Interacting Markov Branching Processes

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by

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Abstract

In engineering, biology and physics, in many systems, the particles or members give birth and die through time. These systems can be modeled by continuous-time Markov Chains and Markov Processes. Applications of Markov Processes are investigated by many scientists, Jagers [1975] for example . In ordinary Markov branching processes, each particles or members are assumed to be identical and independent. However, in some cases, each two members of the species may interact/collide together to give new birth. In considering these cases, we need to have some more general processes. We may use collision branching processes to model such systems. Then, in order to consider an even more general model, i.e. each particles can have branching and collision effect. In this case the branching component and collision component will have an interaction effect. We consider this model as interacting branching collision processes.

In this thesis, in Chapter 1, we firstly look at some background, basic concepts of continuous-time Markov Chains and ordinary Markov branching processes.

After revising some basic concepts and models, we look into more complicated models, collision branching processes and interacting branching collision processes.

In Chapter 2, for collision branching processes, we investigate the basic properties, criteria of uniqueness, and explicit expressions for the extinction probability and the expected/mean extinction time and expected/mean explosion time.

In Chapter 3, for interacting branching collision processes, similar to the struc-

ture in last chapter, we investigate the basic properties, criteria of uniqueness. Because of the more complicated model settings, a lot more details are required in considering the extinction probability. We will divide this section into several parts and consider the extinction probability under different cases and assumptions.

After considering the extinction probability for the interacting branching processes, we notice that the explicit form of the extinction probability may be too complicated. In the last part of Chapter 3, we discuss the asymptotic behavior for the extinction probability of the interacting branching collision processes.

In Chapter 4, we look at a related but still important branching model, Markov branching processes with immigration, emigration and resurrection. We investigate the basic properties, criteria of uniqueness. The most interesting part is that we investigate the extinction probability with our technique/methods using in Chapter 4. This can also be served as a good example of the methods introducing in Chapter 3.

In Chapter 5, we look at two interacting branching models, One is interacting collision process with immigration, emigration and resurrection. The other one is interacting branching collision processes with immigration, emigration and resurrection. we investigate the basic properties, criteria of uniqueness and extinction probability.

My original material starts from Chapter 4. The model used in chapter 4 were introduced by Li and Liu [2011]. In Li and Liu [2011], some calculation in cases of extinction probability evaluation were not strictly defined. My contribution focuses on the extinction probability evaluation and discussing the asymptotic behavior for the extinction probability in Chapter 4. A paper for this model will be submitted in this year. While two interacting branching models are discussed in Chapter 5. Some important properties for the two models are studied in detail.

Contents

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Chapter 1

Introduction

In this first chapter, we introduce the background and basic properties of continuous-time Markov chains and Markov branching processes. These facilitate our discussion models we investigate in later chapters. At the end of this chapter, we will have an outline for this thesis.

1.1 Background

Nowadays, research related with probability plays a great role in science especially for random processes. For example, for a certain species, we may wonder whether the species will eventually be extinct or not. If this is the case, we may want to find the probability and try to take special care towards that species. These questions can be concerned with some random processes.

In probability theory, in 1907, A. Markov introduced an extremely important concept, which is named after him as Markov processes. Markov processes investigate some random phenomena in models with emphasis on the case of finite number of states.

The continuous-time Markov chain is one of the important fields in Markov processes. The first systematic study of continuous-time Markov chains was by

A.N. Kolmogorov(1931). He found that the probability law governing the evolution of the process occurs as the solution of either of two systems of differential equations. The two equations are now called the Kolmogorov backward and forward equations. These investigation continued into the 1940s. Many good references about continuous-time Markov chain are developed, for example, Yamazato [1975], Hou and Guo [1988], Anderson [1991] and Chen [1997] etc.

Markov branching process with denumerable state space is one of the most important subclasses of Markov chains. It originally was used in the analysis of extinction of family names while the original problem was introduced by Galton in 1873. The basic property of Markov branching process is the branching property, i.e., different particles/elements act independently when they give offspring.

However, the independence properties may not be appropriate to all real life examples. Indeed, for example, sometimes branching events may happen with the interaction/collision of two or more particles but not by one particle alone. In other words, the model can be generalized with some interacting properties. The main aim of this thesis is to consider the interacting branching systems, and investigate the basic properties. Moreover, as investigating extinction behavior of the particles in the process is important and challenging, we will look into this area with more details.

1.2 Definitions and Basic Properties of Markov Chains

Definition 1.1 *A stochastic process* $X(t)$, $t \in [0, +\infty)$, *defined on a probability space* $(\Omega, \mathfrak{F}, P)$, *with values in a countable set* E *(to be called the state space of the process), is called a continuous-time parameter Markov chain if for any finite set* $0 \le t_1 \le t_2 \le \cdots \le t_n \le t_{n+1}$ *of 'times', and corresponding set* $i_1, i_2, \cdots, i_{n-1}, i, j$ *of states in* E *such that* $P\{X(t_n) = i, X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1\} > 0$, *we* *have*

$$
\begin{aligned}
& P\{X(t_{n+1}) = j | X(t_n) = i_n | X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1\} \\
= \; & P\{X(t_{n+1}) = j | X(t_n) = i_n\} \qquad (1.1)
\end{aligned}
$$

Equation 1.1 is called the Markov property. If for all s, t such that $0 \le s \le t$ and all $i, j \in E$, the conditional probability $P(X(t) = j|X(s) = i)$ appearing on the right-hand side of 1.1 depends only on $t - s$, and not s and t individually, we say that the process $\{X(t), t \in [0, +\infty)\}$ is homogeneous, or has stationary transition probabilities. In this case, then, $P(X(t) = j|X(s) = i) = P(X(t-s) = j|X(0) = i)$, and we define

$$p_{ij}(t) := P(X(t) = j|X(0) = i), \quad i, j \in E, t \ge 0,$$

as the transition function of the process.

The finite-dimensional probabilities of the process $\{X(t), t \ge 0\}$, that is probabilities of the form $P\{X(t_n) = i_n, X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1\}$, where $0 \le t_1 < t_2 < \cdots < t_n$ and $i_1, i_2, \cdots, i_n \in E$, are all expressible in terms of the transition function $p_{ij}(t)$ and the initial probability distribution $p_i = P(X(0) = i)$, $i \in E$.

Definition 1.2 *Let E be a countable set, to be called the state space. A family of functions $(p_{ij}(t); i, j \in E, t \ge 0)$ is called a transition function on E if*

(i) $p_{ij}(t) \ge 0$ for all $t \ge 0$ and $i, j \in E$; and

$$p_{ij}(0) = \delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \ne j. \end{cases} \qquad (1.2)$$

(ii) For all $t \geq 0$, $i \in E$

$$\sum_{j \in E} p_{ij}(t) \leq 1. \tag{1.3}$$

(iii)For all $s, t \geq 0$ and $i, j \in E$

$$p_{ij}(s+t) = \sum_{k \in E} p_{ik}(s) p_{kj}(t). \tag{1.4}$$

(This is called the Chapman -Kolmogorov equation.)

Furthermore, we call this transition function standard if

(iv)

$$\lim_{t \to 0} p_{ii}(t) = 1 \text{ for all } i \in E, \tag{1.5}$$

(and so, by the inequality $0 \leq \sum_{j \neq i} p_{ij}(t) \leq 1 - p_{ii}(t)$, we have $p_{ij}(t) \to \delta_{ij}$ as $t \to 0$ for all $i, j \in E$).

A transition function is called honest if $\sum_{j \in E} p_{ij}(t) = 1$ for all $t \geq 0$, $i \in E$, and dishonest otherwise.

For a transition function $(p_{ij}(t); i, j \in E, t \geq 0)$, we define

$$q_{ij} = \lim_{t \to 0^+} \frac{p_{ij}(0) - \delta_{ij}}{t} \tag{1.6}$$

Theorem 1.1 *Let $(p_{ij}(t); i, j \in E, t \geq 0)$ be a transition function. Then for all $i, j \in E$, $q_{ij} = p'_{ij}(0)$ exists and*

$$\begin{cases} 0 \leq q_{ij} < +\infty, & \text{if} \quad i \neq j; \\ \sum_{j \neq i} q_{ij} \leq q_i = -q_{ii} \leq +\infty, & \text{for } i \in E. \end{cases} \tag{1.7}$$

The matrix $Q = (q_{ij}; i, j \in E)$ is called a Q-matrix of the transition function $(p_{ij}(t); i, j \in E)$. A state i is called stable if $q_i := -q_{ii} < +\infty$ and instantaneous if $q_i = +\infty$.A state i is called absorbing if $q_i = 0$.

Definition 1.3 *A Q-matrix $Q = (q_{ij}; i, j \in E)$ is called a conservative Q-matrix if $q_{ij} \geq 0 (i \neq j)$ and $\sum_{j \neq i} q_{ij} = -q_{ii}$ for all $i \in E$.*

Lemma 1.1 *Let $(p_{ij}(t))$ be a transition function and suppose state i is stable, then*

$q_{ij} = p'_{ij}(0)$ *exists and is finite.*

Lemma 1.2 *Let $(p_{ij}(t))$ be a transition function and suppose state i is stable, then*

(i) $\sum_{j\in E} p'_{ij}(s) + d'_i(s) = 0$ *for all* $s > 0$*, where* $d_i(s) = 1 - \sum_{j\in E} p_{ij}(s)$*,*

(ii) $\sum_{j\in E} |p'_{ij}(s)| \leq 2q_i$*,*

(iii) $\sum_{j\in E} q_{ij} \leq 0$*,* $\sum_{j\in E} q_{ij} = 0$*, if E is finite and $p_{ij}(t)$ is honest.*

Let $(r_{ij}(\lambda); i, j \in E, \lambda > 0)$ be the Laplace transform of a transition function $(p_{ij}(t); i, j \in E)$. The properties stated in Definition 1.2 become

(i) $r_{ij}(\lambda) \geq 0$ for all $i, j \in E$ and $\lambda > 0$,

(ii) $\lambda \sum_{k\in E} r_{ik}(\lambda) \leq 1$, for all $i \in E$ and $\lambda > 0$,

(iii)$r_{ij}(\lambda) - r_{ij}(\mu) + (\lambda - \mu) \sum_{k\in E} r_{ik}(\lambda) r_{kj}(\mu) = 0$ for all $i, j \in E$ and $\lambda, \mu > 0$,

(iv) $\lim_{\lambda\to\infty} \lambda r_{ii}(\lambda) = 1$ for all $i \in E$.

Lemma 1.3 *A function $r_{ij}(\lambda) \geq 0$ for all $i, j \in E$ and $\lambda > 0$ such that*

$$r_{ij}(\lambda) \geq 0 \text{ for all } i, j \in E \text{ and } \lambda > 0, \tag{1.8}$$

$$\lambda \sum_{k\in E} r_{ik}(\lambda) \leq 1, \text{ for all } i \in E \text{ and } \lambda > 0, \tag{1.9}$$

$$r_{ij}(\lambda) - r_{ij}(\mu) + (\lambda - \mu) \sum_{k\in E} r_{ik}(\lambda) r_{kj}(\mu) = 0 \text{ for all } i, j \in E \text{ and } \lambda, \mu > 0, \tag{1.10}$$

$$\lim_{\lambda\to\infty} \lambda r_{ii}(\lambda) = 1 \text{ for all } i \in E. \tag{1.11}$$

is called a resolvent function, and is called honest if equality holds in (1.9).

Definition 1.4 *Let $(p_{ij}(t))$ be a transition function, and let*

$$r_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)dt, \quad \lambda > 0, i, j \in E \tag{1.12}$$

be the Laplace transform of $p_{ij}(t)$.

Proposition 1.1 *Let $(r_{ij}(\lambda), i, j \in E; \lambda > 0)$ be a resolvent function. Then there is a unique transition function $p_{ij}(t)$ such that*

$$r_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t), \quad \text{for all } \lambda > 0, i, j \in E \tag{1.13}$$

and $p_{ij}(t)$ is honest if $r_{ij}(\lambda)$ is.

Lemma 1.4 *The following equations are equivalent to each other and they are two equivalent forms of the Kolmogorov forward equation:*

$$p'_{ij}(t) = \sum_{k \in E} p_{ik}(t) q_{kj}, \quad t \geq 0, i, j \in E, \tag{1.14}$$

$$\lambda r_{ij}(\lambda) = \delta_{ij} + \sum_{k \in E} r_{ik}(\lambda) q_{kj}, \quad \lambda \geq 0, i, j \in E. \tag{1.15}$$

Lemma 1.5 *The following equations are equivalent to each other and they are two equivalent forms of the Kolmogorov backward equation:*

$$p'_{ij}(t) = \sum_{k \in E} q_{ik}(t) p_{kj}(t), \quad t \geq 0, i, j \in E, \tag{1.16}$$

$$\lambda r_{ij}(\lambda) = \delta_{ij} + \sum_{k \in E} q_{ik} r_{kj}(\lambda), \quad \lambda \geq 0, i, j \in E. \tag{1.17}$$

Suppose $(p_{ij}(t); i, j \in E)$ is the transition function for a continuous-time Markov Chain. When we study the properties of the process, we may look into its corresponding Q- matrix $(q_{ij}; i, j \in E)$. It is because, in most cases, the finding the Q -matrix is much more easier than the transition function $(p_{ij}(t); i, j \in E)$ itself. $(p_{ij}(t); i, j \in E)$ is called Q- function or Q- process as some of its properties can be seen by the Q-matrix. Similarly, the corresponding Laplace transform $(\phi_{ij}(\lambda); i, j \in E)$ is called Q- resolvent. So, we now review some facts and properties for Q-matrix before further analysis.

Theorem 1.2 *Suppose $Q = (q_{ij}; i, j \in E)$ is a stable (but not necessarily conservative) Q- matrix. Then there exists a Q-function $(p_{ij}(t); i, j \in E)$ satisfying both the Kolmogorov backward and forward equations. Moreover, this Q- function $(f_{ij}(t); i, j \in E)$ is minimal in the sense that for any other Q-functions$(p_{ij}(t); i, j \in E)$, $f_{ij}(t) \leq p_{ij}(t) \quad (i, j \in E, t \geq 0)$.*

The minimal solution is called as the Feller minimal Q-function and the corresponding Laplace transform is called as the Feller minimal resolvent function.

The Feller Q-resolvent function $(\phi_{ij}(\lambda); i, j \in E)$ can be obtained either by the backward integral recursion

$$\begin{cases} \phi_{ij}^{(0)}(\lambda) = \frac{\delta_{ij}}{\lambda + q_i}, \\ \phi_{ij}^{(n+1)}(\lambda) = \frac{\delta_{ij}}{\lambda + q_i} + \sum_{k \neq i} \frac{q_{ij}}{\lambda + q_i} \phi_{kj}^{(n)}(\lambda), \quad n \geq 0. \end{cases} \tag{1.18}$$

where $\phi_{ij}^{(n)}(\lambda)$ converge to $\phi_{ij}(\lambda)$ as $n \to \infty$ for all $i, j \in E$, similarly, by the forward integral recursion

$$\begin{cases} \phi_{ij}^{(0)}(\lambda) = \frac{\delta_{ij}}{\lambda + q_j}, \\ \phi_{ij}^{(n+1)}(\lambda) = \frac{\delta_{ij}}{\lambda + q_j} + \sum_{k \neq j} \frac{q_{kj}}{\lambda + q_j} \phi_{ik}^{(n)}(\lambda), \quad n \geq 0. \end{cases} \tag{1.19}$$

where $\phi_{ij}^{(n)}(\lambda)$ converge to $\phi_{ij}(\lambda)$ as $n \to \infty$ for all $i, j \in E$.

The following theorem is related to the uniqueness of Q-function. If Q is conservative, then the Feller minimal Q-function is unique if and only if any one of the following four statements holds.

Theorem 1.3 *The following statements are equivalent.*

(i)The Feller minimal Q-matrix function, $f_{ij}(t)$ is the unique solution of the Kolmogorov backward equations.

(ii) The equation $Qx = \lambda x, 0 \leq x \leq 1$; that is,

$$\sum_{j \in E} q_{ij} x_j = \lambda x_i, \quad 0 \leq x_i \leq 1, i \in E, \tag{1.20}$$

has no nontrivial solution, for some (and therefore for all) $\lambda > 0$.

(iii) The inequality $Qx \geq \lambda x, \quad 0 \leq x_i \leq 1, i \in E$*; that is,*

$$\sum_{j \neq i} q_{ij} x_j \geq (\lambda + q_i) x_i, \quad 0 \leq x_i \leq 1, i \in E, \tag{1.21}$$

has no nontrivial solution, for some (and therefore for all) $\lambda > 0$.

(iv) The equation $Qx = \lambda x, -1 \leq x \leq 1$*; that is,*

$$\sum_{j \in E} q_{ij} x_j = \lambda x_i, \quad -1 \leq x_i \leq 1, i \in E, \tag{1.22}$$

has no nontrivial solution, for some (and therefore for all) $\lambda > 0$.

Definition 1.5 *A conservative* Q*- matrix which satisfies any of the conditions (i)-(iv) of the above theorem is said to be regular. The corresponding* Q*- function is honest and is unique.*

Theorem 1.4 *Suppose that the Feller minimal* Q*- function is dishonest. Then it is the unique* Q*-function satisfying the Kolmogorov forward equation if and only if the equation*

$$\sum_{i \in E} y_i q_{ij} = \lambda y_j, \quad y_j \geq 0, j \in E, \sum_{j \in E} y_j < +\infty \tag{1.23}$$

has no nontrivial solution, for some (and therefore for all) $\lambda > 0$.

The above results can be found in section 2.2 in Anderson [1991]. The uniqueness properties for Q-functions for different models will be discussed individually in detail in the corresponding chapters in this thesis.

After reviewing some fundamental facts about the uniqueness of Q-functions, if Q is regular, we can further discuss the absorbing behavior of the process. We give some related definitions in this chapter and we will look into the details in each model discussed in later chapters.

Let $(p_{ij}(t), i, j \in E)$ be a transition function, and let $(X(t), t > 0)$ denote a continuous-time Markov chain with state space E, and having $p_{ij}(t)$ as its transition function.

Definition 1.6 *Given $i, j \in E$, if $p_{ij}(t) > 0$ for some (and therefore all) $t > 0$, we say that j can be reached from i and we we denote this with $i \hookrightarrow j$. If i and j can be reached from each other, we say that i and j communicate and we denote this with $i \leftrightarrow j$.*

Definition 1.7 *Let $(p_{ij}(t); i, j \in E)$ be a regular transition function.*

(i)A state $i \in E$ is recurrent if $\int_0^\infty p_{ii}(t)dt = +\infty$, and transient if $\int_0^\infty p_{ii}(t)dt < +\infty$. The process $(p_{ij}(t); i, j \in E)$ is called recurrent if all states are recurrent.

(ii) A state i is called positive recurrent if $\lim_{t\to\infty} p_{ii}(t) > 0$. The process $(p_{ij}(t); i, j \in E)$ is called positive recurrent if all the states are positive recurrent.

(iii) A state i is called absorbing if $q_i = 0$.

It is trivial that if i communicates with j, then i is transient if and only if j is transient. So transience and recurrence are class properties.

Definition 1.8 *We can define the average time spent by the process staying in a particular state i*

$$\begin{aligned}\int_0^\infty p_{ii}(t)dt &= \int_0^\infty E(I_{\{X(t)=i\}}|X(0) = i)dt \\ &= E(\int_0^\infty I_{\{X(t)=i\}}dt|X(0) = i) \\ &= E(\textit{ time spent in } i \mid \textit{start in } i\ ).\end{aligned}$$

The criterion of recurrence is not the main focus of our thesis.For more details in this area, we can look at some specialized books, for example, Anderson [1991] and Chen [2004]. After reviewing some background knowledge and properties for continuous-time Makov chain, we now look into a classical model, Markov branching processes as preparation for later chapters for more generalized models.

1.3 Markov Branching Processes

A (one-dimensional) continuous-time Markov branching process (MBP) is a special class of continuous-time Markov chain with the state space $Z_+ = \{0, 1, \cdots\}$ with transition function $P(t) = (p_{ij}(t); i, j \in Z_+)$ and the q-matrix $Q = (q_{ij}; i, j \in Z_+)$ given by

$$q_{ij} = \begin{cases} ib_{j-i+1}, & \text{if } i \geq 1 \\ 0, & \text{otherwise} \end{cases} \tag{1.24}$$

where

$$b_k \geq 0 (k \neq 1), -b_1 = \sum_{k \neq 1} b_k < +\infty. \tag{1.25}$$

Markov branching processes are well-known in probability theory and there are many studies of Markov branching processes in biological and physical sciences. It is known that the basic property which governs the evolution of a Markov branching processes is the branching property, that is, different particles act independently when giving birth or death. As a review of the Markov branching processes, we look into two basic but important theorems. Theorem 1.5 (i) actually is based on the result of Theorem 1.3 in this chapter. Theorem 1.5(ii) explains what the branching property means in mathematical notations. Theorem 1.6 gives the regularity criteria for the Markov Branching Process which can be used to compare with other regularity criteria in other generalized models. Theorem 1.5 and Theorem 1.6 can be found in section 3.3 in Anderson [1991].

Theorem 1.5 *Let Q be given in (1.24) and (1.25), with $(p_{ij}(t))$ being the corresponding Feller minimal Q-function. Then, we know that*

(i) $p_{ij}(t)$ is the unique solution of the Kolmogorov forward equation.

(ii) For all $i \geq 1, j \geq 0$ and $t > 0$,

$$p_{ij}(t) = \sum_{l_1+l_2+\cdots+l_i=j} p_{1l_1}(t) \cdots p_{1l_i}(t) \tag{1.26}$$

With the q-matrix Q defined (1.24) and (1.25), in order to investigate the properties of the branching process, we define a generating function $B(s) = \sum_{k=0}^{\infty} b_k s^k$. There will be lots of discussion and usage of similar generating functions in other generalized models.

Theorem 1.6 *Let Q be given in (1.24) and (1.25), Q is regular if and only if one of the following conditions holds.*

(i) $B'(1) < +\infty$.

(ii)$B'(1) = +\infty$ and for some (therefore for all) $\epsilon \in (q, 1)$,

$$\int_{\epsilon}^{1} \frac{ds}{B(s)} = -\infty \tag{1.27}$$

where q is the smallest positive root of $B(s) = 0$.

It should be noted that the branching property relies on the independence assumption. In most realistic situations, however, this assumption may not be appropriate. Indeed, in practice, particles usually interact with each other. This may explain the reason why there always has been a great effort to generalise the ordinary branching processes to the more general branching models, for example, in the book edited by Athreya and Jagers [1997], Chen and Renshaw [1995] and Chen [2002].

1.4 Outline of the thesis

This thesis mainly concentrates on theoretical study of some generalized Markov models, especially the interacting branching collision processes. In order to study these complex processes, besides the well-known ordinary Markov branching processes, we need to study the collision branching process which is an important component of the interacting branching collision processes. After getting more understanding in collision branching processes, we are then able to move forward to the model most concerned in this thesis. Through studying the interacting branching collision processes, we may apply similar techniques in related models. Markov branching process with immigration-migration and resurrection is chosen to consider. Studying the properties of this model may also be treated as an example for the new techniques that we used in interacting branching collision processes.

Chapter 2 discusses the collision branching processes. This model accounts for the effect of collision or interaction, between particles or individuals. We study the regularity, uniqueness, extinction and explosion behavior in detail.

Chapter 3 discusses the interacting branching collision processes. Studying this model is very challenging as the branching component and the collision component interact with each other. We study the regularity, uniqueness and extinction behavior in detail. When we study the extinction behavior in chapter three, however, in some cases, the closed forms of extinction probability are very complicated. It is difficult to obtain useful information from those complicated expressions. Last section of Chapter 3 deals with this problem and we try to reveal the asymptotic behavior for these complex forms of extinction probabilities and show that the asymptotic behavior for these complicated extinction probabilities actually takes a very simple form.

Chapter 4 discusses the Markov branching processes with immigration-migration and resurrection. We study the regularity, uniqueness, extinction and asymptotic behavior for this model. We try to use similar analytic tools and techniques used in Chapter 3 in tackling some difficult points. Although the two models are quite different, Chapter 4 may still be treated as an example for the techniques introduced in Chapter 3.

Chapter 5 discusses the interacting collision processes with immigration - migration and resurrection and interacting branching collision processes with immigration - migration and resurrection. We study some properties and criteria of regularity, uniqueness and extinction probability for these models. Also, the difficulties for finding the extinction probability are also discussed at the end of the chapter.

My original material starts from Chapter 4. The model used in chapter 4 were introduced by Li and Liu [2011]. In Li and Liu [2011], some calculation in cases of extinction probability evaluation were not strictly defined. My contribution focuses on the extinction probability evaluation and discussing the asymptotic behavior for the extinction probability in Chapter 4. While two interacting branching models are discussed in Chapter 5. Some important properties for the two models are studied in detail.

Finally, a summary of this thesis is presented in Chapter 6.

Chapter 2

Collision Branching Processes

2.1 Introduction

Collision branching processes (CBP) have a great role in the theory of probability especially in physical sciences. Consider a branching process with collisions between particles occurring at random; when any two particles have a collision, they are removed and replaced by k 'offspring' with probability $p_k (k \geq 0)$, independently of other collisions. In any small time interval $(t+\Delta t)$ there is a positive probability $\theta\Delta t + o(\Delta t)$ that a collision occurs, and there is a probability $o(\Delta t)$ of two or more collisions occurring in this time interval. At time t, supposing that there are i particles present. With the assumption of equally likely pairing, after Δt, there will be j particles with probability $\binom{i}{2}\theta p_{j-i+2}\Delta t + o(\delta t)$. We can use $X(t)$, the number of particles living at time t, to be a continuous- time Markov chain with non-zero transition rates $q_{ij} = \binom{i}{2} b_{j-i+2}, (j \geq i-2, i \geq 2)$, where $b_2 = -\theta(1-p_2)$ and $b_j = \theta p_j \ (j \neq 2)$.

Here, we give the formal definition. In this work, we study the model considered in Chen et al. [2004], Chen and Li [2009] in detail. This chapter closely follows Chen et al. [2004].

2.2 Preliminary and Mathematical Model

Definition 2.1 *A q-matrix $Q = (q_{ij};\ i, j \in \mathbf{Z}_+)$ is called a collision branching q-matrix if it has the following form:*

$$q_{ij} = \begin{cases} \binom{i}{2} b_{j-i+2}, & \text{if } j \geq i-2, i \geq 2 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where

$$\begin{cases} b_j \geq 0 (j \neq 2), and - b_2 = \Sigma_{j\neq 2} b_j < +\infty, \\ \text{together with } b_0 > 0, b_1 > 0 \text{ and } \Sigma_{j=3}^{\infty} b_j > 0. \end{cases}$$

Definition 2.2 *A continuous-time Markov chain on the state space $\mathbf{Z}_+$ is called a collision branching process (henceforth referred to as a CBP) if its transition function $P(t) = (p_{ij}(t); i, j \in \mathbf{Z}_+)$ satisfies the forward equation*

$$P'(t) = P(t)Q \tag{2.2}$$

where Q is a CB-q-matrix.

It is obvious that CBPs have two absorbing states which are 0 and 1. Therefore, absorbing/ extinction probabilities for these states require evaluation separately. Also, it can be noted that total rate of leaving each state i is a quadratic function of i, this might lead to the expectation of a more readily explosive behavior. Regularity and uniqueness criteria are considered in section 3. Evaluation of extinction probabilities is discussed in section 4. Expected explosion time is discussed in section 5.

2.3 Uniqueness

Since Q is stable and conservative, by Theorem 1.2, the Feller minimal process for CBP always exists. We need to investigate, under what conditions the process would be unique. In order to investigate these properties of CBPs, it is necessary to define the generating function of a known sequence $\{b_k; k \geq 0\}$ as

$$B(s) = \sum_{k=0}^{\infty} b_k s^k, |s| \leq 1. \tag{2.3}$$

It is obvious that $B(0) = b_0 > 0$ and $B(1) = 0$. Set $m_1 := B'(1) = \sum_{j=1}^{\infty} jb_{j+2} - 2b_0 - b_1$, which satisfies $-\infty < m_1 \leq +\infty$. This quantity, $B'(1)$, measures the drift away from state 0. With normalization with $\sum_{j \neq 2} b_j$, m_1 is the expected jump size from any state i. The above generating function plays an extremely important role in our later analysis. The sign of $B'(1)$ determines the number of roots for the equation $B(s) = 0$ in [0,1]. It is clear that $B(s)$ is well defined at least on $[-1, 1]$. The following simple yet important properties of these functions will be constantly used in this chapter and we state them here for convenience.

Lemma 2.1 (i) *The equation $B(s) = 0$ has at most two roots in $[0, 1]$. More specifically, if $B'(1) \leq 0$ then $B(s) > 0$ for all $s \in [0, 1)$ and 1 is the only root of the equation $B(s) = 0$ in $[0, 1]$, while if $B'(1) > 0$ then $B(s) = 0$ has an additional simple root ρ_b satisfying $0 < \rho_b < 1$ such that $B(s) > 0$ for $s \in (0, \rho_b)$ and $B(s) < 0$ for $s \in (\rho_b, 1)$.*

Proof. We start from proving $B'(s) = 0$ has either one root or two roots in [0,1]. It can be noticed that $B'''(s) > 0$ for all $s \in [0, 1)$, so $B'(s)$ is convex for $s \in [0, 1)$. This means that $B'(s) = 0$ has at most two roots for $s \in [0, 1)$. Furthermore, because of the fact that $B(0) = b_0 > 0$ and $B(1) = 0$, $B'(s)$ cannot be greater than 0 for all $s \in [0, 1)$.This tells that $B'(s) = 0$ must have at least one root in (0,1).

Then, it can be noted that, in (0,1), $B'(s) = 0$ has one root only when $B'(1) \leq 0$.When $B'(1) \leq 0$, since $B'(0) = b_1 > 0$, there exists $\xi \in [0, 1)$ such that $B'(s) > 0$ for all $s \in (0, \xi)$ and $B'(s) < 0$ for all $s \in (\xi, 1)$. This implies that $B(s)$ is strictly increasing on $[0, \xi]$, and strictly decreasing on $[\xi, 1]$.With the end point value of $B(0)$ and $B(1)$, $B(s) > 0$ for all $s \in [0, 1)$, imply that 1 is the only root for $B'(s) = 0$ in [0,1].

Next, if $B'(1) > 0$, (including $B'(1) = +\infty$), there exist ξ_1 and ξ_2 with $0 < \xi_1 < \xi_2 < 1$, such that $B'(\xi_1) = B'(\xi_2) = 0$ and that $B'(s) > 0$ for all $s \in [0, \xi_1) \cup (\xi_2, 1]$ and $B'(s) < 0$ for $s \in (\xi_1, \xi_2)$. $B(s)$ is strictly increasing on $[0, \xi_1] \cup [\xi_2, 1]$ and strictly decreasing on $[\xi_1, \xi_2]$. So, $B(s) = 0$ must have 2 roots, one root is 1 and the other root is within (ξ_1, ξ_2). ■

Lemma 2.2 *Let $(p_{ij}(t); i, j \in Z_+)$ be the Feller minimal Q-function, where Q is a CB q-matrix given in (2.1) and (2.2), then all the states $k \geq 2$ are transient, i.e., for any $i \geq 0$ and $k \geq 2$,*

$$\int_0^\infty p_{ik}(t)dt < +\infty$$

This also means that $\lim_{t\to\infty} p_{ik}(t) = 0$.

Proof. For any fixed $i \geq 0$, from the Kolmogorov forward equations, we have

$$p_{i0}(t) = \delta_{i0} + q_{20} \cdot \int_0^t p_{i2}(t)ds,$$

it implies that $\int_0^\infty p_{i2}(t)dt < +\infty$. Then by mathematical induction with the Kolmogorov forward equations, we can get $\int_0^\infty p_{ik}(t)dt < +\infty$ for all $k \geq 2$. This implies $\lim_{t\to\infty} p_{ik}(t) = 0$. ■

Now, we can continue to deal with the question of uniqueness. It should be noted that if a Feller minimal Q-transition function is honest, the corresponding conservative q-matrix Q is said to be regular and this Q-transition function is unique.

Theorem 2.1 *The CB-q-matrix is regular if and only if $B'(1) \leq 0$.*

Proof. If part: Suppose $B'(1) \leq 0$ and let $P(t) = \{p_{ij}(t)\}$ be the minimal Q-transition function. Substituting (2.1) into the forward equations (2.2) gives

$$p'_{ij}(t) = \sum_{k=2}^{j+2} p_{ik}(t)\binom{i}{2}b_{j-k+2}, i, j \geq 0. \tag{2.4}$$

Then, multiplying s^j on both sides of the above equality and summing over j, for any i and $0 \leq s < 1$,

$$\begin{aligned}
\sum_{j=0}^{\infty} p'_{ij}(t)s^j &= \sum_{j=0}^{\infty}(\sum_{k=2}^{j+2} p_{ik}(t)\binom{k}{2}b_{j-k+2}s^j) \\
&= \sum_{k=2}^{\infty}\sum_{j=k-2}^{\infty} p_{ik}(t)\binom{k}{2}b_{j-k+2}s^j \\
&= \sum_{k=2}^{\infty} p_{ik}(t)\binom{k}{2}s^{k-2}\sum_{j=k-2}^{\infty} b_{j-k+2}s^{j-k+2} \\
&= \sum_{k=2}^{\infty} p_{ik}(t)\binom{k}{2}s^{k-2}\sum_{j=0}^{\infty} b_j s^j \\
&= B(s)\sum_{k=2}^{\infty} p_{ik}(t)\binom{k}{2}s^{k-2}
\end{aligned}$$

We get, for $0 < s < 1$,

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s)\sum_{k=2}^{\infty}\binom{k}{2}p_{ik}(t)s^{k-2}, i \geq 0. \tag{2.5}$$

By Lemma 2.1, together with the above conditions, $B(s) > 0$, for all $s \in (0, 1)$, the right hand side of (2.5) is strictly positive.

By Lemma 1.2, for all $t \geq 0$,

$$\sum_{j=0}^{\infty} |p'_{ij}(t)| \leq 2q_i, \tag{2.6}$$

where $q_i := -q_{ii} = \binom{i}{2}b_2 < \infty$. The series $\sum_{j=0}^{\infty} p'_{ij}(t)s^j$ converges uniformly on $[0, \infty)$ for all $s \in [0, 1)$. Also, as $p'_{ij}(t)$ are continuous , therefore, the derivative of $\sum_{j=0}^{\infty} p_{ij}(t)s^j$ exists and equals to $\sum_{j=0}^{\infty} p'_{ij}(t)s^j$.

After taking integration for the left hand side, we have

$$\sum_{j=0}^{\infty} p_{ij}(t)s^j - s^i \geq 0, \quad i \geq 0, 0 \leq s < 1. \tag{2.7}$$

Letting $s \uparrow 1$ in above equation, $\sum_{j=0}^{\infty} p_{ij}(t) \geq 1$ for all $i \geq 0$, We have shown that the minimal Q-transition function is honest, and Q is regular.

Only if part: Suppose $B'(1) > 0$, our aim is to construct a Q^* matrix which is not regular, then through a comparison of Q^* with the original CB q-matrix Q leads to the conclusion that Q is not regular. Define a (conservative) birth death q-matrix $Q^* = (q^*_{ij}, i, j \in Z_+)$ by

$$q^*_{ij} = \begin{cases} \binom{i}{2} b^*, & \text{if } j = i+1, i \geq 2 \\ \binom{i}{2} a^*, & \text{if } j = i-1, i \geq 2 \\ -\binom{i}{2}(a^* + b^*), & \text{if } j = i \geq 2 \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

where $b^* > a^* > 0$. Here, this Q^* is not regular by Theorem 3.2.2 in Anderson [1991]. By doing so, we need to prove that

$$\begin{aligned} & \sum_{n=2}^{\infty} (\frac{1}{q_{nn+1}}) + \frac{q_{nn-1}}{q_{nn+1} q_{n-1n}} + \cdots + \frac{q_{nn-1} \cdots q_{32}}{q_{nn+1} \cdots q_{23}} \\ = \quad & \frac{2}{b^*} (\sum_{n=2}^{\infty} (\frac{1}{n(n-1)} + \frac{a^*}{b^*} \frac{1}{(n-1)(n-2)} + \cdots + \frac{a^{*n-2}}{b^{*n-2}} \frac{1}{(2)(1)}) \\ = \quad & \frac{2}{b^*} \cdot \sum_{n=2}^{\infty} \sum_{k=2}^{n} (\frac{a^*}{b^*})^{n-k} \frac{1}{k(k-1)} \\ = \quad & \frac{2}{b^*} \cdot \sum_{k=2}^{\infty} \sum_{n=k}^{n} (\frac{a^*}{b^*})^{n-k} \frac{1}{k(k-1)} \\ = \quad & \frac{2}{b^* - a^*} \sum_{k=2}^{\infty} \frac{1}{k(k-1)} < \infty. \end{aligned}$$

Next, we continue our aim, to construct a Q^* matrix which is not regular by choosing suitable a^* and b^* for the CB-q-matrix Q. After doing so, conclusion that Q is not regular then follows.

In order to so, first note that $B'(1) > 0$ is the same as $2b_0 + b_1 < \sum_{j=1}^{\infty} jb_{j+2} (\leq +\infty)$, so we may choose a^* and b^* with

$$2b_0 + b_1 < a^* < b^* < \sum_{j=1}^{\infty} jb_{j+2} \tag{2.9}$$

We have shown that Q^* is not regular, so the equation

$$(\lambda I - Q^*)u = 0 \quad (\lambda > 0) \tag{2.10}$$

has non-trivial solution, we denote this by $\{u^* = u_i(\lambda), i \geq 0\}$.

It is obvious that u^* depends on both value of a^* and b^* and what we want to get the u^* satisfies

$$\lambda u^* \leq Qu^* \tag{2.11}$$

By Theorem 1.3, if such u^* can be found, the conclusion that Q is not regular follows.

We need to find a^* and b^* such that (2.11) holds.

First, we prove that a^* and b^* can be chosen such that both

$$\sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} (\frac{a^*}{b^*})^{k-1} > b^* \tag{2.12}$$

and

$$b_0(\frac{b^*}{a^*}) + (b_0 + b_1) < a^* \tag{2.13}$$

hold. Let x_n be a sequence that strictly decreasing and converging to $2b_0 + b_1$. We denote this relation by $x_n \Downarrow 2b_0 + b_1$. In order words, $\frac{2b_0+b_1}{x_n} \Uparrow 1$,

As $n \to \infty$,

$$\sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} (\frac{2b_0 + b_1}{x_n})^{k-1} \Uparrow \sum_{j=1}^{\infty} jb_{j+2} \tag{2.14}$$

Then it is obvious that b^* can be chosen such that

$$\sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} (\frac{2b_0 + b_1}{b^*})^{k-1} > b^* \tag{2.15}$$

Similarly, let y_n be a sequence that strictly increasing and converging to b^*.

As $n \to \infty$,

$$b_0(\frac{b^*}{y_n}) + b_0 + b_1 \Downarrow 2b_0 + b_1 \tag{2.16}$$

Then it is obvious that a^* can be chosen that

$$b_0(\frac{b^*}{a^*}) + (b_0 + b_1) < a^* \tag{2.17}$$

Next, we replace $(2b_0 + b_1)$ in (2.15) by a^* , which is greater,

$$\sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} (\frac{a^*}{b^*})^{k-1} > b^* \tag{2.18}$$

holds.

To prove $\lambda u^* \leq Qu^*$, we note the solution to $(\lambda I - Q^*)u = 0$ satisfies $u_0(\lambda) = u_1(\lambda) = 0$ and

$$b^*(u_{i+1}(\lambda) - u_i(\lambda)) = a^*(u_i(\lambda) - u_{i-1}(\lambda)) + \lambda u_i(\lambda)\binom{i}{2}^{-1} \quad i \geq 2. \tag{2.19}$$

For example, if $i = 2$, we see that $b^*(u_3(\lambda) - u_2(\lambda)) = (a^* + \lambda)u_2(\lambda) > 0$. This implies that $u_i(\lambda)$ is strictly increasing in i for each fixed λ.

From (2.19), we can

$$u_{i+k}(\lambda) - u_{i+k-1}(\lambda) \geq \frac{a^*}{b^*}(u_{i+k-1}(\lambda) - u_{i+k-2}(\lambda))$$

and

$$u_i(\lambda) - u_{i-1}(\lambda) \leq \frac{b^*}{a^*}(u_{i+1}(\lambda) - u_i(\lambda))$$

After some algebra, we get

$$u_{i+k}(\lambda) - u_{i+k-1}(\lambda) \geq (\frac{a^*}{b^*})^{k-1}(u_{i+1}(\lambda) - u_i(\lambda)) \tag{2.20}$$

and

$$u_{i-1}(\lambda) - u_{i-2}(\lambda) \leq \frac{b^*}{a^*}(u_i(\lambda) - u_{i-1}(\lambda)) \tag{2.21}$$

Equation (2.11) is obviously true for $i = 0$ or $i = 1$.

For $i \geq 2$, we have

$$\begin{aligned}
(Qu)_i &= \sum_{j=0}^{\infty} q_{ij} u_j \\
&= \sum_{j=i-1}^{\infty} q_{ij} u_j \\
&= \binom{i}{2} [b_0 u_{i-2}(\lambda) + b_1 u_{i-1}(\lambda) + \sum_{j=i+1}^{\infty} b_{j-i+2} u_j(\lambda) - \sum_{j=i-2, j\neq i}^{\infty} b_{j-i+2} u_i(\lambda)] \\
&= \binom{i}{2} [b_0 (u_{i-2}(\lambda) - u_i(\lambda)) + b_1 (u_{i-1}(\lambda) - u_i(\lambda)) + \sum_{j=i+1}^{\infty} b_{j-i+2} (u_j(\lambda) - u_i(\lambda))] \\
&= \binom{i}{2} (-I_d + I_b) \qquad (2.22)
\end{aligned}$$

Here, both I_b and I_d , which define below, are positive.

By (2.12) and (2.20), we can get

$$\begin{aligned}
I_b &= \sum_{j=1}^{\infty} b_{j+2} (u_{i+j}(\lambda) - u_i(\lambda)) \\
&= \sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} (u_{i+k}(\lambda) - u_{i+k-1}(\lambda)) \\
&\geq \sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} (\frac{a^*}{b^*})^{k-1} (u_i(\lambda) - u_{i-1}(\lambda) \\
&\geq b^* (u_{i+1}(\lambda) - u_i(\lambda)). \qquad (2.23)
\end{aligned}$$

Similarly, by (2.13) and (2.21), we can get

$$\begin{aligned}
I_d &= b_0 (u_i(\lambda) - u_{i-2}(\lambda)) + b_1 (u_i(\lambda) - u_{i-1}(\lambda)) \\
&= (b_0 + b_1)(u_i(\lambda) - u_{i-1}(\lambda)) + b_0 (u_{i-1}(\lambda) - u_{i-2}(\lambda)) \\
&\leq (b_0 (\frac{b^*}{a^*}) + (b_0 + b_1))(u_i(\lambda) - u_{i-1}(\lambda)) \\
&< a^* (u_i(\lambda) - u_{i-1}(\lambda)). \qquad (2.24)
\end{aligned}$$

From (2.23) and (2.24), together with (2.19) and (2.22), (2.11) is verified. The ‘Only if’ part is thus done.

■

From Theorem 2.1, we can see that if the drift $B'(1)$ is smaller or equal to zero, the CBP is then unique. How about if $B'(1)$ is positive and even if it $B'(1) = +\infty$? We will answer this question in next theorem.

Theorem 2.2 *There exists only one CBP.*

Proof. In this part, we only need to consider the case for $0 < B'(1) < \infty$. To prove that the CBP is unique, we will show that the forward equations have a unique solution. To show this, we will use the Theorem 2.28 in Anderson [1991], i.e we prove that the equation $\mu(\lambda)(\lambda I - Q) = 0$, $0 \leq \mu(\lambda) \in l_1$, has no nontrivial solution for some (and therefore for all) $\lambda > 0$.

Suppose there is such a non-trivial solution when $\lambda = 1$, denoting as $\mu = \mu_i, i \geq 0$. Then, by (2.1), we have

$$\mu_j = \sum_{i=2}^{j+2} \mu_i \binom{i}{2} b_{j-i+2}, \quad j \geq 0, \tag{2.25}$$

with

$$\mu_j \geq 0 \quad (j \geq 0) \quad \textit{and} \quad \sum_{j=0}^{\infty} \mu_j < +\infty \tag{2.26}$$

It is obvious that

$\sum_{j=2}^{\infty} \mu_j > 0$ and $\sum_{j=2}^{\infty} \mu_j s^j < \infty$ for all $s \in [0, 1]$.

By root test, because of having same radius of convergence,

$$\sum_{j=2}^{\infty} \binom{j}{2} \mu_j s^j < \infty, \quad 0 \leq s < 1. \tag{2.27}$$

Together with Fubini's Theorem,

$$\sum_{j=0}^{\infty} \mu_j s^j = B(s) \sum_{i=2}^{\infty} \binom{i}{2} \mu_i s^{i-2}, \quad 0 \leq s < 1. \tag{2.28}$$

From (2.25) and (2.28), it is clear that both $\sum_{j=0}^{\infty} \mu_j s^j$ and $\sum_{i=2}^{\infty} \binom{i}{2} \mu_i s^{i-2}$ are strictly positive for all $s \in (0,1)$ and thus $B(s) > 0$ for all $s \in (0,1)$, which contradicts Lemma 2.1 because $B'(1) \in (0, \infty]$. The proof is then complete. ■

2.4 Extinction and Explosion

Knowing that the CBP is uniquely determined by its q-matrix, we now examine some of its properties. Let $\{X(t), t \geq 0\}$ be the CBP, and let $P(t) = \{p_{ij}(t)\}$ denote the corresponding transition function.

Define the extinction times τ_0 and τ_1 for states 0 and 1 by

$$\tau_0 = \begin{cases} \inf\{t > 0, X(t) = 0\} & \text{if } X(t) = 0 \text{ for some } t > 0 \\ +\infty & \text{if } X(t) \neq 0 \text{ for all } t > 0 \end{cases}$$

$$\tau_1 = \begin{cases} \inf\{t > 0, X(t) = 1\} & \text{if } X(t) = 1 \text{ for some } t > 0 \\ +\infty & \text{if } X(t) \neq 1 \text{ for all } t > 0 \end{cases}$$

and denote the corresponding extinction probabilities by

$$a_{i0} = P\{\tau_0 < +\infty | X(0) = i\} \text{ and } a_{i1} = P\{\tau_1 < +\infty | X(0) = i\}.$$

Theorem 2.3 *The extinction probabilities satisfy*

$$a_{i0} + q a_{i1} = q^i, \tag{2.29}$$

where $q = \rho_b$ is the smallest root of $B(s) = 0$ in $[0,1]$. More specifically,

$$a_{i0} + a_{i1} = 1, \quad \textit{if } B'(1) < 0, \tag{2.30}$$

$$a_{i0} + q a_{i1} = q^i < 1, \quad \textit{if } 0 < B'(1) \leq +\infty. \tag{2.31}$$

Proof. First deal with (2.30), refer to Theorem 2.1 , CB q-matrix is regular if and only if $B'(1) \leq 0$. When $B'(1) \leq 0$, we have the following, from (2.7),

$$\sum_{j=0}^{\infty} p_{ij}(t)s^j - s^i \geq 0, \quad i \geq 0,\, 0 \leq s < 1. \tag{2.32}$$

Also,

$$\lim_{t\to\infty} p_{ij}(t) = 0, \quad \forall\, i, j \geq 2, \text{ since states } i \geq 2 \text{ are transient }. \tag{2.33}$$

Then, taking $t \to \infty$ in (2.32) and using Dominated Convergence Theorem,

$$p_{i0}(t)s^0 + p_{i0}(t)s^1 + \sum_{j=1}^{\infty} p_{ij}s^j - s^i \geq 0, \tag{2.34}$$

$$a_{i0} + s a_{i1} \geq s^i \quad \forall\, s \in [0, 1). \tag{2.35}$$

Taking $s \uparrow 1$ give (2.30) as

$$a_{i0} + a_{i1} \leq 1.$$

Next, we prove (2.31). Since $0 < B'(1) < +\infty$.

from Lemma 2.1, we know the smallest positive root, $q < 1$.

From (2.5),

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s) \sum_{k=2}^{\infty} \binom{k}{2} p_{ik}(t)s^{k-2}, \quad \forall\, s \in [0, 1), \quad i \geq 0. \tag{2.36}$$

Put $s = q$ into (4.2.1)

$$\sum_{j=1}^{\infty} p'_{ij}(t)q^j = B(q) \sum_{k=2}^{\infty} \binom{k}{2} p_{ik}(t)q^{k-2} = 0, \quad \forall\, t > 0. \tag{2.37}$$

$$\Rightarrow \sum_{j=0}^{\infty} p_{ij}(t)q^j = q^i, \forall\, i \geq 2. \tag{2.38}$$

Letting $t \to \infty$,

$$\lim_{t\to\infty} p_{i0}(t) + \lim_{t\to\infty} p_{i1}(t)q + \lim_{t\to\infty} \sum_{j=2}^{\infty} p_{ij}(t)q^j = q^i, \forall\, i \geq 2$$

Then, we apply Dominated Convergence Theorem in last term on left hand side of the above equation gives (2.31):

$$a_{i0} + qa_{i1} = q^i.$$

The proof is then finished. ■

Theorem 2.3 tells us that if $B'(1) < 0$, then the process is absorbed with probability 1 while the the absorption probability is less than 1 if $0 < B'(1) \leq +\infty$. We try to show that, if the absorption has not happened, then the process must explode.

Define probability generating functions $F = \{F_i(t,s), i \geq 0\}$ by $F_i(t,s) = \sum_{j=0}^{\infty} p_{ij}(t)s^j$ from (2.5):

$$\sum_{j=0}^{\infty} p'_(ij)(t)s^j = B(s)\sum_{j=0}^{\infty} \binom{k}{2} p_{ik}(t)s^{k-2}$$

$$\Rightarrow \frac{dF_i(t,s)}{dt} = B(s)\sum_{j=0}^{\infty} \binom{k}{2} p_{ik}(t)s^{k-2}$$

$$\Rightarrow \frac{d^2F_i(t,s)}{ds^2} = \sum_{j=2}^{\infty} j(j-1)p_{ij}(t)s^{j-2}$$

$$\Rightarrow \frac{\partial F_i(t,s)}{\partial t} = \frac{B(s)}{2}\frac{\partial^2 F_i(t,s)}{\partial s^2} \tag{2.39}$$

Lemma 2.3 *The transition function $P(t) = \{p_{ij}(t)\}$ satisfies*

$$\lim_{t\to\infty} \sum_{j=2}^{\infty} p_{ij}(t) = 0, \quad \forall\, i \geq 2. \tag{2.40}$$

Proof. Limit exists because $\sum_{j=2}^{\infty} p_{ij}(t)$ is decreasing in t. It is trivial that

$$p_{i0}(t) + p_{i1}(t) + \sum_{j=2}^{\infty} p_{ij}(t) = \sum_{j=0}^{\infty} p_{ij}(t). \tag{2.41}$$

(i) when $B'(1) \leq 0$, $p(t)$ is honest, and $a_{i0} + a_{i1} = 1$

$$\Rightarrow \lim_{t\to\infty} \sum_{j=2}^{\infty} p_{ij}(t) = 0. \tag{2.42}$$

(ii) $0 < B'(1) < \infty$, from (2.5)

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s) \sum_{k=2}^{\infty} \binom{k}{2} p_{ik}(t)s^{k-2}, \ \forall\ s \in [0,1)$$

and the right hand side is equal to 0 when $s = q$.

$$\frac{1}{B(s)} \sum_{j=0}^{\infty} p'_{ij}(t)s^j = \sum_{k=2}^{\infty} \binom{k}{2} p_{ik}(t)s^{k-2}, \quad \forall\, s \in [0,1). \tag{2.43}$$

Rewriting (2.39),

$$\frac{\partial^2 F_i(t,y)}{\partial y^2} = \frac{2}{B(y)} \frac{\partial F_i(t,y)}{\partial y} = \frac{2}{B(y)} F'_i(t,y),\ i \geq 2. \tag{2.44}$$

Integrating w.r.t. y in $[0,x]$ gives,

$$\sum_{j=1}^{\infty} j p_{ij}(t)x^{j-1} = 2\int_0^x \frac{F'_i(t,y)}{B(y)} dy$$

Integrating w.r.t. x on $[0,s]$ gives,

$$\sum_{j=2}^{\infty} p_{ij}(t)s^j = 2\int_0^s \left(\int_0^x \frac{F'_i(t,y)}{B(y)} dy \right) dx$$

$$\begin{aligned} F_i(t,s) &= p_{i0}(t) + p_{i1}(t)s + 2\int_0^s \left(\int_y^s \frac{F'_i(t,y)}{B(y)} dx \right) dy \\ &= p_{i0}(t) + p_{i1}(t)s + 2\int_0^s \frac{s-y}{B(y)} F'_i(t,y) dy, \end{aligned} \tag{2.45}$$

where $F'_i(t,y) := \partial F_i(t,y)/\partial t$. Letting $s \uparrow 1$, we have

$$\sum_{j=2}^{\infty} p_{ij}(t) = 2\int_0^1 \frac{1-y}{B(y)} F'_i(t,y) dy. \tag{2.46}$$

Our objective is to prove the right hand side of (2.46)is equal to as $t \to \infty$, i.e,

$$\lim_{t\to\infty}\int_0^1 \frac{1-y}{B(y)}F_i'(t,y)dy = 0.$$

Note that for $\varepsilon \in (0,1)$,

$$\lim_{t\to\infty}\int_0^{1-\varepsilon} \frac{1-y}{B(y)}F_i'(t,y) = 0.$$

By (2.43), we can note that the above integrand is dominated by $1/(1-y)^2$ and because the limit as $t \to \infty$ of the left hand side of (2.43) is equal to 0 for $s \in [0,1)$. So, it is suffices to prove that

$$\lim_{t\to\infty}\int_{1-\varepsilon}^1 \frac{1-y}{B(y)}F_i'(t,y)dy = 0.$$

for some suitable ε. Because $B'(1) > 0$, there is a q s.t. $B(q) = 0$, $0 < q < 1$. We get

$$-F_i'(t,s) = |F_i'(t,s)| \le \sum_{j=0}^{\infty} |p_{ij}'(t)|s^j \le \sum_{j=0}^{\infty} |p_{ij}'(t)| \le 2q_i,\ \forall\, q < s < 1.$$

if we take $\varepsilon < 1-q$, note that $\frac{F'(t,s)}{B(s)} > 0\,\forall\, s \in [0,1)$

and $B(s) < 0$ for $s \in (q,1)$,

$$\begin{aligned}\int_{1-\varepsilon}^1 \frac{1-y}{B(y)}F_i'(t,y)dy &= \int_{1-\varepsilon}^1 \frac{(1-y)(-F_i(t,y))}{-B(y)}dy \\ &\le \int_{1-\varepsilon}^1 \frac{(1-y)}{-B(y)}dy(2q_i) < \infty. \qquad (2.47)\end{aligned}$$

By Dominated Convergence Theorem, the proof is completed. ■

Lemma 2.4 *$B(s) = 0$ has a unique root q_* in $(-1,0)$.*

Proof. since $B(-1) < 0$ and $B(0) > 0$, $B(s) = 0$ has at least one root in $(-1,0)$. To prove uniqueness, suppose there are distinct roots q_{*1} and q_{*2} in $(-1,0)$.

Putting

$$q_{ij} = \begin{cases} \binom{i}{2} b_{j-1+2}, & \text{if } i \geq 2, j > i-2. \\ 0, & \text{otherwise,} \end{cases}$$

into $P'(t) = P(t) \cdot Q$, we have

$$p'_{ij}(t) = \sum_{k=2}^{j+2} p_{ik}(t) \binom{k}{2} b_{j-k+2}$$

Multiplying s^j on both sides of the above equality and summing over j yields that for any $i \geq 0$ and $s \in (-1, 1)$,

$$\begin{aligned} \sum_{j=0}^{\infty} p'_{ij} s^j &= \sum_{j=0}^{\infty} \left(\sum_{k=2}^{j+2} p_{ik}(t) \binom{k}{2} b_{j-k+2} \right) s^j \\ &= \sum_{k=2}^{\infty} \sum_{j=k-2}^{\infty} p_{ik}(t) \binom{k}{2} b_{j-k+2} s^j \\ &= \sum_{k=2}^{\infty} p_{ik}(t) \binom{k}{2} s^{k-2} \sum_{j=k-2}^{\infty} b_{j-k+2} s^{j-k+2} \\ &= \sum_{k=2}^{\infty} p_{ik}(t) \binom{k}{2} s^{k-2} \sum_{j=0}^{\infty} b_j s^j \\ &= B(s) \sum_{k=2}^{\infty} p_{ik}(t) \binom{k}{2} s^{k-2}, \end{aligned}$$

$$\Rightarrow \sum_{j=0}^{\infty} p'_{ij} s^j = B(s) \sum_{k=2}^{\infty} p_{ik}(t) \binom{k}{2} s^{k-2}, \ s \in (-1, 1).$$

Therefore, taking integration over $[0, t]$,

$$\sum_{j=0}^{\infty} p_{ij}(t) s^j - s^i = B(s) \sum_{k=2}^{\infty} \int_0^t p_{ik}(u) du \binom{k}{2} s^{k-2}$$

$$\Rightarrow \sum_{j=0}^{\infty} p_{ij}(t) q_{*1}^j = q_{*1}^i$$

and

$$\sum_{j=0} p_{ij}(t) q_{*2}^j = q_{*2}^i$$

.

We can get the following results using similar methods in Theorem 2.3.

$$a_{i0} + q_{*1} a_{i1} = q_{*1}^i$$

and

$$a_{i0} + q_{*2} a_{i1} = q_{*2}^i, \quad \forall i \geq 2$$

Without loss of generality, we set $i = 2$.

$$a_{20} + q_{*1} a_{21} = q_{*1}^2$$

$$a_{20} + q_{*2} a_{21} = q_{*2}^2.$$

$$\begin{aligned} (q_{*2} - q_{*1}) a_{21} &= (q_{*2} - q_{*1})(q_{*2} + q_{*1}) \\ a_{21} &= q_{*1} + q_{*2} \\ &= < 0. \end{aligned}$$

Contradiction happens as the probability should be non-negative. ■

After going through the above theorems and lemmas, we now go to the evaluation of the extinction probability a_{10} and a_{i1}, and the explosion probability $a_{i\infty}$. We will use q_* to denote the unique root of $B(s) = 0$ in $(-1, 0)$.

Theorem 2.4 *(i) If* $B'(1) \leq 0$ *then*

$$a_{i0} = (q_*^i - q_*)/(1 - q_*) \tag{2.48}$$

$$a_{i1} = (1 - q_*^j)/(1 - q_*) \tag{2.49}$$

and $a_{i\infty} = 0$.

(ii) If $0 < B'(1) \leq +\infty$ *then*

$$a_{i0} = (q q_*^i - q_* q^i)/(q - q_*), \tag{2.50}$$

$$a_{i1} = (q^i - q_*^i)/(q - q_*) \tag{2.51}$$

and $a_{i\infty} = \big(q(1 - q_*^i) - q_*(1 - q^i) - (q^i - q_*^i)\big)/(q - q_*)$.

Proof. By Theorem 2.3, when $B'(1) \le 0$, we have

$$a_{i0} + a_{i1} = 1, \quad i \ge 2.$$

Similarly, we can also get

$$\begin{aligned} a_{i0} + q_* a_{i1} &= q_*^i, \\ a_{i1} &= \frac{1 - q_*^i}{1 - q_*}, \\ \Rightarrow a_{i0} &= \frac{q_*^i - q_*}{(1 - q_*)}. \end{aligned}$$

Since $P(t)$ is honest, when $B'(1) \le 0$, we have $a_{i0} + a_{i1} = 1$.

(ii) If $0 < B'(1) \le \infty$, again by Theorem 2.3

$$\begin{cases} a_{i0} + q a_{i1} = q^i \\ a_{i0} + q_* a_{i1} = q_*^i \end{cases} \tag{2.52}$$

a_{i0} and a_{i1} can be evaluated easily. Then, by Lemma 2.2, we have

$$a_{i0} + a_{i1} + a_{i\infty} = 1 \quad , i \ge 2,$$

Therefore, $a_{i\infty}$ can also be found easily.

The proof is then completed.

■

Next, we deal with the mean hitting time. Let

$$\mu_{ik} = E[\tau_k I_{\{\tau_k < \infty\}} | X_0 = i], \quad k = 0, 1$$

denote the expected extinction times starting in state i. Similarly, let $\mu_{i\infty} =$

$E[\tau_\infty I_{\{\tau_\infty<\infty\}}|X_0 = j]$, where τ_∞ is explosion time.

Theorem 2.5 *(i) If $B'(1) \le 0$, the expected extinction times are all finite and are given by*

$$\mu_{i0} = \frac{2}{(1-q_*)^2}\Big[-q_*\int_0^1 \frac{(1-y)^2 f_i(y)}{B(y)}dy + \int_{q_*}^0 \frac{(y-q_*)(1-y)f_i(y)}{B(y)}dy\Big], \tag{2.53}$$

$$\mu_{i1} = \frac{2}{(1-q_*)^2}\Big[\int_0^1 \frac{(1-y)^2 f_i(y)}{B(y)}dy + \int_{q_*}^0 \frac{(y-q_*)(1-y)f_i(y)}{B(y)}dy\Big] \tag{2.54}$$

for $i \ge 2$, where

$$f_i(y) = q_*^i - \frac{q_*(1-y^i)}{1-y} + \frac{y(1-y^{i-1})}{1-y}. \tag{2.55}$$

(ii) If $0 < B'(1) \le +\infty$ then, again, the expected extinction times are all finite. They are given by

$$\mu_{i0} = \frac{2}{(q-q_*)^2}\Big[-q_*\int_0^q \frac{(q-y)^2 f_i(y)}{B(y)}dy + q\int_{q_*}^0 \frac{(y-q_*)(q-y)f_i(y)}{B(y)}dy\Big], \tag{2.56}$$

$$\mu_{i1} = \frac{2}{(q-q_*)^2}\Big[\int_0^q \frac{(q-y)^2 f_i(y)}{B(y)}dy + \int_{q_*}^0 \frac{(y-q_*)(q-y)f_i(y)}{B(y)}dy\Big] \tag{2.57}$$

for $i \ge 2$, where

$$f_i(y) = q_*^i - \frac{q_*(q^i-y^i)}{q-y} + \frac{qy(q^{i-1}-y^{i-1})}{q-y}. \tag{2.58}$$

Proof. Note that the integral in (2.53),(2.54) (2.56) and (2.57) are finite as the function f in (2.55) and (2.58) is bounded on $[-1,1]$. Note that $|(f_i(y))| \le 2i, \quad \forall\, y \in [-1,1]$.

By Lemma 2.1 and Lemma 2.4, we know that $\frac{(q-y)^2}{B(y)}$ and $\frac{(y-q_*)}{B(y)}$ are bounded on $[0,q]$ and $[q_*,0]$ respectively.

Therefore all the integrals are finite.

We prove (ii) first, $0 < B'(1) \le \infty$, $q \in (0,1)$

From (2.39),

$$\frac{\partial F_i(t,s)}{\partial t} = \frac{1}{2}B(s)\frac{\partial^2 F_i(t,s)}{\partial s^2}$$

Integrating (2.39) w.r.t. s and Fubini's theorem, for any $s \in [0, q]$,

$$F_i(t,s) = p_{i0}(t) + p_{i1}(t)s + 2\int_0^s \frac{s-y}{B(y)} F_i'(t,y)dy. \tag{2.59}$$

Similarly, we can do integration along -ve real axis, for any $s \in [q_*, 0]$,

$$F_i(t,s) = p_{i0}(t) + p_{i1}(t)s + 2\int_s^0 \frac{y-s}{B(y)} F_i'(t,s)dy, \tag{2.60}$$

where

$$F_i(t,s) = \sum_{j=1}^{\infty} p_{ij}(t)s^j.$$

Let $s = q$ in (2.59) and $s = q_*$ in (2.60) with

$$\sum_{j=0}^{\infty} p_{ij}(t)q^j = q^i$$

and

$$\sum_{j=0}^{\infty} p_{ij}(t)q_*^j = q_*^i$$

With

$$F_i(t,q) = p_{i0}(t) + p_{i1}(t)q + 2\int_0^q \frac{q-y}{B(y)} F_i'(t,y)dy, \tag{2.61}$$

we get

$$p_{i0}(t) + p_{i1}(t)q = q^i - 2\int_0^q \frac{q-y}{B(y)} F_i'(t,y)dy,$$
$$p_{i0}(t) + p_{i1}(t)q_* = q_*^i - 2\int_{q_*}^0 \frac{y-q_*}{B(y)} F_i'(t,y)dy.$$

Recalling (2.52) together with

$$q^i - p_{i0}(t) - p_{i1}(t)q = 2\int_0^q \frac{q-y}{B(y)} F_i'(t,y)dy,$$

we have

$$
\begin{aligned}
(a_{i0}-p_{i0}(t))+(a_{i1}-p_{i1}(t))q &= 2\int_0^q \frac{q-y}{B(y)}F_i'(t,y)dy,\\
(a_{i0}-p_{i0}(t))+(a_{i1}-p_{i1}(t))q_* &= 2\int_{q_*}^0 \frac{y-q_*}{B(y)}F_i'(t,y)dy.
\end{aligned}
$$

Integrating the above equations w.r.t. t and noting that $F_i(0,y)=y^i$

$$
\begin{aligned}
\int_0^t (a_{i0}-p_{i0}(u))du + q\int_0^t (a_{i1}-p_{i1}(u))du &= 2\int_0^q \frac{q-y}{B(y)}(F_i(t,y)-y^i)dy,\\
\int_0^t (a_{i0}-p_{i0}(u))du + q_*\int_0^t (a_{i1}-p_{i1}(u))du &= 2\int_{q_*}^0 \frac{y-q_*}{B(y)}(F_i(t,y)-y^i)dy.
\end{aligned}
$$

Letting $t\to\infty$ in above two equations and using Dominated Convergence Theorem gives

$$
\begin{aligned}
\mu_{i0}+q\mu_{i1} &= 2\int_0^q \frac{q-y}{B(y)}(F_i(\infty,y)-y^i)dy,\\
\mu_{i0}+q_*\mu_{i1} &= 2\int_{q_*}^0 \frac{(y-q_*)}{B(y)}(F_i(\infty,y)-y^i)dy.
\end{aligned}
$$

We note that $a_{ik}-p_{ik}=P(t<\tau_k<\infty|X_0=i)$, $k=0,1$.

$$
F_i(\infty,y)=\lim_{t\to\infty}F_i(t,y)=a_{i0}+a_{i1}y.
$$

$$
\begin{aligned}
\mu_{i0}+\mu_{i1}q &= 2\int_0^q \frac{(q-y)(a_{i0}+a_{i1}y-y^i)}{B(y)}dy \qquad (2.62)\\
\mu_{i0}+\mu_{i1}q_* &= 2\int_{q_*}^0 \frac{(y-q_*)(a_{i0}+a_{i1}y-y^i)}{B(y)}dy. \qquad (2.63)
\end{aligned}
$$

μ_{i0} and μ_{i1} can be obtained easily.

(i) for $B'(1)\le 0$ (2.63) still holds, let $s\uparrow 1$ in (2.59) gives

$$
1=p_{i0}(t)+p_{i1}(t)+2\int_0^1 \frac{1-y}{B(y)}F_i'(t,y)dy,
$$

$$
p_{i0}(t)+p_{i1}(t)=1-2\int_0^1 \frac{1-y}{B(y)}F_i'(t,y)dy,
$$

Similarly,

$$p_{i0}(t) + p_{i1}(t)q_* = q_*^i - 2\int_{q_*}^{0} \frac{y - q_*}{B(y)} F_i'(t, y)dy.$$

Since $a_{i0} + a_{i1} = 1$, we rewrite,

$$(a_{i0} - p_{io}(t)) + (a_{i1} - p_{i1}(t)) = 2\int_0^1 \frac{1 - y}{B(y)} F_i'(t, y)dy$$

Integrating the above equation w.r.t. t with $F_i(t, 0) = y^i$,

$$\int_0^t (a_{i0} - p_0(u))du + \int_0^t (a_{i1} - p_{i1}(u))du = 2\int_0^1 \frac{1 - y}{B(y)} (F_i(t, y) - y^i)dy.$$

By (2.39) and Lemma 2.3,

$$\lim_{t \to \infty} F_i(t, y) = a_{i0} + a_{iy}, \quad t \to \infty.$$

Using Dominated Convergence Theorem, we get

$$\mu_{i0} + \mu_{i1} = 2\int_0^1 \frac{1 - y}{B(y)} (F_i(\infty, y) - y^i)dy,$$

$$\mu_{i0} + \mu_{i1} = 2\int_0^1 \frac{1 - y}{B(y)} (a_{i0} + a_{i1}y - y^i)dy,$$

together with (2.63),

$$\mu_{i0} + \mu_{i1}q_* = 2\int_{q_*}^{0} \frac{y - q_*}{B(y)} (a_{i0} + a_{i1}y - y^i)dy.$$

μ_{i0} and μ_{i1} can be obtained easily.

The proof is completed. ∎

2.5 Expected Explosion Time

By previous results, for $B'(1) < 0$, the process will certainly be absorbed. So, in this section, we only deal with the expected explosion times for the case $0 < B'(1) \leq +\infty$.

We define

$$p_{i\infty}(t) = 1 - \sum_{j=0}^{\infty} p_{ij}(t) = P(\tau_\infty \le t | X_0 = i) \tag{2.64}$$

as probability of explosion by time t starting from state i, and $p_{i\infty}(t) \to \infty$ as $t \to \infty$.

Theorem 2.6 *If $0 < B'(1) \le +\infty$, then the expected explosion time is finite and is given by*

$$\begin{aligned} &\mu_{i\infty} \\ = \; & \frac{2}{(q-q_*)}\Bigg(\int_0^1 \frac{(1-y)(q-y)}{B(y)} f_i(y)dy - \frac{1-q_*}{q-q_*}\int_0^q \frac{(q-y)^2}{B(y)} f_i(y)dy \\ & + \frac{1-q}{q-q_*}\int_{q_*}^0 \frac{(y-q_*)(q-y)}{B(y)} f_i(y)dy\Bigg) \end{aligned} \tag{2.65}$$

for $i \ge 2$, where $f_i(y)$ is given in (2.55).

Proof. Noting that all of the integrals in (2.65) are finite as discussed above, so $\mu_{i\infty} < \infty$.

$$\mu_{i\infty} = \int_0^\infty (a_{i\infty} - p_{i\infty}(t))dt$$

where

$$\begin{aligned} P(t < \tau_\infty < \infty | X_0 = i) &= a_{i\infty} - p_{i\infty}(t) \\ &= (1 - a_{i0} - a_{i1}) - p_{i\infty}(t). \end{aligned}$$

Together with (2.64)

$$\begin{aligned} \mu_{i\infty} &= \int_0^\infty \sum_{j=0}^{\infty} (p_{ij}(t) - a_0 - a_{i1})dt \\ &= \sum_{j=2}^{\infty} \int_0^\infty p_{ij}(t)dt - \mu_{i0} - \mu_{i1} \end{aligned} \tag{2.66}$$

by (2.5):

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s)\sum_{k=2}^{\infty} \binom{k}{2} p_{ik}(t)s^{k-2},\; s \in [0,1)$$

Integrating (2.5) w.r.t t from 0 to ∞, we get

$$\frac{a_{i0} + a_{i1}s - s^i}{B(s)} = \sum_{k=2}^{\infty} \int_0^{\infty} p_{ik}(t) dt s^{k-2}. \tag{2.67}$$

Considering the coefficient of s^{k-2}, we can obverse that

$$\int_0^{\infty} p_{ik}(t) dt = \frac{2}{k!} G_i^{(k-2)}(0) \tag{2.68}$$

where

$$G_i(s) = \frac{a_{i0} + a_{i1}s - s^i}{B(s)}. \tag{2.69}$$

Integrating (2.67) twice w.r.t. s gives,

$$\sum_{k=2}^{\infty} \int_0^{\infty} p_{ik}(t) dt s^k = 2 \int_0^s (s - y) G_i(y) dy$$

letting $s \uparrow 1$,

$$\sum_{k=2}^{\infty} \int_0^{\infty} p_{ik}(t) dt = 2 \int_0^1 (1 - y) G_i(y) dy \tag{2.70}$$

Substituting (2.70) in (2.66), we have

$$\mu_{i\infty} = 2 \int_0^1 (1 - y) G_i(y) dy - \mu_{i0} - \mu_{i1} \tag{2.71}$$

Then, we substitute a_{i0} and a_{i1} from (2.50) and (2.51) into (2.69). Then, further substituting the intermediate result with (2.56) and (2.57) into (2.40) gives the result we wanted. ■

Finally, we can also look at the time spent in each state over the lifetime of the process.

Let τ_k be the total time spent in state $k \geq 2$, and let $\mu_{ik} = E[\tau_k | X(0) = i]$. Then,

$$\mu_{ik} = E\left(\int_0^{\infty} I_{\{X(t)=k\}} dt | X_0 = i \right) = \int_0^{\infty} p_{ik}(t) dt.$$

Thus

$$\mu_{ik} = \frac{2}{k!} G_i^{(k-2)}(0).$$

This expression is obtained in (2.68) and so we can have our last theorem in this chapter.

Theorem 2.7 *All of the μ_{ik}, $i \geq 2$, $k \geq 2$, are finite and given by*

$$\mu_{ik} = \frac{2}{k!} G_i^{(k-2)}(0), \tag{2.72}$$

where $G_i(s)$ is given in (2.69) and $G_i^{(k-2)}(0)$ is the derivative of $G_i(s)$ near 0.

Remark 2.1 *At this point, we have considered the collision branching processes in detail. We have discussions about the model settings, uniqueness , extinction and explosion behavior of collision branching processes. This certainly helps us in understanding the new challenging model in next chapter, i.e. interacting branching collision processes.*

Chapter 3

Interacting Branching Collision Processes

3.1 Introduction

Last chapter, we have discussed the collision branching processes. This chapter, we will combine the collision branching model with an ordinary Markov branching processes.

In practical cases, particularly in biological sciences, individual particles may interact with each other. In this chapter, to relate this reality, we include a collision component into a ordinary Markov branching process. We call these processes interacting collision branching processes.

In the interacting collision branching processes, the two components, collision branching processes (CBP) and ordinary Markov branching processes (MBP), strongly interact with each other.

For the ordinary Markov branching processes, there are lots of reference already such as Harris [2002], Athreya and Ney [2004], Athreya and Jagers [1997]and Asmussen and Hering [1983]. For the collision branching process, there are much fewer papers discussing in this area, see Chen et al. [2004, 2010] and Kalinkin

[2001, 2002]. Also, we have looked at some properties in the chapter before. For the interacting branching collision process, it is much more troublesome but interesting as the two components interact with each other.

We can use $X(t)$,the number of particles living at time t, to be a continuous-time Markov chain with non-zero transition rates.

Here, we give the formal definition. In this work, we are using the model considered in Chen et al. [2012]. Sections 3.2 - 3.5 closely follow Chen et al. [2012]. Section 3.6 closely follows Chen et al. [2014].

3.2 Preliminary and Mathematical Model

Definition 3.1 *A q-matrix $Q = (q_{ij};\ i, j \in \mathbf{Z}_+)$ is called an interacting branching collision q-matrix (IBC q-matrix) if it has the following form:*

$$q_{ij} = \begin{cases} ib_{j-i+1} + \binom{i}{2} c_{j-i+2}, & \text{if } \ j \geq i-2, i \geq 1 \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

where

$$\begin{cases} b_0 > 0, b_j \geq 0 (j \neq 1), \sum_{k=2}^{\infty} b_k > 0 \text{ and } -b_1 = \sum_{j \neq 1} b_j < +\infty, \\ c_0 > 0, c_j \geq 0 (j \neq 2), \sum_{k=3}^{\infty} c_k > 0 \text{ and } -c_2 = \sum_{j \neq 2} c_j < +\infty, \end{cases} \tag{3.2}$$

together with the conventions $b_{-1} = 0$ and $\binom{1}{2} = 0$.

Definition 3.2 *A Markov interacting branching collision process (henceforth referred to as an IBCP) is a continuous-time Markov chain on the state space $\mathbf{Z}_+$ whose transition function $P(t) = (p_{ij}(t); i, j \in \mathbf{Z}_+)$ satisfies*

$$P'(t) = P(t)Q \tag{3.3}$$

where Q is given in (3.1) - (3.2).

It is obvious that the collision branching processes (CBP) have two absorbing states 0 and 1. However, unlike CBP, with the ordinary Markov branching processes (MBP) component, state 1 is not an absorbing state. So, the absorbing/ extinction probability are considerably different from CBP in the chapter before and this is very challenging.

Regularity and uniqueness criteria are considered in Section 3. Evaluation of extinction probability will be discussed in Section 4 and 5. Section 4 focuses on regular case and Section 5 focuses on irregular case. Furthermore, since we may note that the closed forms for the extinction probabilities of the interacting branching processes are sometimes very complicated, so we try to examine the asymptotic behavior of the extinction probability in Section 6.

Similarly to CBP, if Q is stable and conservative, the Feller minimal process for IBCP always exists. We need to investigate, under what conditions would the processes be unique. In order to investigate these properties of IBCP, it is necessary to define the generating function of two known sequences $\{b_k; k \geq 0\}$ and $\{c_k; k \geq 0\}$ as $B(s) = \sum_{k=0}^{\infty} b_k s^k$ and $C(s) = \sum_{k=0}^{\infty} c_k s^k$.

$B(s)$ and $C(s)$ are well defined at least on $[-1, 1]$. We provide the following lemma which will be used constantly throughout this chapter.

Lemma 3.1 *(i) The equation $B(s) = 0$ has at most two roots in $[0, 1]$. More specifically, if $B'(1) < 0$ then $B(s) > 0$ for all $s \in [-1, 1)$ and 1 is the only root of $B(s) = 0$ in $[0, 1)$. If $0 < B'(1) \leq +\infty$ then $B(s) = 0$ has an additional root in $[0, 1)$, denoted by ρ_b, such that $B(s) > 0$ for all $s \in [-1, \rho_b)$ and $B(s) < 0$ for $s \in (\rho_b, 1)$. Moreover, $B(z) = 0$ has no other root in the complex disk $\{z : |z| \leq 1\}$.*

(ii) The equation $C(s) = 0$ has at most two roots in $[0, 1]$ and exactly one root in $[-1, 0)$. More specifically, if $C'(1) < 0$ then $C(s) > 0$ for all $s \in [0, 1)$ and 1 is the only root of the equation $C(s) = 0$ in $[0, 1]$, which is simple or with multiplicity 2 according to $C'(1) < 0$ or $C'(1) = 0$, while if $0 < C'(1) \leq +\infty$ then $C(s) = 0$ has an additional simple root ρ_c satisfying $0 < \rho_c < 1$ such that $C(s) > 0$ for

$s \in (0, \rho_c)$ and $C(s) < 0$ for $s \in (\rho_c, 1)$. Also $C(s) = 0$ has exactly one root, denoted by $\zeta_c \in [-1, 0]$ such that $C(s) > 0$ for all $s \in (\zeta_c, 0]$ and $|\zeta_c| \le \rho_c$. This root is simple unless $C'(1) = 0$ and $\sum_{k=0}^{\infty} c_{2k+1} = 0$. Also, $|\zeta_c| = \rho_c$ if and only if $\sum_{k=0}^{\infty} c_{2k+1} = 0$. Moreover, $C(z) = 0$ has no other root in the complex disk $\{z; |z| \le 1\}$.

Throughout this chapter, we shall let ρ_b and ρ_c denote the smallest nonnegative root of $B(s) = 0$ and $C(s) = 0$ respectively.

Proof. Proofs are similar in the chapter for CBP and thus omitted. ■

Lemma 3.2 *Suppose that Q is an IBC q-matrix as defined in (3.1) - (3.2) and let $P(t) = (p_{ij}(t); i, j \ge 0)$ and $\Phi(\lambda) = (\phi_{ij}(\lambda); i, j \le 0)$ be a Q-function and its Q-resolvent, respectively. Further assume that the Q-function $P(t)$ and Q-resolvent $\Phi(\lambda)$ satisfy the Kolmogorov forward equation (3.3). Then for any $i \ge 0$, $t \ge 0$, $\lambda > 0$ and $|s| < 1$, we have*

$$\frac{\partial F_i(t,s)}{\partial t} = \frac{C(s)}{2} \cdot \frac{\partial^2 F_i(t,s)}{\partial s^2} + B(s) \cdot \frac{\partial F_i(t,s)}{\partial s} \tag{3.4}$$

or equivalently,

$$\lambda \Phi_i(\lambda, s) - s^i = \frac{C(s)}{2} \cdot \frac{\partial^2 \Phi_i(\lambda, s)}{\partial s^2} + B(s) \cdot \frac{\partial \Phi_i(\lambda, s)}{\partial s} \tag{3.5}$$

where $F_i(t,s) = \sum_{j=0}^{\infty} p_{ij}(t) s^j$ and $\Phi_i(\lambda, s) = \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j$.

Proof. From (3.3),

$$p'_{ij}(t) = \sum_{k=2}^{j+2} p_{ik}(t) \binom{k}{2} c_{j-k+2} + \sum_{k=1}^{j+1} p_{ik}(t) k b_{j-k+1}.$$

Multiplying s^j on both sides of the above equality and summing over $\mathbf{Z}_+$ we immediately obtain (3.4).

$$\begin{aligned}\sum_{j=0}^{\infty} p'_{ij}(t)s^j &= \sum_{j=0}^{\infty}\sum_{k=2}^{j+2} p_{ik}(t)\binom{k}{2}c_{j-k+2}s^j + \sum_{j=0}^{\infty}\sum_{k=1}^{j+1} p_{ik}(t)kb_{j-k+1}s^j \\ &= \sum_{k=2}^{\infty}\binom{k}{2}p_{ik}(t)\sum_{j=k-2}^{\infty} c_{j-k+2}s^j + \sum_{k=1}^{\infty} kp_{ik}(t)\sum_{j=k-1}^{\infty} b_{j-k+1}s^j \\ &= \sum_{k=2}^{\infty}\binom{k}{2}p_{ik}(t)\sum_{j=0}^{\infty} c_j s^{j+k-2} + \sum_{k=1}^{\infty} kp_{ik}(t)\sum_{j=0}^{\infty} b_j s^{j+k-1} \\ &= C(s)\sum_{k=2}^{\infty}\binom{k}{2}p_{ik}(t)s^{k-2} + B(s)\sum_{k=1}^{\infty} kp_{ik}(t)s^{k-1}.\end{aligned}$$

Let $F_i(t,s) = \sum_{j=0}^{\infty} p_{ij}(t)s^j$,

$$\frac{\partial F_i(t,s)}{\partial t} = \frac{C(s)}{2}\frac{\partial^2 F_i(t,s)}{\partial s} + B(s)\frac{\partial F_i(t,s)}{\partial s}.$$

Let $\Phi_i(\lambda) = \{\phi_{ij}(\lambda) : i,j \leq 0\}$ be resolvent function. Thus

$$\sum_{j=0}^{\infty} \lambda\phi_{ij}s^j - s^i = \frac{C(s)}{2}\sum_{k=2}^{\infty} k(k-1)\phi_{ij}(\lambda)s^{k-2} + B(s)\sum_{k=1}^{\infty} k\phi_{ij}(\lambda)s^{k-1}$$

$$\lambda\Phi_i(\lambda,s) - s^i = \frac{C(s)}{2}\frac{\partial^2\Phi_i(\lambda,s)}{\partial s^2} + B(s)\frac{\partial\Phi_i(\lambda,s)}{\partial s},$$

where $\Phi_i(\lambda,s) = \sum_{j=0}^{\infty}\phi_{ij}(\lambda)s^j$. ■

$\int_0^{\infty} p_{ij}(t)dt$ denotes the expected time spent in each state over the lifetime of the process.

Lemma 3.3 *Suppose that Q is an IBC q-matrix as defined in (3.1)- (3.2). Let $P(t) = (p_{ij}(t); i,j \geq 0)$ be a Q-function that satisfies the Kolmogorov forward equations. Then*

(i) $\int_0^{\infty} p_{ij}(t)dt < +\infty (i,j \geq 1)$ and thus $\lim_{t\to\infty} p_{ij}(t) = 0 (i,j \geq 1)$.

(ii) For any $i \geq 1$ and $s \in [0,1)$,

$$G_i(s) = \sum_{j=1}^{\infty}\left(\int_0^{\infty} p_{ij}(t)dt\right)\cdot s^j < +\infty. \tag{3.6}$$

Proof. We will make use of irreducibility of positive states. By Kolmogorov forward equation,

(i)

$$p'_{i0}(t) = p_{i2}(t)c_0 + p_{i1}(t)b_0$$

$$\begin{aligned} &\Rightarrow \int_0^\infty p_{i2}(t)dt < \infty \text{ since } b_0, c_0 > 0 \\ &\Rightarrow \int_0^\infty p_{ij}(t)dt < +\infty \text{ for all } i, j \geq 1. \end{aligned}$$

(i) is proved.

(ii) from (3.4)

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = \frac{C(s)}{2}\sum_{k=2}^{\infty} p_{ik}(t)k(k-1)s^{k-2} + B(s)\sum_{k=1}^{\infty} p_{ik}(t)ks^{k-1} \tag{3.7}$$

which can be rewritten as

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = \sum_{k=1}^{\infty}\left[\frac{(k-1)C(s)}{2} + sB(s)\right]p_{ik}(t)ks^{k-2} \tag{3.8}$$

We separate this problem into two situations, $C'(1) \leq 0$ and $0 < C'(1) \leq +\infty$.

If $C'(1) \leq 0$, we have $C(s) > 0$ for all $s \in [0, 1)$. There exists $\tilde{k} \geq 2$, such that $\frac{(k-1)C(s)}{2} + sB(s) > 0$ for any $k \geq \tilde{k}$. Then by (3.8) we obtain

$$\begin{aligned} \sum_{j=0}^{\infty} p'_{ij}(t)s^j \quad &\geq \quad \sum_{k=1}^{\tilde{k}-1}\left[\frac{(k-1)C(s)}{2} + sB(s)\right]p_{ik}(t)ks^{k-2} \\ &\quad + \left[\frac{(\tilde{k}-1)C(s)}{2} + sB(s)\right]\cdot\sum_{k=\tilde{k}}^{\infty} p_{ik}(t)ks^{k-2}. \end{aligned}$$

Taking integration in the above ineequality with some little algebra, making the second term of the right hand side of the inequality as the subject, we can get (3.6).

On the other hand, if $0 < C'(1) \leq +\infty$, then again by Lemma 3.1 we know that $C(s) = 0$ has a smallest nonnegative root $\rho_c \in [0, 1)$ such that $C(s) < 0$ for any $s \in (\rho_c, 1)$. Now, for any $\tilde{s} \in (\rho_c, 1)$, there exists a $\tilde{k} \geq 2$ such that

$\frac{(k-1)C(\tilde{s})}{2} + \tilde{s}B(\tilde{s}) < 0$ for any $k \geq \tilde{k}$. Then by (3.8)

$$\begin{aligned}
\sum_{j=0}^{\infty} p'_{ij}(t)\tilde{s}^j &= \left[\frac{(k-1)C(\tilde{s})}{2} + \tilde{s}B(\tilde{s})\right]\sum_{k=\tilde{k}}^{\infty} p_{ik}(t)ks^{k-2} \\
&\quad + \sum_{k=1}^{\tilde{k}-1}\left[\frac{(k-1)C(\tilde{s})}{2} + \tilde{s}B(\tilde{s})\right]p_{ik}(t)k\tilde{s}^{k-2} \\
&\leq \left[\frac{(\tilde{k}-1)C(\tilde{s})}{2} + \tilde{s}B(\tilde{s})\right]\sum_{k=\tilde{k}}^{\infty} p_{ik}(t)ks^{k-2} \\
&\quad + \sum_{k=1}^{k-1}\left[\frac{(k-1)C(\tilde{s})}{2} + \tilde{s}B(\tilde{s})\right]p_{ik}(t)k\tilde{s}^{k-2}
\end{aligned}$$

Integrating the above inequality yields that

$$\begin{aligned}
&\left[\frac{(\tilde{k}-1)C(\tilde{s})}{2} + \tilde{s}B(\tilde{s})\right]\cdot\sum_{k=\tilde{k}}^{\infty}\left(\int_0^{\infty} p_{ik}(t)dt\right)k\tilde{s}^{k-2} \\
\geq\ & \lim_{t\to\infty} p_{i0}(t) - \tilde{s}^i - \sum_{k=1}^{\tilde{k}-1}\left[\frac{(k-1)C(\tilde{s})}{2} + \tilde{s}B(\tilde{s})\right]\left(\int_0^{\infty} p_{ik}dt\right)k\tilde{s}^{k-2} \\
>\ & -\infty
\end{aligned}$$

which implies (3.6) since $\frac{(\tilde{k}-1)C(\tilde{s})}{2} + \tilde{s}B(\tilde{s}) < 0$. The proof is then completed. ∎

3.3 Uniqueness

Before discussing the regularity and uniqueness, we need to consider the following lemma.

Lemma 3.4 *Suppose that $Q = (q_{ij}; i, j \geq 0)$ is a conservative q-matrix and $\bar{k} \geq 1$ is an integer. Define a new matrix $Q^* = (q^*_{ij}; i, j \geq 0)$ as*

$$q^*_{ij} = \begin{cases} q_{ij}, & \text{if } i > \bar{k} \\ 0, & \text{otherwise.} \end{cases}$$

Then Q^* is also a conservative q-matrix. Moreover, if Q is regular then so is Q^*.

Proof. We only need to prove the last statement. Suppose that Q^* is not regular, i.e., the following equation

$$Q^*Y \geq \lambda Y$$

must have a nontrivial solution for some $\lambda > 0$, which we denote that as $Y = (y_i; i \geq 0)$.

We can see that $y_i = 0$ for $i \leq \bar{k}$. From the definition, $Y = (y_i; i \geq 0)$ must also a solution of the equation

$$QY \geq \lambda Y$$

So, for $i \leq \bar{k}$,

$$(QY)_i = \sum_{j=0}^{\infty} q_{ij} y_j = \sum_{j=\bar{k}+1}^{\infty} q_{ij} y_j \geq 0 = \lambda y_i$$

since $y_i = 0$ for all $i \leq \bar{k}$.

While for $i > \bar{k}$,

$$(QY)_i = (Q^*Y)_i \geq \lambda y_i.$$

Therefore, Q is not regular and this is a contradiction. We complete the proof.

■

Theorem 3.1 *Let Q is an IBC q-matrix satisfied (3.1) and (3.3). Then Q is regular if and only if $C'(1) \leq 0$.*

Proof. If part: first assume that $C'(1) \leq 0$, if $B'(1) \leq 0$, $B(s)$, $C(s)$ are both positive for all $s \in [0, 1)$. From (3.5),

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j \geq s^i, \quad s \in [0, 1). \tag{3.9}$$

Letting $s \uparrow 1$ in (3.9) yields that $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) = 1$, i.e. Q is regular.

Next, suppose that $C'(1) \leq 0$ and $0 < B'(1) < +\infty$, again from (3.5),

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j - s^i \geq B(s) \sum_{k=1}^{\infty} \phi_{ik} k s^{k-1}, \quad s \in [0, 1). \tag{3.10}$$

If Q is not regular, there exists an $i \geq 0$ and a $\lambda > 0$ such that $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) < 1$. Hence, there exist a $\delta > 0$ and an $\tilde{s} \in (\rho_b, 1)$ such that for all $s \in [\tilde{s}, 1]$ we have

$$s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j > \delta. \tag{3.11}$$

By (3.10) and (3.11) together with noting the fact that $B(s) < 0$ for all $\tilde{s} \in (\rho_b, 1)$ we obtain

$$\begin{aligned} \delta &\leq s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j \leq -B(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) k s^{k-1} \\ \frac{\delta}{-B(s)} &\leq \frac{s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j}{-B(s)} \leq \sum_{k=1}^{\infty} \phi_{ik}(\lambda) k s^{k-1} \end{aligned} \tag{3.12}$$

Therefore,

$$\sum_{k=1}^{\infty} \phi_{ik}(\lambda)(1 - \tilde{s}^k) \geq \int_{\tilde{s}}^{1} \frac{\delta}{-B(s)} ds = +\infty$$

which is a contradiction and hence Q is regular.

Only if part: given Q is regular. Suppose $C'(1) > 0$, by Chen et al. [2004] There exist a^* and b^* such that

$$2c_0 + c_1 < a^* < b^* < \sum_{j=1}^{\infty} j c_{j+2} \tag{3.13}$$

and

$$\begin{aligned} c_0 \left(\frac{b^*}{a^*} \right) + (c_0 + c_1) &< a^* \\ \sum_{j=1}^{\infty} c_{j+2} \sum_{k=1}^{j} \left(\frac{a^*}{b^*} \right)^{k-1} &> b^*. \end{aligned} \tag{3.14}$$

Choose an $\varepsilon \in (b^* - a^*)$ and let $i_0 = [\frac{2b_0}{\varepsilon}] + 1$. Define a q-matrix $Q = (\tilde{q}_{ij}; i, j \geq 0)$ as

$$\tilde{q}_{ij} = \begin{cases} \binom{i}{2} c_{j-i+2} + i b_{j-i+1}, & \text{if } i > i_0 \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 3.4, in order to prove Q is not regular, we only need to prove $\tilde{Q}$ is not regular.

Define a (conservative) birth-death q-matrix $Q^* = (q^*_{ij}; i, j \in \mathbf{Z}_+)$ by

$$q^*_{ij} = \begin{cases} \binom{i}{2} b^* & \text{if } i > i_0, j = i+1, i \geq 2 \\ \binom{i}{2}(a^* + \varepsilon) & \text{if } i > i_0, j = i-1, i \geq 2 \\ -\binom{i}{2}(b^* + a^* + \varepsilon) & \text{if } j = i > i_0 \\ 0 & \textit{otherwise.} \end{cases}$$

Since

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left(\frac{1}{q_{nn-1}} + \frac{q_{nn-1}}{q_{nn+1}q_{n-1n}} + \cdots + \frac{q_{nn-1} \cdots q_{32}}{q_{nn+1} \cdots q_{23}} \right) \\
= \;& \sum_{n=2}^{\infty} \frac{1}{\frac{n(n-1)}{2} b^*} + \frac{\frac{n(n-1)}{2}(a^* + \varepsilon)}{\frac{n(n-1)}{2} b^* \frac{(n-1)(n-2)}{2} b^*} + \cdots + \cdots \\
= \;& \frac{2}{b^*} \left(\sum_{n=2}^{\infty} \frac{1}{n(n-1)} + \frac{a^* + \varepsilon}{b^*} \frac{1}{(n-1)(n-2)} + \cdots + \frac{(a^* + \varepsilon)^{n-2}}{(b^*)^{n-2}} \frac{1}{2 \cdot 1} \right) \\
= \;& \frac{2}{b^*} \sum_{n=2}^{\infty} \sum_{k=2}^{n} \frac{\left((a^* + \varepsilon)/b^*\right)^{n-k}}{k(k-1)} \\
< \;& \frac{2}{b^*} \sum_{n=2}^{\infty} \sum_{k=1}^{n} \frac{(a^*/b^*)^{n-k}}{k(k-1)} \\
= \;& \frac{2}{b^*} \sum_{k=2}^{\infty} \sum_{n=k}^{n-k} \left(\frac{a^*}{b^*} \right)^{n-k} \frac{1}{k(k-1)} \\
= \;& \frac{2}{b^* - a^*} \cdot \sum_{k=2}^{\infty} \frac{1}{k(k-1)} < \infty.
\end{aligned}$$

By Theorem 3.2.2 in Anderson [1991], Q^* is not regular. Hence

$$(\lambda I - Q^*)u = 0 \quad (\lambda > 0) \tag{3.15}$$

has a non-trivial (non-negative) bounded solution denoted by $u^* = (u_1; i \geq 0)$. It can be noted that $u_0 = \cdots = u_{i0} = 0$ and $u_i > 0$ for all $i \geq i_0$ with

$$b^*(u_{i+1} - u_i) = (a^* + \varepsilon)(u_i - u_{i-1}) + \lambda u_i \binom{i}{2}^{-1}, \quad i > i_0. \tag{3.16}$$

It can be seen that $(u_i; i > i_0)$ is strictly increasing in i. From (3.16) for all $k \geq 1$ and $i > i_0$,

$$u_{i+k} - u_{i+k-1} \geq \left(\frac{a^* + \varepsilon}{b^*}\right)^{k-1} (u_{i+1} - u_i) > \left(\frac{a^*}{b^*}\right)^{k-1} (u_{i+1} - u_i) \tag{3.17}$$

and

$$u_{i-1} - u_{i-2} \leq \left(\frac{b^*}{a^* + \varepsilon}\right)(u_i - u_{i-1}) < \left(\frac{b^*}{a^*}\right)(u_i - u_{i-1}). \tag{3.18}$$

For all $i > i_0$, we can get

$$\begin{aligned}
(\tilde{Q}u)_i &= \sum_{j=0}^{\infty} q_{ij} u_j \\
&= \sum_{j=i-2}^{\infty} q_{ij} u_j \\
&= \binom{i}{2}\left(c_0 u_{i-2} + c_1 u_{i-1} + \sum_{j=i+1}^{\infty} c_{j-i+2} u_j - \sum_{j=i-2, j\neq i}^{\infty} c_{j-i+2} u_i\right) \\
&\quad + i\left(b_0 u_{i-1} + \sum_{j=i+1}^{\infty} b_{j-1+1} u_j - \sum_{j=i-1, j\neq i}^{\infty} b_{j-i+1} u_i\right) \\
&= \binom{i}{2}\left[c_0(u_{i-2} - u_i) + c_1(u_{i-1} - u_i) + \sum_{j=i+1}^{\infty} c_{j-i+2}(u_j - u_i)\right] \\
&\quad + i\left[b_0(u_{i-1} - u_i) + \sum_{j=i+1}^{\infty} b_{j-i+1}(u_j - u_i)\right] \\
&= \binom{i}{2}(-I_d + I_b) + i(-J_d + J_b)
\end{aligned}$$

where $I_d = c_0(u_i - u_{i-2}) + c_1(u_i - u_{i-1})$, $I_b = \sum_{j=i+1}^{\infty} c_{j-i+2}(u_j - u_i)$, $J_d = b_0(u_i - u_{i-1})$ and $J_b = \sum_{j=i+1}^{\infty} b_{j-i+1}(u_j - u_i)$.

By (3.16) and (3.17), we get

$$
\begin{aligned}
I_b &= \sum_{j=i+1}^{\infty} c_{j-1+2}(u_j - u_i) \\
&= \sum_{j=1}^{\infty} c_{j+2}(u_{i+j} - u_i) \\
&= \sum_{j=1}^{\infty} c_{j+2} \sum_{k=1}^{j} (u_{i+k} - u_{i+k-1}) \\
&\geq \sum_{j=1}^{\infty} c_{j+2} \sum_{k=1}^{j} \left(\frac{a^*}{b^*}\right)^{k-1} (u_{i+1} - u_i) \\
&> b^*(u_{i+1} - u_i)
\end{aligned}
$$

where b^* is defined in (3.14), and

$$
\begin{aligned}
J_b &= \sum_{j=i+1}^{\infty} b_{j-i+1}(u_j - u_i) \\
&= \sum_{j=1}^{\infty} b_{j+1}(u_{i+j} - u_i) \\
&= \sum_{j=1}^{\infty} b_{j+1} \sum_{k=1}^{j} (u_{i+k} - u_{i+k-1}) \\
&\geq \sum_{j=1}^{\infty} b_{j+1} \sum_{k=1}^{j} \left(\frac{a^*}{b^*}\right)^{k-1} (u_{i+1} - u_i) \\
&= \tilde{b}(u_{i+1} - i_i)
\end{aligned}
$$

where $\tilde{b} = \sum_{j=1}^{\infty} b_{j+1} \sum_{k=1}^{j} (\frac{a^*}{b^*})^{k-1}$.

By (3.18)

$$
\begin{aligned}
I_d &= c_0(u_i - u_{i-2}) + c_1(u_i - u_{i-1}) \\
&= (c_0 + c_1)(u_i - u_{i-1}) + c_0(u_{i-1} - u_{i-2}) \\
&\leq \left(c_0 \frac{b^*}{a^*} + (c_0 + c_1)\right)(u_i - u_{i-1}) \\
&\leq a^*(u_i - u_{i-1}) \qquad (3.19)
\end{aligned}
$$

and

$$J_d = b_0(u_i - u_{i-1}).$$

Therefore,

$$\binom{i}{2} I_b + iJ_b \geq \binom{i}{2} b^*(u_{i+1} - u_i) + i\tilde{b}_0(u_{i+1} - u_i)$$

and

$$\begin{aligned}
\binom{i}{2} I_d + iJ_d &\leq \binom{i}{2} a^*(u_i - u_{i-1}) + ib_0(u_i - u_{i-1}) \\
&= \binom{i}{2} (a^* + \varepsilon)(u_i - u_{i-1}) - \left[\binom{i}{2}\varepsilon - ib_0\right](u_i - u_{i-1}).
\end{aligned}$$

Therefore, $u^* = (u_i; i \geq 0)$ satisfies

$$\tilde{Q}u^* \geq \lambda u^*. \tag{3.20}$$

For all $i \leq i_0$, (3.20) is trivial. For $i > i_0$, by (3.15), (3.18) and (3.19), we get

$$(\tilde{Q}u)_i = \binom{i}{2} I_b + iJ_b - \left[\binom{i}{2} I_d + iJ_d\right] \geq \lambda u_i + \left[\binom{i}{2}\varepsilon - ib_0\right] \geq \lambda u_i$$

Here, $\left[\binom{i}{2}\varepsilon - ib_0\right] \geq 0$ since $i_0 = \left[\frac{2b_0}{\varepsilon}\right] + 1$. Thus $\tilde{Q}$ is not regular and hence by Lemma 3.4, Q is not regular. ■

Theorem 3.2 *There exists only one IBCP which satisfies the Kolmogorov forward equation.*

Proof. Together with Theorem 3.1, we now only need to consider $C'(1) > 0$. To prove that the IBCP is unique, we show that the forward equations have a unique solution. Using similar logic in last chapter, to show this, we use Theorem 2.28 in Anderson [1991], i.e. we prove that the equation

$$Y(\lambda I - Q) = 0, Y \geq 0, Y \cdot I < \infty \tag{3.21}$$

has non-trivial solution for some and (therefore for all) $\lambda > 0$. Where I denotes the column vector on $\mathbf{Z}_+$ whose components are all equal to 1.

Suppose that $\{Y = y_i; i \geq 0\}$ is a solution of (3.21). For $\lambda = 1$, (3.21) can be rewritten as

$$y_n = \sum_{j=1}^{n+2} \left(y_j \binom{j}{2} c_{n-j+2} + j b_{n-j+1}\right), \quad n \geq 0$$

Multiplying both sides of the above equation by s^n, summing over $n \geq 0$. We get

$$\sum_{n=0}^{\infty} y_n s^n = Y_a + Y_b$$

where

$$Y_a = B(s) \sum_{n=1}^{\infty} y_n n s^{n-1}$$

and

$$Y_b = C(s) \sum_{n=2}^{\infty} \binom{n}{2} y_n s^{n-2}.$$

So we have

$$Y(s) = \frac{C(s)}{2} Y'' + B(s) Y'(s) \tag{3.22}$$

First consider $B'(1) > 0$ and $C'(1) > 0$, this means $B(s) < 0$ and $C(s) < 0$ for all $s \in (\rho_b \vee \rho_c, 1)$ and hence the right hand side of (3.22) is negative. However, the left hand side is positive which is a contradiction.

If $B'(1) \leq 0$ and $C'(1) > 0$, then

$$Y(s) \leq B(s) Y'(s), \quad s \in (\rho_c, 1),$$

and

$$\frac{Y'(s)}{Y(s)} \geq B(s).$$

Hence

$$\ln Y(1) - \ln Y(\rho_c) \geq \int_{\rho_c}^{1} \frac{ds}{B(s)} = +\infty.$$

But we know that $0 < Y(1) < \infty$, there is a contraction. ∎

3.4 Extinction Probability

Having established that IBCP is uniquely determined by its q-matrix, we will now examine some of its properties. Let $\{X(t), t \geq 0\}$ be the unique IBCP, and let $P(t) = \{p_{ij}(t)\}$ denote its transition function. Define the extinction time τ_0, by

$$\tau_0 = \begin{cases} \{\inf t > 0, X(t) = 0\} & \text{if } X(t) = 0 \ \text{ for some } \ t > 0 \\ +\infty & \text{if } X(t) \neq 0 \ \text{ for all } \ t > 0 \end{cases}$$

and denote the extinction probability by

$$a_i = P(\tau_0 < \infty | X(0) = i), \quad i \geq 1.$$

Before we consider the absorbing behavior of IBCP, we need to establish two lemmas for this purpose. For the absorbing behavior of IBCP we will separate it into two different cases, regular and irregular.

Denote

$$G_i(s) = \sum_{k=1}^{\infty} \left(\int_0^{\infty} p_{ik}(t)dt \right) s^k, \quad i \geq 1 \tag{3.23}$$

$$H(y) = \int_0^y \frac{B(x)}{C(x)}, \quad y \in (\zeta_c, \rho_c) \tag{3.24}$$

where integral should be taken along the inverse direction if $y < 0$. By Lemma 3.1, we know that $H(y)$ is finite for all $y \in (\zeta_c, \rho_c)$. Also, let

$$A(y) = \exp\{2H(y)\}, \quad y \in (\zeta_c, \rho_c). \tag{3.25}$$

We can also note that

$$\begin{cases} H(0) = 0, \quad H(y) < 0 \text{ if } y \in (\zeta_c, 0) \\ A(0) = 1, \quad A(y) < 1 \text{ if } y \in (\zeta_c, 0) \\ A(y) \to 0, \quad \text{i.f.f. } \ H(y) \to -\infty. \end{cases}$$

Lemma 3.5 *(i)* $\lim_{y\to\zeta_c^+} H(y) = -\infty$ *and* $\lim_{y\to\zeta_c^+} A(y) = 0$.

(ii) $0 < C'(1) \le +\infty$, *i.e.* $\rho_c < 1$. *Suppose, if* $\rho_b = \rho_c < 1$, *then* $0 \le \lim_{y\to\rho_c} H(\rho_c) < +\infty$. *If* $\rho_c < \rho_b \le 1$, *then* $\lim_{y\to\rho_c^-} H(y) = +\infty$.

Proof. It can be seen easily by definition of $A(y)$ and $H(y)$ respectively. ∎

Lemma 3.6 *(i) For any* $i \ge 1$ *and* $|s| < 1$,

$$\frac{C(s)}{2} \cdot G_i''(s) + B(s) \cdot G_i'(s) = a_i - s^i. \tag{3.26}$$

Moreover, for $|s| < \rho_c$, *we have*

$$G_i'(s) \cdot A(s) - G_i'(0) = \int_0^s \frac{2(a_i - y^i)}{c(y)} \cdot A(y)dy \tag{3.27}$$

(ii) For any $i \ge 1$,

$$\lim_{s\to\zeta_c} G_i'(s)A(s) = 0. \tag{3.28}$$

Proof. From (3.4) with Lemma 3.3

$$\sum_{j=0}^{\infty} p_{ij}'(t)s^j = C(s)\sum_{k=2}^{\infty}\binom{k}{2}p_{ik}(t)s^{k-2} + B(s)\sum_{k=1}^{\infty} kp_{ik}(t)s^{k-1}.$$

Integrating above equality with respect to $t \in [0, \infty)$ with

$$G_i(s) = \sum_{k=1}^{\infty}\left(\int_0^{\infty} p_{ik}(t)dt\right)s^k.$$

Therefore,

$$a_i - s^i = \frac{C(s)}{2}G_i''(s) + B(s)G_i'(s) \tag{3.29}$$

Treat (3.29) as a first order differential equation in $G_i'(s)$, after some calculation, we have

$$e^{\int_0^y \frac{2B(x)}{C(x)}dx}\left[G_i''(y) + \frac{2B(y)}{C(y)}G_i'(y)\right] = \left[\frac{2}{C(y)}(a_i - y^i)\right]e^{\int_0^y \frac{2B(x)}{C(x)}dx},$$

$$\left[G_i'(y)e^{\int_0^y \frac{2B(x)}{C(x)}dx}\right]' = \frac{2(a_i - y^i)}{C(y)}A(y),$$

and

$$G_i'(s)\cdot A(s) - G_i'(0) = \int_0^s \frac{2(a_i - y^i)}{C(y)} \cdot A(y)dy.$$

We now try to prove (3.28): $\lim_{s\to\zeta_c} G_i'(s)A(s) = 0$.

If $-1 < \zeta_c < 0$, then $|G_i'(\zeta_c)| < \infty$, by Lemma 3.5 $\lim_{s\to\zeta_c} A(s) = 0$, (3.28) is proved.

If $\zeta_c = -1$, $C(-x) = C(x)$ for all $x \in [0,1]$. For all $x \in (0,1)$,

$$\begin{aligned} B(-x) + B(x) &= 2\sum_{k=0}^{\infty} b_{2k}x^{2k} > 2b_0 \\ -B(-x) &< B(x) - 2b_0 \end{aligned}$$

$$\begin{aligned} A(-s) &= e^{\int_0^{-s} \frac{2B(x)}{C(x)}dx} = e^{-\int_0^{s} \frac{2B(-x)}{C(x)}dx} \\ &\leq e^{\int_0^{s} \frac{2B(x)-4b_0}{C(x)}dx} = A(s)e^{-4b_0\int_0^s \frac{dx}{C(x)}}. \end{aligned}$$

From (3.27) that for $s \in [0,1)$,

$$\begin{aligned} |G_i'(-s)|A(-s) &\leq G_i'(s)A(s)\cdot e^{-4b_0\int_0^s \frac{dx}{C(x)}} \\ &\leq G_i'(0)\cdot e^{-4b_0\int_0^s \frac{dx}{C(x)}} + \left(2\int_0^s \frac{A(y)}{C(y)}dy\right)\cdot e^{-4b_0\int_0^s \frac{dx}{C(x)}} \\ &\leq G_i'(0)\cdot e^{-4b_0\int_0^s \frac{dx}{C(x)}} + \left(2\int_0^s \frac{e^{2b_0\int_0^y \frac{dx}{C(x)}})}{C(y)}dy\right)\cdot e^{-4b_0\int_0^s \frac{dx}{C(x)}} \\ &\leq G_i'(0)\cdot e^{-4b_0\int_0^s \frac{dx}{C(x)}} + \left(2\int_0^s \frac{e^{-b_0\int_0^y \frac{dx}{C(x)}})}{C(y)}dy\right)\cdot e^{-b_0\int_0^s \frac{dx}{C(x)}} \\ &\leq G_i'(0)\cdot e^{-4b_0\int_0^s \frac{dx}{C(x)}} + \frac{2}{b_0}\cdot e^{-b_0\int_0^s \frac{dx}{C(x)}} \end{aligned}$$

Therefore,

$$\lim_{s\to\zeta_c} |G_i'(s)|A(s) = \lim_{s\to 1} |G_i'(-s)|A(-s) = 0.$$

and thus (3.28) is proved. ■

Having the above lemmas and definitions, we are ready to consider the extinction probability. Firstly, looking at the case that Q is regular, i.e. by Theorem 3.2, an IBCP q-matrix Q is regular if and only if $C'(1) \leq 0$.

Theorem 3.3 *Suppose that $C'(1)$ and $B'(1) \leq 0$. Then $a_i = 1$, $(i \geq 1)$.*

Proof. From (3.26)

$$\frac{C(s)}{2} \cdot G_i''(s) + B(s) \cdot G_i'(s) = a_i - s^i,$$

$C(s) > 0$, $B(s) > 0$ for all $s \in [0, 1)$. Therefore, $a_i - s^i \geq 0$. Let $s \to 1$, we have $a_i \geq 1$. But $a_i \leq 1$ is always true and thus $a_i = 1$. ■

Theorem 3.4 *Suppose that $C'(1) \leq 0$ and $0 < B'(1) < +\infty$. Then $a_i = 1 (i \geq 1)$ if and only if $J = +\infty$ where*

$$J = \int_{\zeta_c}^{1} \frac{A(y)}{C(y)} dy \tag{3.30}$$

and $A(y)$ is defined in (3.25). Moreover, if $J < +\infty$ then

$$a_i = J^{-1} \cdot \int_{\zeta_c}^{1} \frac{y^i A(y)}{C(y)} dy, \quad i \geq 1. \tag{3.31}$$

Proof. Suppose $J = +\infty$. Since the positive states communicate to each other, we have either $a_i = 1$ for all $i \geq 1$ and $a_i < 1$ for all $i \geq 1$. We try to prove by contraction.

Assume that $a_i < 1$. In (3.27), we have,

$$G_1'(s) \cdot A(s) - G_1'(0) = 2 \int_0^{a_1} \frac{a_1 - y}{C(y)} \cdot A(y) dy + 2 \int_{a_1}^{s} \frac{a_1 - y}{C(y)} \cdot A(y) dy. \tag{3.32}$$

Let $s \to 1$, first term on right hand side of (3.32) is finite and the second term tends to $-\infty$ since $J = +\infty$. This is a contradiction and we have $a_i = 1$ for all $i \geq 1$. Now suppose $J < +\infty$. By (3.27) and (3.28),

$$-G_i'(0) = -2 \int_{\zeta_c}^{0} \frac{a_i - y^i}{C(y)} \cdot A(y) dy. \tag{3.33}$$

Together with (3.27), we have

$$G_i'(s) \cdot A(s) = \int_{\zeta_c}^{s} \frac{2(a_i - y^i)}{C(y)} \cdot A(y)dy. \tag{3.34}$$

Define

$$x_i = J^{-1} \int_{\zeta_c} \frac{y^i A(y)}{C(y)} dy \quad (i \geq 1).$$

Then by (3.34), $a_i \geq x_i$ $(i \geq 1)$. On the other hand,

$$\begin{aligned}
& \sum_{k=1}^{\infty} q_{ik} x_k + q_{i0} \\
= \; & \sum_{k=i-2}^{\infty} \left[\binom{i}{2} c_{k-i+2} + i b_{k-i+1} \right] x_k \\
= \; & J^{-1} \int_{\zeta_c}^{1} \frac{\sum_{k=i-2}^{\infty} \left[\binom{i}{2} c_{k-i+2} + i b_{k-i+1} \right] y^k}{C(y)} e^{\int_0^y \frac{2B(x)}{C(x)} dx} dy \\
= \; & J^{-1} \int_{\zeta_c}^{1} \frac{\binom{i}{2} C(y) y^{i-2} + i B(y) y^{i-1}}{C(y)} e^{\int_0^y \frac{2B(x)}{C(x)} dx} dy \\
= \; & J^{-1} \left[\int_{\zeta_c}^{1} \binom{i}{2} A(y) y^{i-2} dy + \frac{1}{2} \int_{\zeta_c}^{1} i A'(y) y^{i-1} dy \right] \\
= \; & \frac{i}{2J} \left[\int_{\zeta_c}^{1} (i-1) A(y) y^{i-2} dy + \int_{\zeta_c}^{1} A'(y) y^{i-1} dy \right] \\
= \; & \frac{i}{2J} \left[A(1) - \zeta^{i-1} A(\zeta_c) \right] \\
= \; & 0.
\end{aligned}$$

We have use the assumption of $J < \infty$ and result from Lemma 3.5 for the last step. By Lemma 4.46 of of Chen [2004] or Li and Chen [2006], we know $a_i \leq x_i$ $(i \geq 1)$, $a_i = x_i$ and the proof is then finished. ■

We see that $J < +\infty$ if and only if $J_0 = \int_0^1 \frac{A(y)}{C(y)} dy < +\infty$ where the former J is given in (3.30). In practice, it may be difficult to check whether J is finite or not. So, we try to find some convenient sufficient conditions to check the quantity J.

Corollary 3.1 *Suppose that $C'(1) < 0$ and $0 < B'(1) < +\infty$. Then $a_i = 1$ $(i \geq$*

1).

Proof. Just note that

$$A(1) = e^{\int_0^1 \frac{2B(x)}{C(x)}dx} > 0,$$

$$J = \int_{\zeta_c}^1 \frac{A(y)}{C(y)} dy = +\infty.$$

From Theorem 3.4, conclusion follows. ■

Corollary 3.2 *Suppose $C'(1) = 0$ and $0 < B'(1) < +\infty$ together with $C''(1) < \infty$. If $C''(1) > 4B'(1)$, then $J = +\infty$ and thus $a_i = 1$ $(i \geq 1)$. If $C''(1) < 4B'(1)$, then $J < +\infty$, $a_i < 1$ and a_i is given in (3.31).*

Proof. By using L'Hopital rule two times, we get

$$\lim_{xto1} \frac{2(1-x)B(x)}{C(x)} = -\frac{4B'(1)}{C''(1)} = -\gamma.$$

Denote $\gamma = \frac{4B'(1)}{C''(1)}$ for convenient. Let $g(x) = \frac{2(1-x)B(x)}{C(x)}$, and we can expand $g(x)$ as power series of x in the internal $[0, 1)$ with the form

$$g(x) = \sum_{k=0}^{\infty} g_k x^k, \text{ where } g_k = \frac{g^{(k)}(0)}{k!}$$

by (3.24),

$$\begin{aligned}
H(y) &= \frac{1}{2}\sum_{k=0}^{\infty} g_k \int_0^y \frac{x^k}{1-x} dx \\
&= \frac{1}{2}\sum_{k=0}^{\infty} g_k \int_0^y \frac{[1-(1-x)]^k}{1-x} dx \\
&= \frac{1}{2}\sum_{k=0}^{\infty} g_k \int_0^y \frac{1}{1-x} dx + \frac{1}{2}\sum_{k=1}^{\infty} g_k \int_0^y \sum_{m=1}^{k} (-1)^m (1-x)^{m-1} dx \\
&= -\frac{\ln(1-y)}{2}\sum_{k=0}^{\infty} g_k + H_1(y).
\end{aligned}$$

We know that $H_1(y)$ is bounded and $y \in [0, 1]$,

$$\sum_{k=0}^{\infty} g_k = \lim_{x\uparrow 1} \frac{2(1-x)B(x)}{C(x)} = -\gamma.$$

This gives $A(y) = A_1(y) = (1-y)^\gamma$ where $A_1(y)$ is bounded on $y \in [0,1]$.

From the definition of $J_0 = \int_0^1 \frac{A(y)}{C(y)}dy$, and under the assumptions that $C'(1) = 0$, together with J_1 is finite if and only if the integral $\int_0^1 \frac{dy}{(1-y)^{2-\gamma}}$ is convergent, or equivalently, if and only if $4B'(1) > C''(1)$. The proof is then complete. ■

For regular case, we are now ready to consider the expected extinction time.

Theorem 3.5 *Suppose that Q is given in (3.1) - (3.2)and $P(t) = (p_{ij}(t); i,j \geq 0)$ is the Feller minimal Q-function. If $C'(1) \leq 0$ and $J = +\infty$, then $E_i[\tau_0] < \infty$ $(i \geq 1)$ if and only if*

$$\int_0^1 \left[\frac{1}{A(s)} \int_0^s \frac{(1-y)A(y)}{C(y)} dy\right] ds < \infty \tag{3.35}$$

and $E_i[\tau_0]$ is given by

$$E_i[\tau_0] = \int_0^1 \left[\frac{1}{A(s)} \int_{\zeta_c}^s \frac{2(1-y^i)A(y)}{C(y)} dy\right] ds, \quad i \geq 1 \tag{3.36}$$

where E_i is the conditional expectation when the process starts from $i \geq 1$.

Proof. First note that $E[\tau_0]$ is the conditional expectation for the extinction time, i.e.

$$E[\tau_0] = \int_0^\infty (1 - p_{i0}(t))dt = \sum_{j=1}^\infty p_{ij}(t)dt = G_i(1),$$

our aim is to find $G_i(1)$. By (3.27), we have

$$G_i'(s) \cdot A(s) - G_i'(0) = \int_0^s \frac{2(1-y^i)}{c(y)} \cdot A(y)dy \tag{3.37}$$

and

$$G_i'(0) = \int_{\zeta_c}^0 \frac{2(a_i - y^i)}{c(y)} \cdot A(y)dy \tag{3.38}$$

Integrating (3.37), we get

$$G_i(s) = G_i'(0) \int_0^s \frac{du}{A(u)} + \int_0^s [\frac{1}{A(u)} \int_0^u \frac{2(1-y^i)}{C(y)} A(y)dy]du. \tag{3.39}$$

Substituting (3.38) into (3.39), we get

$$G_i(s) = \int_0^s [\frac{1}{A(u)} \int_{\zeta_c}^u \frac{2(1-y^i)}{C(y)} A(y)dy]du. \tag{3.40}$$

Letting $s \to 1$ in (3.40), we have

$$G_i(1) = \int_0^1 [\frac{1}{A(u)} \int_{\zeta_c}^u \frac{2(1-y^i)}{C(y)} A(y)dy]du. \tag{3.41}$$

(3.36) is then follows from (3.41). It is trivial that (3.36) is finite if and only if (3.35) is true.

In the case of $C'(1) \le 0$ and $J < +\infty$. It is trivial that $E_i[\tau_0] = +\infty$ as $a_i < 1$. As a result, based on the above theorem, we obtain a similar result. ■

Theorem 3.6 *Suppose that Q is an IBC q-matrix as in (3.1) - (3.2) and $P(t) = (p_{ij}(t); i,j \ge 0)$ is the unique IBCP, i.e. the Feller minimal Q-function. If $C'(1) < 0$ and $J < +\infty$, then*

$$a_i E_i[\tau_0 | \tau_0 < \infty] = \int_0^1 \left[\frac{1}{A(s)} \int_{\zeta_c}^s \frac{2(a_i - y^i)A(y)}{C(y)} dy \right] ds, \quad i \ge 1.$$

Proof. The proof is similar to Theorem 3.5 and note that $E_i[\tau_0|\tau_0 < \infty] = a_i^{-1} \int_0^\infty (a_i - p_{i0}(t))dt$. ■

We should note that, in Theorem 3.6, the $E_i[\tau_0|\tau_0 < \infty]$ is based on the conditional mathematical expectation under the condition $\{\tau_0 < \infty\}$.

3.5 Extinction Probability: Irregular Case

We have looked through the regular case, i.e. $C'(1) \le 0$. Now, we go to the irregular case, i.e. $0 < C'(1) < +\infty$, $\rho_c < 1$. In this case, the unique IBCP is still the Feller minimal Q-process and thus dishonest.

Theorem 3.7 *If Q is irregular, i.e. $0 < C'(1) < \infty$. And also $0 < B'(1) < \infty$ with $\rho_c = \rho_b < 1$, then $a_i = \rho_c^i$.*

Proof. Putting $s = \rho_c$ in (3.7), we have

$$\sum_{j=0}^{\infty} p'_{ij}(t)\rho_c^j = 0.$$

Integrating above equality by both sides, we get

$$\sum_{j=0}^{\infty} p_{ij}(t)\rho_c^j = \rho_c^i.$$

Letting $t \to \infty$, and using dominated convergence theorem with Lemma 3.3 (i). The conclusion is reached. ■

Theorem 3.8 *If Q is irregular, i.e. $0 < C'(1) < \infty$. And also $0 < B'(1) < \infty$ with $\rho_c = \rho_b < 1$, then the mean conditioned extinction time $E_i(\tau_0|\tau_0 < \infty)$ is given by*

$$E_i(\tau_0|\tau_0 < \infty) = \rho_c^{-i} \int_0^{\rho_c} \left[\frac{2}{A(s)} \int_{\zeta_c}^{s} \left(1 - \frac{y}{\rho_c}^i\right) \frac{A(y)}{C(y)} dy \right] ds$$

Proof. From (3.26), we have

$$\frac{C(s)}{2} G''_i(s) + B(s)G'_i(s) = \rho_c^i - s^i,$$

and solve this as a first order differential equation. Then, using similar method in Theorem 3.5 will give the desired conclusion.

■

Theorem 3.9 *Suppose that $C'(1) > 0$, i.e. $\rho_c < 1$ and furthermore, $\rho_b < \rho_c < 1$, for all $i \geq 1$, the extinction probability is given by*

$$a_i = \frac{\int_{\zeta_c}^{\rho_c} \frac{y^i A(y)}{C(y)} dy}{\int_{\zeta_c}^{\rho_c} \frac{A(y)}{C(y)} dy}. \tag{3.42}$$

Proof. From (3.26),

$$\frac{C(s)}{2}\cdot G_i''(s)+B(s)\cdot G_i'(s)=a_i-s^i,\ i\geq 1,|s|<1,$$

solving (3.26) on $[0,\rho_c)$ gives,

$$G_i'(s)\cdot e^{\int_0^s\frac{2B(x)}{C(x)}dx}-G_i'(0)=\int_0^s\frac{2(a_i-y^i)}{C(y)}\cdot e^{\int_0^y\frac{2B(x)}{C(x)}dx}dy.$$

Therefore, $G_i'(\rho_c)<+\infty$ and $e^{\int_0^{\rho_c}\frac{2B(x)}{C(x)}dx}=0$.

Since $\int_0^{\rho_c}\frac{2B(x)}{C(x)}dx=-\infty$ with $B(x)<0$ for all (ρ_b,ρ_c). We have

$$-G_i(0)=\int_0^{\rho_c}\frac{2(a_i-y^i)}{C(y)}e^{\int_0^y\frac{2B(x)}{C(x)}dx}dy. \tag{3.43}$$

Solve (3.26) on $(\zeta_c,0]$ gives

$$G_i'(s)\cdot e^{\int_0^s\frac{2B(x)}{C(x)}dx}-G_i'(0)=\int_0^s\frac{2(a_i-y^i)}{C(y)}\cdot e^{\int_0^y\frac{2B(x)}{C(x)}dx}dy.$$

Therefore, $G_i'(\zeta_c)<+\infty$ and $e^{\int_0^{\zeta_c}}\frac{2B(x)}{C(x)}dx=0$. Since $\int_0^{\zeta_c}\frac{2B(x)}{C(x)}dx=-\infty$ with $B(x)<0$ and by Lemma 3.5 we get

$$-G_i'(0)=\int_0^{\zeta_c}\frac{2(a_i-y^i)}{C(y)}\cdot e^{\int_0^y\frac{2B(x)}{C(x)}dx}dy. \tag{3.44}$$

From (3.43) and (3.44), we have

$$\int_{\zeta_c}^{\rho_c}=\frac{2(a_i-y^i)}{C(y)}\cdot e^{\int_0^y\frac{2B(x)}{C(x)}dx}dy=0$$

and thus (3.42) follows. ∎

Next, we will look at the case with $\rho_c<\rho_b\leq 1$. It is surprisingly that the extinction probability for this case is very troublesome. Before we get the closed form for the extinction probability, we try to get a lower and upper bound for this.

Theorem 3.10 *Suppose that $C'(1)>0$, i.e. $\rho_c<1$, and furthermore, $\rho_c<\rho_b\leq$*

1; *then* $\rho_c^i < a_i < \rho_b^i$ *for all* $i \geq 1$.

Proof. This can easily be proved by putting $s = \rho_b$ and ρ_c into (3.26), we have,

$$\begin{aligned} a_i &= B(\rho_c)G_i'(\rho_c) + \rho_c^i \\ a_i &= \frac{C(\rho_b)}{2}G_i'(\rho_b) + \rho_b^i \end{aligned}$$

Note that $B(\rho_c) > 0$ and $C(\rho_b) < 0$. Therefore, $\rho_c^i < a_i < \rho_c^i$. We can also see that when $\rho_c < \rho_b = 1$, $a_i < \rho_b^i = 1$ which follows that IBCP is not honest.

In Theorem 3.10, we haven't calculated the extinction probability a_i, when $\rho_c < \rho_b \leq 1$. To improve the result, we do some mathematical transformation.

From (3.26), be rewrite as

$$A_0 G_i''(s) + B_0(s)G_i'(s) = U_o(s)$$

where $A_0(s) = \frac{C(s)}{2}$, $B_0(s) = B(s)$ and $U_0(s) = a_i - s^i$. Now we define

$$\begin{aligned} A_1(s) &= A_0(s)B_0(s) \\ B_1(s) &= B_0(s)[B_0(s) + A_0'(s)] - A_0(s)B_0'(s) \\ U_1(s) &= B_0(s)U_0'(s) - B_0'(s)U_0(s) \end{aligned}$$

then if we substitute the above equation set recursively, we can get

$$A_n(s)G^{(n+2)}(s) + B_n(s)G_i^{(n+1)}(s) = U_n(s) \tag{3.45}$$

$$A_n(s) = A_{n-1}(s)B_{n-1}(s) \tag{3.46}$$

$$B_n(s) = B_{n-1}(s)[B_{n-1}(s) + A_{n-1}'(s)] - A_{n-1}(s)B_{n-1}'(s) \tag{3.47}$$

$$U_n(s) = B_{n-1}(s)U_{n-1}'(s) - B_{n-1}'(s)U_{n-1}(s) \tag{3.48}$$

Note that all $A_n(s)$ and $B_n(s)$ $(n \geq 0)$ are entirely expressible in terms of the given functions, $B(s)$ and $C(s)$ and also independent of i. Similarly, all $U_n(s)$ are expressible in term s of $B(s)$ and $C(s)$ together with the unknown constant a_i.

By (3.46), we can get

$$A_n(s) = A_0(s) \prod_{k=0}^{n-1} B_k(s),$$

where $A_0(s) = \frac{C(s)}{2}$ which has been defined above.

$$A_n(\rho_c) = A_n(\zeta_c) = 0, \forall\, n \geq 0. \tag{3.49}$$

Prove by mathematical induction, we have

$$B_n(\rho_c) = \Big(B_0(\rho_c) + nA_0'(\rho_c)\Big) \prod_{k=0}^{n-1} B_k(\rho_c), \quad \forall\, n \geq 1 \tag{3.50}$$

$$A_n'(\rho_c) = A_0'(\rho_c) \prod_{k=0}^{n-1} B_k(\rho_c), \quad \forall\, n \geq 1$$

with

$$B_0(\rho_c) = B(\rho_c) > 0 \text{ and } A_0'(\rho_c) = \frac{C'(\rho_b)}{2} < 0.$$

Similarly,

$$B_n(\zeta_c) = \Big(B_0(\zeta_c) + nA_0'(\zeta_c)\Big) \prod_{k=0}^{n-1} B_k(\zeta_c), \quad \forall\, n \geq 1.$$

$$A_n'(\zeta_c) = A_0'(\zeta_c) \prod_{k=0}^{n-1} B_k(\zeta_c), \quad \forall\, n \geq 1$$

with

$$B_0(\zeta_c) = B(\zeta_c) > 0 \text{ and } A_0'(\zeta_c) = \frac{C'(\zeta_c)}{2} > 0. \tag{3.51}$$

■

Lemma 3.7 *Suppose $C'(1) > 0$, i.e. $\rho_c < 1$ and furthermore, $\rho_c < \rho_b \leq 1$. Then we have*

(i) $A_n(\rho_c) = A_n(\zeta_c) = 0, \quad \forall n \geq 0.$

(ii) $B_n(\zeta_c) > 0, \quad \forall n \geq 0.$

(iii) If $-\frac{2B(\rho_c)}{C'(\rho_c)}$ is a positive integer m, then $B_n(\rho_c) > 0$ for all $0 \leq n \leq m-1$ and $B_m(\rho_c) = 0$. If $-\frac{2B(\rho_c)}{C'(\rho_c)}$ is not a positive integer. Setting m as the integer

part of $-\frac{2B(\rho_c)}{C'(\rho_c)}$, *then* $B_n(\rho_c) > 0$ *for all* $0 \le u < m$ *and* $B_m(\rho_c) < 0$.

Proof.

(i) has been shown in the above theorem.

(ii) can be easily proved by considering the above theorem with mathematical induction.

(iii) considering (3.50), noting that $A_0'(\rho_c) = \frac{C'(\rho_c)}{2} < 0$, the result is then trivial.

■

Theorem 3.11 *Suppose that* $\rho_c < \rho_b \le 1$.

(i) If $C'(\rho_c) + 2B(\rho_c) = 0$, *then*

$$a_i = \rho_c^i + i\sigma\rho_c^{i-1} \tag{3.52}$$

(ii) If $C'(\rho_c) + 2B(\rho_c) > 0$, *then*

$$\rho_c^i + i\sigma\rho_c^{i-1} < a_i < \rho_b^i \tag{3.53}$$

(iii) If $C'(\rho_c) + 2B(\rho_c) < 0$, *then*

$$\rho_c^i < a_i < \min\{\rho_b^i, \rho_c^i + i\sigma\rho_c^{i-1}\} \tag{3.54}$$

where σ *is a positive constant which is independent of* i *and given by* $\sigma = -\frac{B(\rho_c)}{B'(\rho_c)}$.

Proof. By putting $n = 1$ in (3.45) - (3.48) together with the fact, we have

$$\begin{aligned} & \frac{B(s)C(s)}{2}G_i''(s) + \frac{B(s)\Big(2B(s) + C'(s)\Big) - C(s)B'(s)}{2}G_i'''(s) \\ = \; & B'(s)(s^i - a_i) - is^{i-1}B(s). \end{aligned} \tag{3.55}$$

Putting $s = \rho_c$ in (3.55), with $C(\rho_c) = 0$, $G_i''(\rho_c) < +\infty$ and $G_i''(\rho_c) < \infty$. We get

$$B(\rho_c)[2B(\rho_c) + C'(\rho_c)]G_i''(\rho_c) = 2B'(\rho_c)(\rho_c^i - a_i) - 2i\rho_c^{i-1}B(\rho_c). \tag{3.56}$$

If $C'(\rho_c) + 2(\rho_c) = 0$, we get

$$\begin{aligned} 2B'(\rho_c)(\rho_0^i - a_i) &= 2i\rho_c^{i-1}B(\rho_c) \\ a_i &= \rho_c^i + i\sigma\rho_c^{i-1} \end{aligned}$$

(ii) If $C'(\rho_c) + 2(\rho_c) > 0$, similar to (i), we get

$$a_i > \rho_c^i + i\sigma\rho_c^{i-1}$$

together with Theorem 3.10, we get

$$\rho_c^i < a_i < \min\{\rho_b^i, \rho_c^i + i\sigma\rho_c^{i-1}\}.$$

■

By Theorem 3.11, we can find out the closed form of a_i if $C'(\rho_c)+2B(\rho_c) = 0$. In the next theorem, we now try to get the a_i if $C'(\rho_c) + 2(\rho_c) < 0$.

Theorem 3.12 *Suppose that $\rho_c < \rho_b \leq 1$ and $C'(\rho_c)+2B(\rho_c) < 0$, then we have*

$$a_i = \frac{\int_{\zeta_c}^{\rho_c} \frac{y^iB'(y)-iy^{i-1}B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy}{\int_{\zeta_c}^{\rho_c} \frac{B'(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy}.$$

Proof. By (3.55), we know that for $|s| < \rho_c$,

$$A_1(s)G_i'''(s) + B_1(s)G_(''s) = U_1(s)$$

where

$$A_1(s) = \frac{B(s)C(s)}{2}, \quad B_1(s) = \frac{B(s)(2B(s) + C'(s)) - C(s)B'(s))}{2}$$

and

$$U_1(s) = B'(s)(s^i - a_i) - is^{i-1}B(s).$$

$$G_i'''(s) + \frac{B_1(s)}{A_1(s)}G_i''(s) = \frac{U_1(s)}{A_1(s)}$$

Solving the above equation on $s \in (\zeta_c, \rho_c)$, with $G_i''(\zeta_c)e^{\int_0^{\zeta_c} \frac{B_1(y)}{A_1(y)}dy} = 0$, we get

$$G_i''(s)e^{\int_0^s \frac{B_1(y)}{A_1(y)}dy} = \int_{\zeta_c}^s \frac{U_1(y)e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx}}{A_1(y)}. \tag{3.57}$$

Letting $s \uparrow \rho_c$ in (3.57), $G_i''(\rho_c)e^{\int_0^{\rho_c} \frac{B_1(y)}{A_1(y)}dy} = 0$ and $B_1(\rho_c) = \frac{1}{2}B(\rho_c)(C'(\rho_c) + 2B(\rho_c)) < 0$, we have

$$\int_{\zeta_c}^{\rho_c} \frac{U_1(y)e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx}}{A_1(y)}dy = 0.$$

i.e.

$$a_i \int_{\zeta_c}^{\rho_c} \frac{B'(y)}{A_1(y)}e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx}dy = \int_{\zeta_c}^{\rho_c} \frac{y^iB'(y) - iy^{i-1}B(y)}{A_1(y)}e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx}dy.$$

$$a_i = \frac{\int_{\zeta_c}^{\rho_c} \frac{y^iB'(y)-iy^{i-1}B(y)}{A_1(y)}e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx}dy}{\int_{\zeta_c}^{\rho_c} \frac{B'(y)}{A_1(y)}e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx}dy}.$$

The proof is then completed. ■

From this point, we only need to investigate the case of $C'(\rho_c) + 2B(\rho_c) > 0$. To do so, we need to do some further transformation. Firstly, we need to know more about the structure of $U_n(s)$ defined in (3.48). We also know that $U_n(s)$ depends on $i \geq 1$. So, we include this factor into our notation and denote it as $U_{ni}(s)$.

Lemma 3.8 *For any $n \geq 1$ and $i \geq 1$, we have*

$$U_{ni}(s) = \sum_{k=0}^{n} D_{n,k}(s)U_{0i}^{(k)}(s) \tag{3.58}$$

where $U_{0i}^{(k)}(s)$ denoted the k'th derivative of $U_{0i}(s) = a_i - s^i$, $\{D_{n,k}(s), 0 \leq k \leq n\}$ are totally expressible by the two known $B(s)$ and $C(s)$. While $D_{n,k}(s)$ does not *depend on* $i \geq 1$. $\{D_{n,k}(s)\}$ *can be written recursively as follows.*

$$D_{1,0}(s) = -B'(s), \quad D_{1,1}(s) = B(s) \tag{3.59}$$

$$D_{n,k}(s) = D_{n-1,n-k}(s)B_{n-1}(s) - D_{n-1,k}(s)B'_{n-1}(s) + D'_{n-1,k}(s)B_{n-1}(s),\ k \leq n-1 \tag{3.60}$$

and

$$D_{n,n}(s) = \prod_{m=0}^{n-1} B_m(s) \tag{3.61}$$

Proof. Using mathematical induction with (3.48), the conclusions will be reached. As an example, when $n = 1$,

$$U_{1i}(s) = B'(s)(s^i - a_i) - is^{i-1}B(s) = D_{1,0}(s)U_{0i}(s) + D_{1,1}(s)U_{0i}^{(1)}(s)$$

where $D_{1,0}(s)$ and $D_{1,1}(s)$ is defined in (3.59). ∎

Remark 3.1 *From the definition of* $U_{0i}(s)$*, it can be noted that*

$$U_{ni}(s) = \sum_{k=0}^{n \wedge i} D_{n,k}(s)U_{0i}^{(k)}(s).$$

We may use the former notation for simplicity.

We now further consider the case that $C'(\rho_c) + 2B(\rho_c) > 0$. Considering $B(\rho_c) > 0$ and $C'(\rho_c) < 0$, there exists a positive integer m such that

$$mC'(\rho_c) + 2B(\rho_c) \leq 0 \text{ and } (m-1)C'(\rho_c) + 2B(\rho_c) > 0.$$

Our Theorem 3.10 and 3.11 are tackling the case $m = 1$. We now consider the case for $m \geq 2$.

Theorem 3.13 *Suppose that* Q *is an IBC q-matrix, with* $\rho_c < \rho_b \leq 1$ *and* $C'(\rho_c) + 2B(\rho_c) > 0$. *Let* $m = \min k \in \mathbf{Z}_+; kC'(\rho_c) + 2B(\rho_c) \leq 0$ *and thus* $m \geq 2$.

(i) If $mC'(\rho_c) + 2B(\rho_c) = 0$*, then* $U_k(\rho_c) > 0$ *for all* $0 \le k \le m-1$ *and* $U_m(\rho_c) = 0$*. Hence by (3.58)*

$$a_i = \rho_c^i + \sum_{k=1}^{m\wedge i} \frac{D_{m,k}(\rho_c)}{D_{m,0}(\rho_c)} \cdot \frac{i!}{(i-k)!} \rho_c^{i-k}. \tag{3.62}$$

In particular, $a_i = \rho_c + \frac{D_{m,1}(\rho_c)}{D_{m,0}(\rho_c)}$.

(ii) If $mC'(\rho_c) + 2B(\rho_c) < 0$*, then* $U_k(\rho_c) > 0$ *for all* $0 \le k \le m-1$ *and* $U_m(\rho_c) < 0$*. Hence*

$$\rho_c^i + \sum_{k=1}^{(m-1)\wedge i} \frac{D_{m-1,k}(\rho_c)}{D_{m-1,0}(\rho_c)} \cdot \frac{i!}{(i-k)!} \cdot \rho_c^{i-k} < a_i < \rho_c^i + \sum_{k=1}^{m\wedge i} \frac{D_{m,k}(\rho_c)}{D_{m,0}(\rho_c)} \cdot \frac{i!}{(i-k)!} \cdot \rho_c^{i-k}. \tag{3.63}$$

In particular, $\rho_c + \frac{D_{m-1,1}(\rho_c)}{D_{m-1,0}(\rho_c)} < a_1 < \rho_c + \frac{D_{m,1}(\rho_c)}{D_{m,0}(\rho_c)}$.

Proof. From (3.50) that

$$B_k(\rho_c) = \frac{1}{2}\Big(2B(\rho_c) + kC'(\rho_c)\Big) \prod_{j=0}^{k-1} B_j(\rho_c), \quad \forall\, k \ge 1. \tag{3.64}$$

From (3.64) and $B_0(\rho_c) = B(\rho_c) > 0$,

$$\begin{aligned} m &= \min\{k \ge 1; kC'(\rho_c) + 2B(\rho_c) \le 0\} \\ &= \min\{k \ge 1; B_k(\rho_c) \le 0\} \end{aligned}$$

From (3.47),

$$A_m(s)G_i^{(m+2)}(s) + B_m(s)G_i^{(m+1)}(s) = U_m(s).$$

(i) Letting $s = \rho_c$ in the above equation,

$$A_m(\rho_c) = 0 \text{ and } B_m(\rho_c) = 0,$$

gives

$$U_m(\rho_c) = 0.$$

(3.62) immediately follows from Lemma 3.8, by noting

$$D_{m,0}(\rho_c)(a_i - \rho_c^i) + \sum_{k=1}^{m\wedge i} D_{m,k}(s)U_{0i}^{(k)}(\rho_c) = 0.$$

(ii) If $mC'(\rho_c)+2B(\rho_c) < 0$, we can prove that $U_k(\rho_c) > 0$ for all $0 \le k \le m-1$ and $U_m(\rho_c) < 0$. and hence that inequalities (3.63) follows by noting

$$D_{m-1,0}(\rho_c)(a_i - \rho_c^i) + \sum_{k=1}^{(m-1)\wedge i} D_{m-1,k}(s)U_{0i}^{(k)}(\rho_c) > 0$$

and

$$D_{m,0}(\rho_c)(a_i - \rho_c^i) + \sum_{k=1}^{m\wedge i} D_{m,k}(s)U_{0i}^{(k)}(\rho_c) < 0.$$

■

Remark 3.2 *In obtaining (3.62), we have assumed that* $D_{m,0}(\rho_c) \neq 0$. *If* $D_{m,0}(\rho_c) = 0$, *using (3.58), we can set* $D_{m,i}(\rho_c) = 0$, *for* $i = 1, 2, 3, \cdots$ *etc. This means that* $A_m(s)$ *and* $B_m(s)$ *are divisible by* $s - \rho_c$. *This also means that*

$$A_m(s)G_i^{(m+2)}(s) + B_m(s)G_l^{(m+1)}(s) = U_m(s)$$

could be reduced by dividing $s-\rho_c$. *Similarly, we have assumed that* $D_{m-1,0}(\rho_c) > 0$ *and* $D_{m,0}(\rho_c) > 0$. *If they are negative, the sign in inequality (3.63) will need to be reversed.*

Assumption 3.1 *Suppose that* Q *is an IBC-q-matrix, with* $\rho_c < \rho_b \le 1$ *and* $C'(\rho_c) + 2B(\rho_c) > 0$, *under the setting in (3.46), we assume that* $A_m(s) > 0$ *for all* $s \in (\xi_c, \rho_c)$ *where* $m = \min\{k \ge 1; kC'(\rho_c) + 2B(\rho_c) < 0\}$.

Theorem 3.14 *Suppose that* Q *is an IBC-q-matrix, with* $\rho_c < \rho_b \le 1$ *and* $C'(\rho_c) + 2B(\rho_c) > 0$ *and that* $-\frac{2B(\rho_c)}{C'(\rho_c)}$ *is not an integer. Let* $m = \min\{k \ge 1; kC'(\rho_c) + 2B(\rho_c) < 0\}$.

(i) with the Assumption 3.1, i.e. $A_m(s) > 0$ *for all* (ξ_c, ρ_c). *The extinction probability is then given by*

$$a_i = \frac{\sum_{k=0}^{m\wedge i} \frac{i!}{(i-k)!} \int_{\xi_c}^{\rho_c} \frac{y^{i-k} D_{m,k}(y)}{A_m(y)} e^{H_m(y)} dy}{\int_{\xi_c}^{\rho_c} \frac{D_{m,0}(y)}{A_m(y)} e^{H_m(s)} dy} \tag{3.65}$$

where $D_{m,k}(y)$ *and* $H_m(y) = \int_0^y \frac{B_m(x)}{A_m(x)} dx$ *are defined in Lemma 3.8.*

Proof. From (3.45), we have

$$A_m(s) G_i^{(m+2)}(s) + B_m(s) G_i^{(m+1)}(s) = U_{mi}(s)$$

under Assumption 3.1, we solve the above equation and have

$$\left[G_i^{(m+1)}(y) e^{H_m(y)} \right]' = \frac{U_{mi}(y)}{A_m(y)} e^{H_m(y)}$$

$$G_i^{(m+1)}(s) e^{H_m(s)} - G_i^{(m+1)}(0) = \int_0^s \frac{U_{mi}(y)}{A_m(y)} e^{H_m(s)} dy \tag{3.66}$$

By (3.50), we have

$$B_n(\rho_c) = [B_0(\rho_c) + nA'(\rho_c)] \prod_{k=0}^{n-1} B_k(\rho_c).$$

With our settings in this case, we know that $B_n(\rho_c) > 0$ for all $0 \le n \le m-1$ and $B_m(\rho_c) < 0$. With $A_m(s) > 0$, for all $s \in (\xi_c, \rho_c)$ and by Lemma 3.7 noticing that ρ_c is a root for $A_m(x) = 0$. We have $e^{H_m(s)} = e^{\int_0^s \frac{B_m(x)}{A_m(x)} dx} \to 0$ as $s \to \rho_c$. So, we can get

$$-G_i^{(m+1)}(0) = \int_0^{\rho_c} \frac{U_{mi}(y)}{A_m(y)} e^{H_m(y)} dy. \tag{3.67}$$

Similarly, by (3.46), we have

$$B_m(\xi_c) = (B_0(\xi_c) + mA_0'(\xi_c)) \prod_{k=0}^{m-1} B_k(\xi_c) > 0$$

with $A_m(s) > 0$, for all $s \in (\xi_c, \rho_c)$ and, by Lemma 3.7, noticing that ξ_c is a root for $A_m(x) = 0$, we have $e^{H_m(s)} = e^{\int_0^s \frac{B_m(x)}{A_m(x)} dx} \to 0$ as $s \to \xi_c$. We get

$$-G_i^{(m+1)}(0) = \int_0^{\xi_c} \frac{U_{mi}(y)}{A_m(y)} e^{H_m(y)} dy. \tag{3.68}$$

From (3.67) and (3.68),

$$\int_{\xi_c}^{\rho_c} \frac{U_{mi}(y)}{A_m(y)} e^{H_m(s)} dy = 0,$$

(3.65) can be obtained with substituting $U_{mi}(y)$ which defined in (3.58).

■

Assumption 3.2 *Suppose that Q is an IBC-q-matrix, with $\rho_c < \rho_b \leq 1$ and $C'(\rho_c) + 2B(\rho_c) > 0$, under the setting in (3.46), we assume that there exists a $s \in (\xi_c, \rho_c)$ such that $A_m(s) \leq 0$, where $m = \min\{k \geq 1; kC'(\rho_c) + 2B(\rho_c) < 0\}$.*

Remark 3.3 *Under Assumption 3.2, we may worry about that $\int_{\xi_c}^{\rho_c} \frac{f(s)}{A_m(s)} ds$, may not be defined at the zero of $A_m(s)$ in (ξ_c, ρ_c). We will consider the problem with complex analysis. Since $A_m(s)$ is an analytic function of s, therefore, without loss of generality, we assume 0 is not a zero for $A(s)$. Otherwise, we can choose another non-zero point of $A(s)$ as the starting point of our discussion. Moreover, we note that there are only finite number of zeros in (ξ_c, ρ_c). Let ξ_{c_k} be the negative zeros, for each ξ_{c_k}, we can find a sufficient small radius r_k, such that $A_m(s)$ has no other zero on the disk $\{z; |z - \xi_{c_k}| \leq r_k\}$. As an example, let ξ_{c_0} as the smallest negative zero, for any $y \in (\xi_c, \xi_{c_0})$, define the complex integral $\int_{C_y} \frac{f(s)}{A_m(y)} dy$ where C_y is the closed curve starting and ending at y and along each circle $\{z; |z - \xi_{c_k}| = r_k\}$ as defined above together with alongside the real number axis. The integral is then a real value. By Theorem of residue, the integral equals to the sum of residues of $\frac{f(s)}{A_m(s)}$ at zeros of $A_m(s)$. If we consider the upper part of C_y, denote by $\tilde{C}_y$, because of symmetric property, $\int_{\tilde{C}_y} \frac{f(s)}{A_m(s)} ds = \frac{1}{2} \int_{C_y} \frac{f(s)}{A_m(s)} dy$. We denote this integral by $(\sim) \oint_0^y \frac{f(s)}{A_m(s)} ds$ for convenience. Similarly, let ρ_{c_0} be the largest positive zero of $A_m(s)$ on (ξ_c, ρ_c), then for any $y \in (\rho_{c_0}, \rho_c)$. $\int_{\tilde{C}_y} \frac{f(s)}{A_m(s)} ds$ is also well-defined and is a real value given that the integral is not passing any zeros of $A_m(s)$.*

Theorem 3.13 Continued *Suppose that Q is an IBC-:*

(ii) with the Assumption 3.2 in Remark 3.3, i.e. there exist $s \in (\xi_c, \rho_c)$ such that $A_m(s) < 0$, where $m = \min\{k \geq 1; kC'(\rho_c) + 2B(\rho_c) < 0\}$. Then

$$a_i = \frac{\sum_{k=0}^{m \wedge i} \frac{i!}{(i-k)!} (\sim) \oint_{\xi_c}^{\rho_c} \frac{y^{i-k} D_{m,k}(y)}{A_m(y)} e^{H_m(y)} dy}{(\sim) \oint_{\xi_c}^{\rho_c} \frac{D_{m,0}(y)}{A_m(y)} e^{H_m(s)} dy} \tag{3.69}$$

where $\tilde{H}_m(y) = (\sim) \oint_0^y \frac{B_m(x)}{A_m(x)} dx$.

Proof. (ii) Again From (3.45), we have

$$A_m(s) G_i^{(m+2)}(s) + B_m(s) G_i^{(m+1)}(s) = U_{mi}(s)$$

under Assumption 3.2, we solve the above equation and have

$$\left[G_i^{(m+1)}(y) e^{\tilde{H}_m(y)} \right]' = \frac{U_{mi}(y)}{A_m(y)} e^{\tilde{H}_m(y)}$$

$$G_i^{(m+1)}(s) e^{\tilde{H}_m(s)} - G_i^{(m+1)}(0) = \int_0^s \frac{U_{mi}(y)}{A_m(y)} e^{\tilde{H}_m(s)} dy$$

similar with (i), we obtain

$$-G_i^{(m+1)}(0) = (\sim) \oint_0^{\rho_c} \frac{U_{mi}(y)}{A_m(y)} e^{\tilde{H}_m(y)} dy$$

and

$$-G_i^{(m+1)}(0) = (\sim) \oint_0^{\xi_c} \frac{U_{mi}(y)}{A_m(y)} e^{\tilde{H}_m(y)} dy.$$

From (3.53) and (3.54), we have

$$(\sim) \oint_{\xi_c}^{\rho_c} \frac{U_{mi}(y)}{A_m(y)} e^{\tilde{H}_m(y)} dy$$

(3.69) can be obtained with substituting $U_{mi}(y)$ which defined in (3.58).

■

3.6 Asymptotic Behavior of Extinction Probability

Though the closed forms for the extinction probabilities of the IBCP are obtained in the theorems above, some of the these closed forms are very complicated. This part will show that, for large i,the asymptotic behavior of these complicated expressions for the extinction probabilities actually takes a very simple form.

As a review and preparation, we first give a lemma about the properties of $A(s)$ given in (3.25).

Recall that $-1 < \zeta_c < 0$ is the negative zero of the generating function $C(s)$ and $0 < \rho_c \leq 1$ is the smallest positive zero of $C(s)$. Also, $\rho_c = 1$ if and only if $0 < C'(1) < +\infty$.

Lemma 3.9 *(i) Suppose* $0 < C'(1) < +\infty$ *and thus* $\rho_c < 1$. *Then*

$$A(y) \sim k(\rho_c - y)^{\alpha} \text{ as } y \to \rho_c^-$$

where $0 < k < +\infty$ *is a constant,. (i.e. independent of* y*) and* $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$,

(ii) $A(y) \sim k(y - \xi_c)^{\beta}$ *as* $y \to \xi_c^+$, *where* $0 < k < +\infty$ *is a constant and* $\beta = \frac{2B(\xi_c)}{C'(\xi_c)}$.

(iii) Suppose $C'(1) = 0$ *and* $C''(1) < 4B'(1) < +\infty$. *Then*

$$A(y) \sim k(1 - y)^{\gamma} \text{ as } y \to 1^-.$$

where $0 < k < +\infty$ *is a constant (i.e. independent of* y*) and* $\gamma = \frac{4B'(1)}{C''(1)}$.

Proof. Under the condition $0 < C'(1) \leq +\infty$, $\rho_c < 1$ is a single zero of $C(s)$ and if we let

$$f(x) = \frac{2B(x)(\rho_c - x)}{C(x)}. \tag{3.70}$$

$f(x)$ could be expanded as a power series of x on the interval $[0, \rho_c)$. Here we view $f(x)$ as a real valued function of x. Suppose the expansion takes the form of

$$f(x) = \sum_{k=0}^{\infty} f_k x^k \tag{3.71}$$

where $f_k = \frac{f^{(k)}(0)}{k!}$ but we shall not use this fact for simplicity as it is not our focus.

By (3.70) and (3.71), for $0 < y < \rho_c$, we have

$$\begin{aligned}
&\int_0^y \frac{2B(x)}{C(x)} dx = \int_0^y \frac{f(x)}{\rho_c - x} dx = \sum_{k=0}^{\infty} f_k \int_0^y \frac{x^k}{\rho_c - x} dx \\
= \ & \sum_{k=0}^{\infty} f_k \int_0^y \frac{[\rho_c - (\rho_c - x)]^k}{\rho_c - x} dx \\
= \ & \left(\sum_{k=0}^{\infty} f_k \rho_c^k \right) \int_0^y \frac{dx}{\rho_c - x} + \sum_{k=1}^{\infty} f_k \sum_{m=1}^{k} (-1)^m \binom{k}{m} \rho_c^{k-m} \int_0^y (\rho_c - x)^{m-1} dx \\
= \ & J_1 + J_2.
\end{aligned} \tag{3.72}$$

We use J_1 and J_2 for convenient and they will be further analyzed below.

From (3.71), J_1 in (3.72) can be further simplified as

$$J_1 = \left(\sum_{k=0}^{\infty} f_k \rho_c^k \right) \int_0^y \frac{dx}{\rho_c - x} = f(\rho_c) \int_0^y \frac{dx}{\rho_c - x}.$$

By L'Hopital's rule, we know that

$$f(\rho_c) = \lim_{x \to \rho_c^+} \frac{2B(x)(\rho_c - x)}{C(x)} = -\frac{2B(\rho_c)}{C'(\rho_c)}$$

which is finite. Similarly, for J_2 in (3.72), it can be rewritten as

$$J_2 = \sum_{k=1}^{\infty} f_k \rho_c^k \sum_{m=1}^{k} (-1)^k \frac{\binom{k}{m}}{m} - \sum_{k=1}^{\infty} f_k \rho_c^k \sum_{m=1}^{k} \frac{\binom{k}{m}}{m} \left(\frac{y}{\rho_c} - 1 \right)^m . \tag{3.73}$$

We can see that the first term in the right hand side of (3.73) is just a constant and the second term of the right hand side is just a rational function of y and clearly bounded for $y \in [0, \rho_c]$. By mean-valued theorem, J_2 can be written as a constant k, such that

$$A(y) = \exp\left(\int_0^y \frac{2B(x)}{C(x)} dx \right) \sim k(\rho_c - y)^{\alpha} \text{ as } y \to \rho_c^- \tag{3.74}$$

where $\alpha = -f(\rho_c) = \frac{2B(\rho_c)}{C'(\rho_c)}$.

For (ii), it is similar to (i), $-1 < \rho_c < 0$ is the single negative zero of $C(s)$ and we let

$$f(x) = \frac{2B(x)(x - \xi_c)}{C(x)}$$

then

$$f(x) = \sum_{k=0}^{\infty} f_k x^k.$$

For $\xi_c < y < 0$, we have

$$\int_y^0 \frac{2B(x)}{C(x)} = \int_y^0 \frac{f(x)}{x - \xi_c} dx = \sum_{k=0}^{\infty} f_k \int_y^0 \frac{x^k}{x - \xi_c} dx.$$

And then the conclusion of (ii) can be drawn similar to (i) with $\beta = -f(\xi_c) = \frac{2B(\xi_c)}{C'(\xi_c)}$.

For (iii), since $C'(1) = 0$, there is no root on $[0, 1)$ and 1 is the zero of $C(s)$ with multiplicity 2. And we can see that $0 < B'(x) < \infty$, and we let

$$h(x) = \frac{2B(x)(1 - x)}{C(x)}.$$

Noting that

$$\lim_{x \to 1^+} h(x) = -\frac{4B'(1)}{C''(x)}.$$

(iii) is also done similar to (i) and (ii).

∎

Lemma 3.10 *For any complex number a, we have*

$$\lim_{z \to \infty} \frac{\Gamma(z + a)}{\Gamma(z)} z^{-a} = 1, \tag{3.75}$$

where real part of a is positive and $\Gamma(\cdot)$ is the gamma function.

Lemma 3.10 is well-known result so the proof is skipped. We are now ready to discuss the core part of this section, the asymptotic behavior of the extinction probability.

Theorem 3.15 *If $C'(1) = 0$ and $0 < B'(1) < +\infty$, then*

(i) the extinction probability $\{a_i\}$, starting from state $i \geq 1$, is less than 1 (for all $i \geq 1$) if and only if $C''(1) < 4B'(1)$.

(ii) In addition, if $C''(1) < 4B'(1)$, then

$$a_i \sim k_1 i^{-\alpha} + k_2 i^{-\beta} \xi_c^i \text{ as } i \to +\infty \tag{3.76}$$

where k_1 and k_2 are constants and $\alpha = \frac{4B'(1)}{C''(1)} - 1 > 0$ and $\beta = \frac{2B(\xi_c)}{C'(\xi_c)} > 0$.

(iii) Furthermore,

$$a_i \sim k i^{-\alpha}, \quad \text{as } i \to \infty \tag{3.77}$$

where k is a constant and $\alpha = \frac{4B'(1)}{C''(1)} - 1 > 0$.

Proof. (i) is proven in this chapter earlier.

(ii) Suppose $C'(1) = 0$ and $C''(1) < 4B'(1) < +\infty$, then by Theorem 3.4 and Corollary 3.2 this chapter before.

$$a_i = \frac{1}{J} \int_{\xi_c}^{1} \frac{y^i A(y)}{C(y)} dy \quad (i \geq 1). \tag{3.78}$$

where $J = \int_{\xi_c}^{1} \frac{A(y)}{C(y)} dy$ is a finite constant which is independent of i. We consider the integral

$$I_1^{(i)} = k_3 \cdot \int_0^1 \frac{y^i (1-y)^\gamma}{(1-y)^2} dy = k_3 \cdot \int_0^1 y^i (1-y)^{\gamma-2} dy$$

where $0 < k_3 < +\infty$ is a constant and $\gamma = \frac{4B'(1)}{C''(1)} > 1$ since we have $0 < C''(1) < 4B'(1) < +\infty$. It can be noted that

$$\int_0^1 y^i (1-y)^{\gamma-2} dy = \frac{\Gamma(i+1)\Gamma(\gamma-1)}{\Gamma(i+\gamma)}$$

with Lemma 3.10, we get

$$I_1^{(i)} \sim k_4 i^{1-\gamma},\ i \to \infty, \text{ for a constant } k_4. \tag{3.79}$$

Thus, we focus of $I_2^{(i)}$, similar to the treatment for $I_1^{(i)}$, but we need to note that ξ_c is a single zero of $C(s)$. Also with Lemma 3.9, we have

$$I_2^{(i)} = k_5 \int_{\xi_c}^{0} y^i (y - \xi_c)^{\beta - 1} dy$$

where $\beta = \frac{2B(\xi_c)}{C'(\xi_c)} > 0$, we have

$$I_2^{(i)} = k_6 \xi_c^n \frac{\Gamma(n+1)\Gamma(\beta)}{\Gamma(n+1+\beta)}.$$

By Lemma 3.10 again, we get

$$I_2^{(i)} \sim k_6 i^{-\beta} \xi_c^i \quad (i \to \infty). \tag{3.80}$$

Combining $I_1^{(i)}$ and $I_2^{(i)}$ in (3.79) and (3.80), together with the constant $1/J$, we get (3.76) and noting that $|\zeta_c| < 1$, (3.77) is given.

■

Now, we can consider the irregular case, i.e. $0 < C'(1) \leq +\infty$. We separate the irregular case into three subcases again similar to what we did in last section, $\rho_a = \rho_c < 1$, $\rho_b < \rho_c < 1$ and $\rho_c < \rho_b \leq 1$, while ρ_c and ρ_c is the smallest positive zero of $B(s)$ and $C(s)$ respectively. For $\rho_b = \rho_c < 1$ is trivial and we get $a_i = \rho_c^i$ $(i \geq 1)$ before in this chapter. We now go to the second case, $\rho_b < \rho_c < 1$.

Theorem 3.16 *If $\rho_b < \rho_c < 1$, then the extinction probability of the IBCP, starting from $i \geq 1$, denoted by $\{a_i\}$, possesses the following asymptotic behavior*

$$a_i \sim k_1 i^{-\alpha} \rho_c^i + k_2 i^{-\beta} \xi_c^i \quad (\text{ as } i \to \infty) \tag{3.81}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} > 0$, $\beta = \frac{2B(\xi_c)}{C'(\xi_c)} > 0$ and k_1 and k_2 are constants which are independent of i. Furthermore, we have

$$a_i \sim k i^{-\alpha} \rho_c^i \quad (\text{ as } i \to \infty). \tag{3.82}$$

where k is a constant and $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} > 0$.

Proof. By Theorem 3.9 in this chapter, we know that the extinction probability $\{a_i\}$, starting from $i \geq 1$, is given by

$$a_i = \frac{\int_{\xi_c}^{\rho_c} \frac{y^i A(y)}{C(y)} dy}{\int_{\xi_c}^{\rho_c} \frac{A(y)}{C(y)} dy}. \tag{3.83}$$

We see that the denominator of the right hand side of (3.83) is a constant and is independent of i. Now, we look at the numerator. We separate it into two integrals,

$$I_1^{(i)} = \int_0^{\rho_c} \frac{y^i A(y)}{C(y)} dy$$

and

$$I_2^{(i)} = \int_{\xi_c}^{0} \frac{y^i A(y)}{C(y)} dy.$$

$I_2^{(i)}$ is already analyzed in last Theorem 3.15, so we now focus on $I_1^{(i)}$, note that $\rho_c < 1$ is the single zero of $C(s)$, for $s \in (0,1)$, by Lemma 3.9 (i), we know there exists a constant k such that

$$I_1^{(i)} = \int_0^{\rho_c} \frac{y^i A(y)}{C(y)} dy = k \int_0^{\rho_c} y^i (\rho_c - y)^{\alpha - 1} dy$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} > 0$ as both $B(\rho_c)$ and $C'(\rho_c)$ are negative.

Doing some simple transformation, we have

$$\begin{aligned} \int_0^{\rho_c} y^i (\rho_c - y)^{\alpha-1} dy &= \rho_c^{i+\alpha} \int_0^1 x^i (1-x)^{\alpha-1} dx \\ &= \rho_c^{i+\alpha} \frac{\Gamma(i+1)\Gamma(\alpha)}{\Gamma(i+\alpha+1)}. \end{aligned}$$

By Lemma 3.10, we have

$$I_1^{(i)} \sim k_{11} i^{-\alpha} \rho_c^i, \quad as \quad i \to \infty$$

and

$$I_2^{(i)} \sim k_{21} i^{-\beta} \xi_c^i, \quad as \quad i \to \infty$$

with some constants k_{11} and k_{21}. (3.81) is proved. Noting that $|\zeta_c| < \rho_c$, (3.82) follows and this completes the proof. ■

Now we can consider the third irregular case, i.e. $\rho_c < \rho_b \leq 1$. Note that by Lemma 3.1, we know that $C'(\rho_c) < 0$ and $B(\rho_c) > 0$. We can further divide our consideration into three cases, as $C'(\rho_c) + 2B(\rho_c) = 0$, $C'(\rho_c) + 2B(\rho_c) < 0$ and $C'(\rho_c) + 2B(\rho_c) > 0$. We first deal with the case of $C'(\rho_c) + 2B(\rho_c) = 0$.

Theorem 3.17 *If $\rho_c < \rho_b \leq 1$ and $C'(\rho_c) + 2B(\rho_c) = 0$. Then the extinction probability $\{a_i\}$, starting from $i \geq 1$, is given by*

$$a_i = \rho_c^i + \sigma i \rho_c^{i-1} \tag{3.84}$$

where $\sigma = -\frac{B(\rho_c)}{B'(\rho_c)}$. Furthermore,

$$a_i \sim k i^{-\alpha} \rho_c^i \quad (i \to +\infty). \tag{3.85}$$

where $k = \frac{\sigma}{\rho_c}$ is a constant and $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} = -1$.

Proof. (3.84) is proved in Theorem 3.11 in this chapter before and then from (3.84),

$$a_i = \rho_c^i + \sigma i \rho_c^{i-1}$$

when $i \to \infty$, we have $a_i \sim k i^{-\alpha} \rho_c^i$ with the condition of $C'(\rho_c) + B(\rho_c) < 0$ which implies $\alpha = -1$. The proof is completed.

■

Now, we move to the second irregular case with $C'(\rho_c) + 2B(\rho_c) < 0$.

Theorem 3.18 *Suppose $\rho_c < \rho_b \leq 1$ and $C'(\rho_c) + 2B(\rho_c) < 0$. Then the extinction probability $\{a_i\}$ of the IBCP, starting from $i \geq 1$, possesses the following* *asymptotic property,*

$$a_i \sim k_1 i^{-\alpha}\rho_c^i + k_2 i^{-\beta}\xi_c^i \quad (i \to +\infty) \tag{3.86}$$

where k_1 *and* k_2 *are constants and* $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < 0$ *and* $\beta = \frac{2B(\xi_c)}{C'(\xi_c)} > 0$*. Furthermore, we have*

$$a_i \sim k\rho_c^i \cdot i^{-\alpha} \quad (i \to +\infty) \tag{3.87}$$

where $-1 < \alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < 0$ *and* k *is a constant.*

Proof. If $\rho_c < \rho_b \leq 1$ and $C'(\rho_c) + 2B(\rho_c) < 0$, then the extinction probability $\{a_i\}$ is given by Theorem 3.12

$$a_i = \frac{\int_{\xi_c}^{\rho_c} \frac{y^i B'(y) - iy^{i-1}B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy}{\int_{\xi_c}^{\rho_c} \frac{B'(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy} \tag{3.88}$$

$$A_0(s) = \frac{C(s)}{2}, \text{ and } B_0(s) = B(s),$$

$$A_1(s) = A_0(s)B_0(s) \text{ and } B_1(s) = B_0(s)[B_0(s) + A_0'(s)] - A_0(s)B_0'(s),$$

where

$$A_1(s) = \frac{B(s)C(s)}{2}, \quad B(s) = \frac{B(s)(2B(s) + C'(s)) - C(s)B'(s)}{2}.$$

Again, we see that the denominator is a constant. i.e.

$$a_i = k \cdot \int_{\zeta_c}^{\rho_c} \frac{y^i B'(y) - iy^{i-1}B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy \tag{3.89}$$

In order to look at the properties of $\{a_1\}$ in (3.89), we try to separate the right hand side of (3.89). Denote

$$\begin{aligned} a_i^+ &= k \cdot \int_0^{\rho_c} \frac{y^i B_0'(y) - iy^{i-1}B_0(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy \\ a_i^- &= k \cdot \int_{\zeta_c}^{0} \frac{y^i B_0'(y) - iy^{i-1}B_0(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy. \end{aligned} \tag{3.90}$$

Furthermore, we need to look at $e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx}$ and therefore $\int_0^y \frac{B_1(x)}{A_1(x)}dx$. By our definition above,

$$\begin{aligned}\int_0^y \frac{B_1(x)}{A_1(x)}dx &= \int_0^y \frac{B_0(x)}{A_0(x)}dx + \int_0^y \frac{A_0'(x)}{A_0(x)}dx - \int_0^y \frac{B_0'(x)}{B_0(x)}dx \\ &= \int_0^y \frac{B_0(x)}{A_0(x)}dx + \ln\frac{A_0(y)}{B_0(y)} + \ln\frac{B_0(0)}{A_0(0)}\end{aligned}$$

where $B_0(0) = b_0 > 0$ and $A_0(0) = \frac{c_0}{2} > 0$.

As a result, we have

$$\exp\left\{\int_0^y \frac{B_1(x)}{A_1(x)}dx\right\} = k_1\frac{A_0(y)}{B_0(y)}\exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}dx\right\} \tag{3.91}$$

where k_1 is a constant which is independent of y. Put (3.91) into (3.90), we see that

$$\begin{aligned}a_i^+ &= k\cdot\int_0^{\rho_c} \frac{y^iB_0'(y) - iy^{i-1}B_0(y)}{A_1(y)}k_1\frac{A_0(y)}{B_0(y)}\exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}dx\right\}dy \\ &= k\cdot\int_0^{\rho_c} \frac{y^iB_0'(y) - iy^{i-1}B_0(y)}{A_0(y)B_0(y)}k_1\frac{A_0(y)}{B_0(y)}\exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}dx\right\}dy \\ &= k_2\int_0^{\rho_c} \frac{iy^{i-1}B_0(y) - y^iB_0'(y)}{(B_0(y))^2}\exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}dx\right\}dy.\end{aligned}$$

$\frac{1}{B_0(s)}$ is bounded on $[0, \rho_c]$ since $\rho_c < \rho_b \le 1$, $B_0(s)$ and $B_0'(s)$ are bounded on $[0, \rho_c]$. By mean value theorem, there exist k_1 and k_2 which are both independent of y and n such that

$$a_i^+ = k_1 i\int_0^{\rho_c} y^{i-1}\exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}dx\right\}dy + k_2\int_0^{\rho_c} y^i\exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}dx\right\}dy \tag{3.92}$$

We note that $\exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}dx\right\}$ actually is defined as $A(y)$ in (3.25)and by using Lemma 3.9, there exists another set of constants k_1 and k_2 that gives

$$a_i^+ = k_1 i\int_0^{\rho_c} y^{i-1}(\rho_c - y)^\alpha dy + k_2\int_0^{\rho_c} y^i(\rho_c - y)^\alpha dy$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < 0$. Since $C'(\rho_c) + 2B(\rho_c) < 0$, we can deduce

$$-1 < \alpha < 0. \tag{3.93}$$

With some easy transformation,

$$\begin{aligned}\int_0^{\rho_c} y^{i-1}(\rho_c - y)^\alpha dy &= \rho_c^i \rho_c^\alpha \int_0^1 x^{i-1}(1-x)^\alpha dx \\ &= \rho_c^{i+\alpha} \int_0^1 x^{i-1}(1-x)^{1+\alpha-1} dx \\ &= \rho_c^{i+\alpha} \frac{\Gamma(i)\Gamma(1+\alpha)}{\Gamma(i+1+\alpha)} \end{aligned} \tag{3.94}$$

Similarly,

$$\int_0^{\rho_c} y^i(\rho_c - y)^\alpha dy = \rho_c^{i+1+\alpha} \frac{\Gamma(i+1)\Gamma(1+\alpha)}{\Gamma(i+2+\alpha)}. \tag{3.95}$$

When $i \to \infty$,

$$\begin{aligned} a_i^+ &= k_1 i \rho_c^{i+\alpha} \frac{\Gamma(i)\Gamma(1+\alpha)}{\Gamma(i+1+\alpha)} + k_2 \rho_c^{i+1+\alpha} \frac{\Gamma(i+1)\Gamma(1+\alpha)}{\Gamma(i+2+\alpha)} \\ &= k_1 i \rho_c^{i+1} i^{-1-\alpha} \Gamma(1+\alpha) + k_2 \rho_c^{i+1+\alpha} i^{-1-\alpha} \Gamma(1+\alpha). \end{aligned}$$

So, for a_i^+ and similar for a_i^-, there exists a k_1 and k_2 such that

$$a_i^+ \sim k_1 i^{-\alpha} \rho_c^i \quad (i \to \infty)$$

$$a_i^- \sim k_2 i^{-\beta} \xi_c^i \quad (i \to \infty)$$

with $\beta = \frac{2B(\zeta_c)}{C'(\zeta_c)} > 0$. Then (3.86) is proved. (3.87) comes from (3.86) directly.

■

Finally we consider the third irregular case, i.e. $C'(\rho_c) + 2B(\rho_c) > 0$. But we know that $C'(\rho_c) < 0$ and $B(\rho_c) > 0$, there exists a smallest positive integer $m \geq 2$, such that $(m-1)C'(\rho_c) + 2B(\rho_c) > 0$ and $mC'(\rho_c) + 2B(\rho_c) \leq 0$. We let $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$, such that $0 < m - 1 < \alpha \leq m$, we now consider the first subcase that $mC'(\rho_c) + 2B(\rho_c) = 0$.

Theorem 3.19 *Suppose $\rho_c < \rho_b \leq 1$ and $C'(\rho_c) + 2B(\rho_c) > 0$. If there exists a positive integer m such that $mC'(\rho_c) + 2B(\rho_c) = 0$, then there exists $m+1$ constants $\{k_0, k_1, \cdots, k_m\}$ such that the extinction probability $\{a_i\}$, starting from*

$i \geq 1$, can be written as

$$a_i = \sum_{i=0}^{\infty} k_l i^l \rho_c^{i-l}. \tag{3.96}$$

In particular, there exists a constant k such that

$$a_i \sim k i^{-\alpha} \rho_c^i \quad (i \to \infty) \tag{3.97}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} = -m$.

Proof. (3.96) follows from (3.62) in Theorem 3.13 in the chapter before. (3.97) can be easily seen by noting that $k_m i^m \rho_c^{i-m}$ is a dominated term in (3.96). Here we have used $m = -\alpha = -\frac{2B(\rho_c)}{C'(\rho_c)}$. ■

It can be seen that Theorem 3.17 is a special case of Theorem 3.19 when $m = 1$.

Next, we consider the other subcase of $mC'(\rho_c) + 2B(\rho_c) < 0$ for some $m \geq 2$. Recall from (3.46) to (3.47), we have define $A_m(s)$ and $B_m(s)$ recursively from $A_n(s)$ and $B_n(s)$ for $(m \geq n \geq 1)$.

$$A_n(s) = A_{n-1}(s)B_{n-1}(s) \tag{3.98}$$

$$B_n(s) = B_{n-1}(s)[B_{n-1}(s) + A'_{n-1}(s) - A_{n-1}(s)B'_{n-1}(s)] \tag{3.99}$$

Explained in Remark 3.3, in order to avoid complex analysis, without loss of generality, we assume that $A_m(s) > 0$ for all $s \in (\xi_c, \rho_c)$.

Theorem 3.20 *Suppose $\rho_c < \rho_b \leq 1$ and $C'(\rho_c) + 2B(\rho_c) > 0$ and that $-\frac{2B(\rho_c)}{C'(\rho_c)}$ is not an integer. Let $m = \min\{k \geq 1, kC'(\rho_c) + 2B(\rho_c) < 0\}$ where $-(m+1) < \alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < -m$ for the positive integer such that $m = \min\{k \geq 1, kC'(\rho_c) + 2B(\rho_c) < 0\}$. Further assume that $A_m(s) > 0$ for all $s \in (\xi_c, \rho_c)$ where $A_m(s)$ is defined sequently as in (3.98) and (3.99). Then the extinction probability $\{a_i\}$ of the IBCP ($i \geq 1$), starting from $i \geq 1$, possesses the asymptotic property that there* *exist* $m+1$ *constants* $\{k_0, k_1, \cdots, k_m\}$ *such that*

$$a_i \sim \sum_{l=0}^{m} k_l \frac{i!}{(i-l)!} \rho_c^{i-l} i^{-\alpha} \quad (i \to +\infty) \tag{3.100}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$. *Furthermore, we have*

$$a_i \sim k \cdot \rho_c^i i^{-\alpha} \quad (i \to +\infty) \tag{3.101}$$

where $-(m+1) < \alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < -m$.

Proof. By Theorem 3.14, we know that for sufficient large i, the extinction probability $\{a_i\}$ is given by

$$a_i = k \sum_{l=0}^{m} \frac{i!}{(i-l)!} \int_{\xi_c}^{\rho_c} \frac{y^{i-l} D_{m,l}(y)}{A_m(y)} \exp\left\{ \int_0^y \frac{B_m(x)}{A_m(x)} dx \right\} dy \tag{3.102}$$

for some constant k that is independent of i where $A_m(s)$ and $B_m(s)$ are defined in (3.98) and (3.99). Recalling from Lemma 3.8, the function $D_{m,l}(s)$ etc are given recursively as

$$D_{1,0}(s) = -B'(s) \quad D_{1,1}(s) = B(s) \tag{3.103}$$

$$D_{i,k}(s) = D_{i-1,k-1}(s)B_{i-1}(s) - D_{i-1,k}(s)B'_{i-1}(s) + D'_{i-1,k}(s)B_{i-1}(s), \ (k \le i-1) \tag{3.104}$$

and

$$D_{i,i}(s) = \prod_{m=0}^{i-1} B_m(s). \tag{3.105}$$

By (3.103) - (3.105), it is easily seen that all $D_{m,l}(s)$ are analytic functions of s, as power series of s, and thus bounded in the interval $[\xi_c, \rho_c]$. It follows that the $\{a_i\}$ in (3.102) could be written as

$$a_i = \sum_{l=0}^{m} k_l \frac{i!}{(i-l)!} \int_{\xi_c}^{\rho_c} \frac{y^{n-1}}{A_m(y)} \exp\left\{ \int_0^y \frac{B_m(x)}{A_m(x)} dx \right\} dy \tag{3.106}$$

where $\{k_0, k_1, \cdots, k_m\}$ are $(m+1)$ constants.

Similar to what we have done in Theorem 3.18, we define

$$\begin{aligned} a_i^+ &= \sum_{l=0}^{m} k_l \frac{i!}{(i-l)!} \int_0^{\rho_c} \frac{y^{n-1}}{A_m(y)} \exp\left\{ \int_0^y \frac{B_m(x)}{A_m(x)} dx \right\} dy \\ a_i^- &= \sum_{l=0}^{m} k_l \frac{i!}{(i-l)!} \int_{\xi_c}^{0} \frac{y^{n-1}}{A_m(y)} \exp\left\{ \int_0^y \frac{B_m(x)}{A_m(x)} dx \right\} dy \end{aligned}$$

where $a_i = a_i^+ + a_i^-$, then by (3.98) and (3.99), we have

$$\frac{B_m(s)}{A_m(s)} = \frac{B_{m-1}(s)}{A_{m-1}(s)} + \frac{A'_{m-1}(s)}{A_{m-1}(s)} - \frac{B'_{m-1}(s)}{B_{m-1}(s)} \tag{3.107}$$

and thus

$$\exp\left\{ \int_0^y \frac{B_m(x)}{A_m(x)} dx \right\} = \exp\left\{ \int_0^y \frac{B_{m-1}(x)}{A_{m-1}(x)} dx \right\} \frac{A_{m-1}(y)}{B_{m-1}(y)} \cdot \frac{B_{m-1}(0)}{A_{m-1}(0)}. \tag{3.108}$$

By repeating (3.107) and (3.108) and noting that $\frac{B_{m-1}(0)}{A_{m-1}(0)}$ is just a constant, we get that

$$\exp\left\{ \int_0^y \frac{B_m(x)}{A_m(x)} dx \right\} = k \cdot \exp\left\{ \int_0^y \frac{B_0(x)}{A_0(x)} dx \right\} \cdot \frac{\prod_{l=0}^{m-1} A_l(y)}{\prod_{l=0}^{m-1} B_l(y)}. \tag{3.109}$$

where k is a constant.

By using (3.98) we could easily see that for any $n \geq 1$, $A_n(s) = A_0(s) \prod_{k=0}^{n-1} B_k(s)$. Using this relation in (3.109) and then substituting the resulting (3.109) into (3.106), we obtain that there exist $(m+1)$ constants, denoted by $\{k_0, k_1, \cdots, k_m\}$ such that

$$a_i^+ = \sum_{l=0}^{m} k_l \frac{i!}{(i-l)!} \int_0^{\rho_c} y^{i-1} \frac{A_0(y) \sum_{k=0}^{m-1} A_k(y)}{(A_m(y))^2} \exp\left\{ \int_0^y \frac{2B(x)}{C(x)} dx \right\} dy. \tag{3.110}$$

Now since m is the minimal k such that $kC'(\rho_c) + 2B(\rho_c) < 0$, and hence ρ_c is not a zero of the function $A_0(y) \sum_{k=0}^{m-1} A_k(y)/(A_m(y))^2$. Thus by applying mean-valued theorem together with (i) and (ii) of Lemma 3.9 we see that $\{a_i\}$ in (3.110) could be written as

$$a_i^+ = \sum_{l=0}^{m} k_1 \cdot \frac{i!}{(i-l)!} \int_0^{\rho_c} y^{i-l} (\rho_c - y)^{\alpha} dy \tag{3.111}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < 0$.

Similar, we have

$$a_i^- = \sum_{l=0}^{m} k_2 \cdot \frac{i!}{(i-l)!} \int_0^{\rho_c} y^{i-l}(\rho_c - y)^\beta dy \tag{3.112}$$

where $\beta = \frac{2B(\rho_c)}{C'(\rho_c)} < 0$.

Using the same transformation we have used before together with using (3.94) and the fact that $|\xi_c| < \rho_c < 1$, we could similarly prove (3.100). Then (3.101) follows directly from (3.100). ■

At this point, we try to use two theorems to conclude the asymptotic behavior of the extinction probability. One is for the regular case and one is for irregular case.

Theorem 3.21 *Suppose* $C'(1) = 0$, $C''(1) < +\infty$, $B'(1) < +\infty$ *and* $J_0 = \int_0^1 \frac{A(y)}{C(y)} dy < +\infty$, *as* $i \to \infty$, *we have*

$$a_i \sim k_l i^{-\alpha} \quad (i \to \infty) \tag{3.113}$$

where $\alpha = \frac{4B'(1)}{C''(1)} - 1 > 0$ *and* k *is a constant which is independent of* i.

Proof. See Theorem 3.15. ■

Theorem 3.22 *Suppose* $0 < C'(1) \leq +\infty$, *i.e.* $0 < \rho_c < 1$. *Further assume* $\rho_b \neq \rho_c$. *Then when* $i \to \infty$,

$$a_i \sim k i^{-\alpha} \rho_c^i \quad (i \to \infty) \tag{3.114}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$ *and* k *is a constant which is independent of* i.

Proof. It can be easily note that Theorem 3.16 to Theorem 3.20 are special cases for Theorem 3.22 with different value of $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$. i.e. $\alpha > 0$, in Theorem 3.16,

$\alpha = -1$ in Theorem 3.17, $-1 < \alpha < 0$ in Theorem 3.18, $\alpha = -m$ for some $m \geq 2$ in Theorem 3.19, $-(m+1) < \alpha < -m$ for some $m \geq 2$ in Theorem 3.19. ■

Remark 3.4 *At this point, we have considered the interacting branching collision processes in detail. We have discussions about the model settings, uniqueness , extinction probability and its asymptotic behavior of the interacting branching collision processes. One may notice that this thesis concentrates mainly on the theoretical study of the models. Till now, we have not made attempts to perform simulations. Actually, we understand the importance of real life applications and simulations of theoretical results. These important parts will be considered in the future.*

From the interacting branching collision processes, we try some new methods in discussing the model properties especially for the part about extinction probability and its asymptotic behavior. In next chapter, we will make use of similar new techniques to discuss another important generalized Markov branching process, i.e. Markov Branching Process with Immigration - Migration and Resurrection. We may also treat this model as an good example in applying related techniques suggested in this chapter.

Chapter 4

Markov Branching Processes with Immigration - Migration and Resurrection

4.1 Introduction

In this chapter, we discuss a generalized Markov branching process with immigration-migration and resurrection.

In this model, both state-independent and state-dependent immigration are considered.

For state-independent case, Adke [1969] and Aksland [1975, 1977] considered state-independent immigration with a birth-death process. For state-dependent case, Yamazato [1975] considered a Markov branching process with immigration only when the process is in state 0. This model is known as Markov branching process with resurrection. See also Foster [1971], Pakes [1971, 1993], Pakes and Tavaré [1981] and Chen and Renshaw [1990, 1993, 1995, 2000]. Li and Chen [2006] considered a branching process with state-independent immigration and resurrection which covered Yamazato's model.

In this chapter, the state-independent immigration-migration and resurrection are considered together which is a generalized model of Li and Chen [2006]. Sections 4.1, 4.2 and 4.3 as far as Theorem 4.5 closely follow Li and Liu [2011].

Definition 4.1 *A q-matrix $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ is called a branching q-matrix with immigration-migration-resurrection, BPIMR q-matrix, if*

$$q_{ij} = \begin{cases} h_j, & \text{if} \quad i = 0, j \geq 0 \\ ib_{j-i+1} + a_{j-i+1}, & \text{if} \quad i \geq 1, j \geq i-1 \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

where

$$\begin{cases} h_j \geq 0\,(j \neq 0), \quad 0 \leq -h_0 = \sum_{j=1}^{\infty} h_j < \infty \\ a_j \geq 0\,(j \neq 1), \quad 0 \leq -a_1 = \sum_{j \neq 1} a_j < \infty \\ b_j \geq 0\,(j \neq 1), \quad 0 \leq -b_1 = \sum_{j \neq 1} b_j < \infty. \end{cases} \tag{4.2}$$

h_j corresponds to the resurrection component, a_j corresponds to the immigration-migration component, and b_j corresponds to the branching component. The generating functions will be defined below.

Definition 4.2 *A Markov branching process with immigration-migration-resurrection is a continuous-time Markov chain on the state space $\mathbf{Z}_+$ whose transition function $P(t) = (p_{ij}; i, j \in \mathbf{Z}_+)$ satisfies*

$$P'(t) = P(t)Q \tag{4.3}$$

where Q is given in (4.1) - (4.2).

The structure of this chapter is as follows. Some preliminary results are firstly obtained in Section 1. Uniqueness and regularity criteria are then obtained in section 2. Section 3 discusses the details related to extinction probability if no resurrection is considered. Asymptotic behavior of extinction probability is discussed in Section 4.

In order to investigate properties of BPIMRs, it is necessary to define the generating functions of the three known sequences $\{h_k; k \geq 0\}$, $\{a_k; k \geq 0\}$, $\{b_k; k \geq 0\}$ as

$$H(s) = \sum_{k=0}^{\infty} h_k s^k, \quad A(s) = \sum_{k=0}^{\infty} a_k s^k \quad \text{and} \quad B(s) = \sum_{k=0}^{\infty} b_k s^k.$$

These three functions play important role in our later analysis. It is clear that $H(s)$, $A(s)$ and $B(s)$ are well defined at least on $[-1, 1]$.

Lemma 4.1 *(i) If $h_0 < 0$, then $H(s) < 0$ for all $s \in [1, -1)$ and $\lim_{s\uparrow 1} H(s) = H(1) = 0$. If $h_0 = 0$, then $H(s) = 0$.*

(ii) $A(s)$ is convex on $[0, 1]$ and $A(s) = 0$ has smallest nonnegative root ρ_a, such that $A(s) > 0$ for all $s \in (0, \rho_a)$ and $A(s) < 0$ for all $s \in (\rho_a, 1)$.

If $A'(1) = 0$, $\rho_a = 1$ with multiplicity 2.

If $A'(1) < 0$, $\rho_a = 1$ with multiplicity 1.

If $0 < A'(1) \leq +\infty$, then $\rho_a < 1$ which is simple. On the other hand, $B(s)$ shares the same properties.

(iii) For any $k > 0$, $kB(s) + A(s)$ is convex on $[0, 1]$ and thus has at most 2 zeros on $[0, 1]$. If $kB'(1) + A'(1) \leq 0$, then $kB(s) + A(s) > 0\ \forall\, s \in [0, 1)$ and $kB(s) + A(s) = 0$ has only one root 1 on $[0, 1]$ which is simple or with multiplicity 2 according to $kB'(1) + A'(1) < 0$ or $= 0$. If $kB'(1) + A'(1) > 0$, then $kB(s) + A(s) = 0$ has exactly two roots s_k and 1 on $[0, 1]$ with $0 < s_k < 1$ such that $kB(s) + A(s) > 0$ for $s \in (0, s_k)$ and $kB(s) + A(s) < 0$ for $s \in (s_k, 1)$. Moreover, both s_k and 1 are simple.

(iv) If $A'(1) = +\infty$ and $B'(1) \leq 0$ or $0 < A'(1) < +\infty$ and $B'(1) = 0$, then for any $k > 0$, the equation $kB(s) + A(s) = 0$ has exactly one root $s_k \in (0, 1)$ such that s_k is increasing with respect to k and $\lim_{k\to\infty} s_k = 1$.

Proof. The proof is skipped here as the result is trivial. ■

Throughout this chapter, we denote ρ_a and ρ_b as the smallest nonnegative roof of $A(s) = 0$ and $B(s) = 0$ respectively.

Lemma 4.2 *Let $P(t) = (p_{ij}; i, j \geq 0)$ and $\Phi(\lambda) = (\phi_{ij}(\lambda); i, j \geq 0)$ be the Feller minimal Q-function and Q-resolvent, respectively, where Q is given in (4.1) - (4.2). Then for any $i \geq 0$ and $s \in [0, 1)$,*

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = H(s)p_{i0}(t) + A(s)\sum_{k=1}^{\infty} p_{ik}(t)s^{k-1} + B(s)\sum_{k=1}^{\infty} p_{ik}(t) \cdot ks^{k-1} \tag{4.4}$$

or equivalently,

$$\lambda\sum_{j=0}^{\infty} \phi_{ij}(\lambda)s^j - s^i = H(s)\phi_{i0}(\lambda) + A(s)\sum_{k=1}^{\infty} \phi_{ik}(\lambda)s^{k-1} + B(s)\sum_{k=1}^{\infty} \phi_{ik}(\lambda)ks^{k-1}. \tag{4.5}$$

Proof. By the Kolmogorov forward equations in (4.3), we have

$$p'_{ij}(t) = p_{i0}(t)h_j + \sum_{k=1}^{j+1} p_{ik}(t) \cdot (a_{j-k+1} + kb_{j-k+1})$$

multiplying s^j and summing over $j \in Z_+$,

$$\begin{aligned}
\sum_{j=0}^{\infty} p'_{ij}(t)s^j &= p_{i0}(t)\sum_{j=0}^{\infty} h_j s^j + \sum_{j=0}^{\infty}\sum_{k=1}^{j+1} p_{ik}(t)(a_{j-k+1} + kb_{j-k+1})s^j \\
&= H(s)p_{i0}(t) + \sum_{k=1}^{\infty}\sum_{j=k-1}^{\infty} p_{ik}(t)s^{k-1}(a_{j-k+1} + kb_{j-k+1})s^{j-k+1} \\
&= H(s)p_{i0}(t) + \sum_{k=1}^{\infty} p_{ik}(t)s^{k-1}\sum_{j=0}^{\infty}(a_j + kb_j)s^j \\
&= H(s)p_{i0}(t) + A(s)\sum_{k=1}^{\infty} p_{ik}(t)s^{k-1} + B(s)\sum_{k=1}^{\infty} p_{ik}(t)ks^{k-1}.
\end{aligned}$$

Using Laplace transform, we get

$$\lambda\sum_{j=0}^{\infty} \phi_{ij}(\lambda)s^j - s^i = H(s)\cdot\phi_{i0}(\lambda) + A(s) + \sum_{k=1}^{\infty} \phi_{ik}(\lambda)s^{k-1} + B(s)\cdot\sum_{k=1}^{\infty} \phi_{ik}(t)\cdot ks^{k-1}.$$

The proof is then completed. ■

Lemma 4.3 *Let $P(t) = \{p_{ij}; i, j \geq 0\}$ be the Feller minimal Q-function where Q is given in (4.1) - (4.2).*

(i) Suppose that $h_0 = 0$. Then for any $i \geq 0$,

$$\int_0^\infty p_{ik}(t)dt < \infty, \quad k \geq 1 \tag{4.6}$$

and thus

$$(ii) \lim_{t\to\infty} p_{ik}(t) = 0, \quad i \geq 0,\, k \geq 1. \tag{4.7}$$

(iii) Moreover, for any $i \geq 0$ and $s \in [0, 1)$, we have

$$\sum_{k=1}^{\infty} \left(\int_0^\infty p_{ik}(t)dt \right) s^k < \infty. \tag{4.8}$$

Proof.

(i) In our settings, $a_0 + b_0 > 0$, from Kolmogorov forward equation

$$p'_{i0}(t) = p_{i1}(t)(a_0 + b_0).$$

Integrating the above equation, we have

$$p_{i0}(t) = \delta_{i0} + (a_0 + b_0) \int_0^t p_{i1}(u)du$$

which implies $\int_0^\infty p_{i1}(t)dt < +\infty$ for all $i \geq 0$. Hence by the irreducibility of positive states we know that

$$\int_0^\infty p_{ij}(t) < +\infty \text{ for all } j \geq 1.$$

(ii) is directly followed from (i).

(iii) Firstly, we consider the case of $0 < B'(1) < \infty$. There exists a $\tilde{k} \geq 1$ such that $\tilde{k}B'(1) + A'(1) > 0$. By Lemma 5.1 (iii), $\tilde{k}B(s) + A(s) = 0$ has a root $s_{\tilde{k}} \in (0, 1)$ such that $\tilde{k}B(s) + A(s) < 0$ in $(s_{\tilde{k}}, 1)$. Note that $H(s) = 0$ by our assumption and $B(s) < 0$ for $s \in (\rho_b, 1)$. Using (4.4), for $s \in (\rho_b \vee s_{\tilde{k}}, 1)$,

$$\begin{aligned}
\sum_{j=0}^{\infty} p'_{ij}(t)s^j &= A(s)\sum_{j=1}^{\infty} p_{ij}(t)s^{j-1} + B(s)\sum_{j=1}^{\infty} p_{ij}(t)js^{j-1} \\
&= A(s)\sum_{j=1}^{\tilde{k}} p_{ij}(t)s^{j-1} + A(s)\sum_{j=\tilde{k}+1}^{\infty} p_{ij}(t)s^{j-1} \\
&\quad + B(s)\sum_{j=1}^{\tilde{k}} p_{ij}(t)js^{j-1} + B(s)\sum_{j=\tilde{k}+1}^{\infty} p_{ij}(t)js^{j-1} \\
&\leq A(s)\sum_{j=1}^{\tilde{k}} p_{ij}(t)s^{j-1} + [\tilde{k}B(s) + A(s)]\sum_{j=\tilde{k}+1}^{\infty} p_{ij}(t)s^{j-1}.
\end{aligned}$$

Integrating the above inequality with respect to t yields,

$$\sum_{j=\tilde{k}+1}^{\infty}\left(\int_0^{\infty} p_{ij}(t)dt\right)s^{j-1} \leq \frac{\lim_{t\to\infty} p_{i0}(t) - s^i - A(s)\sum_{j=1}^{\tilde{k}}\left(\int_0^{\infty} p_{ij}(t)dt\right)s^{j-1}}{\tilde{k}B(s) + A(s)} < +\infty.$$

For the case of $B'(1) \leq 0$, either for the cases $A'(1) = +\infty$ or $0 < A'(1) < +\infty$ with $B'(1) = 0$, from Lemma 5.1 again , we know that for any $k \geq 1$, $kB(s) + A(s) = 0$ has a root $s_k \in (0, 1)$ such that $s_k \uparrow 1$ as $k \to \infty$. So for any $\tilde{s} \in [0, 1)$, there exists a k such that $s_k > \tilde{s}$ with $kB(\tilde{s}) + A(\tilde{s}) > 0$. Again from (4.4),

$$\begin{aligned}
\sum_{j=0}^{\infty} p'_{ij}(t)\tilde{s}^j &= A(\tilde{s})\sum_{j=1}^{\infty} p_{ij}(t)\tilde{s}^{j-1} + B(\tilde{s})\sum_{j=1}^{\infty} p_{ij}(t)j\tilde{s}^{j-1} \\
&\geq A(\tilde{s})\sum_{j=1}^{\infty} p_{ij}(t)\tilde{s}^{j-1} + B(\tilde{s})\sum_{j=k+1}^{\infty} p_{ij}(t)j\tilde{s}^{j-1} \\
&\geq (kB(\tilde{s}) + A(\tilde{s}))\cdot\sum_{j=k+1}^{\infty} p_{ij}(t)\tilde{s}^{j-1} + A(\tilde{s})\cdot\sum_{k=1}^{k} p_{ij}(t)\tilde{s}^{j-1}.
\end{aligned}$$

Integrating the above inequality with respect to t yields,

$$[kB(\tilde{s})+A(\tilde{s})]\cdot\sum_{j=k+1}^{\infty}\left(\int_0^{t} p_{ij}(t)du\right)\tilde{s}^{j-1} \leq \sum_{j=0}^{\infty} p_{ij}\tilde{s}^j - \tilde{s}^i - A(\tilde{s})\sum_{j=1}^{k}\left(\int_0^{t} p_{ij}(u)du\right)\tilde{s}^j.$$

Let $t \to \infty$ and (4.6), we have

$$\sum_{j=k+1}^{\infty} \int_0^{\infty} p_{ij}(u)du \cdot \tilde{s}^{j-1} < \infty.$$

For the case of $B'(1) < 0$ with $0 < A'(1) < +\infty$, there exists a $k \geq 1$ such that $kB'(1) + A'(1) < 0$. Together with the use of Lemma 5.1 (iii), (4.8) can be proved similarly. (4.8) can be proved similarly while for the case of $A'(1)$, $B'(1) \leq 0$, it is trivial with similar proof from above as $A(s)$, $B(s) > 0$ for all $s \in [0, 1)$. The proof is then completed. ∎

4.2 Uniqueness

Now, we are ready to consider the regularity and uniqueness criteria for the Markov branching processes with immigration-migration and resurrection.

Theorem 4.1 *A BIMR q-matrix Q is regular if and only if*

(i) $B'(1) < +\infty$

(ii) $B'(1) = +\infty$ *and* $\int_\varepsilon^1 \frac{1}{-B(s)} ds = +\infty$ *for some (and for all)* $\varepsilon \in (\rho_b, 1)$, *where* $\rho_b < 1$ *is the smallest nonnegative root of* $B(s) = 0$.

Proof. If part:

If $B'(1) \leq 0$, by (4.5), for $s \in [0, 1)$,

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j - s^i \geq H(s) \cdot \phi_{i0}(\lambda) + A(s) \sum_{j=1}^{\infty} \phi_{ij}(\lambda) s^{j-1}.$$

Let $s \uparrow 1$, we immediately see that

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) \quad \geq \quad 1$$

which implies

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) \;=\; 1$$

The Feller minimal Q-process is honest and Q is regular.

Assume $0 < B'(1) \leq +\infty$, the condition $0 < B'(1) < +\infty$, implies $\int_\varepsilon^1 \frac{1}{-B(s)} ds = +\infty$ for some (and then for all) $\varepsilon \in (\rho_b, 1)$. Therefore, we need to prove that Q is regular if and only if $\int_\varepsilon^\infty \frac{1}{-B(s)} ds = +\infty$ for some (and then for all) $\varepsilon \in (\rho_b, 1)$. Suppose that $\int_\varepsilon^1 \frac{1}{-B(s)} ds = +\infty$ for some (and then for all) $\varepsilon \in (\rho_b, 1)$ but Q is not regular. i.e. there exists a $\mu = 1 - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) > 0$ for some $i \geq 0$ and a fixed $\lambda > 0$. Then there exists $\gamma \in (\rho_b, 1)$ such that

$$s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j \geq \frac{\mu}{2} \text{ and } |H(s)| + |A(s)| \leq \frac{\lambda\mu}{3},\ s \in (\gamma, 1]$$

for $s \in (\gamma, 1)$, $B(s) < 0$ together with (4.6)

$$\begin{aligned} \sum_{j=1}^{\infty} \phi_{ij}(\lambda) \cdot j s^{j-1} &= \frac{s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j + H(s)\phi_{i0}(\lambda) + A(s) \cdot \sum_{j=1}^{\infty} \phi_{ij} s^{j-1}}{-B(s)} \\ &\geq \frac{\mu}{-6B(s)}. \end{aligned}$$

Taking integration for the above inequality, we have $\sum_{j=1}^{\infty} \phi_{ij}(\lambda)(1 - \gamma^j) \geq \frac{\mu}{6} \int_\gamma^1 \frac{1}{-B(s)} ds = +\infty$ which is a contradiction.

Only if part: Suppose on the contrary, if Q is regular, with $0 < B'(1) \leq +\infty$ but $\int_\varepsilon^1 \frac{1}{-B(s)} ds < +\infty$ for some (then for all) $\varepsilon \in (\rho_b, 1)$. Since $0 < B'(1) \leq +\infty$, there exists a $\hat{k}$ such that $\hat{k}B'(1) + A'(1) > 0$. By Lemma 5.1 (iii), there is a root $s_{\hat{k}} \in (0, 1)$ for $\hat{k}B(s) + A(s) = 0$. i.e.

$$\begin{aligned} &\hat{k}B(s) + A(s) \leq 0 \text{ for all } \ s \in [\hat{\varepsilon}, 1] \\ \text{and} \quad &jB(s) + A(s) \leq 0 \text{ for all } \ s \in [\hat{\varepsilon}, 1],\ j \geq \hat{k}, \end{aligned}$$

where $\hat{\varepsilon} = \rho_b \vee s_{\hat{k}} < 1$.

We define a q-matrix $Q^* = (q^*_{ij}; i,j \geq 0)$ by

$$q^*_{ij} = \begin{cases} q_{ij}, & \text{if } i > k_0,\ j \geq 0 \\ 0, & \text{if } i \leq k_0,\ j \geq 0. \end{cases}$$

By Lemma 3.4 in last chapter, Q^* is regular since Q is regular. i.e. $\sum_{j=0}^{\infty} p^*_{ij}(t) = 1$ $(i \geq 0)$. Consider the equations

$$u_i = \sum_{j \neq i} \frac{q_{ij}}{\lambda + q_i} u_j,\ 0 \leq u_i \leq 1,\ i \geq 0 \tag{4.9}$$

and

$$u_i = \sum_{j \neq i} \frac{q^*_{ij}}{\lambda + q^*_i} u_j,\ 0 \leq u_i \leq 1,\ i \geq 0. \tag{4.10}$$

Since Q is regular, (4.9) has only trivial solution. Noting that (4.10) is dominated by (4.9) and using Comparison Lemma (see Lemma 3.14 of Chen [2004]) yields that (4.10) has only trivial solution and Q^* is regular. It is similar as in deriving (4.4), we have

$$\sum_{j=\hat{k}}^{\infty} p^{*'}_{ij}(t) s^j = A(s) \cdot \sum_{j=\hat{k}+1}^{\infty} p^*_{ij}(t) s^{j-1} + B(s) \sum_{j=\hat{k}+1}^{\infty} p^*_{ij}(t) \cdot j s^{j-1},\ s \in (0,1), i \geq \hat{k} \tag{4.11}$$

with

$$\lim_{t \to \infty} p^*_{ij}(t) = 0 \quad i,j \geq \tilde{k} + 1$$

Integrating (4.11) with respect to s from ε to 1 where $\varepsilon \in (\hat{\varepsilon}, 1)$,

$$\sum_{j=\hat{k}+1}^{\infty} p^*_{ij}(t)(1 - \varepsilon^j) = \int_{\varepsilon}^{1} \frac{\sum_{j=\hat{k}}^{\infty} p^{*'}_{ij}(t) s^j - A(s) \sum_{j=\hat{k}+1}^{\infty} p^{*'}_{ij}(t) s^{j-1}}{B(s)} ds. \tag{4.12}$$

We know that $B(s) < 0$ in $(\varepsilon_0, 1)$ and $\sum_{j=0}^{\infty} |p'_{ij}(t)| \leq 2q^*_i$ by Lemma 1.2(ii). Consider the right hand side of (4.12), we have

$$\int_{\varepsilon}^{1} \left| \frac{\sum_{j=\hat{k}}^{\infty} p^{*'}_{ij}(t) s^j - A(s) \sum_{j=\hat{k}+1}^{\infty} p^{*'}_{ij}(t) s^{j-1}}{B(s)} \right| ds \leq 2(q^*_i + |a_1|) \cdot \int_{\varepsilon}^{1} \frac{ds}{-B(s)} < \infty.$$

From (4.12), by Dominated Convergence Theorem, we have

$$\lim_{t\to\infty}\sum_{j=\hat{k}+1}^{\infty} p_{ij}^*(t)\cdot(1-\varepsilon^j)=0,\quad i\geq\hat{k}+1. \tag{4.13}$$

We have also used $\lim_{t\to\infty} p'_{ij}(t)=0$ for any $i,j\geq 0$ by Chung [1967]. In other words, by the honesty of $p_{ij}^*(t)$, we have

$$\lim_{t\to\infty}\sum_{j=0}^{\hat{k}} p_{ij}^*(t)=1,\quad i\geq\hat{k}+1. \tag{4.14}$$

However, consider (4.11) again, letting $s=\hat{\varepsilon}<1$, with $jB(\hat{\varepsilon})+A(\varepsilon)\leq 0$ for $j\geq\hat{k}$, let $t\to\infty$,

$$\lim_{t\to\infty}\sum_{j=\hat{k}}^{\infty} p_{ij}^{*'}(t)\hat{\varepsilon}^j=\sum_{j=k_0+1}^{\infty}[A(\hat{\varepsilon})+jB(\hat{\varepsilon})]p_{ij}^{*'}(t)\hat{\varepsilon}^{j-1},\quad i\geq k_0.$$

This means

$$\lim_{t\to\infty}\sum_{j=0}^{\infty} p_{ij}^{*'}(t)\hat{\varepsilon}^j\leq 0,\quad i\geq\hat{k}+1$$

and thus

$$\lim_{t\to\infty}\sum_{j=0}^{\hat{k}} p_{ij}^*(t)\leq\hat{\varepsilon}^{i-\hat{k}}<1.$$

This contradicts (4.14). Our proof is then completed. ■

We now have the regularity criterion of a BPIMR q-matrix Q. How about if the q-matrix Q is not regular? The next theorem shows that there still exists only one Q-function which can satisfy the Kolmogorov forward equations.

Theorem 4.2 *There exists only one BPIMR which satisfies the Kolmogorov forward equations.*

Proof. We only need to consider the case $B'(1)=+\infty$. To prove that the BPIMR is unique, we show that the forward equations have a unique solution. To show this, we will use Theorem 2.8 in Anderson [1991], i.e. we need to prove that the equation

$$Y(\lambda I - Q) = 0 \tag{4.15}$$

has no non-trivial solutions for some and (therefore for all)$Y \geq 0$, $YI < +infty$ and $\lambda > 0$, where I denotes the column vector on $\mathbf{Z}_+$ whose all components are equal to 1. Suppose that $\{Y = y_i; i \geq 0\}$ is a solution of (4.15) for $\lambda = 1$. (4.15) can be rewritten as

$$y_n = y_0 h_n + \sum_{j=1}^{n+1} y_j \cdot (a_{n-j+1} + j b_{n-j+1}), \quad n \geq 0.$$

Multiplying both sides of the above equation by s^n, summing over $n \geq 0$,

$$\begin{aligned}\sum_{n=0}^{\infty} y_n s^n &= \sum_{n=0}^{\infty} y_0 h_n s^n + \sum_{n=0}^{\infty}\sum_{j=1}^{n+1} a_{n-j+1} y_j s^n + \sum_{n=0}^{\infty}\sum_{j=1}^{n+1} j b_{n-j+1} y_i s^n \\ &= Y_a + Y_b + Y_c\end{aligned}$$

$$Y_a = \sum_{n=0}^{\infty} y_0 h_n s^n = y_0 H(s)$$

$$Y_b = A(s) \sum_{n=1}^{\infty} y_n s^{n-1}$$

$$Y_c = B(s) \sum_{n=1}^{\infty} n y_n s^{n-1}$$

We have

$$Y(s) = y_0 H(s) + A(s) \sum_{n=1}^{\infty} y_n s^{n-1} + B(s) \cdot \sum_{n=1}^{\infty} y_n \cdot n s^{n-1} \quad |s| < 1.$$

$$y_0(1 - H(s)) + (s - A(s)) \sum_{n=1}^{\infty} y_n s^{n-1} = B(s) \sum_{n=1}^{\infty} y_n n s^{n-1}, \quad |s| < 1. \tag{4.16}$$

Since $B'(1) = +\infty$, $B(s) = 0$ has a root $\rho_b \in [0, 1)$ and $B(s) < 0$ for all $s \in (\rho_b, 1)$. Since $A(1) = 0$, and $A(s)$ is continuous in $[0, 1]$. There exists an $\varepsilon \in (\rho_b, 1)$ such that $A(s) \leq s$ for all $s \in (\varepsilon, 1)$. We can see (4.16) in $(\varepsilon, 1)$, looking at the sign of coefficient of y_n $(n \geq 0)$. $y_n = 0$ $(n \geq 0)$ is proved and so is this theorem.

■

4.3 Extinction Probability

In this section, Theorem 4.3 to 4.5 closely follows Li and Liu [2011] . Original work starts from Theorem 4.6 onwards. Throughout this section, we will always assume that $h_0 = 0$ and thus the state 0 is an absorbing state. This helps us in considering the property regarding the extinction probability and extinction time.

Let $\{X(t); t \geq 0\}$ be the unique BPIMR, and let $P(t) = \{p_{ij}(t)\}$ denotes its transition function. Define the extinction time by

$$\tau_0 = \begin{cases} \inf\{t > 0, X(t) = 0\}, & \text{if} \quad X(t) = 0 \quad \text{for some} \quad t > 0, \\ +\infty & \text{if} \quad X(t) \neq 0 \quad \text{for all} \quad t > 0. \end{cases}$$

Denote the extinction probability by

$$a_i = P(\tau_0 < \infty | X(0) = i), i \geq 1.$$

We shall consider the absorbing behavior of BPIMR in this section. As a preparation, we first provide some more settings and notations.

From (4.4), with $H(s) = 0$, we have

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = A(s) \sum_{k=1}^{\infty} p_{ik}(t)s^{k-1} + B(s) \sum_{k=1}^{\infty} p_{ik}(t)ks^{k-1}. \tag{4.17}$$

Integrating the above equality with respect to $t \in [0, \infty)$ with
$G_i(s) = \sum_{k=1}^{\infty} (\int_0^{\infty} p_{ik}(t)dt)s^k$, together with Lemma 4.3, we have

$$s(a_i - s^i) = A(s) \cdot G_i(s) + sB(s)G'_i(s), \quad s \in [0, 1) \tag{4.18}$$

Solve the differential equation we have

$$G_i(s)R(s) = \int_0^s \frac{(a_i - y^i)R(y)}{B(y)}dy, \quad s \in [0, \rho_b) \tag{4.19}$$

where $G_i(0) = 0$, $R(y) = e^{\int_{\rho_a}^{y} \frac{A(x)}{xB(x)}dx}$ and

$$R(0) = \lim_{y \to 0^+} e^{\int_{\rho_a}^{y} \frac{A(x)}{xB(x)}dx} = \begin{cases} 0 & \text{if} \quad A(0) > 0, \\ < 1 & \text{if} \quad A(0) = 0. \end{cases}$$

ρ_a and ρ_b is the smallest nonnegative root of $A(s) = 0$ and $B(s) = 0$ respectively.

Theorem 4.3 *Suppose that $\rho_a = \rho_b$. Then $a_i = \rho_b^i, (i \geq 1)$.*

Proof. Putting $s = \rho_b$ into (4.18), we get $a_i = \rho_b^i$. ■

Theorem 4.4 *Suppose $B'(1) \leq 0$ and $0 < A'(1) < +\infty$. Then $a_i = 1$ $(i \geq 1)$ if and only if $J = +\infty$ where*

$$J = \int_0^1 \frac{R(y)}{B(y)}dy \tag{4.20}$$

where $R(y) = e^{\int_{\rho_a}^{y} \frac{A(x)}{xB(x)}dx}$.

Moreover, if $J < +\infty$ then

$$a_i = J^{-1} \int_0^1 \frac{y^i R(y)}{B(y)}dy, \quad i \geq 1. \tag{4.21}$$

Proof. Suppose $j = +\infty$, by the communicating property of the positive states, we know that either $a_i = 1$ for all $i \geq 1$ or $a_i < 1$ for all $i \geq 1$. Now assume that $a_1 < 1$. From (4.19), we have

$$G_1(s)R(s) = \int_0^{a_1} \frac{(a_1 - y)R(y)}{B(y)}dy + \int_{a_1}^s \frac{(a_1 - y)R(y)}{B(y)}dy. \tag{4.22}$$

Let $s \to 1$ in the above equality. Then the first term on the right hand side of (4.22) is obviously a finite constant and the last term tends to $-\infty$ since $J \to +\infty$. However, the left hand side is either finite or $+\infty$. This is a contradiction and thus $a_i = 1$ for all $i \geq 1$. Now suppose that $J < +\infty$. Define $x_i = J^{-1} \int_0^1 \frac{y^i A(y)}{C(y)} dy$ $(i \geq 1)$. Then by (4.19) $a_i \geq x_i$, $(i \geq 1)$.

On the other hand, it can be shown that $(x_i; i \geq 1)$ is a solution of the equation

$$\sum_{k=1}^{\infty} q_{ik} x_k + q_{i0} = 0, \quad 0 \leq x_i \leq 1, \quad i \geq 1.$$

Indeed, for any $i \geq 1$,

$$\begin{aligned}
\sum_{k=1}^{\infty} q_{ik} x_k + q_{i0} &= J^{-1} \int_0^1 \frac{\sum_{t=i-1}^{\infty} (ib_{k-i+1} + a_{k-i+1}) y^k}{B(y)} R(y) dy \\
&= J^{-1} \int_0^1 \frac{iy^{i-1} B(y) + y^{i-1} A(y)}{B(y)} e^{\int_{\rho a}^{y} \frac{A(x)}{xB(x)} dx} dy \\
&= J^{-1} \left[i \int_0^1 y^{i-1} R(y) dy + \int_0^1 y^i R'(y) dy \right] \\
&= J^{-1} [R(1) - R(0)] \\
&= 0.
\end{aligned}$$

Therefore, by Lemma 4.46 of Chen [2004], we know that $a_i \leq x_i$ $(i \geq 1)$. Hence (4.21) is proved. The proof is done.

■

In practice, it may be difficult to check whether J is finite or not. So, we try to find some convenient sufficient conditions to check the quantity J.

Corollary 4.1 *Suppose that* $B'(1) < 0$ *and* $0 < A'(1) < +\infty$. *Then* $a_i = 1$ $(i \geq 1)$.

Proof. We note that $R(1) = e^{\int_{\rho a}^{1} \frac{A(x)}{xB(x)} dx} > 0$ and thus $J = +\infty$, the conclusion is then obvious with Theorem 4.4. ■

Corollary 4.2 *Suppose that* $B'(1) = 0$ *and* $0 < A'(1) < +\infty$ *together with* $B''(1) < +\infty$. *If* $B''(1) > 2A'(1)$, *then* $J = +\infty$ *and thus* $a_i = 1$ $(i \geq 1)$, *while* *if $B''(1) < 2A'(1)$, then $J < +\infty$ and thus $a_i < 1$ $(i \geq 1)$ and then a_i is given by (4.21).*

Proof. Let $g(x) = \frac{(1-x)A(x)}{xB(x)}$, $g(x) = \sum_{k=0}^{\infty} g_k x^k$, $g_k = \frac{g^{(k)}(0)}{k!}$. Define $H(y) = \int_{\rho_a}^{y} \frac{A(x)}{xB(x)} dx$

$$H(y) = \sum_{k=0}^{\infty} g_k \int_{\rho_a}^{y} \frac{x^k}{1-x} dx$$

$$\begin{aligned}
H(y) &= \sum_{k=0}^{\infty} g_k \int_{\rho_a}^{y} \frac{[1-(1-x)]^k}{1-x} dx \\
&= \sum_{k=0}^{\infty} g_k \int_{\rho_a}^{y} \frac{1}{1-x} dx + \sum_{k=1}^{\infty} g_k \int_{\rho_a}^{y} \sum_{m=1}^{k} (-1)^m (1-x)^{m-1} dx \\
&= \left(\frac{\ln(1-\rho_a)}{2} - \frac{\ln(1-y)}{2} \right) \sum_{k=0}^{\infty} g_k + H_1(y) \\
&= \frac{-\ln(1-y)}{2} \sum_{k=0}^{\infty} g_k + \frac{\ln(1-\rho_a)}{2} + H_1(y) \\
&= \frac{-\ln(1-y)}{2} \sum_{k=0}^{\infty} g_k + H_1(y).
\end{aligned}$$

We know that $H_1(y)$ is bounded on $y \in [0, 1]$

$$\sum_{k=0}^{\infty} g_k = \lim_{x \uparrow 1} \frac{(1-x)A(x)}{xB(x)} = -\frac{2A'(1)}{B'(1)} = -\gamma$$

$$\gamma = \frac{2A'(1)}{B''(1)}$$

$$\begin{aligned}
R(y) &= e^{\int_{\rho_a}^{y} \frac{A(x)}{xB(x)} dx} \\
&= R_1(y)(1-y)^{\gamma}
\end{aligned}$$

where $R_1(y)$ is bounded on $y \in [0, 1]$. It then follows that J is finite if and only if the integral $\int_{\rho_a}^{1} \frac{dy}{(1-y)^{2-\gamma}}$ is convergent, or equivalently, if and only if $B''(1) > 2A'(1)$. The proof is now completed. ∎

Theorem 4.5 *Suppose that $\rho_a < \rho_b < 1$. Then for any $i \geq 1$, the extinction probability a_i is given by*

$$a_i = \frac{\int_0^{\rho_b} \frac{y^i R(y)}{B(y)} dy}{\int_0^{\rho_b} \frac{R(y)}{B(y)} dy} < 1 \tag{4.23}$$

where $R(y) = e^{\int_{\rho_a}^{y} \frac{A(x)}{xB(x)} dx}$.

Proof. From (4.18)

$$s(a_i - s^i) = A(s) \cdot G_i(s) + sB(s)G_i'(s), \quad s \in [0, 1)$$

solve on $(0, \rho_b)$, note that $A(x) < 0$ for $x \in (\rho_a, \rho_b)$ and then $\int_{\rho_a}^{\rho_b} \frac{A(x)}{xB(x)} dx = -\infty$

$$\left[G_i(y) e^{\int_{\rho_a}^{y} \frac{A(x)}{xB(x)} dx}\right]' = \frac{a_i - y^i}{B(y)} e^{\int_{\rho_a}^{y} \frac{A(x)}{xB(x)}} dx.$$

$$G_i(\rho_b) e^{\int_{\rho_a}^{\rho_b} \frac{A(x)}{xB(x)} dx} - G_i(0) e^{\int_{\rho_a}^{0} \frac{A(x)}{xB(x)} dx} = \int_0^{\rho_b} \frac{a_i - y^i}{B(y)} e^{\int_{\rho_a}^{y} \frac{A(x)}{xB(x)} dx} dy$$

$$a_i = \frac{\int_0^{\rho_b} \frac{y^i R(y)}{B(y)} dy}{\int_0^{\rho_b} \frac{R(y)}{B(y)} dy} < 1.$$

∎

In Li and Liu [2011], some calculation for the following cases of extinction probability were not strictly defined. So, we try to solved the extinction probability in another way.

Now we focus on the case $\rho_b < \rho_a \leq 1$.

Theorem 4.6 *Suppose that $\rho_b < \rho_a \leq 1$, then $\rho_b^i < a_i < \rho_a^i$ for all $i \geq 1$.*

Proof. Letting $s = \rho_a$ and $s = \rho_b$ into (4.18), we have

$$\begin{aligned} (a_i - \rho_a^i) &= B(\rho_a) G_i'(\rho_a) \\ a_i &= \rho_a^i + B(\rho_a) G_i'(\rho_a) < \rho_a^i \end{aligned}$$

$$
\begin{aligned}
\rho_b(a_i - \rho_b^i) &= A(\rho_b)G_i(\rho_b) \\
a_i &= \rho_b^i + \frac{A(\rho_b)}{\rho_b}G_i(\rho_b) > \rho_b^i.
\end{aligned}
$$

Note that $B(\rho_a) < 0$ and $A(\rho_b) > 0$. We have $\rho_b^i < a_i < \rho_a^i$. The proof is done.

■

From Theorem 4.6, we have obtained a bound for the extinction probability, but actually we could do much better than this. To achieve this, we need to define a simple transformation. From (4.18), we can rewrite as

$$B_0(s)G_i'(s) + A_0(s)G_i(s) = U_0(s)$$

where $B_0(s) = sB(s)$, $A_0(s) = A(s)$ and $U_0(s) = s(a_i - s^i)$

In the transformation, by doing differentiation to the above equation and grouping terms, we can get

$$B_1(s)G_i''(s) + A_1(s)G_i'(s) = U_1(s) \tag{4.24}$$

where

$$
\begin{aligned}
A_1(s) &= A_0(s)[A_0(s) + B_0'(s)] - B_0(s)A_0'(s) && (4.25) \\
B_1(s) &= A_0(s)B_0(s) && (4.26) \\
U_1(s) &= A_0(s)U_0'(s) - A_0'(s)U_0(s) && (4.27)
\end{aligned}
$$

Recursively, we could easily show that for any $n \geq 0$

$$B_n(s)G_i^{(n+1)}(s) + A_n(s)G_i^{(n)}(s) = U_n(s) \tag{4.28}$$

where $\{A_n(s), B_n(s), U_n(s)\}$ are defined recursively as

$$A_n(s) = A_{n-1}(s)[A_{n-1}(s) + B'_{n-1}(s)] - B_{n-1}(s)A'_{n-1}(s) \tag{4.29}$$

$$B_n(s) = A_{n-1}(s)B_{n-1}(s) \tag{4.30}$$

$$U_n(s) = A_{n-1}(s)U'_{n-1}(s) - A'_{n-1}(s)U_{n-1}(s) \tag{4.31}$$

Note that all $A_n(s)$ and $B_n(s)$ $(n \geq 0)$ are entirely expressible in terms of the given functions. $A(s)$ and $B(s)$ are also independent of i. Similarly, $U_n(s)$ are totally expressible in terms of $A(s)$ and $B(s)$ together with the unknown constant a_i.

By (4.30), we can get

$$B_n(s) = B_0(s) \prod_{k=0}^{n-1} A_k(s). \tag{4.32}$$

$B_0(s) = sB(s)$ which has been defined above

$$B_n(\rho_b) = B_n(0) = 0 \quad \forall\, n \geq 0. \tag{4.33}$$

Prove by mathematical induction, we have

$$A_n(\rho_b) = \big(A_0(\rho_b) + nB'_0(\rho_b)\big) \prod_{k=0}^{n-1} A_k(\rho_b) \tag{4.34}$$

$$B'_n(\rho_b) = B'_0(\rho_b) \prod_{k=0}^{n-1} A_k(\rho_b), \quad \forall\, n \geq 1 \tag{4.35}$$

with $A_0(\rho_b) > 0$, $B'_0(\rho_b) < 0$.

$$A_n(0) = [A_0(0) + nB'_0(0)] \prod_{k=0}^{n-1} A_k(0) \tag{4.36}$$

$$B'_n(0) = B'_0(0) \prod_{k=0}^{n-1} A_k(0), \quad \forall\, n \geq 1 \tag{4.37}$$

with $A_0(0) > 0$, $B'_0(0) > 0$.

Lemma 4.4 *Suppose $\rho_b < \rho_a \leq 1$. Then we have*

(i) $B_n(\rho_b) = B_n(0) = 0$, $\forall\, n \geq 0$;

(ii) $A_n(0) > 0, \forall\, n \geq 0$;

(iii) If $-\frac{A(\rho_b)}{\rho B'(\rho_b)}$ *is a positive integer* m, *say, then* $A_n(\rho_b) > 0$ *for all* $0 \leq n \leq m-1$ *and* $A_m(\rho_b) > 0$.

If $-\frac{A(\rho_b)}{\rho_b B'(\rho_b)}$ *is not a positive integer, setting* m *as the integer part of* $-\frac{A(\rho_b)}{\rho_b B'(\rho_b)}$ *then* $A_n(\rho_b) > 0$ *for all* $0 \leq n < m$ *and* $A_m(\rho_b) < 0$.

Proof. (i) has been shown before.

(ii) Since $A_0(0) > 0$ and $B_0'(0) > 0$, the result can be easily obtained by mathematical induction.

(iii) The result directly follows from (4.34). ■

There are some further extinction probability calculation in Li and Liu [2011]. However, some of the complicated settings are not well defined at some particular points. So, the following theorems has solved those difficulties.

Theorem 4.7 *Suppose that* $\rho_b < \rho_a \leq 1$.

(i) If $A(\rho_b) + \rho_b B'(\rho_b) = 0$, *then*

$$a_i = \rho_b^i + i\sigma\rho_b^i, \tag{4.38}$$

while $B_0(s) = sB(s)$, σ *is a constant and is independent of* i, *which defined as* $\sigma = \frac{A(\rho_b)}{A(\rho_b)-\rho_b A'(\rho_b)}$.

(ii) If $A(\rho_b) + \rho_b B'(\rho_b) > 0$, *then*

$$\rho_b^i + i\sigma\rho_b^i < a_i < \rho_a^i, \tag{4.39}$$

(iii) If $A(\rho_b) + \rho_b B'(\rho_b) < 0$, *then*

$$\rho_b^i < a_i < \min\{\rho_a^i, \rho_b^i + i\sigma\rho_b^i\}, \tag{4.40}$$

where σ *is a constant and is independent of* i *which defined as* $\sigma = \frac{A(\rho_b)}{A(\rho_b)-\rho_b A'(\rho_b)}$.

Proof. By putting $n = 1$ in (4.24) - (4.27) together with the fact that $G''(\rho_b) < +\infty$ and $G'(\rho_b) < +\infty$, we have

$$B_1(s)G_i''(s) + A_1(s)G_i'(s) = U_1(s)$$

$$\begin{aligned} & sA(s)B(s)G_i''(s) + \{A(s)[A(s) + B(s) + sB'(s)] - sA'(s)B(s)\}G_i'(s) \\ = \; & A(s)a_i - (i+1)A(s)s^i - sA'(s)a_i + s^{i+1}A'(s). \qquad (4.41) \end{aligned}$$

Let $s = \rho_b$, note that $B_0'(\rho_b) = \rho_b B'(\rho_b)$,

$$A(\rho_b)[A(\rho_b) + B_0'(\rho_b)]G_i'(\rho_b) = A(\rho_b)a_i - (i+1)A(\rho_b)\rho_b^i - \rho_b A'(\rho_b)a_i + \rho_b^{i+1}A'(\rho_b).$$

(i) If $A(\rho_b) + \rho_b B'(\rho_b) = 0$, we have

$$(a_i - \rho_b^i)[A(\rho_b) - \rho_b A'(\rho_b)] = iA(\rho_b)\rho_b^i,$$

then

$$a_i = \rho_b^i + i\sigma\rho_b^i,$$

where σ is a constant and is independent of i which defined as $\sigma = \frac{A(\rho_b)}{A(\rho_b) - \rho_b A'(\rho_b)} > 0$.

(ii) If $A(\rho_b) + \rho_b B'(\rho_b) > 0$, together with Theorem 4.6,

$$\rho_b^i + i\sigma\rho_b^i < a_i < \rho_a^i.$$

(iii) If $A(\rho_b) + \rho_b B'(\rho_b) < 0$, together with Theorem 4.6,

$$\rho_b^i < a_i < \min\{\rho_a^i, \rho_b^i + i\sigma\rho_b^i\}.$$

■

By Theorem 4.7, we see that if $A(\rho_b) + \rho_b B'(\rho_b) = 0$, then the exact value of a_i ($i \geq 1$) are given, while if $A(\rho_b) + \rho_b B'(\rho_b) < 0$, only better bounds are provided.

In fact for this case, an explicit expression for a_i, $(i \geq 1)$ is available.

Theorem 4.8 *Suppose that* $\rho_b < \rho_a \leq 1$*, if* $A(\rho_b) + \rho_b B'(\rho_b) < 0$*, then*

$$a_i = \frac{\int_0^{\rho_b} \frac{(i+1)A(y)y^i - y^{i+1}A'(y)}{B_1(y)} e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx} dy}{\int_0^{\rho_b} \frac{A(y) - yA'(y)}{B_1(y)} e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx} dy}.$$

Proof. By (4.24), $B_1(s)G_i''(s) + A_1(s)G_i'(s) = U_1(s)$. Solve (4.24) as a first order differential equation, we have

$$\left[G_i'(y)e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx}\right]' = \frac{U_1(y)}{B_1(y)} e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx}$$

$$G_i'(\rho_b)e^{\int_{\frac{\rho_b}{2}}^{\rho_b} \frac{A_1(y)}{B_1(y)}dy} - G_i'(0)e^{\int_{\frac{\rho_b}{2}}^{0} \frac{A_1(y)}{B_1(y)}dy} = \int_0^{\rho_b} \frac{U_1(y)}{B_1(y)} e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx} dy$$

Since $A_1(\rho_b) < 0$ and $B_1(\frac{\rho_b}{2}) > 0$, we have $e^{\int_{\frac{\rho_b}{2}}^{\rho_b} \frac{A_1(y)}{B_1(y)}dy} = 0$, since $\int_{\frac{\rho_b}{2}}^{\rho_b} \frac{A_1(y)}{B_1(y)}dy = -\infty$. Similarly, we know that $e^{\int_{\frac{\rho_b}{2}}^{0} \frac{A_1(y)}{B_1(y)}dy} = 0$, we have

$$\int_0^{\rho_b} \frac{U_1(y)}{B_1(y)} e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx} dy = 0$$

Note that

$$U_1(y) = A(y)a_i - (i+1)A(y)y^i - yA'(y)a_i + y^{i+1}A'(y)$$

$$a_i = \frac{\int_0^{\rho_b} \frac{(i+1)A(y)y^i - y^{i+1}A'(y)}{B_1(y)} e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx} dy}{\int_0^{\rho_b} \frac{A(y) - yA'(y)}{B_1(y)} e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx} dy}.$$

The extinction probability is obtained. ■

By Theorem 4.7 and 4.8, we have found the closed form of extinction probability. For the case of $A(\rho_b) + \rho_b B'(\rho_b) > 0$, we need to do some more transformation as preparation. Firstly, we need to know more structure of $U_n(s)$ defined in (4.31). Note that $U_n(s)$ depends upon $i \geq 1$. To emphasize, we shall denote it as $U_{ni}(s)$.

Lemma 4.5 *For any $n \geq 1$ and $i \geq 1$, we have*

$$U_{ni}(s) = \sum_{k=0}^{n} D_{n,k}(s) U_{0i}^{(k)}(s) \tag{4.42}$$

where $U_{0i}^{(k)}(s)$ denoted the k'th derivative of $U_{0i} = s(a_i - s^i)$, $\{D_{n,k}(s), 0 \leq k \leq n$ are totally expressible by the two known $A(s)$ and $B(s)$, while $D_{n,k}(s)$ does not depend on $i \geq 1$. $\{D_{n,k}(s)\}$ can be written recursively as follows.

$$D_{1,0}(s) = -A'(s), \quad D_{1,1}(s) = A(s) \tag{4.43}$$

$$D_{n,k}(s) = D_{n-1,k-1}(s)A_{n-1}(s) - D_{n-1,k}(s)A'_{n-1}(s) + D'_{n-1,k}(s)A_{n-1}(s), 1 \leq k \leq n-1 \tag{4.44}$$

$$D_{n,n}(s) = \prod_{m=0}^{n-1} A_m(s). \tag{4.45}$$

Proof. Using mathematical induction with (4.31), the conclusions will be reached.

As an example, when $n = 1$,

$$\begin{aligned} U_{1i}(s) &= A(s)a_i - (i+1)s^i A(s) - sA'(s)a_i + s^{i+1}A'(s) \\ &= D_{1,0}(s)U_{0i}^{(0)}(s) + D_{1,1}(s)U_{0i}^{(1)}(s) \\ D_{1,0}(s) &= -A'(s) \text{ and } D_{1,1}(s) = A(s), \\ U_{0i}^{(0)}(s) &= s(a_i - s^i) \text{ and } U_{0i}^{(1)}(s) = a_i - (i+1)s^i \end{aligned}$$

■

Remark 4.1 *From the definition of $U_{0i}(s)$, it can be noted that $U_{ni}(s) = \sum_{k=0}^{n \wedge (i+1)} D_{n,k}(s) U_{0i}^{(k)}(s)$.*

We may use the former notation for simplicity. We now further consider the case that $A(\rho_b) + \rho_b B'(\rho_b) > 0$. Considering $A(\rho_b) > 0$ and $\rho_b B'(\rho_b) < 0$, there exists a positive integer m such that $m\rho_b B'(\rho_b) + A(\rho_b) \leq 0$ and $(m-1)\rho_b B'(\rho_b) + A(\rho_b) > 0$. Our Theorem 4.7 and 4.8 are tackling the case $m = 1$. We now consider the case for $m \geq 2$.

Theorem 4.9 *Suppose that Q is a BIMR-q-matrix, with $\rho_b < \rho_a \leq 1$ and $A(\rho_b) + \rho_b B'(\rho_b) > 0$. Let $m = \min\{k \in \mathbf{Z}_+; k\rho_b B'(\rho_b) + A(\rho_b) \leq 0\}$ and thus $m \geq 2$.*

(i) If $m\rho_b B'(\rho_b) + A(\rho_b) = 0$ then $U_k(\rho_b) > 0$ for all $0 \leq k \leq m-1$ and $U_m(\rho_b) = 0$. Hence

$$a_i = \sum_{k=0}^{m\wedge(i+1)} \frac{D_{m,k}(\rho_b)}{\rho_b D_{m,0}(\rho_b) + D_{m,1}(\rho_b)} \frac{(i+1)!}{(i+1-k)!} \rho_b^{i+1-k} \tag{4.46}$$

In particular,

$$a_1 = \frac{D_{m,0}(\rho_0)}{\rho_b D_{m,0}(\rho_b) + D_{m,1}(\rho_b)} \rho_b^2 + \frac{2D_{m,1}(\rho_b)}{\rho_b D_{m,0}(\rho_b) + D_{m,1}(\rho_b)} \rho_b + \frac{2D_{m,2}(\rho_b)}{\rho_b D_{m,0}(\rho_b) + D_{m,1}(\rho_b)} \tag{4.47}$$

(ii) If $m\rho_b B'(\rho_b) + A(\rho_b) < 0$, then $U_k(\rho_b) > 0$ for all $0 \leq k \leq m-1$ and $U_m(\rho_b) < 0$. Hence

$$\sum_{k=0}^{(m-1)\wedge(i+1)} \frac{D_{m-1,k}(\rho_b)}{\rho_b D_{m-1,0}(\rho_b) + D_{m-1,1}(\rho_b)} \frac{(i+1)!}{(i+1-k)!} \rho^{i+1-k} \tag{4.48}$$

$$< \quad a_i < \sum_{k=0}^{m\wedge(i+1)} \frac{D_{m,k}(\rho_b)}{\rho_b D_{m,0}(\rho_b) + D_{m,1}(\rho_b)} \frac{(i+1)!}{(i+1-k)!} \rho^{i+1-k}. \tag{4.49}$$

Proof. From (4.34),

$$A_k(\rho_b) = \big(A(\rho_b) + k\rho_b B'(\rho_b)\big) \prod_{j=0}^{k-1} A_j(\rho_b), \quad \forall\, k \geq 1. \tag{4.50}$$

From (4.28),

$$B_m(s) \cdot G_i^{(m+1)}(s) + A_m(s) G_i^{(m)}(s) = U_m(s),$$

(i) letting $s = \rho_b$ in the above equation, $B_m(\rho_b) = 0$ and $A_m(\rho_b) = 0$ gives $U_m(\rho_b) = 0$. (4.46) immediately follows from Lemma 4.5, by noting

$$D_{m,0}(\rho_b)(\rho_b a_i - \rho_b^{i+1}) + D_{m,1}(\rho_b)[a_i - (i+1)\rho_b^i] + \sum_{k=2}^{m\wedge(i+1)} D_{m,k}(\rho_b)\Big[-\frac{(i+1)!}{(i+1-k)!}\rho_b^{i+1-k}\Big] = 0.$$

(ii) if $A(\rho_b) + m\rho_b B'(\rho_b) < 0$, we can prove that $U_k(\rho_b) > 0$ $\forall 0 \leq k \leq m-1$

and $U_m(\rho_b) < 0$ and hence (4.48) follows immediately by noting

$$D_{m-1,0}(\rho_b)(\rho_b a_i - \rho_b^{i+1}) + D_{m-1,1}(\rho_b)[a_i - (i+1)\rho_b^i] + \sum_{k=2}^{(m-1)\wedge(i+1)} D_{m-1,k}(\rho_b)\left[-\frac{(i+1)!}{(i+1-k)!}\rho_b^{i+1-k}\right] > 0$$

and

$$D_{m,0}(\rho_b)(\rho_b a_i - \rho_b^{i+1}) + D_{m,1}(\rho_b)[a_i - (i+1)\rho_b^i] + \sum_{k=2}^{m\wedge(i+1)} D_{m,k}(\rho_b)\left[-\frac{(i+1)!}{(i+1-k)!}\rho_b^{i+1-k}\right] < 0.$$

In obtaining (4.46), we have assumed that $D_{m,0}(\rho_b)$ and $D_{m,1}(\rho_b)$ is not equal to 0. Similarly, we have assumed that $D_{m-1,0}(\rho_b) > 0$ and $D_{m,0}(\rho_b) > 0$. If they are negative, the inequality will need to be reversed. ■

Remark 4.2 *Actually, if $D_{m,0}(\rho_b) = 0$, we can prove from (4.42) that $D_{m,i}(\rho_b) = 0$ for $i = 1, 2, 3, \cdots$ etc. This means that $A_m(s)$ and $B_m(s)$ are divisible by $s - \rho_b$. This also means that*

$$B_m(s)G_i^{(m+1)}(s) + A_m(s)G_i^{(m)}(s) = U_m(s)$$

could be reduced by dividing $s - \rho_b$.

Assumption 4.1 *Suppose that Q is a BPIMR-q-matrix with $\rho_b < \rho_a \leq 1$ and $A(\rho_b) + \rho_b B'(\rho_b) > 0$, we assume that $B_m(s) > 0$ for all $s \in (0, \rho_b)$ where $m = \min\{k \in \mathbf{Z}_+; k\rho_b B'(\rho_b) + A(\rho_b) \leq 0\}$.*

Assumption 4.2 *Suppose that Q is a BPIMR-q-matrix with $\rho_b < \rho_a \leq 1$ and $A(\rho_b)+\rho_b B'(\rho_b) > 0$, we assume that there exists an $s \in (0, \rho_b)$ such that $B_m(s) \leq 0$ where $m = \min\{k \in \mathbf{Z}_+; k\rho_b B'(\rho_b) + A(\rho_b) \leq 0\}$.*

Theorem 4.10 *Suppose that Q is a BPIMR-q-matrix with $\rho_b < \rho_a \leq 1$ and $A(\rho_b) + \rho_b B'(\rho_b) > 0$ and that $-\frac{A(\rho_b)}{\rho_b B'(\rho_b)}$ is not an integer. Let $m = \min\{k \in \mathbf{Z}_+; k\rho_b B'(\rho_b) + A(\rho_b) \leq 0\}$,*

with Assumption 4.1, i.e. $B_m(s) > 0$ for all $s \in (0, \rho_b)$. The extinction probability is then given by

$$a_i = \frac{\int_0^{\rho_b} \sum_{k=0}^{m\wedge(i+1)} \frac{D_{m,k}(y)}{B_m(y)} \frac{(i+1)!}{(i+1-k)!} y^{i+1-k} e^{H_m(y)} dy}{\int_0^{\rho_b} \left(\frac{yD_{m,0}(y)+D_{m,1}(y)}{B_m(y)}\right) e^{H_m(y)} dy} \tag{4.51}$$

where $H_m(y) = \int_{\frac{\rho_b}{2}}^{y} \frac{A_m(x)}{B_m(x)} dx$.

Proof. From (4.28), we have

$$B_m(s)G_i^{(m+1)}(s) + A_m(s)G_i^{(m)}(s) = U_{mi}(s).$$

Under Assumption 4.1, we solve the above equation and have

$$\begin{aligned} \left[G_i^{(m)}(y)e^{H_m(y)}\right]' &= \frac{U_{mi}(y)}{B_m(y)} e^{H_m(y)} \\ G_i^{(m)}(s)e^{H_m(s)} - G_i^{(m)}(0)e^{H_m(0)} &= \int_0^s \frac{U_{mi}(y)}{B_m(y)} e^{H_m(y)} dy. \end{aligned}$$

By (4.34), $A_n(\rho_b) = \Big(A_0(\rho_b) + nB_0'(\rho_b)\Big) \prod_{k=0}^{n-1} A_k(\rho_b)$ with our setting in the case, we know that $A_n(\rho_b) > 0$ for all $0 \leq n \leq m-1$ and $A_m(\rho_b) < 0$ and by Lemma 4.4, noticing that $B_n(\rho_b) = 0$ we have

$$e^{H_m(s)} = e^{\int_{\frac{\rho_b}{2}}^{s} \frac{A_m(x)}{B_m(x)} dx} \to 0 \text{ as } s \to \rho_b^-.$$

So, we get

$$\lim_{t\to 0^+} G_i^{(m)}(t)e^{H_m(t)} = 0$$

since $H_m(0) = \int_{\frac{\rho_b}{2}}^{0} \frac{A_m(y)}{B_m(y)} = -\infty$. Let $s \to \rho_b^-$, we get

$$\int_0^{\rho_b} \frac{U_{mi}(y)}{B_m(y)} e^{H_m(y)} dy = 0$$

then

$$a_i = \frac{\int_0^{\rho_b} \sum_{k=0}^{m\wedge(i+1)} \frac{D_{m,k}(y)}{B_m(y)} \frac{(i+1)!}{(i+1-k)!} y^{i+1-k} e^{H_m(y)} dy}{\int_0^{\rho_b} \left(\frac{yD_{m,0}(y)+D_{m,1}(y)}{B_m(y)}\right) e^{H_m(y)} dy}$$

The proof is complete. ■

Remark 4.3 *Under Assumption 4.2, we may worry about that* $\int_0^{\rho_b} \frac{f(s)}{B_m(s)} ds$ *may not be defined at the zeros of* $B_m(s)$ *in* $(0, \rho_b)$. *This is not a great problem if we include complex integral when this happens. See Remark 3.3 in last chapter for details.*

Theorem 4.11 *Suppose that* Q *is a BPIMR-q-matrix with* $\rho_b < \rho_a \leq 1$ *and* $A(\rho_b) + \rho_b B'(\rho_b) > 0$ *and that* $-\frac{A(\rho_b)}{\rho_b B'(\rho_b)}$ *is not an integer. Let* $m = \min\{k \in \mathbf{Z}_+; k\rho_b B'(\rho_b) + A'(\rho_b) \leq 0\}$, *with Assumption 4.2, in Remark 4.3, i.e. there exists an* $s \in (0, \rho_b)$ *such that* $B_m(s) \leq 0$. *The extinction probability is then given by*

$$a_i = \frac{(\sim) \oint_0^{\rho_b} \sum_{k=0}^{m\wedge(i+1)} \frac{D_{m,k}(y)}{B_m(y)} \frac{(i+1)!}{(i+1-k)!} y^{i+1-k} e^{\tilde{H}_m(y)} dy}{(\sim) \oint_0^{\rho_b} \left(\frac{yD_{m,0}(y)+D_{m,1}(y)}{B_m(y)}\right) e^{\tilde{H}_m(y)} dy} \tag{4.52}$$

where $\tilde{H}_m(y) = (\sim) \oint_{\frac{\rho_b}{2}}^{y} \frac{A_m(x)}{B_m(x)} dx$.

Proof.

Again from (4.28), we have

$$B_m(s) G_i^{(m+1)}(s) + A_m(s) G_i^{(m)}(s) = U_{mi}(s).$$

Under Assumption 4.2, by the same reasons in proving Theorem 4.10, we solve the above equation

$$\begin{aligned} \left[G_i^{(m+1)}(y) e^{\tilde{H}_m(y)}\right]' &= \frac{U_{mi}(y)}{B_m(y)} e^{\tilde{H}_m(y)} \\ G_i^{(m+1)}(s) e^{\tilde{H}_m(s)} - G_i^{(m)}(0) e^{\tilde{H}_m(0)} &= (\sim) \oint_0^s \frac{U_{mi}(y)}{B_m(y)} e^{\tilde{H}_m(y)} dy. \end{aligned}$$

When $s \to \rho_b^-$,

$$0 = (\sim) \oint_0^{\rho_b} \frac{U_{mi}(y)}{B_m(y)} e^{\tilde{H}_m(y)} dy,$$

therefore,

$$a_i = \frac{(\sim) \oint_0^{\rho_b} \sum_{k=0}^{m \wedge (i+1)} \frac{D_{m,k}(y)}{B_m(y)} \frac{(i+1)!}{(i+1-k)!} y^{i+1-k} e^{\tilde{H}_m(y)} dy}{(\sim) \oint_0^{\rho_b} \left(\frac{y D_{m,0}(y) + D_{m,1}(y)}{B_m(y)} \right) e^{\tilde{H}_m(y)} dy}$$

The proof is then completed. ■

4.4 Asymptotic Behavior of Extinction Probability

Though the closed forms for the extinction probabilities of the BPIMR are obtained in the theorems above, some of the these closed forms are very complicated. This part will show that, for large i,the asymptotic behavior of these complicated expressions for the extinction probabilities actually takes a very simple form.

As a review and preparation, we first give a lemma about the properties of $R(y)$ used in (4.19).

Lemma 4.6 *(i) Suppose* $R(y) = e^{\int_{\rho_a}^{y} \frac{A(x)}{xB(x)} dx}$, $0 < B'(1) \leq +\infty$ *and* $\rho_a < \rho_b < 1$. *Then*

$$R(y) \sim k(\rho_b - y)^{\alpha} \quad as \quad y \to \rho_b^-$$

where $0 < k < +\infty$ *is a constant (i.e. independent of* y*) and* $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)}$.

(ii) Suppose $B'(1) = 0$, $\rho_a < \rho_b = 1$ *and* $B''(1) < 2A'(1) < +\infty$. *Then*

$$R(y) \sim k(1 - y)^{\gamma} \quad as \quad y \to 1^-$$

where $0 < k < +\infty$ *is a constant (i.e. independent of* y*) and* $\gamma = \frac{2A'(1)}{B''(1)}$.

(iii)

$$R(0) = \lim_{y \to 0^+} e^{\int_{\rho_a}^{y} \frac{A(x)}{xB(x)} dx} = \begin{cases} 0 & \text{if} \quad A(0) > 0, \\ < 1 & \text{if} \quad A(0) = 0. \end{cases}$$

(iv) Suppose $R_1(y) = e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A(x)}{xB(x)} dx}$, $\rho_b < \rho_a \leq 1$. *Then*

$$R_1(y) \sim k(\rho_b - y)^{\alpha} \quad \text{as} \quad y \to \rho_b^-$$

where $0 < k < +\infty$ *is a constant (i.e. independent of* y*) and* $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)}$.

Proof.

(i) Under the condition $0 < B'(1) \leq +\infty$ and thus $\rho_b < 1$ is a single zero of $B(s)$ and if we let

$$f(x) = \frac{A(x)}{xB(x)}(\rho_b - x) \tag{4.53}$$

$f(x)$ could be expanded as a power series of x on the interval $[\rho_a, \rho_b)$. Here we view $f(x)$ as a real valued function of x. Suppose the expansion takes the form of

$$f(x) = \sum_{k=0}^{\infty} f_k x^k. \tag{4.54}$$

By (4.53) and (4.54), for $0 < y < \rho_b$, we have

$$\begin{aligned} & \int_{\rho_a}^{y} \frac{A(x)}{xB(x)} dx = \int_{\rho_a}^{y} \frac{f(x)}{\rho_b - x} dx \\ = & \sum_{k=0}^{\infty} f_k \int_{\rho_a}^{y} \frac{x^k}{\rho_b - x} dx \\ = & \sum_{k=0}^{\infty} f_k \int_{\rho_a}^{y} \frac{[\rho_b - (\rho_b - x)]^k}{\rho_b - x} dx \\ = & \left(\sum_{k=0}^{\infty} f_k \rho_b^k \right) \int_{\rho_a}^{y} \frac{dx}{\rho_b - x} + \sum_{k=1}^{\infty} f_k \sum_{m=1}^{k} (-1)^m \binom{k}{m} \rho_b^{k-m} \int_{\rho_a}^{y} (\rho_b - x)^{m-1} dx \\ = & J_1 + J_2 \end{aligned} \tag{4.55}$$

We use J_1 and J_2 for convenience.

From (4.54), J_1 in (4.55) can be further simplified as

$$J_1 = \left(\sum_{k=0}^{\infty} f_k \rho_b^k \right) \int_{\rho_b}^{y} \frac{dx}{\rho_b - x} = f(\rho_b) \int_{\rho_a}^{y} \frac{dx}{\rho_b - x}.$$

By L'Hopital's rule, we know that

$$f(\rho_b) = \lim_{x \to \rho_b} \frac{A(x)}{xB(x)} (\rho_b - x) = \frac{-A(\rho_b)}{\rho_b B'(\rho_b)}$$

which is finite.

Similarly, for J_2 in (4.55), it can be rewritten as

$$J_2 = \sum_{k=1}^{\infty} f_k \rho_b^k \sum_{m=1}^{k} \frac{\binom{k}{m}}{m} \left(\frac{\rho_a}{\rho_b} - 1 \right)^m - \sum_{k=1}^{\infty} f_k \rho_b^k \sum_{m=1}^{k} \frac{\binom{k}{m}}{m} \left(\frac{y}{\rho_b} - 1 \right)^m . \tag{4.56}$$

We can see that the first term in the right hand side of (4.56) is a constant that is independent of y, while the second term in the right hand side of (4.56) is clearly bounded for $y \in [\rho_a, \rho_b]$. From mean-value theorem, J_2 can be written as a constant k_2 such that

$$R(y) = \exp \left\{ \int_{\rho_a}^{y} \frac{A(x)}{xB(x)} dx \right\} \sim k(\rho_b - y)^{\alpha} \quad \text{as} \quad y \to \rho_b^-, \tag{4.57}$$

where $0 < k < +\infty$ is a constant (i.e. independent of y) and $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)}$.

(ii) Since $B'(1) = 0$, there is no root on $[\rho_a, 1)$ and 1 is the zero of $B(s)$ with multiplicity 2, with $0 < A'(1) < \infty$ and we let

$$h(x) = \frac{A(x)(1-x)}{xB(x)}$$

noting that $\lim_{x \to 1^-} h(x) = \frac{-2A'(1)}{B''(1)}$. The conclusion can be obtained similar to (i).

(iii) It is shown in (4.19).

(iv) Under $\rho_b < \rho_a \leq 1$, $f(x) = \frac{A(x)}{xB(x)}(\rho_b - x)$, $f(x)$ could be expanded as a power series of x on the interval $[\frac{\rho_b}{2}, \rho_b)$. Here, we view $f(x)$ as a real valued function of x. Suppose the expansion takes the form of $f(x) = \sum_{k=0}^{\infty} f_k x^k$,

$$\begin{aligned} & \int_{\frac{\rho_b}{2}}^{y} \frac{A(x)}{xB(x)} dx = \int_{\frac{\rho_b}{2}}^{y} \frac{f(x)}{\rho_b - x} dx \\ = & \sum_{k=0}^{\infty} f_k \int_{\frac{\rho_b}{2}}^{y} \frac{x^k}{\rho_b - x} dx \\ = & \sum_{k=0}^{\infty} f_k \rho_b^k \int_{\frac{\rho_b}{2}}^{y} \frac{dx}{\rho_b - x} + \sum_{k=1}^{\infty} f_k \sum_{m=1}^{\infty} (-1)^m \binom{k}{m} \rho_b^{k-m} \int_{\frac{\rho_b}{2}}^{y} (\rho_b - x)^{m-1} dx \\ = & J_1 + J_2 \end{aligned}$$

we use J_1 and J_2 for convenience. From the above equation, J_1 can be simplified as

$$J_1 = \left(\sum_{k=0}^{\infty} f_k \rho_b^k \right) \int_{\frac{\rho_b}{2}}^{y} \frac{dx}{\rho_b - x} = f(\rho_b) \int_{\frac{\rho_b}{2}}^{y} \frac{dx}{\rho_b - x}.$$

By L'Hopital's rule, we know that

$$f(\rho_b) = \lim_{x \to \rho_b^+} \frac{A(x)}{xB(x)} (\rho_b - x) = \frac{-A(\rho_b)}{\rho_b B'(\rho_b)}$$

which is finite.

Similarly,

$$J_2 = \sum_{k=1}^{\infty} f_k \rho_b^k \sum_{m=1}^{k} \frac{\binom{k}{m}}{m} \left(\frac{-1}{2} \right)^m - \sum_{k=1}^{\infty} f_k \rho_b^k \sum_{m=1}^{k} \frac{\binom{k}{m}}{m} \left(\frac{y}{\rho_b} - 1 \right)^m.$$

We can see that the first term in the right hand side of J_2 is a constant that is independent of y, while the second term in the right hand side is clearly bounded for $y \in [\frac{\rho_b}{2}, \rho_b]$. From mean-value theorem, J_2 can be written as a constant k_2, such that

$$R_1(y) = \exp \left\{ \int_{\frac{\rho_b}{2}}^{y} \frac{A(x)}{xB(x)} dx \right\} \sim k(\rho_b - y)^{\alpha} \quad \text{as} \quad y \to \rho_b^-,$$

where $0 < k < +\infty$ is a constant (i.e. independent of y) and$\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)}$.

∎

Lemma 4.7 *For any complex number a, we have*

$$\lim_{z\to\infty}\frac{\Gamma(z+a)}{\Gamma(z)}z^{-a}=1 \tag{4.58}$$

where real part of a is positive and $\Gamma(\cdot)$ is the gamma function.

Lemma 4.7 is well-know result so the proof is skipped.

We are now ready to discuss the core part of this section, the asymptotic behavior of the extinction probability.

Theorem 4.12 *If $B'(1)=0$ and $0<A'(1)<+\infty$, then*

(i) the extinction probability $\{a_i\}$, starting from state $i\geq 1$, is less that 1 (for all $i\geq 1$) if and only if $B''(1)<2A'(1)$.

(ii) In addition, if $B''(1)<2A'(1)$, then

$$a_i\sim ki^{-\alpha}\qquad as\qquad i\to\infty \tag{4.59}$$

where k is a constant and $\alpha=\frac{2A'(1)}{B''(1)}-1>0$

Proof.

(i) is proved in Theorem 4.4 and Corollary 4.2.

(ii) Suppose that $B'(1)=0$ and $\rho_a<\rho_b=1$, then by Theorem 4.4 and Corollary 4.2 in this chapter.

$$a_i=\frac{1}{J}\int_0^1\frac{y^iR(y)}{B(y)}dy\quad(i\geq 1) \tag{4.60}$$

where $J=\int_0^1\frac{R(y)}{B(y)}dy$ is a finite constant which is independent of i.

We consider the integral

$$I_1^{(i)}=\int_0^1\frac{y^iR(y)}{B(y)}dy$$

we know that $\frac{y^iR(y)}{B(y)}$ is bounded on $[0,\varepsilon]$ where $0<\varepsilon<1$ for $i\geq 1$. So, we focus on the behavior of $I_1^{(i)}$ when $y\to 1^-$. $\frac{B(y)}{(1-y)^2}$ is bounded on $[0,1]$ as $B'(1)=0$ and

$B''(1) < +\infty$, with Lemma 4.6 (ii)

$$I_1^{(i)} = k_1 \int_0^1 \frac{y^i(1-y)^\gamma}{(1-y)^2} dy = k_1 \int_0^1 y^i(1-y)^{\gamma-2} dy$$

$0 < k_1 < +\infty$ is constant and $\gamma = \frac{2A'(1)}{B''(1)} > 1$, since we have $0 < B''(1) < 2A'(1) < +\infty$. It can be noted that

$$\int_0^1 y^i(1-y)^{\gamma-2} dy = \frac{\Gamma(i+1)\Gamma(\gamma-1)}{\Gamma(i+\gamma)}$$

where $\Gamma(\cdot)$ is the gamma function, with Lemma 4.7, we get

$$I_1^{(i)} \sim k i^{1-\gamma}, \quad (i \to \infty) \tag{4.61}$$

for some constant k, where $\gamma = 1 + \alpha$ which is (4.59).

■

Theorem 4.13 *Suppose that $\rho_a < \rho_b < 1$, then the extinction probability of the BPIMR, starting from $i \geq 1$, denoted by $\{a_i\}$, possesses the following asymptotic behavior,*

$$a_i \sim k i^{-\alpha} \rho_b^i \quad (i \to \infty) \tag{4.62}$$

when $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)} > 0$.

Proof. By Theorem 4.5, we know that the extinction probability $\{a_i\}$, starting from $i \geq 1$, is given by

$$a_i = \frac{\int_0^{\rho_b} \frac{y^i R(y)}{B(y)} dy}{\int_0^{\rho_b} \frac{R(y)}{B(y)} dy} < 1. \tag{4.63}$$

We see that the denominator of the right hand side of (4.63) is a constant and is independent of i. Now, we look at the numerator, let

$$I_1^{(i)} = \int_0^{\rho_b} \frac{y^i R(y)}{B(y)} dy,$$

Note that $\rho_b < 1$ is the single zero of $B(s)$ for $s \in (0,1)$, by Lemma 4.6 (i), we know there exists a constant k such that

$$I_1^{(i)} = \int_0^{\rho_b} \frac{y^i R(y)}{B(y)} dy = k_1 \int_0^{\rho_b} y^i (\rho_b - y)^{\alpha - 1} dy$$

where $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)} > 0$ since $A(\rho_b)$ and $B'(\rho_b)$ are negative.

$$\begin{aligned} \int_0^{\rho_b} y^i (\rho_b - y)^{\alpha-1} dy &= \rho_b^{i+\alpha} \int_0^1 x^i (1-x)^{\alpha-1} dx \\ &= \rho_b^{i+\alpha} \frac{\Gamma(i+1)\Gamma(\alpha)}{\Gamma(i+\alpha-1)}. \end{aligned}$$

By Lemma 4.7, we have

$$I_1^{(i)} \sim k i^{-\alpha} \rho_b^i, \quad i \to \infty.$$

This completes the proof. ■

Now we turn to consider the case of $\rho_b < \rho_a \leq 1$. We know that $B'(\rho_b) < 0$ and $A(\rho_b) > 0$. We can further divide our consideration into three cases as

$$A(\rho_b) + \rho_b B'(\rho_b) = 0,$$

$$A(\rho_b) + \rho_b B'(\rho_b) < 0 \text{ and}$$

$$A(\rho_b) + \rho_b B'(\rho_b) > 0.$$

We first deal with the case of $A(\rho_b) + \rho_b B'(\rho_b) = 0$.

Theorem 4.14 *If $\rho_b < \rho_a \leq 1$ and $A(\rho_b) + \rho_b B'(\rho_b) = 0$, then the extinction probability $\{a_i\}$ of the BIMRP, starting from $i \geq 1$, given by*

$$a_i = \rho_b^i + i\sigma\rho_b^i, \tag{4.64}$$

where $\sigma = \frac{A(\rho_b)}{A(\rho_b) - \rho_b A'(\rho_b)}$. Furthermore,

$$a_i \sim \sigma i^{-\alpha} \rho_b^i \quad (i \to \infty) \tag{4.65}$$

$\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)} = -1.$

Proof. (4.64) is proved in Theorem 4.7. From (4.64), $a_i = \rho_b^i + i\sigma\rho_b^i$, when $i \to \infty$, we can easily get $a_i \sim \sigma i^{-\alpha}\rho_b^i$ where $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)} = -1$, $\sigma = \frac{A(\rho_b)}{A(\rho_b)-\rho_b A'(\rho_b)}$. ∎

Now, we move to the next subcase for $\rho_b < \rho_a \leq 1$, i.e. $A(\rho_b) + \rho_b B'(\rho_b) < 0$.

Theorem 4.15 *Suppose $\rho_b < \rho_a \leq 1$ and $A(\rho_b) + \rho_b B'(\rho_b) < 0$. Then the extinction probability $\{a_i\}$ of the BIMRP, starting from $i \geq 1$, possesses the following asymptotic property,*

$$a_i \sim k i^{-\alpha} \rho_b^i \tag{4.66}$$

where k is constant, $-1 < \alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)} < 0$.

Proof. If $\rho_b < \rho_a \leq 1$ and $A(\rho_b) + \rho_b B'(\rho_b) < 0$, then the extinction probability $\{a_i\}$ is given by Theorem 4.8

$$a_i = \frac{\int_0^{\rho_b} \frac{(i+1)y^i A(y) - y^{i+1}A'(y)}{B_1(y)} e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx} dy}{\int_0^{\rho_b} \frac{A(y) - yA'(y)}{B_1(y)} e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx} dy} \tag{4.67}$$

$$\begin{aligned}
A_0(s) &= A(s); \quad B_0(s) = sB(s), \\
A_1(s) &= A_0(s)[A_0(s) + B_0'(s)] - B_0(s)A_0'(s) \\
&= A(s)[A(s) + B(s) + sB'(s)] - sA'(s)B(s) \\
B_1(s) &= A_0(s)B_0(s) = sA(s)B(s)
\end{aligned} \tag{4.68}$$

Again, we see that the denominator is a constant, i.e.

$$a_i = k \int_0^{\rho_b} \frac{(i+1)y^i A(y) - y^{i+1}A'(y)}{B_1(y)} e^{\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx} dy. \tag{4.69}$$

In order to look at the properties of $\{a_i\}$ on (4.69), we try to focus on $\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)}dx$.

By our definition above,

$$\begin{aligned}\int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)} dx &= \int_{\frac{\rho_b}{2}}^{y} \frac{A_0(x)}{B_0(x)} dx + \int_{\frac{\rho_b}{2}}^{y} \frac{B_0'(x)}{B_0(x)} dx - \int_{\frac{\rho_b}{2}}^{y} \frac{A_0'(x)}{A_0(x)} dx \\ &= \int_{\frac{\rho_b}{2}}^{y} \frac{A_0(x)}{B_0(x)} dx + \ln \frac{B_0(y)}{A_0(y)} + \ln \frac{A_0(\frac{\rho_b}{2})}{B_0(\frac{\rho_b}{2})}\end{aligned}$$

where $A_0(\frac{\rho_b}{2}) > 0$ and $B_0(\frac{\rho_b}{2}) > 0$.

$$\exp\left\{ \int_{\frac{\rho_b}{2}}^{y} \frac{A_1(x)}{B_1(x)} dx \right\} = k_1 \frac{B_0(y)}{A_0(y)} \exp\left\{ \int_{\frac{\rho_b}{2}}^{y} \frac{A_0(x)}{B_0(x)} dx \right\} \tag{4.70}$$

where k_1 is a constant which is independent of y. Put (4.70) into (4.69), we see that

$$\begin{aligned} a_i &= k \int_0^{\rho_b} \frac{(i+1)y^i A(y) - y^{i+1} A'(y)}{B_1(y)} k_1 \frac{B_0(y)}{A_0(y)} \exp\left\{ \int_{\frac{\rho_b}{2}}^{y} \frac{A_0(x)}{B_0(x)} dx \right\} dy \\ &= k \int_0^{\rho_b} \frac{(i+1)y^i A(y) - y^{i+1} A'(y)}{A_0(y) B_0(y)} k_1 \frac{B_0(y)}{A_0(y)} \exp\left\{ \int_{\frac{\rho_b}{2}}^{y} \frac{A_0(x)}{B_0(x)} dx \right\} dy \\ &= k_2 \int_0^{\rho_b} \frac{(i+1)y^i A(y) - y^{i+1} A'(y)}{(A_0(y))^2} \exp\left\{ \int_{\frac{\rho_b}{2}}^{y} \frac{A_0(x)}{B_0(x)} dx \right\} dy \end{aligned}$$

$\frac{1}{A_0(s)}$ is bounded on $[0, \rho_b]$ since $\rho_b < \rho_a \leq 1$, $A_0(s)$ and $A_0'(s)$ are bounded on $[0, \rho_b]$. By mean value theorem, there exists k_1 and k_2 which are both independent of y and i such that

$$\begin{aligned} a_i &= k_1(i+1) \int_0^{\rho_b} y^i \exp\left\{ \int_{\frac{\rho_b}{2}}^{y} \frac{A_0(x)}{B_0(x)} dx \right\} dy \\ &\quad + k_2 \int_0^{\rho_b} y^{i+1} \exp\left\{ \int_{\frac{\rho_b}{2}}^{y} \frac{A_0(x)}{B_0(x)} dx \right\} dy. \end{aligned} \tag{4.71}$$

We notice that $\exp\left\{ \int_{\frac{\rho_b}{2}}^{y} \frac{A_0(x)}{B_0(x)} dx \right\} dy$ actually is defined as $R_1(y)$ and by using Lemma 4.6, there exists another set of constant k_1 and k_2 that gives

$$a_i = k_1(i+1) \int_0^{\rho_b} y^i (\rho_b - y)^{\alpha} dy + k_2 \int_0^{\rho_b} y^{i+1} (\rho_b - y)^{\alpha} dy$$

where $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)} < 0$, since $A(\rho_b) + \rho_b B'(\rho_b) < 0$, $-1 < \alpha < 0$.

With some easy transformation,

$$\begin{aligned}\int_0^{\rho_b} y^i(\rho_b - y)^\alpha dy &= \rho_b^{i+1}\rho_b^\alpha \int_0^1 x^i(1-x)^\alpha dx \\ &= \rho_b^{i+\alpha+1}\int_0^1 x^i(1-x)^\alpha dx \\ &= \rho_b^{i+\alpha+1}\frac{\Gamma(i+1)\Gamma(\alpha+1)}{\Gamma(i+\alpha+2)}.\end{aligned}$$

Similarly,

$$\int_0^{\rho_b} y^{i+1}(\rho_b - y)^\alpha dy = \rho_b^{i+\alpha+2}\frac{\Gamma(i+2)\Gamma(\alpha+1)}{\Gamma(i+\alpha+3)}.$$

Substituting the above results into (4.71), and applying Lemma 4.7, we will get $a_i \sim k_1 i^{-\alpha}\rho_b^i$ $(i \to \infty)$ and the proof is then completed.

■

Finally, we consider the case for $\rho_b < \rho_a \leq 1$ and $A(\rho_b) + \rho_b B'(\rho_b) > 0$. But we know that $A(\rho_b) > 0$ and $\rho_b B'(\rho_b) < 0$, there exists a smallest positive integer, $m \geq 2$, such that $(m-1)\rho_b B'(\rho_b) + A(\rho_b) > 0$ and $m\rho_b B'(\rho_b) + A(\rho_b) \leq 0$. Firstly, consider the subcase that $m\rho_b B'(\rho_b) + A(\rho_b) = 0$. We let $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)}$ such that $0 < m - 1 < -\alpha \leq m$.

Theorem 4.16 *Suppose $\rho_b < \rho_a \leq 1$ and $A(\rho_b) + \rho_b B'(\rho_b) > 0$. If there exists a positive integer m such that $m\rho_b B'(\rho_b) + A(\rho_b) = 0$, then there exist $(m+1)$ constants, $\{k_0, k_1, \cdots, k_m\}$ with $k_0 = 1$ such that the extinction probability $\{a_i\}$, starting from $i \geq 1$, can be written as*

$$a_i = \sum_{l=0}^{m} k_l (i+1)^l \rho_b^{(i+1)-l}. \tag{4.72}$$

In particular, there exists a constant k such that

$$a_i \sim k i^{-\alpha}\rho_b^i \quad (i \to \infty) \tag{4.73}$$

where $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)} = -m < 0$.

Proof. (4.72) follows from (4.46) in Theorem 4.9, (4.73) can be easily seen by noticing that $k_m i^m \rho_b^{i-m}$ is a dominated term in (4.72). Here $-m = \alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)} < 0$. ■

It can be seen that Theorem 4.16 is a special case of Theorem 4.14 when $m = 1$. Next, we consider the other subcase of $m\rho_b B'(\rho_b) + A(\rho_b) < 0$ for $m \geq 2$. Recall from (4.29) to (4.30). We have define $A_m(s)$ and $B_m(s)$ recursively from $A_n(s)$ and $B_n(s)$ for $(m \geq n \geq 1)$

$$\begin{aligned} A_n(s) &= A_{n-1}(s)[A_{n-1}(s) + B'_{n-1}(s)] - B_{n-1}(s)A'_{n-1}(s) \\ B_n(s) &= A_{n-1}(s)B_{n-1}(s). \end{aligned}$$

Explained in Remark 4.3, in order to avoid complex analysis, with loss of generality, we assume $B_m(s) > 0$ for all $s \in (0, \rho_b)$.

Theorem 4.17 *Suppose $\rho_b < \rho_a \leq 1$ and $A(\rho_b) + \rho_b B'(\rho_b) > 0$ and that $-\frac{A(\rho_b)}{\rho_b B'(\rho_b)}$ is not an integer. Let $m = \min\{k \in \mathbf{Z}_+; k\rho_b B'(\rho_b) + A(\rho_b) < 0\}$ where $-(m+1) < \alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)} < -m$. Further assume that $B_m(s) > 0$ for all $s \in (0, \rho_b)$ where $B_m(s)$ is defined in (4.29) and (4.30). The extinction probability, a_i, of the BIMPR, starting from $i \geq 1$, possesses with asymptotic property that there exist $(m+1)$ constants $\{k_0, k_1, \cdots, k_{m-1}\}$ such that*

$$a_i = \sum_{l=0}^{m} k_l \frac{i!}{(i-l)!} \rho_b^{i-l} i^{-\alpha} \tag{4.74}$$

where $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)}$. Furthermore, we have

$$a_i \sim k_1 i^{-\alpha} \rho_b^i \quad (i \to \infty) \tag{4.75}$$

where $-(m+1) < \alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)} < -m$.

Proof. From Theorem 4.10, we know that for sufficient large i, the extinction probability $\{a_i\}$ is given by

$$a_i = k\sum_{l=0}^{m} \frac{(i+1)!}{(i+1-l)!}\int_0^{\rho_b} \frac{D_{m,l}(y)}{B_m(y)} y^{i+1-l} \exp\left\{\int_{\frac{\rho_b}{2}}^{y} \frac{A_m(x)}{B_m(x)}dx\right\}dy \tag{4.76}$$

for some constant k that is independent of i where $A_m(s)$, $B_m(s)$ are defined in (4.29) and (4.30) and the function $D_{m,l}(s)$ are defined recursively as

$$\begin{aligned} D_{1,0}(s) &= -A'(s); \quad D_{1,1}(s) = A(s), & (4.77)\\ D_{n,k}(s) &= D_{n-1,k-1}(s)A_{n-1}(s) - D_{n-1,k-1}(s)A'_{n-1}(s) \\ &\quad + D'_{n-1,k}(s)A_{n-1}(s) & (4.78)\\ D_{n,n}(s) &= \prod_{m=0}^{n-1} A_m(s) & (4.79) \end{aligned}$$

By (4.77) - (4.79), it is easily seen that all $D_{m,l}(s)$ are analytic function of s, as a power series of s, and thus bounded, particularly within $[0, \rho_b]$. It follows that the $\{a_i\}$ in (4.76) could be written as

$$a_i = \sum_{l=0}^{m} k_l \frac{(i+1)!}{(i+1-l)!}\int_0^{\rho_b} \frac{y^{i+1-l}}{B_m(y)} \exp\left\{\int_{\frac{\rho_b}{2}}^{y} \frac{A_m(x)}{B_m(x)}dx\right\}dy \tag{4.80}$$

where $\{k_0, k_1, \cdots, k_m\}$ are $(m+1)$ constants.

We note that

$$\frac{A_m(x)}{B_m(x)} = \frac{A_{m-1}(x)}{B_{m-1}(x)} + \frac{B'_{m-1}(x)}{B_{m-1}(x)} - \frac{A'_{m-1}(x)}{A_{m-1}(x)}. \tag{4.81}$$

Therefore,

$$\exp\left\{\int_{\frac{\rho_b}{2}}^{y} \frac{A_m(x)}{B_m(x)}dx\right\} = \exp\left\{\int_{\frac{\rho_b}{2}}^{y} \frac{A_{m-1}(x)}{B_{m-1}(x)}dx\right\}\frac{B_{m-1}(y)}{A_{m-1}(y)}\frac{A_{m-1}\left(\frac{\rho_b}{2}\right)}{B_{m-1}\left(\frac{\rho_b}{2}\right)}. \tag{4.82}$$

By repeating (4.81) and (4.82) and noticing that $\frac{A_{m-1}\left(\frac{\rho_b}{2}\right)}{B_{m-1}\left(\frac{\rho_b}{2}\right)}$ is just a constant, we get that

$$\exp\left\{\int_{\frac{\rho_b}{2}}^{y} \frac{A_m(x)}{B_m(x)}dx\right\} = k\exp\left\{\int_{\frac{\rho_b}{2}}^{y} \frac{A_0(x)}{B_0(x)}dx\right\}\frac{\prod_{l=0}^{m-1} B_l(y)}{\prod_{l=0}^{m-1} A_l(y)}. \tag{4.83}$$

By using (4.30). we could easily see that for any $i \geq 1$, $B_n(s) = B_0(s)\prod_{k=0}^{n-1} A_k(s)$.

Substituting this fact into (4.83), then substituting the result into (4.80), we obtain that there exist $(m+1)$ constants, again denoted by $\{k_0, k_1, \cdots, k_m\}$ such that

$$a_i = \sum_{l=0}^{m} k_l \frac{(i+1)!}{(i+1-l)!} \int_0^{\rho_b} y^{i+1-l} \frac{B_0(y) \prod_{k=0}^{m-1} B_k(y)}{(B_m(y))^2} \exp\left\{ \int_{\frac{\rho_b}{2}}^{y} \frac{A_0(x)}{B_0(x)} dx \right\}. \quad (4.84)$$

By mean value theorem and Lemma 4.6 (iv), we have

$$a_i = \sum_{l=0}^{m} k_l \frac{(i+1)!}{(i+l-l)!} \int_0^{\rho_b} y^{i+1-l} (\rho_b - y)^{\alpha} dy \quad (4.85)$$

where $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)} < 0$.

Using some transformation and Lemma 4.7, we can get

$$a_i = \sum_{l=0}^{m} k_l \frac{i!}{(i-l)!} {\rho_b}^{i-l} i^{-\alpha} \quad (i \to \infty)$$

where $\alpha = \frac{A(\rho_b)}{\rho_b B(\rho_b)}$ and (4.75) follows directly from (4.74).

■

At this point, we try to use two theorems to conclude the asymptotic behavior of the extinction probability. One is for the case $\rho_a < \rho_b = 1$ and one is for the case of $\rho_b < 1$.

Theorem 4.18 *Suppose $B'(1) = 0$, $B''(1) < 2A'(1)$, and $\rho_a < \rho_b = 1$ with $J = \int_0^1 \frac{R(y)}{B(y)} dy < +\infty$ where $R(y) = e^{\int_{\rho_a}^{y} \frac{A(x)}{xB(x)} dx}$, as $i \to +\infty$, we have*

$$a_i \sim k i^{-\alpha} \rho_b^i \quad (i \to \infty)$$

where $\alpha = \frac{2A'(1)}{B''(1)} - 1 > 0$ and $\rho_b = 1$ and k is independent of i.

Proof. See Theorem 4.12. ■

Theorem 4.19 *Suppose* $0 < B'(1) \leq +\infty$*, i.e.* $\rho_b < 1$*, then when* $i \to \infty$*,*

$$a_i \sim k i^{-\alpha} \rho_b^i \quad (i \to \infty)$$

where $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)}$ *and* $\rho_b < 1$ *and* k *is independent of* i*.*

Proof. It can be easily note that Theorem 4.13 to 4.17 are special cases for Theorem 4.19 with different values of $\alpha = \frac{A(\rho_b)}{\rho_b B'(\rho_b)}$. i.e. $\alpha > 0$ in Theorem 4.13, $\alpha = -1$ in Theorem 4.14, $-1 < \alpha < 0$ in Theorem 4.15, $\alpha = -m$ for some $m \geq 2$ in Theorem 4.16, $-(m+1) < \alpha < -m$ for some $m \geq 2$ in Theorem 4.17. ∎

Remark 4.4 *At this point, we have considered the Markov branching processes with immigration-migration and resurrection (BPIMR) in detail. BPIMR is an important model in Markov branching processes as it suggests immigration for rescuing a species from extinction. We have discussed about the model settings, uniqueness , extinction probability and its asymptotic behavior of BPIMR. In this chapter, we try the new methods, suggesting by the chapter about interacting branching collision processes (IBCP), in discussing the model properties especially for the part about extinction probability and its asymptotic behavior. Although the results for the two models are not the same, this chapter does give us a better understanding for the properties for BPIMR and the techniques we learnt in the chapter of IBCP.*

Chapter 5

Further Discussion on Markov Branching Processes with Collision, Immigration - Migration and Resurrection

5.1 Introduction

In this chapter, we discuss two generalized Markov branching processes. The first one is Interacting Collision Process with Immigration - Migration and Resurrection (ICIMR). The second one is Interacting Branching Collision Process with Immigration - Migration and Resurrection (IBCIMR).

In these 2 models, both state-independent and state-dependent immigration are considered. From Chapters 2 to 4, we have discussed Collision Branching Process, Interacting Collision Branching Process and Immigration- Migration and Resurrection Process. We will further discuss other related models.

The structure of this chapter is as follows. This chapter is divided into two major parts for ICIMR and IBCIMR. Some preliminary result for ICIMR are obtained in Section 2. Uniqueness and regularity criteria for ICIMR are then obtained in section 3. Section 4 discusses the details related to extinction probability for ICIMR if no resurrection is considered. Then, sections 5-7 follow the same structure for IBCIMR. Some preliminary result for IBCIMR are obtained in Section 5. Regularity criteria for IBCIMR are then obtained in section 6. Section 7 discusses the details related to extinction probability for IBCIMR if no resurrection is considered.

In this chapter, the state-independent immigration-migration and resurrection are considered together which is a generalized model of Li and Chen [2006].

5.2 Preliminary Settings for ICIMR

Definition 5.1 *A q-matrix $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ is called a branching q-matrix with interacting collision process with immigration-migration and resurrection, ICIMR q-matrix, if*

$$q_{ij} = \begin{cases} h_j, & \text{if} \quad i = 0, j \geq 0 \\ \binom{i}{2} c_{j-i+2} + a_{j-i+1}, & \text{if} \quad i \geq 1, j \geq i-2 \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

where

$$\begin{cases} h_j \geq 0\,(j \neq 0), \quad 0 \leq -h_0 = \sum_{j=1}^{\infty} h_j < \infty \\ a_j \geq 0\,(j \neq 1), \quad 0 \leq -a_1 = \sum_{j \neq 1} a_j < \infty \\ c_j \geq 0\,(j \neq 2), \quad 0 \leq -c_1 = \sum_{j \neq 2} c_j < \infty. \end{cases} \tag{5.2}$$

We should assume, through this chapter, $a_{-1} = 0$ and $\binom{1}{2} = 0$.

Definition 5.2 *An interacting collision process with immigration-migration-resurrection is a continuous-time Markov chain on the state space $\mathbf{Z}_+$ whose transition function $P(t) = (p_{ij}; i, j \in \mathbf{Z}_+)$ satisfies*

$$P'(t) = P(t)Q \tag{5.3}$$

where Q is given in (5.1) - (5.2).

In order to investigate properties of ICIMR, it is necessary to define the generating functions of the three know sequences $\{h_k; k \geq 0\}$, $\{a_k; k \geq 0\}$, $\{c_k; k \geq 0\}$ as

$$H(s) = \sum_{k=0}^{\infty} h_k s^k, \quad A(s) = \sum_{k=0}^{\infty} a_k s^k \quad \text{and} \quad C(s) = \sum_{k=0}^{\infty} c_k s^k.$$

These three functions play important role in our later analysis. It is clear that $H(s)$, $A(s)$ and $C(s)$ are well defined at least on $[-1, 1]$.

Lemma 5.1 *(i)If $h_0 < 0$, then $H(s) < 0$ for all $s \in [1, -1)$ and $\lim_{s\uparrow 1} H(s) = H(1) = 0$. If $h_0 = 0$, then $H(s) = 0$.*

(ii) The equation $A(s) = 0$ has at most two roots in $[0, 1]$. More specifically, if $A'(1) < 0$ then $A(s) > 0$ for all $s \in [-1, 1)$ and 1 is the only root of $A(s) = 0$ in $[0, 1)$. If $0 < A'(1) \leq +\infty$ then $A(s) = 0$ has an additional root in $[0, 1)$, denoted by ρ_a, such that $A(s) > 0$ for all $s \in [-1, \rho_b)$ and $A(s) < 0$ for $s \in (\rho_a, 1)$. Moreover, $A(z) = 0$ has no other root in the complex disk $\{z : |z| \leq 1\}$.

(iii) The equation $C(s) = 0$ has at most two roots in $[0, 1]$ and exactly one root in $[-1, 0)$. More specifically, if $C'(1) < 0$ then $C(s) > 0$ for all $s \in [0, 1)$ and 1 is the only root of the equation $C(s) = 0$ in $[0, 1]$, which is simple or with multiplicity 2 according to $C'(1) < 0$ or $C'(1) = 0$, while if $0 < C'(1) \leq +\infty$ then $C(s) = 0$ has an additional simple root ρ_c satisfying $0 < \rho_c < 1$ such that $C(s) > 0$ for $s \in (0, \rho_c)$ and $C(s) < 0$ for $s \in (\rho_c, 1)$. Also $C(s) = 0$ has exactly one root, denoted by $\zeta_c \in [-1, 0]$ such that $C(s) > 0$ for all $s \in (\zeta_c, 0]$ and $|\zeta_c| \leq \rho_c$. This root is simple unless $C'(1) = 0$ and $\sum_{k=0}^{\infty} c_{2k+1} = 0$. Also, $|\zeta_c| = \rho_c$ if and only if $\sum_{k=0}^{\infty} c_{2k+1} = 0$. Moreover, $C(z) = 0$ has no other root in the complex disk $\{z; |z| \leq 1\}$.

Proof. These preliminary proofs are similar in the chapters for CBP and IBP and thus omitted.

■

Throughout this chapter, we denote ρ_a and ρ_c as the smallest nonnegative roof of $A(s) = 0$ and $C(s) = 0$ respectively.

Lemma 5.2 *Let $P(t) = (p_{ij}; i, j \geq 0)$ and $\Phi(\lambda) = (\phi_{ij}(\lambda); i, j \geq 0)$ be the Feller minimal Q-function and Q-resolvent, respectively, where Q is given in (5.1) - (5.2). Then for any $i \geq 0$ and $s \in [0, 1)$,*

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = H(s)p_{i0}(t) + A(s)\sum_{k=1}^{\infty} p_{ik}(t)s^{k-1} + C(s)\sum_{k=2}^{\infty} p_{ik}(t)\binom{k}{2}s^{k-2} \quad (5.4)$$

or equivalently,

$$\lambda\sum_{j=0}^{\infty} \phi_{ij}(\lambda)s^j - s^i = H(s)\phi_{i0}(\lambda) + A(s)\sum_{k=1}^{\infty} \phi_{ik}(\lambda)s^{k-1} + C(s)\sum_{k=2}^{\infty} \phi_{ik}(\lambda)\binom{k}{2}s^{k-2}. \quad (5.5)$$

Proof. By the Kolmogorov forward equations in (5.3), we have

$$p'_{ij}(t) = p_{i0}(t)h_j + \sum_{k=1}^{j+1} p_{ik}(t)a_{j-k+1} + \sum_{k=2}^{j+2} p_{ik}\binom{k}{2}c_{j-k+2}$$

multiplying s^j and summing over $j \in Z_+$,

$$\begin{aligned}
\sum_{j=0}^{\infty} p'_{ij}(t)s^j &= p_{i0}(t)\sum_{j=0}^{\infty} h_j s^j + \sum_{j=0}^{\infty}\sum_{k=1}^{j+1} p_{ik}(t)a_{j-k+1}s^j + \sum_{j=0}^{\infty}\sum_{k=2}^{j+2} p_{ik}(t)\binom{k}{2}c_{j-k+2}s^j \\
&= H(s)p_{i0}(t) + \sum_{k=1}^{\infty}\sum_{j=k-1}^{\infty} p_{ik}(t)s^{k-1}a_{j-k+1}s^{j-k+1} \\
&\quad + \sum_{k=2}^{\infty}\sum_{j=k-2}^{\infty} p_{ik}(t)\binom{k}{2}s^{k-2}c_{j-k+2}s^{j-k+2} \\
&= H(s)p_{i0}(t) + \sum_{k=1}^{\infty} p_{ik}(t)s^{k-1}\sum_{j=0}^{\infty} a_j s^j \\
&\quad + \sum_{k=2}^{\infty} p_{ik}(t)s^{k-2}\sum_{j=0}^{\infty}\binom{k}{2}c_j s^j \\
&= H(s)p_{i0}(t) + A(s)\sum_{k=1}^{\infty} p_{ik}(t)s^{k-1} + C(s)\sum_{k=2}^{\infty} p_{ik}(t)\binom{k}{2}s^{k-2}.
\end{aligned}$$

Using Laplace transform, we get

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j - s^i = H(s)\phi_{i0}(\lambda) + A(s)\sum_{k=1}^{\infty} \phi_{ik}(\lambda) s^{k-1} + C(s)\sum_{k=2}^{\infty} \phi_{ik}(\lambda) \binom{k}{2} s^{k-2}.$$

The proof is then completed. ■

Lemma 5.3 *Let $P(t) = \{p_{ij}; i, j \geq 0\}$ be the Feller minimal Q-function where Q is given in (5.1) - (5.2).*

(i) Suppose that $h_0 = 0$. Then for any $i \geq 1$,

$$\int_0^{\infty} p_{ij}(t)dt < +\infty, (i, j \geq 1) \tag{5.6}$$

and thus

$$(ii) \lim_{t\to\infty} p_{ij}(t) = 0, \quad i \geq 1,\, j \geq 1. \tag{5.7}$$

(iii) For any $i \geq 1$ and $s \in [0, 1)$, we have

$$G_i(s) = \sum_{k=1}^{\infty} \left(\int_0^{\infty} p_{ik}(t)dt \right) s^k < +\infty. \tag{5.8}$$

Proof.

(i) We will make use of the irreducibility of positive states. From Kolmogorov forward equation

$$p'_{i0}(t) = p_{i2}(t)c_0 + p_{i1}(t)a_0.$$

Integrating the above equation, we can get $\int_0^{\infty} p_{i1}(t)dt < +\infty$ and $\int_0^{\infty} p_{i2}(t)dt < +\infty$ for all $i \geq 1$ since

$$a_0, c_0 > 0.$$

Hence by the irreducibility of positive states we know that

$$\int_0^{\infty} p_{ij}(t)dt < +\infty \text{ for all } i, j \geq 1.$$

(ii) is directly followed from (i).

(iii) From (5.4), we have

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = A(s)\sum_{k=1}^{\infty} p_{ik}(t)s^{k-1} + C(s)\sum_{k=2}^{\infty} p_{ik}(t)\binom{k}{2}s^{k-2}$$

which can be rewritten as

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = \sum_{k=1}^{\infty}[sA(s) + \frac{C(s)}{2}k(k-1)]p_{ik}(t)s^{k-2}.$$

We separate this problem into two situations, $C'(1) \leq 0$ and $0 < C'(1) < \infty$.

If $C'(1) \leq 0$, we have $C(s) > 0$ for all $s \in [0, 1)$

There exists a $\tilde{k} \geq 2$ such that $\frac{k(k-1)C(s)}{2} + sA(s) > 0$ for any $k \geq \tilde{k}$. Then, we obtain

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j \geq \sum_{k=1}^{\tilde{k}-1}[sA(s) + \frac{C(s)}{2}k(k-1)]p_{ik}(t)s^{k-2} + \sum_{k=\tilde{k}}^{\infty}[sA(s) + \frac{C(s)}{2}\tilde{k}(\tilde{k}-1)]p_{ik}(t)s^{k-2}$$

Taking integration in the above inequality yields

$$\begin{aligned}[sA(s) + \frac{\tilde{k}(\tilde{k}-1)C(s)}{2}]\sum_{k=\tilde{k}}^{\infty}(\int_0^{\infty} p_{ik}(t)dt)s^{k-2} \quad &\leq \quad \lim_{t\to\infty} p_{i0}(t) - s^i \\ &- \sum_{k=1}^{\tilde{k}-1}[sA(s) + \frac{k(k-1)C(s)}{2}]\int_0^{\infty} p_{ik}(t)dts^{k-2}.\end{aligned}$$

With some simple calculation,we can get the result.

If $0 < C'(1) \leq +\infty$, we know that $C(s) = 0$ has a smallest nonnegative root $\rho_c \in [0, 1)$ such that $C(s) < 0$ for any $s \in (\rho_c, 1)$.

Now, for any $\tilde{s} \in (\rho_c, 1)$, there exists a $\tilde{k} \geq 2$ such that

$$\frac{k(k-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s}) < 0$$

for any $k \geq \tilde{k}$.

Then, we obtain

$$\begin{aligned}
\sum_{j=0}^{\infty} p'_{ij}(t)\tilde{s}^j &= \sum_{k=\tilde{k}}^{\infty}[\frac{k(k-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s})]p_{ik}(t)\tilde{s}^{k-2} \\
&\quad + \sum_{k=1}^{\tilde{k}-1}[\frac{k(k-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s})]p_{ik}(t)\tilde{s}^{k-2} \\
&\le [\frac{\tilde{k}(\tilde{k}-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s})]\sum_{k=\tilde{k}}^{\infty} p_{ik}(t)\tilde{s}^{k-2} \\
&\quad + \sum_{k=1}^{\tilde{k}-1}[\frac{k(k-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s})]p_{ik}(t)\tilde{s}^{k-2}.
\end{aligned}$$

Integrating the above inequality yields

$$[\frac{\tilde{k}(\tilde{k}-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s})]\sum_{k=\tilde{k}}^{\infty}\int_0^{\infty} p_{ik}(t)dt\tilde{s}^{k-2}$$

$$\ge \lim_{t\to\infty} p_{i0}(t) - \tilde{s}^i - \sum_{k=1}^{\tilde{k}-1}[\tilde{s}A(\tilde{s}) + \frac{k(k-1)C(\tilde{s})}{2}]\int_0^{\infty} p_{ik}(t)dt\tilde{s}^{k-2} \ge -\infty$$

which implies (iii) since $\frac{\tilde{k}(\tilde{k}-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s}) < 0$. The proof is then completed.

■

5.3 Uniqueness Criteria for ICIMR

Now, we are ready to consider the regularity and uniqueness criteria for the Interacting Collision Process with Immigration - Migration and Resurrection (ICIMR).

Theorem 5.1 *An ICIMR q-matrix Q is regular if and only if $C'(1) < 0$.*

Proof.

Without loss of generality, $H(s) = 0$ in the following proof.

If part:

Suppose $C'(1) \leq 0$ and $A'(1) \leq 0$, then $A(s)$ and $C(s)$ are both positive for $s \in [0,1)$. By (5.5), for $s \in [0,1)$,

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j \geq s^i.$$

Let $s \uparrow 1$ yields that

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) \quad \geq \quad 1$$

which implies

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) \quad = \quad 1$$

The Feller minimal Q-process is honest and Q is regular.

Next, suppose $C'(1) \leq 0$, if $0 < A'(1) < +\infty$, again by (5.5), for $s \in [0,1)$, $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j - s^i \geq A(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) s^{k-1}$.

If Q is not regular, there exists an $i \geq 0$ and a $\lambda > 0$ such that $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) < 1$.

Hence, there exists a $\delta > 0$ and $\tilde{s} \in (\rho_a, 1)$ such that for $s \in [\tilde{s}, 1]$, we have $s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j > \delta$.

By the above inequalities together and note that $A(s) < 0$ for $\tilde{s} \in (\rho_a, 1)$, we can obtain

$$\begin{aligned} \delta \quad &\leq \quad s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j \leq -A(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) s^{k-1} \\ \frac{\delta}{-A(s)} \quad &\leq \quad \frac{s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j}{-A(s)} \leq \sum_{k=1}^{\infty} \phi_{ik}(\lambda) s^{k-1} \qquad (5.9) \end{aligned}$$

for $s \in [\tilde{s}, 1)$.

Therefore,

$$\sum_{k=1}^{\infty} \phi_{ik}(\lambda) \frac{(1-\tilde{s}^k)}{k} \geq \int_{\tilde{s}}^{1} \frac{\delta}{-A(s)} ds = +\infty$$

which is a contradiction and hence Q is regular.

Only if part: Suppose on the contrary, if Q is regular, with $0 < C'(1) \leq +\infty$. By a similar argument as in Chen et al. [2004] and Chen et al. [2012]. This part can be easily proved.

■

We now have the regularity criterion of a ICIMR q-matrix Q. How about if the q-matrix Q is not regular? The next theorem shows that there still exists only one Q-function which can satisfy the Kolmogorov forward equations.

Theorem 5.2 *There exists only one ICIMR which satisfies the Kolmogorov forward equations.*

Proof. We only need to consider the case $C'(1) > 0$. To prove that the ICIMR is unique, we show that the forward equation has a unique solution. To show this, we will use Theorem 2.8 in Anderson [1991], i.e. we need to prove that the equation

$$Y(\lambda I - Q) = 0 \tag{5.10}$$

has no non-trivial solutions for some and (therefore for all) $\lambda > 0$, $Y \geq 0$ and $YI < +\infty$, where I denotes the column vector on $\mathbf{Z}_+$ whose all components are equal to 1. Suppose that $\{Y = y_i; i \geq 0\}$ is a solution of (5.10) for $\lambda = 1$. (5.10) can be rewritten as

$$y_n = y_0 h_n + \sum_{j=1}^{n+1} y_j (a_{n-j+1}) + \sum_{j=2}^{n+2} y_j \binom{j}{2} c_{n-j+2}, \quad n \geq 0.$$

Multiplying both sides of the above equation by s^n, summing over $n \geq 0$,

$$\begin{aligned}\sum_{n=0}^{\infty} y_n s^n &= \sum_{n=0}^{\infty} y_0 h_n s^n + \sum_{n=0}^{\infty}\sum_{j=1}^{n+1} a_{n-j+1} y_j s^n + \sum_{n=0}^{\infty}\sum_{j=2}^{n+2} \binom{j}{2} c_{n-j+2} y_i s^n \\ &= Y_h + Y_a + Y_c\end{aligned}$$

$$Y_h = \sum_{n=0}^{\infty} y_0 h_n s^n = y_0 H(s)$$

$$\begin{aligned}Y_a &= \sum_{n=0}^{\infty}\sum_{j=1}^{n+1} a_{n-j+1} y_j s^n \\ &= \sum_{j=1}^{\infty}\sum_{n=j-1}^{\infty} a_{n-j+1} y_j s^n \\ &= \sum_{j=1}^{\infty} y_j s^{j-1} \sum_{n=j-1}^{\infty} a_{n-j+1} s^{n-j+1} \\ &= A(s) \sum_{n=1}^{\infty} y_n s^{n-1}\end{aligned}$$

$$\begin{aligned}Y_c &= \sum_{n=0}^{\infty}\sum_{j=2}^{n+2} \binom{j}{2} c_{n-j+2} y_j s^n \\ &= \sum_{j=2}^{\infty}\sum_{n=j-2}^{\infty} \binom{j}{2} c_{n-j+2} y_j s^n \\ &= \sum_{j=2}^{\infty} \binom{j}{2} y_j s^{j-2} \sum_{n=j-2}^{\infty} c_{n-j+2} s^{n-j+2} \\ &= C(s) \sum_{n=2}^{\infty} \binom{n}{2} y_n s^{n-2}\end{aligned}$$

We have

$$Y(s) = y_0 H(s) + A(s) \sum_{n=1}^{\infty} y_n s^{n-1} + \frac{C(s)}{2} \sum_{n=2}^{\infty} y_n n(n-1) s^{n-2} \quad |s| < 1.$$

$$y_0(1 - H(s)) + (s - A(s)) \sum_{n=1}^{\infty} y_n s^{n-1} = \frac{C(s)}{2} \sum_{n=2}^{\infty} y_n n(n-1) s^{n-2}, \quad |s| < 1. \quad (5.11)$$

Since $C'(1) > 0$, $C(s) = 0$ has a root $\rho_c \in [0, 1)$ and $C(s) < 0$ for all $s \in (\rho_c, 1)$. Since $A(1) = 0$, and $A(s)$ is continuous in $[0, 1]$. There exists an $\varepsilon \in (\rho_c, 1)$ such that $A(s) \leq s$ for all $s \in (\varepsilon, 1)$. We can see (5.11) in $(\varepsilon, 1)$, looking at the sign of coefficient of y_n $(n \geq 0)$. $y_n = 0$ $(n \geq 0)$ is proved and so is this theorem.

■

5.4 Extinction Probability for ICIMR

Throughout this section, we will always assume that $h_0 = 0$ and thus the state 0 is an absorbing state. This helps us in considering the property regarding the extinction probability.

Let $\{X(t); t \geq 0\}$ be the unique ICIMR, and let $P(t) = \{p_{ij}(t)\}$ denotes its transition function. Define the extinction time by

$$\tau_0 = \begin{cases} \inf\{t > 0, X(t) = 0\}, & \text{if} \quad X(t) = 0 \quad \text{for some} \quad t > 0, \\ +\infty & \text{if} \quad X(t) \neq 0 \quad \text{for all} \quad t > 0. \end{cases}$$

Denote the extinction probability by

$$a_i = P(\tau_0 < \infty | X(0) = i), i \geq 1.$$

We shall consider some absorbing behavior of ICIMR and the difficulty in evaluation in this section. As a preparation, we first provide some more settings and notations.

From (5.4), with $H(s) = 0$, we have

$$\sum_{j=0}^{\infty} p'_{ij}(t) s^j = A(s) \sum_{k=1}^{\infty} p_{ik}(t) s^{k-1} + \frac{C(s)}{2} \sum_{k=2}^{\infty} p_{ik}(t) k(k-1) s^{k-2}. \tag{5.12}$$

Integrating the above equality with respect to $t \in [0, \infty)$ with
$G_i(s) = \sum_{k=1}^{\infty} (\int_0^{\infty} p_{ik}(t) dt) s^k$, we have

$$\frac{C(s)}{2} G''_i(s) + \frac{A(s)}{s} G_i(s) = a_i - s^i, \quad s \in [0, 1) \tag{5.13}$$

$$\frac{sC(s)}{2}G_i''(s) + A(s)G_i(s) = s(a_i - s^i), \quad s \in [0,1) \tag{5.14}$$

ρ_a and ρ_c is the smallest nonnegative root of $A(s) = 0$ and $C(s) = 0$ respectively.

Theorem 5.3 *Suppose that $C'(1)$ and $A'(1) \leq 0$. Then $a_i = 1$, $(i \geq 1)$.*

Proof. From (5.13)

$$\frac{sC(s)}{2}G_i''(s) + A(s)G_i(s) = s(a_i - s^i),$$

$C(s) > 0$, $A(s) > 0$ for $s \in [0,1)$. Therefore, $a_i - s^i \geq 0$. Let $s \to 1$, we have $a_i \geq 1$. But $a_i \leq 1$ is always true and thus $a_i = 1$. ■

Theorem 5.4 *Suppose that $\rho_a = \rho_c$. Then $a_i = \rho_a^i, (i \geq 1)$.*

Proof. Putting $s = \rho_a$ into (5.13), we get $a_i = \rho_a^i$. ■

At this moment, we have found the extinction probability, a_i, for ICIMR for the case of (1). both $C'(1)$ and $A'(1) \leq 0$ and (2). $\rho_a = \rho_c$.

In order to get the extinction probability for other cases, we need to solve the differential equation (5.13). However, as the order of the differential equation is more than 1. There is a need to know more about the transition function $P(t) = (p_{ij}; i,j \in \mathbf{Z}_+)$. After having such information, we may be able to find the extinction probability, a_i, using numerical methods.

After studying Interacting Collision Process with Immigration - Migration and Resurrection (ICIMR), we now discuss another related model, Interacting Branching Collision Process with Immigration - Migration and Resurrection (IBCIMR)

5.5 Preliminary Settings for IBCIMR

Definition 5.3 *A q-matrix $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ is called a branching q-matrix with interacting branching collision process with immigration-migration and resurrection, IBCIMR q-matrix, if*

$$q_{ij} = \begin{cases} h_j, & \text{if} \quad i = 0, j \geq 0 \\ \binom{i}{2} c_{j-i+2} + a_{j-i+1} + i b_{j-i+1}, & \text{if} \quad i \geq 1, j \geq i-2 \\ 0, & \text{otherwise} \end{cases} \tag{5.15}$$

where

$$\begin{cases} h_j \geq 0 \, (j \neq 0), \quad 0 \leq -h_0 = \sum_{j=1}^{\infty} h_j < \infty \\ a_j \geq 0 \, (j \neq 1), \quad 0 \leq -a_1 = \sum_{j \neq 1} a_j < \infty \\ b_j \geq 0 \, (j \neq 1), \quad 0 \leq -b_1 = \sum_{j \neq 1} b_j < \infty \\ c_j \geq 0 \, (j \neq 2), \quad 0 \leq -c_1 = \sum_{j \neq 2} c_j < \infty. \end{cases} \tag{5.16}$$

We should assume, through this chapter, $a_{-1} = 0$ and $\binom{1}{2} = 0$.

Definition 5.4 *An interacting branching collision process with immigration-migration and resurrection is a continuous-time Markov chain on the state space $\mathbf{Z}_+$ whose transition function $P(t) = (p_{ij}; i, j \in \mathbf{Z}_+)$ satisfies*

$$P'(t) = P(t)Q \tag{5.17}$$

where Q is given in (5.15) - (5.16).

In order to investigate properties of IBCIMR, it is necessary to define the generating functions of the four know sequences $\{h_k; k \geq 0\}$, $\{a_k; k \geq 0\}$, $\{b_k; k \geq 0\}$ and $\{c_k; k \geq 0\}$ as

$$H(s) = \sum_{k=0}^{\infty} h_k s^k, \quad A(s) = \sum_{k=0}^{\infty} a_k s^k, \quad B(s) = \sum_{k=0}^{\infty} b_k s^k \quad \text{and} \quad C(s) = \sum_{k=0}^{\infty} c_k s^k.$$

These four functions play important role in our later analysis. It is clear that $H(s)$, $A(s)$, $B(s)$ and $C(s)$ are well defined at leat on $[-1, 1]$.

Lemma 5.4 *(i) if $h_0 < 0$, then $H(s) < 0$ for all $s \in [1, -1)$ and $\lim_{s\uparrow 1} H(s) = H(1) = 0$. If $h_0 = 0$, then $H(s) = 0$.*

(ii) The equation $A(s) = 0$ has at most two roots in $[0, 1]$. More specifically, if $A'(1) < 0$ then $A(s) > 0$ for all $s \in [-1, 1)$ and 1 is the only root of $A(s) = 0$ in $[0, 1)$. If $0 < A'(1) \leq +\infty$ then $A(s) = 0$ has an additional root in $[0, 1)$, denoted by ρ_a, such that $A(s) > 0$ for all $s \in [-1, \rho_b)$ and $A(s) < 0$ for $s \in (\rho_a, 1)$. Moreover, $A(z) = 0$ has no other root in the complex disk $\{z : |z| \leq 1\}$. Same property holds for $B(s)$.

(iii) The equation $C(s) = 0$ has at most two roots in $[0, 1]$ and exactly one root in $[-1, 0)$. More specifically, if $C'(1) < 0$ then $C(s) > 0$ for all $s \in [0, 1)$ and 1 is the only root of the equation $C(s) = 0$ in $[0, 1]$, which is simple or with multiplicity 2 according to $C'(1) < 0$ or $C'(1) = 0$, while if $0 < C'(1) \leq +\infty$ then $C(s) = 0$ has an additional simple root ρ_c satisfying $0 < \rho_c < 1$ such that $C(s) > 0$ for $s \in (0, \rho_c)$ and $C(s) < 0$ for $s \in (\rho_c, 1)$. Also $C(s) = 0$ has exactly one root, denoted by $\zeta_c \in [-1, 0]$ such that $C(s) > 0$ for all $s \in (\zeta_c, 0]$ and $|\zeta_c| \leq \rho_c$. This root is simple unless $C'(1) = 0$ and $\sum_{k=0}^{\infty} c_{2k+1} = 0$. Also, $|\zeta_c| = \rho_c$ if and only if $\sum_{k=0}^{\infty} c_{2k+1} = 0$. Moreover, $C(z) = 0$ has no other root in the complex disk $\{z; |z| \leq 1\}$.

Proof. These preliminary proofs are similar in the chapters for CBP and IBP and thus omitted.

■

Throughout this chapter, we denote ρ_a , ρ_b and ρ_c as the smallest nonnegative roof of $A(s) = 0$, $B(s) = 0$ and $C(s) = 0$ respectively.

Lemma 5.5 *Let $P(t) = (p_{ij}; i, j \geq 0)$ and $\Phi(\lambda) = (\phi_{ij}(\lambda); i, j \geq 0)$ be the Feller minimal Q-function and Q-resolvent, respectively, where Q is given in (5.15) -*

(5.16). Then for any $i \geq 0$ and $s \in [0, 1)$,

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = H(s)p_{i0}(t) + A(s)\sum_{k=1}^{\infty} p_{ik}(t)s^{k-1} + B(s)\sum_{k=1}^{\infty} kp_{ik}(t)s^{k-1} + C(s)\sum_{k=2}^{\infty} p_{ik}(t)\binom{k}{2}s^{k-2} \tag{5.18}$$

or equivalently,

$$\begin{aligned} \lambda\sum_{j=0}^{\infty} \phi_{ij}(\lambda)s^j - s^i &= H(s)\phi_{i0}(\lambda) + A(s)\sum_{k=1}^{\infty} \phi_{ik}(\lambda)s^{k-1} \\ &\quad + B(s)\sum_{k=1}^{\infty} \phi_{ik}(\lambda)ks^{k-1} + C(s)\sum_{k=2}^{\infty} \phi_{ik}(\lambda)\binom{k}{2}s^{k-2} \end{aligned} \tag{5.19}$$

Proof. By the Kolmogorov forward equations in (5.17), we have

$$p'_{ij}(t) = p_{i0}(t)h_j + \sum_{k=1}^{j+1} p_{ik}(t)a_{j-k+1} + \sum_{k=1}^{j+1} p_{ik}(t)kb_{j-k+1} + \sum_{k=2}^{j+2} p_{ik}\binom{k}{2}c_{j-k+2}$$

multiplying s^j and summing over $j \in Z_+$,

$$\begin{aligned} \sum_{j=0}^{\infty} p'_{ij}(t)s^j &= p_{i0}(t)\sum_{j=0}^{\infty} h_j s^j + \sum_{j=0}^{\infty}\sum_{k=1}^{j+1} p_{ik}(t)a_{j-k+1}s^j \\ &\quad + \sum_{j=0}^{\infty}\sum_{k=1}^{j+1} p_{ik}(t)kb_{j-k+1}s^j + \sum_{j=0}^{\infty}\sum_{k=2}^{j+2} p_{ik}(t)\binom{k}{2}c_{j-k+2}s^j \\ &= H(s)p_{i0}(t) + \sum_{k=1}^{\infty}\sum_{j=k-1}^{\infty} p_{ik}(t)s^{k-1}a_{j-k+1}s^{j-k+1} \\ &\quad + \sum_{k=1}^{\infty}\sum_{j=k-1}^{\infty} p_{ik}(t)s^{k-1}kb_{j-k+1}s^{j-k+1} + \sum_{k=2}^{\infty}\sum_{j=k-2}^{\infty} p_{ik}(t)\binom{k}{2}s^{k-2}c_{j-k+2}s^{j-k+2} \\ &= H(s)p_{i0}(t) + \sum_{k=1}^{\infty} p_{ik}(t)s^{k-1}\sum_{j=0}^{\infty} a_j s^j \\ &\quad + \sum_{k=1}^{\infty} kp_{ik}(t)s^{k-1}\sum_{j=0}^{\infty} b_j s^j + \sum_{k=2}^{\infty} p_{ik}(t)s^{k-2}\sum_{j=0}^{\infty}\binom{k}{2}c_j s^j \\ &= H(s)p_{i0}(t) + A(s)\sum_{k=1}^{\infty} p_{ik}(t)s^{k-1} + B(s)\sum_{k=1}^{\infty} kp_{ik}(t)s^{k-1} + C(s)\sum_{k=2}^{\infty} p_{ik}(t)\binom{k}{2}s^{k-2}. \end{aligned}$$

Using Laplace transform, we get

$$\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)s^j - s^i = H(s)\phi_{i0}(\lambda) + A(s)\sum_{k=1}^{\infty}\phi_{ik}(\lambda)s^{k-1}$$
$$+B(s)\sum_{k=1}^{\infty}\phi_{ik}(\lambda)ks^{k-1} + C(s)\sum_{k=2}^{\infty}\phi_{ik}(\lambda)\binom{k}{2}s^{k-2}.$$

The proof is then completed. ■

Lemma 5.6 *Let $P(t) = \{p_{ij}; i,j \geq 0\}$ be the Feller minimal Q-function where Q is given in (5.15) - (5.16).*

(i) Suppose that $h_0 = 0$. Then for any $i \geq 1$,

$$\int_0^{\infty} p_{ij}(t)dt < +\infty, (\quad i,j \geq 1) \tag{5.20}$$

and thus

$$(ii) \lim_{t\to\infty} p_{ij}(t) = 0, \quad i \geq 1,\ j \geq 1. \tag{5.21}$$

(iii) For any $i \geq 1$ and $s \in [0,1)$, we have

$$G_i(s) = \sum_{k=1}^{\infty}\left(\int_0^{\infty} p_{ik}(t)dt\right)s^k < +\infty. \tag{5.22}$$

Proof.

(i) We will make use of the irreducibility of positive states. From Kolmogorov forward equation

$$p'_{i0}(t) = p_{i1}(t)a_0 + p_{i1}(t)b_0 + p_{i2}(t)c_0.$$

Integrating the above equation, we can get

$$\int_0^{\infty} p_{i1}(t)dt < +\infty \text{ and } \int_0^{\infty} p_{i2}(t)dt < +\infty$$

for all $i \geq 1$ since

$$a_0, b_0, c_0 > 0.$$

Hence by the irreducibility of positive states we know that

$$\int_0^\infty p_{ij}(t)dt < +\infty \text{ for all } i, j \geq 1.$$

(ii) is directly followed from (i).

(iii) From (5.18), we have

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = A(s)\sum_{k=1}^{\infty} p_{ik}(t)s^{k-1} + B(s)\sum_{k=1}^{\infty} kp_{ik}(t)s^{k-1} + C(s)\sum_{k=2}^{\infty} p_{ik}(t)\binom{k}{2}s^{k-2}$$

which can be rewritten as

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = \sum_{k=1}^{\infty}[sA(s) + sB(s)k + \frac{C(s)}{2}k(k-1)]p_{ik}(t)s^{k-2}.$$

We separate this problem into two situations, $C'(1) \leq 0$ and $0 < C'(1) < \infty$.

If $C'(1) \leq 0$, we have $C(s) > 0$ for $s \in [0, 1)$.

There exists a $\tilde{k} \geq 2$ such that $\frac{k(k-1)C(s)}{2} + sA(s) + sB(s)k > 0$ for any $k \geq \tilde{k}$. Then, we obtain

$$\begin{aligned}\sum_{j=0}^{\infty} p'_{ij}(t)s^j \geq & \sum_{k=1}^{\tilde{k}-1}[sA(s) + sB(s)k + \frac{C(s)}{2}k(k-1)]p_{ik}(t)s^{k-2} \\ & + \sum_{k=\tilde{k}}^{\infty}[sA(s) + sB(s)\tilde{k} + \frac{C(s)}{2}\tilde{k}(\tilde{k}-1)]p_{ik}(t)s^{k-2}\end{aligned}$$

Taking integration in the above inequality yields

$$\begin{aligned}&[sA(s) + sB(s)\tilde{k} + \frac{\tilde{k}(\tilde{k}-1)C(s)}{2}]\sum_{k=\tilde{k}}^{\infty}(\int_0^\infty p_{ik}(t)dt)s^{k-2} \\ \leq & \lim_{t\to\infty} p_{i0}(t) - s^i - \sum_{k=1}^{\tilde{k}-1}[sA(s) + sB(s)k + \frac{k(k-1)C(s)}{2}](\int_0^\infty p_{ik}(t)dt)s^{k-2}.\end{aligned}$$

With some simple calculation,we can get the result.

If $0 < C'(1) \leq +\infty$, we know that $C(s) = 0$ has a smallest nonnegative root $\rho_c \in [0, 1)$ such that $C(s) < 0$ for any $s \in (\rho_c, 1)$.

Now, for any $\tilde{s} \in (\rho_c, 1)$, there exists a $\tilde{k} \geq 2$ such that

$$\frac{k(k-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s}) + \tilde{s}B(\tilde{s})k < 0$$

for any $k \geq \tilde{k}$.

Then, we obtain

$$\begin{aligned}\sum_{j=0}^{\infty} p'_{ij}(t)\tilde{s}^j &= \sum_{k=\tilde{k}}^{\infty}[\frac{k(k-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s}) + \tilde{s}B(\tilde{s})k]p_{ik}(t)\tilde{s}^{k-2} \\ &\quad + \sum_{k=1}^{\tilde{k}-1}[\frac{k(k-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s}) + \tilde{s}B(\tilde{s})k]p_{ik}(t)\tilde{s}^{k-2} \\ &\leq [\frac{\tilde{k}(\tilde{k}-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s}) + \tilde{s}B(\tilde{s})\tilde{k}]\sum_{k=\tilde{k}}^{\infty} p_{ik}(t)\tilde{s}^{k-2} \\ &\quad + \sum_{k=1}^{\tilde{k}-1}[\frac{k(k-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s}) + \tilde{s}B(\tilde{s})k]p_{ik}(t)\tilde{s}^{k-2}.\end{aligned}$$

Integrating the above inequality yields

$$[\frac{\tilde{k}(\tilde{k}-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s}) + \tilde{s}B(\tilde{s})k]\sum_{k=\tilde{k}}^{\infty}\int_0^{\infty} p_{ik}(t)dt\tilde{s}^{k-2}$$

$$\geq \lim_{t\to\infty} p_{i0}(t) - \tilde{s}^i - \sum_{k=1}^{\tilde{k}-1}[\frac{k(k-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s}) + \tilde{s}B(\tilde{s})k]\int_0^{\infty} p_{ik}(t)dt\tilde{s}^{k-2} \geq -\infty$$

which implies (iii) since $\frac{\tilde{k}(\tilde{k}-1)C(\tilde{s})}{2} + \tilde{s}A(\tilde{s}) + \tilde{s}B(\tilde{s})k < 0$. The proof is then completed. ∎

5.6 Regularity Criteria for IBCIMR

Now, we are ready to consider the regularity criteria for the Interacting Branching Collision Process with Immigration - Migration and Resurrection (IBCIMR).

Theorem 5.5 *An IBCIMR q-matrix Q is regular if and only if $C'(1) < 0$.*

Proof.

Without loss of generality, $H(s) = 0$ in the following proof.

If part: Suppose $C'(1) \leq 0$,if $A'(1) \leq 0$ and $B'(1) \leq 0$, then $A(s)$,$B(s)$ and $C(s)$ are both positive for all $s \in [0, 1)$. By (5.19), for $s \in [0, 1)$,

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j \geq s^i.$$

Let $s \uparrow 1$ yields that

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) \quad \geq \quad 1$$

which implies

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) \quad = \quad 1$$

The Feller minimal Q-process is honest and Q is regular.

Next, suppose $C'(1) \leq 0$, if $0 < A'(1) < +\infty$ and $0 < B'(1) < +\infty$, again by (5.19), for $s \in [0, 1)$,

$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j - s^i \geq A(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) s^{k-1} + B(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) k s^{k-1}$.

If Q is not regular, there exists an $i \geq 0$, and a $\lambda > 0$ such that $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) < 1$.

Hence, there exists a $\delta > 0$ and $\tilde{s} \in (\rho_a \vee \rho_b, 1)$ such that for $s \in [\tilde{s}, 1]$, we have

$s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j > \delta$.

By the above inequalities together and note that $A(s) < 0$ and $B(s) < 0$ for all $\tilde{s} \in (\rho_a \vee \rho_b, 1)$, we can obtain

$$\delta \quad \leq \quad s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j \leq -A(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) s^{k-1} - B(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) k s^{k-1} \\ . \tag{5.23}$$

With some simple calculation, we can get $\lambda \sum_{j=0}^{\infty} \phi_{ij} \geq +\infty$ which is a contradiction and hence Q is regular.

Next,suppose $C'(1) \leq 0$, if $B'(1) \leq 0$ and $0 < A'(1) < +\infty$, again by (5.19), for $s \in [0, 1)$,

$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j - s^i \geq A(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) s^{k-1}$.

If Q is not regular, there exists an $i \geq 0$, and a $\lambda > 0$ such that $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) <$ 1.

Hence, there exists a $\delta > 0$ and $\tilde{s} \in (\rho_a, 1)$ such that for $s \in [\tilde{s}, 1]$, we have

$s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j > \delta$.

By the above inequalities together and note that $A(s) < 0$ for $\tilde{s} \in (\rho_a, 1)$, we can obtain

$$\begin{aligned} \delta &\leq s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j \leq -A(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) s^{k-1} \\ \frac{\delta}{-A(s)} &\leq \frac{s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j}{-A(s)} \leq \sum_{k=1}^{\infty} \phi_{ik}(\lambda) s^{k-1} \end{aligned} \tag{5.24}$$

for all $s \in [\tilde{s}, 1)$.

Therefore,

$$\sum_{k=1}^{\infty} \phi_{ik}(\lambda) \frac{(1 - \tilde{s}^k)}{k} \geq \int_{\tilde{s}}^{1} \frac{\delta}{-A(s)} ds = +\infty$$

which is a contradiction and hence Q is regular.

Similarly, suppose $C'(1) \leq 0$, if $A'(1) \leq 0$ and $0 < B'(1) < +\infty$, again by (5.19), for $s \in [0, 1)$,

$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j - s^i \geq B(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) k s^{k-1}$.

If Q is not regular, there exists an $i \geq 0$, and a $\lambda > 0$ such that $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) <$ 1.

Hence, there exists a $\delta > 0$ and $\tilde{s} \in (\rho_b, 1)$ such that for $s \in [\tilde{s}, 1]$, we have

$s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j > \delta$.

By the above inequalities together and note that $B(s) < 0$ for $\tilde{s} \in (\rho_b, 1)$, we can obtain

$$\begin{aligned} \delta &\leq s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j \leq -B(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) k s^{k-1} \\ \frac{\delta}{-B(s)} &\leq \frac{s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j}{-B(s)} \leq \sum_{k=1}^{\infty} \phi_{ik}(\lambda) k s^{k-1} \end{aligned} \quad (5.25)$$

for $s \in [\tilde{s}, 1)$.

Therefore,

$$\sum_{k=1}^{\infty} \phi_{ik}(\lambda)(1 - \tilde{s}^k) \geq \int_{\tilde{s}}^{1} \frac{\delta}{-B(s)} ds = +\infty$$

which is a contradiction and hence Q is regular.

Only if part: Suppose on the contrary, if Q is regular, with $0 < C'(1) \leq +\infty$. By a similar argument as in Chen et al. [2004] and Chen et al. [2012]. This part can be easily proved.

■

5.7 Extinction Probability for IBCIMR

Throughout this section, we will always assume that $h_0 = 0$ and thus the state 0 is an absorbing state. This helps us in considering the property regarding the extinction probability.

Let $\{X(t); t \geq 0\}$ be the IBCIMR, and let $P(t) = \{p_{ij}(t)\}$ denotes its transition function. Define the extinction time by

$$\tau_0 = \begin{cases} \inf\{t > 0, X(t) = 0\}, & \text{if} \quad X(t) = 0 \quad \text{for some} \quad t > 0, \\ +\infty & \text{if} \quad X(t) \neq 0 \quad \text{for all} \quad t > 0. \end{cases}$$

Denote the extinction probability by

$$a_i = P(\tau_0 < \infty | X(0) = i), i \geq 1.$$

We shall consider some absorbing behavior of IBCIMR and the difficulty in evaluation in this section. As a preparation, we first provide some more settings and notations.

From (5.18), with $H(s) = 0$, we have

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = A(s)\sum_{k=1}^{\infty} p_{ik}(t)s^{k-1} + B(s)\sum_{k=1}^{\infty} kp_{ik}(t)s^{k-1} + \frac{C(s)}{2}\sum_{k=2}^{\infty} p_{ik}(t)k(k-1)s^{k-2}. \tag{5.26}$$

Integrating the above equality with respect to $t \in [0, \infty)$ with $G_i(s) = \sum_{k=1}^{\infty}(\int_0^{\infty} p_{ik}(t)dt)s^k$, we have

$$\frac{C(s)}{2}G''_i(s) + B(s)G'_i(s) + \frac{A(s)}{s}G_i(s) = a_i - s^i, \quad s \in [0, 1) \tag{5.27}$$

$$\frac{sC(s)}{2}G''_i(s) + sB(s)G'_i(s) + A(s)G_i(s) = s(a_i - s^i), \quad s \in [0, 1) \tag{5.28}$$

ρ_a, ρ_b and ρ_c is the smallest nonnegative root of $A(s) = 0$, $B(s) = 0$ and $C(s) = 0$ respectively.

Theorem 5.6 *Suppose that $A'(1)$, $B'(1)$ and $C'(1) \leq 0$. Then $a_i = 1$, $(i \geq 1)$.*

Proof. From (5.27)

$$\frac{sC(s)}{2}G''_i(s) + sB(s)G'_i(s) + A(s)G_i(s) = s(a_i - s^i),$$

$A(s) > 0$, $B(s) > 0$ and $C(s) > 0$ for $s \in [0, 1)$. Therefore, $a_i - s^i \geq 0$. Let $s \to 1$, we have $a_i \geq 1$. But $a_i \leq 1$ is always true and thus $a_i = 1$. ∎

Theorem 5.7 *Suppose that $\rho_a = \rho_b = \rho_c$. Then $a_i = \rho_a^i, (i \geq 1)$.*

Proof. Putting $s = \rho_a$ into (5.13), we get $a_i = \rho_a^i$. ■

At this moment, we have found the extinction probability ,a_i, for IBCIMR for the case of both $A'(1)$,$B'(1)$,$C'(1) \leq 0$ and $\rho_a = \rho_b = \rho_c$.

In order to get the extinction probability for other case, we need to solve the differential equation (5.27. However, as the order of the differential equation is more than 1. There is a need to know more about the transition function $P(t) = (p_{ij}; i, j \in \mathbf{Z}_+)$. After having such information, we may be able to find the extinction probability,a_i, using numerical methods.

Remark 5.1 *At this point, we have considered two related Markov branching processes in detail. As we know the importance of ICBP and BPIMR in chapters before. Since ICIMR and IBCIMR introduced in this chapter are new models having related intuitive meaning, there is a need to study their properties. We have discussed the model settings, some properties about uniqueness , regularity and extinction probability and of ICIMR and IBCIMR. However, as the differential equations involved in the above models are even more complicated. We only know the extinction probability for certain cases. Numerical methods may be introduced to help our understanding for the two models, especially for evaluation of extinction probability.*

Chapter 6

Conclusions and Future Work

6.1 Conclusions

In the previous chapters, we have studied some generalized branching models, especially for the interacting branching collision processes and discussed some of the important characteristics of the the corresponding Q-processes. This mainly includes the uniqueness and regularity criteria, the extinction and the corresponding asymptotic behavior.

In Chapter 2, the collision branching processes are studied in detail. We review the uniqueness and regularity criteria for this model. Then, extinction and explosive behavior are also considered in detail. This chapter mainly serves as some background for our next model considered, the interacting branching collision processes.

In Chapter 3, the interacting branching collision processes are considered in detail. We study the uniqueness and regularity criteria for the model. Then, we focus on the evaluation of the extinction probability for the processes. In order to do so, we separate our problem into different subcases. We can see that the collision component takes a relatively important part than the branching component. After applying different transformations, we finally have the explicit forms of extinction probability. However, we note that some of the explicit forms are very complicated. To deal with this problem, we consider the asymptotic behavior of the extinction probability.

In Chapter 4, the Markov branching processes with immigration-migration and resurrection are studied in detail. We studied the uniqueness and regularity criteria for the model. Then, we go to the evaluation of the extinction probability for the processes. In order to do so, we separate our problem into different subcases. We see that the state-dependent branching component takes a relatively important part than the state-independent immigration component. This is actually quite trivial as it means that, for a particular species, when the overall death rate is larger than or equal to the birth rate, immigration and resurrection is necessary to rescue the species. After applying different transformations, we finally have the explicit forms of extinction probability. However, again, some of the explicit forms are very complicated. To deal with this problem, the asymptotic behavior of the extinction probability are then considered.

In Chapter 5, the interacting collision processes with immigration - migration and resurrection and interacting branching collision processes with immigration - migration and resurrection are considered in detail. We study some of the uniqueness and regularity criteria for the models. We can see that the collision component takes a relatively important part than the branching component and immigration component. Then, we focus on the evaluation of the extinction probability for the processes. Some cases for extinction probability are solved. While some cases are still needed further consideration as the differential equations are too complicated to solved. Numerical methods may help for this purpose.

My original material starts from Chapter 4. The model used in chapter 4 were introduced by Li and Liu [2011]. In Li and Liu [2011], some calculation in cases of extinction probability evaluation were not strictly defined. My contribution focuses on the extinction probability evaluation and discussing the asymptotic behavior for the extinction probability in Chapter 4. While two interacting branching models are discussed in Chapter 5. Some important properties for the two models are studied in detail.

6.2 Future Work

The models studied in the previous chapters play an important role in the study of generalized branching processes. The models can be fitted in different practical cases, particular in biological science, individuals usually interact with each other. Immigration or protection of species are always a hot issue. The following are some related further work that we would like to investigate in the future.

(i) We have included a branching component into the collision processes. Actually, according to different situations, we may include a migration component into the collision processes, etc. Various combinations may fit in different real life applications.

(ii) In this thesis, we have only considered at most 2 absorbing states in our models. However, we may also considered that the case for having $n(> 2)$ absorbing states. Evaluating the extinction behavior will be much more challenging, but there may be such models in realistic situations.

(iii) It can be noted that our discussion in this thesis, theoretical study of the models are focused. We have not made attempts to perform simulations. Actually, we understand the importance of real life applications and simulations of theoretical results. These important parts will be considered in the future.

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