Time-like reductions of supergravity and black string solutions

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by

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Abstract

We consider various geometrical and physical aspects of the supergravity q-maps, which are induced by dimensional reduction to three dimensions of five-dimensional $\mathcal{N} = 2$ supergravity theories coupled to vector multiplets. In this way, the q-maps can be thought of as a composition of the r-maps and c-maps. We treat in parallel the case of reduction over two space-like directions and over one space-like and one time-like direction. We observe that in the latter case, surprisingly, the order in which the time-like and space-like reductions are performed is relevant for some geometrical properties of the resulting reduced theories. For the simplest example of pure supergravity in five dimensions, we show indeed that the target manifolds obtained from the two reductions correspond to inequivalent open submanifolds in the pseudo-Riemannian symmetric space $G_{2(2)}/(SL_2 \cdot SL_2)$. Moreover, each submanifold is endowed with a different integrable structure which makes one a complex manifold and the other a para-complex manifold.

As an application we investigate how the q-map can be used to generate new non-extremal and extremal non-BPS static black string solutions in five dimensions. We also make progress towards constructing new stationary solutions. The generic nature of these constructions, which don't rely on the target manifolds being symmetric spaces, allow us to gain a more systematic understanding of various properties of black objects in supergravity.

Declaration

I hereby declare that all work described in this thesis is the result of my own research unless reference to others is given. None of this material has previously been submitted to this or any other university. All work was carried out in the Theoretical Physics Division of the Department of Mathematical Sciences, University of Liverpool, UK, during the period of October 2010 until April 2014.

Publication list

This thesis contains material that has appeared in the following publications by the author:

(1) P. Dempster and T. Mohaupt, *"Non-extremal and non-BPS extremal five-dimensional black strings from generalized special real geometry,"* Class. Quant. Grav. **31** (2014) 045019 [arXiv:1310.5056].

(2) V. Cortés, P. Dempster and T. Mohaupt, *"Time-like reductions of five-dimensional supergravity,"* JHEP04 (2014) 190 [arXiv:1401.5672].

Unpublished material will also be presented, some of which is scheduled to appear in the following publications:

(3) V. Cortés, P. Dempster, T. Mohaupt and O. Vaughan, *"Special geometry of Euclidean supersymmetry IV: the local c-map,"* to appear.

(4) V. Cortés, P. Dempster and T. Mohaupt, *"Time-like reductions of five-dimensional supergravity with vector multiplets,"* to appear.

There are also two publications by the author, completed during the period of the degree, that will not be presented in this thesis:

(5) P. Dempster and M. Tsulaia, *"On the structure of quartic vertices for massless higher spin fields on Minkowski background,"* Nucl. Phys. **B865** (2012) 353-375 [arXiv:1203.5597].

(6) I. L. Buchbinder, P. Dempster and M. Tsulaia, *"Massive higher spin fields coupled to a scalar: aspects of interaction and causality,"* Nucl. Phys. **B877** (2013) 260-289 [arXiv:1308.5539].

Contents

Chapter 1

Introduction

String theory has long been considered the most likely candidate to provide a consistent theory of quantum gravity. Since the "second superstring revolution" we have been able to go beyond string perturbation theory and understand many of the non-perturbative aspects of string theories, both through various dualities and through the emergence of the eleven-dimensional theory known as M-theory [1–4]. An important tool in this regard is provided by supergravity theories, which arise naturally as low-energy effective theories of string and M-theory.

By considering strings propagating in backgrounds which involve different compact manifolds, the number of supercharges, as well as the matter content, admitted by the effective supergravity theories can be varied. An important aspect of such supergravity theories, which we will make use of throughout this thesis, is the fact that they come equipped with two types of geometry: spacetime geometry and the geometry of the scalar target manifold. For theories with 16 or more supercharges ($\mathcal{N} \geq 4$ in four dimensions), the geometry of the scalar manifold is completely fixed once the matter content of the theory is specified. On the other hand, for theories with 8 or 4 supercharges ($\mathcal{N} = 2$ or $\mathcal{N} = 1$ in four dimensions) the matter content does not completely fix the scalar geometry, and so such theories can possess a much richer structure.

Whilst the scalar manifolds of $\mathcal{N} = 1$ theories in four dimensions are only required to be Kähler [5], those of the $\mathcal{N} = 2$ theories (which we will concentrate on in this thesis) must satisfy more restrictive conditions, and are controlled by the properties of *special geometry*. As well as being interesting for purely mathematical reasons [6, 7], such geometry plays an important role in the understanding of non-perturbative aspects of gauge theories [8, 9], string compactifications [10, 11], and the microscopic understanding of black hole physics [12–15]. More recently, special geometry has been used to construct new solutions to $\mathcal{N} = 8$ supergravity [16,17], as well as to obtain new asymptotically-AdS solutions in gauged supergravities [18–20] with applications to the AdS/CFT correspondence.

We now present an introduction to the main results in this thesis:

The r-map, c-map and q-map

For the $\mathcal{N} = 2$ supergravity theories in which we will be interested in this thesis one can understand the effect of both space-like and time-like dimensional reduction via a series of maps between the target manifolds of such theories: the r-maps [21, 22], c-maps [23, 24] and q-maps. This latter can be understood as the composition $q = c \circ r$ of an r-map and a c-map. These maps provide us with important tools both in a mathematical and physical context, and remain an active area of research.

Mathematically the r-map, c-map, and by extension the q-map, all preserve completeness [25]. Therefore, by classifying complete projective special real manifolds, a project initiated in [26], one can obtain large classes of complete, and generically non-homogeneous, quaternionic-Kähler manifolds.

Physically, each of these maps can be understood as a relation between the target manifolds of two rigid or locally supersymmetric field theories. In this thesis we will be interested in taking as our starting point a five-dimensional $\mathcal{N} = 2$ theory of n vector multiplets.

In the case of rigid supersymmetry, the target manifold of this theory is an affine special real manifold M_n. By space-like reduction, we obtain a four-dimensional theory of n vector multiplets, with target space an affine special Kähler manifold. A further space-like reduction gives a three-dimensional theory, for which the degrees of freedom can be packaged into n hypermultiplets, with target space a hyperkähler manifold

Q_{4n} [27]. This provides us with maps

$$M_n \xrightarrow{r} N_{2n} \xrightarrow{c} Q_{4n}.$$

If instead one reduces over a time-like direction, then the target manifolds of the resulting Euclidean theories are equipped with a split-signature metric, and can be described using para-complex geometry [28, 29]. Indeed, time-like reduction of the five-dimensional theory gives rise to a target space which is affine special para-Kähler [28]. Likewise, both time-like reduction of the four-dimensional Minkowski theory and space-like reduction of the four-dimensional Euclidean theory give rise to target spaces which are para-hyperkähler [29].

In the case of local supersymmetry, the target manifold of the five-dimensional $\mathcal{N} = 2$ supergravity theory with n vector multiplets is a projective special real manifold $\bar{M}_n$ [30]. By space-like reduction we obtain a four-dimensional theory of $(n + 1)$ vector multiplets, the extra degrees of freedom compared to the rigid case coming from the reduction of the gravity multiplet. The relevant target manifold is a projective special Kähler manifold $\bar{N}_{2n+2}$. A further space-like reduction gives a three-dimensional supergravity theory, for which the degrees of freedom can this time be packaged into $(n + 2)$ hypermultiplets, with target space a quaternionic-Kähler manifold $\bar{Q}_{4n+8}$ [24], as required by supersymmetry [31]. This provides us with maps

$$\bar{M}_n \xrightarrow{\bar{r}} \bar{N}_{2n+2} \xrightarrow{\bar{c}} \bar{Q}_{4n+8}.$$

If instead one reduces over a time-like direction, then the target manifolds of the resulting Euclidean-signature supergravity theories are again equipped with split-signature metrics, and can be described using para-complex geometry. Time-like reduction of the five-dimensional theory gives rise to a target space which is projective special para-Kähler [32].

The study of the time-like versions of the c-map has only been undertaken fairly recently, in work by the author and collaborators [33, 34]. Here it is shown that both time-like reduction of the four-dimensional Minkowski theory and space-like reduction of the four-dimensional Euclidean theory give rise to a three-dimensional Eulclidean theory whose target manifold is para-quaternionic-Kähler.

In Chapter 4 we analyse in detail the structure of those (para-)quaternionic-Kähler manifolds which are in the image of one of the q-maps.

Commutativity of space-like and time-like reductions

A central theme throughout this thesis is the use of time-like dimensional reduction as a solution-generating technique. In particular, instanton solutions to the dimensionally-reduced Euclidean theories can be dimensionally lifted to provide us with stationary solitonic solutions, e.g. black holes, to our original supergravity theories [35]. Such solutions play a crucial role as testing ground for various conjectures in string theory and other theories of quantum gravity [1].

The structure of Euclidean supergravity theories is much less well understood than Minkowski-signature ones, and they often present us with surprising features. For example, in [36] it was shown that time-like reductions of IIA and IIB supergravities give rise to two inequivalent (i.e. not related by a real field redefinition) nine-dimensional Euclidean supergravities. This is in contrast to the case of space-like reduction, where the two nine-dimensional Minkowski theories are related by real field redefinitions: an artefact of T-duality. One of the primary motivations for this thesis is to better understand the structure of Euclidean-signature field theories (both with rigid and local supersymmetry), continuing the work of [28, 29, 32].

An important question that arises in the study of Euclidean supergravity theories obtained from both space-like and time-like reductions is whether the order of the reduction matters. In [36] it was argued that, for the case where the scalar manifold of the Euclidean theory is a homogeneous (coset) space, this order does not matter. However, their argument rested on being able to parametrise the scalar coset manifold using the Borel gauge, which for the cosets appearing in Euclidean supergravity theories (which have non-compact stability group [35]) does not provide a *global* parametrisation of the manifold.

In this thesis, we concentrate on the reductions of five-dimensional $\mathcal{N} = 2$ supergravity coupled to an arbitrary number of vector multiplets. Upon reduction to a three-dimensional Euclidean theory, the scalar target manifolds are generically non-homogeneous. However, one can use properties of the special geometry underlying these theories to get a handle on the admissible geometric structures which are present for all such spaces.

A surprising result of this thesis, which has been presented in a publication by the author [37], is that, even for the simplest case of pure five-dimensional supergravity, some of these geometric structures are sensitive to the order in which the space-like and time-like reductions occur. Note that this is in contrast to the situation for rigid supersymmetry, where it was shown in [29] that the three-dimensional Euclidean theories obtained by time-then-space (TS) and space-then-time (ST) reductions could be related by a real field redefinition.

In Chapter 5 we show the two scalar manifolds obtained from TS and ST reduction of the five-dimensional pure supergravity theory can be distinguished by the existence of different integrable structures on each. Indeed, for TS reduction, the scalar manifold comes equipped with an integrable para-complex structure, whilst for ST reduction we find an integrable complex structure. This fact is generic for all target manifolds obtained after dimensional reduction of the five-dimensional $\mathcal{N} = 2$ theories to three Euclidean dimensions, and so this non-commutativity of time-like and space-like reductions is expected to carry over to the general five-dimensional theory with an arbitrary number of vector multiplets. This will be analysed further in a future publication by the author [38].

Black string solutions

In four dimensions, no-go theorems [39] forbid the existence of extended black objects in general relativity with horizon topology different to S^2, which is the case for black holes.

However, higher dimensions are less restrictive, and one can construct solutions with more exotic horizon topology, see e.g. [40]. The general class of such solutions that we are interested in in this thesis are black branes, which possess translational symmetry in some spacetime directions. Such solutions are of particular interest in the context of string theory.

Such black objects come in two main classes: extremal and non-extremal. These are distinguished thermodynamically by the fact that extremal black objects have vanishing horizon temperature. In theories of extended supergravity, extremal objects can be further separated into BPS and non-BPS. The BPS objects are characterised by the preservation of some degree of the supersymmetry of their parent theory, and saturate the Bogomolnyi bound relating their mass to the central charge of the underlying supersymmetry algebra.

The advantage of studying BPS solutions is that they can be obtained and classified using the so-called Killing spinor equations [41]. These are a set of first-order differential equations, and are therefore generally easier to solve than the full second-order field equations. Such objects are by now well understood, both at a macroscopic and microscopic level. Constructing non-BPS and non-extremal solutions is significantly more involved, since these do not satisfy the Killing spinor equations, and as such there is still a lot to discover.

In this thesis we will concentrate on the case of stationary black strings (1-branes) in five dimensions. These possess translational symmetry in both a time-like and space-like direction, which constitute the worldvolume directions of the string. By dimensionally reducing over these isometric directions we obtain a three-dimensional Euclidean-signature supergravity theory, to which we can find instanton solutions. These can then be lifted back to five dimensions, and to the sought-after black string solutions. This procedure is generally known as 'diagonal' dimensional reduction and oxidation [42].

This dimensional reduction provides us with the link between the spacetime geometry and the target space geometry. In particular, reduction from five to three dimensions implements the q-map at the level of the target space. All of the five-dimensional degrees of freedom then parametrise the para-quaternionic-Kähler manifold in the image of the q-map. In fact, for the static extremal black strings we will meet in Chapter 6, the truncation of certain five-dimensional fields means that we are restricted to a specific para-Kähler submanifold of the full para-quaternionic-Kähler manifold.

Our method for constructing instanton solutions to the three-dimensional Euclidean theory follows the philosophy of [43], and can be interpreted from the point-of-view of the target space geometry in the following manner. Single-centred extremal BPS black strings correspond to null geodesic curves contained within the eigendistributions of the integrable para-complex structure, whilst non-BPS black strings correspond to null geodesic curves *not* contained within these eigendistributions. Indeed, we can use such a geometrical characterisation of extremal solutions whenever the underlying target manifold admits an integrable para-complex structure. In Chapter 7 we utilise this fact to construct new BPS and non-BPS stationary solutions for which we relax the condition of staticity.

The construction of non-extremal solutions is less formulaic. Since non-extremal black objects are non-BPS, we are unable to use Killing spinor equations. However, various methods have been used to construct non-extremal black holes and black branes, which often involve reducing the equations of motion to first-order [44–46].

In Chapter 6, we extend the formalism of [47, 48] and construct non-extremal solutions directly at the level of the equations of motion. We first impose that the solutions be spherically symmetric in the three-dimensional space transverse to the string. We are then able to integrate the second-order equations of motion directly and obtain a general solution, before finding conditions on the integration constants which ensure that the five-dimensional solutions correspond to physical black strings with finite scalar fields. These solutions have appeared previously in a publication by the author [49].

This formalism can also be extended to more general non-extremal stationary solutions, and we make progress in this direction in Chapter 7.

Outline

This thesis is organised as follows: in Chapters 2 and 3 we introduce the main mathematical and physical background needed to understand the bulk of this thesis. In Chapter 4 we analyse the supergravity q-maps for the case of an arbitrary number of vector multiplets coupled to supergravity, and motivate the question of ST vs. TS reductions. We then concentrate, in Chapter 5, on the example of pure supergravity in five dimensions, and show that dimensional reduction over one space-like and one time-like direction provides us with two inequivalent open submanifolds of the symmetric space $G_{2(2)}/(SL_2 \cdot SL_2)$. In Chapters 6 and 7 we then turn to spacetime geometry. In Chapter 6 we construct new non-extremal and extremal non-BPS static black string solutions to five-dimensional supergravity with vector multiplets. Then, in Chapter 7, we relax the condition of staticity and construct more general stationary solutions. We finish in Chapter 8 with conclusions and ideas for future work.

Notation and conventions

We use the conventions of [5].

Chapter 2

Preliminary mathematics

In this chapter we introduce the various mathematical ideas which will be important throughout this thesis. We begin by simply requiring the existence of a differentiable manifold, and add further structure (metric, complex structure, quaternionic structure, etc.) to this as we go.

We begin in Section 2.1 with an overview of differential geometry, introducing the elementary material upon which the rest of the chapter builds and working up to special real geometry, which will play an important role in this thesis. In Section 2.2 we introduce the dual notions of complex and para-complex differential geometry, with the aim of describing projective special (para-)Kähler manifolds, before moving on to discuss (para-)quaternionic-Kähler manifolds in Section 2.3. Finally, in Section 2.4, we turn to the subject of Lie algebras, presenting a number of key results which will be of use in Chapter 5.

2.1 Differential geometry

In this section we introduce a number of fundamental concepts in differential geometry, with the aim of familiarising the reader with the tools needed to approach the bulk of this thesis. We do not claim to give a completely rigorous account of the material, and refer liberally throughout to the relevant source material for further information.

After some basics, we define in Section 2.1.2 the notion of an (affine) connection on a differentiable manifold , and use this to discuss geodesy, holonomy and curvature.

We then equip our manifold with a pseudo-Riemannian metric in Section 2.1.3 and use this to discuss the Levi-Civita connection, orthonomal frames, pseudo-Riemannian symmetric spaces, and finish with Berger's theorem on Riemannian holonomy. We end with Section 2.1.4 on the subject of special real geometry, which contains many of the results we will use throughout this thesis.

2.1.1 Basics of differential geometry

Throughout this section we take M to be some arbitrary m-dimensional differentiable manifold. We denote by $\Gamma(TM)$ the set of smooth vector fields on M, i.e. the set of sections of the tangent bundle TM, and by $\mathfrak{F}(M)$ the set of smooth functions on M.

Integral curves

Let $X \in \Gamma(TM)$ be some smooth vector field. Then we define an **integral curve** generated by X to be a function

$$\gamma : [a, b] \to M,$$

parametrized by $t \in [a, b]$ such that the tangent to the curve at a point $\gamma(t) \in M$ is $X_{\gamma(t)}$, i.e. the value of the vector field at that point.

In particular, take some local coordinate patch $U \subset M$ on which we choose a set of local coordinates $\{x^\mu\}$. Then the integral curve is defined by the property

$$\frac{dx^\mu}{dt} = X^\mu(x(t)). \tag{2.1}$$

2.1.2 Connections on the tangent bundle

We now introduce the notion of an affine connection on a differentiable manifold M. Note that no extra structure, e.g. existence of a metric, is required in this subsection.

Affine connections

We define an **affine connection** ∇ via the map

$$\begin{aligned} \nabla : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X, Y) &\mapsto \nabla_X Y. \end{aligned} \tag{2.2}$$

In order to be a connection, ∇ must satisfy some properties, namely linearity in its first argument and the Leibniz identity in its second, viz.

$$\nabla_{fX+gY} Z = f \nabla_X Y + g \nabla_Y Z,$$

for any $X, Y, Z \in \Gamma(TM)$ and $f, g \in \mathfrak{F}(M)$ some smooth functions on M, and

$$\nabla_X (fY) = X(f)Y + f \nabla_X Y,$$

for any $X, Y \in \Gamma(TM)$ and $f \in \mathfrak{F}(M)$.

Another way of formulating this is via the language of **derivations**. In particular, we choose some smooth vector field $X \in \Gamma(TM)$. Then define a linear map

$$\nabla_X : \Gamma(TM) \to \Gamma(TM),$$

which acts as a derivation on the module of smooth vector fields $\Gamma(TM)$ over the ring of smooth functions $\mathfrak{F}(M)$. In this context, saying that ∇_X acts as a derivation just tells us that it is linear $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$ and satisfies the Leibniz property as before.

Affine connection in a coordinate basis

Suppose we now choose some coordinate basis for TM, so $X = X^\mu \partial_\mu$. Then

$$\begin{aligned} \nabla_X Y &= X^\mu \nabla_\mu (Y^\lambda \partial_\lambda) \\ &= X^\mu (\partial_\mu Y^\lambda) \partial_\lambda + X^\mu Y^\nu (\nabla_\mu \partial_\nu) \end{aligned}$$

$$= \quad X^\mu \left[\partial_\mu Y^\lambda + \Gamma^\lambda_{\mu\nu} Y^\nu\right] \partial_\lambda,$$

where we have used both linearity (first line) and the Leibniz property (second line) of the connection. The **connection coefficients** are defined via.

$$\nabla_\mu \partial_\nu =: \Gamma^\lambda_{\mu\nu} \partial_\lambda,$$

i.e. they are the components of the vector field $\nabla_\mu \partial_\nu$ in the coordinate basis.

The affine connection on functions

We extend the action of the affine connection ∇_X to the space of smooth functions $\mathfrak{F}(M)$ on M by

$$\nabla_X f = X(f) = \pounds_X f,$$

where $\pounds_X f$ is the Lie derivative of f along the curve X. In a coordinate basis $X = X^\mu \partial_\mu$ we have

$$X(f) = X^\mu \frac{\partial f}{\partial x^\mu},$$

which is just the usual directional derivative of f along the vector field X.

The affine connection on tensor fields

We can further extend the definition of ∇_X to arbitrary rank tensor fields by simply requiring it to act as a derivation on the algebra of tensor fields $\mathfrak{D}(M)$ over $\mathbb{R}$. In particular, we want it to preserve tensor-type (i.e. map tensors of rank (r, s) to tensors of rank (r, s)), to commute with contractions, and to satisfy

$$\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2). \tag{2.3}$$

As a quick example, we compute the action of ∇_X on a smooth 1-form $\omega \in \Gamma(T^*M)$. Take some vector field $Y \in \Gamma(TM)$. Then on the one hand the definition of ∇_X on $\mathfrak{F}(M)$ tells us that we should have

$$\nabla_X(\omega(Y)) = X(\omega(Y)).$$

On the other hand, the property (2.3) tells us that

$$\nabla_X(\omega(Y)) = (\nabla_X\omega)(Y) + \omega(\nabla_X Y),$$

i.e. the connection acts first on one argument then the other. Putting this together we find the action of ∇_X on cotangent vector fields to be

$$(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y). \tag{2.4}$$

Parallel transport and geodesics

Choose some smooth vector field $V \in \Gamma(TM)$, and consider the integral curve $c(t)$ generated by V.

Definition 1. *We say that a vector field $X \in \Gamma(TM)$ is* **parallel transported** *along a curve $c(t)$ if*

$$\nabla_V X = 0.$$

Using this, we define the notion of a **geodesic curve** $\gamma(t)$ as an integral curve generated by a vector field $V \in \Gamma(TM)$ which satisfies

$$\nabla_V V = 0,$$

at least up to some reparametrization of the curve.

Given a point $p \in M$, we can define the notion of a **geodesic symmetry** as a map $s_p : M \to M$ which fixes p and reverses geodesics through p, i.e. $s_p(\gamma(t)) = \gamma(-t)$. In other words $s_p(p) = p$ and $(s_p)_* = -\mathbb{1}_{T_pM}$, where $(s_p)_* : T_pM \to T_pM$ is the **differential** of s at p. Note that the map s_p need only be defined in a neighbourhood of p. Geodesic symmetries will be important later when we talk about symmetric spaces.

Holonomy

Given a connection ∇ on M, we can use the notion of parallel transport to define a transformation group $\mathrm{Hol}(\nabla, p)$ acting on the tangent space T_pM.

In particular, take $p \in M$ and consider the set of all closed loops in M based at p:

$$C_p(M) = \{\gamma : [0,1] \to M \,|\, \gamma(0) = \gamma(1) = p\}.$$

Now, take a vector $X \in T_pM$ and parallel transport it around some loop $c(t) \in C_p(M)$. The result will be a new vector $X_c \in T_pM$. Hence, to each loop we can associate a map $P_c : T_pM \to T_pM$ which takes $X \mapsto X_c$.

The set of all such transformations, obtained by considering all possible loops in $C_p(M)$, gives a group $\mathrm{Hol}(\nabla, p) \subset GL(m, \mathbb{R})$ called the **holonomy group at** p. Explicitly, the action of an element $\mathrm{Hol}(\nabla, p)$ on T_pM is given by

$$P_c X = Xh = X^\mu {h_\mu}^\nu e_\nu,$$

where $h \in \mathrm{Hol}(\nabla, p)$ and $\{e_\mu\}$ is a basis of T_pM.

Note that if M is arcwise connected, then $\mathrm{Hol}(\nabla, p) \cong \mathrm{Hol}(\nabla, q)$ for any $p, q \in M$. Hence the holonomy group is independent of the base point, and we simply refer to $\mathrm{Hol}(\nabla)$. This will be the case for all of the manifolds we encounter in this thesis.

Moreover, if we consider only the subset $C_p^0(M) \subset C_p(M)$ of loops that are homotopic to the identity (can be shrunk to a point) then we obtain the **restricted holonomy group at** p, denoted $\mathrm{Hol}^0(\nabla, p)$. For simply-connected manifolds this of course coincides with $\mathrm{Hol}(\nabla, p)$.

The torsion tensor

A particularly important tensor field that can be constructed from the affine connection ∇ is the **torsion tensor**

$$\begin{aligned} T : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X, Y) &\mapsto \nabla_X Y - \nabla_Y X - [X, Y]\,. \end{aligned} \tag{2.5}$$

In a coordinate basis $\{\partial_\mu\}$ we have

$$T(X, Y) = \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right) X^\mu Y^\nu \partial_\lambda.$$

Definition 2. *We call an affine connection* ∇ **torsion-free** *if* $T(X,Y)=0$ *for any* $X,Y\in\Gamma(TM)$.

In terms of a coordinate basis, the torsion-free condition just tells us that the connection components $\Gamma^\lambda_{\mu\nu}$ are symmetric in their lower indices.

The curvature tensor

Another important tensor field constructed from the connection is the **curvature tensor**

$$\begin{aligned} R:\Gamma(TM)\times\Gamma(TM)\times\Gamma(TM) &\to \Gamma(TM) \\ (X,Y,Z) &\mapsto \nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z. \end{aligned} \tag{2.6}$$

In a coordinate basis $\{\partial_\mu\}$ we have

$$R(X,Y)Z = X^\mu Y^\nu Z^\rho\left[\partial_\mu\Gamma^\lambda_{\nu\rho}-\partial_\nu\Gamma^\lambda_{\mu\rho}+\Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho}-\Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}\right]\partial_\lambda.$$

Definition 3. *We call an affine connection* ∇ **flat** *if* $R(X,Y)Z=0$ *for any* $X,Y,Z\in\Gamma(TM)$.

Note that for manifolds with flat connection, the holonomy group $\mathrm{Hol}(\nabla)$ is trivial (see 10.25 of [50]).

2.1.3 Pseudo-Riemannian geometry

We now move on to consider differentiable manifolds endowed with additional structure, namely the existence of a metric g.

Let (M,g) be a pseudo-Riemannian manifold of signature (p,q). We call an affine connection ∇ on M **metric compatible** if $\nabla g=0$. It turns out that there exists a unique metric compatible torsion-free connection D on (M,g), which we call the **Levi-Civita connection**.

We can compute the Levi-Civita connection explicitly using the Koszul formula [51]

$$2g(D_XY,Z) \;=\; Xg(Y,Z)+Yg(X,Z)-Zg(X,Y)$$

$$+g([X,Y],Z) - g(X,[Y,Z]) - g(Y,[X,Z]), \qquad (2.7)$$

where $X, Y, Z \in \Gamma(TM)$. Note that when taking X, Y, Z to be coordinate vector fields, and hence to commute, the final three terms on the right hand side of (2.7) vanish, and one recovers the usual form of the Levi-Civita connection in terms of Christoffel symbols. The complementary case, where X, Y, Z are left-invariant vector fields, will be useful in Chapter 4.

Orthonormal frame

Given a pseudo-Riemannian manifold (M, g), we can define an **orthonormal frame** $\{e_a\}$, which spans TM, satisfying

$$g(e_a, e_b) = \eta_{ab},$$

where

$$\eta_{ab} = \begin{pmatrix} -\mathbb{1}_p & 0 \\ 0 & \mathbb{1}_q \end{pmatrix}.$$

We can relate the basis $\{e_a\}$ to a coordinate basis $\{\partial_\mu\}$ by

$$e_a = e_a{}^\mu \partial_\mu.$$

The **vielbeins** $e_a{}^\mu$ are $SL(m, \mathbb{R})$ matrices and satisfy

$$e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab},$$

where $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ are the components of the metric g in the coordinate basis. The inverse vielbein $e_\mu{}^a$ is given by

$$e_\mu{}^a = g_{\mu\nu} \eta^{ab} e_b{}^\nu.$$

Any two orthonormal frames $\{e_a\}$ and $\{e'_a\}$ are related by an $SO(p, q)$ transformation, i.e. $e'_a = M_a{}^b e_b$ for $M \in SO(p, q)$. Hence the natural language for describing orthonormal frames is to introduce a principal $SO(p,q)$ bundle, called the **frame bundle**, over M [52].

We can define a **dual basis** $\{\theta^a\}$ of T^*M satisfying $\theta^a(e_b) = \delta^a_b$. In terms of the coordinate basis $\{dx^\mu\}$, we have

$$\theta^a = e^a{}_\mu dx^\mu,$$

and so the metric g can be written

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} \theta^a \otimes \theta^b.$$

For most of the applications in this thesis, it will be convenient to work in such an orthonormal frame.

Connection 1-form and Cartan's structure equations

Given an affine connection ∇ on a pseudo-Riemannian manifold (M,g), we can write $\nabla_X Y$ in terms of an orthonormal frame $\{e_a\}$, as we did with the coordinate basis in Section 2.1.2. In particular,

$$\nabla_X Y = X^a \left(e_a Y^c + Y^b \gamma^c_{ab} \right) e_c,$$

where we have defined the connection coefficients with respect to the basis $\{e_a\}$ as

$$\nabla_a e_b =: \gamma^c_{ab} e_c.$$

These can be related to the connection coefficients $\Gamma^\rho_{\mu\nu}$ of Section 2.1.2 by

$$\gamma^c_{ab} = e_\lambda{}^c e_a{}^\mu \left(\partial_\mu e_b{}^\lambda + e_b{}^\nu \Gamma^\lambda_{\mu\nu} \right).$$

We can likewise write the components of the torsion tensor T and curvature tensor R in the orthonormal basis [52].

We define the **connection 1-form** as

$$\omega^a{}_b = \gamma^a_{bc}\,\theta^c. \tag{2.8}$$

This satisfies **Cartan's structure equations**

$$d\theta^a + \omega^a{}_b \wedge \theta^b = T^a, \tag{2.9}$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_b, \tag{2.10}$$

where we have defined the **torsion 2-form** $T^a = \frac{1}{2}T^a_{bc}\,\theta^b \wedge \theta^c$, and the **curvature 2-form** $R^a{}_b = \frac{1}{2}R^a{}_{bcd}\,\theta^c \wedge \theta^d$.

Pseudo-Riemannian symmetric spaces

An important class of pseudo-Riemannian manifolds, which will appear in Chapter 5, are symmetric spaces. In order to define these, we follow Theorem 10.72 of [50] and Corollary 8.16 of [53].

Theorem 1. *Let (M,g) be a pseudo-Riemannian manifold. Then the following are equivalent:*

(i) $DR = 0$, where D is the Levi-Civita connection on (M,g).

(ii) The geodesic symmetry s_p around any point $p \in M$ acts isometrically, i.e. $(s^*g)_p = g_p$.

Recall [51] that the **pullback** $(s_p)^*$ acts on the $(0,2)$ tensor field g as

$$(s^*g)_p(X,Y) = g\left((s_p)_*X, (s_p)_*Y\right), \quad X, Y \in T_pM.$$

Definition 4. *A pseudo-Riemannian manifold satisfying (i) or (ii) in Theorem 1 is called **locally symmetric**.*

If, in addition, (M,g) is complete and simply-connected, then it is a **symmetric space**. In this latter case, the manifold is a homogeneous space[1] G/H for which there

[1] A topological space X is called a **homogeneous space** if there exists a transitive action of some Lie group G on X. The Lie subgroup $H \subset G$ is the stabilizer of the 'origin' $o \in X$.

exists a certain involutive automorphism σ of G. In particular, let $G^\sigma = \{g \in G | \sigma(g) = g\}$ be the fixed-point set of the involution, and G^σ_e the component connected to the identity. Then $G^\sigma_e \subset H \subset G^\sigma$. A consequence of this is that G^σ_e, H and G^σ all have the same Lie algebra $\mathfrak{h}$. We will have more to say on this in Section 2.4.

Symmetric spaces have an important role to play in the context of extended supergravity theories, as they often turn up as the target manifolds of certain non-linear sigma models [35]. Indeed, it is in just such a setting that we will meet them in Chapter 5.

Berger's list of Riemannian holonomies

The notion of holonomy gives rise to a classification of Riemannian manifolds, known as Berger's theorem, which will prove illuminating in the remainder of this thesis. We follow 10.92 of [50].

Theorem 2. *Let (M, g) be a Riemannian manifold which is not locally symmetric and for which the restricted holonomy group* $\mathrm{Hol}^0(D)$ *is irreducible*[2]*. Then* $\mathrm{Hol}^0(D)$ *is contained within one of the groups in Table 2.1.*

Holonomy	**Dimension**	**Manifold**
$SO(n)$	n	Orientable
$U(n)$	$2n$	Kähler
$SU(n)$	$2n$	Calabi-Yau
$Sp(n)$	$4n$	Hyperkähler
$Sp(n) \cdot Sp(1)$	$4n$	Quaternionic-Kähler
G_2	7	G_2-manifold
Spin(7)	8	Spin(7) manifold

Table 2.1: Berger's list of Riemannian holonomies

As an example of what Table 2.1 means, we show:

Theorem 3. *For a generic orientable Riemannian manifold (M, g) of dimension n, the holonomy group* $\mathrm{Hol}(D)$ *is contained within $SO(n)$.*

Proof: We first note that, since $Dg = 0$, the connection D preserves the length of a vector. Hence, $g_p(P_c X, P_c X) = g_p(X, X)$ for any $X \in T_p M$. Expanding in an

[2]From a theorem of de Rham (10.43 of [50]), this is equivalent to saying that M is not locally a product manifold.

orthonormal frame $\{e_a\}$ for T_pM, we find

$$\eta_{ab} = h_a{}^c h_b{}^d \eta_{cd},$$

and so $h \in SO(n) \subset GL(n, \mathbb{R})$. Hence, the holonomy group is reduced to a subgroup $\mathrm{Hol}(D) \subset SO(n)$. □

For the case where M is a (irreducible, simply-connected) symmetric space G/H, the holonomy group is simply equal to the isotropy subgroup H. This will be important when we look at the symmetric space $G_{2(2)}/(SL_2 \cdot SL_2)$ in Chapter 5.

Manifolds of special holonomy play an important role in string compactifications (see [54] and references therein) due to the fact that they admit covariantly constant spinors (Theorem 3.6.1 of [55]). In particular, this means that strings propagating on a background including one of these compact manifolds (e.g. heterotic strings on $\mathbb{R}^{3,1} \times X_6$ where X_6 is a Calabi-Yau threefold [56]) preserve some number of supercharges in the lower-dimensional non-compact space .

2.1.4 Special real geometry

Special real manifolds are central to the main subject matter of this thesis: five-dimensional $\mathcal{N} = 2$ supergravity theories. We will see in Chapter 3 that the consistent supersymmetric coupling of five-dimensional $\mathcal{N} = 2$ vector multiplets with gravity requires the presence of a so-called "projective special real" manifold. In this section we will introduce the relevant mathematical framework to understand this result. We follow [48, 57] and unpublished work by V. Cortés and T. Mohaupt.

We start with an intrinsic definition of a Hessian manifold, and add further structure until we reach a conic affine special real (CASR) manifold. We then define a projective special real (PSR) manifold as a certain quotient of, or equivalently a hypersurface in, the CASR manifold.

Definition 5. *A pseudo-Riemannian manifold (M, g, ∇) endowed with a flat, torsion-free* ***special connection*** *∇ is called* ***Hessian*** *if ∇g is completely symmetric.*

Given a flat torsion-free connection, ∇, we can cover M with a set of 'normal coordinates' h^I such that the connection components Γ^I_{JK} vanish. This tells us that

$$\nabla_X Y = X^I(\partial_I Y^J)\partial_J, \quad \text{with} \quad \partial_I = \frac{\partial}{\partial h^I}.$$

Then the requirement that ∇g be completely symmetric, i.e. that $(\nabla_X g)(Y, Z)$ be symmetric in X, Y, Z, can be shown to be equivalent to the condition

$$\partial_I g_{JK} = \partial_J g_{IK},$$

which tells us that locally the metric g can be written as

$$g = \frac{\partial^2 H}{\partial h^I \partial h^J}\, dh^I \otimes dh^J,$$

for some **Hesse potential** $H(h)$. For the particular case in which the function H is a cubic polynomial, we have

Definition 6. *A Hessian manifold (M, g, ∇) with Hesse potential a cubic polynomial is called an **affine special real manifold**.*

Affine special real manifolds appear in supersymmetric field theory when one wants to write down a consistent interacting Lagrangian for rigid $\mathcal{N} = 2$ vector multiplets in five dimensions. In this case, one finds that the scalar fields contained within the vector multiplets should parametrise an affine special real manifold [28, 58].

Definition 7. *A d-**conic Hessian manifold** (M, g, ∇, ξ) is a Hessian manifold (M, g, ∇) endowed with a vector field ξ such that:*

(i) $D\xi = \frac{d}{2}\mathbb{1}$, *where D is the Levi-Civita connection on (M, g).*

(ii) $\nabla\xi = \mathbb{1}$.

The first of these conditions can be analysed by recourse to the Koszul formula (2.7) with $Y = \xi$. The left hand side then reads

$$2g(D_X\xi, Z) = d\, g(X, Z),$$

using the first conic condition. Since this is now symmetric in X, Z we need only concentrate on those terms on the right hand side of (2.7) which are likewise symmetric in X, Z. Hence, we have

$$d\, g(X,Z) = \xi g(X,Z) + g([X,\xi],Z) + g([Z,\xi],X),$$

which in components reads

$$d\, g_{IJ} = \xi^K \partial_K g_{IJ} + g_{KJ}\partial_I \xi^K + g_{IK}\partial_J \xi^K = \pounds_\xi g_{IJ}. \tag{2.11}$$

Thus ξ acts as a homothety on the metric g.

We turn next to the second condition defining the d-conic Hessian manifold, which tells us that $\nabla_X \xi = X$ for any smooth vector field $X \in \Gamma(TM)$. In terms of special coordinates h^I this becomes

$$X^I(\partial_I \xi^J)\partial_J = X^J \partial_J,$$

or in other words $\partial_I \xi^J = \delta_I^J$. Hence, the vector field ξ is an Euler vector field

$$\xi = h^I \partial_I,$$

with respect to the special coordinates. Plugging this into (2.11) we find

$$h^K \partial_K g_{IJ} = (d-2) g_{IJ},$$

which tells us that the components of the metric g on a d-conic Hessian manifold, with respect to the special coordinate basis, are homogeneous of degree $d-2$. Since we saw already that $g_{IJ} = \partial^2_{I,J} H$, then the Hesse potential for a d-conic Hessian manifold should be homogeneous of degree d. Combining this with the definition of an affine special real manifold, we have

Definition 8. *A **conic affine special real (CASR) manifold** (M, g, ∇, ξ) is a 3-conic Hessian manifold with Hesse potential a cubic polynomial. In particular,*

$H(h) = c_{IJK}h^I h^J h^K$ *for some constants* c_{IJK}.

In the physics literature, CASR manifolds play an important role in moving between theories with rigid and local supersymmetry. In particular, the additional assumption that the Hesse potential be homogeneous means that the rigid theory is invariant under superconformal transformations [57]. One can therefore use the superconformal calculus in five dimensions [59,60] to construct a locally supersymmetric theory. Geometrically, this corresponds to performing the so-called 'superconformal quotient', which we now discuss.

From CASR to PSR manifolds

We start with a CASR manifold (M, g, ∇, ξ), which we take to be some domain $M \subset \mathbb{R}^n$ parametrized by special coordinates h^I, with $I = 1, \ldots, n$. The vector field ξ induces an $\mathbb{R}^+$ action on M given by

$$h^I \mapsto \lambda h^I, \quad \lambda \in \mathbb{R}^+.$$

Hence, the domain M should be chosen such that it is invariant under multiplication by positive numbers.

There are a number of equivalent ways to define a projective special real (PSR) manifold. The most familiar is to define a PSR manifold as a hypersurface $\mathcal{H} \subset M$ given by

$$\mathcal{H} = \{x \in M | H(x) = 1\}.$$

The full CASR manifold can then be obtained from $\mathcal{H}$ by the $\mathbb{R}^+$ action generated by ξ. We can equip $\mathcal{H}$ with the metric $g_{\mathcal{H}} = -\frac{1}{3}g|_{\mathcal{H}}$, which is assumed to be Riemannian. Away from $\mathcal{H}$ the tensor field $-\frac{1}{3}g$ on M gives rise to a Lorentzian metric on M, since it is negative definite in the direction of ξ [57].

We can also think of $\mathcal{H}$ as the set of cosets of M under the $\mathbb{R}^+$ action. In other words, we can obtain $\mathcal{H}$ by 'projecting' along the orbits of the vector field ξ, i.e. $\mathcal{H} \cong M/\mathbb{R}^+$.

We now want to equip $\mathcal{H}$ with a Riemannian metric. Since $\pounds_\xi g = 3g$, we see that the metric g on the original CASR manifold changes along the orbits of ξ. In order to define $\mathcal{H}$ as a Riemannian coset space, we want a tensor field on M upon which ξ acts isometrically. To this end, we define

$$a = \frac{\partial^2 \tilde{H}}{\partial h^I \partial h^J} dh^I \otimes dh^J, \quad \text{where} \quad \tilde{H} = -\frac{1}{3} \log H. \tag{2.12}$$

One can show that the tensor field a satisfies $\mathcal{L}_\xi a = 0$, i.e. it is constant along orbits of ξ. Since it will be useful in Chapter 3, we write explicitly the components of a in terms of the special coordinates:

$$a_{IJ} = -2 \left(\frac{(ch)_{IJ}}{chhh} - \frac{3}{2} \frac{(chh)_I (chh)_J}{(chhh)^2} \right), \tag{2.13}$$

where we have defined $chhh := c_{IJK} h^I h^J h^K$, $(chh)_I := c_{IJK} h^J h^K$, etc.

We can therefore define a PSR manifold $\mathcal{H}$ as the quotient

$$\mathcal{H} \cong M/\mathbb{R}^+,$$

with quotient metric obtained from a. Note that for the hypersurface embedding $i : \mathcal{H} \to M$, the pullback of $-\frac{1}{3}g$ and a induce the same metric on $\mathcal{H}$:

$$g_{\mathcal{H}} = i^* \left(-\frac{1}{3} g \right) = i^*(a).$$

Projective special real manifolds will be important in Chapter 3, where we will see that they naturally appear when one tries to consistently couple $\mathcal{N} = 2$ vector multiplets to gravity in five dimensions.

2.2 (Para-)Complex differential geometry

This section deals in parallel with the subjects of complex and para-complex differential geometry, which have many important applications in supergravity. Whilst complex differential geometry is fairly familiar, the use of para-complex differential geometry in physics was only formalised fairly recently [28]. By introducing the notion of ϵ-complex manifolds, where $\epsilon = -1$ refers to complex manifolds and $\epsilon = 1$ to para-complex manifolds, we are able to treat both types of geometry on the same footing.

The primary aim of this section is to provide the necessary material to understand projective special ϵ-Kähler manifolds, which appear in the study of four-dimensional $\mathcal{N}=2$ vector multiplets coupled to gravity. However, along the way we will introduce a host of material which will prove useful throughout this thesis. We begin in Section 2.2.1 by defining almost ϵ-complex structures, and treat their integrability in Sections 2.2.2–2.2.3, before moving on in Section 2.2.4 to ϵ-Kähler manifolds. Finally, the main bulk of this section is contained in Section 2.2.5 where we introduce the notion of special ϵ-Kähler manifolds.

We mostly follow the treatment of [52] for complex geometry, and [28, 32] for its para-complex cousin.

2.2.1 Almost ϵ-complex structure

An ϵ-complex structure J is a tensor field such that at each point $p \in M$, the endomorphism $J_p \in \mathrm{End}(T_pM)$ satisfies

$$J_p^2 = \epsilon \, \mathrm{Id}_{T_pM}.$$

For $\epsilon = -1$ this is the usual definition of an almost complex structure. For $\epsilon = 1$ we also impose that the two eigenspaces $T_pM^{\pm} = \ker(\mathrm{Id}_p \mp J_p)$ have the same dimension [28]. In this case J is called an **almost para-complex structure**.

Given an almost ϵ-complex structure J on M, we can define a dual structure $J^* \in \Gamma(\mathrm{End}(T^*M))$, which is defined by

$$(J^*\xi)(X) = \xi(JX),$$

for any $\xi \in \Gamma(T^*M)$, $X \in \Gamma(TM)$.

2.2.2 Integrability of an almost complex structure

We follow Section 2.11 of [50], which gives three equivalent criteria for an almost complex structure J on M to be integrable. We investigate them in turn.

Vector fields

Let $Z \in T_pM^{\mathbb{C}} \cong T_pM \otimes \mathbb{C}$ be a vector in the complexified tangent space at some point $p \in M$, and $J_p \in \mathrm{End}(T_pM^{\mathbb{C}})$ an almost complex structure.

Since J_p is an anti-involution, we can split the complexified tangent space into two disjoint subspaces [52]

$$T_pM^{\mathbb{C}} = T_pM^{+} \oplus T_pM^{-},$$

where

$$T_pM^{\pm} = \left\{ Z \in T_pM^{\mathbb{C}} | J_pZ = \pm iZ \right\}. \tag{2.14}$$

We call T_pM^{+} the **holomorphic tangent space** and T_pM^{-} the **anti-holomorphic tangent space**. These are also sometimes denoted $T_pM^{(1,0)}$ and $T_pM^{(0,1)}$, which is the notation we use in the following. Hence, any almost complex structure J gives rise, at each point $p \in M$, to a decomposition of $T_pM^{\mathbb{C}}$ into a holomorphic and an anti-holomorphic tangent space.

We decompose the generators $T_pM^{\mathbb{C}} = \mathrm{span}\{X_a\}$ into $T_pM^{(1,0)} = \mathrm{span}\{X_i\}$ and $T_pM^{(0,1)} = \mathrm{span}\{\bar{X}_{\bar{i}}\}$. Then

Definition 9. *The almost complex structure J is* ***integrable*** *if $[X_i, X_j] \in T_pM^{(1,0)}$ for any $X_i, X_j \in T_pM^{(1,0)}$.*

Since the generators $X_a \in T_pM^{\mathbb{C}}$ satisfy the relations

$$[X_a, X_b] = f^c_{ab} X_c,$$

the statement that J is integrable is equivalent to the statement that the structure constants f^c_{ab} satisfy

$$f^{\bar{k}}_{ij} = 0, \qquad f^{k}_{\bar{i}\bar{j}} = 0. \tag{2.15}$$

Using this information we can find an equivalent condition for an almost complex structure J to be integrable based on 1-forms.

1-forms

Given the basis $\{X_a\}$ of $T_pM^{\mathbb{C}}$, we can define a dual basis $\{\theta^a\}$ of the complexified cotangent space $T_p^*M^{\mathbb{C}} \cong T_p^*M \otimes \mathbb{C}$ in the usual way. One can then show [51] that the θ^a satisfy

$$d\theta^a = -f^a_{bc}\,\theta^b \wedge \theta^c. \tag{2.16}$$

The decomposition, via J, of $T_pM^{\mathbb{C}}$ into a holomorphic and anti-holomorphic tangent space induces a similar decomposition of $T_p^*M^{\mathbb{C}}$ into $T_p^*M^{(1,0)}$ and $T_p^*M^{(0,1)}$. We can then split $\{\theta^a\}$ into disjoint sets $\{\theta^i\}$ and $\{\bar{\theta}^{\bar{i}}\}$.

The exterior derivative decomposes as

$$d\theta^i = -f^i_{jk}\theta^j \wedge \theta^k - 2f^i_{j\bar{k}}\theta^j \wedge \bar{\theta}^{\bar{k}} - f^i_{\bar{j}\bar{k}}\bar{\theta}^{\bar{j}} \wedge \bar{\theta}^{\bar{k}},$$

so that J being integrable in the vector field sense given above is equivalent to the vanishing of the final term here. As such, we have

Theorem 4. *The almost complex structure J is integrable in the sense of Definition 9 iff the exterior derivative $d\theta$ of any holomorphic 1-form $\theta \in T_p^*M^{(1,0)}$ contains no $(0,2)$ form.*

Newlander-Nirenberg theorem

The final condition for J to be integrable is the subject of the famous Newlander-Nirenberg theorem. We first define the **Nijenhuis tensor**[3]

$$N(X,Y) := -J^2\,[X,Y] + J\,[JX,Y] + J\,[X,JY] - [JX,JY]\,. \tag{2.17}$$

Then the **Newlander-Nirenberg theorem** states:

Theorem 5. *The almost complex structure J is integrable in the sense of Definition 9 iff $N(X,Y) = 0$ for any $X, Y \in \Gamma(TM)$.*

The proof can be found in Theorem 8.12 of [52]. Note that the vanishing of the Nijenhuis tensor is a local condition, so only needs to hold on open sets of M. This

[3]The first term is just $[X,Y]$ for J an almost complex structure, but becomes important in the case where J is an almost para-complex structure.

will be important in Chapter 5, when we see that two open orbits contained within the same pseudo-Riemannian symmetric space admit different integrable structures.

2.2.3 Integrability of an almost para-complex structure

The condition for integrability of an almost para-complex structure can be derived similarly to the complex case in the previous subsection. However, in this case the eigendistributions $T_pM^{\pm}$ are real vector spaces. Since J_p is an involution on T_pM, we can make the decomposition

$$T_pM = T_pM^+ \oplus T_pM^-,$$

where

$$T_pM^{\pm} = \{X \in T_pM | J_pX = \pm X\}.$$

Then Frobenius' theorem guarantees that the almost para-complex structure J is integrable if $[X, Y] \in T_pM^+$ for any $X, Y \in T_pM^+$. Again, this is equivalent to the condition that the Nijenhuis tensor (2.17) vanishes [28, 61].

The integrability of an almost ϵ-complex structure J defines on M the structure of an ϵ-complex manifold. The simplest examples are $\mathbb{C}_\epsilon^n$, where $\mathbb{C}_\epsilon$ is the ring of ϵ-complex numbers

$$\mathbb{C}_\epsilon := \mathbb{R}[i_\epsilon], \quad i_\epsilon^2 = \epsilon.$$

We call i_ϵ the ϵ-complex unit. For $\epsilon = -1$ this is the usual complex unit i, while for $\epsilon = 1$ we have a para-complex unit, generally denoted e. Note that the set of para-complex numbers is not a field, since it admits zero divisors. Hence, one needs to be careful when using certain facts from linear algebra. For an overview, see [61].

Note (Proposition 2 of [28]) that a para-complex manifold (M, J) of real dimension $2m$ is locally a product manifold, so that we can find local coordinates $(z_+^i, z_-^i)_{i=1,\dots,m}$ for which $dz_\pm^i \circ J = \pm dz_\pm^i$. For further details and examples of para-complex manifolds, and para-complex geometry in general, see [62] and references therein.

A function $f : M \to \mathbb{C}_\epsilon$ is called **ϵ-holomorphic** if $df\, J = i_\epsilon df$. This is a specific example of the more general notion of an ϵ-holomorphic map between ϵ-complex manifolds (M, J) and (M', J'), which satisfies $df\, J = J'\, df$.

2.2.4 Almost ϵ-hermitian and ϵ-Kähler manifolds

An almost ϵ-complex manifold (M, J) equipped with a pseudo-Riemannian metric g is said to be **almost ϵ-hermitian** if g is compatible with J, i.e.

$$g_p(J_pX, J_pY) = -\epsilon g_p(X, Y),$$

for any $X, Y \in T_pM$. Then we can define a 2-form ω_J, called the **fundamental 2-form**, on (M, g, J) via

$$\omega_J(X, Y) = g(JX, Y).$$

We have

Definition 10. *An almost ϵ-hermitian manifold (M, g, J) is* ***almost ϵ-Kähler*** *if the fundamental 2-form ω_J is closed, $d\omega_J = 0$.*

If J is integrable, then we obtain the usual condition for (M, g, J) to be an ϵ-hermitian or ϵ-Kähler manifold. In particular, in this case $d\omega_J = 0$ can be shown to be equivalent to $DJ = 0$, where D is the Levi-Civita connection on (M, g). Then locally one can find a set of ϵ-holomorphic coordinates z^a such that the metric g is given by

$$g = \mathrm{Re}\left(\frac{\partial^2 K}{\partial z^a \partial \bar{z}^b} dz^a \otimes d\bar{z}^b\right),$$

for some function $K(z, \bar{z})$ called the **ϵ-Kähler potential**.

2.2.5 Special ϵ-Kähler manifolds

Just as the special real manifolds we met in Section 2.1.4 will prove to be useful for five-dimensional supergravity theories, so the special Kähler manifolds we will introduce next will be important for the study of four-dimensional $\mathcal{N} = 2$ supergravity theories, as we will see in Chapter 3.

We begin this section by defining the notion of an affine special ϵ-Kähler (ASϵK) manifold. These are important for the study of supersymmetric field theories in four dimensions. In particular, in order to consistently couple rigid $\mathcal{N} = 2$ vector multiplets in four-dimensional Minkowski ($\epsilon = -1$) or Euclidean ($\epsilon = 1$) space(time), the scalar fields should parametrise an affine special ϵ-Kähler manifold [5, 28].

Definition 11. *An ϵ-Kähler manifold (M, g, J) equipped with a flat torsion-free 'special' connection ∇ is called* ***affine special ϵ-Kähler*** *if:*

(i) ∇g is completely symmetric, i.e. (M, g, ∇) is Hessian.

(ii) $\nabla\omega = 0$, where ω is the fundamental 2-form on (M, g, J).

Since (M, g, ∇) is Hessian, we can use the arguments of Section 2.1.4 to introduce a set of 'special' coordinates X^I on M. The second condition then implies [28] that the special coordinates are compatible with the ϵ-Kähler structure. That is, the metric g can be obtained from the ϵ-Kähler potential

$$K(X, \bar{X}) = i_\epsilon (X^I \bar{F}_I - \bar{X}^I F_I), \tag{2.18}$$

where $F(X)$ is an ϵ-holomorphic function called the **prepotential**. Here we have used the notation $F_I(X) = \partial_I F(X)$, etc. The components of the metric on the ASϵK manifold are then given by[4]

$$N_{IJ}(X, \bar{X}) := \frac{\partial^2 K}{\partial X^I \partial \bar{X}^J} = -i_\epsilon \left(F_{IJ} - \bar{F}_{IJ}\right). \tag{2.19}$$

Following the story of special real geometry in Section 2.1.4, we next define a conic affine special ϵ-Kähler (CASϵK) manifold. This plays the same role for the four-dimensional theory as the CASR manifold did for the five-dimensional theory. Namely, it allows us to move between theories with rigid and local supersymmetry by means of the superconformal calculus (see [5] and references therein).

Definition 12. *A* ***conic affine special ϵ-Kähler manifold*** *(N, g_N, J, ∇, ξ) is an affine special ϵ-Kähler manifold (N, g_N, J, ∇) endowed with a vector field ξ satisfying:*

(i) $D\xi = \mathrm{Id}$, where D is the Levi-Civita connection with respect to g_N.

[4] Here we write N_{IJ} for the components of the metric g to be consistent with the literature.

(ii) $\nabla\xi = \mathrm{Id}$.

Comparing to Definition 7 we see that CASϵK manifolds are a subclass of 2-conic Hessian manifolds. We saw there that the vector field ξ was an Euler vector field with respect to the special real coordinates. Likewise, using special ϵ-holomorphic coordinates X^I on N, we find that the conditions (i) and (ii) of Definition 12 imply that

$$\xi = X^I \frac{\partial}{\partial X^I} + \bar{X}^I \frac{\partial}{\partial \bar{X}^I}. \tag{2.20}$$

We can also introduce a second privileged vector field $J\xi$ on N, given by

$$J\xi = i_\epsilon X^I \frac{\partial}{\partial X^I} - i_\epsilon \bar{X}^I \frac{\partial}{\partial \bar{X}^I}. \tag{2.21}$$

Together ξ and $J\xi$ generate a $\mathbb{C}^*_\epsilon$ action on the CASϵK manifold N. In the case $\epsilon = -1$, we see that ξ generates dilatations $X^I \to |\lambda| X^I$, while $J\xi$ generates a $U(1)$ transformation

$$X^I \mapsto e^{i\phi} X^I, \quad \bar{X}^I \mapsto e^{-i\phi} \bar{X}^I.$$

In the case $\epsilon = 1$, $\mathbb{C}^*_\epsilon$ is the group of invertible para-complex numbers, which is disconnected [61]. We can therefore choose whether to allow the full action of $\mathbb{C}^*_{\epsilon=1} \equiv C^*$, or just the component connected to the identity C^*_0. Note that the action of $e \in C^*$ induces an anti-isometry on the CASϵK manifold which takes $g_N \mapsto -g_N$.

Performing an analysis similar to that in Section 2.1.4, we find that ξ acts homothetically on g_N, while $J\xi$ acts isometrically

$$\pounds_\xi g_N = 2 g_N, \quad \pounds_{J\xi} g_N = 0.$$

Moreover, the components (2.19) of the metric g_N should be homogeneous of degree $d - 2 = 0$, which implies that the prepotential $F(X)$ should be homogeneous of degree 2.

Motivated by analogy with CASR manifolds, we define a tensor field

$$g = \frac{\partial^2 \mathcal{K}}{\partial X^I \partial \bar{X}^J} dX^I \otimes d\bar{X}^J, \quad \mathcal{K} = -\log K(X, \bar{X}). \tag{2.22}$$

The components of the tensor field g are given by

$$g_{I\bar{J}} = -\frac{N_{IJ}}{XN\bar{X}} + \frac{(N\bar{X})_I(NX)_J}{(XN\bar{X})^2}, \tag{2.23}$$

where N_{IJ} are the components of the metric g_N on N (2.19). We note that $X^I g_{I\bar{J}} = g_{I\bar{J}}\bar{X}^J = 0$, which implies that the tensor field g is degenerate along the directions spanned by $\{\xi, J\xi\}$

$$g(\xi, \cdot) = g(J\xi, \cdot) = 0.$$

Hence g *does not* provide a metric on the CASϵK manifold. However, both ξ and $J\xi$ act isometrically on g

$$\pounds_\xi g = \pounds_{J\xi} g = 0.$$

Definition 13. *A* ***projective special*** ϵ***-Kähler*** *(PSϵK) manifold $(\bar{N}, \bar{g}, \bar{J})$ is defined to be the quotient manifold $N/\mathbb{C}_\epsilon^*$ of a conic affine special ϵ-Kähler manifold (N, g_N, J, ∇, ξ). The metric $\bar{g}$ on $\bar{N}$ is induced by the tensor field g on N, while the ϵ-complex structure $\bar{J}$ on $\bar{N}$ is induced from J on N.*

The $\mathbb{C}_\epsilon^*$ action by which we quotient to get the PSϵK manifold is that generated by the vector fields $\{\xi, J\xi\}$. Since the tensor field g is isometric along these directions, it gives a natural metric on the quotient space $N/\mathbb{C}_\epsilon^*$. Note that the action of $e \in C^*$ mentioned above, which acts as an anti-isometry on the CASϵK manifold, has no effect on the metric g of the PSϵK manifold.

There are a number of ways to think of this construction. It is often convenient to consider a codimension 1 hypersurface $S \subset N$ inside the CASϵK manifold, given by $g_N(\xi, \xi) = \text{const}$. This fixes the homothety ξ acting on g_N and defines a so-called **Sasakian manifold**. A particularly useful choice is to consider the hypersurface $\{K = 1\}$, where K is the ϵ-Kähler potential for g_N. Since we still have the $U(1)$ action generated by $J\xi$, we can then think of the Sasakian manifold S as a $U(1)$ principal bundle over the PSϵK manifold $\bar{N}$, so $\bar{N} = S/U(1)$.

The PSϵK manifold $\bar{N}$ can be parametrized by a set of projective coordinates

$$z^A = \frac{X^A}{X^0}.$$

Since the prepotential $F(X)$ is homogeneous of degree 2, we write

$$F(X^0, X^1, \ldots) = (X^0)^2 F\left(1, \frac{X^1}{X^0}, \ldots\right) := (X^0)^2 \mathcal{F}(z^1, \ldots),$$

which defines a non-homogeneous prepotential $\mathcal{F}(z)$. Using the notation $\mathcal{F}_A := \frac{\partial \mathcal{F}}{\partial z^A}$, etc. we can write the metric $\bar{g}$ on the PSϵK manifold as

$$\bar{g} = \mathrm{Re}\left(\frac{\partial^2 \mathcal{K}(z,\bar{z})}{\partial z^A \partial \bar{z}^B}\, dz^A \otimes d\bar{z}^B\right), \tag{2.24}$$

where

$$\mathcal{K}(z,\bar{z}) = -\log\left(i_\epsilon\left[2(\mathcal{F} - \bar{\mathcal{F}}) - (\mathcal{F}_A + \bar{\mathcal{F}}_A)(z^A - \bar{z}^A)\right]\right). \tag{2.25}$$

Projective special ϵ-Kähler manifolds will appear in Chapter 3 when we consider the coupling of local $\mathcal{N} = 2$ vector multiplets. We will see that our use of a factor ϵ will allow us to treat the case of such theories living in four Minkowski or Euclidean dimensions simultaneously. Indeed, the coupling of vector multiplets in 3+1 dimensions will naturally be described by using projective special Kähler geometry ($\epsilon = -1$), while in $4 + 0$ dimensions they will be described by projective special para-Kähler geometry ($\epsilon = 1$).

2.3 (Para-)Quaternionic-Kähler and hyperkähler geometry

In this section we introduce the material necessary for understanding quaternionic-Kähler and hyperkähler geometry. Although fairly complicated mathematically, such manifolds appear naturally in physics, as we will see in Chapter 3, when one considers theories of rigid or local $\mathcal{N} = 2$ hypermultiplets.

We first go through some of the preliminaries (Section 2.3.1) for defining ϵ-quaternionic-Kähler manifolds. In Section 2.3.2 we define hyperkähler manifolds, before finishing with (pseudo-)quaternionic-Kähler manifolds in Section 2.3.3 and para-quaternionic-Kähler manifolds in Section 2.3.4.

The material in this section is collected from a number of disparate sources, predominantly [50, 55, 63].

2.3.1 Preliminaries

Symplectic groups

The definitions of hyperkähler and quaternionic-Kähler manifolds involve the symplectic groups, as we saw in Table 2.1. We define the **symplectic group** $Sp(2n, F) \subset SL(2n, F)$ over a field F (generally taken to be $\mathbb{R}$ or $\mathbb{C}$) to be the group of $2n \times 2n$ matrices M with coefficients in F which satisfy

$$M^T \Omega M = \Omega,$$

where Ω is the skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}.$$

We can use this to define[5] the **pseudo-unitary-symplectic group** $Sp(k, l)$ via

$$Sp(k, l) = U(2k, 2l) \cap Sp(2k + 2l, \mathbb{C}) \subset SO(4k, 4l).$$

A particularly useful low-dimensional example is the case $k + l = 1$, in which case $Sp(2, \mathbb{C})$ can be identified with $SL(2, \mathbb{C})$, and we see that

$$Sp(1) = SU(2).$$

Finally, it will be useful to define a product of the groups $Sp(k, l)$ and $Sp(1)$ via

$$Sp(k, l) \cdot Sp(1) = (Sp(k, l) \times Sp(1)) / \mathbb{Z}_2,$$

where the $\mathbb{Z}_2$ factor corresponds to multiplication by $\pm$Id. This product has a natural

[5] We follow the conventions of Section 2.4 of [34]. In other places, $Sp(n)$ is written as $USp(2n)$.

action on $\mathbb{H}^m$, where $m = k + l$, via

$$\begin{aligned} Sp(k,l)\cdot Sp(1)\times\mathbb{H}^m &\rightarrow \mathbb{H}^m \\ (A^\mu{}_\nu, \lambda; x^\mu) &\mapsto A^\mu{}_\nu x^\nu \lambda^\dagger, \end{aligned}$$

which makes obvious the action of $\mathbb{Z}_2$ described earlier.

Almost ϵ-hypercomplex structures

Suppose M admits two anti-commuting almost complex structures J_1, J_2, such that

$$J_1 J_2 = -J_2 J_1, \qquad J_1^2 = J_2^2 = -\mathrm{Id}.$$

Then there is a third almost complex structure $J_3 = J_1 J_2$ with $J_3^2 = -\mathrm{Id}$. In fact, we can define a family $\mathcal{I}$ of almost complex structures on M parametrized by the two-sphere S^2 of unit imaginary quaternions. That is, take $ai + bj + ck \in \mathrm{Im}(\mathbb{H}) = S^2$. Then

$$(aJ_1 + bJ_2 + cJ_3)^2 = -\mathrm{Id}.$$

The triple $\{J_1, J_2, J_3\}$ we call an **almost hypercomplex structure** on M [64].

Suppose now that M admits an almost complex structure J_1 and an almost para-complex structure J_2, which anti-commute. Then there exists a second para-complex structure $J_3 = J_1 J_2$. In this case we call the triple $\{J_1, J_2, J_3\}$ an **almost para-hypercomplex structure** on M [65].

Almost ϵ-quaternionic manifolds

A manifold M is said to be **almost ϵ-quaternionic** if there exists a sub-bundle $Q \subset \mathrm{End}(TM)$ such that for any $x \in M$ there exists some open neighbourhood $U \ni x$ such that

$$Q|_U = \mathrm{span}\{J_1, J_2, J_3\},$$

where the J_α are a basis of almost ϵ-hypercomplex structures on M. We will call Q an **ϵ-quaternionic structure** on M.

For $\epsilon = -1$ this is the usual definition of a quaternionic structure (Section 1.2 of [63]), while for $\epsilon = 1$ we call it a **para-quaternionic structure** [61].

ϵ-Quaternionic hermitian manifolds

Let (M, g) be a pseudo-Riemannian manifold such that (M, Q) is an almost ϵ-quaternionic manifold for which the basis of almost ϵ-hypercomplex structures are metric compatible, i.e.

$$g_p(JX, JY) = -\epsilon g_p(X, Y),$$

for any $X, Y \in T_pM$ and $J \in Q_p$. Then (M, Q, g) is **quaternionic hermitian** for $\epsilon = -1$ and **para-quaternionic hermitian** for $\epsilon = 1$.

2.3.2 Hyperkähler manifolds

We now move on to define hyperkähler manifolds. In the same way that the affine special real/Kähler manifolds which we met earlier encode the couplings of rigid $\mathcal{N} = 2$ vector multiplets, so hyperkähler manifolds encode the couplings of rigid $\mathcal{N} = 2$ hypermultiplets, as we will see in Chapter 3.

As we saw in Theorem 2, the statement that a Riemannian manifold (M, g) be hyperkähler is a statement about its Riemannian holonomy group. In particular, let D be the Levi-Civita connection on a $4n$-dimensional Riemannian manifold (M, g). Then

Definition 14. *(M, g) is **hyperkähler** if the Riemannian holonomy group $Hol(D)$ is contained within $Sp(n)$.*

Note that as Lie groups $Sp(n) \subset SU(2n)$ so hyperkähler manifolds represent a subclass of Kähler manifolds. In particular, they are Ricci flat (see Proposition 10.29 of [50]).

Relation to almost complex structures

We can make the relation between hyperkähler and almost Kähler manifolds more explicit with the following theorem (Definition 1.1.1 of [63]):

Theorem 6. *A Riemannian manifold (M, g) is hyperkähler iff $\exists$ an almost hypercomplex structure $\{J_1, J_2, J_3\}$ such that (M, g, J_α) is an almost Kähler manifold with respect to each of the J_α.*

We can package these conditions into one by defining the quaternion-valued 2-form $\omega \in \Omega^2(M, \mathbb{H})$ via

$$\omega = i\omega_1 + j\omega_2 + k\omega_3,$$

and requiring $d\omega = 0$.

This has so far only said that hyperkähler manifolds are almost Kähler. However, there is a theorem by Hitchin (Lemma 1.1.3 of [63]) which tells us that the almost complex structures J_α are integrable if $d\omega = 0$, i.e. if (M, g, J_α) is hyperkähler. The existence of such a triplet of integrable complex structures in fact makes this a hypercomplex manifold (Section 7.5.1 of [55]).

Examples

In the case $n = 1$ we have $Sp(1) = SU(2)$, so a four-dimensional hyperkähler manifold is Kähler and Ricci flat, and vice versa. As such, a compact four-dimensional Kähler manifold with vanishing first Chern class (i.e. a Calabi-Yau 2-fold) is hyperkähler. Such manifolds are either a torus or a $K3$ surface.

Non-compact four-dimensional hyperkähler manifolds are important examples of gravitational instantons, i.e. solutions to the four-dimensional Einstein equations in Euclidean signature [66, 67].

2.3.3 Quaternionic-Kähler manifolds

If we were following directly the discussion of special real and special Kähler geometry above, the natural progression would be to define some notion of "conic-hyperkähler" manifolds, before taking a superconformal quotient to obtain quaternionic-Kähler[6] manifolds which could then be used to construct theories of local $\mathcal{N} = 2$ hypermultiplets.

[6] If we were to continue our naming conventions, these might be called "projective hyperkähler", but thankfully the mathematicians got there first.

We do not enter into details on this subject, however, as it would remove us too far from the central narrative. In terms of the superconformal calculus for four-dimensional hypermultiplets [68], we want to be able to construct a hyperkähler cone, or Swann bundle [69], over a quaternionic-Kähler manifold. This admits a certain $\mathbb{H}^*$ action which allows us to perform the superconformal quotient as in the real and Kähler cases above. The mathematical details of this procedure goes under the name "QK/HK correspondence" [70] and is the subject of much recent study [71–76].

As we saw in Theorem 2, the statement that a Riemannian manifold (M, g) be quaternionic-Kähler is a statement about its Riemannian holonomy group. In particular, let D be the Levi-Civita connection on a $4n$-dimensional Riemannian manifold (M, g) for $n \geq 2$. Then

Definition 15. *(M, g) is **quaternionic-Kähler** if the Riemannian holonomy group Hol(D) is contained within $Sp(n) \cdot Sp(1)$.*

In the case $n = 1$, we have $Sp(1) \cdot Sp(1) \cong SO(4)$ so any oriented four-dimensional manifold would automatically be quaternionic-Kähler in this sense. However, one generally applies a more restrictive condition on (M, g) in this case, namely (Section 7.5.1 of [55]):

Definition 16. *A four-dimensional Riemannian manifold (M, g) of signature $(4, 0)$ is quaternionic-Kähler if it is oriented, Einstein, and has self-dual Weyl tensor.*

Relation to quaternionic structure

There is an important theorem (Proposition 14.36 of [50]) which relates the definition of a quaternionic-Kähler manifold given above to the existence of a quaternionic structure Q on (M, g). In particular

Theorem 7. *A Riemannian manifold (M, g) is quaternionic-Kähler iff $\exists$ a quaternionic structure Q such that (M, Q, g) is quaternionic hermitian and:*

(i) The Levi-Civita connection preserves Q.

(ii) For any $p \in U_i \cap U_j$, the quaternionic structures Q_p on U_i and U_j agree.

Let's have a closer look at what each of these statements mean. The action of D on $J \in Q$ is given as a function $DJ : TM \to \mathrm{End}(TM)$ with

$$DJ(X) = [D_X, J] \, .$$

To say that D preserves the quaternionic structure is to say that the endomorphism $DJ(X)$ is again part of the quaternionic structure, i.e. it is generated by the basis $\{J_1, J_2, J_3\}$. Concentrating on J_1, for example, we would have

$$[D_X, J_1] = a(X)J_1 + b(X)J_2 + c(X)J_3,$$

where $a, b, c \in \Gamma(T^*M)$ are 1-forms. However, taking into account the fact that we have $J_1^2 = -\mathrm{Id}$, one can show

$$0 = \left[D_X, J_1^2\right] = -2a(X)\mathrm{Id},$$

which sets $a(X) = 0$. As such, $DJ_1(X)$ is a linear combination of J_2, J_3. A similar argument works for $DJ_2(X)$ and $DJ_3(X)$. Moreover, the condition $J_1J_2 = J_3$ implies that we should have

$$\begin{aligned} DJ_1(X) &= & &+ \alpha(X)J_2 + \beta(X)J_3 \\ DJ_2(X) &= -\alpha(X)J_1 & & \qquad\quad + \gamma(X)J_3 \\ DJ_3(X) &= -\beta(X)J_1 - \gamma(X)J_2 & & \qquad\qquad\qquad . \end{aligned}$$

For the second condition, consider open sets $U_i \cap U_j \neq \emptyset$ in M with quaternionic structures Q_i and Q_j respectively. Then at $p \in U_i \cap U_j$, we'll have two subsets of $\mathrm{End}(T_pM)$: one generated by $(Q_i)_p$ and one by $(Q_j)_p$. The consistency condition above then states that the two vector spaces thus obtained should agree. Note that this *does not* mean that the individual complex structures need agree.

One way to think of this is that the bundle $\mathrm{End}(TM)$ admits a decomposition

$$\mathrm{End}(TM) = Q \oplus Q',$$

where Q is invariant under $SO(3)$ transformations rotating the S^2 of almost complex structures.

The parallel 4-form

One of the lessons to be learnt from the study of Riemannian holonomy (see, e.g. Section 10C of [50]) is that whenever we have a reduction of the holonomy representation $\mathrm{Hol}(D)$ there is some corresponding tensor field on TM which is parallel with respect to D.

For the case of quaternionic-Kähler manifolds the relevant tensor field is a 4-form given by [63]

$$\Omega = \sum_{\alpha=1}^{3} \omega_\alpha \wedge \omega_\alpha,$$

which is non-degenerate and globally well-defined. Then we have (Definition 1.2.1 of [63]):

Theorem 8. *Let (M, Q, g) be a $4n$-dimensional quaternionic hermitian manifold with $n \geq 2$. Then it is quaternionic-Kähler if $D\Omega = 0$.*

Curvature

We now turn to look at the curvature of quaternionic-Kähler manifolds. The main result, presented as Theorem 14.39 of [50], is:

Theorem 9. *A quaternionic-Kähler manifold of dimension $4n \geq 8$ is Einstein, i.e. $R_{\mu\nu} = cg_{\mu\nu}$ for some constant c.*

Two proofs of this theorem can be found in [50]. Recall that in the four-dimensional case, a quaternionic-Kähler manifold was defined to be an Einstein manifold.

In fact, quaternionic-Kähler manifolds are naturally split into those with positive Ricci curvature and those with negative Ricci curvature, due to the following theorem (Theorem 14.45 of [50]):

Theorem 10. *A quaternionic-Kähler manifold is Ricci-flat, i.e. $c = 0$, iff it is locally hyperkählerian.*

We will see in Section 3.4 that in order for a quaternionic-Kähler manifold to be admissible as the target space of local hypermultiplets it should have negative Ricci curvature.

How to check a manifold is quaternionic-Kähler

For many applications in supergravity it is useful to know, given a particular Riemannian manifold (M, g), whether it is quaternionic-Kähler. Above we have given a number of equivalent ways of formulating this condition. However in practice, at least for the applications we have in mind, it is easiest to use the definition in terms of holonomy representations. That is, we want to show $\mathrm{Hol}(D) \subset Sp(n) \cdot Sp(1)$.

For this, we can make use of the **Ambrose-Singer theorem** (Theorem 10.58 of [50]), which tells us that, given a metric g, we need only to compute the Levi-Civita connection 1-form ω of (2.8). If we can show that

$$\omega \in T^*M \otimes \mathfrak{h},$$

where $\mathfrak{h} \subset \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$, then Ambrose-Singer will ensure that (M, g) is quaternionic-Kähler.

We follow section 2.4 of [34]. In order for the Levi-Civita connection 1-form ω to lie in $(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) \otimes T^*M$, one should be able to make the decomposition

$$\omega = p \otimes \mathbb{1}_{2n} + \mathbb{1}_2 \otimes \begin{pmatrix} q & t \\ -\bar{t} & \bar{q} \end{pmatrix}, \tag{2.26}$$

where the components of p, q, t are 1-forms satisfying

$$\begin{pmatrix} q & t \\ -\bar{t} & \bar{q} \end{pmatrix} \in \mathfrak{sp}(n),$$

and $p \in \mathfrak{sp}(1)$. The first of these conditions can be shown to be equivalent to $q^\dagger = -q$ and $t^T = -t$, while the second is equivalent to $\mathrm{tr}(p) = 0$, $p^\dagger = -p$.

Pseudo-quaternionic-Kähler manifolds

Let D be the Levi-Civita connection on a $4n$-dimensional pseudo-Riemannian manifold (M, g), where g is a metric of signature $(4k, 4l)$ with $k + l = n$ and $n \geq 2$. Then

Definition 17. *(M, g) is **pseudo-quaternionic-Kähler** if the Riemannian holonomy group Hol(D) is contained within $Sp(k,l) \cdot Sp(1)$.*

The theory of pseudo-quaternionic-Kähler manifolds runs fairly parallel to that for quaternionic manifolds outlined above, with a suitable change in the signature of the metric. Further details are given in, e.g. [77].

The only difference for technical applications comes in the check that a given pseudo-Riemannian manifold is pseudo-quaternionic-Kähler, which becomes [34]

$$q^\dagger \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_l \end{pmatrix} = - \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_l \end{pmatrix} q,$$

$$t^T \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_l \end{pmatrix} = - \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_l \end{pmatrix} t,$$

and

$$\mathrm{tr}(p) = 0, \qquad p^\dagger = -p.$$

2.3.4 Para-quaternionic-Kähler manifolds

We turn our attention now to para-quaternionic-Kähler manifolds [77], which will appear throughout this thesis as the target spaces in the image of the time-like or Euclidean c-maps, as we will see in Chapter 3.

Let D be the Levi-Civita connection on a $4n$-dimensional pseudo-Riemannian manifold (M, g), where g is a metric of split-signature $(2n, 2n)$ with $n \geq 2$. Then

Definition 18. *(M, g) is **para-quaternionic-Kähler** if the Riemannian holonomy group Hol(D) is contained within $Sp(2n, \mathbb{R}) \cdot Sp(2, \mathbb{R})$.*

As with quaternionic-Kähler manifolds, the case $n = 1$ should be treated separately since again this only implies that the manifold be oriented. We require instead the stricter condition that it be oriented, Einstein and have a self-dual Weyl tensor.

Relation to para-quaternionic structure

One can relate the definition of a para-quaternionic-Kähler manifold given in terms of holonomy to the existence of a para-quaternionic structure Q on (M, g). In particular,

Theorem 11. *A Riemannian manifold (M, g) is para-quaternionic-Kähler iff $\exists$ a para-quaternionic structure Q such that (M, Q, g) is para-quaternionic hermitian and:*

(i) The Levi-Civita connection preserves Q.

(ii) For any $p \in U_i \cap U_j$, the para-quaternionic structures Q_p on U_i and U_j agree.

We can treat the first of these conditions similarly to the quaternionic case. In particular, saying that the Levi-Civita connection D preserves the para-quaternionic structure Q is equivalent to the existence of 1-forms $\alpha, \beta, \gamma \in \Gamma(T^*M)$ such that

$$\begin{aligned} DJ_1(X) &= && +\alpha(X)J_2 + \beta(X)J_3 \\ DJ_2(X) &= -\alpha(X)J_1 && \qquad\qquad +\gamma(X)J_3 \\ DJ_3(X) &= \beta(X)J_1 && +\gamma(X)J_2 \qquad\qquad . \end{aligned}$$

The second condition is just analogous to the corresponding condition for quaternionic-Kähler manifolds.

How to check a manifold is para-quaternionic-Kähler

To check whether (M, g) is para-quaternionic-Kähler, we use the Ambrose-Singer theorem as before. This time we require that the Levi-Civita connection 1-form ω lies in $(\mathfrak{sp}(2n, \mathbb{R}) \oplus \mathfrak{sp}(2, \mathbb{R})) \otimes T^*M$. For this to be the case one should be able to make the decomposition [34]

$$\omega = p \otimes \mathbb{1}_{2n} + \mathbb{1}_2 \otimes \begin{pmatrix} q & t \\ -t^T & -q^T \end{pmatrix}, \tag{2.27}$$

where

$$t^T \begin{pmatrix} -\mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n \end{pmatrix} = \begin{pmatrix} -\mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n \end{pmatrix} t,$$

and

$$\operatorname{tr}(p) = 0.$$

2.4 Lie algebras

In this section we introduce the material necessary for understanding the structure of Lie algebras and symmetric spaces. This material will be important in Chapter 5 when we study the pseudo-Riemannian symmetric space $G_{2(2)}/(SL_2 \cdot SL_2)$.

For brevity we will content ourselves with simply stating the various results that we need, referring to the literature for proofs and further details where necessary.

We begin in Section 2.4.1 by providing a number of definitions and terminology needed for discussing Lie algebras. We then turn, in Section 2.4.2, to the study of real forms of a complex semi-simple Lie algebra, and relate their classification to the classification of (pseudo-)Riemannian symmetric spaces. In Section 2.4.3 we introduce the root space decomposition of a given real form, before using this in Section 2.4.4 to describe the Iwasawa decomposition of a Lie algebra.

2.4.1 Preliminaries

There are many excellent texts on the basics of Lie algebras. We mainly follow [78, 79].

Definitions

Recall that a **Lie algebra** $\mathfrak{g}$ is a vector space over some field F which comes equipped with a bilinear antisymmetric 'bracket' relation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

A **Lie subalgebra** $\mathfrak{h} \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ which is closed under the bracket operation, that is $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. An **ideal** $\mathfrak{h}$ in $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$.

Adjoint representation

Associated to any Lie algebra $\mathfrak{g}$ there is a linear map $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}\mathfrak{g}$ called the **adjoint representation** which associates to any $X \in \mathfrak{g}$ the endomorphism $\operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}$ given by

$$(\operatorname{ad} X)Y = [X, Y]\,. \tag{2.28}$$

In terms of this endomorphism, the Jacobi identity becomes

$$(\operatorname{ad} Z)\,[X, Y] = [(\operatorname{ad} Z)X, Y] + [X, (\operatorname{ad} Z)Y]\,,$$

which is just the property that $\operatorname{ad} Z$ acts as a derivation on the Lie algebra $\mathfrak{g}$.

Examples

A familiar example of a Lie algebra, which we have already used when talking about integrability of vector fields in Section 2.2, is the set of smooth vector fields $\Gamma(TM)$ on a manifold M, with bracket $[X, Y] = XY - YX$ for any $X, Y \in \Gamma(TM)$.

The second important example is the Lie algebra $\mathfrak{g}$ of a Lie group G, which is generated by left-invariant vector fields. That is, for any group element $a \in G$ we can associate a map $L_a : G \to G$ acting by left translations $L_a(g) = ag$. Then left-invariant vector fields on G are precisely those for which $(L_a)_* X_g = X_{ag}$ for $X_g \in T_g G$. We can then identify the set of left-invariant vector fields $\mathfrak{g} \subset \Gamma(TG)$ with the tangent space $T_e G$ at the identity $e \in G$. The Lie algebra $\mathfrak{g}$ therefore has the same dimension as the group G.

Solvable, nilpotent and semi-simple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Write

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k].$$

The sequence

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \dots\,, \tag{2.29}$$

is called the **derived series** for $\mathfrak{g}$. We say that $\mathfrak{g}$ is **solvable** if $\mathfrak{g}^j = 0$ for some j.

We next define

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_{k+1}] .$$

The sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \ldots, \tag{2.30}$$

is called the **lower central series** for $\mathfrak{g}$. We say that $\mathfrak{g}$ is **nilpotent** if $\mathfrak{g}_j = 0$ for some j. Note that $\mathfrak{g}^j \subseteq \mathfrak{g}_j$, so any nilpotent Lie algebra is automatically solvable. Both nilpotent and solvable algebras will be important when we discuss Iwasawa subalgebras in Section 2.4.4.

Many of the most elegant classification theorems for Lie algebras only hold for the so-called **semi-simple** Lie algebras, which can be defined as those $\mathfrak{g}$ with no nonzero solvable ideals. Throughout the remainder of this thesis, we will always take $\mathfrak{g}$ to be semi-simple.

Killing form

The **Killing form** of a Lie algebra $\mathfrak{g}$ is a bilinear function $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}\, X \mathrm{ad}\, Y). \tag{2.31}$$

Clearly the Killing form is symmetric via cyclicity of the trace. Moreover, let $\alpha : \mathfrak{g} \to \mathfrak{g}$ be an automorphism of $\mathfrak{g}$, i.e.

$$[\alpha(X), \alpha(Y)] = [X, Y] \quad \forall\, X, Y \in \mathfrak{g}.$$

Then we have

$$B(\alpha(X), \alpha(Y)) = B(X, Y) \quad \forall\, \alpha \in \mathrm{Aut}(\mathfrak{g}),\ X, Y \in \mathfrak{g}. \tag{2.32}$$

In other words, the Killing form is preserved by automorphisms of $\mathfrak{g}$.

The Killing form is an important tool in a number of classification theorems for Lie algebras. In particular, Cartan's criterion (Theorem 1.42 of [79]) states that the

Lie algebra $\mathfrak{g}$ is semisimple iff the Killing form for $\mathfrak{g}$ is nondegenerate. In this case, we can diagonalise B in a suitable basis. The eigenvectors with negative eigenvalue are the **compact** generators, whist those with positive eigenvalue are the **non-compact** generators.

Lie algebra involution

An **involution** $\alpha : \mathfrak{g} \to \mathfrak{g}$ is an automorphism[7] of the Lie algebra $\mathfrak{g}$ such that $\alpha^2 = \mathbb{1}$ is the identity map.

With any involution α of $\mathfrak{g}$, we can decompose $\mathfrak{g}$ into disjoint subspaces, being the $+1$ and -1 eigenspaces of α. We denote by

$$\mathfrak{k} = \{X \in \mathfrak{g} : \alpha(X) = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g} : \alpha(X) = -X\},$$

the $+1$ and -1 eigenspaces respectively. We then have the Lie algebra decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Note that the algebraic structure of $\mathfrak{g}$ simplifies as follows. Let $X, Y \in \mathfrak{k}$. Then

$$\alpha\,[X, Y] = [\alpha(X), \alpha(Y)] = [X, Y]\,,$$

so for $X, Y \in \mathfrak{k}$, the Lie bracket $[X, Y] \in \mathfrak{k}$. Likewise, for $X, Y \in \mathfrak{p}$, we find that $[X, Y] \in \mathfrak{k}$, whereas for $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$, we have $[X, Y] \in \mathfrak{p}$. Hence, the algebraic structure of $\mathfrak{g}$ decomposes as

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{2.33}$$

We see that the $+1$ eigenspace $\mathfrak{k} \subset \mathfrak{g}$ forms a Lie subalgebra of $\mathfrak{g}$, which normalizes $\mathfrak{p}$.

Alternatively, we could think of the involution as providing a $\mathbb{Z}_2$**-grading** $\mathfrak{g} =$

[7] Generally we find the cases where $\alpha = \pm\mathbb{1}$ uninteresting.

$\mathfrak{g}_{ev} \oplus \mathfrak{g}_{odd}$ of $\mathfrak{g}$, where

$$[\mathfrak{g}_{ev}, \mathfrak{g}_{ev}] \subset \mathfrak{g}_{ev}, \quad [\mathfrak{g}_{ev}, \mathfrak{g}_{odd}] \subset \mathfrak{g}_{odd}, \quad [\mathfrak{g}_{odd}, \mathfrak{g}_{odd}] \subset \mathfrak{g}_{ev}. \tag{2.34}$$

This is the notation we will use in Chapter 5.

In fact, the conditions (2.33) are precisely equivalent to the condition that G/K is a symmetric space, where $G = \exp \mathfrak{g}$ and $K = \exp \mathfrak{k}$ [80]. The triple $(\mathfrak{g}, \mathfrak{k}, \alpha)$ is called a **symmetric Lie algebra**.

2.4.2 Real forms of Lie algebras

Given a complex Lie algebra $\mathfrak{g}$, we say that a **real form** $\mathfrak{g}_0$ exists if there is a basis such that the structure constants are real. This is equivalent to the statement that $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$:

$$\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}_0 \oplus i\mathfrak{g}_0.$$

Any real form $\mathfrak{g}_0$ of $\mathfrak{g}$ defines a notion of **conjugation** on $\mathfrak{g}$. In particular, we write

$$\mathfrak{g} \ni Z = X_0 + iY_0 \in \mathfrak{g}_0 \oplus i\mathfrak{g}_0,$$

then the conjugation with respect to $\mathfrak{g}_0$ is given by $Z \mapsto \bar{Z} = X_0 - iY_0$.

Theorem 6.11 of [79] guarantees that any complex semi-simple Lie algebra $\mathfrak{g}$ admits a **compact real form** $\mathfrak{u}_0$, that is, a real Lie algebra on which the Killing form is negative definite.

Classification of real forms

Given a complex semi-simple Lie algebra $\mathfrak{g}$, a natural question to ask is: what are the allowed real forms?

In order to answer this, we proceed as follows. First, we know that any such $\mathfrak{g}$ admits a compact real form $\mathfrak{u}_0$. We then look for the possible involutions $\alpha : \mathfrak{u}_0 \to \mathfrak{u}_0$.

Each such involution gives rise to a decomposition of $\mathfrak{u}_0$ into eigenspaces

$$\mathfrak{u}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0. \tag{2.35}$$

Note that both subspaces are compact, since $\mathfrak{u}_0$ is compact.

We now perform the 'Weyl unitarity trick' on the (-1) eigenspace, and replace $\mathfrak{p}_0 \mapsto i\mathfrak{p}_0$. This defines a real form

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0, \tag{2.36}$$

of $\mathfrak{g}$. It turns out that as α runs over all involutions of $\mathfrak{u}_0$, so (2.36) gives rise to all real forms of $\mathfrak{g}$ [78]. The results of this classification can be found in Table 9.3 of [78], and can be represented graphically through the use of Tits-Satake diagrams [81].

Since $\mathfrak{p}_0$ was compact, we find now that $i\mathfrak{p}_0$ is non-compact. Such a decomposition of a Lie algebra into compact and non-compact subspaces is known as a **Cartan decomposition** [79].

Riemannian symmetric spaces

The real forms (2.35) and (2.36) provide us with examples of symmetric Lie algebras $(\mathfrak{u}_0, \mathfrak{k}_0, \alpha)$ and $(\mathfrak{g}_0, \mathfrak{k}_0, \theta)$, which give rise to symmetric spaces G/K and G^*/K with maximally compact stability group $K = \exp \mathfrak{k}_0$. Since the generators of $\mathfrak{p}_0$ (resp. $i\mathfrak{p}_0$) are compact (resp. non-compact), the coset G/K (resp. G^*/K) will be a compact (resp. non-compact) symmetric space.

Moreover, we can endow each of these with a negative (resp. positive) definite metric derived from the Killing form B restricted to $\mathfrak{p}_0$ (resp. $i\mathfrak{p}_0$), giving them the structure of a Riemannian symmetric space.

Classification of real forms of symmetric spaces

The non-compact symmetric spaces G^*/K are important in many supergravity applications [35]. However, when dealing with time-like dimensional reduction, one encounters pseudo-Riemannian symmetric spaces, for which the stability group K is a non-compact subgroup of G^* [82]. From our discussion thus far, we would expect that such non-compact subgroups should appear by considering non-Cartan involutions of a given non-compact real form $\mathfrak{g}_0$. A natural question to ask is then: given a non-compact real form $\mathfrak{g}_0$ of some complex semi-simple Lie algebra $\mathfrak{g}$, what are the admissible involutions of $\mathfrak{g}_0$?

The classification is similar to that for determining the real forms of a given complex Lie algebra. The results can be found in Table 9.7 of [78], from which we can read off the possible pseudo-Riemannian symmetric spaces G^*/K^*.

2.4.3 Root spaces

From now on we change notation slightly and take $\mathfrak{g}$ to be a *real* semi-simple Lie algebra of dimension n.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ (which always exists) and take $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subalgebra of $\mathfrak{p}$, which we take to have dimension $\dim \mathfrak{a} = r = \operatorname{rank} \mathfrak{g}$. Let $(H_i) = \{H_1, \ldots, H_r\}$ be a basis for $\mathfrak{a}$. The matrices $\operatorname{ad} H_i$ will be simultaneously diagonalisable (Lemma 7.5.9 of [83]) and defined by their action on the basis (T_a), $a = 1, \ldots, n$ of $\mathfrak{g}$.

In particular, we have

$$\operatorname{ad} H_i(T_a) = \lambda_a(H_i) T_a, \tag{2.37}$$

which defines a set of n r-component vectors λ_a, spanning the dual space $\mathfrak{a}^*$. The non-zero λ_a are called the **roots** of $\mathfrak{g}$, the set of which we denote Σ.

Given a particular basis (H_i) for the Cartan subalgebra, we can choose a subset $\Sigma^+ \subset \Sigma$ of roots, which we call **positive roots**, by taking those vectors whose first non-zero entry is positive[8]. Finally, the **simple roots** are those positive roots which cannot be written as the sum of two other positive roots. It is a theorem (Proposition 2.49 of [79]) that the number of simple roots of a Lie algebra $\mathfrak{g}$ is equal to its rank.

An example is in order, which we take from [37].

[8]This definition of positivity is sometimes called **lexicographic ordering** [79]. However, one is free to choose some other definition of positivity, provided (i) for any nonzero root λ, one of λ and $-\lambda$ is positive; and (ii) any sum and positive multiple of positive roots is positive.

Example: G_2

Consider the Lie algebra $\mathfrak{g}$ of the group $G = G_{2(2)}$, being the noncompact group of type $G2$. This has dimension 14 and rank 2. We will meet this Lie algebra in Chapter 5, so in order to prepare ourselves we will use it as our toy example for understanding root spaces.

In the notation of Chapter 5 we take the basis $(b_a) = \{b_1, \ldots, b_{14}\}$ of $\mathfrak{g}$ given in (5.14) and choose

$$H_1 = b_1 + b_2, \quad H_2 = b_1 - b_2.$$

The matrices $\operatorname{ad} H_1$ and $\operatorname{ad} H_2$ in the basis (b_a) are given by

$$\begin{aligned} \operatorname{ad} H_1 &= \operatorname{diag}(0,0,1,2,1,-1,-2,-1,1,0,-1,-1,0,1), \\ \operatorname{ad} H_2 &= \operatorname{diag}(0,0,3,0,-3,-3,0,3,1,-2,1,-1,2,-1). \end{aligned}$$

Hence, the non-zero roots of G_2 are given by

$$\begin{gathered} \lambda_3 = (1,3), \quad \lambda_4 = (2,0), \quad \lambda_5 = (1,-3), \quad \lambda_6 = (-1,-3), \\ \lambda_7 = (-2,0), \quad \lambda_8 = (-1,3), \quad \lambda_9 = (1,1), \quad \lambda_{10} = (0,-2), \\ \lambda_{11} = (-1,1), \quad \lambda_{12} = (-1,-1), \quad \lambda_{13} = (0,2), \quad \lambda_{14} = (1,-1), \end{gathered}$$

where the first element of λ_a denotes the eigenvalue of b_a under H_1, the second under H_2. These roots can be plotted on a plane with axes corresponding to H_1 and H_2 eigenvalues, and one sees that they form the usual root diagram of the Lie algebra of type $G2$, presented in Figure 2.1.

The positive roots, i.e. those with first non-zero entry positive, are given by $\Sigma^+ = \operatorname{span}\{\lambda_3, \lambda_4, \lambda_5, \lambda_9, \lambda_{13}, \lambda_{14}\}$. In Figure 2.1 these are denoted with an open diamond. Moreover, one can show that each of $\{\lambda_3, \lambda_4, \lambda_9, \lambda_{14}\}$ can be written as the sum of other positive roots. Hence, the simple roots are $\{\lambda_5, \lambda_{13}\}$.

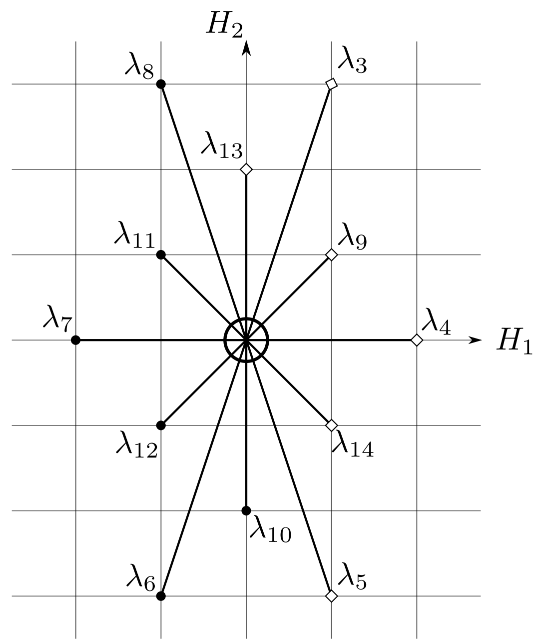

Figure 2.1: Root system for $G2$

Restricted root spaces

Any non-zero root $\lambda \in \mathfrak{a}^*$ defines a **restricted root space**

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \operatorname{ad} H(X) = \lambda(H)X \quad \forall\, H \in \mathfrak{a}\}\,. \tag{2.38}$$

The non-zero $\lambda \in \mathfrak{a}^*$ for which $\mathfrak{g}_\lambda$ is nontrivial[9] are then precisely the (restricted) roots of $\mathfrak{g}$. Furthermore, we put

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} : \operatorname{ad} H(X) = 0 \quad \forall\, H \in \mathfrak{a}\}\,. \tag{2.39}$$

From the construction of $\mathfrak{g}_\lambda$ and $\mathfrak{g}_0$, we see that $\mathfrak{g}$ is the direct sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda, \tag{2.40}$$

which just says that any $X \in \mathfrak{g}$ either has zero eigenvalue under the action of $\operatorname{ad} H_i$ for all i, or it has non-zero eigenvalue for some $\operatorname{ad} H_i$.

[9]Note that $\mathfrak{g}_\lambda$ always contains the zero element $0 \in \mathfrak{g}$.

Let $X \in \mathfrak{g}_\lambda$ and $Y \in \mathfrak{g}_\mu$ be elements of the restricted root space for some roots λ, μ. Then

$$\mathrm{ad}\, H\, [X, Y] = [H, [X, Y]] = [[H, X], Y] + [X, [H, Y]] = (\lambda + \mu)(H)\, [X, Y],$$

where we've used the Jacobi identity. Hence, $[X, Y] \in \mathfrak{g}_{\lambda+\mu}$ (it could be the zero element), and so we have

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}. \tag{2.41}$$

Positive root spaces and nilpotent Lie subalgebras

We now restrict ourselves to the positive root space $\Sigma^+ \subset \Sigma$. Define

$$\mathfrak{n}^+ = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda. \tag{2.42}$$

Take $X, Y \in \mathfrak{n}^+$ with $X \in \mathfrak{g}_\lambda$ and $Y \in \mathfrak{g}_\mu$ for some positive roots $\lambda, \mu \in \Sigma^+$. Then $[X, Y] \in \mathfrak{g}_{\lambda+\mu} \subset \mathfrak{n}^+$, and so $\mathfrak{n}^+$ is a Lie subalgebra of $\mathfrak{g}$. One can show further that $\mathfrak{n}^+$ is a nilpotent Lie algebra, in the sense that the lower central series (2.30) terminates [83].

2.4.4 The Iwasawa decomposition for Lie algebras

We now have all the necessary information in place to study the **Iwasawa decomposition** for real semi-simple Lie algebras. The question of how this decomposition lifts to the level of the Lie group we study in detail in Chapter 5.

Theorem 12. *Any real semi-simple Lie algebra $\mathfrak{g}$ can be written as the direct sum*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+, \tag{2.43}$$

where $\mathfrak{a}$ is abelian and $\mathfrak{n}^+$ nilpotent.

Proof: The proof can be found as Proposition 6.43 of [79]. □

The Iwasawa subalgebra

We denote by

$$\mathfrak{l} := \mathfrak{a} \oplus \mathfrak{n}^+, \tag{2.44}$$

the **Iwasawa subalgebra** appearing in the Iwasawa decomposition (2.43). One can show that $\mathfrak{l} \subset \mathfrak{g}$ is a solvable subalgebra, which relies on the fact that $\mathfrak{n}^+$ is nilpotent [79].

It is a fact, which we will use in Chapter 5, that any two Iwasawa subalgebras of $\mathfrak{g}$ are conjugated via some element of $\mathfrak{g}$. This is explained in Section VI.5 of [79].

Example: G_2 revisited

We return now to look at the restricted root spaces for the case $\mathfrak{g} = G_{2(2)}$. We have $\mathfrak{g}_0 = \text{span}\{b_1, b_2\} = \mathfrak{a}$ for the maximal abelian subalgebra. Taking $\lambda \in \Sigma^+$ gives

$$\mathfrak{n}^+ = \text{span}\{b_3, b_4, b_5, b_9, b_{13}, b_{14}\} \subset \mathfrak{g}.$$

This is precisely the nilpotent subalgebra $\mathfrak{n}$ appearing in the Iwasawa subalgebra of Proposition 2 in [37].

Chapter 3

Preliminary physics

We now introduce the necessary background physics needed for the main body of this thesis, concentrating on the role of $\mathcal{N} = 2$ supersymmetry.

We start in Section 3.1 with a general discussion of black objects in supergravity theories, specifically concentrating on those which can be derived from string theoretic considerations. We then turn to the subject of $\mathcal{N} = 2$ theories in five dimensions (Section 3.2) and describe the coupling of $\mathcal{N} = 2$ vector multiplets to gravity, relating this to the geometric structures of special real manifolds introduced in the previous chapter. We perform a similar analysis in Section 3.3 for four-dimensional vector multiplets in both Minkowski and Euclidean signature, and relate this to the special (para-)Kähler manifolds, before mentioning the structure of hypermultiplets coupled to gravity in Section 3.4. We then move on to the technique of dimensional reduction (Section 3.5), which we use to relate the matter-coupled supergravity theories in five, four and three dimensions. Finally, in Section 3.6, these reductions are interpreted as providing maps between the various scalar target spaces, known as the r- and c-maps.

3.1 Black objects in supergravity theories

We concentrate in this section on the general structure of supergravity theories, and investigate the type of solutions which they admit. We begin in Section 3.1.1 by outlining how matter-coupled supergravity theories naturally emerge from looking at string theory at low energies. We then turn to look at the type of solutions that can be found to such theories, concentrating on the 'black p-branes' in Section 3.1.2 and then investigating some of their physical properties in Section 3.1.3. Section 3.1.4 contains a number of useful definitions pertaining to general field configurations. Finally, in Sections 3.1.5 and 3.1.6, we look at the scalar manifolds associated to supergravity theories containing scalar fields, and introduce a number of useful geometrical notions which will help us to find and classify solutions.

3.1.1 From strings to supergravity

One of the primary motivations for studying supergravity as an extension of general relativity (GR) comes from string theory. In particular, the massless bosonic sector of the type II superstring contains the graviton, a 2-form potential, a scalar field called the dilaton, and various other $(n-1)$-form gauge potentials, where the precise values of n depend on which of the type II string theories (IIA or IIB) we're considering [84].

At low energies then (where the massive modes of the string become irrelevant) we naturally obtain gravitational theories in some D-dimensional background containing various gauge potentials. Moreover, by considering suitable geometries on which the strings propagate, we can add various 'matter' fields to this content.

The effective action

Consider the worldsheet action describing a string moving in some background (G_{MN}, B_{MN}, ϕ) which is sourced by its massless modes [85]

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma} \left[\left(\gamma^{\alpha\beta} G_{MN}(X) + i\epsilon^{\alpha\beta} B_{MN}(X) \right) \partial_\alpha X^M \partial_\beta X^N + \alpha' R(\gamma)\phi(X) \right] . \tag{3.1}$$

The condition that the worldsheet theory be Weyl invariant at the quantum level is equivalent to the vanishing of a set of beta functions $\beta_{MN}(G) = \beta_{MN}(B) = \beta(\phi) = 0$ [85]. These can in turn be interpreted as the equations of motion coming from an effective action for the fields (G_{MN}, B_{MN}, ϕ) which constitute the NS-NS sector of type II strings

$$S_{NS-NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G}\, e^{-2\phi} \left(R + 4\partial_M\phi\, \partial^M\phi - \frac{1}{12} H_{MNP} H^{MNP} \right) , \tag{3.2}$$

where $H_{(3)} = dB_{(2)}$ is the field strength of the B-field. Note that the action (3.2) has a non-canonical factor $e^{-2\phi}$ multiplying the Ricci scalar, which tells us that the metric G_{MN} is really the 'string frame' metric.

The string frame action (3.2) has the advantage that it is tailored to string perturbation theory. In particular, the asymptotic value of the dilaton ϕ_∞ is related to the string coupling constant g_S via

$$g_S = e^{\phi_\infty}. \tag{3.3}$$

The factor of $e^{-2\phi_\infty} = g_S^{-2}$ in (3.2) is due to the fact that the action was computed at tree level in string perturbation theory.

One can go to the 'Einstein frame' metric, for which we recover the canonical Einstein-Hilbert term, by absorbing the dilaton into the string-frame metric via an appropriate field redefinition. Indeed, making the field redefinition $G_{MN} = e^{\phi/2} g_{MN}$ we find

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{2}\partial_M \phi \, \partial^M \phi - \frac{1}{12} e^{-\phi} H_{MNP} H^{MNP} \right), \tag{3.4}$$

which takes a form amenable to relativists. For Type II theories, one should add to this the contribution from RR-sector fields, which schematically take the same form, with the 3-form field strengths replaced by suitable n-form field strengths.

The advantage of coming to such gravitational theories from string theory is that, by including also the contribution from massless fermions in the string spectrum, one can embed the bosonic actions into a fully supersymmetric action. Within theories of extended supersymmetry, one can identify a particular class of "BPS" states, which saturate a supersymmetric mass bound and can therefore be compared between different theories. This has led to a fuller understanding of non-perturbative aspects of string and M-theory by studying the spectrum of allowed BPS states in a given supergravity theory [4].

3.1.2 Black p-branes

In dimension greater than four, one can encounter spacetime solutions with horizon topology differing from the S^{D-2} one expects for black holes [86, 87]. In Chapter 6 we will construct charged solutions in five-dimensional supergravity which have topology $\mathbb{R} \times S^2$, i.e. they have infinite spatial extent in one direction. Such objects are referred to as black strings, and are a specific case of a general class of black p-brane solutions which appear in supergravity theories.

In this thesis, the class of D-dimensional supergravity theories we want to consider are those comprising the metric g_{MN}, a set of scalar fields φ^a, and $(n-1)$-form gauge potentials with n-form field strengths. As we saw above, these constitute the general field contents one obtains by a suitable truncation of the spectrum of string or M-theory.

In most of the applications in this thesis we will only be interested in the case $n = 2$, but for the moment we continue more generally.

Extremal p-branes

The simplest class of solutions we can look for are the so-called (extremal) 'p-brane' solutions [42, 88], which have an $ISO(p,1) \times SO(D-p-1)$ symmetry. That is, the spacetime has translational and Lorentz symmetry along the $(p+1)$-dimensional world-volume of the brane, and isotropic symmetry in the directions transverse to it. Splitting our D spacetime coordinates into x^μ ($\mu = 0, 1, \ldots, p$) along the brane and y^m ($m = p+1, \ldots, D-1$) in the transverse directions, the most general metric ansatz respecting the $ISO(p,1) \times SO(D-p-1)$ symmetry is given by

$$ds^2_{(D)} = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(r)} \delta_{mn} dy^m dy^n, \tag{3.5}$$

where $r = (y^m y^m)^{\frac{1}{2}}$ is the isotropic radial coordinate in the transverse space. The corresponding ansatz for the scalar fields is $\varphi^a = \varphi^a(r)$.

Charged solutions

For the gauge fields, we need to check which $(n-1)$-form gauge potentials couple to the p-brane worldvolume. We can do this via a simple dimension-counting argument. Given an $(n-1)$-form gauge potential with field strength $F_{[n]}$, the electric charge associated with a given configuration is

$$Q = \int_{S^{D-n}_{\infty}} *F_{[n]}.$$

Counting dimensionality we have an S^{D-n}, taking up $(D-n)$ spatial dimensions, placed at a large radial distance (a further spatial dimension) from some source. Hence, the source of the $(n-1)$-form gauge potential should be extended in $p = n-2$ spatial directions. Turning this around, we see that a p-brane is coupled electrically to a $(p+1)$-form gauge potential. Moving now to the magnetic charge

$$P = \int_{S^{n}_{\infty}} F_{[n]},$$

we can use the same arguments to show that the source should extend in $\tilde{p} = D-n-2$ spatial directions. Hence, a p-brane is coupled magnetically to a $(D-p-3)$-form gauge potential.

Note that in the case $p+1 = D-p-3$, i.e. when $p+2 = \frac{D}{2}$ is half the dimension of the spacetime, a given p-brane solution can couple both electrically and magnetically to the $(p+1)$-form gauge potential, giving rise to dyonic solutions. This is the case for, e.g. black holes (0-branes) in four dimensions and 3-branes in ten dimensions, which appear in IIB supergravity [88].

Concentrating on the case $D = 5$, $n = 1$, which is relevant for five-dimensional supergravity coupled to vector multiplets, we see that the electrically charged solutions will have $p = 0$, while magnetically charged solutions will have $\tilde{p} = 1$. The static electric black hole solutions have been well studied in [43, 47] using a formalism akin to that presented in this thesis. Here we will focus on the magnetic black string solutions in Chapter 6, before going on to consider more general classes of both electric and magnetic black objects in Chapter 7.

An important point, which we will make use of later, is that for the p-brane solutions

(3.5) the fields are all independent of the worldvolume coordinates x^μ. Hence we can hope to construct such solutions by dimensional reduction of the D-dimensional theory to a $(D-p-1)$-dimensional Euclidean theory [35]. We will clarify what we mean by the notion of 'dimensional reduction' in Section 3.5.

3.1.3 Properties of the solutions

The black brane solutions that we wish to consider generally have infinite spatial extent in some directions, and so the total energy of the spacetime as measured by some surface integral at infinity would diverge. Instead, then, we should consider the energy density, or **tension** of the solution. One can achieve this by suitably modifying the usual expression for ADM mass. Writing $g_{MN} = \eta_{MN} + h_{MN}$, we have [42]

$$\mathcal{E} = \frac{1}{16\pi} \int_{\partial M_T} d^{D-p-1}\Sigma^i \left(\partial^j h_{ij} - \partial_i h^b_b \right), \tag{3.6}$$

where $i, j = 1, \ldots p-1$ are the spatial coordinates of the brane, and $b = 1, \ldots, D-1$ runs over all spatial directions. The integral is performed over the boundary ∂M_T of the transverse space.

The entropy density per unit worldvolume S_{BH} of the solution is given as usual by the Bekenstein-Hawking formula

$$S_{BH} = \frac{A}{4},$$

where the area A is taken to be the area of the $(\tilde{p}+2)$-sphere at the event horizon. In other words, it is the area of a surface of constant time, radial distance, and p-brane spatial volume.

3.1.4 Field configurations

In this section we give a number of definitions pertaining to the structure of field configurations in theories with gravity, following the treatment in [35]. This will enable us to make sensible modelling assumptions about our spacetime geometry in order to simplify the search for solutions.

Recall that a spacetime (M, g) is said to be **stationary** if it admits a Killing vector

K which is time-like near infinity. We extend this to the concept of a stationary field configuration:

Definition 19. *A field configuration is called stationary if the spacetime is stationary and the Lie derivative with respect to K of the scalars and the vector field strengths vanishes.*

Definition 20. *A stationary spacetime is **static** if the Killing vector K is hypersurface orthogonal, i.e. $K_{[\mu}\nabla_{\nu}K_{\rho]} = 0$.*

For the purposes of dimensional reduction as we will meet them in this thesis we generally require the existence of a pair of commuting Killing vectors K and $\tilde{K}$ on our spacetime M, at least one of which should be space-like.

The orbits generated by the space-like Killing vector will tell us about the geometry of the solution. For example, if $\tilde{K}$ generates orbits isomorphic to $\mathbb{R}$ then the spacetime is translation invariant. Likewise, the spacetime is axisymmetric if all orbits are closed [39, 89].

3.1.5 Non-linear sigma models

As well as the spacetime geometry, on which we have been concentrating thus far, the supergravity theories we will meet in this thesis are all endowed with a second type of geometry: that of the scalar target space. In particular, consider a matter-coupled supergravity action of the schematic form

$$S = \int d^D x \sqrt{g} \left(\frac{R}{2} - G_{ab}(\Phi) \partial_\mu \Phi^a \partial^\mu \Phi^b + \dots \right), \tag{3.7}$$

where the coupling $G_{ab}(\Phi)$ depends on the n scalar fields Φ^a, and the dots could stand for additional terms involving field strengths, fermions, etc. Such theories are referred to as gravity-coupled **non-linear sigma models**.

It will be instructive to derive the equations of motion for such theories. The Einstein equation, after taking a trace and back-substituting, is

$$\frac{1}{2} R_{\mu\nu} - G_{ab}(\Phi) \partial_\mu \Phi^a \partial_\nu \Phi^b = 0. \tag{3.8}$$

The equations of motion for the scalar fields Φ^a are

$$\nabla^2\Phi^a + \Gamma^a_{bc}(\Phi)\partial_\mu\Phi^b\partial^\mu\Phi^c = 0, \tag{3.9}$$

where the coefficients $\Gamma^a_{bc}(\Phi)$ are given by

$$\Gamma^a_{bc}(\Phi) = \frac{1}{2}G^{ad}\left(\partial_b G_{dc} + \partial_c G_{bd} - \partial_d G_{bc}\right),$$

and $\partial_a \equiv \frac{\partial}{\partial\Phi^a}$. This affords the following interpretation. A particular field configuration $\Phi^a(x)$ can be thought of as a map from a D-dimensional spacetime (X, g) to an n-dimensional pseudo-Riemannian manifold $(\mathcal{M}_D, G)$, which we call the **target space**. The metric on the target space is precisely the coupling matrix $G_{ab}(\Phi)$ appearing in the non-linear sigma model, which gives us the natural identification of $\Gamma^a_{bc}(\Phi)$ with the components of the Levi-Civita connection on $(\mathcal{M}_D, G)$. A given field configuration should satisfy the scalar equations of motion (3.9). Maps $\Phi : X \to \mathcal{M}_D$ satisfying (3.9) we call **harmonic**.

Hence, we see that finding scalar field configurations $\Phi^a(x)$ satisfying the equations of motion is equivalent to finding harmonic maps from spacetime into an n-dimensional target space $\mathcal{M}_D$. Given such a map, the Einstein equation (3.8) then determines the three-dimensional spacetime geometry.

3.1.6 Totally geodesic submanifolds

In practice, it is often convenient to work with only a small subset of all the fields in a given theory, enabling us to find solutions to a simpler set of field equations. However, we need to ensure that any such truncation of the field content is 'consistent' in the following sense:

Definition 21. *A **consistent truncation** is a truncation of the field content for which the solutions of the truncated theory are solutions of the full untruncated theory.*

In other words, we should be able to truncate a given field either at the level of the action or at the level of the field equations.

In many of the applications we'll be interested in in this thesis, we would like to truncate certain scalar fields of the theory. In terms of the target manifolds, this would correspond to restricting ourselves to maps $\Phi : X \to M' \subset \mathcal{M}_D$ from spacetime X into some submanifold M' of the full target space. We can then turn the condition that the truncation is 'consistent' into a geometrical condition on the submanifold M', namely that it be a **totally geodesic submanifold**. We start with the mathematical definition before relating this to the notion of consistent truncation of scalar fields in which we are primarily interested.

We follow Definition 8 of [32]. Let $\iota : M' \to (M, D)$ be an embedding of M' into M, where the manifold M is equipped with a connection D.

Definition 22. *The embedding $\iota : M' \to (M, D)$ is called **totally geodesic** if for any two vector fields $X, Y \in \Gamma(TM)$ which are tangent to M', $D_X Y$ is again tangent to M'.*

Indeed, let $n = \dim M$ and $m = \dim M' < \dim M$. Take $\{X_1, \ldots, X_n\}$ to be a local frame for TM in some neighbourhood of $p \in M'$, such that $\{X_1, \ldots, X_m\}$ restricted to M' is a local frame for TM'. Then the condition that M' be totally geodesic is equivalent to the statement that

$$D_{X_i} X_j = \sum_{k=1}^{m} \Gamma^k_{ij} X_k,$$

along M' for all $i, j = 1, \ldots, m$. Splitting the indices as $i = 1, \ldots, m$ and $x = m + 1, \ldots, n$, this is just the statement that the coefficients Γ^x_{ij} vanish.

We saw above that finding solutions to the scalar field equations is equivalent to looking for harmonic maps from spacetime to some target space $\mathcal{M}_D$. If we instead restrict ourselves to maps $\Phi : X \to M' \subset \mathcal{M}_D$ into totally geodesic submanifolds, then we can make use of the following (Proposition 12 in [32]):

Proposition 1. *Let $\iota : M' \to M$ be a totally geodesic embedding. Then a map $\varphi : N \to M'$ is harmonic iff $f = \iota \circ \varphi : N \to M$ is harmonic.*

For the cases we'll be interested in, N here is taken to be a D-dimensional spacetime, while $M = \mathcal{M}_D$ is taken to be the target manifold of the D-dimensional theory.

Generally we will look for totally geodesic submanifolds $M' \subset \mathcal{M}_D$ obtained by truncating some of the scalar fields in our theory. This would correspond to the case where M' is a hypersurface in $\mathcal{M}_D$. From the definition of totally geodesic submanifolds in Definition 22, it is not immediately clear how the case of such truncations could be treated. However, we can make use of the following [90]:

Proposition 2. *Let (M, G) be a pseudo-Riemannian manifold with metric G. If there exists an involution $\tau : M \to M$ which acts isometrically on G, then the fixed-point set of τ is a totally geodesic submanifold $M' \subset M$.*

We will see a number of examples of Proposition 2 when we look at constructing solutions, where truncating certain scalar fields will correspond to turning off certain charges in the supergravity theory. The 'consistent truncations' of Definition 21 are then precisely those which give rise to totally geodesic submanifolds.

3.2 The five-dimensional theory

This thesis is primarily concerned with the study of five-dimensional $\mathcal{N} = 2$ supergravity theories coupled to supersymmetric matter multiplets. Such theories, apart from being interesting in their own right, can give us insights into the non-perturbative structure of string and M-theory through study of the various solitonic objects which they admit.

In this section we first introduce the relevant field content for studying supersymmetric field theories in five dimensions, before analysing the action describing an arbitrary number of vector multiplets coupled to supergravity. This action will be our starting point for much of the work presented in this thesis.

3.2.1 The field content

We start with the fermionic content [28]. In five dimensions (assuming Lorentzian spacetime signature) the minimal spinor representations are symplectic Majorana spinors λ^i, with $i = 1, 2$, which transform as a doublet under the $SU(2)$ R-symmetry group of the five-dimensional $\mathcal{N} = 2$ superalgebra. Introducing the totally antisymmetric tensor ϵ_{ij}, the symplectic Majorana condition just tells us that the spinors λ^i satisfy the reality condition [28]

$$(\lambda^i)^* = -B\lambda^j \epsilon_{ij},$$

where B is the charge conjugation matrix.

Counting the degrees of freedom, we see that the minimal spinor in five dimensions has 8 real components. Hence, the "$\mathcal{N} = 2$" theory, which admits 8 real supercharges, is really the minimal allowed in five dimensions.

We now turn to the on-shell massless multiplets of the $\mathcal{N} = 2$ theory. The ones we will need for our present purposes are the $\mathcal{N} = 2$ vector and gravity multiplets.

The $\mathcal{N} = 2$ vector multiplet in five dimensions consists of a $U(1)$ gauge field $A_{\hat{\mu}}$, a real scalar ϕ, and an $SU(2)$ doublet of symplectic Majorana spinors λ^i. Counting the on-shell degrees of freedom, we find $3 + 1 = 4$ on the bosonic side, and 4 on the fermionic side, which matches as required. For the off-shell multiplets, we would need to add a further auxiliary bosonic field $Y^{ij} = Y^{ji}$ transforming as a triplet of $SU(2)$ [28]. However, for our purposes the on-shell multiplets will suffice.

The $\mathcal{N} = 2$ gravity multiplet in five dimensions consists of the vielbein $e_{\hat{\mu}}^{\hat{m}}$, a gauge field $\mathcal{A}_{\hat{\mu}}$ called the 'graviphoton', and an $SU(2)$ doublet of symplectic Majorana gravitini $\psi_{\hat{\mu}}^i$. Again, counting the on-shell degrees of freedom we find $3 + 5 = 8$ on the bosonic side and $2 \times 4 = 8$ on the fermionic side.

For the matter-coupled supergravity theories that we will be interested in we want to take $n_V^{(5)}$ copies of the vector multiplets and a single copy of the gravity multiplet. Indexing the vector multiplets by $x = 1, \ldots, n_V^{(5)}$, our full field content is

$$(e_{\hat{\mu}}^m, \psi_{\hat{\mu}}^i, \mathcal{A}_{\hat{\mu}}, A_{\hat{\mu}}^x, \lambda^{i|x}, \phi^x).$$

At this stage we should point out a bit of a sleight-of-hand. The ϕ^x, it will turn out, parametrise an $n_V^{(5)}$-dimensional manifold $\mathcal{H}$, which is not necessarily flat. Therefore the spinors $\lambda^{i|x}$ should really be vectors $\lambda^{i|a}$ with $a = 1, \ldots, n_V^{(5)}$ a flat $SO(n_V^{(5)})$ tangent space index introduced via the vielbeins f_x^a of $\mathcal{H}$ [30]. However, since we will only be interested in the bosonic part of this multiplet, we leave such technicalities aside and refer the reader to the appropriate places in the literature for a full treatment.

The most general supersymmetry transformation rules for this field content are given in equation (2.6) of [30], and we do not repeat them here. Taking them as granted we now turn our attention to constructing a five-dimensional action describing the dynamics of this field content, which should be gauge-invariant and supersymmetric.

3.2.2 The five-dimensional action

The action describing the coupling of $n_V^{(5)}$ five-dimensional $\mathcal{N}=2$ vector multiplets to supergravity was first analysed in [30]. We here concentrate only on the bosonic sector, since this will be all we need to construct black brane solutions, and refer to [30] for the fermionic completion. Using the conventions of [32], the five-dimensional action is given by

$$\begin{aligned} S_5 &= \int d^5x \left[\sqrt{\hat{g}} \left(\frac{\hat{R}}{2} - \frac{3}{4} g_{xy}(\phi) \partial_{\hat{\mu}} \phi^x \partial^{\hat{\mu}} \phi^y - \frac{1}{4} a_{ij}(h) \mathcal{F}^i_{\hat{\mu}\hat{\nu}} \mathcal{F}^{j|\hat{\mu}\hat{\nu}} \right) \right. \\ &\qquad \left. + \frac{1}{6\sqrt{6}} c_{ijk} \varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}} \mathcal{F}^i_{\hat{\mu}\hat{\nu}} \mathcal{F}^j_{\hat{\rho}\hat{\sigma}} \mathcal{A}^k_{\hat{\lambda}} \right]. \end{aligned} \tag{3.10}$$

Here $x = 1, \ldots, n_V^{(5)}$ labels the vector multiplet scalars, while $i = 1, \ldots, n_V^{(5)} + 1$ labels[1] the gauge fields: $n_V^{(5)}$ from the vector multiplets, and one (the graviphoton) from the supergravity multiplet. Our convention is to use 'hats' for the five-dimensional spacetime indices $\hat{\mu} = 0, \ldots, 4$.

In [30] the authors showed that requirements of gauge invariance (which imposes that the coefficients c_{ijk} be constant) and supersymmetry restricts the scalar target space of the five-dimensional theory to be an $n_V^{(5)}$-dimensional projective special real (PSR) manifold, as described in Section 2.1.4. That is, the scalar fields ϕ^x parametrise the hypersurface $H(h) = 1$ within a conic affine special real (CASR) manifold with Hesse potential

$$H(h) = c_{ijk} h^i h^j h^k.$$

The gauge coupling matrix $a_{ij}(h)$ is then given by the components (2.13) of the tensor

[1] Note that since we've dropped the fermionic terms, i no longer stands for a symplectic index.

field a on the CASR manifold, while

$$g_{xy}(\phi) = a_{ij}(h)\frac{\partial h^i}{\partial \phi^x}\frac{\partial h^j}{\partial \phi^x},$$

defines the metric on the PSR manifold. Hence, the full dynamics of the Lagrangian (3.10) is determined once we specify the number of vector multiplets, $n_V^{(5)}$, and the coefficients c_{ijk}.

The five-dimensional matter-coupled supergravity theories described (for the vector multiplet sector) by (3.10) can be obtained from 11-dimensional supergravity [91] by compactification on a Calabi-Yau three-fold X [92]. In this description, the Hodge numbers $(h^{1,1}, h^{2,1})$ of the Calabi-Yau determine the number of vector and hypermultiplets present in the five-dimensional theory, while the coefficients c_{ijk} are given by the intersection matrix

$$c_{ijk} \equiv \int_X V_i \wedge V_j \wedge V_k,$$

where $i, j, k = 1, \ldots, h^{1,1}$ and V_i is a basis of the cohomology $H^{1,1}(X)$. We will come to the structure of the hypermultiplet sector in Section 3.4.

3.3 The four-dimensional theory

We next turn our attention to the four-dimensional $\mathcal{N} = 2$ theories. Within this thesis, these will arise predominantly in the dimensional reduction of the five-dimensional theory over a space-like or time-like direction. We will therefore find it useful to treat the four-dimensional Minkowski and Euclidean theories simultaneously.

3.3.1 The field content

The $\mathcal{N} = 2$ supersymmetry algebra in four-dimensional Minkowski (resp. Euclidean) space can be obtained from dimensional reduction of the five-dimensional supersymmetry algebra over a space-like (resp. time-like) direction [28]. We will discuss the issue of dimensional reduction in more detail in Section 3.5. For the moment, we will take it to mean that representations of the five-dimensional tangent space group $SO(1, 4)$ are decomposed as representations of the four-dimensional tangent space group $SO(1, 3)$

or $SO(4)$ as appropriate.

The minimal spinors in four Minkowski dimensions can be taken as either Majorana or Weyl [5]. However, in order to treat both the Minkowski and Euclidean theories on the same footing, we follow [28] and take them to be Majorana. Then reduction of the five-dimensional fermions gives four-dimensional Majorana fermions Ω_i transforming under the $SU(2)$ R-symmetry group of the four-dimensional superalgebra.

Continuing in this manner, we can derive the four-dimensional supermultiplets from dimensional reduction of their five-dimensional counterparts. The reduction of the five-dimensional vector multiplet gives the four-dimensional vector multiplet

$$(A_\mu, \Omega_i, z),$$

where z is a scalar field made up of the five-dimensional scalar field and the fifth component of the five-dimensional gauge field. For space-like reduction, these combine to form a complex scalar field, while for time-like reduction it is para-complex, as we will see in Section 3.5.3. One can of course derive this field content from the multiplet calculus of massless $\mathcal{N} = 2$ representations in four dimensions [93].

The $\mathcal{N} = 2$ supergravity multiplet in four dimensions consists, as in the five-dimensional case, of the vielbein, an $SU(2)$ doublet of gravitini, and the graviphoton. The extra degrees of freedom coming from reduction of the five-dimensional supergravity multiplet arrange themselves into a further four-dimensional vector multiplet. This will be important in Section 3.5.3.

To summarise, our four-dimensional field content is

$$(e_\mu^m, \psi_\mu^i, A_\mu^I, \Omega_i^A, z^A),$$

where here $A = 1, \ldots, n_V^{(4)}$ labels the four-dimensional vector multiplets, and $I = (A, n_V^{(4)} + 1)$ the gauge fields.

3.3.2 The four-dimensional action

The action describing the coupling of $n_V^{(4)}$ four-dimensional $\mathcal{N} = 2$ vector multiplets to supergravity in Minkowski signature was described in [94, 95]. Here we follow [32] and treat the four-dimensional Euclidean and Minkowski theories in parallel. This will be important in Section 3.5.3 where we want to consider both space-like and time-like reductions of the five-dimensional supergravity theory (3.10). Introducing the parameter ϵ_1 via

$$\epsilon_1 = \begin{cases} -1 & \text{if } d = 1+3 \\ +1 & \text{if } d = 0+4 \, , \end{cases}$$

the bosonic part of the action is given by [32]

$$S_4 = \int d^4x\sqrt{g}\left[\frac{R}{2} - g_{A\bar{B}}(z,\bar{z})\partial_\mu z^A \partial^\mu \bar{z}^B + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}\right]. \quad (3.11)$$

The fermionic completion of (3.11) can be found for the Minkowski case in [95]. Our conventions for the dual field strengths are $\tilde{F}^{I|\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F^I_{\rho\sigma}$. The dependence of (3.11) on ϵ_1 enters through the scalar fields z^A, which are ϵ_1-complex, and the couplings $g_{A\bar{B}}, \mathcal{R}_{IJ}, \mathcal{I}_{IJ}$. In order that all kinetic terms have the correct sign in the Minkowski theory we require $g_{A\bar{B}}$ to be positive definite and $\mathcal{I}_{IJ}$ to be negative definite. For the Euclidean theory, we take $g_{A\bar{B}}$ to have split signature $(n_V^{(4)}, n_V^{(4)})$.

Gauge invariance and supersymmetry of the action (3.11) restrict the $2n_V^{(4)}$-dimensional target space parametrized by the z^A to be a projective special ϵ_1-Kähler (PSϵ_1K) manifold with metric $g_{A\bar{B}}(z,\bar{z})$, as described in Section 2.2.5.

In particular, we take X^I to be the special ϵ_1-holomorphic coordinates on the CASϵ_1K manifold N, such that

$$z^A = \frac{X^A}{X^0},$$

are coordinates on the PSϵ_1K manifold $\bar{N}$. Then the couplings $\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i_{\epsilon_1}\mathcal{I}_{IJ}$ can be determined from the prepotential $F(X)$ via [32]

$$\mathcal{N}_{IJ}(X,\bar{X}) = \bar{F}_{IJ}(\bar{X}) - \epsilon_1 i_{\epsilon_1}\frac{(\bar{N}X)_I(\bar{N}X)_J}{X\bar{N}X}, \quad (3.12)$$

where

$$N_{IJ}(X,\bar{X}) = -i_{\epsilon_1}(F_{IJ} - \bar{F}_{IJ}),$$

are the components of the metric on the CASϵ_1K manifold.

The dynamics of the four-dimensional theory are therefore encoded in a single ϵ_1-holomorphic function $F(X)$, which is homogeneous of degree 2.

3.4 The hypermultiplet sector

The other $\mathcal{N} = 2$ supermultiplet that we will encounter in this thesis is the hypermultiplet, which consists of four scalar fields and two spinors. Since this field content remains unchanged by dimensional reduction, the bosonic part of the hypermultiplet sector is identical in five, four or three dimensions. The only difference comes in the spinor representations involved [5].

The full action for n hypermultiplets coupled to supergravity is given in [31]. In four dimensions the Lagrangian for the bosonic sector reads

$$e^{-1}\mathcal{L} = \frac{R}{2} - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - h_{uv}(q)\partial_\mu q^u \partial^\mu q^v, \tag{3.13}$$

where $u, v = 1, \ldots, 4n$ labels the scalar fields in the hypermultiplets. In [31], the authors show that imposing that the full Lagrangian be locally supersymmetric restricts the $4n$-dimensional target manifold for the scalar fields to be quaternionic-Kähler, as described in Section 2.3.3, with negative scalar curvature

$$R = -8n(n+2).$$

The three-dimensional action can be obtained from (3.13) by dimensional reduction. In this case, the bosonic degrees of freedom contained in the four-dimensional gravity multiplet can be packaged into an extra hypermultiplet, using the fact that three-dimensional vector fields can be dualized to scalars. We will see this in more detail in Section 3.5.4.

3.5 Dimensional reduction

The procedure of dimensional reduction, which we have touched upon at various points so far, is one of the central tools we will use in this thesis to understand the structure of supergravity theories.

In this section we will give an overview of the philosophy and procedure of Kaluza-Klein dimensional reduction (Sections 3.5.1 and 3.5.2) in a general setting. We then move on in Section 3.5.3 to reduce the five-dimensional supergravity theory of Section 3.2 to four dimensions, and in Section 3.5.4 we reduce the four-dimensional supergravity of Section 3.3 to three dimensions.

3.5.1 Kaluza-Klein dimensional reduction

In this section we will elaborate on the procedure of Kaluza-Klein dimensional reduction, which plays a central role in much of this thesis. There are many excellent reviews on this subject, so we will be fairly schematic and simply present the main results and techniques, referring to the literature for more detailed calculations.

Our main interest in this thesis is dimensional reduction over a circle S^1 or torus T^n, which we can simply treat as successive S^1 reductions. Therefore we concentrate on the case of reduction on a circle. Kaluza-Klein reduction on more exotic compact manifolds, e.g. S^n, can be found in [96].

For the S^1 reductions, we first want to expand the $(D+1)$-dimensional fields $\Phi(x^M) = \Phi(x^\mu, z)$ as a sum of Fourier modes

$$\Phi(x^\mu, z) = \sum_n \phi_n(x^\mu) e^{inz/R}, \qquad (3.14)$$

where R is the radius of the compact S^1. This simply provides us with a rewriting of the original $(D+1)$-dimensional field in terms of an infinite tower of D-dimensional fields $\phi_n(x^\mu)$ with masses of order $|n|/R$. The Kaluza-Klein procedure then amounts to truncating the massive spectrum of this D-dimensional theory and keeping only the massless fields, in this case ϕ_0. For the S^1 example this always amounts to a consistent truncation of the theory. However, for dimensional reduction on a general compact manifold, one needs to take care that the interactions between zero modes do not give rise to non-zero modes [96].

Note that for the decomposition (3.14), truncating the modes with $n \neq 0$ is equivalent to requiring that the $(D+1)$-dimensional field be independent of the internal coordinate z. Indeed, for S^1 reductions such an ansatz always provides us with a consistent truncation to the massless spectrum in D dimensions, and so *we will take this as our 'definition' of Kaluza-Klein reduction for the remainder of this thesis.*

We next need to consider the representations under which fields transform in various dimensions. In $D+1$ dimensions, fields are classified by some choice of representation of the tangent space group $SO(D,1)$. Dimensional reduction then corresponds to expressing representations of $SO(D,1)$ as representations of $SO(D-d,1) \times SO(d)$, where we take the compact manifold to be d-dimensional[2]. For example, consider a gauge field A_M in $4+1$ dimensions, which transforms in the vectorial representation of $SO(4,1)$. From the four-dimensional point-of-view, this looks like a vector A_μ and singlet A_z of $SO(3,1)$.

3.5.2 Dimensional reduction

We can now move on to consider the dimensional reduction of a $(D+1)$-dimensional action describing a p-form gauge field coupled to gravity. The relevant action is

$$S = S_{\text{EH}} + S_{\text{gauge}} = \int d^{D+1}x\, \hat{\mathbf{e}} \left[\frac{\hat{R}}{2} - \frac{1}{2(p+1)!} \mathcal{F}_{\hat{\mu}_1 \ldots \hat{\mu}_{p+1}} \mathcal{F}^{\hat{\mu}_1 \ldots \hat{\mu}_{p+1}} \right], \tag{3.15}$$

where the 'hats' refer to $(D+1)$-dimensional objects, and $\hat{\mathbf{e}} = \det(\hat{e})$ is the determinant of the $(D+1)$-dimensional vielbein. We wish to dimensionally reduce the action (3.15) over a circle which we take to be either space-like or time-like. In order to treat both cases simultaneously, we take x^0 to be the compact direction and introduce the parameter ϵ_1 via

$$\epsilon_1 = \begin{cases} -1 & \text{if } x^0 \text{ space-like} \\ +1 & \text{if } x^0 \text{ time-like.} \end{cases}$$

[2] In the case where one of the internal directions is time-like this would be $SO(D-d+1) \times SO(d-1,1)$.

This appears in the $(D+1)$-dimensional tangent space metric as

$$\eta_{\hat{a}\hat{b}} = (-\epsilon_1, \eta_{ab}),$$

where $\eta_{ab} = \mathrm{diag}(\epsilon_1, \mathbb{1})$.

In order to perform the Kaluza-Klein reduction of (3.15) we need to make suitable ansätze for the various fields involved. From the discussion in Section 3.5.1, we should require that both the metric and the field strengths are independent of the compact coordinate x^0. In addition, we make the following ansatz for the $(D+1)$-dimensional vielbein:

$$\hat{e}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} e^{\beta\phi} & 0 \\ e^{\beta\phi} V_\mu & e^{-\alpha\phi} e_\mu{}^a \end{pmatrix}. \tag{3.16}$$

Here we have split the metric degrees of freedom into a scalar ϕ (generally referred to as the Kaluza-Klein scalar or dilaton), the Kaluza-Klein vector V_μ, and the D-dimensional vielbein. In terms of the $(D+1)$-dimensional line element we have

$$ds^2_{D+1} = -\epsilon_1 e^{2\beta\phi}(dx^0 + V_\mu dx^\mu)^2 + e^{-2\alpha\phi} ds^2_D\,. \tag{3.17}$$

The constants α and β, which for now need only satisfy $\beta \neq 0$, will be determined by the requirement that we pass from the $(D+1)$-dimensional Einstein frame[3] to the D-dimensional Einstein frame upon reduction.

Symmetries

Let us take a step back for a second and consider the symmetries in $(D+1)$ and D dimensions. This will help us understand how to make a sensible reduction ansatz. In particular, consider $(D+1)$-dimensional general coordinate transformations (GCTs)

$$\delta_\xi g_{\hat{\mu}\hat{\nu}} = \xi^{\hat{\rho}}\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}} + g_{\hat{\rho}\hat{\nu}}\partial_{\hat{\mu}}\xi^{\hat{\rho}} + g_{\hat{\rho}\hat{\mu}}\partial_{\hat{\nu}}\xi^{\hat{\rho}}.$$

[3]Recall that the Einstein frame is defined such that the Einstein-Hilbert term in the Lagrangian has a constant prefactor.

The most general form for the transformation parameters $\xi^{\hat{\mu}}$ which preserves the Kaluza-Klein form of the metric (3.17) is [96]

$$\xi^{\mu} = \xi^{\mu}(x), \quad \xi^{0} = cx^{0} + \lambda(x).$$

From the D-dimensional point-of-view, the parameter c gives rise to a constant shift of the dilaton accompanied by a scaling of the gauge field, whilst $\xi^{\mu}(x)$ and $\lambda(x)$ parametrise, respectively, D-dimensional GCTs and local gauge transformations. Indeed, focussing on the parameter $\lambda(x)$, we find $\delta_{\lambda}\phi = \delta_{\lambda}g_{\mu\nu} = 0$ and $\delta_{\lambda}V_{\mu} = \partial_{\mu}\lambda(x)$.

We turn our attention now to the $(D+1)$-dimensional n-form gauge potentials. In fact, since we will only meet forms with $n = 1$ in this thesis, we restrict to this case for the moment. Under $(D+1)$-dimensional GCTs, the gauge field $\mathcal{A}_{\hat{\mu}}$ transforms as

$$\delta_{\xi}\mathcal{A}_{\hat{\mu}} = \xi^{\hat{\rho}}\partial_{\hat{\rho}}\mathcal{A}_{\hat{\mu}} + \mathcal{A}_{\hat{\rho}}\partial_{\hat{\mu}}\xi^{\hat{\rho}}.$$

Decomposing the gauge field as $\mathcal{A}_{\hat{\mu}} = (\mathcal{A}_0, \mathcal{A}_{\mu})$ we find

$$\delta_{\xi}\mathcal{A}_0 = \xi^{\mu}\partial_{\mu}\mathcal{A}_0 + c\mathcal{A}_0, \quad \delta_{\xi}\mathcal{A}_{\mu} = \xi^{\rho}\partial_{\rho}\mathcal{A}_{\mu} + \mathcal{A}_{\rho}\partial_{\mu}\xi^{\rho} + \mathcal{A}_0\partial_{\mu}\lambda.$$

This tells us, as expected, that $\mathcal{A}_0$ and $\mathcal{A}_{\mu}$ transform as a scalar and vector respectively under D-dimensional GCTs. However, the extra parameter ξ^0 has introduced further transformations. In particular, the gauge field $\mathcal{A}_{\mu}$ is not invariant under ξ^0 transformations. It is therefore useful to define a new gauge field

$$A_{\mu} = \mathcal{A}_{\mu} - \mathcal{A}_0 V_{\mu},$$

which is invariant, $\delta_{\xi^0}A_{\mu} = 0$. Hence we can write the 1-form $\mathcal{A}$ in five dimensions as

$$\mathcal{A}_{\hat{\mu}}dx^{\hat{\mu}} = \mathcal{A}_0(dx^0 + V_{\mu}dx^{\mu}) + A_{\mu}dx^{\mu}.$$

We are now in a position to proceed with the dimensional reduction of (3.15). We here simply present the results. Details of the calculation can be found in [34].

Reduction of the Einstein-Hilbert term

We first concentrate on reduction of the Einstein-Hilbert piece, $S_{\rm EH}$, of (3.15). It turns out to be convenient to do this in two stages: first, perform the reduction using the vielbein ansatz (3.16) with $\alpha = 0$ and $\beta = 1$, and then perform a Weyl rescaling of the D-dimensional metric at the end to ensure that we reduce to the D-dimensional Einstein frame. We can then read off the appropriate values of α and β that would reduce straight to the Einstein frame. We find [34]

$$S_{\rm EH} = \int d^D x\, \mathbf{e} \left[e^{\phi} \frac{R}{2} + \frac{1}{8} \epsilon_1 e^{3\phi} H_{\mu\nu} H^{\mu\nu} \right],$$

where here $H = dV$ is the field strength of the Kaluza-Klein vector. In order to remove the non-canonical factor in front of the Ricci scalar we make a conformal rescaling, $g_{\mu\nu} = e^{2A\phi} \tilde{g}_{\mu\nu}$. The Ricci scalar in this case transforms as [89]

$$R = e^{-2A\phi} \left[\tilde{R} - 2(D-1)A\, \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi - (D-1)(D-2)A^2\, \tilde{g}^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi \right].$$

Choosing $A = -\frac{1}{D-2}$ and throwing away a total derivative piece gives (we have dropped the 'tilde' from the D-dimensional metric)

$$S_{\rm EH} = \int d^D x\, \mathbf{e} \left[\frac{R}{2} - \frac{D-1}{2(D-2)} \partial_\mu \phi\, \partial^\mu \phi + \frac{1}{8} \epsilon_1 e^{\frac{2D-2}{D-2}\phi} H_{\mu\nu} H^{\mu\nu} \right]. \qquad (3.18)$$

We note that the conformal rescaling is equivalent to choosing

$$\alpha = \frac{1}{D-2}, \quad \beta = 1, \qquad (3.19)$$

in the reduction ansatz (3.16).

Reduction of the gauge term

We now reduce the term $S_{\rm gauge}$ in (3.15) involving the p-form gauge potentials. Following the discussion above we define the p-form

$$A_{[p]} = \mathcal{A}_{[p]} - \mathcal{A}_{[p-1]} \wedge V,$$

where $(\mathcal{A}_{[p-1]})_{\mu_1\ldots\mu_{p-1}} = (\mathcal{A}_{[p]})_{0\mu_1\ldots\mu_{p-1}}$. We also introduce the field strengths $F_{[p+1]} = dA_{[p]}$ and $G_{[p]} = d\mathcal{A}_{[p-1]}$. Following [34] we find

$$\begin{aligned} S_{\text{gauge}} &= \int d^D x\, \mathbf{e} \left[-\frac{1}{2(p+1)!} e^{(2p+2-D)\alpha\phi+\beta\phi} F_{\mu_1\ldots\mu_{p+1}} F^{\mu_1\ldots\mu_{p+1}} \right. \\ &\qquad \left. + \frac{1}{2p!} \epsilon_1 e^{(2p-D)\alpha\phi-\beta\phi} G_{\mu_1\ldots\mu_p} G^{\mu_1\ldots\mu_p} \right], \end{aligned} \tag{3.20}$$

where the coefficients α and β should be fixed as in (3.19).

3.5.3 Reduction of five-dimensional supergravity

We can now put the results of Section 3.5.2 into practice and use them to dimensionally reduce the action (3.10) of five-dimensional vector multiplets coupled to supergravity, which would correspond to $D = 4$, $p = 1$.

Following [32] we denote by σ the Kaluza-Klein scalar in the five-dimensional vielbein ansatz, and $\mathcal{A}^0_\mu$ the Kaluza-Klein vector. Moreover, we should set $\alpha = \frac{1}{2}$ and $\beta = 1$ in (3.16) in order to reduce to the four-dimensional Einstein frame. Hence, the Kaluza-Klein ansatz for our five-dimensional line element becomes

$$ds^2_{(5)} = -\epsilon_1 e^{2\sigma} (dx^0 + \mathcal{A}^0)^2 + e^{-\sigma} ds^2_{(4)}. \tag{3.21}$$

The dimensional reduction of (3.10) is carried out in [32]. The result is that the four-dimensional action is given by

$$S_4 = \int d^4x\, \mathbf{e} \left[\frac{R}{2} - g_{ij}(z,\bar{z}) \partial_\mu z^i \partial^\mu \bar{z}^j + \frac{1}{4} \mathcal{I}_{IJ} F^I_{\mu\nu} F^{J|\mu\nu} + \frac{1}{4} \mathcal{R}_{IJ} F^I_{\mu\nu} \tilde{F}^{J|\mu\nu} \right], \tag{3.22}$$

where $i = 1, \ldots, n_V^{(5)} + 1$ and $I = (0, i)$. The explicit expressions for the fields and couplings appearing in (3.22) are given in [32]. Since we will make use of them at various points in the thesis, we reproduce them here. The scalar fields z^i are ϵ_1-complex and given by $z^i = x^i + i_{\epsilon_1} y^i$, where

$$x^i = 2 \cdot 6^{-\frac{1}{6}} \mathcal{A}^i_0, \quad y^i = 6^{\frac{1}{3}} e^\sigma h^i, \tag{3.23}$$

and i_{ϵ_1} is the ϵ_1-complex unit. The hypersurface condition $H(h) = 1$ satisfied by the scalar fields h^i parametrising the CASR manifold then gives the relation $H(y) = 6e^{3\sigma}$.

The four-dimensional gauge fields A^I_μ are given in terms of the five-dimensional gauge fields $\mathcal{A}^i_{\hat{\mu}}$ and the Kaluza-Klein vector $\mathcal{A}^0_\mu$ as

$$A^0_\mu = -\frac{1}{\sqrt{2}}\mathcal{A}^0_\mu, \quad A^i_\mu = \sqrt{2}\cdot 6^{-\frac{1}{6}}\left(\mathcal{A}^i_\mu - \mathcal{A}^i_0\,\mathcal{A}^0_\mu\right). \tag{3.24}$$

The remaining quantities in (3.22) are given by

$$\begin{aligned}
g_{ij} &= \frac{3}{2}\epsilon_1\left(\frac{(cy)_{ij}}{cyyy} - \frac{3}{2}\frac{(cyy)_i(cyy)_j}{(cyyy)^2}\right) := \epsilon_1\hat{g}_{ij}(y),\\
\mathcal{I}_{00} &= \epsilon_1(cyyy)\left(\frac{1}{6} + \frac{2}{3}gxx\right),\\
\mathcal{I}_{0i} &= -\frac{2}{3}\epsilon_1(cyyy)(gx)_i,\\
\mathcal{I}_{ij} &= \frac{2}{3}\epsilon_1(cyyy)g_{ij},\\
\mathcal{R}_{00} &= -\frac{1}{3}(cxxx),\\
\mathcal{R}_{0i} &= \frac{1}{2}(cxx)_i,\\
\mathcal{R}_{ij} &= -(cx)_{ij}.
\end{aligned} \tag{3.25}$$

Here we have used the shorthand $cyyy \equiv c_{ijk}\,y^iy^jy^k$, $gxx \equiv g_{ij}\,x^ix^j$, etc. It will be necessary in Chapter 4 to consider also the components of the inverse matrix $\mathcal{I}^{IJ}$. These are given by

$$\begin{aligned}
\mathcal{I}^{00} &= 6\epsilon_1(cyyy)^{-1},\\
\mathcal{I}^{0i} &= 6\epsilon_1(cyyy)^{-1}x^i,\\
\mathcal{I}^{ij} &= 6\epsilon_1(cyyy)^{-1}\left(x^ix^j + \frac{1}{4}g^{ij}\right).
\end{aligned} \tag{3.26}$$

The reduced action (3.22) is of the form appropriate to describe four-dimensional Minkowski ($\epsilon_1 = -1$) or Euclidean ($\epsilon_1 = 1$) supergravity coupled to $n_V^{(4)} = n_V^{(5)} + 1$ vector multiplets, as in (3.11). Indeed it was shown in [30, 32] that the action (3.22) corresponds to a theory with a 'very special' prepotential

$$F(X) = -\frac{1}{6}\epsilon_1 c_{ijk}\frac{X^i X^j X^k}{X^0},$$

which only depends on the data c_{ijk} of the five-dimensional theory. We will revisit this in Section 3.6.1. In the case of space-like reduction ($\epsilon_1 = -1$), such prepotentials can also be obtained from compactification of type II supergravity on a Calabi-Yau threefold with intersection numbers c_{ijk}.

3.5.4 Reduction of four-dimensional supergravity

We now turn to the reduction of the four-dimensional Minkowski ($\epsilon_1 = -1$) or Euclidean ($\epsilon_1 = 1$) action (3.11) describing vector multiplets coupled to supergravity. This corresponds to setting $D = 3$, $p = 1$ in Section 3.5.2, so that $\alpha = \beta = 1$ in the Kaluza-Klein ansatz (3.16). We denote by x^4 the compact direction, and introduce the parameter ϵ_2 which takes the value $\epsilon_2 = -1$ ($\epsilon_2 = 1$) for space-like (time-like) reduction. The full ansatz for the four-dimensional line element is then

$$ds^2_{(4)} = -\epsilon_2 e^{2\phi}(dx^4 + B)^2 + e^{-2\phi}ds^2_{(3)}. \tag{3.27}$$

We further introduce the quantity $\epsilon := -\epsilon_1\epsilon_2 = (-1)^t$, which keeps track of the number of time-like directions in the three-dimensional theory.

For the space-like reduction of the four-dimensional Minkowski theory (corresponding to $\epsilon_1 = \epsilon_2 = -1$ in our notation), the dimensional reduction was carried out in [24]. For time-like reduction of the Minkowski theory ($\epsilon_1 = -\epsilon_2 = -1$), the corresponding calculation was presented in [34]. The full calculation with arbitrary $\epsilon_{1,2}$ will appear in a future publication by the author [33]. We present this here.

We denote the field strength of the Kaluza-Klein vector as $H = dB$, and decompose the four-dimensional gauge fields as $A^I_\mu = (A^I_m + B_m\zeta^I, \zeta^I)$, where m is a three-dimensional space(time) index. Then the three-dimensional action is given by

$$\begin{aligned} S_3 &= \int d^3x\, \mathbf{e} \left[\frac{R}{2} - \partial_m \phi\, \partial^m \phi - g_{ij} \partial_m z^i \partial^m \bar{z}^j + \frac{1}{8} \epsilon_2 e^{4\phi} H_{mn} H^{mn} \right. \\ &\quad + \frac{1}{4} e^{2\phi} \mathcal{I}_{IJ} \left(F^I_{mn} + H_{mn} \zeta^I \right) \left(F^{J|mn} + H^{mn} \zeta^J \right) \\ &\quad \left. - \frac{1}{2} \epsilon_2 e^{-2\phi} \mathcal{I}_{IJ} \partial_m \zeta^I \partial^m \zeta^J - \frac{1}{2} \epsilon_2 \mathcal{R}_{IJ} \varepsilon^{mnp} \left(F^I_{mn} + H_{mn} \zeta^I \right) \partial_p \zeta^J \right]. \end{aligned} \tag{3.28}$$

Recall now that in three dimensions a vector field can be dualised to a scalar field. This is a specific example ($D = 3$, $p = 1$) of the fact that in D dimensions we can dualise a p-form gauge potential into a $(D - p - 2)$-form, using the Hodge-$*$ operator.

In the case at hand, we introduce the dual field strengths

$$G_m = -\frac{1}{2} \epsilon_2 \varepsilon_{mnp} G^{np}, \quad G_{mn} = \epsilon_1 \varepsilon_{mnp} G^p,$$

where G denotes any 2-form field strength. Note that we have used here the definition of ϵ given above, which appears via e.g. $\varepsilon_{mnp} \varepsilon^{mnq} = 2! \epsilon\, \delta^q_p$. Using this in (3.28) we obtain

$$\begin{aligned} S_3 &= \int d^3x\, \mathbf{e} \left[\frac{R}{2} - \partial_m \phi\, \partial^m \phi - g_{ij} \partial_m z^i \partial^m \bar{z}^j - \frac{1}{4} \epsilon_1 e^{4\phi} H_m H^m \right. \\ &\quad + \frac{1}{2} \epsilon\, e^{2\phi} \mathcal{I}_{IJ} \left(F^I_m + H_m \zeta^I \right) \left(F^{J|m} + H^m \zeta^J \right) \\ &\quad \left. - \frac{1}{2} \epsilon_2 e^{-2\phi} \mathcal{I}_{IJ} \partial_m \zeta^I \partial^m \zeta^J + \mathcal{R}_{IJ} \left(F^I_m + H_m \zeta^I \right) \partial^m \zeta^J \right]. \end{aligned} \tag{3.29}$$

The dualised field strengths F^I_m and H_m are not completely arbitrary, however, but must satisfy the Bianchi identities $\partial^m F^I_m = \partial^m H_m = 0$. We can encode this in the three-dimensional Lagrangian through the use of Lagrange multipliers $\tilde{\zeta}_I$ and $\tilde{\phi}$. In particular, we add the following term to the Lagrangian in (3.29):

$$\mathbf{e}^{-1} \mathcal{L}_{\rm LM} = - \left(F^I_m + H_m \zeta^I \right) \partial^m \tilde{\zeta}_I + \frac{1}{2} H^m \left(\partial_m \tilde{\phi} + \zeta^I \overleftrightarrow{\partial}_m \tilde{\zeta}_I \right). \tag{3.30}$$

The equations of motion for F^I_m and H_m coming from the combined action (3.29) and

(3.30) are

$$H_m = \epsilon_1 \, e^{-4\phi} \left(\partial_m \tilde{\phi} + \zeta^I \overleftrightarrow{\partial}_m \tilde{\zeta}_I \right) , \tag{3.31}$$

$$F_m^I + H_m \zeta^I = \epsilon \, e^{-2\phi} \, \mathcal{I}^{IJ} \left(\partial_m \tilde{\zeta}_J - \mathcal{R}_{JK} \partial_m \zeta^K \right) . \tag{3.32}$$

Finally, we can substitute these expressions back into the action to obtain the reduction of (3.11) in terms of the $4(n_V^{(4)} + 1)$ scalar fields $(z^i, \phi, \tilde{\phi}, \zeta^I, \tilde{\zeta}_I)$:

$$\begin{aligned} S_3 = \int d^3x \, \mathbf{e} \Bigg[& \frac{R}{2} - g_{ij} \partial_m z^i \partial^m \bar{z}^j - \partial_m \phi \, \partial^m \phi \\ & + \epsilon_1 e^{-4\phi} \left(\partial_m \tilde{\phi} + \zeta^I \overleftrightarrow{\partial}_m \tilde{\zeta}_I \right) \left(\partial^m \tilde{\phi} + \zeta^J \overleftrightarrow{\partial}^m \tilde{\zeta}_J \right) - \frac{1}{2} \epsilon_2 e^{-2\phi} \mathcal{I}_{IJ} \partial_m \zeta^I \partial^m \zeta^J \\ & - \frac{1}{2} \epsilon e^{-2\phi} \mathcal{I}^{IJ} \left(\partial_m \tilde{\zeta}_I - \mathcal{R}_{IK} \partial_m \zeta^K \right) \left(\partial^m \tilde{\zeta}_J - \mathcal{R}_{JL} \partial^m \zeta^L \right) \Bigg] . \end{aligned} \tag{3.33}$$

In Section 3.6.2 we will describe the geometry of the scalar target space for this theory, and argue that it is ϵ-quaternionic-Kähler. It is therefore admissible as the target manifold for a theory of $n_H = n_V^{(4)} + 1$ hypermultiplets coupled to three-dimensional (Minkowski or Euclidean) gravity, albeit with couplings dependent on the prepotential $F(X)$ of the four-dimensional theory. Moreover, we will see that this four-dimensional origin endows the target manifold with additional structure, making it ϵ_1-complex.

3.6 r-maps and c-maps

In the final section of this chapter we'll review the current state of the art with regards the use of dimensional reduction to provide maps between target spaces.

In Section 3.5 we carried out the reduction of five-dimensional (resp. four-dimensional) $\mathcal{N} = 2$ supergravity coupled to vector multiplets over a space-like or time-like direction, and presented the resulting four-dimensional (resp. three-dimensional) action.

In this section we will concentrate on the scalar sectors of these theories, which are described by certain non-linear sigma models. Dimensional reduction induces maps between the corresponding target spaces, which we term the r- and c-maps.

We begin in Section 3.6.1 with the five-to-four reduction. We prove that the target space of the reduced theory is given by a certain projective special ϵ_1-Kähler manifold with prepotential dependent on the data of the five-dimensional theory. In Section 3.6.2 we then look at the four-to-three reduction. The target space in this case is given by an ϵ-quaternionic-Kähler manifold equipped with an integrable ϵ_1-complex structure.

3.6.1 The supergravity r-maps

We saw in Section 3.5.3 that dimensional reduction of five-dimensional $\mathcal{N} = 2$ supergravity coupled to $n_V^{(5)}$ vector multiplets over a space-like (resp. time-like) direction gives rise to a four-dimensional Minkowski (resp. Euclidean) theory describing supergravity coupled to $(n_V^{(5)} + 1)$ vector multiplets. Moreover, the prepotential $F(X)$ encoding the couplings of the four-dimensional theory is given in terms of the data c_{ijk} of the five-dimensional theory.

Concentrating solely on the target space geometry, which is important for characterising supergravity solutions, we see that dimensional reduction induces a pair of maps

$$\bar{r}^{\epsilon_1} : \mathcal{H} \to \bar{N},$$

which associate to a given PSR manifold $\mathcal{H}$ of dimension $n_V^{(5)}$ a $2(n_V^{(5)}+1)$-dimensional manifold $\bar{N}$ equipped with a (pseudo-)Riemannian metric $g_{\bar{N}}$:

$$g_{\bar{N}} = -\hat{g}_{ij}(y) \left(dy^i dy^j - \epsilon_1 dx^i dx^j\right). \tag{3.34}$$

For $\epsilon_1 = -1$, which corresponds to space-like reduction, the r-map was studied in [30]. The case $\epsilon_1 = 1$, corresponding to time-like reduction, was studied extensively in [32], where it was termed the time-like r-map.

As a simple application of some of the material we introduced in Chapter 2, we prove, combining the results of [30, 32] the following:

Proposition 3. *The (pseudo)-Riemannian manifold $(\bar{N}, g_{\bar{N}})$ admits an integrable ϵ_1-complex structure J, with respect to which $(\bar{N}, g_{\bar{N}}, J)$ is an ϵ_1-Kähler manifold.*

Proof: We introduce the frame $F = (\partial_{y^i}, \partial_{x^i})$ for $T\bar{N}$ and co-frame $F^* = (dy^i, dx^i)$ for

$T^*\bar{N}$. With respect to F the metric $g_{\bar{N}}$ is given by

$$g_{\bar{N}} = \begin{pmatrix} -\hat{g} & 0 \\ 0 & \epsilon_1\hat{g} \end{pmatrix}.$$

Introduce an endomorphism $J \in \mathrm{End}(T\bar{N})$ via

$$J = \epsilon_1 \partial_{x^i} \otimes dy^j + \partial_{y^i} \otimes dx^j,$$

which has the matrix representation

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ \epsilon_1\mathbb{1} & 0 \end{pmatrix},$$

with respect to the frame F. Clearly J defines an almost ϵ_1-complex structure on $\bar{N}$, and one can easily show that $g_{\bar{N}}(JX, JY) = -\epsilon_1 g_{\bar{N}}(X, Y)$, so that $(\bar{N}, g_{\bar{N}}, J)$ is almost ϵ_1-hermitian.

To show integrability of J, it is easiest to note that for $\epsilon_1 = -1$ (resp. $\epsilon_1 = 1$) the basis of $T^*\bar{N}^{(1,0)}$ (resp. $T^*\bar{N}^+$) is made up of exact forms. Hence, J is integrable in the sense of Theorem 4.

Finally, the fundamental 2-form is given by

$$\omega = -\frac{1}{2}\hat{g}_{ij}(y)dx^i \wedge dy^j,$$

which has exterior derivative

$$d\omega = -\frac{1}{2}(\partial_k\hat{g}_{ij})dx^i \wedge dy^j \wedge dy^k = 0,$$

where we have used the fact that $\hat{g}_{ij}(y)$ is Hessian. This completes the proof. □

Hence we see that the manifolds in the image of the local r-map are ϵ_1-Kähler. Indeed, it is further shown in [30, 32] that such manifolds are projective special ϵ_1-Kähler, as required for the target manifolds of four-dimensional supergravity coupled to vector multiplets, with ϵ_1-holomorphic prepotential

$$F(X) = -\frac{1}{6}\epsilon_1 c_{ijk}\frac{X^i X^j X^k}{X^0}. \tag{3.35}$$

As an example of how such calculations progress, we unify the results of [30] and [32]. We begin by introducing the ϵ_1-holomorphic coordinates

$$z^i = \frac{X^i}{X^0} = x^i + i_{\epsilon_1} y^i.$$

Then, following Section 2.2.5, we find the inhomogeneous prepotential $\mathcal{F}(z) = -\frac{1}{6}\epsilon_1(czzz)$. The Kähler potential $\mathcal{K}(z,\bar{z})$ is given by

$$\mathcal{K}(z,\bar{z}) = -\log K(z,\bar{z}),$$

for

$$K(z,\bar{z}) = \frac{1}{6}\epsilon_1 i_{\epsilon_1} c(z-\bar{z})(z-\bar{z})(z-\bar{z}) = \frac{4}{3}\epsilon_1(cyyy).$$

Hence, the components of the metric on the manifold $\bar{N}$ parametrized by z^i are

$$\frac{\partial^2\mathcal{K}}{\partial z^i \partial \bar{z}^j} = \frac{3}{2}\epsilon_1\left(\frac{(cy)_{ij}}{cyyy} - \frac{3}{2}\frac{(cyy)_i(cyy)_j}{(cyyy)^2}\right) = \epsilon_1 \hat{g}_{ij}(y),$$

and we find

$$g_M = \epsilon_1 \hat{g}_{ij}(y) dz^i d\bar{z}^j = -\hat{g}_{ij}(y)\left(dy^i dy^j - \epsilon_1 dx^i dx^j\right).$$

This proves that the metric $g_{\bar{N}}$ is projective special ϵ_1-Kähler with prepotential (3.35).

3.6.2 The supergravity c-maps

We saw in Section 3.5.4 that dimensional reduction of Euclidean or Minkowski four-dimensional supergravity coupled to vector multiplets gave rise to a three-dimensional theory of hypermultiplets coupled to gravity, with all the data of the three-dimensional theory determined by the ϵ_1-holomorphic prepotential $F(X)$ of the four-dimensional theory.

Concentrating on the target spaces of the theories, dimensional reduction induces a family maps

$$\bar{c}^{(\epsilon_1,\epsilon_2)} : \bar{N} \to \bar{Q},$$

which associate to each projective special ϵ_1-Kähler manifold of dimension $2n_V^{(4)}$ a $4(n_V^{(4)}+1)$-dimensional manifold $\bar{Q}$ equipped with a (pseudo-)Riemannian metric $g_{\bar{Q}}$:

$$\begin{aligned} g_{\bar{Q}} &= g_{i\bar{j}}(z,\bar{z})dz^i d\bar{z}^j + (d\phi)^2 - \epsilon_1 e^{-4\phi}\left(d\tilde{\phi} + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)^2 \\ &+ \frac{1}{2}\epsilon_2 e^{-2\phi}\mathcal{I}_{IJ} d\zeta^I d\zeta^J + \frac{1}{2}\epsilon e^{-2\phi}\mathcal{I}^{IJ}\left(d\tilde{\zeta}_I - \mathcal{R}_{IK}d\zeta^K\right)\left(d\tilde{\zeta}_J - \mathcal{R}_{JL}d\zeta^L\right). \end{aligned} \tag{3.36}$$

We will go through each of these maps in turn. The nomenclature is that of [33].

The spatial c-map

The spatial c-map corresponds to the case $\epsilon_1 = \epsilon_2 = -1$, and is obtained from the space-like reduction of the four-dimensional Minkowski theory.

It was shown in [24] that the Riemannian manifold $(\bar{Q}, g_{\bar{Q}}^{(-1,-1)})$ admits a quaternionic structure Q for which $(\bar{Q}, Q, g_{\bar{Q}}^{(-1,-1)})$ is a quaternionic-Kähler manifold. The idea is to show that the Levi-Civita 1-form admits a decomposition as in (2.26). In this case the Ambrose-Singer theorem ensures that $\mathrm{Hol}(D) \subset Sp(n)\cdot Sp(1)$ for $n = n_V^{(4)}+1$, and hence that $(\bar{Q}, Q, g_{\bar{Q}}^{(-1,-1)})$ is quaternionic-Kähler.

The c-map can be understood physically from the point-of-view of T-duality between type IIA and IIB strings on $M_4 \times X_6$, where X_6 is a Calabi-Yau threefold [23].

The temporal c-map

The temporal c-map corresponds to the case $\epsilon_1 = -\epsilon_2 = -1$, and is obtained from a time-like reduction of the four-dimensional Minkowski theory.

It was shown in [34] that the pseudo-Riemannian manifold $(\bar{Q}, g_{\bar{Q}}^{(-1,1)})$ admits a para-quaternionic structure Q for which $(\bar{Q}, Q, g_{\bar{Q}}^{(-1,1)})$ is a para-quaternionic-Kähler manifold. Again, the idea is to show that the Levi-Civita 1-form admits a decomposition as in (2.27). In this case the Ambrose-Singer theorem ensures that $\mathrm{Hol}(D) \subset Sp(2n,\mathbb{R})\cdot Sp(2,\mathbb{R})$ for $n = n_V^{(4)}+1$, and hence that $(\bar{Q}, Q, g_{\bar{Q}}^{(-1,-1)})$ is para-quaternionic-Kähler.

The Euclidean c-map

The remaining case of interest is $\epsilon_1 = -\epsilon_2 = 1$, which corresponds to space-like reduction of a four-dimensional Euclidean theory. In this case, one can again show [33] that the resulting pseudo-Riemannian scalar manifold is para-quaternionic-Kähler.

In summary then, we have:

Proposition 4. *The (pseudo-)Riemannian manifold $(\bar{Q}, g_{\bar{Q}}^{(\epsilon_1,\epsilon_2)})$ admits an ϵ-quaternionic structure Q, with respect to which $(\bar{Q}, Q, g_{\bar{Q}}^{(\epsilon_1,\epsilon_2)})$ is an ϵ-quaternionic-Kähler manifold, where $\epsilon = -\epsilon_1\epsilon_2$.*

The proofs of these assertions are best presented after a rewriting of the three-dimensional theory (3.33) using the so-called 'real formulation' of special geometry developed in [97]. Since this will require a host of additional background material we do not attempt to prove these results in this thesis, and instead refer the reader to the literature already mentioned, in particular [24, 33].

In addition to being ϵ-quaternionic-Kähler, the manifolds $(\bar{Q}, Q, g_{\bar{Q}}^{(\epsilon_1,\epsilon_2)})$ in the image of the local c-maps admit some additional structure. In particular:

Proposition 5. *The ϵ-quaternionic-Kähler manifolds $(\bar{Q}, Q, g_{\bar{Q}}^{(\epsilon_1,\epsilon_2)})$ in the image of the c-maps admit an integrable ϵ_1-complex structure J compatible with the ϵ-quaternionic structure. This makes $(\bar{Q}, g_{\bar{Q}}^{(\epsilon_1,\epsilon_2)}, J)$ into an ϵ_1-complex manifold.*

Proof: For the case $\epsilon_1 = \epsilon_2 = -1$, the proof can be found in Propositions 1 and 2 of [98]. The remaining cases will be presented in [33].

We will prove Propositions 4 and 5 for the case of pure five-dimensional supergravity in Chapter 5.

Chapter 4

The supergravity q-maps

In this chapter we will combine the r-maps and c-maps by performing the dimensional reduction to three dimensions of five-dimensional $\mathcal{N} = 2$ supergravity coupled to vector multiplets. In the notation of Section 3.6, the resulting map

$$\bar{q}^{(\epsilon_1,\epsilon_2)} = \bar{c}^{(\epsilon_1,\epsilon_2)} \circ \bar{r}^{\epsilon_1},$$

would take an $n_V^{(5)}$-dimensional PSR manifold to a $4(n_V^{(5)} + 2)$-dimensional ϵ-quaternionic-Kähler manifold. This is represented in Figure 4.1.

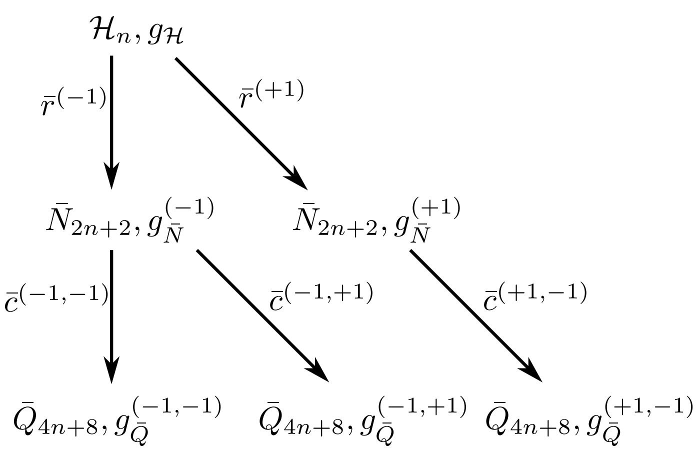

Figure 4.1: The supergravity q-maps.

We begin this chapter in Section 4.1 with a derivation of the three-dimensional Lagrangian obtained by dimensional reduction of the five-dimensional theory of $\mathcal{N} = 2$

supergravity coupled to vector multiplets, and note that the metric on the corresponding target manifold locally separates into a product metric. In Section 4.2 we investigate the structure of the group manifold L fibered over the PSR manifold defining the original theory, and compare this to the solvable subgroup of isometries for a generic q-map space. We then turn to the question of whether the time-then-space and space-then-time reductions commute in Section 4.3, and identify a map between the corresponding target spaces which is then used to find the 'hidden' symmetry generator present for all q-map spaces. Finally, in Section 4.4, we calculate the connection and curvature tensors on the group manifold L.

The material from this chapter will appear in a future publication [38] by the author.

4.1 The q-maps from dimensional reduction

We start with the bosonic part of the action for five-dimensional $\mathcal{N} = 2$ supergravity coupled to n vector multiplets[1], which we saw in Section 3.2. The action, which we reproduce here for convenience, is given by

$$\begin{aligned} S_5 \quad = \quad & \int d^5x \left[\sqrt{\hat{g}} \left(\frac{\hat{R}}{2} - \frac{3}{4} g_{xy}(\phi) \partial_{\hat{\mu}} \phi^x \partial^{\hat{\mu}} \phi^y - \frac{1}{4} a_{ij}(h) \mathcal{F}^i_{\hat{\mu}\hat{\nu}} \mathcal{F}^{j|\hat{\mu}\hat{\nu}} \right) \right. \\ & \left. + \frac{1}{6\sqrt{6}} c_{ijk} \varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}} \mathcal{F}^i_{\hat{\mu}\hat{\nu}} \mathcal{F}^j_{\hat{\rho}\hat{\sigma}} \mathcal{A}^k_{\hat{\lambda}} \right] . \end{aligned} \tag{4.1}$$

We look for field configurations $(g_{\hat{\mu}\hat{\nu}}, \phi^x, \mathcal{A}^i_{\hat{\mu}})$ admitting two commuting isometries, which we take to be along the x^0 and x^4 directions. We consider in parallel the cases where x^0, x^4 are both space-like, and those where one of the directions is time-like. This latter case will be important in Chapters 6 and 7 when we look for stationary solutions of the five-dimensional equations of motion.

Combining the ansätze for the five-to-four and four-to-three reductions (3.21) and (3.27), we make the metric ansatz $M_5 = S^1 \times S^1 \times M_3$ with

$$ds^2_{(5)} = -\epsilon_1 e^{2\sigma} \left(dx^0 + \mathcal{A}^0 \right)^2 - \epsilon_2 e^{2\phi-\sigma} \left(dx^4 + B \right)^2 + e^{-2\phi-\sigma} ds^2_{(3)}, \tag{4.2}$$

[1] Throughout this chapter we use, for convenience, $n = n_V^{(5)}$ to denote the number of five-dimensional vector multiplets.

where $\epsilon_{1,2}$ take the values -1 for reduction over a space-like direction and $+1$ for a time-like reduction. We also introduce the variable $\epsilon := -\epsilon_1\epsilon_2 = (-1)^t$, where t is the number of time-like directions in the three-dimensional theory. The Kaluza-Klein vectors have components $\mathcal{A}^0 = \mathcal{A}^0_4 dx^4 + \mathcal{A}^0_\mu dx^\mu$, and $B = B_\mu dx^\mu$. Note that here we are using $\mu = 1, 2, 3$ as a three-dimensional space(time) index.

Using the ansatz (4.2) we can perform a two-step dimensional reduction of the five-dimensional action (4.1), following the procedure outlined in Section 3.5. In particular, we first reduce along the x^0 direction, which is space-like (resp. time-like) for $\epsilon_1 = -1$ (resp. $\epsilon_1 = 1$), and then further over the x^4 direction, which depends on the sign of ϵ_2.

Indeed, all of the ingredients for this reduction were already given in Section 3.5. We simply need to plug the expressions (3.25)–(3.26) describing the four-dimensional theory reduced from five dimensions into the expression (3.33) for the three-dimensional Lagrangian obtained by reduction of the four-dimensional theory. Performing the reduction in this manner, and suitably rearranging, we find that the three-dimensional theory takes the form of a gravity-coupled non-linear sigma model with Lagrangian

$$\begin{aligned}
\mathcal{L}_3 \;=\;& \frac{R}{2} - \frac{3}{4} g_{xy}(\phi)\partial\phi^x\partial\phi^y + \frac{3}{4\sigma^2}\epsilon_1 a_{ij}(h)\partial x^i\partial x^j - \frac{3}{4\sigma^2}(\partial\sigma)^2 - \frac{1}{4\phi^2}(\partial\phi)^2 \\
&+ \frac{1}{4\phi^2}\epsilon_1\left(\partial\tilde{\phi} + \zeta^I \overleftrightarrow{\partial}\tilde{\zeta}_I\right)^2 + \frac{\sigma^3}{12\phi}\epsilon(\partial\zeta^0)^2 \\
&+ \frac{\sigma}{4\phi}\epsilon_2 a_{ij}(h)\left(\partial\zeta^i - x^i\partial\zeta^0\right)\left(\partial\zeta^j - x^j\partial\zeta^0\right) \\
&+ \frac{3}{\sigma^3\phi}\epsilon_2\left(\partial\tilde{\zeta}_0 + x^i\partial\tilde{\zeta}_i + \frac{1}{2}(cxx)_i\partial\zeta^i - \frac{1}{6}(cxxx)\partial\zeta^0\right)^2 \\
&+ \frac{1}{\sigma\phi}\epsilon a^{ij}(h)\left(\partial\tilde{\zeta}_i + (cx)_{ik}\partial\zeta^k - \frac{1}{2}(cxx)_i\partial\zeta^0\right) \\
&\times\left(\partial\tilde{\zeta}_j + (cx)_{jl}\partial\zeta^l - \frac{1}{2}(cxx)_j\partial\zeta^0\right).
\end{aligned} \tag{4.3}$$

The relations between the five-dimensional and three-dimensional field contents can be determined from the formulae in Chapter 3. In particular, we saw already that

$$\mathcal{A}^i_0 = \frac{6^{1/6}}{2}x^i, \tag{4.4}$$

are the components of the five-dimensional gauge fields along the x^0 direction. The scalars (σ, ϕ) appearing in (4.3) are related to the Kaluza-Klein scalars appearing in the metric ansatz (4.2) by

$$e^{2\phi} \mapsto \phi, \quad 6^{1/3} e^{\sigma} \mapsto \sigma. \tag{4.5}$$

The Kaluza-Klein vectors can be determined from the relations

$$H_\mu = \frac{1}{\phi^2}\left(\partial_\mu \tilde{\phi} + \zeta^I \partial_\mu \tilde{\zeta}_I - \tilde{\zeta}_I \partial_\mu \zeta^I\right), \tag{4.6}$$

$$\mathcal{F}^0_\mu + \zeta^0 H_\mu = -\frac{6\epsilon_2}{\sigma^3 \phi}\left(\partial_\mu \tilde{\zeta}_0 + x^i \partial_\mu \tilde{\zeta}_i + \frac{1}{2}(cxx)_i \partial_\mu \zeta^i - \frac{1}{6}(cxxx)\partial_\mu \zeta^0\right), \tag{4.7}$$

and

$$\mathcal{A}^0_4 = -\sqrt{2}\zeta^0. \tag{4.8}$$

Finally, the remaining components of the five-dimensional gauge fields can be determined from

$$\begin{aligned} \mathcal{F}^i_\mu + \zeta^i H_\mu \;=\;& \frac{2\epsilon}{\sigma\phi} a^{ij}(h)\left(\partial_\mu \tilde{\zeta}_j + (cx)_{jk}\partial_\mu \zeta^k - \frac{1}{2}(cxx)_j \partial_\mu \zeta^0\right) \\ & -\frac{6\epsilon_2}{\sigma^3\phi} x^i \left(\partial_\mu \tilde{\zeta}_0 + x^i \partial_\mu \tilde{\zeta}_i + \frac{1}{2}(cxx)_i \partial_\mu \zeta^i - \frac{1}{6}(cxxx)\partial_\mu \zeta^0\right), \end{aligned} \tag{4.9}$$

and

$$\mathcal{A}^i_4 = \frac{6^{1/6}}{\sqrt{2}}(\zeta^i - x^i \zeta^0). \tag{4.10}$$

The scalar manifolds obtained by SS ($\epsilon_1 = \epsilon_2 = -1$), ST ($\epsilon_1 = -\epsilon_2 = -1$) and TS ($\epsilon_1 = -\epsilon_2 = 1$) reduction are denoted, respectively, $\bar{Q}^{(SS)}$, $\bar{Q}^{(ST)}$ and $\bar{Q}^{(TS)}$ and are parametrized by the $4(n_V^{(5)} + 2)$ scalar fields $(\phi^x, x^i, \sigma, \phi, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i)$. Each of these manifolds is equipped with a metric $g_{\bar{Q}}^{(\epsilon_1,\epsilon_2)}$ given by

$$\begin{aligned} g_{\bar{Q}}^{(\epsilon_1,\epsilon_2)} \;=\;& \frac{3}{4} g_{xy}(\phi) d\phi^x d\phi^y - \frac{3}{4\sigma^2}\epsilon_1 a_{ij}(h) dx^i dx^j + \frac{1}{4\phi^2} d\phi^2 + \frac{3}{4\sigma^2} d\sigma^2 \\ & -\frac{1}{4\phi^2}\epsilon_1 \left(d\tilde{\phi} + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)^2 - \frac{\sigma^3}{12\phi}\epsilon (d\zeta^0)^2 \\ & -\frac{\sigma}{4\phi}\epsilon_2 a_{ij}(h)\left(d\zeta^i - x^i d\zeta^0\right)\left(d\zeta^j - x^j d\zeta^0\right) \\ & -\frac{3}{\sigma^3\phi}\epsilon_2 \left(d\tilde{\zeta}_0 + x^i d\tilde{\zeta}_i + \frac{1}{2}(cxx)_i d\zeta^i - \frac{1}{6}(cxxx) d\zeta^0\right)^2 \end{aligned}$$

$$-\frac{1}{\sigma\phi}\epsilon a^{ij}(h)\left(d\tilde{\zeta}_i + (cx)_{ik}d\zeta^k - \frac{1}{2}(cxx)_i d\zeta^0\right)$$
$$\times\left(d\tilde{\zeta}_j + (cx)_{jl}d\zeta^l - \frac{1}{2}(cxx)_j d\zeta^0\right). \quad (4.11)$$

The metric $g_{\bar{Q}}$ is positive definite for SS reduction, while it has split signature for both ST and TS reductions, albeit with a different distribution of signs in each case.

Since it lies in the image of one of the c-maps, Proposition 4 guarantees that the pseudo-Riemannian manifolds[2] $(\bar{Q}^{(\epsilon_1,\epsilon_2)}, g_{\bar{Q}}^{(\epsilon_1,\epsilon_2)})$ are ϵ-quaternionic-Kähler. Moreover, Proposition 5 tells us that in each case $\bar{Q}^{(\epsilon_1,\epsilon_2)}$ admits an integrable ϵ_1-complex structure.

From the structure of (4.11), we see that the metric on the (para-)quaternionic-Kähler manifold in the image of the q-map can be written locally as a bundle metric [25]

$$g_{\bar{Q}}(p,a) = g_{\mathcal{H}}(p) + g_L(p,a), \quad (p,a) \in \mathcal{H} \times L,$$

where

$$g_{\mathcal{H}} = \frac{3}{4}g_{xy}(\phi)d\phi^x d\phi^y,$$

is the metric on the PSR manifold $\mathcal{H}$ with coordinates (ϕ^x) and g_L is a family of metrics on the manifold L which is fibred over $\mathcal{H}$. We will investigate the structure of the manifold L in the next section, where we will see that it can be identified with the orbit of a certain solvable Lie group G appearing as the generic isometry group for spaces in the image of the q-map.

4.2 The group manifold

We now turn our attention to a description of the manifolds $L^{(\epsilon_1,\epsilon_2)} \cong \mathbb{R}^{3n+6} \times \mathbb{R}^{>0} \times \mathbb{R}^{>0}$, which have real dimension $3n + 8$ and are parametrized by the coordinates $(\sigma, \phi, x^i, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i)$.

[2] We write $\bar{Q}^{(SS)} = \bar{Q}^{(-1,-1)}$, etc.

We define the following co-frame (θ^A) on $T^*L^{(\epsilon_1,\epsilon_2)}$:

$$\begin{aligned}
\eta^{n+2} &= \frac{1}{\phi}\left(d\tilde{\phi} + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right), \quad \xi_{n+2} = \frac{d\phi}{\phi}, \\
\alpha^i &= \frac{\sqrt{3}}{\sigma} dx^i, \quad \beta = \frac{\sqrt{3}}{\sigma} d\sigma, \\
\eta^0 &= \sqrt{\frac{\sigma^3}{3\phi}} d\zeta^0, \quad \eta^i = \sqrt{\frac{\sigma}{\phi}}\left(d\zeta^i - x^i d\zeta^0\right), \\
\xi_0 &= 2\sqrt{\frac{3}{\sigma^3\phi}}\left(d\tilde{\zeta}_0 + x^i d\tilde{\zeta}_i + \frac{1}{2}(cxx)_i d\zeta^i - \frac{1}{6}(cxxx)d\zeta^0\right), \\
\xi_i &= \frac{2}{\sqrt{\sigma\phi}}\left(d\tilde{\zeta}_i + (cx)_{ij} d\zeta^j - \frac{1}{2}(cxx)_i d\zeta^0\right).
\end{aligned} \tag{4.12}$$

With respect to this basis, the metric g_L can be written as

$$\begin{aligned}
4g_L &= -\epsilon_1 \eta^{n+2} \otimes \eta^{n+2} + \xi_{n+2} \otimes \xi_{n+2} - \epsilon_1 \delta_{ij} \alpha^i \otimes \alpha^j + \beta \otimes \beta \\
&\quad -\epsilon \eta^0 \otimes \eta^0 - \epsilon_2 \delta_{ij} \eta^i \otimes \eta^j - \epsilon_2 \xi_0 \otimes \xi_0 - \epsilon \delta^{ij} \xi_i \otimes \xi_j,
\end{aligned} \tag{4.13}$$

where we have made use of the freedom to choose a basis of h^i such that $a_{ij}(p) = \delta_{ij}$ at any point $p \in \mathcal{H}$ on the PSR base space. Hence the 1-forms (4.12) form an orthonormal basis of $T^*L^{(\epsilon_1,\epsilon_2)}$ with respect to the scalar product (4.13).

The exterior derivatives of (4.12) are given by

$$\begin{aligned}
& d\eta^{n+2} = -\xi_0 \wedge \eta^0 - \xi_i \wedge \eta^i - \xi_{n+2} \wedge \eta^{n+2}, \\
& d\xi_{n+2} = 0, \\
& d\alpha^i = \frac{1}{\sqrt{3}} \alpha^i \wedge \beta, \\
& d\beta = 0, \\
& d\eta^0 = \frac{\sqrt{3}}{2} \beta \wedge \eta^0 - \frac{1}{2} \xi_{n+2} \wedge \eta^0, \\
& d\eta^i = \frac{1}{2\sqrt{3}} \beta \wedge \eta^i - \frac{1}{2} \xi_{n+2} \wedge \eta^i - \alpha^i \wedge \eta^0, \\
& d\xi_0 = -\frac{\sqrt{3}}{2} \beta \wedge \xi_0 - \frac{1}{2} \xi_{n+2} \wedge \xi_0 + \alpha^i \wedge \xi_i, \\
& d\xi_i = -\frac{1}{2\sqrt{3}} \beta \wedge \xi_i - \frac{1}{2} \xi_{n+2} \wedge \xi_i + \frac{2}{\sqrt{3}} c_{ijk} \alpha^j \wedge \eta^k,
\end{aligned} \tag{4.14}$$

which shows that (4.12) generate a (dual) Lie algebra $\mathfrak{g}^*$. We can then interpret g_L

as a left-invariant pseudo-Riemannian metric on the corresponding Lie group. The structure constants of $\mathfrak{g}$ are determined from the relation

$$d\theta^A = -c^A_{BC}\,\theta^B \wedge \theta^C.$$

The corresponding Lie brackets amongst the left-invariant vector fields

$$(T_A) = (V_{n+2}, U^{n+2}, A_i, B, V_0, V_i, U^0, U^i), \tag{4.15}$$

dual to

$$(\theta^A) = (\eta^{n+2}, \xi_{n+2}, \alpha^i, \beta, \eta^0, \eta^i, \xi_0, \xi_i),$$

are then given by

$$\begin{aligned}
[B, A_i] &= \frac{1}{\sqrt{3}}A_i, \quad \left[U^{n+2}, V_{n+2}\right] = V_{n+2},\\
\left[V_0, U^0\right] &= -V_{n+2}, \quad \left[V_i, U^j\right] = -\delta_i^j V_{n+2},\\
\left[U^{n+2}, V_I\right] &= \frac{1}{2}V_I, \quad \left[U^{n+2}, U^I\right] = \frac{1}{2}U^I,\\
[B, V_0] &= -\frac{\sqrt{3}}{2}V_0, \quad [B, V_i] = -\frac{1}{2\sqrt{3}}V_i,\\
\left[B, U^0\right] &= \frac{\sqrt{3}}{2}U^0, \quad \left[B, U^i\right] = \frac{1}{2\sqrt{3}}U^i,\\
[A_i, V_0] &= V_i, \quad \left[A_i, U^j\right] = -\delta_i^j U_0, \quad [A_i, V_j] = -\frac{2}{\sqrt{3}}c_{ijk}U^k.
\end{aligned} \tag{4.16}$$

One can use these relations to show that the derived series (2.29) terminates, and hence that $\mathfrak{g}$ is a solvable Lie algebra. We will see in Section 4.2.3 that the corresponding Lie group can be identified with a solvable subgroup of the generic isometry group of the (para-)quaternionic-Kähler manifolds in the image of the q-map.

The explicit expressions for the vector fields (T_A) in a coordinate basis on $L^{(\epsilon_1,\epsilon_2)}$ are:

$$V_{n+2} = \phi\,\partial_{\tilde{\phi}}, \quad U^{n+2} = \phi\,\partial_\phi, \quad A_i = \frac{1}{\sqrt{3}}\sigma\partial_{x^i}, \quad B = \frac{1}{\sqrt{3}}\sigma\partial_\sigma,$$

$$V_0 = \sqrt{\frac{3\phi}{\sigma^3}}\left[(\partial_{\zeta^0} + \tilde{\zeta}_0\partial_{\tilde{\phi}}) + x^i(\partial_{\zeta^i} + \tilde{\zeta}_i\partial_{\tilde{\phi}}) - \frac{1}{2}(cxx)_i(\partial_{\tilde{\zeta}_i} - \zeta^i\partial_{\tilde{\phi}}) + \frac{1}{6}(cxxx)(\partial_{\tilde{\zeta}_0} - \zeta^0\partial_{\tilde{\phi}})\right],$$

$$V_i = \sqrt{\frac{\phi}{\sigma}}\left[(\partial_{\zeta^i} + \tilde{\zeta}_i\partial_{\tilde{\phi}}) - (cx)_{ij}(\partial_{\tilde{\zeta}_j} - \zeta^j\partial_{\tilde{\phi}}) + \frac{1}{2}(cxx)_i(\partial_{\tilde{\zeta}_0} - \zeta^0\partial_{\tilde{\phi}})\right],$$
$$U^0 = \frac{\sqrt{\sigma^3\phi}}{2\sqrt{3}}(\partial_{\tilde{\zeta}_0} - \zeta^0\partial_{\tilde{\phi}}),$$
$$U^i = \frac{\sqrt{\sigma\phi}}{2}\left[(\partial_{\tilde{\zeta}_i} - \zeta^i\partial_{\tilde{\phi}}) - x^i(\partial_{\tilde{\zeta}_0} - \zeta^0\partial_{\tilde{\phi}})\right]. \tag{4.17}$$

So far then we have the following picture. The three dimensional reductions provide us with scalar manifolds locally having the form $\bar{Q}^{(\epsilon_1,\epsilon_2)} = \mathcal{H} \times L^{(\epsilon_1,\epsilon_2)}$, where in each case $L^{(\epsilon_1,\epsilon_2)}$ can be identified with the group manifold L of a solvable subgroup of the generic isometry group of spaces in the image of the q-map. For each of the three reductions this manifold is equipped with a different left-invariant metric. In the ordering of (4.15) the signature is

$$\mathrm{sign}(g_L) = (-\epsilon_1, 1, -\epsilon_1\mathbb{1}_{n+1}, 1, -\epsilon, -\epsilon_2\mathbb{1}_{n+1}, -\epsilon_2, -\epsilon\mathbb{1}_{n+1}). \tag{4.18}$$

We now move on to study the isometry group of the ϵ-quaternionic-Kähler manifold $\bar{Q} = \bar{c} \circ \bar{r}(\mathcal{H})$ obtained by applying the supergravity q-map to the PSR manifold $\mathcal{H}$. We'll do this in three steps: first we look at the generic isometry group of the projective special ϵ_1-Kähler manifolds in the image of the r-maps; then the generic isometry group of the ϵ-quaternionic-Kähler manifolds in the image of the c-maps; and finally put these together to look at the generic isometry group of spaces in the image of the q-map.

4.2.1 Isometries generated by r-maps

Recall from Section 2.1.4 that a projective special real (PSR) manifold can be thought of as a homogeneous cubic hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ defined by the equation

$$H := c_{ijk}h^ih^jh^k = 1,$$

and comes equipped with a positive definite metric

$$g_{\mathcal{H}} = -\frac{1}{3}\,\partial^2 \log H\big|_{\mathcal{H}}.$$

Let

$$\mathrm{Aut}(H) = \{\varphi \in GL(n+1,\mathbb{R}) | \varphi^* H = H\},$$

be the group of automorphisms of H. Then clearly any automorphism of H preserves the metric $g_{\mathcal{H}}$, so we have $\mathrm{Aut}(H) \subset \mathrm{Isom}(\mathcal{H})$. It may in addition be possible for the isometry group to be further enhanced by invariances which are not symmetries of the full five-dimensional action [99]. Applying the local r-map to $(\mathcal{H}, g_{\mathcal{H}})$ we obtain a projective special ϵ_1-Kähler manifold $(\bar{N}, g_{\bar{N}})$, where $g_{\bar{N}}$ is ϵ_1-Kähler with potential

$$\mathcal{K} = -\log\left(H\left(\mathrm{Im}(z)\right)\right).$$

Since this is invariant under automorphisms of H, we see automatically that $\mathrm{Aut}(H) \subset \mathrm{Isom}(\bar{N})_{\mathrm{holom}}$, where the latter is the group of ϵ_1-holomorphic isometries, i.e. automorphisms which preserve the ϵ_1-complex structure J.

We are interested here in the further isometries of $\bar{N}$ generated by the r-map, which constitute the 'generic' isometry group of any r-map space and exist independently of the choice of c_{ijk}. These are given by

- Translations in the real parts of z,

  $$x^i \to x^i + w, \quad w \in \mathbb{R}^{n+1},$$

  which form an abelian subgroup, isomorphic to $\mathbb{R}^{n+1}$, of the full isometry group of $\bar{N}$.

- Real dilatations

  $$z^i \to \lambda z^i, \quad \lambda \in \mathbb{R}^*,$$

  which induce a Kähler transformation on $\mathcal{K}$ and therefore leave the metric $g_{\bar{N}}$ invariant.

Hence, the 'generic' isometry group of any space in the image of the r-map is given by the solvable group

$$\mathcal{L} = \mathbb{R}^* \ltimes \mathbb{R}^{n+1}.$$

The Lie algebra $\mathfrak{l}$ of $\mathcal{L}$ takes the form

$$\mathfrak{l} = V \oplus \mathbb{R}H,$$

where $V = \mathbb{R}^{n+1}$ is the abelian Lie algebra of dimension $n+1$ on which H acts as a derivation. The non-zero commutators are given by

$$[H, X] = -X, \tag{4.19}$$

for any $X \in V$.

4.2.2 Isometries generated by c-maps

The ϵ-quaternionic-Kähler spaces $\bar{Q}$ in the image of the c-map are locally a product $\bar{Q} = \bar{N} \times G_c$, where $\bar{N}$ is a $2m$-dimensional PSϵ_1K manifold, and G_c a solvable group, which can be identified with an Iwasawa subgroup of the Lie group $SU(m+1,1)$ [25]. The metric on $\bar{Q}$ is locally of the form

$$g_{\bar{Q}}(p,a) = g_{\bar{N}}(p) + g_{G_c}(p,a), \quad (p,a) \in \bar{N} \times G_c,$$

where g_{G_c} are a family of left-invariant metrics on G_c depending on a parameter $p \in \bar{N}$.

We look now at the description of the Lie algebra $\mathfrak{g}_c = \mathrm{iwa}\,\mathfrak{su}(m+1,1)$ of G_c. We follow unpublished notes by Thomas Mohaupt. We can decompose the generators of $\mathfrak{g}_c$ into

$$\mathfrak{g}_c = V \oplus \mathbb{R}Z_0 \oplus \mathbb{R}D,$$

where $V = \mathbb{R}^{2m+2}$ is the abelian Lie algebra of dimension $2m+2$, on which $\mathbb{R}Z_0$ acts as a central extension. This makes $V \oplus \mathbb{R}Z_0$ a Heisenberg algebra [25], on which D acts as a derivation. The non-zero commutators are given by

$$[X, Y] = \omega(X,Y)Z_0, \quad [D, X] = \frac{1}{2}X, \quad [D, Z_0] = Z_0, \tag{4.20}$$

where $X \in V$ and ω is a non-degenerate symplectic form on V.

In [25] the authors realise the Lie group G_c as a group of affine transformations of $\mathbb{R}^{2m+4}$, where for our case $m = n+1$. Each element $(\lambda, \alpha, v^I, \tilde{v}_I) \in G_c = \mathbb{R}^{2n+6}$ can be identified with the affine transformation

$$\begin{aligned} \phi &\mapsto e^{\lambda}\phi, \\ \zeta^I &\mapsto e^{\frac{1}{2}\lambda}\zeta^I + v^I, \\ \tilde{\zeta}_I &\mapsto e^{\frac{1}{2}\lambda}\tilde{\zeta}_I + \tilde{v}_I, \\ \tilde{\phi} &\mapsto e^{\lambda}\tilde{\phi} + e^{\frac{1}{2}\lambda}\left(\tilde{v}_0\zeta^0 + \tilde{v}_i\zeta^i - v^i\tilde{\zeta}_i - v^0\tilde{\zeta}_0\right) + \alpha. \end{aligned} \tag{4.21}$$

We can therefore identify G_c with the orbit of the point $(1, 0, 0, 0)$, which is $L_c = \mathbb{R}^{>0} \times \mathbb{R}^{2n+5} \subset \mathbb{R}^{2n+6}$. The identification is given by

$$G_c \ni (\lambda, \alpha, v^I, \tilde{v}_I) \mapsto (e^{\lambda}, \alpha, v^I, \tilde{v}_I) \in L_c.$$

We have thus identified the generic isometry group of c-map spaces. It may still be the case that for certain choices of prepotential the duality group $\mathcal{D}(\bar{N})$ of the four-dimensional theory, defined as the subgroup of symplectic transformations $X^I \mapsto \tilde{X}^I$ which leave the prepotential invariant $\tilde{F}(\tilde{X}) = F(\tilde{X})$, is enhanced by so-called "hidden" symmetries [100]. These then descend to isometries of the special quaternionic-Kähler manifolds in the image of the c-map.

4.2.3 Isometries generated by the q-map

The automorphism group of spaces in the image of the c-map contains both the solvable group of isometries G_c described in the previous section and the duality group $\mathcal{D}(\bar{N})$ of the four-dimensional theory:

$$\mathrm{Aut}(\bar{Q}, g_{\bar{Q}}, Q) \supset \mathcal{D}(\bar{N}) \ltimes G_c.$$

For those spaces contained also in the image of the q-map, i.e. for which $\bar{Q} = \bar{c}(\bar{N}) = \bar{c} \circ \bar{r}(\mathcal{H})$, the duality group of the four-dimensional theory contains the solvable group $\mathcal{L} = \mathbb{R}^* \ltimes \mathbb{R}^{n+1}$, where n is the dimension of $\mathcal{H}$. Hence, any q-map space has at least the isometry group

$$\mathrm{Aut}(\bar{Q}, g_{\bar{Q}}, Q) \supset (\mathbb{R}^* \ltimes \mathbb{R}^{n+1}) \ltimes G_c.$$

As we did in Section 4.2.2, we now seek to elucidate the structure of the Lie group $G = \mathcal{L} \ltimes G_c$ appearing as part of the isometry group for generic q-map spaces by realising it as a group of affine transformations of $\mathbb{R}^{3n+8}$, which we parametrise by $(\sigma, \phi, x^i, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i)$. We do this in multiple steps, building up the full form of the transformations by combining those descending from both the r-map and c-map.

We concentrate first on the scaling symmetry

$$\Phi \to e^{n_\Phi \lambda + m_\Phi \mu} \Phi, \tag{4.22}$$

where the weights (m_Φ, n_Φ) are given by

Φ	σ	ϕ	x^i	$\tilde{\phi}$	ζ^0	ζ^i	$\tilde{\zeta}_i$	$\tilde{\zeta}_0$
m_Φ	1	0	1	0	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
n_Φ	0	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

This can be considered a 'basis' of the two dilatations present in the q-map isometry group: the scaling of ϕ coming from the c-map and that of σ from the r-map.

We next consider how the $(n+1)$-dimensional group of translations $x^i \mapsto x^i + w^i$ coming from the r-map descend to isometries of the q-map space. We find

$$\begin{aligned}
\sigma &\mapsto \sigma, \\
\phi &\mapsto \phi, \\
x^i &\mapsto x^i + w^i, \\
\zeta^0 &\mapsto \zeta^0, \\
\zeta^i &\mapsto \zeta^i + w^i \zeta^0, \\
\tilde{\zeta}_i &\mapsto \tilde{\zeta}_i - (cw)_{ij}\zeta^j - \frac{1}{2}(cww)_i\, \zeta^0, \\
\tilde{\zeta}_0 &\mapsto \tilde{\zeta}_0 - w^i \tilde{\zeta}_i + \frac{1}{2}(cww)_i \zeta^i + \frac{1}{6}(cwww)\zeta^0, \\
\tilde{\phi} &\mapsto \tilde{\phi}.
\end{aligned} \tag{4.23}$$

These combine with the dilatations with weights m_Φ above to give the action of the isometry group generated by the r-map that we met in Section 4.2.1.

Note that the r-map isometries act non-trivially on fields which only appear after a second reduction. This is due to the fact that we have made field redefinitions upon reduction in order to maintain general covariance [100].

Finally, the isometries coming from the c-map act as in (4.21), namely:

$$\begin{aligned}
\sigma &\mapsto \sigma, \\
\phi &\mapsto e^{\lambda}\phi, \\
x^i &\mapsto x^i, \\
\zeta^0 &\mapsto e^{\frac{1}{2}\lambda}\zeta^0 + v^0, \\
\zeta^i &\mapsto e^{\frac{1}{2}\lambda}\zeta^i + v^i, \\
\tilde{\zeta}_i &\mapsto e^{\frac{1}{2}\lambda}\tilde{\zeta}_i + \tilde{v}_i, \\
\tilde{\zeta}_0 &\mapsto e^{\frac{1}{2}\lambda}\tilde{\zeta}_0 + \tilde{v}_0, \\
\tilde{\phi} &\mapsto e^{\lambda}\tilde{\phi} + e^{\frac{1}{2}\lambda}\left(\tilde{v}_0\zeta^0 + \tilde{v}_i\zeta^i - v^i\tilde{\zeta}_i - v^0\tilde{\zeta}_0\right) + \alpha.
\end{aligned} \tag{4.24}$$

The next step is to turn on all of the transformation parameters, those descending from both the r-map and c-map, and determine the action of the full $(3n+8)$-dimensional isometry group. We can effectively write this down by inspection by combining (4.23) and (4.24), making sure that we include the correct scaling under dilatations. One can indeed check that the following transformations leave the 1-forms (4.12) invariant:

$$\begin{aligned}
\sigma &\mapsto e^{\mu}\sigma, \\
\phi &\mapsto e^{\lambda}\phi, \\
x^i &\mapsto e^{\mu}x^i + w^i, \\
\zeta^0 &\mapsto e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}\zeta^0 + v^0, \\
\zeta^i &\mapsto e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}\zeta^i + e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}w^i\zeta^0 + v^i, \\
\tilde{\zeta}_i &\mapsto e^{\frac{1}{2}\lambda+\frac{1}{2}\mu}\tilde{\zeta}_i - e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}(cw)_{ij}\,\zeta^j - \frac{1}{2}e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}(cww)_i\,\zeta^0 + \tilde{v}_i,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
\tilde{\zeta}_0 \quad &\mapsto \quad e^{\frac{1}{2}\lambda+\frac{3}{2}\mu}\tilde{\zeta}_0 - e^{\frac{1}{2}\lambda+\frac{1}{2}\mu}w^i\tilde{\zeta}_i + \frac{1}{2}e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}(cww)_i\,\zeta^i \\
&\qquad +\frac{1}{6}e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}(cwww)\zeta^0 + \tilde{v}_0, \\
\tilde{\phi} \quad &\mapsto \quad e^{\lambda}\tilde{\phi} + \tilde{v}_0 e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}\zeta^0 + \tilde{v}_i\left(e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}\zeta^i + e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}w^i\zeta^0\right) \\
&\qquad -v^i\left(e^{\frac{1}{2}\lambda+\frac{1}{2}\mu}\tilde{\zeta}_i - e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}(cw)_{ij}\,\zeta^j - \frac{1}{2}e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}(cww)_i\,\zeta^0\right) \\
&\qquad -v^0\left(e^{\frac{1}{2}\lambda+\frac{3}{2}\mu}\tilde{\zeta}_0 - e^{\frac{1}{2}\lambda+\frac{1}{2}\mu}w^i\tilde{\zeta}_i + \frac{1}{2}e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}(cww)_i\,\zeta^i\right. \\
&\qquad\quad \left.+\frac{1}{6}e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}(cwww)\zeta^0\right) + \alpha.
\end{aligned}$$

Looking back to the five-dimensional field content (4.4)–(4.10), we note that all fields involve components of the 1-forms (4.12). Hence, since these 1-forms are invariant under the action of the q-map isometries (4.25), we see that none of these transformations alter the physical field content of the theory. The most they can do is scale various of the field strengths by positive constants, which can be reabsorbed into the electric and magnetic charges of the field configurations.

The transformations (4.25) give us the action of a finite group transformation with parameters $(\mu, \lambda, w^i, v^I, \tilde{v}_I, \alpha) \in \mathbb{R}^{3n+8}$, and allow us to identify G with the orbit of the canonical base point $(\sigma, \phi, x^i, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i) = (1, 1, 0, 0, 0, 0, 0, 0)$ in $L = \mathbb{R}^{>0} \times \mathbb{R}^{>0} \times \mathbb{R}^{3n+6} \subset \mathbb{R}^{3n+8}$ via

$$G \ni (\mu, \lambda, w^i, \alpha, v^0, v^i, \tilde{v}_0, \tilde{v}_i) \mapsto (e^{\lambda}, e^{\mu}, w^i, \alpha, v^0, v^i, \tilde{v}_0, \tilde{v}_i) \in L.$$

Using this, we can write down the group multiplication on $\mathbb{R}^{3n+8}$ which defines G. In particular, we see that the composition of two group transformations

$$(\mu, \lambda, w^i, \alpha, v^0, v^i, \tilde{v}_0, \tilde{v}_i) \circ (\mu', \lambda', w'^i, \alpha', v'^0, v'^i, \tilde{v}'_0, \tilde{v}'_i),$$

produces the transformation $(\mu'', \lambda'', w''^i, \alpha'', v''^0, v''^i, \tilde{v}''_0, \tilde{v}''_i)$ given by

$$\begin{aligned}
\mu'' &= \mu + \mu', \\
\lambda'' &= \lambda + \lambda', \\
w''^i &= e^{\mu}w'^i + w^i,
\end{aligned}$$

$$\begin{aligned}
v''^0 &= e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}v'^0 + v^0, \\
v''^i &= e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}v'^i + e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}w^i v'^0 + v^i, \\
\tilde{v}''_i &= e^{\frac{1}{2}\lambda+\frac{1}{2}\mu}\tilde{v}'_i - e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}(cw)_{ij}v'^j - \frac{1}{2}e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}(cww)_i v'^0 + \tilde{v}_i, \\
\tilde{v}''_0 &= e^{\frac{1}{2}\lambda+\frac{3}{2}\mu}\tilde{v}'_0 - e^{\frac{1}{2}\lambda+\frac{1}{2}\mu}w^i\tilde{v}'_i + \frac{1}{2}e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}(cww)_i v'^i \\
&\quad + \frac{1}{6}e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}(cwww)v'^0 + \tilde{v}_0, \\
\alpha'' &= e^{\lambda}\alpha' + \tilde{v}_0 e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}v'^0 + \tilde{v}_i\left(e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}v'^i + e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}w^i v'^0\right) \\
&\quad - v^i\left(e^{\frac{1}{2}\lambda+\frac{1}{2}\mu}\tilde{v}'_i - e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}(cw)_{ij}v'^j - \frac{1}{2}e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}(cww)_i v'^0\right) \\
&\quad - v^0\left(e^{\frac{1}{2}\lambda+\frac{3}{2}\mu}\tilde{v}'_0 - e^{\frac{1}{2}\lambda+\frac{1}{2}\mu}w^i\tilde{v}'_i + \frac{1}{2}e^{\frac{1}{2}\lambda-\frac{1}{2}\mu}(cww)_i v'^i \right. \\
&\quad \left. + \frac{1}{6}e^{\frac{1}{2}\lambda-\frac{3}{2}\mu}(cwww)v'^0\right) + \alpha.
\end{aligned} \tag{4.26}$$

Infinitesimally, the transformations (4.25) are generated by

$$\delta = \mu H + \lambda D + w^i T_i + v^I Q_I + \tilde{v}_I P^I + \alpha Z_0,$$

where the vector fields are given by[3]

$$\begin{aligned}
H &= -\frac{1}{\sqrt{3}}\left[\sigma\partial_\sigma + x^i\partial_{x^i} - \frac{3}{2}\zeta^0\partial_{\zeta^0} - \frac{1}{2}\zeta^i\partial_{\zeta^i} + \frac{1}{2}\tilde{\zeta}_i\partial_{\tilde{\zeta}_i} + \frac{3}{2}\tilde{\zeta}_0\partial_{\tilde{\zeta}_0}\right], \\
D &= -\phi\partial_\phi - \tilde{\phi}\partial_{\tilde{\phi}} - \frac{1}{2}\zeta^I\partial_{\zeta^I} - \frac{1}{2}\tilde{\zeta}_I\partial_{\tilde{\zeta}_I}, \\
T_i &= -\frac{1}{\sqrt{3}}\left[\partial_{x^i} + \zeta^0\partial_{\zeta^i} - c_{ijk}\zeta^j\partial_{\tilde{\zeta}_k} - \tilde{\zeta}_i\partial_{\tilde{\zeta}_0}\right], \\
Q_0 &= \sqrt{3}\left(\partial_{\zeta^0} - \tilde{\zeta}_0\partial_{\tilde{\phi}}\right), \\
Q_i &= \partial_{\zeta^i} - \tilde{\zeta}_i\partial_{\tilde{\phi}}, \\
P^0 &= \frac{1}{2\sqrt{3}}\left(\partial_{\tilde{\zeta}_I} + \zeta^I\partial_{\tilde{\phi}}\right), \\
P^i &= \frac{1}{2}\left(\partial_{\tilde{\zeta}_i} + \zeta^i\partial_{\tilde{\phi}}\right), \\
Z_0 &= -\partial_{\tilde{\phi}}.
\end{aligned} \tag{4.27}$$

One can then show that the frame

$$(V_A) = \left(Z_0, D, T_i, H, Q_0, Q_i, P^0, P^i\right), \tag{4.28}$$

[3]The factors have been chosen for later convenience.

on TL gives rise to the same non-trivial brackets as the frame (T_A) in (4.15). Hence we can identify the two Lie algebras.

These generators can be plotted on a root diagram with respect to the Cartan basis

$$H_1 = \frac{\sqrt{3}}{2}(H + \sqrt{3}D), \quad H_2 = \frac{1}{2}(D - \sqrt{3}H),$$

which is shown in Figure 4.2. The solvable subgroup generated by (4.27) is represented by root vectors ending in open diamonds.

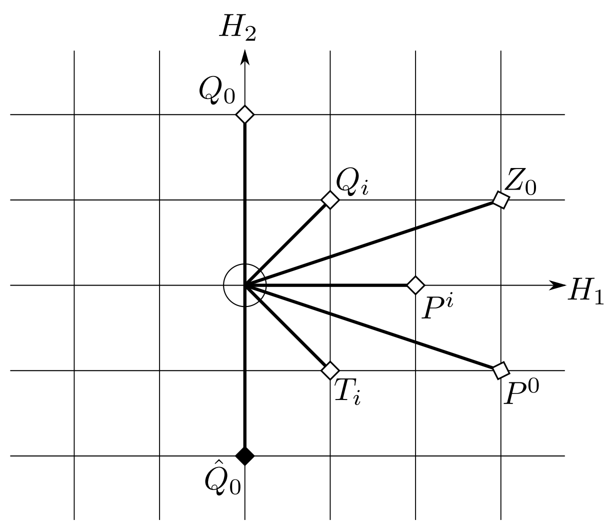

Figure 4.2: Root system for the generic isometry group of q-map spaces. Open diamonds represent roots associated to the generators of the solvable subgroup of isometries, while the closed diamond corresponds to the additional 'hidden symmetry'.

On top of the $3n+8$ isometries described above which describe the group manifold L, the (para-)quaternionic-Kähler spaces in the image of the q-map may admit additional "hidden symmetries". In general, the existence of such hidden symmetries depends on the model chosen. However, for spaces in the image of the q-map we are always guaranteed at least one additional hidden symmetry, independently of the choice of c_{ijk} [100]. Since we have already exhausted the group of transformations which leave the 1-forms (4.12) invariant, this hidden symmetry should act non-trivially on these 1-

forms, and hence on the five-dimensional fields (4.4)–(4.10) which are built from their components. We should then be able to use the action of this hidden symmetry on suitable 'seed' solutions to 'switch-on' extra charges. This solution-generating technique has already proven to be of great success in the case where the target manifold of the three-dimensional theory is a symmetric space (see, e.g. [16] and references therein). However, we are working here in the completely generic case, making no assumptions about the data c_{ijk} of the original five-dimensional theory.

Although in theory the relevant hidden symmetry transformation can be deduced from the expressions in [100], we take a more roundabout approach by first identifying a particular transformation which can be interpreted as a Weyl reflection of the completed root space in Figure 4.2. This will also play an important role when looking at solutions of the five-dimensional theory in Chapter 7.

4.3 Time-space vs. Space-time reductions

In this section we restrict ourselves to the case $\epsilon = 1$, i.e. $\epsilon_1 = -\epsilon_2$, which corresponds to ST or TS reduction.

Consider the metric $g_{\bar{Q}}$ given in equation (4.11), and denote the two metrics obtained from ST and TS reduction by $g_{\bar{Q}}^{(-1,1)}$ and $g_{\bar{Q}}^{(1,-1)}$ respectively. We want to investigate the question: how can the pseudo-Riemannian manifolds $(\bar{Q}, g_{\bar{Q}}^{(-1,1)})$ and $(\bar{Q}, g_{\bar{Q}}^{(1,-1)})$ be related?

Note first that one can always find an analytic continuation relating the three dimensional reductions [101]. In our conventions, the continuation from SS reduction to TS reduction is given by

$$(\phi^x, \sigma, \phi, x^i, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i) \mapsto (\phi^x, \sigma, \phi, ix^i, i\tilde{\phi}, -i\zeta^0, \zeta^i, -\tilde{\zeta}_0, i\tilde{\zeta}_i),$$

while that from SS reduction to ST reduction is

$$(\phi^x, \sigma, \phi, x^i, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i) \mapsto (\phi^x, \sigma, \phi, x^i, -\tilde{\phi}, i\zeta^0, i\zeta^i, i\tilde{\zeta}_0, i\tilde{\zeta}_i).$$

These substitutions change the relative signs in (4.11) in the same way as making the corresponding changes in ϵ_1, ϵ_2. In the case where the scalar manifold is a symmetric space, this can be interpreted as choosing different real forms of the underlying complex symmetric space.

In the case of pure supergravity in five dimensions, the authors of [101] also constructed a map, which they called the "(t, ψ) flip"[4], mapping the ST and TS reductions into each other. In particular, as we will clarify in Chapter 5, pure five-dimensional supergravity reduced to three Euclidean dimensions gives rise to scalar manifolds which can be described as open orbits of the pseudo-Riemannian symmetric space $G_{2(2)}/(SL_2 \cdot SL_2)$, which is a real form of the complex-Riemannian symmetric space $G_2^{\mathbb{C}}/SO(4, \mathbb{C})$. In [101] the scalar manifolds obtained by ST and TS reductions were described by equipping each with the same Riemannian metric but choosing different reality conditions for the complexified fields parametrising $G_2^{\mathbb{C}}/SO(4, \mathbb{C})$. The (t, ψ) flip of [101] then acts isometrically on the Riemannian metric, but exchanges the reality conditions associated with the ST and TS reductions.

Note that this differs to our approach, which is to use the same real coordinates on each of the scalar manifolds but to equip them with different pseudo-Riemannian metrics. Generalizing to the case of an arbitrary number n of vector multiplets coupled to five-dimensional supergravity, we therefore require that the (t, ψ) flip provide us with a real map $\varphi : \mathbb{R}^{4n+8} \to \mathbb{R}^{4n+8}$ acting isometrically:

$$\varphi^* g_{\bar{Q}}^{(-1,1)} = g_{\bar{Q}}^{(1,-1)}.$$

Such a map would then give us a way of exchanging the order of TS and ST reductions. In particular, if this map is globally well-defined on $\bar{Q}$ then we can identify the two scalar manifolds obtained by ST and TS reductions and deduce that space-like and time-like reductions commute, at least for the case at hand.

[4] The naming comes from the two directions parametrising the reduction torus in [101].

4.3.1 The (t, ψ) flip

Using our notation, we generalise the argument of [101] and begin by making the following real field redefinitions:

$$
\begin{aligned}
V &= \sigma^{\frac{1}{2}}\phi^{\frac{1}{2}}, \\
\rho_2 &= \sigma^{-\frac{3}{2}}\phi^{\frac{1}{2}}, \\
\rho_1 &= \frac{1}{\sqrt{3}}a\zeta^0, \\
\mu_2^i &= bx^i, \\
\mu_1^i &= \frac{b}{\sqrt{3}}\left(\zeta^i - x^i\zeta^0\right), \\
\nu_i &= e\left(\tilde{\zeta}_i + \frac{e}{2}(cx)_{ij}\,\zeta^j\right), \\
\tilde{\mu}_2 &= g\left(\tilde{\zeta}_0 - \frac{1}{6}(cxx)_i\,\zeta^i\right), \\
\tilde{\mu}_1 &= -\frac{g}{2\sqrt{3}}\left(\tilde{\phi} - \zeta^0\tilde{\zeta}_0 - \zeta^i\tilde{\zeta}_i - \frac{2}{3}(cx)_{ij}\,\zeta^i\zeta^j + \frac{1}{3}(cxx)_i\,\zeta^0\zeta^i\right), \\
y^i &= Vh^i,
\end{aligned}
\tag{4.29}
$$

where a, b, e, g are arbitrary constant coefficients with $a^2 = 1$. We take the $4n + 8$ real scalar fields parametrising $\bar{Q}$ to be $(y^i, \rho_1, \rho_2, \mu_1^i, \mu_2^i, \nu_i, \tilde{\mu}_1, \tilde{\mu}_2)$. The reason for introducing the variables (4.29) is that they turn out to have extremely simple transformation properties under the interchange of ST and TS reductions, as we will now clarify.

In terms of the new variables (V, ρ_2) the metric ansatz (4.2) for the case of ST or TS reductions ($\epsilon_2 = -\epsilon_1$) becomes

$$
ds^2_{(5)} = 6^{-\frac{1}{6}}V\left[-\epsilon_1\frac{1}{\sqrt{6}\,\rho_2}(dx^0 + \mathcal{A}^0)^2 + \epsilon_1\sqrt{6}\,\rho_2(dx^4 + B)^2\right] + \frac{1}{V^2}ds^2_{(3)}, \tag{4.30}
$$

Hence we see that V acts as a volume modulus for the direct $5 \to 3$ reduction over a (split-signature) torus, and therefore should be unaffected by which toroidal direction we first reduce over.

Let us focus on the two-dimensional (world-volume) part of the metric (4.30):

$$
ds^2_{int} = -\epsilon_1\frac{1}{\sqrt{6}\,\rho_2}(dx^0 + \mathcal{A}^0)^2 + \epsilon_1\sqrt{6}\,\rho_2(dx^4 + B)^2. \tag{4.31}
$$

From (4.8) and (4.29) we can write

$$\mathcal{A}^0 = -\sqrt{2}\,\zeta^0\,dx^4 + \mathcal{A}^0_\mu dx^\mu = -a\sqrt{6}\,\rho_1\,dx^4 + \mathcal{A}^0_\mu dx^\mu,$$

so that the 'diagonal' part of (4.31) becomes

$$ds^2_{int} \supset -\epsilon_1\left[\frac{1}{\sqrt{6}\,\rho_2}(dx^0)^2 + \sqrt{6}\left(\frac{\rho_1^2-\rho_2^2}{\rho_2}\right)(dx^4)^2\right].$$

This hints that the transformation

$$\rho_2 \to \frac{-\rho_2}{\rho_1^2-\rho_2^2}, \quad \rho_1 \to \frac{-\rho_1}{\rho_1^2-\rho_2^2}, \tag{4.32}$$

would do the job (at least up to some factor) of flipping the sign $\epsilon_1 \to -\epsilon_1$ in the diagonal part of the world-volume metric.

To further justify this, we rewrite the metric (4.11) on the para-quaternionic-Kähler manifold in terms of the variables (4.29). Defining

$$\hat{g}_{ij}(y) = \frac{3}{4V^2}a_{ij}(h),$$

we have

$$\begin{aligned}
g_{\bar{Q}} \;=\;& \hat{g}_{ij}(y)dy^i dy^j - \frac{V\rho_2}{b^2}\epsilon_1\hat{g}_{ij}(y)d\mu_2^i d\mu_2^j - \frac{1}{4\rho_2^2}d\rho_1^2 + \frac{1}{4\rho_2^2}d\rho_2^2 \\
&-\frac{1}{4V^3\rho_2}\epsilon_1\left(\frac{1}{j}d\tilde{\mu}_1 - \frac{a}{j}\rho_1 d\tilde{\mu}_2 + \frac{2\sqrt{3}}{be}\mu_1^i d\nu_i + \frac{2\sqrt{3}a}{be}\rho_1\mu_2^i d\nu_i\right.\\
&\left.+\frac{1}{b^3}(c\mu_1\mu_2)_i d\mu_1^i - \frac{1}{b^3}(c\mu_1\mu_1)_i d\mu_2^i - \frac{a}{b^3}\rho_1(c\mu_1\mu_2)_i d\mu_2^i + \frac{a}{b^3}\rho_1(c\mu_2\mu_2)_i d\mu_1^i\right)^2\\
&+\frac{V}{b^2\rho_2}\epsilon_1\hat{g}_{ij}(y)\left(d\mu_1^i + a\rho_1 d\mu_2^i\right)\left(d\mu_1^j + a\rho_1 d\mu_2^j\right)\\
&+\frac{\rho_2}{V^3}\epsilon_1\left(-\frac{1}{2j}d\tilde{\mu}_2 + \frac{\sqrt{3}}{be}\mu_2^i d\nu_i + \frac{1}{2b^3}(c\mu_2\mu_2)_i d\mu_1^i - \frac{1}{2b^3}(c\mu_1\mu_2)_i d\mu_2^i\right)^2\\
&-\frac{3}{4V^4}\hat{g}^{ij}(y)\left(\frac{1}{e}d\nu_i + \frac{\sqrt{3}}{2b^2}(c\mu_2)_{ik}d\mu_1^k - \frac{\sqrt{3}}{2b^2}(c\mu_1)_{ik}d\mu_2^k\right)\\
&\quad\times\left(\frac{1}{e}d\nu_j + \frac{\sqrt{3}}{2b^2}(c\mu_2)_{jl}\,d\mu_1^l - \frac{\sqrt{3}}{2b^2}(c\mu_1)_{jl}\,d\mu_2^l\right).
\end{aligned} \tag{4.33}$$

We see that there is a two-dimensional subspace spanned by (ρ_1, ρ_2) with metric

$$g_2 = \frac{d\rho_1^2 - d\rho_2^2}{4\rho_2^2},$$

which is just the metric on the two-dimensional para-Kähler symmetric space $SO(2,1)/SO(1,1)$. Clearly the transformation (4.32) acts isometrically on g_2, as we can see by writing everything in terms of the para-complex variable $\rho = \rho_1 + e\rho_2$.

The remaining terms in (4.33) are slightly more complicated to deal with. However, after some calculation, one can show that the transformations

$$\begin{aligned}
&V \mapsto V, \quad \nu_i \mapsto \nu_i, \quad y^i \mapsto y^i, \\
&(\mu_1^i, \mu_2^i) \mapsto (\mu_2^i, -\mu_1^i), \\
&(\tilde{\mu}_1, \tilde{\mu}_2) \mapsto (-\tilde{\mu}_2, \tilde{\mu}_1), \\
&\rho_2 \to \frac{-\rho_2}{\rho_1^2 - \rho_2^2}, \quad \rho_1 \to \frac{-\rho_1}{\rho_1^2 - \rho_2^2},
\end{aligned} \tag{4.34}$$

have the effect of flipping $\epsilon_1 \to -\epsilon_1$ in (4.33). Hence, the transformation (4.34) has precisely the effect of taking us from the manifold $(\bar{Q}, g_{\bar{Q}}^{(-1,1)})$ to $(\bar{Q}, g_{\bar{Q}}^{(1,-1)})$, i.e. it interchanges the ST and TS reductions.

We note from the form of (4.34), however, that the transformation is singular along the loci $\rho_1 = \pm\rho_2$. This implies that we still have no *global* isometry relating the manifolds obtained from time-space and space-time reduction.

In the next chapter we will investigate in more detail the global structure of the scalar manifolds obtained by dimensional reduction in the simplest case of pure five-dimensional supergravity. The general case will be treated in a future publication by the author [38].

4.3.2 The hidden symmetry

We now move on to use the (t, ψ) flip to generate the additional 'hidden' symmetry of the para-quaternionic-Kähler manifolds in the image of the q-map.

First, we write the generators (4.27) of the solvable Lie algebra $\mathfrak{g}$ in terms of the variables (4.29):

$$
\begin{aligned}
H &= -\frac{1}{\sqrt{3}}\left[\frac{1}{2}V\partial_V - \frac{3}{2}\rho_1\partial_{\rho_1} - \frac{3}{2}\rho_2\partial_{\rho_2} - \frac{1}{2}\mu_1^i\partial_{\mu_1^i} + \mu_2^i\partial_{\mu_2^i} + \frac{1}{2}\nu_i\partial_{\nu_i} + \frac{3}{2}\tilde{\mu}_2\partial_{\tilde{\mu}_2}\right], \\
D &= -\frac{1}{2}V\partial_V - \frac{1}{2}\rho_1\partial_{\rho_1} - \frac{1}{2}\rho_2\partial_{\rho_2} - \frac{1}{2}\mu_1^i\partial_{\mu_1^i} - \frac{1}{2}\nu_i\partial_{\nu_i} - \tilde{\mu}_2\partial_{\tilde{\mu}_2} - \tilde{\mu}_1\partial_{\tilde{\mu}_1}, \\
T_i &= -\frac{1}{\sqrt{3}}\left[b\partial_{\mu_2^i} - \frac{\sqrt{3}e}{2b}(c\mu_1)_{ij}\partial_{\nu_j} - \left(\frac{g}{e}\nu_i - \frac{g}{2\sqrt{3}b^2}(c\mu_1\mu_2)_i\right)\partial_{\tilde{\mu}_2}\right. \\
&\qquad\left. - \frac{g}{2\sqrt{3}b^2}(c\mu_1\mu_1)_i\partial_{\tilde{\mu}_1}\right], \\
Q_0 &= a\partial_{\rho_1} - \mu_2^i\partial_{\mu_1^i} + \tilde{\mu}_2\partial_{\tilde{\mu}_1}, \\
Q_i &= \frac{b}{\sqrt{3}}\partial_{\mu_1^i} + \frac{e}{2b}(c\mu_2)_{ij}\partial_{\nu_j} - \frac{g}{6b^2}(c\mu_2\mu_2)_i\partial_{\tilde{\mu}_2} + \left(\frac{g}{\sqrt{3}e}\nu_i + \frac{g}{6b^2}(c\mu_1\mu_2)_i\right)\partial_{\tilde{\mu}_1}, \\
P^0 &= \frac{g}{2\sqrt{3}}\partial_{\tilde{\mu}_2}, \\
P^i &= \frac{e}{2}\partial_{\nu_i}, \\
Z_0 &= \frac{g}{2\sqrt{3}}\partial_{\tilde{\mu}_1}.
\end{aligned} \tag{4.35}
$$

The action of the (t, ψ) flip (4.34) then leaves P^i invariant, while exchanging

$$P^0 \leftrightarrow Z_0, \quad T_i \leftrightarrow Q_i.$$

The remaining root, Q_0, however, is mapped to

$$\hat{Q}_0 = a(\rho_1^2 + \rho_2^2)\,\partial_{\rho_1} + 2a\rho_1\rho_2\,\partial_{\rho_2} + \mu_1^i\partial_{\mu_2^i} - \tilde{\mu}_1\partial_{\tilde{\mu}_2}, \tag{4.36}$$

which completes the root diagram in Figure 4.2 describing the symmetries of a generic q-map space. Hence we see that the (t, ψ) flip acts as a Weyl reflection of the root space in Figure 4.2 in the hypersurface orthogonal to P^i.

One can show that the generators

$$Y_0 = \frac{1}{2}(D - \sqrt{3}H), \quad Y_+ = Q_0, \quad Y_- = \hat{Q}_0,$$

generate an $\mathfrak{sl}(2)$ subalgebra $[Y_-, Y_+] = 2Y_0$, $[Y_0, Y_\pm] = \pm Y_\pm$ of the full isometry group. We will comment on some possible applications of this in Chapter 8.

The finite form of the transformation generated by $\hat{Q}_0$ is given by its action on the fields (4.29) by

$$\begin{aligned} \mu_2^i &\mapsto \mu_2^i + \hat{v}^0 \mu_1^i, \\ \tilde{\mu}_2 &\mapsto \tilde{\mu}_2 - \hat{v}^0 \tilde{\mu}_1, \\ \rho &\mapsto \frac{\rho}{1 - a\hat{v}^0 \rho}, \end{aligned} \tag{4.37}$$

where $\rho = \rho_1 + e\rho_2$. One can show explicitly that this provides an isometry of the metric (4.33). Again, this is a fairly lengthy but straightforward calculation, so we omit the details.

4.4 Geometrical data on G

In this section we calculate some useful geometric data (connection, curvature) on the group manifold L. We use this to determine the conditions necessary on the PSR manifold for which L is a locally symmetric space.

We first want to calculate the Levi-Civita connection on $L \cong G$, which we equip with the left-invariant metric g_L given by (4.13). The associated scalar product on $\mathfrak{g}$ is denoted $\langle \cdot, \cdot \rangle$ and is defined by the Gram matrix

$$\mathcal{G} = \mathrm{diag}(-\epsilon_1, 1, -\epsilon_1 \mathbb{1}_{n+1}, 1, -\epsilon, -\epsilon_2 \mathbb{1}_{n+1}, -\epsilon_2, -\epsilon \mathbb{1}_{n+1}). \tag{4.38}$$

That is, we consider the Lie algebra $\mathfrak{g}$ with the basis (T_A) of (4.15) and structure constants (4.16), and equip it with a pseudo-Euclidean scalar product $\langle \cdot, \cdot \rangle$ defined by (4.38).

The Levi-Civita connection can be calculated from this data using the Koszul formula (2.7). In this manner we find

$$\begin{aligned} D_{U^{n+2}} &= 0, \\ D_{V_{n+2}} &= -U^{n+2} \wedge V_{n+2} - \frac{1}{2} U^0 \wedge V_0 - \frac{1}{2} U^i \wedge V_i, \\ D_B &= 0, \\ D_{A_i} &= -\frac{1}{\sqrt{3}} B \wedge A_i - \frac{1}{2} \epsilon V_0 \wedge V_i - \frac{1}{2} \epsilon \delta_{ij} U^0 \wedge U^j \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{\sqrt{3}}\epsilon_2 c_{ij}{}^k U^j \wedge V_k, \\
D_{V_0} &= \frac{1}{2}V_0 \wedge U^{n+2} + \frac{1}{2}\epsilon_2 U^0 \wedge V_{n+2} - \frac{\sqrt{3}}{2}V_0 \wedge B \\
& -\frac{1}{2}\epsilon_1 \delta^{ij} V_i \wedge A_j, \\
D_{V_i} &= \frac{1}{2}V_i \wedge U^{n+2} + \frac{1}{2}\epsilon\delta_{ij} U^j \wedge V_{n+2} - \frac{1}{2\sqrt{3}}V_i \wedge B \\
& +\frac{1}{2}V_0 \wedge A_i + \frac{1}{\sqrt{3}}\epsilon_1 c_{ij}{}^k U^j \wedge A_k, \\
D_{U^0} &= \frac{1}{2}U^0 \wedge U^{n+2} - \frac{1}{2}\epsilon V_0 \wedge V_{n+2} + \frac{\sqrt{3}}{2}U^0 \wedge B \\
& -\frac{1}{2}U^i \wedge A_i, \\
D_{U^i} &= \frac{1}{2}U^i \wedge U^{n+2} - \frac{1}{2}\epsilon_2 \delta^{ij} V_j \wedge V_{n+2} + \frac{1}{2\sqrt{3}}U^i \wedge B \\
& +\frac{1}{2}\epsilon_1 \delta^{ij} U^0 \wedge A_j - \frac{1}{\sqrt{3}} c^{ijk} V_j \wedge A_k, \qquad (4.39)
\end{aligned}
$$

where all indices on the c_{ijk} are raised and lowered with δ_{ij}, e.g. $c_{ij}{}^k = c_{ijl}\delta^{lk}$. Here we use the identification of bi-vectors with skew-symmetric endomorphisms

$$
(X \wedge Y)(Z) = X\langle Y, Z\rangle - \langle X, Z\rangle Y \ , \quad X, Y, Z, \in \mathfrak{g} \ . \qquad (4.40)
$$

We next proceed to compute the curvature of the connection D, using the formula

$$
R(X,Y) = [D_X, D_Y] - D_{[X,Y]} \ ,
$$

for vector fields $X, Y \in \mathfrak{g}$. In this way $R(X,Y)$ is considered as a skew-symmetric endomorphism of $\mathfrak{g}$, given by

$$
\begin{aligned}
R(U^{n+2}, V_{n+2}) &= -D_{V_{n+2}}, \quad R(U^{n+2}, B) = R(U^{n+2}, A_i) = 0, \\
R(U^{n+2}, V_0) &= -\frac{1}{2}D_{V_0}, \quad R(U^{n+2}, V_i) = -\frac{1}{2}D_{V_i}, \quad R(U^{n+2}, U^0) = -\frac{1}{2}D_{U^0}, \\
R(U^{n+2}, U^i) &= -\frac{1}{2}D_{U^i}, \\
R(V_{n+2}, B) &= R(V_{n+2}, A_i) = 0, \quad R(V_{n+2}, V_0) = \frac{1}{2}\epsilon D_{U^0}, \\
R(V_{n+2}, V_i) &= \frac{1}{2}\epsilon_2 \delta_{ij} D_{U^j}, \quad R(V_{n+2}, U^0) = -\frac{1}{2}\epsilon_2 D_{V_0}, \\
R(V_{n+2}, U^i) &= -\frac{1}{2}\epsilon \delta^{ij} D_{V_j},
\end{aligned}
$$

$$
\begin{aligned}
R(B,A_i) &= -\frac{1}{\sqrt{3}}D_{A_i}, \quad R(B,V_0)=\frac{\sqrt{3}}{2}D_{V_0}, \quad R(B,V_i)=\frac{1}{2\sqrt{3}}D_{V_i},\\
R(B,U^0) &= -\frac{\sqrt{3}}{2}D_{U^0}, \quad R(B,U^i)=-\frac{1}{2\sqrt{3}}D_{U^i},\\
R(A_i,V_0) &= -\frac{1}{2}D_{V_i},\\
R(A_i,V_j) &= \frac{1}{2}\epsilon_1\delta_{ij}D_{V_0}+\frac{1}{\sqrt{3}}c_{ijk}D_{U^k}+F_{ij}{}^{lm}V_l\wedge A_m,\\
R(A_i,U^0) &= -\frac{1}{2}\epsilon_1\delta_{ij}D_{U^j},\\
R(A_i,U^j) &= \frac{1}{2}\delta_i^jD_{U^0}-\frac{1}{\sqrt{3}}\epsilon_1c_i{}^{jk}D_{V_k}+F_{il}{}^{jm}U^l\wedge A_m,\\
R(V_0,V_i) &= -\frac{1}{2}\epsilon D_{A_i}, \quad R(V_0,U^0)=\frac{1}{2}D_{V_{n+2}}, \quad R(V_0,U^i)=0,\\
R(V_i,U^0) &= 0, \quad R(V_i,U^j)=\frac{1}{2}\delta_i^jD_{V_{n+2}}+\frac{1}{\sqrt{3}}\epsilon_2c_i{}^{jk}D_{A_k}+F_{il}{}^{jm}U^l\wedge V_m,\\
R(U^0,U^i) &= -\frac{1}{2}\epsilon\delta^{ij}D_{A_j},
\end{aligned}
\tag{4.41}
$$

where the constant model-dependent quantity $F_{ij}{}^{kl}$ is given by

$$
F_{ij}{}^{lm}\equiv\frac{1}{3}\left(c_{ijk}c^{klm}+c_{ik}{}^lc_j{}^{km}\right)-\frac{1}{4}\left(\delta_{ij}\delta^{lm}+\delta_i^l\delta_j^m\right)-\frac{1}{6}\delta_j^l\delta_i^m. \tag{4.42}
$$

Using (4.39) and (4.41), one can show that $DR=0$, i.e. L is locally symmetric, iff the $F_{ij}{}^{kl}$ vanish identically. This clearly holds for $n=0$, $c_{111}=1$, which corresponds to the case of pure five-dimensional supergravity. It is a subject for future work to classify the possible c_{ijk} for which $DR=0$.

Chapter 5

Time-like reductions of pure supergravity

In Chapter 4 we wrote down the five-dimensional Lagrangian (4.1) describing $\mathcal{N} = 2$ supergravity coupled $n_V^{(5)}$ vector multiplets, as well as its dimensional reduction (4.3) to three (Euclidean or Minkowski) dimensions. We also motivated the question of whether the two target spaces obtained by space-then-time (ST) and time-then-space (TS) reduction are "the same" in any sense. The aim of this section is to concentrate on the case where the five-dimensional theory is restricted to be pure five-dimensional supergravity. This theory has been well studied in the context of generating five-dimensional solutions in [101–105].

We begin in Section 5.1 with a description of five-dimensional pure supergravity and its reduction to three dimensions. In Section 5.2 we turn to a mathematical description of the scalar manifolds of the three-dimensional theory, which in each case can be identified with a certain solvable group manifold contained within the symmetric space $G_2^{\mathbb{C}}/SO(4,\mathbb{C})$, and follow this up in Section 5.3 with a description of the automorphism group of this solvable Lie group. Then, in Section 5.4, we use this formalism to determine how the scalar manifolds obtained by ST and TS reduction fit into the symmetric space $G_{2(2)}/(SL_2 \cdot SL_2)$, which is the main result of this chapter. Finally, in Section 5.5 we describe the geometric structures on the solvable subgroup, and prove that the scalar manifolds obtained by dimensional reduction of pure supergravity are

ϵ-quaternionic-Kähler, and admit integrable ϵ_1-complex structures.

This chapter is based on the paper [37] by the author.

5.1 Dimensional reduction of pure five-dimensional supergravity

We start with the bosonic action for pure five-dimensional supergravity. This can be obtained from (4.1) by considering the case with no five-dimensional vector multiplets ($n_V^{(5)} = 0$) and Hesse potential $H = (h^0)^3$. The action then becomes

$$S = \int d^5x \left[\sqrt{\hat{g}} \left(\frac{\hat{R}}{2} - \frac{1}{4} \mathcal{F}_{\hat{\mu}\hat{\nu}} \mathcal{F}^{\hat{\mu}\hat{\nu}} \right) + \frac{1}{6\sqrt{6}} \varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}} \mathcal{F}_{\hat{\mu}\hat{\nu}} \mathcal{F}_{\hat{\rho}\hat{\sigma}} \mathcal{A}_{\hat{\lambda}} \right], \tag{5.1}$$

which is simply the Einstein-Maxwell action supplemented by a Chern-Simons term.

Performing the dimensional reduction as in Section 3.5.2, we obtain the three-dimensional Lagrangian

$$\begin{aligned} \mathcal{L}_3 \;=\; & \frac{R}{2} + \frac{3}{4\sigma^2}\epsilon_1 (\partial x)^2 - \frac{3}{4\sigma^2}(\partial\sigma)^2 - \frac{1}{4\phi^2}(\partial\phi)^2 + \frac{1}{4\phi^2}\epsilon_1 \left(\partial\tilde{\phi} + \zeta^I \overleftrightarrow{\partial} \tilde{\zeta}_I\right)^2 \\ & + \frac{\sigma^3}{12\phi}\epsilon (\partial\zeta^0)^2 + \frac{\sigma}{4\phi}\epsilon_2 \left(\partial\zeta^1 - x\partial\zeta^0\right)^2 \\ & + \frac{3}{\sigma^3\phi}\epsilon_2 \left(\partial\tilde{\zeta}_0 + x\partial\tilde{\zeta}_1 + \frac{1}{2}x^2\partial\zeta^1 - \frac{1}{6}x^3\partial\zeta^0\right)^2 \\ & + \frac{1}{\sigma\phi}\epsilon \left(\partial\tilde{\zeta}_1 + x\partial\zeta^1 - \frac{1}{2}x^2\partial\zeta^0\right)^2, \end{aligned} \tag{5.2}$$

where $I = 0, 1$. The relations between the five- and three-dimensional field contents can be read off from the corresponding expressions (4.4)–(4.10) in Chapter 4.

For this chapter the scalar manifolds obtained by SS ($\epsilon_1 = \epsilon_2 = -1$), ST ($\epsilon_1 = -\epsilon_2 = -1$) and TS ($\epsilon_1 = -\epsilon_2 = 1$) reduction are denoted, respectively, $M^{(SS)}$, $M^{(ST)}$ and $M^{(TS)}$, and are parametrized by the eight scalar fields $(x, \sigma, \phi, \tilde{\phi}, \zeta^0, \zeta^1, \tilde{\zeta}_0, \tilde{\zeta}_1)$.

Comparing with the analysis in Chapter 4 we see that, since $n_V^{(5)} = 0$, the PSR manifold appearing as the target space of the five-dimensional theory is zero-dimensional. Hence, the eight-dimensional target manifold of the three-dimensional theory can be identified with the group fiber of the q-map spaces that we met previously. We can therefore use the formulae of Section 4.2, after putting $n_V^{(5)} = 0$ and $H(h) = (h^0)^3$, to describe the target manifold of the theory with Lagrangian (5.2). In Section 5.5 this will provide us with a direct proof that the target manifold of the three-dimensional theory is ϵ-quaternionic-Kähler and equipped with a pair of integrable ϵ_1-complex structures.

Although many of the following expressions could be read directly from those in Chapter 4 we reproduce them here for convenience. We first introduce the basis of 1-forms

$$\begin{aligned}
\eta^2 &= \frac{1}{\phi}\left(d\tilde{\phi} + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right), \qquad \xi_2 = \frac{d\phi}{\phi}, \\
\alpha &= \frac{\sqrt{3}}{\sigma}dx, \qquad \beta = \frac{\sqrt{3}}{\sigma}d\sigma, \\
\eta^0 &= \sqrt{\frac{\sigma^3}{3\phi}}d\zeta^0, \qquad \eta^1 = \sqrt{\frac{\sigma}{\phi}}\left(d\zeta^1 - xd\zeta^0\right), \\
\xi_0 &= 2\sqrt{\frac{3}{\sigma^3\phi}}\left(d\tilde{\zeta}_0 + xd\tilde{\zeta}_1 + \frac{1}{2}x^2 d\zeta^1 - \frac{1}{6}x^3 d\zeta^0\right), \\
\xi_1 &= \frac{2}{\sqrt{\sigma\phi}}\left(d\tilde{\zeta}_1 + xd\zeta^1 - \frac{1}{2}x^2 d\zeta^0\right),
\end{aligned} \tag{5.3}$$

which we also denote

$$(\theta^a) = (\eta^2, \xi_2, \alpha, \beta, \eta^0, \eta^1, \xi_0, \xi_1)\ . \tag{5.4}$$

In terms of (5.3) the metric $g^{(\epsilon_1,\epsilon_2)}$ on the scalar manifold[1] $M^{(\epsilon_1,\epsilon_2)}$ can be written as

$$4g^{(\epsilon_1,\epsilon_2)} = -\epsilon_1\eta^2\otimes\eta^2 + \xi_2\otimes\xi_2 - \epsilon_1\alpha\otimes\alpha + \beta\otimes\beta - \epsilon\eta^0\otimes\eta^0 - \epsilon_2\eta^1\otimes\eta^1 - \epsilon_2\xi_0\otimes\xi_0 - \epsilon\xi_1\otimes\xi_1. \tag{5.5}$$

The exterior derivatives $d\theta^a$ are given by

$$\begin{aligned}
d\eta^2 &= -\xi_0\wedge\eta^0 - \xi_1\wedge\eta^1 - \xi_2\wedge\eta^2, \\
d\xi_2 &= 0, \\
d\alpha &= \frac{1}{\sqrt{3}}\alpha\wedge\beta, \\
d\beta &= 0, \\
d\eta^0 &= \frac{\sqrt{3}}{2}\beta\wedge\eta^0 - \frac{1}{2}\xi_2\wedge\eta^0,
\end{aligned} \tag{5.6}$$

[1] We write $M^{(SS)} = M^{(-1,-1)}$, etc.

$$
\begin{aligned}
d\eta^1 &= \frac{1}{2\sqrt{3}}\beta\wedge\eta^1 - \frac{1}{2}\xi_2\wedge\eta^1 - \alpha\wedge\eta^0,\\
d\xi_0 &= -\frac{\sqrt{3}}{2}\beta\wedge\xi_0 - \frac{1}{2}\xi_2\wedge\xi_0 + \alpha\wedge\xi_1,\\
d\xi_1 &= -\frac{1}{2\sqrt{3}}\beta\wedge\xi_1 - \frac{1}{2}\xi_2\wedge\xi_1 + \frac{2}{\sqrt{3}}\alpha\wedge\eta^1,
\end{aligned}
$$

which shows that they form a Lie algebra with structure constants given by the relation $d\theta^a = -c^a_{bc}\theta^b\wedge\theta^c$, on which $g^{(\epsilon_1,\epsilon_2)}$ is a left-invariant metric.

Denoting the basis of vector fields dual to (θ^a) by

$$
(T_a) = (V_2, U^2, A, B, V_0, V_1, U^0, U^1), \tag{5.7}
$$

we find equivalently

$$
\begin{aligned}
[B,A] &= \frac{1}{\sqrt{3}}A, \quad \left[U^2,V_2\right] = V_2,\\
\left[V_0,U^0\right] &= -V_2, \quad \left[V_1,U^1\right] = -V_2,\\
\left[U^2,V_I\right] &= \frac{1}{2}V_I \quad \text{for } I=0,1, \quad \left[U^2,U^I\right] = \frac{1}{2}U^I \quad \text{for } I=0,1,\\
[B,V_0] &= -\frac{\sqrt{3}}{2}V_0, \quad [B,V_1] = -\frac{1}{2\sqrt{3}}V_1, \quad \left[B,U^0\right] = \frac{\sqrt{3}}{2}U^0,\\
&\quad \left[B,U^1\right] = \frac{1}{2\sqrt{3}}U^1,\\
[A,V_0] &= V_1, \quad \left[A,U^1\right] = -U^0, \quad [A,V_1] = -\frac{2}{\sqrt{3}}U^1.
\end{aligned} \tag{5.8}
$$

This Lie algebra is seen to be solvable. Indeed, we will see below that it is an Iwasawa subalgebra of the Lie algebra of $G_{2(2)}$.

The explicit expressions for the vector fields (T_a) in a coordinate basis on $M^{(\epsilon_1,\epsilon_2)}$ are

$$
\begin{aligned}
&V_2 = \phi\,\partial_{\tilde\phi}, \quad U^2 = \phi\,\partial_\phi, \quad A = \frac{1}{\sqrt{3}}\sigma\partial_x, \quad B = \frac{1}{\sqrt{3}}\sigma\partial_\sigma,\\
&V_0 = \sqrt{\frac{3\phi}{\sigma^3}}\left[(\partial_{\zeta^0} + \tilde\zeta_0\partial_{\tilde\phi}) + x(\partial_{\zeta^1} + \tilde\zeta_1\partial_{\tilde\phi}) - \frac{1}{2}x^2(\partial_{\tilde\zeta_1} - \zeta^1\partial_{\tilde\phi}) + \frac{1}{6}x^3(\partial_{\tilde\zeta_0} - \zeta^0\partial_{\tilde\phi})\right],\\
&V_1 = \sqrt{\frac{\phi}{\sigma}}\left[(\partial_{\zeta^1} + \tilde\zeta_1\partial_{\tilde\phi}) - x(\partial_{\tilde\zeta_1} - \zeta^1\partial_{\tilde\phi}) + \frac{1}{2}x^2(\partial_{\tilde\zeta_0} - \zeta^0\partial_{\tilde\phi})\right],\\
&U^0 = \frac{1}{2\sqrt{3}}\sqrt{\sigma^3\phi}(\partial_{\tilde\zeta_0} - \zeta^0\partial_{\tilde\phi}),
\end{aligned}
$$

$$U^1 = \frac{1}{2}\sqrt{\sigma\phi}\left[(\partial_{\tilde{\zeta}_1} - \zeta^1\partial_{\tilde{\phi}}) - x(\partial_{\tilde{\zeta}_0} - \zeta^0\partial_{\tilde{\phi}})\right]. \tag{5.9}$$

The curvature calculations (4.41) for the q-map group manifold adapted to the case at hand tell us that $DR = 0$, so that the target manifolds $M^{(\epsilon_1,\epsilon_2)}$ are locally symmetric.

In the case of SS reduction, the target manifold can be identified with the Riemannian symmetric space $G_{2(2)}/SO(4)$ [36, 102], which is quaternionic-Kähler. On the other hand, dimensional reduction of pure five-dimensional supergravity over one time-like and one space-like direction gives rise to a scalar target space locally isometric to the pseudo-Riemannian symmetric space (see [104] and references therein)

$$S = \frac{G_{2(2)}}{SL_2 \cdot SL_2}, \tag{5.10}$$

which is para-quaternionic-Kähler [77].

In [105] it was shown that the spaces obtained from ST and TS reduction are locally isometric by relating them to a standard parametrization of the space (5.10). Geometrically, however, we have seen that the target spaces M^{ST} and M^{TS} are distinguished by the integrability properties of the left-invariant almost ϵ_1-complex structures within the para-quaternionic structure. This hints that the local isometry of [105] does not capture the full description of these manifolds.

In particular, we will show in this chapter that the spaces M^{ST} and M^{TS} can be realised as open orbits of the Iwasawa subgroup L of $G_{2(2)}$ acting on S. While these orbits are locally isometric, they are not related by an automorphism of L.

Before moving on to a description of $G_{2(2)}$, we note that both of the symmetric spaces $G_{2(2)}/SO(4)$ and $G_{2(2)}/(SL_2 \cdot SL_2)$ are different real forms of

$$\frac{G^{\mathbb{C}}_{2(2)}}{SO(4)^{\mathbb{C}}}.$$

In [101] this was used to relate all three scalar manifolds obtained by dimensional reduction to each other via suitable analytic continuations, as we discussed in Chapter 4.

5.2 Aspects of $G_{2(2)}$ and its subgroups

Throughout this section we write $G = G_{2(2)}$ for the simply connected noncompact form of the Lie group of type $G2$.

5.2.1 Description of the Lie algebra of $G_{2(2)}$

We now seek to describe the Lie algebra $\mathfrak{g}$ of the Lie group G. We follow Ch. 5, Section 1.2 of [106].

The Lie algebra $\mathfrak{g}$ admits a $\mathbb{Z}_3$-grading

$$\mathfrak{g} = V + \mathfrak{sl}(V) + V^*,$$

where the subspaces have degree $(1, 0, -1) \in \mathbb{Z}_3$ respectively under the grading, and $V = \mathbb{R}^3$. Note that $\mathfrak{sl}(V) = \mathfrak{sl}(3, \mathbb{R})$. The Lie brackets are given by

$$\begin{aligned} [A, x] &= Ax, \quad [A, \xi] = -A^T \xi, \\ [x, y] &= -2x \times y, \\ [\xi, \eta] &= 2\xi \times \eta, \\ [x, \xi] &= 3x \otimes \xi - \xi(x)\mathrm{Id} \in \mathfrak{sl}(V) \subset \mathfrak{gl}(V) \cong V \otimes V^*, \end{aligned} \tag{5.11}$$

for all $x, y \in V$, $\xi, \eta \in V^*$ and $A \in \mathfrak{sl}(V)$. The cross products are defined by

$$x \times y = \det(x, y, \cdot) \in V^*, \quad \xi \times \eta = \det{}^{-1}(\xi, \eta, \cdot) \in V^{**} = V,$$

where $\det^{-1} \in \wedge^3 V$ is the inverse of $\det \in \wedge^3 V^*$. We shall denote by $(e_i) = (e_1, e_2, e_3)$ the standard basis of V, by (e^i) its dual basis and by e_i^j the endomorphism $e_i \otimes e^j$ of V. Then, e.g. $e_1 \times e_2 = e^3$.

The Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}$ consisting of all diagonal matrices in $\mathfrak{sl}(V)$ is given by

$$\mathfrak{a} = \left\{ \sum_i \lambda_i e_i^i \;\middle|\; \sum_i \lambda_i = 0 \right\}.$$

5.2.2 The solvable Iwasawa subgroup $L \subset G$

Choose a basis

$$H_1 = e_1^1 - e_3^3, \quad H_2 = e_1^1 - 2e_2^2 + e_3^3,$$

of the Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}$. Then, following Section 2.4.3 we can construct a maximal nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{g}$

$$\mathfrak{n} = \mathrm{span}\{e_1, e^2, e^3, e_1^2, e_1^3, e_2^3\} \subset \mathfrak{g},$$

as the sum of the positive root spaces of $\mathfrak{a}$. We then define $\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{n} \subset \mathfrak{g}$, which is the[2] solvable Lie algebra (Iwasawa subalgebra) appearing in the Iwasawa decomposition of $\mathfrak{g}$, as described in Section 2.4.4.

Define a basis $(\mathcal{V}_1, \ldots, \mathcal{V}_8)$ of $\mathfrak{l}$ by

$$\begin{aligned} &\mathcal{V}_1 = -3e_1^3,\ \mathcal{V}_2 = \frac{1}{2}(e_1^1 - e_3^3),\ \mathcal{V}_3 = \frac{1}{\sqrt{3}}e^2,\ \mathcal{V}_4 = \frac{1}{2\sqrt{3}}(e_1^1 - 2e_2^2 + e_3^3), \\ &\mathcal{V}_5 = \sqrt{3}e_2^3,\ \mathcal{V}_6 = e^3,\ \mathcal{V}_7 := -\sqrt{3}e_1^2,\ \mathcal{V}_8 := -e_1. \end{aligned} \tag{5.12}$$

Using the relations (5.11) we can show that the Iwasawa subalgebra $\mathfrak{l} \subset \mathfrak{g}$ with this basis has precisely the same nontrivial brackets (5.8) as the basis (T_a) of the Lie algebra obtained from each of the three dimensional reductions.

We have seen so far then that the three reductions of pure five-dimensional supergravity (SS, ST and TS) provide us with scalar manifolds $M^{(SS)}$, $M^{(ST)}$ and $M^{(TS)}$ which can all be identified with the group manifold L of an Iwasawa subgroup of $G_{2(2)}$, parametrized by $(x, \sigma, \phi, \tilde{\phi}, \zeta^0, \zeta^1, \tilde{\zeta}_0, \tilde{\zeta}_1)$. For each of these reductions the scalar manifold is equipped with a different left-invariant metric $g^{(\epsilon_1, \epsilon_2)}$ with signature

$$\mathrm{sign}(g) = (-\epsilon_1, +, -\epsilon_1, +, -\epsilon, -\epsilon_2, -\epsilon_2, -\epsilon)\ . \tag{5.13}$$

Hence, for SS reduction the metric is positive definite, while for ST and TS reductions we obtain split-signature metrics, albeit with a different distribution of signs.

In the case of the Riemannian symmetric space G/SO_4, we know that SO_4 is a max-

[2] Recall that any two Iwasawa subalgebras of $\mathfrak{g}$ are related to each other by conjugation.

imally compact subgroup of G, and so the Iwasawa decomposition is exact (Theorem 6.46 of [79]). In particular, this means that the Iwasawa subgroup L acts transitively on the symmetric space G/SO_4. Hence, we can identify *globally* the Riemannian manifold $(M^{(SS)}, g^{(SS)})$ obtained from SS reduction with the Riemannian symmetric space G/SO_4, or alternatively with the L-orbit $L \cdot o$ of the canonical base point $o \in G/SO_4$.

For the split-signature case, we need to study the pseudo-Riemannian real form $G/(SL_2 \cdot SL_2)$, which we turn to next.

5.2.3 The symmetric space $S = G_{2(2)}/(SL_2 \cdot SL_2)$

In order to describe the pseudo-Riemannian symmetric space appearing in the ST and TS reductions of five-dimensional supergravity, we introduce the $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_{ev} + \mathfrak{g}_{odd}$ of the Lie algebra $\mathfrak{g}$ with

$$\begin{aligned} \mathfrak{g}_{ev} &= \mathfrak{a} + \operatorname{span}\{e_3, e^3, e_1^2, e_2^1\} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2, \\ \mathfrak{g}_{odd} &= \operatorname{span}\{e_1, e_2, e^1, e^2, e_1^3, e_2^3, e_3^1, e_3^2\}. \end{aligned}$$

That this is indeed a $\mathbb{Z}_2$-grading of $\mathfrak{g}$ can be seen simply by using the Lie brackets (5.11). The two $\mathfrak{sl}_2$ factors are generated by the $\mathfrak{sl}_2$-triples

$$(\mathbf{h}^{(1)} = [e_1^2, e_2^1] = e_1^1 - e_2^2,\ \mathbf{e}^{(1)} = e_1^2,\ \mathbf{f}^{(1)} = e_2^1),$$

and

$$(\mathbf{h}^{(2)} = [e_3, e^3] = -e_1^1 - e_2^2 + 2e_3^3,\ \mathbf{e}^{(2)} = e_3,\ \mathbf{f}^{(2)} = e^3).$$

That is,

$$\left[\mathbf{h}^{(a)}, \mathbf{e}^{(a)}\right] = 2\mathbf{e}^{(a)}, \quad \left[\mathbf{h}^{(a)}, \mathbf{f}^{(a)}\right] = -2\mathbf{f}^{(a)}, \quad \left[\mathbf{e}^{(a)}, \mathbf{f}^{(a)}\right] = \mathbf{h}^{(a)},$$

and $\left[X^{(1)}, X^{(2)}\right] = 0$, for any $X^{(a)} \in \mathfrak{sl}_2^{(a)}$.

The corresponding pseudo-Riemannian symmetric space $S = G/G_{ev}$ is

$$S = \frac{G_{2(2)}}{SL_2 \cdot SL_2}.$$

Proposition 6. *The symmetric space S admits a G-invariant para-quaternionic-Kähler structure (g, Q), where the metric g is induced by a multiple of the Killing form. Hence (S, g, Q) is a para-quaternionic-Kähler manifold.*

Proof: The proof is given as Proposition 1 of [37]. □

We saw earlier that the standard Iwasawa subgroup $L \subset G$ acts transitively on the Riemannian symmetric space G/SO_4. However, the orbit $L \cdot o$ of the canonical base point $o \in S$ if not even open (in the topological sense). To see this we note that, given a subgroup $U \subset G$, the orbit $U \cdot o \subset S = G/G_{ev}$ of the canonical base point $o \in S$ is open iff $\mathfrak{g}_{ev} + \mathrm{Lie}(U) = \mathfrak{g}$, i.e. $\dim(U) = \dim(G/G_{ev})$. For an Iwasawa subalgebra $\mathfrak{l}' \subset \mathfrak{g}$, this is the case iff $\mathfrak{g}_{ev} \cap \mathfrak{l}' = 0$.

Clearly, for the Iwasawa subalgebra $\mathfrak{l}$ in Section 5.2.2, we see $\mathfrak{l} \cap \mathfrak{g}_{ev} \neq 0$, and so the orbit $L \cdot o \subset S$ is not open. This means that we cannot globally describe the pseudo-Riemannian symmetric space S simply by picking some representative in the standard Iwasawa subgroup $L \subset G$, as was the case for the Riemannian symmetric space G/SO_4. This is related to the failure of the Iwasawa decomposition to provide a global covering of G/H for H non-compact. However, it is still possible to find a decomposition of the form HL for an open subset $U \subset G$, thus providing a local parametrization of the symmetric space G/H. In this case, the orbit $L \cdot o$ of the canonical base point will be open.

This failure of the Iwasawa decomposition has important consequences for stationary supergravity solutions, which in many applications can be described by cosets G/H for H non-compact [35]. In [107] it was shown that solutions with regular event horizons correspond to complete geodesics contained within an open orbit of the Iwasawa subgroup, whereas those not contained in a single orbit lift to singular spacetimes. In [108] it was argued that elements of G for which the Iwasawa decomposition fails to hold correspond to the so-called 'active duality transformations' mapping BPS to non-BPS solutions.

5.2.4 Open orbits in the symmetric space

The Iwasawa decomposition is not unique. In fact, given some Iwasawa subgroup $L \subset G$, we can find a conjugate Iwasawa subgroup $L' = C_a(L) := aLa^{-1}$ for $a \in G$ with the same Lie algebra[3]. In the case where L' acts with open orbit in S, we obtain a left-invariant locally symmetric para-quaternionic-Kähler structure on $L' \cong L$ induced from the symmetric para-quaternionic-Kähler structure on S.

Our strategy in this section is to look for conjugate Iwasawa subgroups $L' = C_a(L)$ such that the orbit $M_1 = L' \cdot o$ is open in S, and then show that M_1 is isometrically covered by one of the scalar manifolds $M^{(ST)}$ or $M^{(TS)}$. We do this at the level of the algebra: we seek $a \in G$ such that the conjugate Iwasawa subalgebra $\mathfrak{l}' = \mathrm{Ad}_a(\mathfrak{l}) \subset \mathfrak{g}$ is transversal to $\mathfrak{g}_{ev}$.

To proceed, we first pick some element $a = \exp(\xi) \in G$, where $\xi \in \mathfrak{g}$, and calculate $X' = \mathrm{Ad}_a X = e^{\mathrm{ad}\,\xi} X$ for each $X \in \mathfrak{l}$. One could then compare a generic element of $\mathfrak{l}'$ with a generic element of $\mathfrak{g}_{ev}$ to see whether there exist any non-trivial members of their intersection. However, we already know that $\mathfrak{g}_{odd} \cap \mathfrak{g}_{ev} = 0$. Hence, if we can find some vector space isomorphism $\varphi : \mathfrak{l} \to \mathfrak{g}_{odd}$, then $\mathfrak{l}' \cong \mathfrak{g}_{odd}$ will be transversal to $\mathfrak{g}_{ev}$. To construct such a map, let us denote by $\pi : \mathfrak{g} \to \mathfrak{g}_{odd}$ the projection along $\mathfrak{g}_{ev}$, and by $\varphi : \mathfrak{l} \to \mathfrak{g}_{odd}$ the map

$$X \mapsto \pi(X').$$

If the vectors $\varphi(\mathcal{V}_b)$ are linearly independent, then φ is a vector space isomorphism.

From the isomorphism $\varphi : \mathfrak{l} \to \mathfrak{g}_{odd}$ we compute the left-invariant metric g_1 on $L \cong L'$ from the scalar product $\langle \cdot, \cdot \rangle_1 = \varphi^* \langle \cdot, \cdot \rangle_B$ on $\mathfrak{l}$, where $\langle \cdot, \cdot \rangle_B$ is the scalar product on $\mathfrak{g}_{odd}$ obtained by restricting $\frac{1}{8}B$ to $\mathfrak{g}_{odd}$. Here B is the usual Killing form on $\mathfrak{g}$, as defined in (2.31). It may still be the case that g_1 in the basis $(\mathcal{V}_a)$ does not correspond to either $g^{(ST)}$ or $g^{(TS)}$, which are both diagonal with respect to this basis. However, the metrics could still be equivalent if they are related by the action of some automorphism of L (up to a positive scale factor). This will lead us in the next section to analyse the automorphism group of L, or equivalently of $\mathfrak{l}$ (since L is connected).

[3] That is, $\mathfrak{l} = \mathrm{Lie}(L)$ and $\mathfrak{l}' = \mathrm{Lie}(L')$ have the same non-trivial brackets.

In what follows we use the basis of $\mathfrak{g}$ given by

$$b_1 = e_1^1 - e_2^2, \quad b_2 = e_2^2 - e_3^3, \quad b_3 = e_1^2, \quad b_4 = e_1^3, \quad b_5 = e_2^3, \quad b_6 = e_2^1, \quad b_7 = e_3^1,$$
$$b_8 = e_3^2 \ , \quad b_9 = e_1 \ , \quad b_{10} = e_2 \ , \quad b_{11} = e_3 \ , \quad b_{12} = e^1 \ , \quad b_{13} = e^2 \ , \quad b_{14} = e^3. \quad (5.14)$$

In this notation, $\mathfrak{l} = \mathrm{span}\{b_1, b_2, b_3, b_4, b_5, b_9, b_{13}, b_{14}\}$, while $\mathfrak{g}_{odd}$ we take to have the basis $(f_1, \ldots, f_8) := (b_9, b_{10}, b_{12}, b_{13}, b_4, b_5, b_7, b_8)$.

For future use, we note that the non-trivial scalar products $\langle \cdot, \cdot \rangle_B$ between elements of $\mathfrak{g}_{odd}$ are given by

$$\langle f_1, f_3 \rangle_B = \langle f_2, f_4 \rangle_B = 3, \quad \langle f_6, f_8 \rangle_B = \langle f_5, f_7 \rangle_B = 1, \tag{5.15}$$

from which we can read off the corresponding Gram matrix representing $\langle \cdot, \cdot \rangle_B$:

$$\mathcal{G} = \begin{pmatrix} 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} . \tag{5.16}$$

5.3 Automorphisms of the solvable algebra

In anticipation of what is to come, we now turn our attention to the automorphism group of the solvable Lie group L. Since L is connected, we have $\mathrm{Aut}(L) = \mathrm{Aut}(\mathfrak{l})$, and so we concentrate on finding automorphisms at the level of the Lie algebra $\mathfrak{l}$. Recall that $\mathfrak{l}$ has non-trivial brackets (5.8) with respect to the basis (T_a) defined in (5.7). Automorphisms of $\mathfrak{l}$ are characterized by the existence of an invertible linear map $\Lambda : \mathfrak{l} \to \mathfrak{l}$ satisfying

$$[\Lambda(X), \Lambda(Y)] = \Lambda\,[X, Y] \quad \forall\, X, Y \in \mathfrak{l}.$$

In fact, it will be convenient to use the dual formulation of the Lie algebra $\mathfrak{l}$ defined by the dual basis (θ^a) of $\mathfrak{l}^*$ given in (5.4), which has exterior algebra (5.6). The important point is that a map $\Lambda : \mathfrak{l} \to \mathfrak{l}$ is an automorphism of $\mathfrak{l}$ iff the dual map $\Lambda^* : \mathfrak{l}^* \to \mathfrak{l}^*$ satisfies

$$d\Lambda^*\theta = \Lambda^* d\theta \quad \forall\, \theta \in \mathfrak{l}^*.$$

For practical purposes, then, we want to look for all invertible linear maps Λ^* satisfying

$$d(\Lambda^*\theta)^a = -c^a_{bc}\,(\Lambda^*\theta)^b \wedge (\Lambda^*\theta)^c \quad \forall\, a = 1, \ldots, 8, \tag{5.17}$$

where c^a_{bc} are the structure constants of the algebra $\mathfrak{l}$.

We find that the most general automorphism of $\mathfrak{l}$ depends on eight real parameters $(a, \ldots, h)$, and is given by its action on the basis of $\mathfrak{l}^*$ via

$$\begin{aligned}
\Lambda^*(\xi_2) &= \xi_2, \\
\Lambda^*(\beta) &= \beta, \\
\Lambda^*(\alpha) &= b\alpha + c\beta, \\
\Lambda^*(\eta^0) &= d\xi_2 - \sqrt{3}d\beta + e\eta^0, \\
\Lambda^*(\eta^1) &= f\xi_2 + 2bd\alpha + \left(2cd - \frac{1}{\sqrt{3}}f\right)\beta - \sqrt{3}ce\eta^0 \\
&\quad + be\eta^1, \\
\Lambda^*(\xi_1) &= h\xi_2 - \frac{4}{\sqrt{3}}bf\alpha + \left(\frac{1}{\sqrt{3}}h - \frac{4}{\sqrt{3}}cf\right)\beta - \sqrt{3}c^2e\eta^0 \\
&\quad + 2bce\eta^1 + b^2e\xi_1, \\
\Lambda^*(\xi_0) &= g\xi_2 - 2bh\alpha + \left(\sqrt{3}g - 2ch\right)\beta - c^3e\eta^0 \\
&\quad + \sqrt{3}bc^2e\eta^1 + b^3e\xi_0 + \sqrt{3}b^2ce\xi_1, \\
\Lambda^*(\eta^2) &= b^3e^2\eta^2 + a\xi_2 - \left(4bdh + \frac{4}{\sqrt{3}}bf^2\right)\alpha
\end{aligned}$$

$$+\left(2\sqrt{3}dg - 4cdh + \frac{2}{\sqrt{3}}fh - \frac{4}{\sqrt{3}}cf^2\right)\beta$$
$$+\left(2\sqrt{3}ceh - 2\sqrt{3}c^2ef - 2c^3de - 2eg\right)\eta^0$$
$$+\left(2\sqrt{3}bc^2de + 4bcef - 2beh\right)\eta^1$$
$$+ 2b^3de\xi_0 + \left(2\sqrt{3}b^2cde + 2b^2ef\right)\xi_1.$$

The details of the proof that this gives the general form of automorphisms of $\mathfrak{l}^*$ is fairly straightforward, and we refer the reader to Theorem 1 of [37]. We can now read off the matrix $\mathcal{M}$ representing Λ^* with respect to the basis (θ^a), that is, $\Lambda^*(\theta^a) = \mathcal{M}^a{}_b\theta^b$:

$$\mathcal{M} = \begin{pmatrix} b^3e^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & d & f & g & h \\ m_{3,1} & 0 & b & 0 & 0 & 2\,bd & -2\,bh & -4/3\,\sqrt{3}bf \\ m_{4,1} & 0 & c & 1 & -\sqrt{3}d & m_{4,5} & \sqrt{3}g - 2\,ch & 1/3\,\sqrt{3}\,(h - 4\,cf) \\ m_{5,1} & 0 & 0 & 0 & e & -\sqrt{3}ce & -c^3e & -\sqrt{3}c^2e \\ m_{6,1} & 0 & 0 & 0 & 0 & be & \sqrt{3}bc^2e & 2\,bce \\ 2\,b^3de & 0 & 0 & 0 & 0 & 0 & b^3e & 0 \\ m_{8,1} & 0 & 0 & 0 & 0 & 0 & \sqrt{3}b^2ce & b^2e \end{pmatrix}, \quad (5.18)$$

where

$$m_{3,1} = -4\,b\left(dh + 1/3\,\sqrt{3}f^2\right), \quad m_{4,1} = 2\,\sqrt{3}dg - 4\,cdh + 2/3\,\sqrt{3}fh - 4/3\,\sqrt{3}cf^2,$$

$$m_{4,5} = 2\,cd - 1/3\,\sqrt{3}f\ , \quad m_{5,1} = 2\,\sqrt{3}ceh - 2\,\sqrt{3}c^2ef - 2\,c^3de - 2\,eg,$$

$$m_{6,1} = 2\,b\left(\sqrt{3}c^2de + 2\,cef - eh\right)\ , \quad m_{8,1} = 2\,b^2\left(\sqrt{3}cde + ef\right).$$

In order for Λ^* to be invertible, we require $\det(\mathcal{M}) = b^{10}e^6 \neq 0$, and so both b and e should be non-zero. This decomposes the eight-parameter family of automorphisms of $\mathfrak{l}$ into four connected components. Note that the matrices $\mathcal{M}$ with $a = c = d = f = g = h = 0$ and $b, e \in \{\pm 1\}$ form a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{Aut}(\mathfrak{l})$ acting by the diagonal matrices

$$\mathrm{diag}(-1,1,-1,1,-1,1,1,-1) \quad \text{and} \quad \mathrm{diag}(1,1,1,1,-1,-1,-1,-1).$$

We note finally that the matrix (5.18) is the transpose of the matrix representing the automorphism Λ of $\mathfrak{l}$ with respect to the basis (T_a).

5.4 Identifying the open orbits corresponding to ST and TS reductions

In this section, we determine the appropriate group elements $a \in G$ for which the conjugated Iwasawa subalgebra $\mathfrak{l}' = \mathrm{Ad}_a(\mathfrak{l}) \subset \mathfrak{g}$ is transversal to $\mathfrak{g}_{ev}$. This gives us an Iwasawa subgroup $L' \subset G$ with open orbit in S.

5.4.1 An open orbit corresponding to Time-Space reduction

We claim that the element $a = \exp \xi$ with $\xi = e^1 + e^1_3 = b_7 + b_{12} \in \mathfrak{g}$ defines an Iwasawa subalgebra $\mathfrak{l}' = \mathrm{Ad}_a(\mathfrak{l})$ transversal to $\mathfrak{g}_{ev}$.

To see this, compute $X' = \mathrm{Ad}_a X = e^{\mathrm{ad}\,\xi} X$ for every basis element $X \in \mathfrak{l}$. We have (Proposition 4 of [37])

$$\begin{aligned}
&b_1' = b_1 + b_{12} + b_7, \quad b_2' = b_2 + b_7, \quad b_3' = b_3 - b_7 + b_8 + b_{11} + b_{13},\\
&b_4' = b_4 - b_1 - b_2 + b_6 - b_7 - b_{10} - b_{12} + b_{14}, \quad b_5' = b_5 - b_6,\\
&b_9' = b_9 - 2b_1 - b_2 - 3b_7 + b_{11} - b_{12}, \quad b_{13}' = b_{13} - 3b_7 + 2b_{11},\\
&b_{14}' = b_{14} + 3b_6 - 2b_{10} - b_{12}.
\end{aligned}$$

From this we can read off the action of φ:

$$\begin{aligned}
&\varphi(b_1) = b_7 + b_{12}, \quad \varphi(b_2) = b_7, \quad \varphi(b_3) = -b_7 + b_8 + b_{13},\\
&\varphi(b_4) = b_4 - b_7 - b_{10} - b_{12}, \quad \varphi(b_5) = b_5, \quad \varphi(b_9) = b_9 - 3b_7 - b_{12},\\
&\varphi(b_{13}) = b_{13} - 3b_7, \quad \varphi(b_{14}) = -2b_{10} - b_{12}.
\end{aligned}$$

As a map from $\mathfrak{l}$ with basis $(\mathcal{V}_a)$ to $\mathfrak{g}_{odd}$ with basis (f_a), the matrix representing φ, that is $\varphi(\mathcal{V}_a) = A_{ab} f_b$, is given by

$$A = \begin{pmatrix} 0 & 3 & 3 & 0 & -3 & 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & \sqrt{3} & -\sqrt{3} \\ -1 & 0 & 1 & 0 & 0 & 0 & 3 & 0 \end{pmatrix}.$$

One can check that $\det(A) = 3 \neq 0$, and therefore the vectors $\varphi(\mathcal{V}_b)$ are linearly independent. Hence the Iwasawa subalgebra $\mathfrak{l}' \cong \mathfrak{g}_{odd}$ is transversal, and the orbit $M_1 = L' \cdot o \subset S$ is open in $S = G/G_{ev}$.

The Gram matrix $\mathcal{G}_1$ of the scalar product $\langle \cdot, \cdot \rangle_1 = \varphi^* \langle \cdot, \cdot \rangle_B$ on $\mathfrak{l}$ with respect to the basis $(\mathcal{V}_b)$ is given by

$$\mathcal{G}_1 = A \mathcal{G} A^T,$$

where $\mathcal{G}$ is the Gram matrix (5.16) of the scalar product $\langle \cdot, \cdot \rangle_B$ on $\mathfrak{g}_{odd}$ with respect to the basis (f_b). The resulting matrix is

$$\mathcal{G}_1 = \begin{pmatrix} -18 & -3 & 6\sqrt{3} & 0 & 0 & 0 & -12\sqrt{3} & -18 \\ -3 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \\ 6\sqrt{3} & 0 & 0 & 0 & 0 & -2\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & -2\sqrt{3} & 0 & 0 & 0 & 6\sqrt{3} & 3 \\ -12\sqrt{3} & 0 & 0 & 0 & -3 & 6\sqrt{3} & 0 & 0 \\ -18 & -\frac{3}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & 3 & 0 & -6 \end{pmatrix}. \tag{5.19}$$

In order to compare this with the metrics obtained from dimensional reduction, we seek an automorphism of $\mathfrak{l}$ which brings (5.19) to a diagonal matrix. We impose that $\mathcal{M}\mathcal{G}_1\mathcal{M}^T$ is diagonal, where $\mathcal{M}$ is the matrix (5.18) representing the transpose of a general automorphism of $\mathfrak{l}$. One finds that this constrains the parameters of $\mathcal{M}$ as

$$d = -\frac{1}{\sqrt{3}}, \quad f = -\frac{1}{2}, \quad h = g = 0, \quad a = -\frac{1}{6}, \quad c = -\frac{1}{2}, \quad b = \pm\frac{1}{2}, \quad e = \pm\frac{2}{\sqrt{3}} \; .$$

Thus there is a unique inner automorphism diagonalizing the Gram matrix (5.19), and precisely one such automorphism in each component of $\mathrm{Aut}(\mathfrak{l})$. The diagonalized Gram matrix is in all cases

$$\mathcal{G}_1^{\mathrm{diag}} = \frac{1}{2}\mathrm{diag}(-1, 1, -1, 1, -1, 1, 1, -1) \; , \tag{5.20}$$

which agrees, up to an overall (positive) scale factor, with the metric $g^{(TS)}$ of the scalar manifold $M^{(TS)}$ obtained by dimensional reduction with $\epsilon_1 = 1, \epsilon_2 = -1$.

To summarise, we have shown

Proposition 7. *The left-invariant metric $g^{(TS)} = g^{(1,-1)}$ on L obtained by time-space reduction of pure five-dimensional supergravity is related by a unique inner automorphism of L, combined with a scaling by $\frac{1}{2}$, to the left-invariant metric g_1 on L obtained from the open orbit $M_1 = L' \cdot o \subset S$, where $L' = C_a(L)$ and $a = \exp(e^1 + e^1_3) \in G$ is an Iwasawa subgroup of G.*

5.4.2 An open orbit corresponding to Space-Time reduction

We claim that the element $a = \exp \xi$ with $\xi = e^1 + e^2_3 = b_8 + b_{12} \in \mathfrak{g}$ defines an Iwasawa subalgebra $\mathfrak{l}'' = \mathrm{Ad}_a(\mathfrak{l})$ transversal to $\mathfrak{g}_{ev}$.

To see this, compute $X' = \mathrm{Ad}_a X = e^{\mathrm{ad}\,\xi} X$ for every basis element $X \in \mathfrak{l}$. We have [37]

$$b'_1 = b_1 - b_8 + b_{12}, \quad b'_2 = b_2 + 2b_8, \quad b'_3 = b_3 - b_7 + b_{11} + b_{13},$$

$$b'_4 = -b_3 + b_4 + b_6 + b_7 - b_{10} - b_{11} - b_{13} + b_{14},$$

$$b'_5 = -b_2 + b_5 - b_8, \quad b'_9 = -2b_1 - b_2 + b_9 - b_{12},$$

$$b'_{13} = -3b_7 + 2b_{11} + b_{13}, \quad b'_{14} = 3b_6 + 3b_7 - 2b_{10} - 2b_{11} - b_{13} + b_{14},$$

from which we find

$$\varphi(b_1) = -b_8 + b_{12}, \quad \varphi(b_2) = 2b_8, \quad \varphi(b_3) = -b_7 + b_{13},$$
$$\varphi(b_4) = b_4 + b_7 - b_{10} - b_{13}, \quad \varphi(b_5) = b_5 - b_8, \quad \varphi(b_9) = b_9 - b_{12},$$
$$\varphi(b_{13}) = b_{13} - 3b_7, \quad \varphi(b_{14}) = 3b_7 - 2b_{10} - b_{13},$$

corresponding to the matrix $\varphi(\mathcal{V}_a) = A_{ab} f_b$ given by

$$A = \begin{pmatrix} 0 & 3 & 0 & 3 & -3 & 0 & -3 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & -\sqrt{3} \\ 0 & -2 & 0 & -1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & \sqrt{3} & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

One can check that $\det(A) = -12 \neq 0$, and therefore the vectors $\varphi(\mathcal{V}_b)$ are linearly independent. Hence the Iwasawa subalgebra $\mathfrak{l}'' \cong \mathfrak{g}_{odd}$ is transversal, and the orbit $M_2 = L'' \cdot o \subset S$ is open in $S = G/G_{ev}$.

The Gram matrix $\mathcal{G}_2$ of the scalar product $\langle \cdot, \cdot \rangle_2 = \varphi^* \langle \cdot, \cdot \rangle_B$ on $\mathfrak{l}$ with respect to the basis $(\mathcal{V}_b)$ is given by

$$\mathcal{G}_2 = A \mathcal{G} A^T,$$

where $\mathcal{G}$ is the Gram matrix (5.16) of the scalar product $\langle \cdot, \cdot \rangle_B$ on $\mathfrak{g}_{odd}$ with respect to the basis (f_b). The resulting matrix is

$$\mathcal{G}_2 = \begin{pmatrix} 72 & 0 & 6\sqrt{3} & 0 & 0 & -36 & -12\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{3}{2} \\ 6\sqrt{3} & 0 & 0 & 0 & 0 & -2\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{3}{2} & -6 & 0 & 0 & 0 \\ -36 & 0 & -2\sqrt{3} & 0 & 0 & 12 & 6\sqrt{3} & 0 \\ -12\sqrt{3} & 0 & 0 & 0 & 0 & 6\sqrt{3} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & -6 \end{pmatrix} . \tag{5.21}$$

In order to compare this with the metrics obtained from dimensional reduction, we seek an automorphism of $\mathfrak{l}$ which brings (5.21) to a diagonal matrix. We impose that $\mathcal{M}\mathcal{G}_2\mathcal{M}^T$ is diagonal, where $\mathcal{M}$ is the matrix (5.18) representing the transpose of a general automorphism of $\mathfrak{l}$. One finds that this constrains the parameters of $\mathcal{M}$ as

$$f = 0, \quad d = \frac{1}{4\sqrt{3}}, \quad h = -\frac{1}{4}, \quad g = 0, \quad a = 0, \quad c = 0, \quad b = \pm 1, \quad e = \pm\frac{1}{2\sqrt{3}} .$$

Thus there is a unique inner automorphism diagonalizing the Gram matrix (5.21), and precisely one such automorphism in each component of $\mathrm{Aut}(\mathfrak{l})$. The diagonalized Gram matrix is in all cases

$$\mathcal{G}_2^{\mathrm{diag}} = \frac{1}{2}\mathrm{diag}(1,1,1,1,-1,-1,-1,-1) , \tag{5.22}$$

which agrees, up to an overall (positive) scale factor, with the metric $g^{(ST)}$ of the scalar manifold $M^{(ST)}$ obtained by dimensional reduction with $\epsilon_1 = -1, \epsilon_2 = 1$.

To summarise, we have shown:

Proposition 8. *The left-invariant metric $g^{(ST)} = g^{(-1,1)}$ on L obtained by space-time reduction of pure five-dimensional supergravity is related by a unique inner automorphism of L, combined with a scaling by $\frac{1}{2}$, to the left-invariant metric g_2 on L obtained* *from the open orbit* $M_2 = L'' \cdot o \subset S$*, where* $L'' = C_a(L)$ *and* $a = \exp(e^1 + e_3^2) \in G$ *is an Iwasawa subgroup of* G*.*

5.4.3 Disjoint open L-orbits on S

In the previous subsections we have constructed open orbits $M_i = L^{(i)} \cdot o$ $(i = 1, 2)$ of the canonical base point $o \in S = G/G_{ev}$ under subgroups $L^{(i)} = a_i L a_i^{-1}$, with $a_i \in G$, conjugate to the standard Iwasawa subgroup $L \subset G$. We can take a different point of view, however, and consider the L-orbit $L \cdot o^{(i)}$ of different points $o^{(i)} \in S$. We have

$$M_i = L^{(i)} \cdot o = a_i L a_i^{-1} o = a_i L o^{(i)} = L_{a_i}(L \cdot o^{(i)}),$$

where we have defined $o^{(i)} = a_i^{-1} o$, and $L_a : S \to S$ is the diffeomorphism given by the left G-action on S.

Proposition 9. *The open* L*-orbits* $L \cdot o^{(i)}$ *are disjoint.*

Proof: We refer to Proposition 14 of [37] for the proof, which relies on the assertion that if two such L-orbits, $L \cdot o^{(1)}$ and $L \cdot o^{(2)}$, are not disjoint then there exists some $a \in L$ such that the left-invariant metrics g_1 and g_2 on L are related by

$$g_2 = C^*_{a^{-1}} g_1,$$

i.e. by a specific automorphism of L. However, there exists no element $\mathcal{M}$ of $\mathrm{Aut}(\mathfrak{l}) \cong \mathrm{Aut}(L)$ such that $\mathcal{G}_1 = \mathcal{M}\mathcal{G}_2\mathcal{M}^T$, as can be shown by explicit calculation using the matrices (5.18), (5.19) and (5.21). □

Corollary 1. *The Iwasawa subgroup* $L \subset G$ *acts with at least two open orbits on* $S = G/(SL_2 \cdot SL_2)$*.*

5.5 Geometric structures on the Iwasawa subgroup

We end this chapter by exploring some of the geometric structures carried by the Iwasawa subalgebra $\mathfrak{l}$ equipped with the three metrics $g^{(\epsilon_1,\epsilon_2)}$ related to the different dimensional reductions of pure five-dimensional supergravity.

To be concrete, we use the basis (T_a) of $\mathfrak{l}$ given in (5.7), and equip $\mathfrak{l}$ with a scalar product $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle^{\epsilon_1,\epsilon_2}$ defined by the Gram matrix

$$\mathcal{G}^{\mathrm{diag}} = \mathrm{diag}(-\epsilon_1, 1, -\epsilon_1, 1, -\epsilon, -\epsilon_2, -\epsilon_2, -\epsilon), \tag{5.23}$$

with respect to the basis (T_a).

We define the skew-symmetric endomorphisms

$$\begin{aligned} J_1 &= \epsilon_2 U^2 \wedge V_2 - B \wedge A + \epsilon\frac{\sqrt{3}}{2}U^1 \wedge U^0 - \epsilon_2\frac{1}{2}U^1 \wedge V_1 \\ &\quad +\epsilon_2\frac{1}{2}U^0 \wedge V_0 + \epsilon\frac{\sqrt{3}}{2}V_1 \wedge V_0 \,, \qquad (5.24) \\ J_2 &= \epsilon_2\frac{\sqrt{3}}{2}U^1 \wedge V_2 + \epsilon\frac{1}{2}V_0 \wedge V_2 - \frac{1}{2}U^0 \wedge U^2 - \epsilon_1\frac{\sqrt{3}}{2}V_1 \wedge U^2 \\ &\quad -\frac{1}{2}U^1 \wedge A - \epsilon_1\frac{\sqrt{3}}{2}V_0 \wedge A - \frac{\sqrt{3}}{2}U^0 \wedge B + \epsilon_1\frac{1}{2}V_1 \wedge B \,, \qquad (5.25) \\ J_3 &= \epsilon_2\frac{1}{2}U^0 \wedge V_2 - \epsilon\frac{\sqrt{3}}{2}V_1 \wedge V_2 - \epsilon_1\frac{\sqrt{3}}{2}U^1 \wedge U^2 + \frac{1}{2}V_0 \wedge U^2 \\ &\quad -\frac{\sqrt{3}}{2}U^0 \wedge A + \epsilon_1\frac{1}{2}V_1 \wedge A - \epsilon_1\frac{1}{2}U^1 \wedge B - \frac{\sqrt{3}}{2}V_0 \wedge B \,, \qquad (5.26) \\ \tilde{J}_1 &= -\epsilon_2 U^2 \wedge V_2 - B \wedge A + \epsilon\frac{\sqrt{3}}{2}U^1 \wedge U^0 - \epsilon_2\frac{1}{2}U^1 \wedge V_1 \\ &\quad +\epsilon_2\frac{1}{2}U^0 \wedge V_0 + \epsilon\frac{\sqrt{3}}{2}V_1 \wedge V_0 \,, \qquad (5.27) \end{aligned}$$

where we have used the notation of Section 4.4 to write $J_\alpha \in \mathrm{End}(TM)$ as bivectors.

One can check that the J_α $(\alpha = 1, 2, 3)$ are pairwise anti-commuting, and satisfy the following relations:

$$(J_1)^2 = \epsilon_1 \mathrm{Id} \,, \quad (J_2)^2 = \epsilon_2 \mathrm{Id} \,, \quad (J_3)^2 = \epsilon_3 \mathrm{Id} := \epsilon \mathrm{Id} \,, \quad J_3 = J_1 J_2 \,.$$

Hence, the endomorphisms J_α define left-invariant almost ϵ_α-complex structures on L. For $\epsilon_\alpha = -1$ we have complex structures, whereas for $\epsilon_\alpha = 1$ we have para-complex structures. Note moreover that $\tilde{J}_1$, which only differs from J_1 via its action on the subspace spanned by $\{U^2, V_2\}$, is ϵ_1-complex. We put $Q := \mathrm{span}\{J_\alpha | \alpha = 1, 2, 3\}$, which at this stage is an almost ϵ-hypercomplex structure.

In this section we want to prove that $(L, \bar{g}, Q)$ is an ϵ-quaternionic-Kähler manifold with left-invariant ϵ-quaternionic structure Q, and that J_1 and $\tilde{J}_1$ are integrable left-invariant ϵ_1-complex structures on L.

The integrability of the structures J_1 and $\tilde{J}_1$ is proven by computation of the corresponding Nijenhuis tensor (2.17), which can be shown to vanish for $J = J_1$ and $J = \tilde{J}_1$ by using the definitions (5.24) and (5.27), as well as the algebra (5.8). From Section 2.2, this shows that the structures J_1 and $\tilde{J}_1$ are integrable. One can also show that neither N_{J_2} nor N_{J_3} vanish.

In order to show that Q is an ϵ-quaternionic structure on L, we need to check that Q is parallel with respect to the Levi-Civita connection D. For the case of pure five-dimensional supergravity the group fiber coincides with the whole scalar manifold, so that the Levi-Civita connection can be read simply from (4.39):

$$\begin{aligned}
D_{V_2} &= U^2 \wedge V_2 + \frac{1}{2}U^0 \wedge V_0 + \frac{1}{2}U^1 \wedge V_1,\\
D_{U^2} &= 0,\\
D_A &= \frac{1}{\sqrt{3}}B \wedge A + \frac{1}{2}\epsilon V_0 \wedge V_1 + \frac{1}{2}\epsilon U^0 \wedge U^1 + \frac{1}{\sqrt{3}}\epsilon_2 U^1 \wedge V_1,\\
D_B &= 0,\\
D_{V_0} &= -\frac{1}{2}V_0 \wedge U^2 - \frac{1}{2}\epsilon_2 U^0 \wedge V_2 + \frac{\sqrt{3}}{2}V_0 \wedge B + \frac{1}{2}\epsilon_1 V_1 \wedge A,\\
D_{V_1} &= -\frac{1}{2}V_1 \wedge U^2 - \frac{1}{2}\epsilon U^1 \wedge V_2 + \frac{1}{2\sqrt{3}}V_1 \wedge B\\
&\quad -\frac{1}{2}V_0 \wedge A - \frac{1}{\sqrt{3}}\epsilon_1 U^1 \wedge A,\\
D_{U^0} &= -\frac{1}{2}U^0 \wedge U^2 + \frac{1}{2}\epsilon V_0 \wedge V_2 - \frac{\sqrt{3}}{2}U^0 \wedge B + \frac{1}{2}U^1 \wedge A,\\
D_{U^1} &= -\frac{1}{2}U^1 \wedge U^2 + \frac{1}{2}\epsilon_2 V_1 \wedge V_2 - \frac{1}{2\sqrt{3}}U^1 \wedge B\\
&\quad -\frac{1}{2}\epsilon_1 U^0 \wedge A + \frac{1}{\sqrt{3}}V_1 \wedge A.
\end{aligned} \tag{5.28}$$

We can now compute DJ_α for each of the skew-symmetric endomorphisms $J_\alpha \in Q$. Using the explicit expressions (5.24)–(5.26) and (5.28), we find

$$\begin{aligned}
[D_X, J_1] &= \hat{\alpha}(X)J_2 + \hat{\beta}(X)J_3,\\
[D_X, J_2] &= \epsilon\,\hat{\alpha}(X)J_1 + \hat{\gamma}(X)J_3,\\
[D_X, J_3] &= \epsilon_2\,\hat{\beta}(X)J_1 + \epsilon_1\,\hat{\gamma}(X)J_2,
\end{aligned} \tag{5.29}$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in \mathfrak{l}^*$ are 1-forms related to the basis (5.3) of $\mathfrak{l}^*$ by

$$\hat{\alpha} = -\frac{1}{2}\eta^0 - \frac{\sqrt{3}}{2}\xi_1, \quad \hat{\beta} = -\frac{\sqrt{3}}{2}\epsilon_1\,\eta^1 - \frac{1}{2}\xi_0, \quad \hat{\gamma} = \frac{1}{2}\epsilon_2\,\eta^2 - \frac{\sqrt{3}}{2}\alpha. \tag{5.30}$$

This shows that the ϵ-quaternionic structure Q is parallel with respect to the Levi-Civita connection. Hence, $(L, \bar{g}, Q)$ is an ϵ-quaternionic-Kähler manifold.

Chapter 6

Five-dimensional black string solutions

The five-dimensional Lagrangian (4.1) describing $\mathcal{N} = 2$ supergravity coupled to $n_V^{(5)}$ vector multiplets gives rise to the three-dimensional Lagrangian (4.3) upon dimensional reduction. The aim of this chapter is to look for a particular class of solutions to the five-dimensional theory which have one time-like and one space-like isometry, thus enabling us to use the dimensional reduction results. The main idea is that instanton solutions of the dimensionally-reduced Euclidean-signature supergravity theory can be lifted to solitonic solutions of the original five-dimensional supergravity theory.

The main result of this chapter is the construction of static non-extremal black strings in $\mathcal{N} = 2$ supergravity coupled to vector multiplets. We refrain from working in a specific model (i.e. choice of Hesse potential for the five-dimensional theory) and try to make the discussion as generic as possible.

We begin in Section 6.1 by identifying the class of solutions (static, magnetically charged black strings) that we want to look for, and derive the relevant three-dimensional field equations in Section 6.2. We then analyse the three-dimensional Einstein equations (Section 6.3) and scalar equations of motion (Section 6.4), thus determining the form of the three-dimensional instanton solutions. We then lift these instantons to five dimensions in Section 6.5 and analyse their properties. We also investigate the structure of BPS and non-BPS extremal solutions. Finally, in Section 6.6, we consider the lift of our instantons to four dimensions, and identify new non-extremal generalisations of small black holes.

The work in this chapter appeared in the publication [49] by the author.

6.1 The ansätze

We look for solutions to the five-dimensional field equations coming from (4.1) which:

(i) admit "stationary configurations" in the sense of Definition 19.

(ii) admit a space-like Killing vector field $\tilde{K}$ which commutes with K and with respect to which $\pounds_{\tilde{K}}\Phi = 0$ for any scalar or field strength Φ (see Definition 19). Orbits of $\tilde{K}$ should be isomorphic to $\mathbb{R}$, i.e. $\tilde{K}$ generates translations along some space-like direction.

Choosing adapted coordinates $\{x^0, x^4\}$ such that the Killing vectors K and $\tilde{K}$ act simply as translations (we shan't for the moment specify which of x^0 and x^4 correspond to time-like and which to space-like directions), the general Kaluza-Klein form of the five-dimensional metric is given by (4.2), which we reproduce here for convenience:

$$ds^2_{(5)} = -\epsilon_1 e^{2\sigma}(dx^0 + \mathcal{A}^0)^2 - \epsilon_2 e^{2\phi-\sigma}(dx^4 + B)^2 + e^{-2\phi-\sigma} ds^2_{(3)}. \tag{6.1}$$

All fields in (6.1) depend only on the transverse coordinates x^μ, with $\mu = 1, 2, 3$. Since K is taken to be a time-like Killing vector, one of $\{x^0, x^4\}$ corresponds to a time-like direction. Hence, we concentrate on the case of time-space or space-time reduction, for which $\epsilon_2 = -\epsilon_1$.

In this chapter we will be interested in static field configurations, i.e. those for which the Killing vector K is hypersurface orthogonal (see Definition 20). We will relax this condition in Chapter 7 in order to find more general classes of stationary solutions. In the case of TS reduction staticity of the metric restricts $\mathcal{A}^0_4 = \mathcal{A}^0_\mu = 0$, whilst for ST reduction we require $\mathcal{A}^0_4 = B_\mu = 0$. In order to treat our two Killing vectors K and $\tilde{K}$ on the same footing, we likewise impose that $\tilde{K}$ be hypersurface orthogonal. This allows us to set the remaining components of the Kaluza-Klein 1-forms to zero. Hence, our metric ansatz (6.1) becomes

$$ds^2_{(5)} = -\epsilon_1 e^{2\sigma}(dx^0)^2 - \epsilon_2 e^{2\phi-\sigma}(dx^4)^2 + e^{-2\phi-\sigma} ds^2_{(3)}. \tag{6.2}$$

We focus on the case where the metric (6.2) describes a five-dimensional black 1-brane. From the dimension-counting arguments of Section 3.1.2 we deduce that a 1-brane (string) in five dimensions can be magnetically charged under a 1-form gauge field. Hence, we make the 'magnetic' ansatz for the $n_V^{(5)} + 1$ gauge fields

$$\mathcal{A}^i_{\hat{\mu}} dx^{\hat{\mu}} = \mathcal{A}^i_\mu dx^\mu, \tag{6.3}$$

i.e. we impose that the 'electric' components vanish, $\mathcal{A}^i_0 = \mathcal{A}^i_4 = 0$.

It is important that these conditions correspond to a "consistent truncation" of the field content in the sense Definition 21. We will show this explicitly in Chapter 7.

6.2 The three-dimensional action

For the class of static magnetically charged black 1-brane solutions that we focus on in this chapter, we can truncate (4.3) to obtain the following three-dimensional Lagrangian:

$$e^{-1}\mathcal{L}_3 = \frac{R}{2} - \hat{g}_{ij}(y)\partial_\mu y^i \partial^\mu y^j - (\partial\phi)^2 + e^{-2\phi-3\sigma}\hat{g}^{ij}(y)\partial_\mu \tilde{\zeta}_i \, \partial^\mu \tilde{\zeta}_j. \tag{6.4}$$

The three-dimensional field content consists of the $n_V^{(5)}$ original (unconstrained) scalar fields ϕ^x parametrising the PSR manifold; the $n_V^{(5)} + 1$ scalars $\tilde{\zeta}_i$ dual to the magnetic field strengths in three dimensions; and the two Kaluza-Klein scalars σ, ϕ coming from the metric decomposition[1]. This gives a total of $2n_V^{(5)} + 3$ scalar fields. In (6.4) we have combined the constrained scalar fields h^i parametrising the CASR manifold with the Kaluza-Klein scalar σ via $y^i = 6^{\frac{1}{3}} e^\sigma h^i$.

[1]Note that the scalars σ and ϕ in (6.4) differ from those in (4.3) in that we have avoided making the field redefinition (4.5).

The Lagrangian (6.4) can be further simplified by making the field redefinition

$$w^i = e^{-\phi-\frac{3}{2}\sigma}y^i, \quad \xi = \phi - \frac{3}{2}\sigma, \tag{6.5}$$

after which (6.4) becomes

$$e^{-1}\mathcal{L}_3 = \frac{R}{2} - \hat{g}_{ij}(w)\partial_\mu w^i\partial^\mu w^j + \hat{g}^{ij}(w)\partial_\mu\tilde{\zeta}_i\partial^\mu\tilde{\zeta}_j - \frac{1}{4}(\partial\xi)^2. \tag{6.6}$$

Note that this Lagrangian has no explicit dependence on ϵ_1 or ϵ_2, meaning that the subsector of fields relevant for static black string solutions is manifestly insensitive to whether we first reduce over space or over time.

In terms of the metric degrees of freedom (ξ, σ), the metric ansatz (6.2) reads

$$ds^2_{(5)} = e^{\xi+2\sigma}\left[-\epsilon_1 e^{-\xi}(dx^0)^2 - \epsilon_2 e^{\xi}(dx^4)^2\right] + e^{-2(\xi+2\sigma)}ds^2_{(3)}. \tag{6.7}$$

This makes it clear, as we shall confirm later, that ξ somehow encodes the 'non-extremality' of the solution. Recall from Section 3.1.2 that an extremal black p-brane has a $ISO(p,1)$ group of isometries acting on its worldvolume directions. For the line element (6.7) this is only the case if ξ vanishes identically.

Returning to (6.6), we can read off that the $2n_V^{(5)}+3$ scalars $(w^i, \tilde{\zeta}_i, \xi)$ parametrize the totally geodesic submanifold $S \subset \mathcal{M}_3$ of the full para-quaternionic-Kähler manifold appearing as the target space of the full reduced theory, equipped with the metric

$$g_S = \hat{g}_{ij}(w)dw^i dw^j - \hat{g}^{ij}(w)d\tilde{\zeta}_i d\tilde{\zeta}_j + \frac{1}{4}d\xi^2.$$

We see that the metric on $S = N \times \mathbb{R}$ is the product of a one-dimensional factor, parametrized by ξ, and the $2(n_V^{(5)}+1)$-dimensional manifold N with metric

$$g_N = \hat{g}_{ij}(w)dw^i dw^j - \hat{g}^{ij}(w)d\tilde{\zeta}_i\, d\tilde{\zeta}_j.$$

In fact, we can identify N with the cotangent bundle $N \cong T^*M$ of the PSR manifold M from the five-dimensional theory [49]. Moreover, since $g_M = \hat{g}_{ij}(w)dw^i dw^j$ is Hessian, it follows that g_N is a para-Kähler metric on N. The proof is similar to that in Section 3.6.1 to show that TM was para-Kähler. Full details can be found in Appendix A of [49].

Before moving on to look at the equations of motion coming from (6.6), we should take a step back and remind ourselves of how to relate our three-dimensional field content to our five-dimensional field content. This is the basis of "dimensional oxidation". For the metric, we already have the relation (6.7), while for the remaining fields we find

$$h^i = e^{\xi+2\sigma} w^i, \qquad F^i_{\mu\nu} = -\frac{1}{\sqrt{2}} \epsilon_{\mu\nu\rho}\, \hat{g}^{ij}(w) \partial^\rho \tilde{\zeta}_j. \tag{6.8}$$

6.3 The three-dimensional Einstein equations

Now that we have a field content and action for our three-dimensional theory, we can go about solving the relevant Euclidean field equations. The field configuration thus obtained will correspond to an instanton solution, which can be dimensionally lifted to a five-dimensional soliton using the relations (6.7) and (6.8) between the three-dimensional and five-dimensional field contents.

We start by solving the three-dimensional Einstein equations coming from (6.6). After taking a trace and back-substituting they read:

$$\frac{1}{2} R_{\mu\nu} - \hat{g}_{ij}(w) \partial_\mu w^i \partial_\nu w^j + \hat{g}^{ij}(w) \partial_\mu \tilde{\zeta}_i \, \partial_\nu \tilde{\zeta}_j - \frac{1}{4} \partial_\mu \xi \, \partial_\nu \xi = 0. \tag{6.9}$$

For now we will concentrate on looking for solutions describing a single static black string, and which therefore possess spherical symmetry in the transverse space.

Note that here we are imposing spherical symmetry at the level of the equations of motion. A slightly different approach, taken by, e.g. [109–111], is to impose spherical symmetry at the level of the action, and use this to perform a further dimensional reduction to a one-dimensional theory depending only on a single radial coordinate. A particular advantage of our approach, as we will see later, is that for extremal solutions one can dispense with the assumption of spherical symmetry and allow for the possibility of multi-centred black strings.

Any spherically symmetric line element in three (Euclidean) dimensions can be brought to the form [112]

$$ds^2_{(3)} = e^{4A(\tau)}d\tau^2 + e^{2A(\tau)}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{6.10}$$

where τ is a radial coordinate defined in the range $0 < \tau < \infty$. The asymptotic region here corresponds to $\tau \to 0$ while the near horizon limit corresponds to $\tau \to \infty$. Imposing spherical symmetry on the full field configuration means that the scalar fields $(w^i, \tilde{\zeta}_i, \xi)$ depend only on τ.

Plugging the metric ansatz (6.10) into (6.9) for $\mu, \nu \neq \tau$, and solving as in [49], we find

$$e^{A(\tau)} = \frac{c}{\sinh(c\tau)}, \tag{6.11}$$

where the constant c is chosen positive for concreteness. Hence we find that the three-dimensional line element (6.10) is given by

$$ds^2_{(3)} = \frac{c^4}{\sinh^4(c\tau)}d\tau^2 + \frac{c^2}{\sinh^2(c\tau)}d\Omega^2_2. \tag{6.12}$$

This is the metric of our instanton solution, which we will later lift to five dimensions.

Turning our attention to the remaining Einstein equations (6.9), namely those with $\mu = \nu = \tau$, we find the Hamiltonian constraint

$$c^2 - \hat{g}_{ij}(w)\dot{w}^i\dot{w}^j + \hat{g}^{ij}(w)\dot{\tilde{\zeta}}_i\dot{\tilde{\zeta}}_j - \frac{1}{4}\dot{\xi}^2 = 0, \tag{6.13}$$

where $\dot{}$ denotes differentiation with respect to τ.

In terms of the geometry of the target space S, the Hamiltonian constraint has the following interpretation. Consider some geodesic curve C on S parametrized by $\tau \in \mathbb{R}$. The tangent vector to C at some point $p \in S$ with coordinates $(w^i, \tilde{\zeta}_i, \xi)$ is given by $X_p = (\dot{w}^i, \dot{\tilde{\zeta}}_i, \dot{\xi})$. Then the Hamiltonian constraint (6.13) is just the statement that this tangent vector has constant norm, i.e. $g_S(X_p, X_p) = c^2$, which in turn tells us that the radial coordinate τ is an affine curve parameter. Curves C with $c^2 > 0$ are space-like, while those with $c = 0$ are null. We will see later that space-like geodesics on S

correspond to three-dimensional instantons which lift to non-extremal black strings in five dimensions, whereas null geodesics correspond to instantons which lift to extremal black strings.

We can write the metric (6.12) in a more 'revealing' form by introducing the new radial coordinate

$$\rho = \frac{ce^{c\tau}}{\sinh(c\tau)}, \tag{6.14}$$

which is no longer an affine coordinate for the geodesic curve C. The range $0 < \tau < \infty$ of the affine radial coordinate corresponds to the range $\infty > \rho > 2c$ for (6.14). In particular, we will see that the asymptotic region of the five-dimensional solution is situated at $\rho \to \infty$ while the outer horizon is situated at $\rho = 2c$. One can in fact use the coordinate ρ to continue the solution to the region $0 < \rho < 2c$ between the inner and outer horizons [47].

In terms of (6.14) the metric (6.12) takes the form

$$ds^2_{(3)} = d\rho^2 + W(\rho)\rho^2 d\Omega_2^2, \tag{6.15}$$

where

$$W := 1 - \frac{2c}{\rho} = e^{-2c\tau}, \tag{6.16}$$

is harmonic in the three-dimensional transverse space.

6.4 The three-dimensional scalars

We now turn our attention to the remainder of the three-dimensional field content, namely the $2n_V^{(5)} + 3$ scalar fields $(w^i, \tilde{\zeta}_i, \xi)$. Again, we impose spherical symmetry at the level of the equations of motion, so that they only depend on the coordinate τ. The goal is to solve the remaining field equations coming from (6.6) in order to determine the three-dimensional field configuration, and try to relate this geometrically to the scalar manifold S.

6.4.1 Determining ξ

We first consider the equation of motion for the scalar ξ. This is simply $\ddot{\xi} = 0$ and can be solved by $\xi(\tau) = a\tau + b$ for some integration constants a, b. Since ξ appears as a metric degree of freedom, we expect these constants to be fixed by considering suitable boundary conditions for the five-dimensional line element. In particular, we require that the line element becomes flat in the asymptotic ($\tau \to 0$) limit, and has a finite (non-zero) horizon area per unit length in the near horizon ($\tau \to \infty$) limit.

Looking at (6.7) in the $\tau \to 0$ limit, we see that both σ and ξ should tend to zero to ensure transverse asymptotic flatness, which fixes $b = 0$.

The horizon of the black string has topology $\mathbb{R} \times S^2$, with the $\mathbb{R}$ factor corresponding to the spatial worldvolume of the string. In order to have a finite horizon size per unit length of the string, we need the coefficients of both the S^2 and $\mathbb{R}$ factors in the metric to be finite in the limit $\tau \to \infty$.

The S^2 factor appearing in the five-dimensional line element can be read off by taking the $\tau \to \infty$ limit of (6.12) and plugging it into (6.7) to find

$$ds^2_{(5)} \supset (2c)^2 e^{-4\sigma(\tau) - 2(a+c)\tau} d\Omega^2_2.$$

In order that this integrates up to give a finite non-zero result as $\tau \to \infty$, the scalar field $\sigma(\tau)$ must have the asymptotic expansion

$$2\sigma(\tau) = 2\sigma_{\text{hor}} - (a + c)\tau \quad \text{as } \tau \to \infty.$$

Substituting this into the part of (6.7) containing the worldvolume directions we have

$$ds^2_{(5)} \supset e^{2\sigma_{\text{hor}} - c\tau} \left[-\epsilon_1 e^{-a\tau} (dx^0)^2 - \epsilon_2 e^{a\tau} (dx^4)^2 \right].$$

Precisely which factor corresponds to the spatial part of the horizon depends on whether we are considering a time-space ($\epsilon_1 = 1, \epsilon_2 = -1$) or space-time ($\epsilon_1 = -1, \epsilon_2 = 1$) reduction of the five-dimensional theory. For the former, where x^4 is a spatial direction, the requirement of finite (non-zero) horizon size imposes $a = c$, while for the latter, where x^0 is a spatial direction, we must take $a = -c$. Hence, in general, $a = \epsilon_1 c$, and we find

$$\xi(\tau) = \epsilon_1 c\tau. \tag{6.17}$$

We can thus understand in what sense the scalar ξ encodes the non-extremality of the spacetime solution.

We have seen, then, that the physical requirements that our metric describes an asymptotically flat black string with finite horizon area per unit length determines exactly the profile of ξ in terms of the parameter c, thereby reducing the number of independent integration constants by 2.

6.4.2 Determining $\tilde{\zeta}_i$

We now move on to the equations of motion for the scalars $\tilde{\zeta}_i$ dual to the three-dimensional field strengths. From (6.6) we find

$$\frac{d}{d\tau}\left(\hat{g}^{ij}(w)\dot{\tilde{\zeta}}_j\right) = 0,$$

which integrates to give

$$\dot{\tilde{\zeta}}_i = \hat{g}_{ij}(w)\tilde{p}^j, \tag{6.18}$$

where the $\tilde{p}^i$ are integration constants proportional to the magnetic charge of the solution under the five-dimensional gauge fields $\mathcal{A}^i_{\hat{\mu}}$. To see this, we use (6.8) to relate the three-dimensional scalars (6.18) to the non-zero components of the five-dimensional field strength. In particular, we find

$$F^i_{\theta\varphi} = -\frac{1}{\sqrt{2}}\,\tilde{p}^i\sin\theta. \tag{6.19}$$

Note that constant shifts in the scalars $\tilde{\zeta}_i$ simply correspond to gauge transformations of the five-dimensional gauge fields, and so further integrating (6.18) would not give rise to additional physical degrees of freedom.

Before moving on, we substitute (6.17) and (6.18) into the Hamiltonian constraint

(6.13), which now becomes

$$\frac{3}{4}c^2 - \hat{g}_{ij}(w)\left(\dot{w}^i\dot{w}^j - \tilde{p}^i\tilde{p}^j\right) = 0. \tag{6.20}$$

6.4.3 Determining w^i

Finally, the equations of motion for the scalars w^i read, after using (6.18) and the fact that the metric $\hat{g}_{ij}$ is Hessian,

$$\hat{g}_{ij}(w)\ddot{w}^j + \frac{1}{2}\partial_i\hat{g}_{jk}\left(\dot{w}^j\dot{w}^k - \tilde{p}^j\tilde{p}^k\right) = 0. \tag{6.21}$$

Due to the explicit dependence of (6.21) on $\hat{g}_{ij}$ and its derivatives, it is difficult to solve (6.21) in a model-independent way. For now we content ourselves with finding a class of explicit solutions which always contains the standard Reissner-Nordström black string.

Following [49] we first contract (6.21) with w^i, and use homogeneity of the metric $\hat{g}_{ij}$, along with the Hamiltonian constraint (6.20), to arrive at

$$\hat{g}_{ij}(w)w^i(\ddot{w}^j - c^2w^j) = 0. \tag{6.22}$$

We can obtain our class of universal solutions by setting $\ddot{w}^j - c^2w^j = 0$, from which we obtain

$$w^i(\tau) = A^i\cosh(c\tau) + \frac{B^i}{c}\sinh(c\tau), \tag{6.23}$$

where the A^i and B^i are integration constants, and we have chosen the prefactors for later convenience. We can find relations between the integration constants by ensuring that the solutions (6.23) still satisfy the Hamiltonian constraint (6.20) and the full equations of motion (6.22), which become, respectively,

$$\hat{g}_{ij}\left(c^2A^iA^j - B^iB^j + \tilde{p}^i\tilde{p}^j\right) = 0, \quad \text{and} \quad \partial_k\hat{g}_{ij}\left(c^2A^iA^j - B^iB^j + \tilde{p}^i\tilde{p}^j\right) = 0.$$

It is convenient at this point to introduce the quantities

$$p^i = B^i - cA^i, \quad \bar{p}^i = B^i + cA^i, \tag{6.24}$$

in terms of which the Hamiltonian constraint and equations of motion become

$$\hat{g}_{ij}(p^i\bar{p}^j - \tilde{p}^i\tilde{p}^j) = 0, \quad \partial_k\hat{g}_{ij}(p^i\bar{p}^j - \tilde{p}^i\tilde{p}^j) = 0. \tag{6.25}$$

It will turn out when we look at the full five-dimensional solution that the $p^i, \bar{p}^i$ encode the behaviour of the five-dimensional scalar fields at the inner and outer horizons respectively.

Writing the ansatz (6.23) in terms of the radial coordinate ρ defined in (6.14) we have

$$w^i(\rho) = \left(A^i + \frac{p^i}{\rho}\right)W^{-\frac{1}{2}} := \mathcal{H}^i(\rho)W^{-\frac{1}{2}}, \tag{6.26}$$

where W is given in (6.16), and the functions $\mathcal{H}^i(\rho)$ are harmonic in the three-dimensional transverse space.

This completes the determination of the three-dimensional instanton solution. To recap, the three-dimensional line element is given in terms of the affine coordinate τ by (6.12), and in terms of the isotropic radial coordinate ρ by (6.15). The scalar field ξ is fixed in terms of the parameter c by (6.17) or, using (6.16), by

$$\xi(\rho) = -\frac{1}{2}\epsilon_1 \log W(\rho). \tag{6.27}$$

The scalar fields $\tilde{\zeta}_i$ satisfy (6.18) and encode the magnetic charges of the five-dimensional solution via (6.19): in this sense we have already ‘dimensionally lifted’ these fields. Finally, the scalar fields w^i are given in terms of τ by (6.23) and in terms of ρ by (6.26). However, the w^i as they stand still depend on $2(n_V^{(5)} + 1)$ undetermined integration constants, which should be fixed in terms of the ‘physical charges’ of the solution. Given the relations (6.25) this will necessarily have to be achieved in a model-dependent fashion.

6.5 Five-dimensional solutions

In order to make contact with our initial goal of constructing black strings in five dimensions, we simply need to ‘dimensionally lift’ the three-dimensional instanton constructed in Section 6.4 using the relationship between the five-dimensional and three-dimensional field contents. In section 6.5.1 we derive the functional form of the five-dimensional line element describing static non-extremal black strings, and note some of its properties in section 6.5.2. In section 6.5.3 we then further analyse the solutions and determine the remaining integration constants in terms of the physical 'moduli' of the solution. Finally, in sections 6.5.4 and 6.5.5 we turn to the extremal limit and analyse the underlying geometrical structure of extremal BPS and non-BPS solutions in terms of data on the target manifold.

6.5.1 Non-extremal black strings

Using (6.8) and the hypersurface condition $H(h) = 1$, we find

$$e^{\xi+2\sigma} = H(w)^{-\frac{1}{3}} = H(\mathcal{H})^{-\frac{1}{3}} W^{\frac{1}{2}}. \tag{6.28}$$

In addition, we see from (6.27) that

$$e^{\xi} = W^{-\frac{1}{2}\epsilon_1}. \tag{6.29}$$

Substituting (6.28), (6.29), and the three-dimensional line element (6.15), into the five-dimensional line element (6.7) we find

$$ds^2_{(5)} = H(\mathcal{H})^{-\frac{1}{3}} \left(-W dt^2 + dy^2\right) + H(\mathcal{H})^{\frac{2}{3}} \left(\frac{d\rho^2}{W} + \rho^2 d\Omega_2^2\right), \tag{6.30}$$

where $\{t, y\}$ are the worldvolume directions of the string. Note that transverse asymptotic flatness of the metric (6.30) implies that $H(\mathcal{H}) \to 1$ as $\rho \to \infty$. We can also read off from (6.30) that the black string has two horizons. An outer (event) horizon at $\rho = 2c$, and an inner (Cauchy) horizon at $\rho = 0$. As $c \to 0$, we see that both horizons 'coalesce', confirming the identity of c as the non-extremality parameter.

The constrained five-dimensional scalar fields $h^i(\rho)$ are found from (6.8) to be

$$h^i(\rho) = H(\mathcal{H})^{-\frac{1}{3}} \mathcal{H}^i(\rho). \tag{6.31}$$

Looking at the $\rho \to \infty$ limit of this we find that the integration constants A^i are simply given by the asymptotic values of the scalar fields $h^i_\infty \equiv h^i(\infty)$. Asymptotic flatness of (6.30) is then just the statement that $H(h_\infty) = 1$, i.e. that the scalars are restricted to the hypersurface $\{H = 1\}$ defining the original PSR manifold. This gives us $n_V^{(5)}$ independent integration constants associated to the boundary values of the physical scalars parametrizing the five-dimensional target manifold.

To interpret the remaining integration constants (which can be taken either as B^i, p^i or $\bar{p}^i$), we look at the behaviour of the scalars $h^i(\rho)$ at the outer and inner horizons:

$$h^i \xrightarrow[\rho \to 2c]{} H(\bar{p})^{-\frac{1}{3}} \bar{p}^i, \qquad h^i \xrightarrow[\rho \to 0]{} H(p)^{-\frac{1}{3}} p^i.$$

This motivates calling p^i (resp. $\bar{p}^i$) the inner (resp. outer) 'horizon charges'. In the full solution we should be able to use (6.25) to determine these in terms of the asymptotic charges h^i_∞ and $\tilde{p}^i$. We will come back to this point in Subsection 6.5.3.

To recap, then, the line element of our five-dimensional solution is given by (6.30), where the functions $\mathcal{H}^i(\rho)$ are harmonic in the space transverse to the worldvolume of the string. The five-dimensional field strengths are given in terms of the charges $\tilde{p}^i$ by (6.19). Finally, the constrained scalar fields, which satisfy $H(h) = 1$, are given in terms of the harmonic functions $\mathcal{H}^i$ by (6.31). This data determines the functional form of our five-dimensional magnetic black string. The constants p^i (equivalently $\bar{p}^i$) appearing in $\mathcal{H}^i$ should be chosen such that the equations (6.25) are satisfied, subject to the restriction that the line element (6.30) be regular outside the outer horizon.

6.5.2 Properties of the solution

Before continuing, let us calculate some properties of the solution constructed in Subsection 6.5.1. The entropy of the inner and outer horizons are, respectively

$$S_- = \pi H(p)^{\frac{2}{3}}, \qquad S_+ = \pi H(\bar{p})^{\frac{2}{3}},$$

while the temperature associated to each horizon is

$$T_- = \frac{\sqrt{2c}}{4\pi} H(p)^{-\frac{1}{2}}, \quad T_+ = \frac{\sqrt{2c}}{4\pi} H(\bar{p})^{-\frac{1}{2}},$$

which vanishes in the extremal limit $c \to 0$.

Using the normalization of Section 3.1.3 (see also [111]) we find that the tension of the black string is

$$\mathcal{T} = \frac{1}{2} h_i(\infty) \bar{p}^i \equiv \frac{1}{2} c_{ijk} h^i_\infty h^j_\infty \bar{p}^k, \tag{6.32}$$

while the magnetic central charge is

$$\mathcal{Z}_m = h_i(\infty) \tilde{p}^i \equiv c_{ijk} h^i_\infty h^j_\infty \tilde{p}^k. \tag{6.33}$$

6.5.3 The remaining conditions

In order that our ansatz (6.23) for the scalars w^i gives rise to a spacetime solution corresponding to a five-dimensional black string, we need to ensure that the integration constants satisfy the model-dependent conditions (6.25), which we recall correspond to the Hamiltonian constraint and the w^i equations of motion.

Our strategy here is to look for configurations of charges which allow us to construct regular black string solutions in a relatively model-independent fashion.

To see how this might work, let us focus on the Hamiltonian constraint, which we rewrite in terms of $\bar{p}^i$, the 'horizon charge' associated with the outer horizon:

$$\hat{g}_{ij}(\mathcal{H}) \left(\bar{p}^i \bar{p}^j - 2c h^i_\infty \bar{p}^j - \tilde{p}^i \tilde{p}^j \right) = 0. \tag{6.34}$$

Completing the square, this becomes

$$\hat{g}_{ij}(\mathcal{H}) \left(\bar{p}^i - c h^i_\infty \right) \left(\bar{p}^j - c h^j_\infty \right) = \hat{g}_{ij}(\mathcal{H}) \left(c^2 h^i_\infty h^j_\infty + \tilde{p}^i \tilde{p}^j \right). \tag{6.35}$$

If we consider a diagonal model, i.e. choose a Hesse potential $H(h)$ such that $\hat{g}_{ij}$ and its derivatives (when evaluated on the solution) are diagonal, then (6.35) can be solved by

$$\bar{p}^i - ch^i_\infty = \pm\sqrt{c^2(h^i_\infty)^2 + (\tilde{p}^i)^2}\,, \tag{6.36}$$

for $i = 0, \ldots, n_V^{(5)}$. This determines the integration constants ('horizon charges') $\bar{p}^i$, or equivalently $p^i = \bar{p}^i - 2ch^i_\infty$, in terms of the quantities h^i_∞ and $\tilde{p}^i$. The sign in (6.36) should be chosen such that the line element (6.30) is 'regular' in the sense of having no metric singularities outside the event horizon. In particular, we should require that the $\mathcal{H}^i(\rho)$ do not vanish for any $\rho > 2c$. Writing

$$\mathcal{H}^i(\rho) = h^i_\infty + \frac{p^i}{\rho} = \frac{(\rho - 2c)h^i_\infty + \bar{p}^i}{\rho},$$

we see that the harmonic function remains non-zero for $\rho > 2c$ provided $\text{sign}(h^i_\infty) = \text{sign}(\bar{p}^i)$. Hence, if h^i_∞ is positive we should choose the $+$ sign in (6.36), whereas if it is negative we should choose the $-$ sign. One can show, moreover, that with this sign choice the functions $\mathcal{H}^i(\rho)$ remain non-zero up to the inner horizon at $\rho = 0$.

Hence, for diagonal models we have identified a class of solutions depending on $2n_V^{(5)} + 2$ independent parameters: the $n_V^{(5)} + 1$ magnetic charges $\tilde{p}^i$; the $n_V^{(5)} + 1$ asymptotic values of the scalar fields h^i_∞ subject to the constraint $H(h_\infty) = 1$; and the non-extremality parameter c.

One method of constructing solutions for a completely general class of models is to impose the condition (6.36) and determine what restrictions the off-diagonal components of (6.35) put on the various parameters. First, we define $\beta^i \equiv \text{sign}(h^i_\infty)\sqrt{c^2(h^i_\infty)^2 + (\tilde{p}^i)^2}$. Then substituting (6.36) into (6.35) we find

$$\hat{g}_{ij}(\mathcal{H})\left(c^2 h^i_\infty h^j_\infty + \tilde{p}^i\tilde{p}^j - \beta^i\beta^j\right) = 0. \tag{6.37}$$

Certainly for diagonal elements, i.e. $i = j$, the LHS of this expression vanishes identically. For off-diagonal elements, $i \neq j$, a model-independent way of satisfying (6.37) can be found by imposing the stronger condition that

$$c^2 h^i_\infty h^j_\infty + \tilde{p}^i\tilde{p}^j - \beta^i\beta^j = 0,$$

for each i, j with $\hat{g}_{ij}(\mathcal{H}) \neq 0$, which can be shown to be equivalent to

$$h^i_\infty \tilde{p}^j - h^j_\infty \tilde{p}^i = 0.$$

Hence, the ratios

$$\frac{h^i_\infty}{h^j_\infty} = \frac{\tilde{p}^i}{\tilde{p}^j} \equiv \mu^i_j,$$

are constant. In the generic case, where all elements of $\hat{g}_{ij}(\mathcal{H})$ are non-zero, this implies that all h^i_∞ and $\tilde{p}^i$ are proportional to each other, and can be written simply as multiples of, say, h^0_∞ and $\tilde{p}^0$. In particular, we write $h^i_\infty = \mu^i h^0_\infty$, $\tilde{p}^i = \mu^i \tilde{p}^0$. Note that h^0_∞ is not itself an independent constant, but should be chosen such that $H(h_\infty) = 1$. Furthermore, the ansatz (6.36) fixes $\bar{p}^i = \mu^i \bar{p}^0$, and so all of the functions $\mathcal{H}^i(\rho)$ should be proportional to $\mathcal{H}^0(\rho)$. This in turn tells us that $H(\mathcal{H}) \propto (\mathcal{H}^0)^3$, and so the scalar fields (6.31) are constant and equal to their asymptotic values.

Going back to the five-dimensional line element (6.30), we see that in this case the solution takes the form

$$ds^2_{(5)} = \frac{1}{\mathcal{H}^0}\left(-W dt^2 + dy^2\right) + (\mathcal{H}^0)^2 \left(\frac{d\rho^2}{W} + \rho^2 d\Omega^2_2\right),$$

which is simply the Reissner-Nordström (RN) black string [88]. Hence, for the generic case, our ansatz (6.23) with (6.36) produces the magnetically-charged five-dimensional RN black string.

Between these two extremes ('diagonal' models vs. generic models) there are those for which $\hat{g}_{ij}$ and its derivatives admit a block decomposition. In this case, the ansatz (6.36) restricts each of the integration constants within a block to be proportional to one another. For k blocks we therefore obtain a solution depending on k harmonic functions, as in [47, 48].

It turns out, however, that there exist a large class of models for which (6.36) is not the most general ansatz one can make [43]. In particular, suppose that we can find some constant matrix $R^i{}_j \neq \pm\delta^i_j$ which leaves $\hat{g}_{ij}$ invariant, i.e.

$$R^T \hat{g} R = \hat{g},$$

as matrices. Then we can solve (6.35) by setting

$$\bar{p}^i - ch^i_\infty = \pm R^i{}_j \sqrt{c^2(h^j_\infty)^2 + (\tilde{p}^j)^2}, \tag{6.38}$$

where the sign is chosen such that $\text{sign}(h^i_\infty) = \text{sign}(\bar{p}^i)$, and by taking the integration constants to again be proportional to each other within each block of the metric. However, we now see that having all of h^i_∞ and $\tilde{p}^i$ proportional to each other does not necessarily imply that all of the $\bar{p}^i$ should be proportional, and hence that the scalar fields $h^i(\rho)$ need not be constant.

Hence, for models admitting an 'R-matrix', we can find solutions depending on a reduced number of integration constants but which nevertheless have non-constant scalar fields.

6.5.4 Extremal black strings

In Subsection 6.5.1 we have shown how to construct static non-extremal magnetic black strings to five-dimensional $\mathcal{N} = 2$ supergravity theories, which depend upon some non-extremality parameter c. By taking the extremal limit $c \to 0$ of such solutions, we are able to construct both BPS and non-BPS extremal black strings. In this case the function $W(\rho)$ tends to unity and the line element (6.30) becomes

$$ds^2_{(5)} = H(\mathcal{H})^{-\frac{1}{3}} \left(-dt^2 + dy^2\right) + H(\mathcal{H})^{\frac{2}{3}} \left(d\rho^2 + \rho^2 d\Omega^2_2\right), \tag{6.39}$$

which has a single horizon at $\rho \equiv \frac{1}{\tau} = 0$. Moreover, the Hamiltonian constraint (6.34) becomes simply

$$\hat{g}_{ij}(\mathcal{H}) \left(p^i p^j - \tilde{p}^i \tilde{p}^j\right) = 0. \tag{6.40}$$

For a generic model, this can be solved by taking $p^i = \pm\tilde{p}^i$, where the sign is chosen such that the harmonic function $\mathcal{H}^i(\rho)$ remains non-zero for $\rho > 0$, i.e. we choose $\text{sign}(p^i) = \text{sign}(h^i_\infty)$. However, if the metric admits a constant 'R-matrix', then we can solve (6.40) by taking

$$p^i = R^i{}_j \tilde{p}^j.$$

In each case, we have

$$h^i(\rho) \to H(p)^{-1/3} p^i \quad \text{for} \quad \rho \to 0,$$

which now depends only on the charges $\tilde{p}^i$ of the solution. This is the attractor mechanism for extremal black strings [111].

Let us look now at how different choices of an R-matrix lead to different classes of extremal solutions. The tension (6.32) of the extremal black string is

$$\mathcal{T} = \frac{1}{2} h_i(\infty) p^i = \frac{1}{2} h_i(\infty) R^i{}_j \tilde{p}^j,$$

so that

$$\mathcal{T} \mp \frac{1}{2} \mathcal{Z}_m = \frac{1}{2} h_i(\infty) \left(R^i{}_j \mp \delta^i_j \right) \tilde{p}^j.$$

Hence we see that solutions with $R^i{}_j = \pm \delta^i_j$ are BPS, i.e. $\mathcal{T} = \frac{1}{2} |\mathcal{Z}_m|$, while those with $R^i{}_j \neq \pm \delta^i_j$ are non-BPS. We see that if a given model (i.e. choice of Hesse potential) admits the existence of a non-trivial R-matrix, we can explicitly construct extremal non-BPS black string solutions. In [49] this is done for the case of the ST^2 and STU models.

In the consideration so far, we have obtained single-centred black strings from considering the extremal limit of the non-extremal solutions constructed in Subsection 6.5.1. However, looking back at the metric ansatz (6.7), we see that concentrating on extremal solutions should be equivalent to truncating the field ξ, since this enhances the isometry group of the worldvolume directions to the full $\mathrm{Iso}(1,1)$. Hence, we proceed by setting $\xi = 0$ at the level of the action (6.6). This is a consistent truncation, for which the target space of the three-dimensional theory reduces to the para-Kähler submanifold $N = T^*M \subset \mathcal{M}_3$.

The equations of motion for w^i (making no assumptions about the three-dimensional geometry) are given by

$$\Delta w^i + \hat{\Gamma}^i_{jk} \left(\partial_\mu w^j \partial^\mu w^k - \hat{g}^{jl} (\partial_\mu s_l) \hat{g}^{km} (\partial^\mu s_m) \right) = 0, \qquad (6.41)$$

where Δ is the three-dimensional Laplacian and

$$\hat{\Gamma}^i_{jk} = \frac{1}{2}\hat{g}^{il}(\partial_l \hat{g}_{jk}), \tag{6.42}$$

are the connection coefficients for the Hessian metric $\hat{g}$ on the PSR manifold.

Then, if $\hat{g}$ admits an R-matrix, we can make the 'extremal instanton ansatz'

$$\partial_\mu w^i = R^i{}_j \hat{g}^{jk} \partial_\mu \tilde{\zeta}_k, \tag{6.43}$$

after which the equations of motion (6.41) become

$$\Delta w^i = 0.$$

Consistency with the three-dimensional Einstein equations (6.9) requires moreover that $R_{\mu\nu} = 0$, i.e. that the three-dimensional spacetime be Ricci flat, and hence flat.

We can solve the w^i equations of motion in this case with arbitrary multi-centred harmonic functions

$$w^i(\vec{x}) = \mathcal{H}^i(\vec{x}) \equiv h^i_\infty + \sum_n \frac{p^i_n}{|\vec{x} - \vec{x}_n|},$$

where $\vec{x} = (x^\mu) = (x^1, x^2, x^3)$. In this manner we obtain static multi-centred black string solutions corresponding to parallel strings with horizons located at $\vec{x}_n$ in the transverse space. Again, solutions with $R = \pm\mathbb{1}$ correspond to BPS configurations, while those with $R \neq \pm\mathbb{1}$ correspond to non-BPS configurations.

We finish this section by returning to the extremal instanton ansatz (6.43) and elucidating its geometrical meaning.

6.5.5 The geometry of extremal solutions

The pseudo-Riemannian manifold (N, g_N) which is the target space relevant for extremal black strings admits an integrable para-complex structure J with respect to which g_N is para-Kähler.

Given a frame $F = (\partial_{w^i}, \partial_{\tilde{\zeta}_i})$ for TN, and a co-frame $F^* = (dw^i, d\tilde{\zeta}_i)$ for T^*N, we define an endomorphism $J \in \mathrm{End}(TN)$ by

$$J = \hat{g}^{ij}(w)\partial_{w^i} \otimes d\tilde{\zeta}_j + \hat{g}_{ij}(w)\partial_{\tilde{\zeta}_i} \otimes dw^j \,.$$

This has the matrix representation

$$J = \begin{pmatrix} 0 & \hat{g} \\ \hat{g}^{-1} & 0 \end{pmatrix},$$

with respect to the frame F, and so J defines a para-complex structure on N. The dual endomorphism $J^* \in \mathrm{End}(T^*N)$ acts on the co-frame F^* as

$$J^*(dw^i) = \hat{g}^{ij} d\tilde{\zeta}_j \ , \quad J^*(d\tilde{\zeta}_i) = \hat{g}_{ij} dw^j \ .$$

It is useful to consider a second co-frame for T^*N with respect to which the endomorphism J^* acts diagonally. In particular, we take

$$F'^* = \left(dw^i + \hat{g}^{ij} d\tilde{\zeta}_j, dw^i - \hat{g}^{ij} d\tilde{\zeta}_j\right),$$

with respect to which J^* acts as

$$J^*(dw^i \pm \hat{g}^{ij} d\tilde{\zeta}_j) = \pm(dw^i \pm \hat{g}^{ij} d\tilde{\zeta}_j).$$

For the case of single-centred extremal black strings, we see that the tangent vectors $(\dot{w}^i, \pm\hat{g}^{ij}\dot{\tilde{\zeta}}_j)$ correspond to null geodesic curves contained within the eigendistributions of the integrable para-complex structure J, whereas the tangent vectors $(\dot{w}^i, R^i{}_j \hat{g}^{jk}\dot{\tilde{\zeta}}_k)$ with non-trivial R-matrix correspond to null geodesic curves not contained within the eigendistributions of J. This gives a geometrical characterisation of BPS vs. non-BPS solutions in the single-centred case.

For the case of multi-centred black strings, solutions no longer correspond to null geodesics, but rather to totally isotropic submanifolds of N [35]. Again we can characterise BPS vs. non-BPS solutions by whether these submanifolds are contained within the eigendistributions of the para-complex structure.

This geometrical characterisation of extremal solutions carries over to more general situations where the target space is para-Kähler. We will see more of this in Chapter 7.

6.6 Four-dimensional solutions

In Section 6.5 we were able to dimensionally lift the three-dimensional instanton solution constructed in Sections 6.3 and 6.4 back to five dimensions in order to find static black string solutions to our five-dimensional supergravity theory. We saw that our ability to do this relied on relating the five-dimensional Minkowski theory and the three-dimensional Euclidean theory via dimensional reduction over one time-like and one space-like dimension. One advantage of splitting the dimensional reduction up into two stages like this is that we can now lift our three-dimensional instanton solutions to four Minkowski dimensions, thereby constructing a solitonic solution to the four-dimensional supergravity theory obtained by space-like reduction of the five-dimensional theory. In the language of four-dimensional $\mathcal{N} = 2$ theories, we would be considering a theory with prepotential

$$F(X) = \frac{1}{6} c_{ijk} \frac{X^i X^j X^k}{X^0},$$

which we met in Section 3.5.3.

Since we want to lift only over the time-like direction, we need to concentrate on the case of space-then-time reduction, i.e. take $\epsilon_1 = -1 = -\epsilon_2$. Then we can rewrite the metric ansatz (6.7) as

$$ds^2_{(5)} = e^{2\sigma} dy^2 + e^{-\sigma} ds^2_{(4)},$$

where

$$ds^2_{(4)} = -e^{2\xi+3\sigma} dt^2 + e^{-2\xi-3\sigma} ds^2_{(3)}.$$

Using (6.28), (6.29), and the explicit form of the three-dimensional part of the line element (6.15), we find that the four-dimensional solution has line element

$$ds^2_{(4)} = -H(\mathcal{H})^{-\frac{1}{2}} W dt^2 + H(\mathcal{H})^{\frac{1}{2}} \left(\frac{d\rho^2}{W} + \rho^2 d\Omega_2^2 \right). \qquad (6.44)$$

The four-dimensional black hole thus obtained has an inner horizon at $\rho = 0$ and an outer horizon at $\rho = 2c$. The area of the outer horizon is

$$A_+ = 4\pi\sqrt{2c}\, H(\bar{p})^{\frac{1}{2}},$$

whereas the area of the inner horizon vanishes. In the extremal limit $c \to 0$, the outer horizon shrinks to zero size, and we are left with a 'small' black hole. The line element (6.44) should therefore be thought of as a non-extremal deformation of a small black hole.

In Chapter 7 we will see that by turning on additional fields in the five-dimensional theory, we can construct black holes in four dimensions which in the extremal limit remain physical, i.e. their horizon area remains non-zero.

Chapter 7

Stationary five-dimensional black objects

In Chapter 6 we were able to use the target space geometry appearing upon dimensional reduction from five to three dimensions to construct static black string solutions to five-dimensional supergravity. In this chapter, we relax the condition of staticity, and look for more general classes of five-dimensional solutions.

Recall that in Section 6.5.5, a useful geometrical characterisation of extremal solutions was given in terms of the eigendistributions of a given integrable para-complex structure on a para-Kähler scalar manifold. Motivated by this, in this chapter we will look for the possible maximal totally geodesic submanifolds admitting a para-Kähler structure.

In Section 7.1 we identify three consistent truncations of the three-dimensional theory, corresponding to three totally geodesic submanifolds of the full scalar target space $\mathcal{M}_3$. We then treat each of these in turn (Sections 7.2–7.4), commenting on the relevant geometry and finding explicit black hole and black string solutions.

The work presented in this chapter is ongoing and as yet unpublished, but will form the basis of a number of future publications by the author.

7.1 Consistent truncations

We start with the scalar manifold $\mathcal{M}_3$ of the dimensionally reduced theory of five-dimensional supergravity coupled to vector multiplets, which is the target manifold associated to the non-linear sigma model with Lagrangian (4.3).

For the TS and ST reductions (which we will concentrate on in this chapter) the resulting scalar manifolds $(\bar{Q}, g_{\bar{Q}})$ are para-quaternionic-Kähler. Here we look for totally geodesic para-Kähler submanifolds of $\bar{Q}$, from which we can hope to construct instantons which lift to solutions of the five-dimensional field equations.

Recall from Proposition 2 in Chapter 3 that we can identify totally geodesic submanifolds of $\bar{Q}$ by finding an involution on $\bar{Q}$ which acts isometrically on $g_{\bar{Q}}$. The fixed-point set of this involution will then define a totally geodesic submanifold.

We look for totally geodesic submanifolds of half the dimension of $\bar{Q}$, that is, of dimension $2n+4$. These correspond to turning off half of the three-dimensional fields. We first identify three isometric involutions of $\bar{Q}$, which give rise to three totally geodesic submanifolds: two 'electric' and one 'magnetic'. In Sections 7.2–7.4 we then treat each in turn and identify classes of five-dimensional solutions to the truncated theories.

The three involutions are:

(i) The 'electric' truncation, corresponding to the involution

$$(\phi^x, \sigma, \phi, x^i, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i) \mapsto (\phi^x, \sigma, \phi, x^i, \tilde{\phi}, -\zeta^0, -\zeta^i, -\tilde{\zeta}_0, -\tilde{\zeta}_i). \tag{7.1}$$

(ii) The second 'electric' truncation, corresponding to the involution

$$(\phi^x, \sigma, \phi, x^i, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i) \mapsto (\phi^x, \sigma, \phi, -x^i, -\tilde{\phi}, -\zeta^0, \zeta^i, \tilde{\zeta}_0, -\tilde{\zeta}_i). \tag{7.2}$$

(iii) The 'magnetic' truncation, corresponding to the involution

$$(\phi^x, \sigma, \phi, x^i, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i) \mapsto (\phi^x, \sigma, \phi, -x^i, -\tilde{\phi}, \zeta^0, -\zeta^i, -\tilde{\zeta}_0, \tilde{\zeta}_i). \tag{7.3}$$

Given the expression (4.11) for the metric $g_{\bar{Q}}$, it is a simple exercise to show explicitly that the involutions (7.1)–(7.3) act isometrically.

7.2 Electric truncation I

The first 'electric truncation' corresponds to the involution (7.1) which acts isometrically on $g_{\bar{Q}}$. Hence the fixed-point set of (7.1) is a totally geodesic submanifold $S_e \subset \bar{Q}$, parametrized by the $2(n_V^{(5)}+2)$ scalars $(\phi^x, \sigma, \phi, x^i, \tilde{\phi})$. The associated Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{R}{2} - \frac{3}{4} g_{xy}(\phi)\partial\phi^x\partial\phi^y + \frac{3}{4\sigma^2}\epsilon_1 a_{ij}(h)\partial x^i \partial x^j - \frac{1}{4\phi^2}(\partial\phi)^2 - \frac{3}{4\sigma^2}(\partial\sigma)^2 \\ &\quad + \frac{1}{4\phi^2}\epsilon_1(\partial\tilde{\phi})^2. \end{aligned} \tag{7.4}$$

The corresponding metric on S_e is

$$g_{S_e} = \frac{3}{4}g_{xy}(\phi)d\phi^x d\phi^y - \frac{3}{4\sigma^2}\epsilon_1 a_{ij}(h)dx^i dx^j + \frac{1}{4\phi^2}(d\phi)^2 + \frac{3}{4\sigma^2}(d\sigma)^2 - \frac{1}{4\phi^2}\epsilon_1(d\tilde{\phi})^2. \tag{7.5}$$

Note that for SS or ST reduction (i.e. $\epsilon_1 = -1$), this metric is positive definite, while for TS reduction ($\epsilon_1 = 1$) it has split signature $(n_V^{(5)}+2, n_V^{(5)}+2)$. Motivated by the method for finding solutions in, e.g. [47], we therefore concentrate on the TS reduction. Indeed, we will show that in this case (S_e, g_{S_e}) admits an integrable para-complex structure J which gives (S_e, g_{S_e}, J) the structure of a para-Kähler manifold.

The consistent truncation defined by the involution (7.1) restricts us to the subset of field configurations with $\zeta^0 = \zeta^i = \tilde{\zeta}_0 = \tilde{\zeta}_i = 0$. We also focus on the case of TS reduction. In terms of the five-dimensional fields, this corresponds to setting both the 'magnetic' and one of the 'electric' components of the five-dimensional gauge fields to zero, $\mathcal{A}^i_\mu = \mathcal{A}^i_4 = 0$, as well as truncating the first Kaluza-Klein vector $\mathcal{A}^0$.

Therefore, the remaining five-dimensional field content is

$$\mathcal{A}^i = \frac{6^{1/6}}{2} x^i \, dt, \tag{7.6}$$

for the gauge fields, and

$$ds^2_{(5)} = -6^{-\frac{2}{3}}\sigma^2 dt^2 + 6^{\frac{1}{3}}\frac{\phi}{\sigma}(dz + B_\mu dx^\mu)^2 + 6^{\frac{1}{3}}\frac{1}{\sigma\phi}ds^2_{(3)}, \tag{7.7}$$

for the metric. The Kaluza-Klein vector B_μ is determined from the three-dimensional scalars via its field strength

$$H_{\mu\nu} = \frac{1}{\phi^2}\epsilon_{\mu\nu\rho}\,\partial^\rho\tilde{\phi}. \tag{7.8}$$

Introducing coordinates h^i parametrising the CASR manifold, so that $g_{xy}(\phi)d\phi^x d\phi^y = a_{ij}(h)dh^i dh^j$, it is convenient to make the field redefinition

$$y^i = \sigma h^i, \quad \hat{g}_{ij}(y) := \frac{3}{4}a_{ij}(y) = \frac{3}{4\sigma^2}a_{ij}(h),$$

so that (7.4) becomes

$$\mathcal{L} = \frac{R}{2} - \hat{g}_{ij}(y)\partial y^i\partial y^j + \hat{g}_{ij}(y)\partial x^i\partial x^j - \frac{1}{4\phi^2}(\partial\phi)^2 + \frac{1}{4\phi^2}(\partial\tilde{\phi})^2. \tag{7.9}$$

In terms of the metric on the scalar manifold we find

$$g_{S_e} = \hat{g}_{ij}(y)\left(dy^i dy^j - dx^i dx^j\right) + \frac{1}{4\phi^2}\left(d\phi^2 - d\tilde{\phi}^2\right). \tag{7.10}$$

This is just the metric on the para-Kähler manifold

$$S_e = \bar{N} \times \frac{SL(2,\mathbb{R})}{SO(1,1)},$$

which is the product of the projective special para-Kähler manifold $\bar{N} \cong TM$ appearing in the image of the time-like r-map, which we met in Section 3.6.1, and a two-dimensional factor parametrized by $(\phi, \tilde{\phi})$. Since it is the product of two para-Kähler manifolds, (S_e, g_{S_e}) is automatically para-Kähler.

7.2.1 Equations of motion

We first look at the equations of motion coming from (7.9). The Einstein equations read

$$\frac{1}{2}R_{\mu\nu} - \hat{g}_{ij}(y)\partial_\mu y^i\partial_\nu y^j + \hat{g}_{ij}(y)\partial_\mu x^i\partial_\nu x^j - \frac{1}{4\phi^2}\partial_\mu\phi\,\partial_\nu\phi + \frac{1}{4\phi^2}\partial_\mu\tilde{\phi}\,\partial_\nu\tilde{\phi} = 0, \tag{7.11}$$

while the equations of motion for the scalars read

$$\nabla^2 y^i + \hat{\Gamma}^i_{jk}\partial^\mu y^j \partial_\mu y^k + \hat{\Gamma}^i_{jk}\partial_\mu x^j \partial^\mu x^k = 0, \tag{7.12}$$

$$\nabla^2 x^i + 2\hat{\Gamma}^i_{jk}\partial_\mu y^j \partial^\mu x^k = 0, \tag{7.13}$$

$$\nabla^2 \phi - \frac{1}{\phi}(\partial\phi)^2 - \frac{1}{\phi}(\partial\tilde{\phi})^2 = 0, \tag{7.14}$$

$$\nabla^2 \tilde{\phi} - \frac{2}{\phi}\partial_\mu \phi \, \partial^\mu \tilde{\phi} = 0, \tag{7.15}$$

where $\hat{\Gamma}^i_{jk}$ are given by (6.42). Note that the equations of motion for (y^i, x^i) and $(\phi, \tilde{\phi})$ decouple.

7.2.2 Extremal BPS and non-BPS solutions

We look for solutions of (7.11) with flat transverse space, i.e. $R_{\mu\nu} = 0$. We leave the case of non-extremal solutions to future work. In this case, (7.11) is solved if we make the ansätze

$$\phi = \pm\tilde{\phi}, \quad y^i = R^i{}_j x^j, \tag{7.16}$$

where the "R-matrix" satisfies $R^T \hat{g} R = \hat{g}$. With this, (7.14) and (7.15) tell us that $\Box(\phi^{-1}) = 0$, so

$$\phi = \frac{1}{f(\vec{x})}, \tag{7.17}$$

where $f(\vec{x})$ is harmonic in the three-dimensional transverse space. Likewise, (7.12) and (7.13) reduce to

$$\Box y^i + 2\hat{\Gamma}^i_{jk}\partial_\mu y^j \partial^\mu y^k = 0,$$

which is equivalent to

$$\partial_\mu \left(\hat{g}_{ij}(y) \partial^\mu y^j \right) = 0.$$

Following [47] we introduce the field y_i via

$$\partial_\mu y_i = \hat{g}_{ij}(y) \partial_\mu y^j, \tag{7.18}$$

in terms of which the equation of motion for y^i becomes $\Box y_i = 0$. Hence, we have

$$y_i = \mathcal{H}_i(\vec{x}), \tag{7.19}$$

where the functions $\mathcal{H}_i(\vec{x})$ are harmonic in the three-dimensional transverse space.

The Kaluza-Klein vector has field strength

$$H_{\mu\nu} = \pm f^2(\vec{x})\epsilon_{\mu\nu\rho}\,\partial^\rho \left(\frac{1}{f(\vec{x})}\right) = \mp\epsilon_{\mu\nu\rho}\,\partial^\rho f(\vec{x}),$$

which takes the form $dB = *_{(3)}df$. The five-dimensional metric takes the schematic form

$$ds^2_{(5)} = -\sigma^2 dt^2 + \sigma^{-1}\left[f^{-1}(dz + \vec{B}\cdot d\vec{x})^2 + f d\vec{x}^2\right] = -\sigma^2 dt^2 + \sigma^{-1} ds^2_{GH},$$

which corresponds to a non-rotating black hole with four-dimensional Gibbons-Hawking base [41]. The remaining scalar field σ can be determined from y^i by $\sigma^3 = H(y^i)$. However, one first needs to invert the relation (7.18), which can only be achieved in a model-dependent fashion, as in the study of five-dimensional black holes [43, 47, 48].

Example: The STU model

As a brief example of the form such stationary solutions could take, we concentrate on the STU-model [113]. This has $n_V^{(5)} = 2$ and Hesse potential

$$H(h) = h^1 h^2 h^3.$$

We also restrict ourselves to the single-centered spherically symmetric case. Given a solution $y_i = \mathcal{H}_i(\rho) = A_i + \frac{Q_i}{\rho}$, we can solve (7.18) to find

$$y^i = -\frac{1}{4\mathcal{H}_i(\rho)}, \quad i = 1, 2, 3.$$

The function $f(\rho) = a + \frac{2n}{\rho}$ gives $B = 2n\cos\theta\, d\phi$. Hence, we find the five-dimensional line element

$$ds^2_{(5)} = -(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-2/3}dt^2 + (\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{1/3}\left[f^{-1}(dz + 2n\cos\theta\, d\phi)^2 + f(d\rho^2 + \rho^2 d\Omega_2^2)\right].$$

Depending on the values of the constants a and n, the four-dimensional part of this solution has a different interpretation. For example, the case $a = 0$, $2n = 1$ would correspond to a flat base space

$$ds^2_{(4)} = dr^2 + \frac{r^2}{4}(d\psi^2 + d\phi^2 + 2\cos\theta\, d\psi d\phi + d\theta)^2,$$

written in terms of the Euler angles (θ, ϕ, ψ) on S^3. Here we have put $\rho = r^2/4$ and $\psi = z$, which is taken to have periodicity 4π [66].

The solution is electrically charged under the five-dimensional gauge fields, with

$$q_i = R_i{}^j Q_j.$$

The possible R-matrices for the STU-model are given by

$$R = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix},$$

where the choice of sign for each diagonal element is independent. For $R = \pm\mathbb{1}$ we obtain BPS solutions, while for $R \neq \pm\mathbb{1}$ we obtain non-BPS solutions.

7.3 Electric truncation II

The second 'electric truncation' corresponds to the involution (7.2) which acts isometrically on $g_{\bar{Q}}$. Hence the fixed-point set of (7.2) is a totally geodesic submanifold $S'_e \subset \bar{Q}$, parametrized by the $2(n_V^{(5)} + 2)$ scalars $(\phi^x, \sigma, \phi, \zeta^i, \tilde{\zeta}_0)$. The associated Lagrangian is

$$\mathcal{L} = \frac{R}{2} - \frac{3}{4}g_{xy}(\phi)\partial\phi^x\partial\phi^y - \frac{1}{4\phi^2}(\partial\phi)^2 - \frac{3}{4\sigma^2}(\partial\sigma)^2$$

$$+\frac{\sigma}{4\phi}\epsilon_2 a_{ij}(h)\partial\zeta^i\partial\zeta^j + \frac{3}{\phi\sigma^3}\epsilon_2(\partial\tilde{\zeta}_0)^2, \tag{7.20}$$

and the corresponding metric on S'_e is

$$g_{S'_e} = \frac{3}{4}g_{xy}(\phi)d\phi^x d\phi^y + \frac{1}{4\phi^2}(d\phi)^2 + \frac{3}{4\sigma^2}(d\sigma)^2 - \frac{\sigma}{4\phi}\epsilon_2 a_{ij}(h)d\zeta^i d\zeta^j - \frac{3}{\phi\sigma^3}\epsilon_2 d\tilde{\zeta}_0^2. \tag{7.21}$$

Note that for SS or TS reduction (i.e. $\epsilon_2 = -1$), this metric is positive definite, while for ST reduction ($\epsilon_2 = 1$) it has split signature $(n_V^{(5)}+2, n_V^{(5)}+2)$. For the same reasons as above we therefore concentrate on the ST reduction.

The consistent truncation defined by the involution (7.2) restricts us to the subset of field configurations with $x^i = \tilde{\phi} = \zeta^0 = \tilde{\zeta}_i = 0$. We also focus on the case of ST reduction. In terms of the five-dimensional fields, this corresponds to setting both the 'magnetic' and one of the 'electric' components of the five-dimensional gauge fields to zero, $\mathcal{A}^i_\mu = \mathcal{A}^i_0 = 0$, as well as truncating the second Kaluza-Klein vector $B_\mu = 0$ and one component of the first, $\mathcal{A}^0_4 = 0$.

Therefore, the remaining five-dimensional field content is

$$\mathcal{A}^i = \frac{6^{1/6}}{\sqrt{2}}\zeta^i\, dt, \tag{7.22}$$

for the gauge fields, and

$$ds^2_{(5)} = -6^{1/3}\left(\frac{\phi}{\sigma}\right)dt^2 + 6^{-2/3}\sigma^2(dz + \mathcal{A}^0_\mu dx^\mu)^2 + \frac{6^{1/3}}{\sigma\phi}ds^2_{(3)}, \tag{7.23}$$

for the metric. The Kaluza-Klein vector $\mathcal{A}^0_\mu$ is determined from the three-dimensional scalars via its field strength

$$\mathcal{F}^0_{\mu\nu} = -\frac{6\sqrt{2}\epsilon}{\sigma^3\phi}\epsilon_{\mu\nu\rho}\partial^\rho\tilde{\zeta}_0. \tag{7.24}$$

It is convenient to make the field redefinitions

$$\sigma = u^{-\frac{1}{2}}v^{-\frac{1}{2}}, \quad \phi = u^{-\frac{1}{2}}v^{\frac{3}{2}}, \quad y^i = vh^i, \quad \hat{g}_{ij}(y) = \frac{3}{4v^2}a_{ij}(h), \tag{7.25}$$

so (7.20) becomes

$$\mathcal{L} = \frac{R}{2} - \hat{g}_{ij}(y)\partial y^i \partial y^j + \frac{1}{3}\hat{g}_{ij}(y)\partial\zeta^i\partial\zeta^j - \frac{1}{4u^2}(\partial u)^2 + \frac{3}{u^2}(\partial\tilde{\zeta}_0)^2, \tag{7.26}$$

or in terms of the metric on the scalar manifold

$$g_{S'_e} = \hat{g}_{ij}(y)dy^i dy^j - \frac{1}{3}\hat{g}_{ij}(y)d\zeta^i d\zeta^j + \frac{1}{4u^2}du^2 - \frac{3}{u^2}d\tilde{\zeta}_0^2. \tag{7.27}$$

7.3.1 Relating the electric truncations

Using the (t,ψ) flip of Chapter 4, we can relate the two electric truncations to each other. In particular, if we start with the truncation in Section 7.2, the variables $(V, \rho_1, \rho_2, \mu_1^i, \mu_2^i, \nu_i, \tilde{\mu}_1, \tilde{\mu}_2)$ given in (4.29) become

$$V = \sigma^{\frac{1}{2}}\phi^{\frac{1}{2}}, \quad \rho_2 = \sigma^{-\frac{3}{2}}\phi^{\frac{1}{2}}, \quad \mu_2^i = bx^i, \quad \tilde{\mu}_1 = -\frac{g}{2\sqrt{3}}\tilde{\phi},$$

with all others vanishing. Applying the (t,ψ) flip to this configuration gives us a theory with

$$\phi' = \sigma^{\frac{3}{2}}\phi^{\frac{1}{2}}, \quad \sigma' = \sigma^{-\frac{1}{2}}\phi^{\frac{1}{2}}, \quad (\zeta^i)' = \sqrt{3}\,x^i, \quad (\tilde{\zeta}_0)' = -\frac{1}{2\sqrt{3}}\tilde{\phi},$$

and $(x^i)' = (\tilde{\phi})' = (\zeta^0)' = (\tilde{\zeta}_i)' = 0$. This is precisely the field content of the second electric truncation.

Moreover, we see that the (t,ψ) flip really maps the Lagrangian (7.4), which is relevant for TS reduction, to the Lagrangian (7.20) relevant for ST reduction. This provides us with a global isometry between the totally geodesic submanifolds (S_e, g_{S_e}) and $(S'_e, g_{S'_e})$. We will comment on the possible significance of this in Chapter 8.

7.4 Magnetic truncation

We now concentrate on the 'magnetic truncation', corresponding to the involution (7.3) which acts isometrically on $g_{\bar{Q}}$. Hence the fixed-point set of (7.3) is a totally geodesic submanifold $S_m \subset \bar{Q}$, parametrized by the $2(n_V^{(5)}+2)$ scalars $(\phi^x, \sigma, \phi, \zeta^0, \tilde{\zeta}_i)$. The associated Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{R}{2} - \frac{3}{4}g_{xy}(\phi)\partial\phi^x\partial\phi^y - \frac{1}{4\phi^2}(\partial\phi)^2 - \frac{3}{4\sigma^2}(\partial\sigma)^2 + \frac{\sigma^3}{12\phi}(\partial\zeta^0)^2 \\ &\quad + \frac{1}{\phi\sigma}a^{ij}(h)\partial\tilde{\zeta}_i\partial\tilde{\zeta}_j, \end{aligned} \tag{7.28}$$

and the corresponding metric on S_m is

$$g_{S_m} = \frac{3}{4}g_{xy}(\phi)d\phi^x d\phi^y + \frac{1}{4\phi^2}(d\phi)^2 + \frac{3}{4\sigma^2}(d\sigma)^2 - \frac{\sigma^3}{12\phi}(d\zeta^0)^2 - \frac{1}{\phi\sigma}a^{ij}(h)d\tilde{\zeta}_i\, d\tilde{\zeta}_j. \tag{7.29}$$

Note that for both ST and TS reductions this metric has split signature $(n_V^{(5)}+2, n_V^{(5)}+2)$.

The consistent truncation defined by the involution (7.3) restricts us to the subset of field configurations with $x^i = \tilde{\phi} = \zeta^i = \tilde{\zeta}_0 = 0$. In terms of the five-dimensional fields, this corresponds to setting the 'electric' components of the five-dimensional gauge fields to zero, $\mathcal{A}^i_0 = \mathcal{A}^i_4 = 0$, as well as truncating the second Kaluza-Klein vector $B_\mu = 0$, and the three-dimensional components of the first Kaluza-Klein vector $\mathcal{A}^0_\mu = 0$.

Therefore, the remaining five-dimensional field content is

$$\mathcal{F}^i_{\mu\nu} = \frac{6^{1/6}\sqrt{2}\,\epsilon_2}{\sigma\phi}\epsilon_{\mu\nu\rho}\, a^{ij}(h)\partial^\rho\tilde{\zeta}_j, \tag{7.30}$$

for the gauge fields, and

$$ds^2_{(5)} = -\epsilon_1\, 6^{-2/3}\sigma^2(dx^0 + \mathcal{A}^0_4 dx^4)^2 - \epsilon_2\, 6^{1/3}\left(\frac{\phi}{\sigma}\right)(dx^4)^2 + \frac{6^{1/3}}{\sigma\phi}ds^2_{(3)}, \tag{7.31}$$

for the five-dimensional line element. The Kaluza-Klein vector is given by

$$\mathcal{A}^0 = -\sqrt{2}\zeta^0\, dx^4. \tag{7.32}$$

It is convenient to make the field redefinitions

$$\sigma = u^{-\frac{1}{2}}v^{-\frac{1}{2}}, \quad \phi = u^{\frac{1}{2}}v^{-\frac{3}{2}}, \quad y^i = vh^i, \quad \hat{g}_{ij}(y) = \frac{3}{4v^2}a_{ij}(h), \tag{7.33}$$

so (7.28) becomes

$$\mathcal{L} = \frac{R}{2} - \hat{g}_{ij}(y)\partial_\mu y^i \partial^\mu y^j - \frac{1}{4u^2}(\partial u)^2 + \frac{1}{12u^2}(\partial \zeta^0)^2 + \frac{3}{4}\hat{g}^{ij}(y)\partial_\mu \tilde{\zeta}_i \partial^\mu \tilde{\zeta}_j, \qquad (7.34)$$

or in terms of the metric g_{S_m}:

$$g_{S_m} = \hat{g}_{ij}(y)dy^i\, dy^j + \frac{1}{4u^2}(du)^2 - \frac{1}{12u^2}(d\zeta^0)^2 - \frac{3}{4}\hat{g}^{ij}(y)d\tilde{\zeta}_i\, d\tilde{\zeta}_j. \qquad (7.35)$$

The five-dimensional gauge field strength (7.30) then becomes

$$\mathcal{F}^i_{\mu\nu} = -\frac{3 \cdot 6^{-1/6}\epsilon_2}{\sqrt{2}}\epsilon_{\mu\nu\rho}\hat{g}^{ij}(y)\partial^\rho \tilde{\zeta}_j, \qquad (7.36)$$

and the line element (7.31) is

$$ds^2_{(5)} = -\epsilon_1 \frac{6^{-2/3}}{uv}(dx^0 + \mathcal{A}^0_4 dx^4)^2 - \epsilon_2 6^{1/3}\left(\frac{u}{v}\right)(dx^4)^2 + 6^{1/3}v^2 ds^2_{(3)}. \qquad (7.37)$$

We can consider a further truncation of this model to field configurations with $\zeta^0 = 0$. This is induced by the involution $(y^i, u, \zeta^0, \tilde{\zeta}_i) \mapsto (y^i, u, -\zeta^0, \tilde{\zeta}_i)$ which acts isometrically on (7.35). The resulting scalar target space is then precisely that which we studied in Chapter 6 in the context of static black string solutions. This proves that the truncation we made in Chapter 6 is consistent, since it corresponds to restricting ourselves to a totally geodesic submanifold of a totally geodesic submanifold.

The manifold S_m is para-Kähler since it is the product

$$S_m = T^*M \times \frac{SL(2,\mathbb{R})}{SO(1,1)},$$

of the para-Kähler manifold T^*M, which was relevant for static extremal black strings in Section 6.5.5, and a two-dimensional factor parametrised by (u, ζ^0).

7.4.1 Equations of motion

We first look at the equations of motion coming from (7.34). The Einstein equations read

$$\frac{1}{2}R_{\mu\nu} - \hat{g}_{ij}(y)\partial_\mu y^i \partial_\nu y^j - \frac{1}{4u^2}\partial_\mu u\,\partial_\nu u + \frac{1}{12u^2}\partial_\mu \zeta^0 \partial_\nu \zeta^0 + \frac{3}{4}\hat{g}^{ij}(y)\partial_\mu \tilde{\zeta}_i \partial_\nu \tilde{\zeta}_j = 0, \quad (7.38)$$

while the equations of motion for the scalars read

$$\nabla^2 y^i + \hat{\Gamma}^i_{jk}\partial^\mu y^j \partial_\mu y^k - \frac{3}{4}\hat{\Gamma}^i_{jk}\hat{g}^{jm}\hat{g}^{kn}\partial_\mu s_m \partial^\mu s_n = 0, \quad (7.39)$$

$$\nabla^2 \tilde{\zeta}_i - 2\hat{\Gamma}^k_{ij}\partial_\mu y^j \partial^\mu \tilde{\zeta}_k = 0, \quad (7.40)$$

$$\nabla^2 u - \frac{1}{u}(\partial u)^2 - \frac{1}{3u}(\partial \zeta^0)^2 = 0, \quad (7.41)$$

$$\nabla^2 \zeta^0 - \frac{2}{u}\partial_\mu u\,\partial^\mu \zeta^0 = 0. \quad (7.42)$$

7.4.2 Extremal BPS and non-BPS solutions

We look for solutions of (7.38) with flat transverse space, i.e. $R_{\mu\nu} = 0$. In this case, (7.38) is solved if we make the ansätze[1]

$$\zeta^0 = \pm\sqrt{3}u, \quad \partial_\mu y^i = \frac{\sqrt{3}}{2}R^i{}_j\,\hat{g}^{jk}(y)\partial_\mu \tilde{\zeta}_k, \quad (7.43)$$

where the "R-matrix" satisfies $R^T\hat{g}R = \hat{g}$.

With this, (7.41) and (7.42) tell us that $\Box(u^{-1}) = 0$, so

$$u = \frac{1}{f(\vec{x})}, \quad (7.44)$$

where $f(\vec{x})$ is harmonic in the three-dimensional transverse space. Likewise, (7.39) and (7.40) reduce to $\Box y^i = 0$, so

$$y^i = \mathcal{H}^i(\vec{x}), \quad (7.45)$$

[1]Note that, in terms of the variables (4.29), the first of these conditions becomes $\rho_1 = \pm\rho_2$, which corresponds to the singular locus of the (t,ψ)-flip (4.34).

is harmonic. Returning to (7.33), we see then that

$$v = H(\mathcal{H})^{\frac{1}{3}}.$$

The remaining five-dimensional fields are given by

$$\mathcal{A}^0 = \mp\frac{\sqrt{6}}{f}dx^4, \quad \mathcal{F}^i_{\mu\nu} = -\frac{3\cdot 6^{-1/6}\epsilon_2}{\sqrt{2}}\epsilon_{\mu\nu\rho}(R^{-1})^i{}_j\partial^\rho\mathcal{H}^j.$$

Plugging these expressions into the line element (7.37) we find

$$ds^2_{(5)} = -\epsilon_1\frac{6^{-2/3}f}{H(\mathcal{H})^{1/3}}\left(dx^0 + \frac{\sqrt{6}}{f}dx^4\right)^2 - \epsilon_2\frac{6^{1/3}}{fH(\mathcal{H})^{1/3}}(dx^4)^2 + 6^{1/3}H(\mathcal{H})^{2/3}d\vec{x}^2, \quad (7.46)$$

where we have fixed the sign of $\mathcal{A}^0$. Note that choosing the opposite sign would simply correspond to a coordinate redefinition $x^4 \to -x^4$.

In order to interpret these solutions physically we concentrate on the case of ST reduction and introduce the coordinates

$$t = -x^4, \quad z = x^4 + 6^{-1/2}x^0, \quad (7.47)$$

in terms of which the line element (7.46) becomes

$$ds^2_{(5)} = \frac{6^{1/3}}{H(\mathcal{H})^{1/3}}\left[-dt^2 + dz^2 + (f-1)(dt+dz)^2\right] + 6^{1/3}H(\mathcal{H})^{2/3}d\vec{x}^2.$$

In [41] such solutions are called ‘plane-fronted waves’. This particular solution can be thought of as taking the extremal black string of Chapter 6 and putting a pp-wave along its worldvolume.

7.4.3 Non-extremal solutions

We now turn our attention to spherically symmetric solutions of (7.38)–(7.42). We take coordinates (τ, θ, ϕ), and insist that all fields are independent of both θ and ϕ, as in Chapter 6. Recall that the most general form for the three-dimensional transverse space consistent with the Einstein equations (which for $\mu, \nu \neq \tau$ are just $R_{\mu\nu} = 0$) is

$$ds^2_{(3)} = \frac{c^4}{\sinh^4(c\tau)} d\tau^2 + \frac{c^2}{\sinh^2(c\tau)} d\Omega_2^2, \tag{7.48}$$

where we take $c > 0$. The remaining components of (7.38), namely those with $\mu = \nu = \tau$, give us the Hamiltonian constraint

$$c^2 - \hat{g}_{ij}(y)\dot{y}^i\dot{y}^j + \frac{3}{4}\hat{g}^{ij}(y)\dot{\tilde{\zeta}}_i\dot{\tilde{\zeta}}_j - \frac{\dot{u}^2}{4u^2} + \frac{(\dot{\zeta}^0)^2}{12u^2} = 0. \tag{7.49}$$

We turn now to the scalar equations of motion (7.39)–(7.42). The equations of motion for $(y^i, \tilde{\zeta}_i)$ are the same as in Chapter 6[2]. For $\tilde{\zeta}_i$ we have

$$\frac{d}{d\tau}\left(\hat{g}^{ij}(y)\dot{\tilde{\zeta}}_j\right) = 0,$$

which we solve with

$$\dot{\tilde{\zeta}}_i = \frac{2}{\sqrt{3}}\,\hat{g}_{ij}(y)\tilde{p}^j, \tag{7.50}$$

where the integration constants $\tilde{p}^i$ are proportional to the magnetic charge of the solution under the gauge field $\mathcal{A}^i$.

Using (6.18), the equation of motion (7.39) for the y^i becomes

$$\ddot{y}^i + \frac{1}{2}\hat{g}^{il}(y)(\partial_l\hat{g}_{jk})\left(\dot{y}^j\dot{y}^k - \tilde{p}^j\tilde{p}^k\right) = 0.$$

As in Section 6.4.3 we solve this with

$$y^i(\tau) = A^i\cosh(c\tau) + \frac{B^i}{c}\sinh(c\tau). \tag{7.51}$$

Introducing the radial coordinate ρ via (6.14) we can rewrite (7.51) as

$$y^i(\rho) = \left(A^i + \frac{B^i - cA^i}{\rho}\right)W^{-\frac{1}{2}} := W^{-\frac{1}{2}}\mathcal{H}^i(\rho), \tag{7.52}$$

where the function $W(\rho)$ was introduced in (6.16).

[2]Note that there we used w^i for the scalar fields.

We next consider the equations for (u, ζ^0). For ζ^0, (7.42) becomes

$$\ddot{\zeta}^0 - \frac{2}{u}\dot{u}\dot{\zeta}^0 = u^2 \frac{d}{d\tau}\left(\frac{\dot{\zeta}^0}{u^2}\right) = 0,$$

which has solution

$$\dot{\zeta}^0 = \sqrt{3}Du^2, \tag{7.53}$$

for some integration constant D. With this, (7.41) becomes

$$\ddot{u} - \frac{1}{u}\dot{u}^2 - D^2u^3 = 0.$$

Introducing $w = u^{-1}$, we have

$$w\ddot{w} - \dot{w}^2 + D^2 = 0,$$

which can be solved by taking

$$w(\tau) = \alpha\cosh(c\tau) + \frac{\beta}{c}\sinh(c\tau), \tag{7.54}$$

provided $D^2 = \beta^2 - c^2\alpha^2$. In terms of the radial coordinate ρ of (6.14) we have

$$w(\rho) = \left(\alpha + \frac{\Delta}{\rho}\right) W^{-\frac{1}{2}} := f(\rho)W^{-\frac{1}{2}},$$

where $f(\rho)$ is harmonic in the three-dimensional transverse space and we have defined $\Delta := \beta - c\alpha$. Hence,

$$u(\rho) = W^{\frac{1}{2}}f(\rho)^{-1}. \tag{7.55}$$

Returning to the Hamiltonian constraint and using the solutions we've found so far, we see that (7.49) reduces to

$$\frac{3}{4}c^2 - \hat{g}_{ij}(y)\left(\dot{y}^i\dot{y}^j - \tilde{p}^i\tilde{p}^j\right) = 0, \tag{7.56}$$

which is the same as we had for the non-extremal black string case (6.20). Hence the class of models for which we can find solutions is identical.

Summarising, we have

$$u = \frac{1}{f(\rho)} W(\rho)^{\frac{1}{2}}, \quad v = H(\mathcal{H})^{\frac{1}{3}} W(\rho)^{-\frac{1}{2}},$$

for the Kaluza-Klein scalars.

Turning to the Kaluza-Klein vector $\mathcal{A}^0$, we have

$$\dot{\zeta}^0 = \sqrt{3\Delta(\Delta + 2c\alpha)}\, u^2.$$

In terms of the radial coordinate ρ, this becomes

$$\frac{d\zeta^0}{d\tau} = -\rho^2 W(\rho) \frac{d\zeta^0}{d\rho} = -\sqrt{3\Delta(\Delta + 2c\alpha)}\, \frac{W(\rho)}{f^2(\rho)}.$$

Hence

$$\frac{d\zeta^0}{d\rho} = -\frac{\sqrt{3\Delta(\Delta + 2c\alpha)}}{(\alpha\rho + \Delta)^2},$$

the solution to which is

$$\zeta^0(\rho) - \zeta^0(\infty) = \frac{1}{\alpha} \frac{\sqrt{3\Delta(\Delta + 2c\alpha)}}{\alpha\rho + \Delta}. \tag{7.57}$$

Substituting these expressions into (7.37) results in the asymptotically-flat five-dimensional line element

$$\begin{aligned} ds^2_{(5)} &= -\epsilon_1 \frac{6^{-2/3} f}{H(\mathcal{H})^{1/3}} (dx^0 - \sqrt{2}\zeta^0\, dx^4)^2 - \epsilon_2 \frac{6^{1/3} W}{f H(\mathcal{H})^{1/3}} (dx^4)^2 \\ &\quad + 6^{1/3} H(\mathcal{H})^{2/3} \left(\frac{d\rho^2}{W} + \rho^2 d\Omega_2^2 \right). \end{aligned} \tag{7.58}$$

We can determine the integration constant $\zeta^0(\infty)$ by imposing that the line element (7.58) approaches the extremal single-centered solution (7.46) as $c \to 0$. In particular, we want

$$\zeta^0(\rho) \xrightarrow{c \to 0} -\frac{\sqrt{3}\rho}{\alpha\rho + \Delta},$$

which requires $\zeta^0(\infty) = -\sqrt{3}\,\alpha^{-1}$. Hence we find

$$\zeta^0(\rho) = -\frac{\sqrt{3}}{\alpha}\left(\frac{\alpha\rho - \sqrt{\Delta}(\sqrt{\Delta+2c}-\sqrt{\Delta})}{\alpha\rho+\Delta}\right). \tag{7.59}$$

Note that for $\Delta = 0$, we have $f(\rho) = \alpha$, and (7.58) matches with the static case (6.30) provided $\alpha = 1$. This provides the interpretation of Δ as encoding the deviation of the solution from statiticity.

Reading off the metric coefficients we have

$$g_{00} = -\epsilon_1 \frac{6^{-2/3}}{H(\mathcal{H})^{1/3}}\left(1+\frac{\Delta}{\rho}\right), \tag{7.60}$$

$$g_{04} = -\epsilon_1 \frac{6^{-1/6}}{H(\mathcal{H})^{1/3}}\left(1-\frac{\sqrt{\Delta}(\sqrt{\Delta+2c}-\sqrt{\Delta})}{\rho}\right), \tag{7.61}$$

$$g_{44} = -\epsilon_1 \frac{6^{1/3}}{H(\mathcal{H})^{1/3}}\frac{(\sqrt{\Delta+2c}-\sqrt{\Delta})^2}{\rho}, \tag{7.62}$$

from which we see that the (x^0, x^4) part of the metric degenerates at

$$g_{00}\,g_{44} - (g_{04})^2 \equiv -\frac{6^{-1/3}W}{H(\mathcal{H})^{2/3}} = 0,$$

i.e. at $\rho = 2c$.

Let us now concentrate on the case of ST reduction, which will be relevant for constructing four-dimensional black hole solutions in Section 7.4.4. Making the coordinate change $(x^0, x^4) \mapsto (t, z)$ given in (7.47), the line element (7.58) can be written in the form

$$\begin{aligned} ds^2_{(5)} = {} & \frac{6^{1/3}}{H(\mathcal{H})^{1/3}}\left[-W\left(\sqrt{\frac{\Delta+2c}{2c}}\,dt+\sqrt{\frac{\Delta}{2c}}\,dz\right)^2+\left(\sqrt{\frac{\Delta}{2c}}\,dt+\sqrt{\frac{\Delta+2c}{2c}}\,dz\right)^2\right] \\ & +6^{1/3}H(\mathcal{H})^{2/3}\left(\frac{d\rho^2}{W}+\rho^2 d\Omega_2^2\right), \end{aligned} \tag{7.63}$$

which corresponds to a static black string (6.30) boosted in the z-direction with boost parameter given by

$$t \to \sqrt{\frac{\Delta+2c}{2c}}\,t+\sqrt{\frac{\Delta}{2c}}\,z, \quad z \to \sqrt{\frac{\Delta}{2c}}\,t+\sqrt{\frac{\Delta+2c}{2c}}\,z. \tag{7.64}$$

In this form, the metric (7.63) is asymptotically flat in the directions transverse to the worldvolume of the string. Hence, we can read off the ADM mass and momentum, which are given by

$$M = c_{ijk} h^i_\infty h^j_\infty \hat{p}^k, \quad P_z = \sqrt{\Delta(\Delta + 2c)},$$

where $\hat{p}^i := p^i + (\Delta + 2c) h^i_\infty$.

Before moving on to consider the reduction of (7.63) to four dimensions, we comment on an interesting observation. Namely, the boosted black string solution can be generated from the static black string (6.30) by a certain action of the 'hidden' $SL(2,\mathbb{R})$ symmetry described in Section 4.3.2.

To see this, we start with the static black string (6.30) which, in terms of the variables (4.29), has

$$\rho_1 = 0, \quad \rho_2 = W^{-\epsilon_1/2},$$

and perform the transformation

$$\rho \mapsto \rho' = e^\sigma \frac{1+\rho}{\cosh\sigma + \rho \sinh\sigma},$$

where

$$\sinh\sigma = -\sqrt{\frac{\Delta}{2c}}, \quad \cosh\sigma = \sqrt{\frac{\Delta + 2c}{2c}}.$$

For ρ_1' we get the correct expression to match $\zeta^0(\rho)$, whereas for ρ_2' we find $\rho_2' = W^{1/2} f^{-1}$, which reproduces the metric (7.58).

7.4.4 Four-dimensional solutions

We now consider the space-like reduction of (7.58) to four Minkowski dimensions. To proceed we write (7.58) in a form suitable for reduction:

$$ds^2_{(5)} = \frac{6^{-2/3} f}{H(\mathcal{H})^{1/3}} \left(dx^0 - \sqrt{2}\zeta^0 \, dx^4 \right)^2 + \frac{6^{1/3} H(\mathcal{H})^{1/6}}{f^{1/2}} ds^2_{(4)}.$$

We can then read off the expression for the four-dimensional part of the line element, which is (putting $x^4 = -t$)

$$ds^2_{(4)} = -\frac{W}{f^{1/2}H(\mathcal{H})^{1/2}}dt^2 + f^{1/2}H(\mathcal{H})^{1/2}\left(\frac{d\rho^2}{W} + \rho^2 d\Omega_2^2\right). \quad (7.65)$$

This describes a static non-extremal four-dimensional black hole with outer horizon at $\rho = 2c$ and inner horizon at $\rho = 0$. Unlike the four-dimensional solutions we met in Section 6.6, however, we find that (7.65) has finite area at both the inner and outer horizons, which remain finite in the extremal limit $c \to 0$. Indeed we find the entropy

$$S_+ = \pi\sqrt{(\Delta + 2c)H(\bar{p})}, \quad S_- = \pi\sqrt{\Delta H(p)}.$$

Our four-dimensional solution is electrically charged under

$$\mathcal{A}^0 = 6^{1/2}\left(1 - \frac{\sqrt{(\Delta + 2c)\Delta}}{\rho}\right) dt,$$

with charge

$$\tilde{q}_0 = -\sqrt{6(\Delta + 2c)\Delta}. \quad (7.66)$$

We can invert this relation to find Δ in terms of the physical electric charge $\tilde{q}_0$ of the four-dimensional black hole. Indeed, we have

$$\Delta = -c + \sqrt{c^2 + \frac{1}{6}(\tilde{q}_0)^2},$$

where we have chosen the sign such that $\Delta > 0$. The relation between the horizon charges $p^i, \bar{p}^i$ and the magnetic charges $\tilde{p}^i$ should be solved model-by-model, as explained in Chapter 6.

These four-dimensional solutions take precisely the same form as the non-extremal black holes found in the static axion-free truncations of four-dimensional $\mathcal{N} = 2$ supergravity [114]. We have therefore identified the five-dimensional lift of such solutions as being boosted black strings (7.63) with momentum P_z proportional to the electric charge $\tilde{q}_0$ of the four-dimensional solution.

Chapter 8

Conclusion and outlook

In this thesis we developed our understanding of the q-map from a geometrical point-of-view, specifically in relation to time-like dimensional reduction, and used this to generate new stationary solutions of five-dimensional $\mathcal{N} = 2$ supergravity coupled to vector multiplets.

We first derived the three-dimensional action obtained from dimensional reduction of our five-dimensional theory of vector multiplets coupled to supergravity, treating both space-like and time-like reductions on equal footing. This provided us with three maps, which we called $q^{(SS)}$, $q^{(ST)}$ and $q^{(TS)}$, depending on whether the first and second reduction steps were taken to be over a space-like or time-like direction.

We argued that the target manifolds in the image of each of these maps was ϵ-quaternionic-Kähler. Moreover, they admitted an integrable ϵ_1-complex structure compatible with the ϵ-quaternionic structure. This meant that, surprisingly, the target manifolds to the two Euclidean-signature theories obtained by ST and TS reductions were equipped with distinct geometrical structures: one was a complex manifold, the other para-complex.

Locally, however, the two manifolds seem to be isometric to each other. We demonstrated this by generalising the (t, ψ)-flip obtained in [101], between the scalar manifolds $\bar{Q}^{(ST)}$ and $\bar{Q}^{(TS)}$ obtained by ST and TS reductions respectively. Although this map did not provide a *global* isometry, so that we were still unable to identify the two spaces, it did prove useful in explicitly constructing a 'hidden' symmetry generator present in the isometry group of every q-map space. This hidden symmetry completes an $SL(2, \mathbb{R})$ global symmetry group of the three-dimensional theory, which we again stress is completely generic and does not rely on the target manifold being homogeneous. Analysing how such $SL(2, \mathbb{R})$ transformations act on a given asymptotically flat five-dimensional solution could provide an indication as to how one can generalise the recent work on 'subtracted geometries' [115–118] and the Kerr/CFT correspondence (see [119] and references therein).

We then turned to the question of the global structure of the scalar manifolds $\bar{Q}^{(ST)}$ and $\bar{Q}^{(TS)}$ for the example of pure five-dimensional supergravity. In this case we found that the two spaces could be realised as inequivalent open orbits of the Iwasawa subgroup L of the Lie group $G_{2(2)}$, lying inside the pseudo-Riemannian symmetric space $S = G_{2(2)}/(SL_2 \cdot SL_2)$. Although again they were found to be locally isometric, the orbits were inequivalent in the sense that there is no automorphism of L that relates them. We will investigate similar questions for the general case of five-dimensional supergravity coupled to n vector multiplets in a future publication [38]. In particular, it still remains to be determined whether or not there exists a global isometry relating the general q-map spaces.

This question of whether time-like and space-like reductions commute could have implications for the moduli space of string compactifications that involve time-like directions. For space-like reductions, it is conjectured that points on the classical moduli space should be identified under the action of the discrete U-duality group [1]. However, in the case of time-like reductions, such duality groups do not act properly discontinuously [120], and so it is still unclear how such an identification should take place. In particular, the resulting space will not necessarily be Hausdorff [120]. By analysing the global structure of such moduli spaces, we hope to be able to clarify the role of the duality group in the case of compactifications including a time-like direction.

As an application of the q-map, we constructed five-dimensional solutions admitting a time-like and space-like Killing vector: black strings. By dimensionally reducing to a three-dimensional Euclidean supergravity theory, we were able to construct explicit instanton solutions, which then lifted to non-extremal static black strings in five dimensions. We also investigated the extremal limit of such solutions, and argued that both BPS and non-BPS black strings could be obtained for any model admitting a suitable 'R-matrix', before giving a geometric meaning of such solutions in terms of null geodesics falling within the eigendistributions of an integrable para-complex structure.

We then made progress towards extending this formalism to non-static solutions. By modifying the techniques used previously to construct extremal and non-extremal black holes and black strings in five-dimensional supergravity, we found that it was straightforward to explicitly construct new classes of stationary solutions.

In the case of black holes, we found that it was possible to relate directly the theories relevant for ST and TS reductions via the action of the (t, ψ) flip. This can be understood in the context of the 4d/5d correspondence of [121, 122]. A charged four-dimensional black hole can be lifted to a five-dimensional black hole. This is the ST side of the reduction. Applying the (t, ψ) flip to this configuration gives another five-dimensional solution which looks like a black hole sitting over a four-dimensional Gibbons-Hawking base. Although we only saw this for extremal solutions, it should work equally for non-extremal solutions, which we leave for future work.

In terms of constructing five-dimensional solutions, the goal is to use the formalism developed in this thesis to obtain the most general charged rotating black holes and black rings in five-dimensional supergravity with an arbitrary number of vector multiplets. Although progress has been made using group-theoretic techniques (see [16] and references therein), we would like to exploit only those geometrical structures which are generic, and can be utilised irrespective of whether the target manifolds are homogeneous spaces. A first step towards this could be to understand how our more general constructions fit into the framework provided by the aforementioned group-theoretic methods.

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