b,c}$ corresponding to Fig. 12(b,c) are given by

$$
\begin{aligned}
\rho_b &= \frac{1}{2}(C_1 + C_2) = \frac{1}{2}[(N^2+1)k_1^2 + (M^2+1)k_2^2 + 4MNk_1k_2], \\
\rho_c &= \frac{1}{2}[X_1Nk_1^2 + X_2Mk_2^2 + X_{12}k_1k_2],
\end{aligned}
\tag{6.17}
$$

with

$$
\begin{aligned}
C_1 &= N^2k_1^2 + M^2k_2^2 + 2MNk_1k_2, \\
C_2 &= k_1^2 + k_2^2 + 2MNk_1k_2, \\
X_1 &= 4M - \frac{N^2-1}{N}, \\
X_2 &= 4N - \frac{M^2-1}{M}X_2, \\
X_{12} &= 8.
\end{aligned}
\tag{6.18}
$$

$C_{1,2}$ correspond to the two different symmetrisations of the gauge lines in Fig. 12(b), while the X_1, X_2 and X_{12} correspond to the contributions from the "blob" in Fig. 12(c) which represents the three one-loop diagrams depicted in Fig. 13 (the dashed line representing a ghost propagator).

(We note here that the two-loop results for general Chern-Simons theories obtained in Ref. [43] are not directly comparable since they were computed in the $\mathcal{N} = 1$ framework.)

As mentioned in the Introduction, we shall consider two classes of model in some detail;

the first without, and the second with, multitrace deformations. We therefore first consider the case $H_1 = H_2 = H_{12} = 0$ in Eq. (6.4), and in fact we start with the even simpler example of

$$H_1 = H_2 = H_{12} = 0, \quad h_1 = h_2 = 0, \quad h_3 + h_4 = 0, \tag{6.19}$$

with $h_3 = -h_4 = h$ real. This is a class of theories considered in Ref. [59], which reduces to the ABJ model on setting $K_1 = -K_2$ (or $k_1 = -k_2$) and to the ABJM model on further setting $M = N$.

The four-loop diagrams contributing to the anomalous dimensions are depicted in Figs. 14, 15.

The contributions to $F^{-4} Z_{A^1}^{(4)}$ from these diagrams are given by

$$\begin{aligned}
G_a &= 3\rho_h^2 I_4, \\
G_b &= 2(MN^3 + NM^3 - 4M^2 - 4N^2 + 10MN - 4)h^4 I_{4bbb}, \\
G_c &= -3\rho_h\rho_b I_4, \\
G_d &= 3\rho_h C_2 I_{4bbb}, \\
G_e &= -\rho_h\rho_b I_4, \\
G_f &= 2\rho_h C_2 I_5, \\
G_g &= -T_2 I_5, \\
G_h &= -T_2 I_{4bbb}, \\
G_i &= -T_1 I_5, \\
G_j &= 4T_2\left(I_4 - \frac{1}{2}I_{4bbb}\right), \\
G_k &= -2T_2 I_4, \\
G_l &= -2T_1(-2I_4 + 2I_{4bbb} + I_5), \\
G_m &= -2\rho_h\rho_c\left(I_4 - \frac{1}{2}I_{4bbb}\right),
\end{aligned}$$

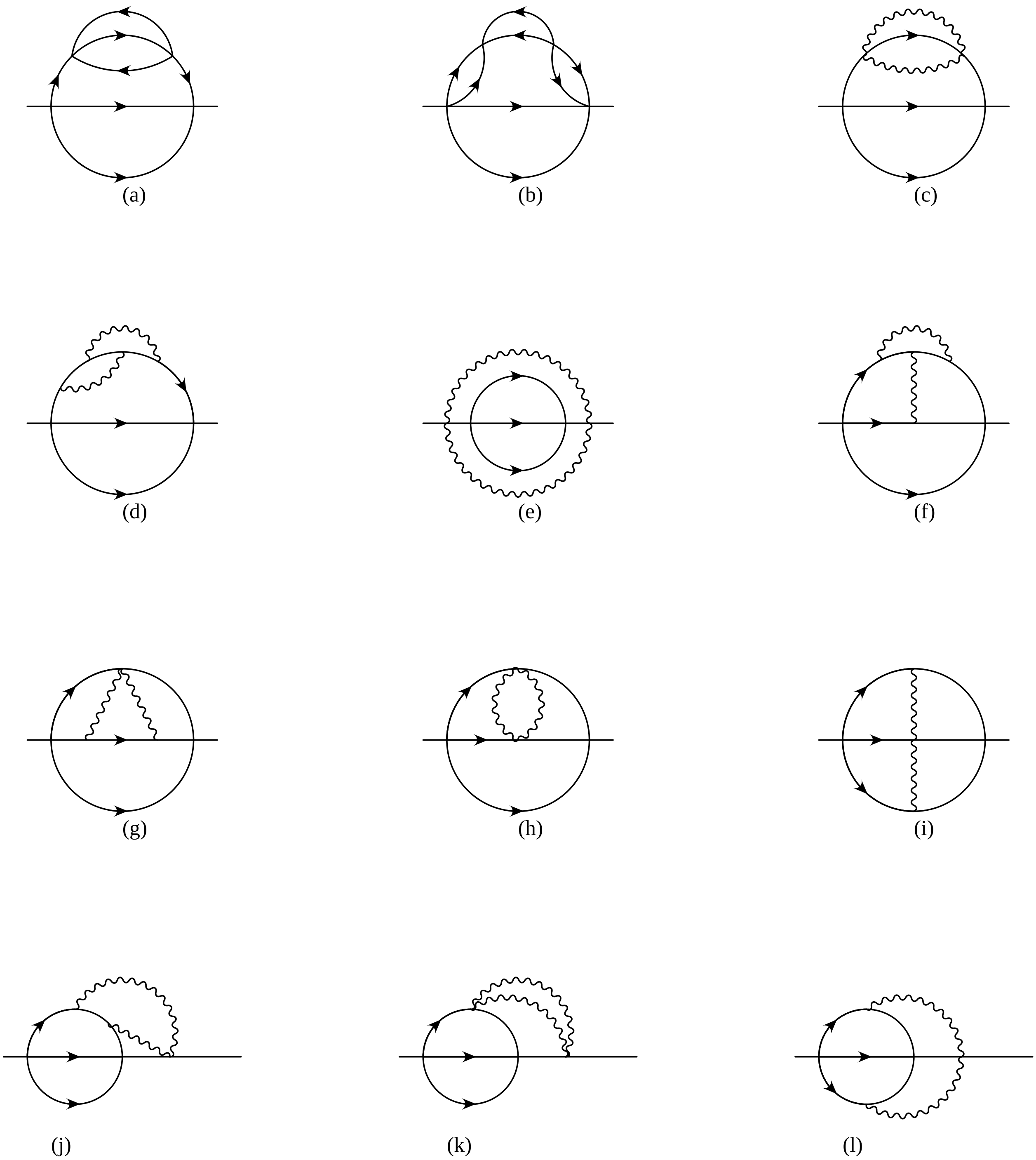

Figure 14: Four-loop diagrams.

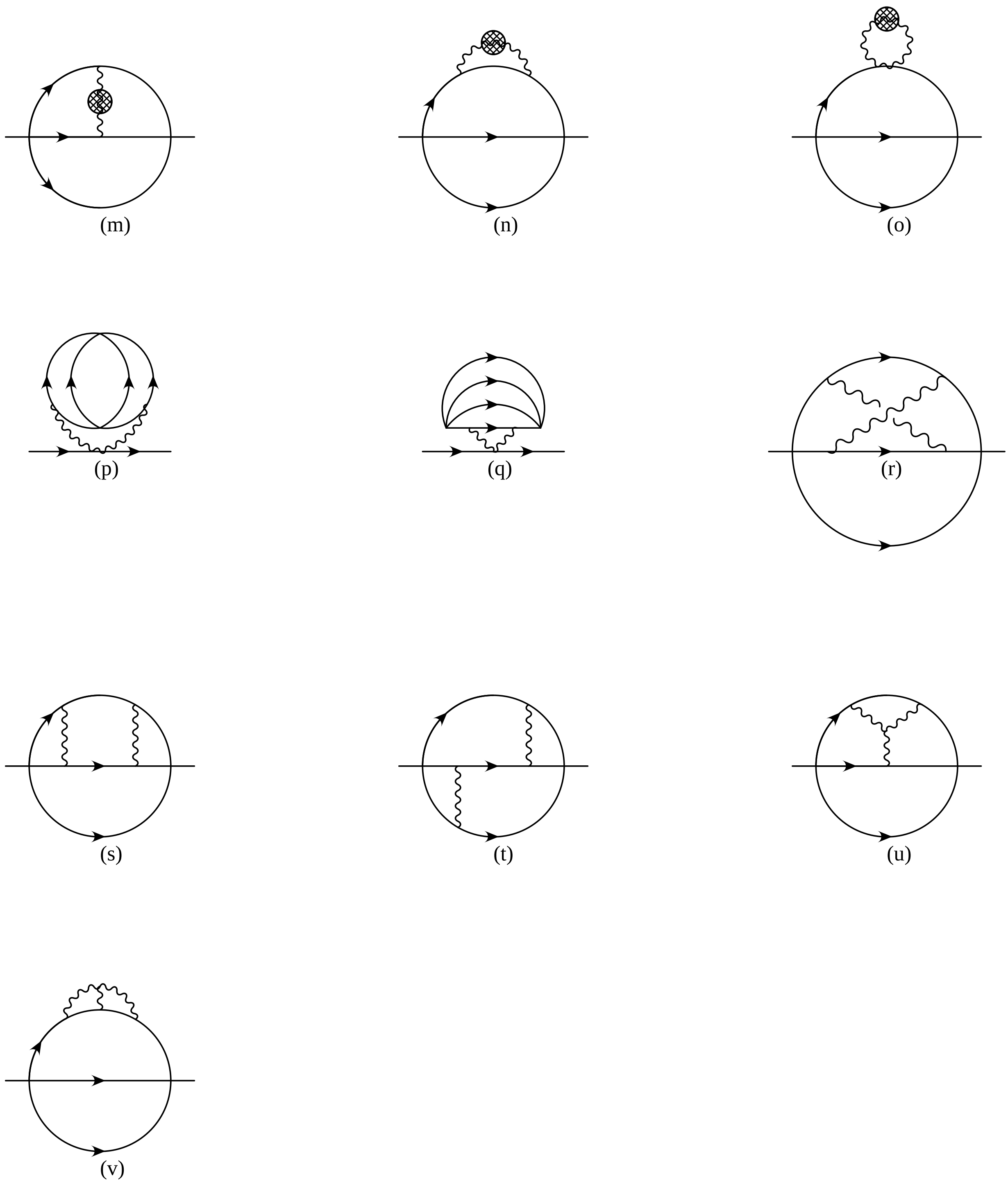

Figure 15: Four-loop diagrams (continued).

$$
\begin{aligned}
G_n &= 3\rho_h \rho_c I_{22}, \\
G_o &= -3\rho_h \rho_c I_{22}, \\
G_p &= 2\rho_h T_3 (I_4 - J_4 - I_{42bbc}), \\
G_q &= -2\rho_h T_3 (I_4 - J_4), \\
G_r &= \frac{1}{\epsilon} h^2 T_4 (a + b\pi^2)
\end{aligned}
\tag{6.20}
$$

where

$$
\rho_h = 2(MN-1)h^2 \tag{6.21}
$$

is the common value of $\rho_{A^{1,2}}$, $\rho_{B_{1,2}}$ upon imposing Eq. (6.19) and

$$
\begin{aligned}
T_1 &= h^2[(N^2 - 2MN + 1)k_1^2 + (M^2 - 2MN + 1)k_2^2 \\
&\quad + 2(M^2N^2 + 2MN - M^2 - N^2 - 1)k_1 k_2], \\
T_2 &= h^2[(N^3M + 5MN - 3N^2 - 3)k_1^2 + (M^3N + 5MN - 3M^2 - 3)k_2^2 \\
&\quad + 4(M^2 + N^2 - 3MN + 1)k_1 k_2], \\
T_3 &= 4[(k_1^2 + k_2^2)MN + 2k_1 k_2], \\
T_4 &= [(3MN - N^2 - 2)k_1^2 + (3MN - M^2 - 2)k_2^2 + 2(N^2 + M^2 - 3MN + 1)k_1 k_2].
\end{aligned}
\tag{6.22}
$$

The results are expressed in terms of a basis of momentum integrals defined and computed in Ref. [54]. The divergent contributions from these momentum integrals are listed in Appendix E. The contributions from Fig. 15(s)-(u) are all finite or zero and therefore not listed explicitly.

The full result obtained by summing the individual contributions in Eq. (6.20),

$$
F^{-4} Z_{A^1}^{(4)} = G^{(4)},
$$

$$G^{(4)} = G_a + \ldots + G_r, \tag{6.23}$$

may be divided into transcendental and rational contributions (according to whether the contribution on the right-hand side of Eq. (C.6) contains a factor of π^2 or not, respectively) as

$$G^{(4)} = G_{\text{rat}} + G_{\text{trans}}\pi^2. \tag{6.24}$$

The transcendental contribution is given by

$$\begin{aligned} G_{\text{trans}} =& \frac{1}{\epsilon}\Big\{h^4(MN^3 + NM^3 - 4M^2 - 4N^2 + 10MN - 4) \\ &+ \frac{1}{6}h^2[k_1^2(-11MN - 5N^3M + 6M^2N^2 + 1 + 9N^2) \\ &+ k_2^2(-11MN - 5M^3N + 6M^2N^2 + 1 + 9M^2) \\ &+ k_1k_2(-8N^2 - 8M^2 + 4M^2N^2 + 32MN - 20)] + bh^2T_4\Big\}. \end{aligned} \tag{6.25}$$

We shall postpone comment on this until later, and focus on the rational contribution, which is given by

$$G_{\text{rat}} = 3\rho_h^2 I_4 - 2\rho_h\rho_k I_4 - 2\rho_h T_3 I_{42bbc} + \frac{a}{\epsilon}h^2T_4, \tag{6.26}$$

where ρ_k is given by Eq. (6.14). We have used here the fact that I_5 as defined in Eq. (C.6) gives only a transcendental simple pole. Since T_4 is $O(N^2)$, the a term from the non-planar graph G_r certainly gives no contribution at leading order $O(N^4)$; and based on experience with non-planar graphs, we believe it is likely that $a = 0$. Upon imposing the one-loop superconformality condition

$$\rho_h = \rho_k, \tag{6.27}$$

(using Eqs. (6.21), (6.14)), we find

$$G_{\text{rat}} = \rho_k^2 I_4 - 2\rho_k T_3 I_{42bbc} + \frac{a}{\epsilon} h_c^2 T_4. \tag{6.28}$$

The value of $h = h_c(k_1, k_2)$ in the 3rd term in Eq. (6.28) will be determined by solving Eq. (6.27) and clearly depends on the particular form of the superpotential. However the remaining terms in Eq. (6.28) are independent of h and thus (since we see from Eq. (6.22) that the 3rd term is subleading in N, M) the form of G_{rat} is independent of the form of the superpotential to leading order. In fact, it is straightforward to see that this result is more general and applies to any theory of the form Eq. (5.2) with a superpotential Eq.(6.4) but without the multi-trace terms. Firstly, the I_4 terms in Eq. (6.20) supply the double pole contributions of the form h^4 and h^2k^2; and this will remain the case for a general theory. The h^4 terms are given according to Eqs. (6.12), (6.14) and (6.13) by

$$\begin{aligned} & 4(\rho_{A^1} + \rho_{B_1})h_1^2(MN+1) \\ + \; & 2(\rho_{A^1} + \rho_{B_1} + \rho_{A^2} + \rho_{B_2})[(|h_3|^2 + |h_4|^2)MN + h_3\overline{h}_4 + h_4\overline{h}_3] - 2\rho_{A^1}^2, \end{aligned} \tag{6.29}$$

which reduces to $6\rho_k^2$ upon imposing the two-loop superconformal invariance condition, now from Eqs. (6.13)

$$\rho_{A^{1,2}} = \rho_{B_{1,2}} = \rho_k. \tag{6.30}$$

This reproduces exactly the contribution of the first term in Eq. (6.26) to Eq. (6.28). The h^2k^2 terms are given according to Eq. (6.12) by $-4\rho_k\rho_{A^1}$ which of course reduces to $-4\rho_k^2$ upon imposing $\rho_{A^1} = \rho_k$. This reproduces exactly the contribution of the second term in Eq. (6.26) to Eq. (6.28). Furthermore, in the general case, the coefficient in G_p in Eq. (6.20) becomes

$$2(\rho_{A^1} + \rho_{A^2} + \rho_{B_1} + \rho_{B_2})[(k_1^2 + k_2^2)MN + 2k_1k_2], \tag{6.31}$$

which reduces to

$$8\rho_k[(k_1^2 + k_2^2)MN + 2k_1k_2] = 2\rho_k T_3 \tag{6.32}$$

upon imposing Eq. (6.30); now reproducing the contribution of the third term in Eq. (6.26) to Eq. (6.28). Finally, the contribution from the non-planar graph G_r is subleading in N, M for any theory with superpotential of the form Eq. (6.4) with $H_1 = H_2 = H_{12} = 0$; in fact the only reason we have had to exclude multi-trace deformations from the present discussion is that otherwise this is no longer true. Therefore the form of G_{rat} in Eq. (6.28) is in general independent of the form of the potential at leading order in M, N upon imposing the conformal invariance condition, as long as multi-trace deformations are excluded. Since we believe it likely that $a = 0$, this result may well also hold at lower orders and in the presence of multi-trace deformations.

The results from the remaining diagrams with no Yukawa couplings are of course also independent of the form of the potential, and the rational contribution from these graphs must take the form

$$-\rho_k^2 I_4 + \frac{1}{\epsilon}\delta(k_1, k_2), \tag{6.33}$$

so that upon adding the rational results from all the graphs in Eqs. (6.28), (6.33) we obtain

$$F^{-4}Z^{(4)}_{\text{rat}A^1} = -2\rho_k T_3 I_{42bbc} + \frac{a}{\epsilon}h^2 T_4 + \frac{1}{\epsilon}\delta(k_1, k_2). \tag{6.34}$$

In other words the double pole has cancelled, as it must due to the lower order superconformal invariance, and we are left with a four-loop rational contribution to the anomalous dimension

$$F^{-4}\gamma^{(4)}_{\text{rat}} = -16\rho_k T_3 + 4ah^2 T_4 + 4\delta(k_1, k_2). \tag{6.35}$$

which is independent of the form of the superpotential at leading order in M, N, and (if $a = 0$) at lower orders too. *A fortiori*, it has the same value for each field A^1, A^2, B_1, B_2.

Therefore, upon imposing the superconformal invariance conditions at lowest order, either the rational contribution to the four-loop anomalous dimension vanishes for any theory, or else a universal non-vanishing result is obtained. If such a non-vanishing result is indeed obtained, then a coupling redefinition is required to restore superconformal invariance. Such redefinitions may not be unique when there are several couplings; but one simple possibility is to redefine each coupling as

$$\delta_{\text{rat}} h_i = \frac{\gamma^{(4)}_{\text{rat}}}{2c_i \rho_k}, \tag{6.36}$$

where c_i is the coefficient of h_i^2 in the two-loop anomalous dimension–fortunately the same coefficient in each anomalous dimension it appears, for each of $h_{1\ldots4}$. In Eq. (6.36), the h_i are chosen as solutions of Eq. (6.30). As we explained earlier, we believe there is strong evidence that there is no N-leading rational correction at four loops for the ABJ model; and our result therefore implies that no rational correction is expected at this order for any theory in the class considered. In fact we believe that our result will also extend to the superconformal theories with flavour matter discussed in Ref. [21]; and, if $a = 0$ in G_r in Eq. (6.20), to theories with multi-trace deformations as well. However, a slight complication here is that the additional couplings in this case do *not* appear with the same coefficient in each two-loop anomalous dimension, and moreover the flavour matter fields would have different "ρ_k" from the A, B fields. The obvious extension of Eq. (6.36) to the full set of couplings would therefore not be appropriate. In this case it is not clear how to give a simple universal form for the required redefinition such as Eq. (6.36); nevertheless the terms required to be cancelled by the redefinitions, analogous to Eq. (6.35), would still be universal in the sense of being independent of the potential.

We shall not consider further here the transcendental contribution for this class of models, since we can draw a stronger conclusion for the case of the second class of models;

suffice it to say that the result given in Eq. (6.25) for the simplest example in this class, Eq. (6.19) clearly gives a model-dependent result upon imposing two-loop superconformality, Eq. (6.27).

We therefore now turn to the second class of models, containing multi-trace deformations, considering the simplest example of such a model, taking in Eq. (6.4)

$$M = N, \quad k_1 = -k_2 = k, \quad h_3 = -h_4, \quad H_{12} = H_1 = H_2. \tag{6.37}$$

In this case the two-loop result in Eq. (6.13) reduces to

$$F^{-2}\gamma^{(2)}_{A^1} = \rho_H - \rho_k, \tag{6.38}$$

where

$$\rho_H = 2h^2(N^2-1) + H^2(2N^2+1), \tag{6.39}$$

with ρ_k given according to Eq. (6.17) but with now in Eq.(6.18)

$$C_1 = 0, \quad C_2 = -2(N^2-1)k^2, \tag{6.40}$$

so that

$$\begin{aligned} \rho_b = -(N^2-1)k^2, \qquad & \rho_c = 3(N^2-1)k^2, \\ \rho_k \quad = \quad & 2(N^2-1)k^2. \end{aligned} \tag{6.41}$$

The results for the diagrams in Figs 14, 15 are now given by

$$\begin{aligned} G'_a =& 3\rho_H^2, \\ G'_b =& [4(N^2-1)(N^2+2)h^4 + 36(N^2-1)h^2H^2 \end{aligned}$$

$$
\begin{aligned}
&+(2N^2+1)(4N^4+6N^2+5)H^4)I_{4bbb},\\
G'_c =& -3\rho_b\rho_H I_4,\\
G'_d =& 3C_2\rho_H I_{4bbb},\\
G'_e =& -\rho_b\rho_H I_4,\\
G'_f =& 2C_2\rho_H I_5,\\
G'_g =& -T'_2 I_5,\\
G'_h =& -T'_2 I_{4bbb},\\
G'_i =& -T'_1 I_5,\\
G'_j =& 4T'_2\left(I_4-\frac{1}{2}I_{4bbb}\right),\\
G'_k =& -2T'_2 I_4,\\
G'_l =& -2T'_1(-2I_4+2I_{4bbb}+I_5),\\
G'_m =& -2\rho_H\rho_c\left(I_4-\frac{1}{2}I_{4bbb}\right),\\
G'_n =& 3\rho_c\rho_H I_{22},\\
G'_o =& -3\rho_c\rho_H I_{22},\\
G'_p =& 2T'_3(I_4-J_4-I_{42bbc}),\\
G'_q =& -2T'_3(I_4-J_4),\\
G'_r =& \frac{2}{\epsilon}(N^2-1)[3h^2-(N^2-1)H^2]k^2(a+b\pi^2),
\end{aligned}
\tag{6.42}
$$

where

$$
\begin{aligned}
T'_1 &= -(N^2-1)k^2[2(N^2+2)^2h^2+12H^2],\\
T'_2 &= (N^2-1)k^2[2(N^2+5)h^2-4(2N^2-5)H^2],\\
T'_3 &= 8(N^2-1)\rho_H k^2.
\end{aligned}
\tag{6.43}
$$

The case $M = N$ and $k_1 = -k_2$ can be expressed in terms of 3-algebras [26]; this lends itself to automation and the results in Eq. (6.42) were obtained using FORM [60].

For this class of models we shall start by discussing the transcendental contributions to the anomalous dimension, since the results are more striking than for the rational case. The transcendental contribution (from graphs with Yukawa couplings) is given by summing G'_r together with the contributions involving I_5 in Eq. (6.42)and using Eq. (C.6):

$$\begin{aligned} G_{\text{trans}} = \frac{1}{2\epsilon}\Big\{&4(N^2-1)(N^2+2)h^4 + 36(N^2-1)h^2H^2 \\ &+ (2N^2+1)(4N^4+6N^2+5)H^4 \\ &+ 2(N^2-1)\Big[-\frac{2}{3}(N^2+11)h^2 + (2N^2-3)H^2 \\ &+ 3bh^2 - (N^2-1)bH^2\Big]k^2\Big\}. \end{aligned} \tag{6.44}$$

To lowest order the vanishing of the anomalous dimensions now requires (using Eqs. (6.38), (6.39), (6.41)) that the couplings h and H must be chosen to satisfy

$$2(N^2-1)h^2 + (2N^2+1)H^2 = 2(N^2-1)k^2. \tag{6.45}$$

In order for G_{trans} to adopt a universal form upon imposing two-loop superconformal invariance as in Eq. (6.45), we would require Eq. (6.44) to adopt the form

$$G_{\text{trans}} = f(2(N^2-1)h^2 + (2N^2+1)H^2). \tag{6.46}$$

This is clearly not the case. We shall therefore consider the two cases $H = 0$ and $H \neq 0$ separately, and find that they are very different. If $H = 0$ then the superconformal invariance condition $\gamma = 0$ becomes to lowest order simply

$$h^2 = k^2 \tag{6.47}$$

and imposing Eq. (6.47), Eq. (6.44) reduces to

$$G_{\text{trans}} = \frac{1}{\epsilon}g(N) \equiv \frac{1}{\epsilon}(N^2-1)\left[\frac{2}{3}(2N^2-5)+6b\right]k^2. \tag{6.48}$$

If the graphs we have not computed (i.e. those with no Yukawa couplings) are assumed to give a transcendental contribution

$$\frac{1}{\epsilon}\{(N^2-1)[\sigma_1+\sigma_2N^2]k^2-g(N)\}, \tag{6.49}$$

then using Eq. (5.16) the total transcendental contribution to the anomalous dimension at this order is

$$F^{-4}\gamma^{(4)}_{\text{trans}} = 4(N^2-1)[\sigma_1+\sigma_2N^2]k^2 \tag{6.50}$$

and so we need to make a redefinition

$$F^{-2}\delta_{\text{trans}}h = -2(\sigma_1+\sigma_2N^2)h_c \tag{6.51}$$

to restore superconformal invariance at this order. The "superconformal" value for h is (from Eq. (6.47)) simply $h_c = k$. (The factor N^2-1 in Eq. (6.49) may be inferred from the fact that for $K_1 = -K_2$, all the contributions vanish identically in the abelian case due to the "quiver" structure.) Note that in Eq. (6.51) we are still suppressing the "transcendental" factor of π^2.

Returning to the case of $H \neq 0$, it is clear that the redefinition of Eq. (6.51) is not enough, since it could not cancel the N^6 terms in Eq. (6.44). In fact, we shall now require the redefinition of Eq. (6.51) (where now h_c is a solution of Eq. (6.45), together with a corresponding H_c), supplemented by a further redefinition of H, given by

$$
\begin{aligned}
F^{-2}\delta_{\text{trans}}H = &-\frac{1}{96(N^2-1)(2N^2+1)}H_c[16(N^2-1)(8N^4+23N^2-22)\\
&-24(N^2-1)(2N^4-10N^2-1)b)\\
&+3H_c^3(2N^2+1)(4N^6+4N^4+22N^2-21].
\end{aligned}
\tag{6.52}
$$

This is easily derived by using Eq. (6.45) to substitute for h in terms of H in Eq. (6.44). It will be noted that $\delta_{\text{trans}}H$ is higher order in N^2 than $\delta_{\text{trans}}h$ owing to the N^6 term in Eq. (6.44). We therefore conclude that even in the best-case scenario where in Eq. (6.49), $\sigma_{1,2}=0$ so that $\delta_{\text{trans}}h=0$ in Eq. (6.51), and consequently no "transcendental" redefinition is required in the ABJM model, a redefinition will nevertheless inevitably be required as soon as multitrace deformations are included. Finally turning to the rational contribution to the anomalous dimension for these models, the discussion would largely follow that for the previous class of models. However as mentioned there, for $H_1, H_2, H_{12}\neq 0$ we would find a model-dependent contribution from G'_r upon imposing two-loop superconformal invariance, and this would require a model-dependent redefinition of $H_1, H_2, H_{12}\neq 0$ akin to Eq. (6.52).

6.4 Summary

We have shown that on the one hand, superconformal invariance of Chern-Simons theories requires transcendental corrections beyond leading order for fairly simple generalisations of the ABJ model (namely those with "multi-trace deformations"); and on the other hand, that at leading order (and likely beyond) in N, M, the rational corrections for a wide class of theories have a universal form. We have also argued (based on the results of Ref. [54] that it follows that there is in fact no rational correction required for any theory of the form discussed. Our conclusions could be extended beyond leading order in N by the computation of the non-planar diagram in Fig. (15)(r). It would also be of considerable interest to complete the calculation of the non-Yukawa diagrams, even if only at leading order in N, M (which would avoid the need for further non-planar diagram calculations) and only for the ABJ model. It now seems very likely that the majority of superconformal theories will require a ("transcendental") coupling redefinition beyond leading order in order to retain the superconformal invariance. One might *a priori* entertain the possibility that there might exist a special "superconformal" renormalisation scheme in which all these theories were superconformal beyond leading order; analogous to the "holomorphic" scheme in four dimensions in which the gauge β-function adopts the NSVZ form. In this case, there would be expected to be a universal redefinition in terms of a general superpotential which would effect the transformation to this scheme. However, we now see that this redefinition is likely to depend in a highly non-trivial way on the form of the superconformal theory, and will probably need to be computed independently for each theory. This reduces considerably the constraints on the form of the four-loop anomalous dimension which might follow from requiring superconformality. Regrettably we are forced to conclude that there is no alternative to computing the full anomalous dimension if we wish to pursue further checks on superconformality.

7 Conclusions

Having calculated results for the anomalous dimension and β-functions at two loops for a softly broken version of $\mathcal{N}=2$ Chern-Simons matter theories we showed that our results were consistent with those obtained from the superfield formalism. We then considered looking at possible extensions to the theory, for instance the maximal supersymmetric Chern-Simons theory which for a theory with a single gauge group is $\mathcal{N}=3$. The component formulation of this theory was presented in Ref. [61]. The quantum properties of this theory were discussed in Ref. [62] based on the $d=3$ $\mathcal{N}=3$ harmonic superspace formalism developed in Ref. [63], and it was shown that this theory is all-orders finite. It would be interesting to investigate whether the softly-broken version of this theory is also finite. Theories with higher degrees of supersymmetry (up to $\mathcal{N}=8$) [27] may be obtained in the case of direct product groups and matter in the bi-fundamental representation. A rich variety of these theories [64, 65] are expected to be superconformal by virtue of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, originally stated in Ref [29]. These theories can be expressed in terms of $\mathcal{N}=2$ superfields and are obtained by a judicious choice of field content and also a particular choice of Yukawa couplings (as a function of the gauge couplings). The conformal properties of a range of these models was checked explicitly at the two-loop level in Refs. [21,26]. It would be quite straightforward to extend our results to the case of direct product gauge groups and thereby derive exact results for the softly broken versions of these theories. One could then ask whether there were a choice of soft couplings which would maintain finiteness. In the case of β_h this would entail arranging for γ_1 to vanish; this is not guaranteed by the vanishing of γ, since the derivative in Eq. (4.11) would be taken before specialising to the special form for the Yukawa couplings which guarantees the extended supersymmetry. Nevertheless it was shown in the four-dimensional case [31] that there was a choice of soft couplings which would guarantee $\gamma_1=0$. However this relied on the existence of the gaugino mass as a soft coupling and a similar choice is not possible here; there is therefore no obvious way to guarantee the vanishing of β_h. The same argument applied to Eq. (4.14) would imply that we could not render β_{m^2} zero. The softly-broken versions of these superconformal theories would therefore not be finite. Finally, it would be interesting to address the question of gauge groups with a $U(1)$ factor, where, as we have noted, there are additional technical subtleties.

However we decided to look at the next to leading order calculation for the general $\mathcal{N} = 2$ supersymmetric Chern-Simons theory with the idea of exploiting the four loop results in order to verify the superconformality properties of various superconformal theories, as given explicitly either in terms of a "quiver" description or using 3-algebra structures.

We calculated a large portion of the four-loop calculation and presented the results in exhaustive detail and although we were unable to finish it we gave many details that any prospective parties who choose to continue the calculation could take advantage of although not for the non-planar case. Our initial idea of exploiting the superconformality constraints to determine the rest of the coefficients now seems very unlikely, in light of the results we calculated relating to the superconformality of the Yukawa terms beyond leading order. While it is true that the rational corrections did not require a coupling redefinition to preserve superconformality the "transcendental" corrections did. This was to such a degree for the ABJ model that it seems very likely that each theory would require its own specific redefinition to the Yukawa terms. The most likely case for determining these would require the completion of the four-loop calculation. This would be greatly simplified if it were possible to determine the rule for the three point vertex diagrams canceling as this would vastly reduce the number of planar diagrams to be calculated and could even provide for the cancellation of many of the non-planar diagrams.

A Useful Group Theory Identities

The table of group structures may be readily obtained using the following easily derived but useful group identities:

$$
\begin{aligned}
R_B R_A R_B &= \left(C_2(R) - \frac{1}{2} C_2(G) \right) R_A, \\
f^{ABE} f^{CDE} R_A R_C R_B R_D &= 0, \\
R_A R_B R_C R_A R_B R_C &= C_{30} - \frac{3}{2} C_{21} + \frac{1}{2} C_{12}, \\
R_A R_B R_C R_D R_A R_B R_C R_D &= C_{40} - 3C_{31} + \frac{11}{4} C_{22} - \frac{3}{4} C_{13} + F_4, \\
R_A R_B R_C R_A R_D R_B R_C R_D &= C_{40} - \frac{5}{2} C_{31} + 2C_{22} - \frac{1}{2} C_{13} + F_4, \\
R_A R_B R_C R_D R_A R_C R_B R_D &= C_{40} - 2C_{31} + \frac{3}{2} C_{22} - \frac{3}{8} C_{13} + F_4, \\
f^{ABC} R_A R_D R_C R_E R_D R_B R_E &= -i \left(\frac{1}{2} C_{31} - \frac{3}{4} C_{22} + \frac{1}{4} C_{13} - F_4 \right), \\
f^{ABF} f^{CDF} R_A R_C R_E R_D R_B R_E &= \frac{1}{4} C_{22} - \frac{1}{8} C_{13} + F_4, \\
f^{ABF} f^{CDF} R_A R_C R_E R_B R_D R_E &= F_4, \\
f^{ABC} f^{DEF} R_A R_D R_B R_E R_C R_F &= -\frac{1}{4} C_{22} + \frac{1}{8} C_{13} - F_4
\end{aligned} \tag{A.1}
$$

B Useful formulae for Dimensional Regularisation

The following formulae where used in the component calculation

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^m[(k-p)^2]^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(m+n-\frac{d}{2})}{\Gamma(m)\Gamma(n)} \frac{\Gamma(\frac{d}{2}-n)\Gamma(\frac{d}{2}-m)}{\Gamma(d-m-n)} \frac{1}{(p^2)^{(m+n-\frac{d}{2})}} \tag{B.1}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k_\mu}{(k^2)^m[(k-p)^2]^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(m+n-\frac{d}{2})}{\Gamma(m)\Gamma(n)} \frac{\Gamma(\frac{d}{2}-n)\Gamma(\frac{d}{2}-m+1)}{\Gamma(d-m-n+1)} \frac{p_\mu}{(p^2)^{(m+n-\frac{d}{2})}} \tag{B.2}$$

$$\begin{aligned}\int \frac{d^dk}{(2\pi)^d} \frac{k_\mu k_\nu}{(k^2)^m[(k-p)^2]^n} &= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{\Gamma(d-m-n+2)} \frac{1}{(p^2)^{(m+n-\frac{d}{2})}} \\ &\times [\frac{1}{2}p^2\delta_{\mu\nu}\Gamma(m+n-\frac{d}{2}-1)\Gamma(\frac{d}{2}-m+1)\Gamma(\frac{d}{2}-n+1) \\ &+ p_\mu p_\nu \Gamma(m+n-\frac{d}{2})\Gamma(\frac{d}{2}-m+2)\Gamma(\frac{d}{2}-n)]\end{aligned} \tag{B.3}$$

C Momentum Integrals

Here is the list of results for the divergences of our basis of subtracted momentum integrals [54, 55]:

$$I_4 = \frac{1}{(8\pi)^4}\left(-\frac{1}{2\epsilon^2} + \frac{2}{\epsilon}\right) \tag{C.1}$$

$$I_{22} = -\frac{1}{(8\pi)^4\epsilon^2} \tag{C.2}$$

$$I_{4bbb} = \frac{1}{(8\pi)^4}\frac{\pi^2}{2\epsilon} \tag{C.3}$$

$$I_{42bbc} = \frac{1}{(8\pi)^4}\frac{2}{\epsilon} \tag{C.4}$$

$$I_{422qAbBd} = \frac{1}{(8\pi)^4}\left[\frac{1}{4\epsilon^2} + \frac{1}{\epsilon}\left(\frac{5}{4} - \frac{\pi^2}{12}\right)\right] \tag{C.5}$$

$$I_5 = \frac{1}{4}I_4 - \frac{5}{8}I_{22} - I_{4bbb} + I_{42bbc} - 2I_{422qAbBd} \tag{C.6}$$

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