Fundamental Domains for Left-right Actions in Lorentzian Geometry

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by

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Abstract

We consider $\tilde{G} = \widetilde{\mathrm{SU}(1,1)} \cong \widetilde{\mathrm{SL}(2,\mathbb{R})}$. The aim of this thesis is to compute the fundamental domains for two series of groups of the form $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ acting on $\tilde{G}$ by left-right multiplication, i.e. $(g,h) \cdot x = gxh^{-1}$, where $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are discrete subgroups of $\tilde{G}$ of the same finite level and $\tilde{\Gamma}_2$ is cyclic. The level of a subgroup $\tilde{\Gamma}$ in $\tilde{G}$ is defined as the index of the group $\tilde{\Gamma} \cap Z(\tilde{G})$ in the center $Z(\tilde{G}) \cong \mathbb{Z}$. From computing the fundamental domain we can describe the biquotients $\tilde{\Gamma}_1 \backslash \tilde{G} / \tilde{\Gamma}_2$ which are diffeomorphic to the links of certain quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularities, i.e. the intersections of the singular variety with sufficiently small spheres around the isolated singular point as shown in [16].

Dedication

All my thanks to Allah for helping me bring this work to a successful completion

To

My parent and family for supporting me constantly by their unlimited care and prayers

To

My lovely wife and kids who were my source of enthusiasm

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Chapter 1

Introduction

In this thesis we compute the fundamental domains for two series of discrete subgroups of finite level in $\widetilde{\mathrm{SU}(1,1)}$ acting by left and right multiplication. The level is the index of the intersection of a subgroup $\tilde{\Gamma}$ of $\widetilde{\mathrm{SU}(1,1)}$ with the center of $\widetilde{\mathrm{SU}(1,1)}$. We consider a discrete subgroup $\tilde{\Gamma}_1$ and a cyclic discrete subgroup $\tilde{\Gamma}_2$ in $\tilde{G} = \widetilde{\mathrm{SU}(1,1)}$ of the same level k. We consider the product $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ acting on $\tilde{G}$ by $(g,h) \cdot x = gxh^{-1}$. We study the corresponding fundamental domains in more detail by determining the faces and edges and describing the gluing of the faces under the group action. Some of the figures of corresponding fundamental domains are given as well. The choice of the discrete subgroups which we study is motivated by the singularity theory.

We consider the universal covering $\tilde{G}$ of the Lie group $\mathrm{PSU}(1,1)$ which is equivalent to $\mathrm{PSL}(2,\mathbb{R})$, the group of orientation-preserving isometries of the hyperbolic plane. Let the unit disc $\mathbb{D}$ in $\mathbb{C}$ be our model of the hyperbolic plane. The group $\tilde{G}$ is considered as a hypersurface embedded in the bundle $\tilde{L} = \mathbb{R}_+ \times \tilde{G}$. The Killing form on $\tilde{G}$ induces a Lorentz metric of signature $(2,1)$ on $\tilde{G}$ and a pseudo-Riemannian metric of signature $(2,2)$ on $\tilde{L}$.

We define the fundamental domains as follows:

Definition 1. *Let G be a group. Let X be a set. Let G act on X. A subset F of X is a fundamental domain for the action of G on X if*

- $\bigcup_{g \in G} g(F) = X$.

- $(F^\circ) \cap g(F^\circ) = \emptyset$ *for any* $g \in G - \{Id\}$.

In 1992, Fischer [20] suggested how to construct fundamental domains for the action of a discrete subgroup of PSU(1,1) by left multiplication. Hence, Fischer studied the case where the discrete subgroup $\tilde{\Gamma}_1$ is of level 1 and the subgroup $\tilde{\Gamma}_2$ is trivial. After that, the case of level 2 was studied by Pratoussevitch in 1998 [14] . Subsequently, in 2000 Pratoussevitch [15], [17] introduced the main construction of polyhedral fundamental domains for a discrete subgroup $\tilde{\Gamma}_1$ which acts on $\widetilde{\mathrm{SU}(1,1)}$ by left translations. Then, in 2011 Pratoussevitch [18] generalised the corresponding construction to the case of an action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\widetilde{\mathrm{SU}(1,1)}$. Therefore, a special case for the construction given in [18] is when the subgroup $\tilde{\Gamma}_2$ is trivial where the construction becomes the same as in [15] and [17].

The resulting fundamental domains that come from the above construction are polyhedra in the Lorentz manifold $\tilde{G}$ with totally geodesic faces. In particular, we use tangent half-spaces on the submanifold $\widetilde{\mathrm{SU}(1,1)}$ in points of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ to construct a 4-dimensional polytope P. Under the radial projection, the images of the 3-dimensional faces of the polytope P are the fundamental domains for the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\widetilde{\mathrm{SU}(1,1)}$. In other words, we obtain the fundamental domains for the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ by projecting the faces of the polyhedron P onto $\tilde{G}$. The fundamental domains are compact if the subgroup $\tilde{\Gamma}_1$ is co-compact.

Milnor, Dolgachev and Neumann in [11], [6], [12] and [13] studied the connection between discrete subgroups of finite level and quasi-homogeneous isolated singularities of complex surfaces. There is a correspondence between the class of subgroups for which we compute the fundamental domains and an interesting class of singularities. The subgroups from this class correspond one-to-one to quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularities. In particular, Pratoussevitch [16] proved that there is a diffeomorphism between the bi-quotients of the form $\tilde{\Gamma}_1 \backslash \tilde{G} / \tilde{\Gamma}_2$ and the links of quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularities. The links of quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularities are the intersections of the singular varieties with sufficiently small spheres around the isolated singular point. Bi-quotients of the form $\tilde{\Gamma}_1\backslash\tilde{G}/\tilde{\Gamma}_2$ are standard Lorentz space forms which were studied by Kulkarni and Raymond [9].

As we mentioned at the beginning of the introduction the main aim of this thesis is to compute the fundamental domains for two series of discrete subgroups of finite level in $\widetilde{\mathrm{SU}(1,1)}$ acting by left and right multiplication. Let $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ be a product of two discrete subgroups of $\tilde{G}$, one of them cyclic. This product is acting on $\tilde{G}$ via $(g,h)\cdot x = gxh^{-1}$. Let $\tilde{\Gamma}_1 = \tilde{\Gamma}_1(p_1,q,r)^k$ and $\tilde{\Gamma}_2 = (C_{p_2})^k$. The group $\tilde{\Gamma}_1(p_1,q,r)^k$ is a subgroup of level k whose image in $\mathrm{PSU}(1,1)$ is a triangle group $\Gamma(p_1,q,r)$ (For more details see section 3.1). The group $\tilde{\Gamma}_2 = (C_{p_2})^k$ is a subgroup of level k whose image in $\mathrm{PSU}(1,1)$ is a cyclic group of order p_2. Particularly, we construct the fundamental domains for the following two series of discrete subgroups of finite level in $\tilde{G}$: $\tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$ and $\tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$.

The motivation for choosing these particular discrete subgroups $\tilde{\Gamma}_1$ of finite level in $\widetilde{\mathrm{SU}(1,1)}$ correspond to the singularities in the series E and Z according to the classification by V. I. Arnold. See table 6.1 for more details.

Our method to compute the fundamental domains for $\tilde{\Gamma}_1\times\tilde{\Gamma}_2$ is as follows: First we use the main Theorem which was introduced by Pratoussevitch in [18] to construct the fundamental domains. By applying this Theorem we obtain infinitely many prisms (prism-like polyhedra) and we know that in the co-compact case the fundamental domains are already obtained using finitely many prisms. Therefore, we use an estimate to exclude infinitely many prisms which are far from the fundamental domain. We show that for the products of groups: $\tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$ and $\tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$ there are such finite sets of finitely many prisms which yield the fundamental domain. After that, we look at such prisms and how they intersect with the tangent space for the identity element $\tilde{e}$. Then we choose several faces of each prism and obtain polyhedron which is contained in the fundamental domain. After that, we use the combinatorial criterium to show the that polyhedron is the fundamental domain. Furthermore, we explicitly study the faces of the fundamental domain and the way how the faces are glued together to obtain the links of quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularities.

This thesis is organized as follows:

In **Chapter 2** we describe some basic concepts such as linear fraction transformations and the covering spaces of the Lie group $\mathrm{PSU}(1,1)$. Then we look at the embedding of this Lie group in the 4-dimensional pseudo-Euclidean space and the embedded tangent spaces and half-spaces. The main aim of this chapter is to introduce the main Theorem for the construction of the fundamental domains for $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$.
In **Chapter 3** we describe the discrete subgroups $\tilde{\Gamma}_1$ of $\widetilde{\mathrm{SU}(1,1)}$ whose images in $\mathrm{PSU}(1,1)$ are triangle groups. We state the conditions for the existence of a subgroup of $\tilde{G}$ of level k and we find the generators for such groups $\tilde{\Gamma}_1$. We define the edge crown and the vertex crown of a triangle group.
In **Chapter 4** we discuss the methods of finding finite representations of the fundamental domains. First, we prove some estimates and apply them to exclude infinitely many prisms. After that, we describe a combinatorial criterium similar to the Poincaré-Maskit Theorem.
In **Chapter 5** we compute the fundamental domains for the two series of products of discrete subgroups of finite level in $\tilde{G}$: $\tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$ and $\tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$. First, we find the generators of the groups $\tilde{\Gamma}(p_1,3,3)^k \times (C_3)^k$. Then, we study the shape of the fundamental domains and describe the faces and the edges of the fundamental domains. We also look at the gluing of the faces of the fundamental domains. The main results for the case $\tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$ are contained in Definition 69, Theorem 75 and Proposition 74. The main results for the case $\tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$ are contained in Definition 76, Theorem 83 and Proposition 82.
In **Chapter 6** we look at the link spaces of $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities. We show that there are diffeomorphisms between the link spaces of $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities and the biquotients of $\tilde{\Gamma}_1 \backslash \tilde{G} / \tilde{\Gamma}_2$. We explain our motivation for choosing the subgroups $\tilde{\Gamma}_1$.

Chapter 2

Construction of fundamental domains

The material of this chapter follows [15], [17] and [18].

2.1 Linear fractional transformations

Definition 2. *(Linear fractional transformations): A linear fractional transformation is a mapping* $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ *(where* $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$*) of the form*

$$f(z) = \frac{az+b}{cz+d} \quad \text{for } a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

The group of all linear fractional transformations is denoted by $LF(2, \mathbb{C})$.

We can consider two subgroups of the set of linear fractional transformations. Each of these subgroups is isomorphic to the group of orientation-preserving isometries of the hyperbolic plane:

a) The group $\mathrm{PSL}(2, \mathbb{R})$ which consists of all linear fractional transformations of the form

$$f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1.$$

We can think of $\mathrm{PSL}(2,\mathbb{R})$ as the group of orientation-preserving isometries of the upper half-plane model $\mathbb{H}$ of the hyperbolic plane.

Now consider the isomorphism between the group of all such transformations $\mathrm{PSL}(2,\mathbb{R})$ and the group $\mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$, where

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a,b,c,d \in \mathbb{R},\ ad - bc = 1 \right\}.$$

So, we obtain

$$\mathrm{Isom}^+(\mathbb{H}) = \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}.$$

b) The group $\mathrm{PSU}(1,1)$ which consists of all linear fractional transformations of the form

$$f(z) = \frac{az+b}{\bar{b}z+\bar{a}}, \quad a,b \in \mathbb{C}, \quad a\bar{a} - b\bar{b} = 1.$$

We can think of $\mathrm{PSU}(1,1)$ as the group of orientation-preserving isometries of the disk model $\mathbb{D}$ of the hyperbolic plane.

We have

$$\mathrm{Isom}^+(\mathbb{D}) = \mathrm{PSU}(1,1) = \mathrm{SU}/\{\pm 1\},$$

where

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : (a,b) \in \mathbb{C}^2, |a|^2 - |b|^2 = 1 \right\}.$$

There are three types of elements in the group $\mathrm{PSU}(1,1)$, distinguished by their fixed point behavior. A hyperbolic element has in $\bar{\mathbb{D}}$ two fixed points, which are in $\partial\mathbb{D}$, a parabolic element has in $\bar{\mathbb{D}}$ one fixed point, which is in $\partial\mathbb{D}$. Moreover, an elliptic element has in $\mathbb{D}$ one fixed point, which is in $\mathbb{D}$. We can tell the difference between those types by looking at the traces of the corresponding matrices. The trace of the matrix is the sum of its diagonal elements. For $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $\mathrm{tr}(T) = a + d$. We denote the absolute value of the trace by $\mathrm{Tr}(T) = |a+b|$. An element T is *elliptic* if $\mathrm{Tr}(T) < 2$, *parabolic* if $\mathrm{Tr}(T) = 2$ and *hyperbolic* if $\mathrm{Tr}(T) > 2$.

2.2 PSU$(1,1)$ and its covering space $\widetilde{\mathrm{PSU}(1,1)}$

From the topological point of view the group SU$(1,1)$ is homeomorphic to the open solid torus $S^1 \times \mathbb{C} \simeq S^1 \times \mathbb{D}$ via the map $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mapsto (\frac{a}{|a|}, b)$. So, to find the fundamental group for the group SU$(1,1)$ we just need to find the fundamental group of $S^1 \times \mathbb{C}$, which is $\pi_1(S^1 \times \mathbb{C}) \cong \mathbb{Z}$. The matrix group SU(1,1) is a 2-fold covering of PSU(1,1), hence we obtain

$$\pi_1(\mathrm{PSU}(1,1)) \cong \pi_1(\mathrm{PSL}(2,\mathbb{R})) \cong \pi_1(S^1 \times \mathbb{C}) \cong \mathbb{Z}.$$

A connected covering of PSU$(1,1)$ (as a Lie group) is uniquely determined up to an isomorphism by the number of the sheets. The covering with the countably infinite number of sheets is the universal covering. The pre-image of the identity element of PSU$(1,1)$ under this covering map is the centre of the covering Lie group of PSU$(1,1)$.

2.3 Embeddings of PSU(1,1) and $\widetilde{\mathrm{PSU}(1,1)}$

The bilinear form $((w_1,z_1),(w_2,z_2)) := \mathrm{Re}(w_1\bar{w}_2 - z_1\bar{z}_2)$ induces a pseudo-Riemannian metric on $\mathbb{C}^2$. The totally geodesic submanifolds with respect to this metric are the affine subspaces. We consider the pseudo-sphere with respect to this bilinear form:

$$G = \{(a,b) \in \mathbb{C}^2 : |a|^2 - |b|^2 = 1\}.$$

The identification of G with SU$(1,1)$ gives a group structure on G with identity element $e = (1,0)$. We consider the pseudo-sphere G and the positive cone L over G,

$$L = \{(w,z) \in \mathbb{C}^2 : |w| > |z|\} = \mathbb{R}_+ \cdot G$$

as pseudo-Riemannian manifolds with the (induced) metric of signature $(+,-,-)$ and $(+,+,-,-)$ respectively. The radial projection $\Psi : L \to G$ is a trivialisable $\mathbb{R}_+$-bundle over G. The nonempty intersections of G with linear subspaces of (real) dimension 2 and 3 in $\mathbb{C}^2$ are the totally geodesic submanifolds of G of dimension 1 and 2 respectively.

Let $\tilde{G}$ and $\tilde{L}$ be the universal coverings of G and L with the pseudo-Riemannian metrics of signature $(+,-,-)$ and $(+,+,-,-)$ which is obtained as the pull-back of the metric on G and L respectively. The projection $\tilde{G} \to G$ induces a group structure on $\tilde{G}$. Let $\tilde{e}$ be the identity element with respect to this group structure.
We can describe $\tilde{L}$ as

$$\tilde{L} = \{(\alpha, r, z) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{C} \mid |z| < r\},$$

$$\pi : \tilde{L} \to L, \quad i.e. \;\; \pi(\alpha, r, z) = (re^{i\alpha}, z).$$

We call the number $\alpha \in \mathbb{R}$ the argument of the element $(\alpha, r, z) \in \tilde{L}$. The real number r is the absolute value of w if $(w, z) = \pi(\alpha, r, z)$. Moreover, we can describe $\tilde{G}$ as

$$\tilde{G} = \{(\alpha, r, z) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{C} \mid |z|^2 = r^2 - 1\} \subset \tilde{L}.$$

Let $\arg : \tilde{L} \to \mathbb{R}$ be the lift with $\arg(\tilde{e}) = 0$ of the map $L \to \mathbb{S}^1$ given by $(a, b) \mapsto \frac{a}{|a|}$. We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\arg} & \mathbb{R} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ L & \longrightarrow & \mathbb{S}^1, \end{array}$$

where the map $\mathbb{R} \to \mathbb{S}^1$ is the universal covering of $\mathbb{S}^1$ given by $t \mapsto e^{it}$. We have $\arg(\tilde{g}^{-1}) = -\arg(\tilde{g})$, where $\tilde{g} \in \tilde{G}$.

The following two Propositions are taken from [15], Proposition 1 and 2.

Proposition 3. *For $g \in G$ let $L_g, R_g, K_g : G \mapsto G$ be the left resp. right multiplication resp. the conjugation with the element g, i.e. $L_g(h) = gh$, $R_g(h) = hg$, $K_g(h) = ghg^{-1}$ and $K_g = L_g \circ R_g^{-1}$. Let η, $\varepsilon : G \mapsto G$ be defined by $\eta((a, b)) = (\bar{a}, \bar{b})$ and $\varepsilon((a, b)) = (\bar{a}, -b)$. The map ε is the taking of the inverse in G, i.e. $\varepsilon(g) = g^{-1}$. The maps $L_g, R_g, K_g, \eta, \varepsilon$ are isometries of G. The maps K_g and η are automorphisms of G, the map ε is an anti-automorphism of G. We have*

$$\mathrm{Isom}_0(G) = \langle L_a, R_b \mid a, b \in G \rangle \cong (G \times G)/\{(z, z) : z \in Z(G)\}$$

and

$$\mathrm{Isom}^+(G) = \langle \mathrm{Isom}_0(G), \eta \rangle, \quad \mathrm{Isom}(G) = \langle \mathrm{Isom}_0(G), \varepsilon \rangle = \langle \mathrm{Isom}^+(G), \eta, \varepsilon \rangle.$$

Proposition 4. *For $\tilde{g} \in \tilde{G}$ let $L_{\tilde{g}}, R_{\tilde{g}}, K_{\tilde{g}} : \tilde{G} \mapsto \tilde{G}$ be the left resp. right multiplication resp. the conjugation with the element $\tilde{g}$, i.e. $L_{\tilde{g}}(\tilde{h}) = \tilde{g}\tilde{h}$, $R_{\tilde{g}}(\tilde{h}) = \tilde{h}\tilde{g}$, $K_{\tilde{g}}(\tilde{h}) = \tilde{g}\tilde{h}\tilde{g}^{-1}$ and $K_{\tilde{g}} = L_{\tilde{g}} \circ R_{\tilde{g}}^{-1}$. Let $\tilde{\eta}$, $\tilde{\varepsilon} : \tilde{G} \mapsto \tilde{G}$ be the lifts of the isometries η resp. ε of G with $\tilde{\eta}(\tilde{e}) = \tilde{e}$ and $\tilde{\varepsilon}(\tilde{e}) = \tilde{e}$. We have $\arg \circ \tilde{\varepsilon} = -\arg$ and $\arg \circ \tilde{\eta} = -\arg$. The maps $L_{\tilde{g}}, R_{\tilde{g}}, K_{\tilde{g}}, \tilde{\eta}, \tilde{\varepsilon}$ are isometries of $\tilde{G}$. The maps $K_{\tilde{g}}$ and $\tilde{\eta}$ are automorphisms of $\tilde{G}$, the map $\tilde{\varepsilon}$ is an anti-automorphism of $\tilde{G}$. We have*

$$\mathrm{Isom}_0(\tilde{G}) = \langle L_a, R_b \mid a, b \in \tilde{G} \rangle \cong (\tilde{G} \times \tilde{G}) / \{(z,z) : z \in Z(\tilde{G})\}$$

and

$$\mathrm{Isom}^+(\tilde{G}) = \langle \mathrm{Isom}_0(\tilde{G}), \tilde{\eta} \rangle, \quad \mathrm{Isom}(\tilde{G}) = \langle \mathrm{Isom}^+(\tilde{G}), \tilde{\varepsilon} \rangle = \langle \mathrm{Isom}_0(\tilde{G}), \tilde{\eta}, \tilde{\varepsilon} \rangle.$$

2.4 Embedded tangent spaces and half-spaces

The hyperplane $\hat{E}_g = \{y \in \mathbb{C}^2 : (y,g) = 1\}$ with $g \in G$ divides $\mathbb{C}^2$ in two half-spaces

$$\hat{H}_g = \{y \in \mathbb{C}^2 : (y,g) \le 1\} \;\text{ and }\; \hat{I}_g = \{y \in \mathbb{C}^2 : (y,g) \ge 1\}.$$

We can see clearly from the formula of the half-space $\hat{H}_g$ that the point 0 is in $\hat{H}_g$. Let

$$E_g = \hat{E}_g \cap L, \;\; H_g = \hat{H}_g \cap L, \;\; I_g = \hat{I}_g \cap L.$$

The submanifolds E_g and $\hat{E}_g$ are the embedded tangent spaces of G at the point g in L and $\mathbb{C}^2$ respectively. By this we mean that the submanifolds E_g and $\hat{E}_g$ are the maximal totally geodesic submanifolds of L and $\mathbb{C}^2$ respectively which are tangent on G at the point g.

Proposition 5. *We have for* $e = (1,0)$

$$\begin{aligned}\hat{E}_e &= \{(w,z) \in \mathbb{C}^2 : Re(w) = 1\},\\ \hat{I}_e &= \{(w,z) \in \mathbb{C}^2 : Re(w) \geq 1\},\\ \hat{H}_e &= \{(w,z) \in \mathbb{C}^2 : Re(w) \leq 1\}.\end{aligned}$$

Moreover

$$\begin{aligned}E_e &= \{(w,z) \in \mathbb{C}^2 : Re(w) = 1, |z| < |w|\},\\ I_e &= \{(w,z) \in \mathbb{C}^2 : Re(w) \geq 1, |z| < |w|\},\\ H_e &= \{(w,z) \in \mathbb{C}^2 : Re(w) \leq 1, |z| < |w|\}.\end{aligned}$$

Proposition 6. *For* $a, g \in G$ *we have* $a \cdot E_g = E_{ag},\ a \cdot I_g = I_{ag},\ a \cdot H_g = H_{ag}$.

Proposition 7. *Let* $g \in G$. *Then* E_g *and* I_g *are star-convex with respect to* g, *i.e. any segment between* g *and any point in* E_g *or* I_g *stays in* E_g *or* I_g *respectively.*

Corollary 8. *For any* $g \in G$ *the sets* E_g *and* I_g *are contractible.*

We use the projection maps $\pi : \tilde{G} \to G$ and $\pi : \tilde{L} \to L$ to define $\tilde{E}_{\tilde{g}}, \tilde{I}_{\tilde{g}}$ and $\tilde{H}_{\tilde{g}}$, where $\tilde{g} \in \tilde{G}$ and $\pi(\tilde{g}) = g \in G$. Let $\tilde{E}_{\tilde{g}}$ and $\tilde{I}_{\tilde{g}}$ be the connected components of $\pi^{-1}(E_g)$ and $\pi^{-1}(I_g)$ respectively which contain g. Similarly to E_g, the submanifold $\tilde{E}_{\tilde{g}}$ is the embedded totally geodesic submanifold in $\tilde{L}$ which is tangent to $\tilde{G}$ at the point $\tilde{g}$. $\tilde{E}_{\tilde{g}}$ divides $\tilde{L}$ into two halfspaces which are $\tilde{I}_{\tilde{g}}$ and $\tilde{H}_{\tilde{g}} = \tilde{L} - (\tilde{I}_{\tilde{g}})^{\circ}$.

Proposition 9. *For* $\tilde{a}, \tilde{g} \in \tilde{G}$ *we have* $\tilde{a} \cdot \tilde{E}_{\tilde{g}} = \tilde{E}_{\tilde{a}\tilde{g}},\ \tilde{a} \cdot \tilde{I}_{\tilde{g}} = \tilde{I}_{\tilde{a}\tilde{g}},\ \tilde{a} \cdot \tilde{H}_{\tilde{g}} = \tilde{H}_{\tilde{a}\tilde{g}}$.

Proposition 10. *We have*

$$\begin{aligned}\tilde{E}_{\tilde{e}} &= \{\tilde{g} \in \tilde{L} : \pi(\tilde{g}) \in E_e, |\arg(\tilde{g})| < \pi/2\},\\ \tilde{I}_{\tilde{e}} &= \{\tilde{g} \in \tilde{L} : \pi(\tilde{g}) \in I_e, |\arg(\tilde{g})| < \pi/2\},\\ \tilde{H}_{\tilde{e}} &= \{\tilde{g} \in \tilde{L} : \pi(\tilde{g}) \in H_e, |\arg(\tilde{g})| < \pi/2\} \cup \{\tilde{g} \in \tilde{L} : |\arg(\tilde{g})| \geq \pi/2\}.\end{aligned}$$

Moreover

$$\tilde{H}_{\tilde{e}} = \pi^{-1}(H_e) \cup \bigcup_{\tilde{g} \in \pi^{-1}(e) - \tilde{e}} \tilde{I}_{\tilde{g}}.$$

Proposition 11. *For* $\tilde{g} \in \tilde{G}$ *and* $g = \pi(\tilde{g}) \in G$ *we have* $\pi(\tilde{E}_{\tilde{g}}) = E_g, \pi(\tilde{I}_{\tilde{g}}) = I_g$ *and* $\pi(\tilde{H}_{\tilde{g}}) = L$*. The maps* $\pi|_{\tilde{E}_{\tilde{g}}} : \tilde{E}_{\tilde{g}} \to E_g$ *and* $\pi|_{\tilde{I}_{\tilde{g}}} : \tilde{I}_{\tilde{g}} \to I_g$ *are homeomorphisms.*

2.5 Elliptic elements and their lifts

Let x be a base point in $\mathbb{D}$ and $t \in \mathbb{R}$. We consider the map $\rho_x : \mathbb{R} \to \mathrm{PSU}(1,1)$, where $\rho_x(t)$ is the rotation at x through the angle t, i.e the element of $\mathrm{PSU}(1,1)$ with the fixed point x and the derivative at x equal to e^{it}. If $t \notin 2\pi\times\mathbb{Z}$ then the element $\rho_x(t)$ is elliptic whereas if $t \in 2\pi\times\mathbb{Z}$ then $\rho_x(t) = \mathrm{id}$. There exists a map $r_x : \mathbb{R} \to G$ s.t. for all t the projection of $r_x(t)$ to $\mathrm{PSU}(1,1)$ is $\rho_x(t)$ and $r_x(0) = (1,0) = e$. The map $r_x : \mathbb{R} \to G$ is given by

$$r_x(t) = \Big(\cos\frac{t}{2} + i\frac{1+|x|^2}{1-|x|^2}\sin\frac{t}{2}, -i\frac{2x}{1-|x|^2}\sin\frac{t}{2}\Big).$$

Then $r_x(2\pi) = (-1,0) = -e$. There exists a map $\tilde{r}_x : \mathbb{R} \to \tilde{G}$ s.t. for all t the projection of $\tilde{r}_x(t)$ to $\mathrm{PSU}(1,1)$ is $\rho_x(t)$ and $\tilde{r}_x(0) = \tilde{e}$. The maps ρ_x, r_x and $\tilde{r}_x$ are homomorphisms. The element $\tilde{r}_x(t)$ acts on $\mathbb{D}$ as the rotation at x through the angle t. If $x = 0$ then $r_0(t) = (e^{it/2}, 0)$ and $\arg\,(\tilde{r}_0(2t)) = t$.

Proposition 12. *If* $g(0) = 0$ *or* $h(0) = 0$ *then* $\arg(g\cdot h) = \arg(g) + \arg(h)$.

Proposition 13. *We have* $\tilde{r}_x(2\pi) = c$*, where* c *does not depend on* $x \in \mathbb{D}$*. The element* c *is one of the two generators of the center* $Z(\tilde{G})$*. Moreover, we have* $\arg(\tilde{r}_x(2t)) \in (0,\pi)$ *for any* $t \in (0,\pi)$ *and* $x \in \mathbb{D}$.

2.6 Discrete subgroups of $\widetilde{\mathrm{SU}(1,1)}$

A *Fuchsian group* is a discrete subgroup of $\mathrm{PSU}(1,1)$. Let $Z(\tilde{G})$ be the center of $\tilde{G}$. Note that $Z(\tilde{G}) \cong \mathbb{Z}$. The *level* of a discrete subgroup $\tilde{\Gamma}$ of $\tilde{G}$ is the index of $\tilde{\Gamma} \cap Z(\tilde{G})$ in $Z(\tilde{G})$. Equivalently, the level of $\tilde{\Gamma}$ is the smallest $k \neq 0$ such that $c^k \in \tilde{\Gamma}$, where c is the generator of $Z(\tilde{G})$ defined in Proposition 13.

2.7 Elements of the construction

Let $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ be discrete subgroups of finite level k in $\tilde{G}$ and let $\tilde{\Gamma}_2$ be cyclic. For $i = 1,2$, let Γ_i be the image of $\tilde{\Gamma}_i$ in $\mathrm{PSU}(1,1)$. We assume the existence of a joint fixed point $u \in \mathbb{D}$ of Γ_1 and Γ_2.

For $i = 1,2$, the isotropy group $(\Gamma_i)_u$ of u in Γ_i is a finite cyclic group generated by $\rho_u(2\pi/p_i)$, where $p_i = |(\Gamma_i)_u|$. The isotropy group $(\tilde{\Gamma}_i)_u$ of u in $\tilde{\Gamma}_i$ is an infinite cyclic group generated by $\tilde{d}_i = \tilde{r}_u(4\vartheta_i)$, where $\vartheta_i = \frac{\pi k}{2p_i}$. We can assume without loss of generality that $u = 0 \in \mathbb{D}$. The element

$$\tilde{d}_1 = \tilde{r}_u\left(\frac{2\pi k}{p_1}\right)$$

is a generator of $(\tilde{\Gamma}_1)_u$, the stabiliser of u in $\tilde{\Gamma}_1$. The element

$$\tilde{d}_2 = \tilde{r}_u\left(\frac{2\pi k}{p_2}\right)$$

is a generator of $(\tilde{\Gamma}_2)_u = \tilde{\Gamma}_2$.

$\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ acts on $\tilde{G}$ by left-right multiplication

$$(g,h)\cdot x = gxh^{-1}.$$

Remark 14. *Note that in fact we are considering the action of the group* $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2/\langle(c^k,c^k)\rangle$*, which means that* $(\tilde{g}_1,\tilde{g}_2) \sim (\tilde{g}_1c^k,\tilde{g}_2c^k)$. *This is since*

$$\begin{aligned}(\tilde{g}_1,\tilde{g}_2) : x &\mapsto \tilde{g}_1 x\tilde{g}_2^{-1}\\ (\tilde{g}_1c^k,\tilde{g}_2c^k) : x &\mapsto (\tilde{g}_1c^k)x(\tilde{g}_2c^k)^{-1} = \tilde{g}_1c^kxc^{-k}\tilde{g}_2^{-1} = \tilde{g}_1x\tilde{g}_2^{-1}c^kc^{-k} = \tilde{g}_1x\tilde{g}_2^{-1}.\end{aligned}$$

We define for a point x in the orbit $\tilde{\Gamma}_1(u)$

$$T(x) = \{(\tilde{g}_1,\tilde{g}_2) \in \tilde{\Gamma}_1 \times \tilde{\Gamma}_2 \mid \tilde{g}_1(u) = x\}.$$

Let

$$Q_x = \bigcap_{(\tilde{g}_1,\tilde{g}_2)\in T(x)} \tilde{H}_{\tilde{g}_1\tilde{g}_2} \quad \text{and} \quad R_x = \bigcup_{(\tilde{g}_1,\tilde{g}_2)\in T(x)} \tilde{I}_{\tilde{g}_1\tilde{g}_2}.$$

For instance, for $x = u$ we have that

$$T(u) = (\tilde{\Gamma}_1)_u \times \tilde{\Gamma}_2 = \{(\tilde{d}_1^{m_1}, \tilde{d}_2^{m_2}) \mid m_1, m_2 \in \mathbb{Z}\} = \langle (\tilde{d}_1, e), (e, \tilde{d}_2) \rangle.$$

The generator $(\tilde{d}_1, e)$ acts on $\tilde{G}$ by left multiplication with $\tilde{d}_1$ and the generator $(e, \tilde{d}_2)$ acts on $\tilde{G}$ by right multiplication with $\tilde{d}_2^{-1}$.
Let $p = \operatorname{lcm}(p_1, p_2)$ be the least common multiple of p_1 and p_2. Let

$$\tilde{d} = \tilde{r}_u\left(\frac{2\pi k}{p}\right).$$

Then $\langle \tilde{d}_1, \tilde{d}_2 \rangle = \langle \tilde{d} \rangle$. Let

$$\vartheta_k = \frac{\pi k}{2p} = \frac{\pi k}{2 \operatorname{lcm}(p_1, p_2)}.$$

Then

$$\tilde{d} = \tilde{r}_u(4\vartheta_k).$$

The element $\tilde{d}$ acts on $\tilde{G}$ by left multiplication:

$$\tilde{d} \cdot (\alpha, r, z) = (\alpha + 2\vartheta_k, r, z e^{2i\vartheta_k}),$$

and it acts on the (α, r)−half-plane by the translation mapping

$$\tau(\alpha, r) = (\alpha + 2\vartheta_k, r).$$

Remark 15. *If* $\gcd(p_1, p_2) = 1$ *then* $p = \operatorname{lcm}(p_1, p_2) = p_1 \cdot p_2$. *Therefore,* $\tilde{d} = \tilde{r}_u\left(\frac{2\pi k}{p_1 \cdot p_2}\right)$. *So,*

$$\tilde{d}^{p_1} = \tilde{r}_u\left(\frac{2\pi k}{p_2}\right) = \tilde{d}_2,$$

$$\tilde{d}^{p_2} = \tilde{r}_u\left(\frac{2\pi k}{p_1}\right) = \tilde{d}_1.$$

We have

$$Q_u = \bigcap_{(\tilde{g}_1, \tilde{g}_2) \in T(u)} \tilde{H}_{\tilde{g}_1 \tilde{g}_2} = \bigcap_{m_1, m_2 \in \mathbb{Z}} \tilde{H}_{\tilde{d}_1^{m_1} \tilde{d}_2^{m_2}} = \bigcap_{m \in \mathbb{Z}} \tilde{H}_{\tilde{d}^m}$$

since $\langle \tilde{d}_1, \tilde{d}_2 \rangle = \langle \tilde{d} \rangle$.

Proposition 16. *We have* $T(u) = (\tilde{\Gamma}_1)_u \times \tilde{\Gamma}_2$*. For* $x \in \tilde{\Gamma}_1(u)$ *and* $a \in T(x)$ *we have* $T(x) = a \cdot T(u), Q_x = a \cdot Q_u$ *and* $R_x = a \cdot R_u$*.*

2.8 Prisms Q_x

We are going to mention how Pratoussevitch in [18] described the set Q_x. Recall that $Q_u = \bigcap_{m \in \mathbb{Z}} \tilde{H}_{\tilde{d}^m}$. The image of the set $\tilde{H}_{\tilde{e}}$ under the projection $(\alpha, r, z) \mapsto (\alpha, r)$ is

$$X_{\tilde{e}} = \{(\alpha, r) \in \mathbb{R} \times \mathbb{R}_+ \mid r \cdot \cos\alpha \le 1 \quad \text{or} \quad |\alpha| \ge \frac{\pi}{2}\}.$$

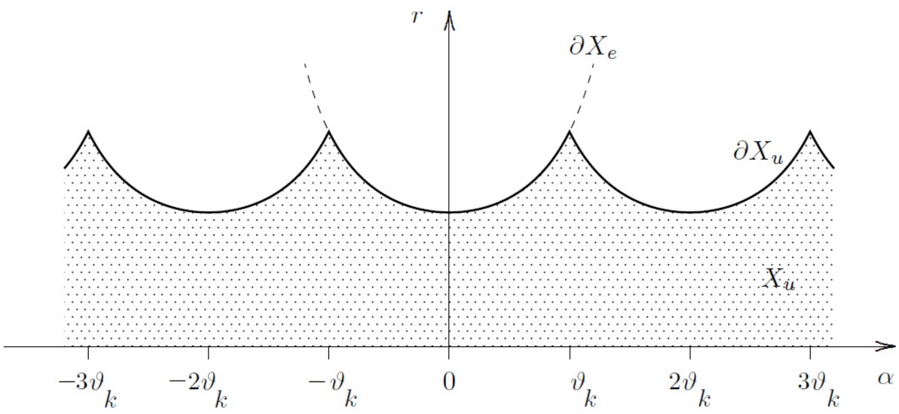

Figure 2.1: The image X_u of Q_u in the $(\alpha, r)-$ half-plane. (Adapted from [15]).

The images of the sets $\tilde{H}_{\tilde{d}^m} = \tilde{d}^m \cdot \tilde{H}_{\tilde{e}}$ under the projection $(\alpha, r, z) \mapsto (\alpha, r)$ are the translates $\tau^m(X_{\tilde{e}})$ of the set $X_{\tilde{e}}$, where τ is the translation $\tau(\alpha, r) = (\alpha + 2\vartheta_k, r)$. The manifold Q_u is a disc bundle over its image $X_u = \bigcap_{m \in \mathbb{Z}} \tau^m(X_{\tilde{e}})$ in the (α, r)-plane. The shaded area in figure 2.1 is X_u. (The real line is not part of X_u.) An important condition for the construction to work will be $p > k$. This condition ensures that $\vartheta_k = \frac{\pi k}{2p} < \frac{\pi}{2}$ and hence the sets $\tau^m(\partial X_{\tilde{e}})$ intersect and Q_u is a prism, compare with figure 2.1. The subsets Q_x are images of the subset Q_u under the action of the group $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$. For any $x \in \tilde{\Gamma}_1(u)$ there is an element $g \in \tilde{\Gamma}_1$ such that $g(x) = u$. Then $Q_x = g \cdot Q_u$.

The sets $g\cdot Q_u$ play a central role in our construction. We want to explain the geometric nature of these objects. We have described Q_u as a disc bundle over the set X_u in the (α, r)-half-plane $\mathbb{R}\times\mathbb{R}_+$. We may describe $Q_u \subset \tilde{L} \subset \mathbb{C}\times\mathbb{R}\times\mathbb{R}_+$ as

$$Q_u = (\mathbb{C} \times X_u) \cap \tilde{L}.$$

We think of X_u as a universal covering of a punctured plane polygon. The projection $(\alpha, r) \mapsto re^{i\alpha}$ projects the boundary $\partial X_{\tilde{e}}$ into a star polygon. For more details see [18] pp 7 and 8.

The following estimates are obtained by approximating the prisms Q_x via their inscribed and circumcribed cylinders:

Proposition 17. *Let x be a point in $\mathbb{D}$ that belongs to the orbit $\Gamma_1(u)$ of the point u. Let $(w, z) \in L$. Then*

i) If $(w, z) \in \pi(Q_x)$ then $|w| - |z| < |w| - |x| \cdot |z| \leq |w - \bar{z}x| \leq \frac{\sqrt{1-|x|^2}}{\cos\vartheta_k}$.

ii) If $|w - \bar{z}x| \leq \sqrt{1 - |x|^2}$ then $\pi^{-1}((w, z)) \subset Q_x$.

Proof. Let us first consider the case $x = u = 0$. In this case (i) reduces to: $(w, z) \in \pi(Q_u) \Rightarrow |w| \leq \frac{1}{\cos\vartheta_k}$. This is clear from figure 2.1: If $(w, z) \in \pi(Q_u)$ then $w = re^{i\alpha}$, where (α, r) belongs to the shaded area in figure 2.1. For (α, r) in the shaded area, r is the largest when α is an odd multiple of ϑ_k and is then equal to $r = \frac{1}{\cos\vartheta_k}$. Hence for $(w = re^{i\alpha}, z) \in \pi(Q_u)$ we have

$$|w| = r \leq \frac{1}{\cos\vartheta_k}.$$

For (ii), in the case $x = u = 0$ it reduces to: $|w| \leq 1 \Rightarrow \pi^{-1}((w, z)) \subset Q_u$. Let $g = (w, z) \in L$ with $|w| \leq 1$. Then we have $g \in H_{\pi(\tilde{a})}$ for all $\tilde{a} \in (\tilde{\Gamma}_1)_u$ and therefore $\pi^{-1}(g) \subset \tilde{H}_{\tilde{a}}$ for all $\tilde{a} \in (\tilde{\Gamma}_1)_u$. Thus

$$\pi^{-1}(g) \subset Q_u.$$

In the general case $x \in \tilde{\Gamma}_1(u) \setminus \{u\}$, let $\tilde{g} \in \tilde{\Gamma}_1$ be an element such that

$\tilde{g}(x) = u$ and let $(a,b) = \pi(\tilde{g})$. The element $(a,b) \in G$ acts on $\mathbb{D}$ by

$$(a,b) \cdot x = \frac{ax+b}{\bar{b}x+\bar{a}}.$$

The property $(a,b) \cdot x = u = 0$ implies $b = -ax$. From $(a,b) \in G$ we conclude

$$1 = |a|^2 - |b|^2 = |a|^2 - |a|^2|x|^2 = |a|^2(1-|x|^2)$$

and hence $|a| = \frac{1}{\sqrt{1-|x|^2}}$. Let us consider $(w,z) \in L$. Let $(w',z') = \pi(\tilde{g}) \cdot (w,z)$. Then

$$|w'| = |aw + b\bar{z}| = |aw - ax\bar{z}| = |a| \cdot |w - x\bar{z}| = \frac{|w - x\bar{z}|}{\sqrt{1-|x|^2}}.$$

To show (i), assume that $(w,z) \in \pi(Q_x)$. Then

$$(w',z') = \pi(\tilde{g}) \cdot (w,z) \in \pi(\tilde{g}) \cdot \pi(Q_x) = \pi(\tilde{g} \cdot Q_x) = \pi(Q_u),$$

hence $|w'| \leq \frac{1}{\cos\vartheta_k}$. On the other hand, $|w - x\bar{z}| = |w'| \cdot \sqrt{1-|x|^2}$, hence $|w - x\bar{z}| \leq \frac{\sqrt{1-|x|^2}}{\cos\vartheta_k}$. The other inequalities in (i) follow using the triangle inequality:

$$|w - \bar{x}z| \geq ||w| - |\bar{x}z|| = ||w| - |x| \cdot |z||.$$

We know that $(w,z) \in L \Rightarrow |w| > |z|$ and $|x| < 1 \Rightarrow |w| > |x| \cdot |z|$. Thus

$$||w| - |x| \cdot |z|| = |w| - |x| \cdot |z| > |w| - |z|.$$

To show (ii), assume that

$$|w - x\bar{z}| \leq \sqrt{1-|x|^2}.$$

Then $|w'| = \frac{|w-xz|}{\sqrt{1-|x|^2}} \leq 1$, hence $\pi^{-1}((w',z')) \subset Q_u$ and

$$\pi^{-1}((w,z)) = \pi^{-1}(\pi(\tilde{g})^{-1} \cdot (w',z')) = \tilde{g}^{-1} \cdot \pi^{-1}((w',z')) \subset \tilde{g}^{-1} \cdot Q_u = Q_x.$$

∎

2.9 Construction

We are going to state the main Theorem which describes the construction of the fundamental domains for $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ given by Pratoussevitch [18].

Theorem 18. *Let $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ be discrete subgroups of finite level k in $\tilde{G}$, let $\tilde{\Gamma}_2$ be cyclic. Let Γ_1 and Γ_2 be the images of $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ in $PSU(1,1)$. Let $p_1 = |(\Gamma_1)_u|$, $p_2 = |\Gamma_2| = |(\Gamma_2)_u|$ and $p = \text{lcm}(p_1, p_2)$. Consider $p > k$. For a point x in the orbit $\tilde{\Gamma}_1(u)$ let*

$$T(x) = \{(\tilde{g}_1, \tilde{g}_2) \in \tilde{\Gamma}_1 \times \tilde{\Gamma}_2 \mid \tilde{g}_1(u) = x\}.$$

Let

$$Q_x = \bigcap_{(\tilde{g}_1,\tilde{g}_2)\in T(x)} \tilde{H}_{\tilde{g}_1\tilde{g}_2}.$$

We consider in $\tilde{L}$ the four-dimensional polytope

$$P = \bigcup_{x\in\tilde{\Gamma}_1(u)} Q_x = \bigcup_{x\in\tilde{\Gamma}_1(u)} \bigcap_{(\tilde{g}_1,\tilde{g}_2)\in T(x)} \tilde{H}_{\tilde{g}_1\tilde{g}_2}.$$

The boundary of P is invariant with respect to the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$. The subset

$$F_{\tilde{e}} = Cl_{\partial P}(Int(\partial \tilde{H}_{\tilde{e}} \cap \partial P))$$

is a fundamental domain for the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on ∂P. The family

$$(F_{\tilde{g}_1\tilde{g}_2})_{\tilde{g}_1\in\tilde{\Gamma}_1,\tilde{g}_2\in\tilde{\Gamma}_2}$$

is locally finite in ∂P. The projection $\tilde{L} \mapsto \tilde{G}$ induces a $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$-equivalent homeomorphism

$$\partial P \mapsto \tilde{G}.$$

The images $\mathcal{F}_{\tilde{e}}$ of $F_{\tilde{e}}$ under the projection are fundamental domains for the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$. The family $(\mathcal{F}_{\tilde{g}_1\tilde{g}_2})_{\tilde{g}_1\in\tilde{\Gamma}_1,\tilde{g}_2\in\tilde{\Gamma}_2}$ is locally finite.

If $\tilde{\Gamma}_1$ is co-compact, then the fundamental domain $F_{\tilde{g}}$ is a compact polyhedron, i.e. a finite union of finite compact intersections of half-spaces $\tilde{I}_{\tilde{a}}$.

Remark 19. *Int and Cl denote the interior and the closure with respect to*

∂P.

Proposition 20.

$$F_{\tilde{e}} = Cl\ Int\left((\tilde{E}_{\tilde{e}} \cap \partial Q_u) - \left(\bigcup_{x \in \tilde{\Gamma}_1(u) \smallsetminus \{u\}} Int\ Q_x \right) \right).$$

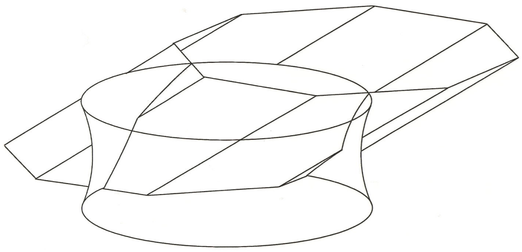

Figure 2.2: Q and L. (Taken from [15]).

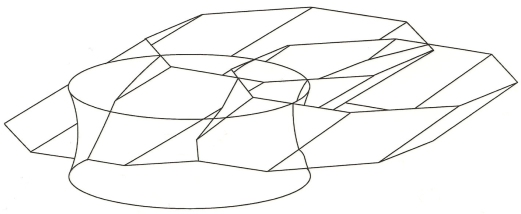

Figure 2.3: Q_x and L. (Taken from [15]).

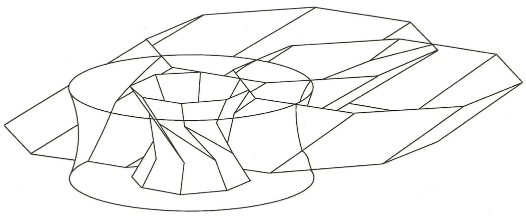

Figure 2.4: F, L and Q_x. (Taken from [15]).

Definition 21. *Let* $\tilde{S}_{\tilde{e}} = \tilde{E}_{\tilde{e}} \cap \partial Q_u$ *and for a subset* N *of* $\tilde{\Gamma}_1(u) \smallsetminus \{u\}$ *let* $F_N = \tilde{S}_{\tilde{e}} \cap \bigcap_{x \in N} R_x = \tilde{E}_{\tilde{e}} \cap \partial Q_u \cap \bigcap_{x \in N} R_x$. *Using this notation we can rewrite*

Proposition 20 as

$$F_{\tilde{e}} = CI\ Int\Big(F_{\tilde{\Gamma}_1(u)\setminus\{u\}}\Big).$$

Moreover, let $S_e = \pi(\tilde{S}_{\tilde{e}}) = \{(1 + i\omega, z) \in L : |\omega| \le \tan\vartheta_k\}$.

Remark 22. *The set* $\tilde{S}_{\tilde{e}}$ *can be seen in projection to* E_e *as the horizontal layer* S_e *between two horizontal planes* $\omega = \pm\tan\vartheta_k$*, intersecting with the complement* R_x *of* Q_x *amounts to cutting out* Q_x*. Figures 2.2, 2.3 and 2.4 illustrate the construction in the case* $k = 1$, $\tilde{\Gamma}_1 = \Gamma(4,3,3)^1, \tilde{\Gamma}_2 =< c >= Z(\tilde{G})$. *We identify the tangent space* E_e *with* $\mathbb{R}^3$*. We can think of the sets* Q_x *as prisms. The layer* $\tilde{S}_{\tilde{e}}$ *is indicated in the figures by the top and bottom of the hyperboloid and the prism. In particular, figure 2.2 shows how* $\tilde{S}_{\tilde{e}}$ *intersects with* R_x *in order to cut out one prism* Q_x*. Also, figure 2.3 shows how two prisms* Q_x *go together and how* $\tilde{S}_{\tilde{e}}$ *intersects them. Moreover, if we think of all the prisms* Q_x *and intercutting with* $\tilde{S}_{\tilde{e}}$ *that will give us the faces of the fundamental domain* F *as in figure 2.4.*

Chapter 3

Triangle groups

This chapter is mostly based on [15], [17] and [18].

3.1 Triangle groups

The cases we are going to consider are where $\tilde{\Gamma}_1$ is related to a triangle group. A triangle group $\Gamma(p,q,r)$ is the subgroup of orientation-preserving isometries in a discrete group of isometries $\Gamma(p,q,r)^*$ of the hyperbolic plane, where the group $\Gamma(p,q,r)^* \subset \mathrm{Isom}(\mathbb{D})$ is generated by the reflections in the edges of a triangle with angles $\frac{\pi}{p}, \frac{\pi}{q}$ and $\frac{\pi}{r}$ for $p,q,r \in \mathbb{N}$. $\Gamma(p,q,r)$ is a subgroup of index 2 in $\Gamma(p,q,r)^*$ which is generated by the rotations through $\frac{2\pi}{p}, \frac{2\pi}{q}, \frac{2\pi}{r}$ in the vertices of the triangle (each rotation is a product of 2 reflections). We have

$$\Gamma(p,q,r) = \Gamma(p,q,r)^* \cap \mathrm{Isom}^+(\mathbb{D}).$$

The sum of angles of a hyperbolic triangle is less than π, hence we obtain that $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$. Two triangle groups are conjugate as subgroups in the group of orientation-preserving isometries if and only if the corresponding hyperbolic triangles are isometric.

Let $\Delta(p,q,r)$ be the hyperbolic triangle in $\mathbb{D}$ with vertices u, v and w such that

$$u = 0, \quad v \in \mathbb{R}, v > 0 \text{ and } \mathrm{Im}(w) > 0,$$

with the angles at the vertices u, v, w being

$$\alpha_u = \frac{\pi}{p}, \; \alpha_v = \frac{\pi}{q} \text{ and } \alpha_w = \frac{\pi}{r},$$

see figure 3.1. Note that $\arg(w) = \frac{\pi}{p}$, hence $w = |w| \cdot e^{i\frac{\pi}{p}}$.

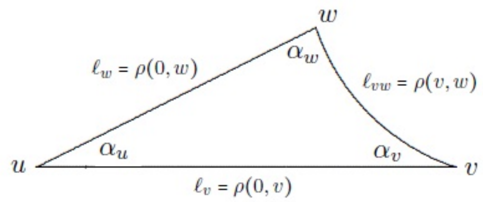

Figure 3.1: Hyperbolic triangle.

Let $\ell_v = \rho(u,v) = \rho(0,v), \ell_w = \rho(u,w) = \rho(0,w)$ and $\ell_{vw} = \rho(v,w)$ be the lengths of the edges of the above triangle $\Delta(p,q,r)$.

Using the general formulae for hyperbolic triangles, we can compute the length of the edges as follows:

$$\begin{aligned}
\cosh \ell_v &= \frac{\cos\frac{\pi}{p}\cos\frac{\pi}{q} + \cos\frac{\pi}{r}}{\sin\frac{\pi}{p}\sin\frac{\pi}{q}}, \\
\cosh \ell_w &= \frac{\cos\frac{\pi}{p}\cos\frac{\pi}{r} + \cos\frac{\pi}{q}}{\sin\frac{\pi}{p}\sin\frac{\pi}{r}}, \\
\cosh \ell_{vw} &= \frac{\cos\frac{\pi}{q}\cos\frac{\pi}{r} + \cos\frac{\pi}{p}}{\sin\frac{\pi}{q}\sin\frac{\pi}{r}}.
\end{aligned}$$

Also, using $\cosh^2 x - \sinh^2 x = 1$, we obtain

$$\sinh \ell_v = \frac{W}{\sin\frac{\pi}{p}\sin\frac{\pi}{q}}, \quad \sinh \ell_w = \frac{W}{\sin\frac{\pi}{p}\sin\frac{\pi}{r}}, \quad \sinh \ell_{vw} = \frac{W}{\sin\frac{\pi}{q}\sin\frac{\pi}{r}},$$

where

$$W = \sqrt{\cos^2\frac{\pi}{p} + \cos^2\frac{\pi}{q} + \cos^2\frac{\pi}{r} + 2\cos\frac{\pi}{p}\cos\frac{\pi}{q}\cos\frac{\pi}{r} - 1}.$$

Let

$$\rho_u = \rho_u(\frac{2\pi}{p}), \quad \rho_v = \rho_v(\frac{2\pi}{q}) \quad \text{and} \quad \rho_w = \rho_w(\frac{2\pi}{r}),$$

where $\rho_x(t)$ is the rotation at x through the angle t as defined in section 2.5.

The triangle group $\Gamma(p,q,r)$ is generated by the elements ρ_u, ρ_v, ρ_w and has the presentation

$$\Gamma(p,q,r) = < \rho_u, \rho_v, \rho_w | \rho_u^p = \rho_v^q = \rho_w^r = \rho_u\rho_v\rho_w = 1 > .$$

The third generator ρ_w can be expressed in terms of the other two, hence we also have the presentation

$$\Gamma(p,q,r) = < \rho_u \rho_v | \rho_u^p = \rho_v^q = (\rho_u\rho_v)^r = 1 > .$$

Let

$$\tilde{r}_u = \tilde{r}_u(\frac{2\pi}{p}), \quad \tilde{r}_v = \tilde{r}_v(\frac{2\pi}{q}), \quad \text{and} \quad \tilde{r}_w = \tilde{r}_w(\frac{2\pi}{r}),$$

where $\tilde{r}_x(t)$ is as defined in section 2.5.

Proposition 23. *(Taken from [15] Proposition 36)*
We have $\tilde{r}_u^p = \tilde{r}_v^q = \tilde{r}_w^r = \tilde{r}_u\tilde{r}_v\tilde{r}_w = c$.

Proposition 24. *In the case* $\tilde{\Gamma}_1 = \tilde{\Gamma}(p,3,3)$ *we have* $(\tilde{r}_v^2\tilde{r}_u^{-1})^2 = \tilde{r}_u\tilde{r}_v^{-2}c$ *and* $\tilde{r}_u\tilde{r}_v\tilde{r}_u = \tilde{r}_v^2\tilde{r}_u^{-1}\tilde{r}_v^2$.

Proof. We have $q = r = 3$, hence $\tilde{r}_u^p = \tilde{r}_v^3 = \tilde{r}_w^3 = \tilde{r}_u\tilde{r}_v\tilde{r}_w = c$ by Proposition 23. Then $\tilde{r}_w = \tilde{r}_v^{-1}\tilde{r}_u^{-1}c$ and $c = \tilde{r}_w^3 = (\tilde{r}_v^{-1}\tilde{r}_u^{-1}c)^3$, hence $c = (\tilde{r}_v^{-1}\tilde{r}_u^{-1})^3c^3$ and $(\tilde{r}_v^{-1}\tilde{r}_u^{-1})^3 = c^{-2}$. Now $\tilde{r}_v^3 = c$ implies $\tilde{r}_v^{-1} = \tilde{r}_v^2c^{-1}$. Hence $c^{-2} = (\tilde{r}_v^{-1}\tilde{r}_u^{-1})^3 = (\tilde{r}_v^2c^{-1}\tilde{r}_u^{-1})^3 = (\tilde{r}_v^2\tilde{r}_u^{-1})^3c^{-3}$ and therefore $c = (\tilde{r}_v^2\tilde{r}_u^{-1})^3 = (\tilde{r}_v^2\tilde{r}_u^{-1})^2\tilde{r}_v^2\tilde{r}_u^{-1}$. So, $\tilde{r}_u\tilde{r}_v^{-2}c = (\tilde{r}_v^2\tilde{r}_u^{-1})^2$.
For $\tilde{r}_u\tilde{r}_v\tilde{r}_u = \tilde{r}_v^2\tilde{r}_u^{-1}\tilde{r}_v^2$ we have $(\tilde{r}_v^2\tilde{r}_u^{-1})^2 = \tilde{r}_v^2\tilde{r}_u^{-1}\tilde{r}_v^2\tilde{r}_u^{-1} = \tilde{r}_u\tilde{r}_v^{-2}c$. Now $\tilde{r}_v^3 = c$ implies $\tilde{r}_v^{-2} = \tilde{r}_vc^{-1}$ and therefore $\tilde{r}_u\tilde{r}_v^{-2}c = \tilde{r}_u\tilde{r}_vc^{-1}c = \tilde{r}_u\tilde{r}_v$. Hence, $\tilde{r}_v^2\tilde{r}_u^{-1}\tilde{r}_v^2 = \tilde{r}_u\tilde{r}_v\tilde{r}_u$.

∎

Theorem 25. *(Taken from [15] Theorem 38)*
If the condition

$$\gcd(k,p) = \gcd(k,q) = \gcd(k,r) = 1, \qquad pqr - pq - qr - rp \equiv 0 \quad mod\ k \quad (3.1)$$

is satisfied, then there exists a unique subgroup of $\tilde{G}$ of level k with image $\Gamma(p,q,r)$ in $PSU(1,1)$. We denote this subgroup by $\Gamma(p,q,r)^k$. If the condition (3.1) is satisfied, then there exist integers n_u and n_v such that

$$pn_u + 1 \equiv qn_v + 1 \equiv 0 \quad mod\ k,$$

and then a set of generators of $\Gamma(p,q,r)^k$ is given by the elements $\tilde{r}_u c^{n_u}, \tilde{r}_v c^{n_v}$, c^k. If the condition (3.1) is not satisfied, then there is no subgroup of $\tilde{G}$ of level k with image $\Gamma(p,q,r)$ in $PSU(1,1)$.
In particular:
If $q = r = 3, k \not\equiv 0 \mod 3$ and $p \equiv 3 \mod k$, then the condition (3.1) is satisfied. A set of generators of the subgroup $\Gamma(p,3,3)^k$ is then given by the elements $\tilde{r}_u c^n, \tilde{r}_v c^n, c^k$, where n is the unique integer such that $3n + 1 \equiv 0 \mod k$ and $0 \leq n < k$.

Example 26. *Let $k = 2$ then by applying Theorem 25 we have that the generators of the subgroup $\Gamma(p,3,3)^2$ are given by $\tilde{r}_u c, \tilde{r}_v c, c^2$ since $k = 2$ and $n = 1$. We know from Proposition 13 that $c = \tilde{r}_x(2\pi)$ for any $x \in \mathbb{D}$. The projection from $\tilde{G}$ to $SU(1,1)$ gave us that $c = \tilde{r}_x(2\pi) \mapsto r_x(2\pi) = -1$. So, the generators of the subgroup $\Gamma(p,3,3)^2$ are given by:*

$$\tilde{r}_u c \mapsto -r_u, \quad \tilde{r}_v c \mapsto -r_v, \quad c^2 \mapsto (-1)^2 = 1.$$

Remark 27. *A lift of the triangle group $\Gamma(p,q,r)$ into the k-fold covering G_k of $PSU(1,1)$ is a subgroup of G_k such that the restriction of the projection $G_k \to PSU(1,1)$ is an isomorphism of the subgroup and $\Gamma(p,q,r)$.*
There is a 1-1-corresponding between lifts of $\Gamma(p,q,r)$ into G_k and subgroups of level k in $\tilde{G}$ whose projection is $\Gamma(p,q,r)$.

3.2 Crowns of triangle groups

A tiling of the hyperbolic plane $\mathbb{D}$ is given by the orbit of the hyperbolic triangle $\Delta(p,q,r)$ under the action of the group $\Gamma(p,q,r)^*$.
Let $\Gamma = \Gamma(p,q,r)$ be a triangle group.

Definition 28. *(Dirichlet region): Dirichlet region for the group Γ centered* *at $x \in \mathbb{D}$ is the set*

$$D_x(\Gamma) = \{z \in \mathbb{D} \mid \rho(z,x) \leq \rho(z,T(x)) \text{ for all } T \in \Gamma\}.$$

The Dirichlet region of Γ centered at u can also be described as the union of those images of the triangle $\Delta(p,q,r)$ under the elements of $\Gamma(p,q,r)^*$ which contain the point u.

Definition 29. *(Vertex crown $\mathcal{V}_x$): The vertex crown $\mathcal{V}_x$ of $x \in \Gamma(u)$ consists of all those point in $\Gamma(u)\backslash\{x\}$ whose Dirichlet region shares at least one point with the Dirchlet region of x.*

Definition 30. *(Edge crown $\mathcal{E}_x$): The edge crown $\mathcal{E}_x$ of $x \in \Gamma(u)$ consists of all those points in $\Gamma(u)\backslash\{x\}$ whose Dirichlet region shares at least an edge with the Dirchlet region of x.*

Let $\mathcal{V} = \mathcal{V}_u$ and $\mathcal{E} = \mathcal{E}_u$. We have

$$\mathcal{V} = \{x_{m,l} \mid 0 \leq m < p, 1 \leq l < q\} \cup \{x'_{m,l} \mid 0 \leq m < p, 1 \leq l < r\}$$

and

$$\mathcal{E} = \{x_{0,1}, \cdots, x_{p-1,1}, x'_{0,1}, \cdots, x'_{p-1,1}\},$$

where $x_{m,l} = (\rho_u^m \rho_v^l)(u)$ and $x'_{m,l} = (\rho_u^m \rho_w^l)(u)$ for $m, l \in \mathbb{Z}$.

Proposition 31. *In the case $\Gamma = \Gamma(p,3,3)$ we have*

$$\mathcal{V} = \mathcal{E} = \{x_{m,1}, x_{m,2} | 0 \leq m < p\}.$$

The figures 3.2 and 3.3 show the edge crown and the vertex crown for the triangle groups $\Gamma(5,3,3)$ and $\Gamma(7,3,3)$.
The red region is the Dirichlet region for the triangle group $\Gamma(p,3,3)$ centered at $u = 0$, where $p = 5, 7$, and the green region is the union of Dirichlet regions for the edge crown and the vertex crown for this group.

Proposition 32. *For all $x \in \mathcal{E}$ we have*

$$\cosh \rho(u,x) = 2B^2 + 1 \quad \textit{and} \quad |x| = \frac{B}{\sqrt{B^2+1}},$$

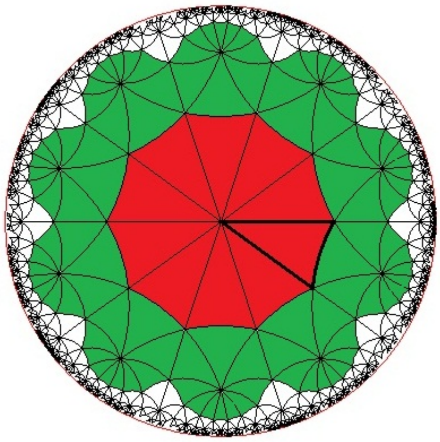

Figure 3.2: The edge crown and the vertex crown for $\Gamma(5, 3, 3)$.

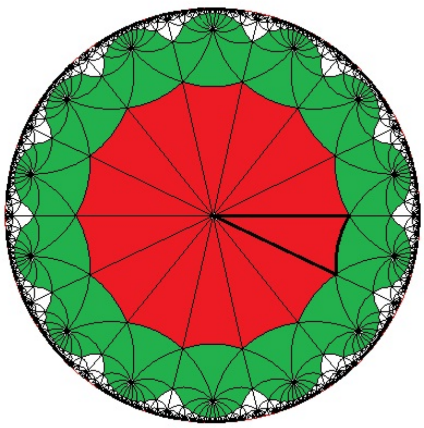

Figure 3.3: The edge crown and the vertex crown for $\Gamma(7, 3, 3)$.

where $B = \sinh \ell_v \sin \frac{\pi}{q}$.

Proposition 33. *For all* $m \in \mathbb{Z}$ *we have*

$$\begin{aligned}
\cosh \rho(x_{m,1}, x_{m,q-1}) &= 2\sinh^2 \ell_v \sin^2 \frac{2\pi}{q} + 1, \\
\cosh \rho(x_{m,q-1}, x_{m+1,1}) &= 2\sinh^2 \ell_w \sin^2 \frac{2\pi}{r} + 1, \\
|x_{m,1} - x_{m,q-1}| &= \frac{\sinh \ell_v \sin \frac{2\pi}{q}}{\sinh^2 \ell_v \sin^2 \frac{\pi}{q} + 1}, \\
|x_{m,q-1} - x_{m+1,1}| &= \frac{\sinh \ell_w \sin \frac{2\pi}{r}}{\sinh^2 \ell_w \sin^2 \frac{\pi}{r} + 1}.
\end{aligned}$$

Moreover

$$\frac{|x_{m,1} - x_{m,q-1}|}{\cos \frac{\pi}{q}} = \frac{|x_{m,q-1} - x_{m+1,1}|}{\cos \frac{\pi}{r}}.$$

3.3 Lifting elliptic elements

To describe isometries in $\mathrm{PSU}(1,1)$ via matrices, we consider their lifts into $\mathrm{SU}(1,1)$. Note that the pre-image of an elliptic element $\rho_x(t)$ under the projection $\mathrm{SU}(1,1) \to \mathrm{PSU}(1,1)$ is $\pm r_x(t)$. We will determine these lifts for the generators

$$\rho_u = \rho_u(\frac{2\pi}{p}), \quad \rho_v = \rho_v(\frac{2\pi}{q}) \text{ and } \rho_w = \rho_w(\frac{2\pi}{r})$$

of the triangle group $\Gamma(p,q,r)$.

Recall that $u = 0, v \in \mathbb{R}, v > 0, w = |w| \cdot e^{i\frac{\pi}{p}}$ (see figure 3.1). Using formula for $r_x(t)$ in section 2.5 we obtain:

$$r_u = r_0\left(\frac{2\pi}{p}\right) = \left(\cos\frac{\pi}{p} + i \sin\frac{\pi}{p}, 0\right)$$

$$= (e^{i\frac{\pi}{p}}, 0).$$

$$
\begin{aligned}
r_v &= r_v\left(\frac{2\pi}{q}\right) \\
&= \left(\cos\frac{\pi}{q} + i\cosh\rho(0,v)\sin\frac{\pi}{q}, -i\frac{v}{|v|}\sinh\rho(0,v)\sin\frac{\pi}{q}\right) \\
&= \left(\cos\frac{\pi}{q} + i\cosh\rho(0,v)\sin\frac{\pi}{q}, -i\sinh\rho(0,v)\sin\frac{\pi}{q}\right).
\end{aligned}
$$

$$
\begin{aligned}
r_w &= r_w\left(\frac{2\pi}{r}\right) \\
&= \left(\cos\frac{\pi}{r} + i\cosh\rho(0,w)\sin\frac{\pi}{r}, -i\frac{w}{|w|}\sinh\rho(0,w)\sin\frac{\pi}{r}\right) \\
&= \left(\cos\frac{\pi}{r} + i\cosh\rho(0,w)\sin\frac{\pi}{r}, -ie^{i\frac{\pi}{p}}\sinh\rho(0,w)\sin\frac{\pi}{r}\right).
\end{aligned}
$$

Finally we can write these elements as matrices as follows:

$$
r_u = \begin{pmatrix} e^{i\frac{\pi}{p}} & 0 \\ 0 & e^{-i\frac{\pi}{p}} \end{pmatrix},
$$

$$
r_v = \begin{pmatrix} \cos\frac{\pi}{q} + i\cosh\rho(0,v)\sin\frac{\pi}{q} & -i\sinh\rho(0,v)\sin\frac{\pi}{q} \\ i\sinh\rho(0,v)\sin\frac{\pi}{q} & \cos\frac{\pi}{q} - i\cosh\rho(0,v)\sin\frac{\pi}{q} \end{pmatrix},
$$

$$
r_w = \begin{pmatrix} \cos\frac{\pi}{r} + i\cosh\rho(0,w)\sin\frac{\pi}{r} & -ie^{i\frac{\pi}{p}}\sinh\rho(0,w)\sin\frac{\pi}{r} \\ ie^{i\frac{\pi}{p}}\sinh\rho(0,w)\sin\frac{\pi}{r} & \cos\frac{\pi}{r} - i\cosh\rho(0,w)\sin\frac{\pi}{r} \end{pmatrix}.
$$

Proposition 34. *For $k, l \in \mathbb{Z}$ we have*

$$
r_u^k = (e^{i\frac{k\pi}{p}}, 0), \quad r_v^l = \left(\cos\frac{l\pi}{q} + iC\sin\frac{l\pi}{q}, -iB(l)\right),
$$

where $C = \cosh\rho(0,v), S = \sinh\rho(0,v)$ *and* $B(l) = S\sin\frac{l\pi}{q} = \sinh\rho(0,v)\sin\frac{l\pi}{q}$.

Chapter 4

Finding finite representation of fundamental domains for the groups $\tilde{\Gamma}(p_1, q, r)^k \times (C_{p_2})^k$

4.1 Net estimate

Let $\tilde{\Gamma}_1 = \tilde{\Gamma}(p_1, q, r)^k$, with $p_1 \geq k + 3$. Let $\tilde{\Gamma}_2 = (C_{p_2})^k$ be the covering of a cyclic group of order p_2 in PSU(1,1). Recall that $\vartheta_k = \frac{\pi k}{2\,\mathrm{lcm}(p_1, p_2)}$. In the case $\gcd(p_1, p_2) = 1$ we have $\vartheta_k = \frac{\pi k}{2 p_1 p_2}$. In this section we generalise the net estimate described in [15] to the case $\tilde{\Gamma}(p_1, q, r)^k \times (C_{p_2})^k$.

4.1.1 Net estimate

We are going to show an estimate for the distance from the vertical axis in the (w, z)–space to the points in $\pi(F_{\mathcal{E}})$, where $F_{\mathcal{E}} = (\tilde{E}_{\tilde{e}} \cap \partial Q_u)^\circ - (\bigcup_{x \neq u} Q_x)$, by approximating the sets $\pi(Q_x)$ through their inscribed cylinders.

Notation: In a metric space X let us denote by $U(x, r)$ resp. $B(x, r)$ the open resp. closed ball of radius r with center at $x \in X$.

Definition 35. *An $s-$net of radius d is a finite subset of the circle of radius d with center at 0 in $\mathbb{C}$ such that for some numbering $x_1, ..., x_m, x_{m+1} = x_1$ of the points clockwise we have max$_i$ $|x_i - x_{i+1}| = s > 0$. (Then the inequality $s \leq 2d$ must hold).*

Definition 36. *Let $N \subset \tilde{\Gamma}_1(u) \setminus \{u\}$ be an s-net of radius $d < 1$. Recall that $s < 2d$. Let*

$$R_N = \frac{2d^2}{\sqrt{4d^2 - s^2}}.$$

If $R_N < 1$ then $s < 2d\sqrt{1-d^2}$, hence there exists $\vartheta_N \in (0, \frac{\pi}{2})$ such that

$$\cos\vartheta_N = \frac{s}{2d\sqrt{1-d^2}}.$$

For $\vartheta \in [-\vartheta_N, \vartheta_N]$ let

$$\ell_N^{\pm}(\vartheta) = \frac{1}{R_N} \pm \frac{1}{2d^2}\sqrt{4d^2(1-d^2)\cos^2\vartheta - s^2}.$$

Remark 37. *$R_N < 1$ implies $\ell_N^+(\vartheta) \geq \frac{1}{R_N} > 1$. The functions $\ell_N^{\pm}$ are even. The functions ℓ_N^- and ℓ_N^+ are monotone increasing and decreasing respectively on $[0, \vartheta_N]$.*

Theorem 38. *Let $N \subset \tilde{\Gamma}_1(u)$ be an s-net of radius $d < 1$ such that $R_N < 1$ and $\vartheta_N \geq \vartheta_k$. The inequality $\vartheta_k \leq \vartheta_N$ implies that the functions $\ell_N^{\pm}$ are defined on $[-\vartheta_k, \vartheta_k]$. Assume that $\ell_N^-(\vartheta_k) \leq 1$. Then for $w = 1 + i\tan\vartheta$ such that $|\vartheta| \leq \vartheta_k$ we have that if the point (w, z) is in $\pi(F_N)$ then*

$$|z| < \ell_N^-(\vartheta) \cdot |w|.$$

Proof. Let $x_1, \cdots, x_m$ be a numbering of the points of the net N such that

$$\max_{i=1,\cdots,m}|x_i - x_{i+1}| = s,$$

where $x_{m+1} = x_1$. We have $|x_i| = d$. Let $z_i = \frac{\bar{w}}{\bar{x}_i}$ and $r = \frac{\sqrt{1-d^2}}{d}$. The geometric meaning of z_i and r is as follows: the point z_i corresponds to the axis of the prism Q_{x_i} and r is the inscribed radius of Q_{x_i}. This can be made precise using Proposition 17, which implies that $(w, z) \in \pi(Q_{x_i})$ provided that

$$|w - x_i\bar{z}| \leq \sqrt{1 - |x_i|^2} \quad \text{and} \quad |w| < |z|.$$

We can rewrite the first inequality as

$$\left|\bar{z} - \frac{w}{x_i}\right| \leq \frac{\sqrt{1 - |x_i|^2}}{|x_i|}, \quad \left|z - \frac{\bar{w}}{\bar{x}_i}\right| \leq \frac{\sqrt{1 - |x_i|^2}}{|x_i|}.$$

With $z_i = \frac{\bar{w}}{\bar{x}_i}$ and $|x_i| = d$ we can rewrite this inequality as

$$|z - z_i| \leq \frac{\sqrt{1 - d^2}}{d} = r.$$

The points $z_i = \frac{\bar{w}}{\bar{x}_i}$ are obtained from the points x_i through inversion, rotation and scaling, hence they also form a net, say a $2\hat{s}$-net of radius $\hat{d}$. We have $\hat{d} = |z_i| = |\frac{w}{x_i}| = \frac{|w|}{d}$. In order to find $\hat{s}$ we compute:

$$|z_i - z_{i+1}| = \left|\frac{\bar{w}}{\bar{x}_i} - \frac{\bar{w}}{\bar{x}_{i+1}}\right| = |\bar{w}| \cdot \left|\frac{\bar{x}_{i+1} - \bar{x}_i}{\bar{x}_i \bar{x}_{i+1}}\right| = \frac{|w| \cdot s}{d^2},$$

hence $2\ \hat{s} = \frac{|w| \cdot s}{d^2}$. We know that $w = 1 + i \tan\vartheta$ and $|w| = \sqrt{1 + \tan^2\vartheta} = \sqrt{\frac{\cos^2\vartheta + \sin^2\vartheta}{\cos^2\vartheta}} = \frac{1}{\cos\vartheta}$ for $\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. The inequality $|\vartheta| \leq \vartheta_k \leq \vartheta_N$ implies that $|w| = \frac{1}{\cos\vartheta} \leq \frac{1}{\cos\vartheta_N} = \frac{2d\sqrt{1-d^2}}{s}$. Hence $\hat{s} \leq r$ because $\hat{s} = \frac{s \cdot |w|}{2d^2} \leq \frac{\sqrt{1-d^2}}{d}$. We have $|z_i - z_{i+1}| = 2\hat{s} \leq 2r$, hence the circles $\partial B(z_i, r)$ and $\partial B(z_{i+1}, r)$ intersect each other in two points at the distance to the origin equal to

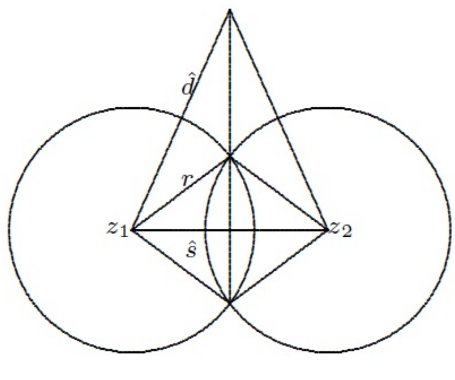

Figure 4.1: The subscribed circle estimate.(Taken from [15]).

$$
\begin{aligned}
&\sqrt{\hat{d}^2 - \hat{s}^2} \pm \sqrt{r^2 - \hat{s}^2} \\
= \ & \sqrt{\frac{|w|^2}{d^2} - \frac{s^2|w|^2}{4d^4}} \pm \sqrt{\frac{1-d^2}{d^2} - \frac{s^2|w|^2}{4d^4}} \\
= \ & |w| \cdot \left(\sqrt{\frac{4d^2 - s^2}{4d^4}} \pm \sqrt{\frac{1-d^2}{d^2|w|^2} - \frac{s^2}{4d^4}} \right) \\
= \ & |w| \cdot \left(\frac{1}{R_N} \pm \frac{1}{2d^2} \sqrt{\frac{4d^2(1-d^2)}{|w|^2} - s^2} \right) \\
= \ & |w| \cdot \ell_N^{\pm}(\vartheta),
\end{aligned}
$$

see figure 4.1. Thus the annulus $\{z \in \mathbb{C} : \ell_N^-(\vartheta) \le \frac{|z|}{|w|} \le \ell_N^+(\vartheta)\}$ is contained in the union of the disks $B(z_i, r)$. Taking into account the assumption $\ell_N^-(\vartheta_k) \le 1$, we have

$$
\ell_N^-(\vartheta) \le \ell_N^-(\vartheta_k) \le 1 < \ell_N^+(\vartheta_k) \le \ell_N^+(\vartheta).
$$

Hence $\ell_N^-(\vartheta)|w| \le |z| < |w|$ implies $\ell_N^-(\vartheta) \le \frac{|z|}{|w|} < 1 \le \ell_N^+$ and therefore $(w,z) \in \pi(\tilde{E}_{\tilde{e}} \cap \bigcup_{x \in N} Q_x)$. Thus $|z| < \ell_N^-(\vartheta) \cdot |w|$ implies $(w,z) \in \pi(F_N)$. ∎

Proposition 39. *Let Γ be the triangle group $\Gamma(p,q,r)$ with $p \ge q \ge r$. Let $B = \sinh \ell_v \cdot \sin \frac{\pi}{q}$, where ℓ_v is defined in subsection 3.1. Then the edge crown $\mathcal{E}$ is an s-net of radius d, where*

$$
s = \frac{2B \cos \frac{\pi}{q}}{B^2 + 1} \quad \textit{and} \quad d = \frac{B}{\sqrt{B^2+1}}.
$$

For this net, we have $d < 1$, $R_{\mathcal{E}} = \tanh \ell_v < 1$, $\vartheta_{\mathcal{E}} = \frac{\pi}{q}$ and

$$
\ell_{\mathcal{E}}^{\pm}(\vartheta) = \frac{1}{R_{\mathcal{E}}} \pm \frac{1}{B} \cdot \sqrt{\cos^2 \vartheta - \cos^2 \frac{\pi}{q}}.
$$

Proof. Proposition 32 implies that $\mathcal{E}$ is an s-net of radius d, where

$$
s = \max\{|x_{m,1} - x_{m,q-1}|, |x_{m,q-1} - x_{m+1,1}|\} \ \text{ and } \ d = \frac{B}{\sqrt{B^2+1}}.
$$

Proposition 33 and the fact that $q \geq r$ imply

$$|x_{m,1} - x_{m,q-1}| = \frac{\cos\frac{\pi}{q}}{\cos\frac{\pi}{r}} \cdot |x_{m,q-1} - x_{m+1,1}| \geq |x_{m,q-1} - x_{m+1,1}|,$$

hence $s = |x_{m,1} - x_{m,q-1}| = \frac{\sinh \ell_v \sin\frac{2\pi}{q}}{\sinh^2 \ell_v \sin^2\frac{\pi}{q}+1} = \frac{2B\cos\frac{\pi}{q}}{B^2+1}$. From the formula for s and d one can derive the formula for $R_{\mathcal{E}}, \vartheta_{\mathcal{E}}$ and $\ell^{\pm}_{\mathcal{E}}(\vartheta)$. ∎

Now we can specialise Theorem 38 for the case $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(p_1, 3, 3)^k \times (C_{p_2})^k$ and $N = \mathcal{E}$:

Theorem 40. *Let* $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(p_1, 3, 3)^k \times (C_{p_2})^k$. *We assume that* $\vartheta_k \leq \frac{\pi}{q}$ *and* $\ell^-_{\mathcal{E}}(\vartheta_k) \leq 1$. *Let* $w = 1 + i\tan\vartheta$ *such that* $|\vartheta| \leq \vartheta_k$. *Then if the point* (w, z) *is in* $\pi(F_{\mathcal{E}})$ *then*

$$|z| < \ell^-_{\mathcal{E}}(\vartheta) \cdot |w|.$$

4.1.2 Application of the net estimate

We shall apply the net estimate to the edge crown $\mathcal{E}$. The following Proposition was taken from [15].

Proposition 41. *Let* x *be a point in* $\tilde{\Gamma}_1(u)\backslash\{u\}$ *such that* $|x| \geq R > 0$. *Then for all* $(w, z) \in \pi(Q_x)$ *we have*

$$|z| \geq \left(\frac{1}{R} - \frac{\sqrt{1 - R^2}}{R|w|\cos\vartheta_k}\right) \cdot |w|.$$

Proof. Let (w, z) be a point in $\pi(Q_x)$. Proposition 17 implies that $|z| \geq f(|x|) \cdot |w|$, where $f(t) = \frac{1}{t}(1 - c\sqrt{1 - t^2})$ and $c = \frac{1}{|w|\cos\vartheta_k} \geq 1$. The function f is monotone increasing on $(0, 1)$ because $f'(t) = \frac{c - \sqrt{1-t^2}}{t^2\sqrt{1-t^2}}$. Thus

$$|z| \geq f(|x|) \cdot |w| \geq f(R) \cdot |w|.$$

∎

In subsection 4.1.1 we proved an estimate on the distance from the vertical axis to the points of the set $\pi(F_{\mathcal{E}})$. On the other hand, in Proposition 41 the lower bound was determined for the distance from the vertical axis to the points of the set $\pi(Q_x)$. Combining these two estimates we can show under certain conditions that the sets $\pi(Q_x)$ with $x \notin \mathcal{E} \cup \{u\}$ share no points with $\pi(F_{\mathcal{E}})$ and therefore with $\pi(F_{\tilde{e}}) \subset \pi(F_{\mathcal{E}})$, i.e. $\pi(F_{\tilde{e}}) = \pi(F_{\mathcal{E}})$ and hence $F_{\tilde{e}} = F_{\mathcal{E}}$.

Notation: For $R \in (0,1)$ let

$$M_R = \frac{1 - \sqrt{1 - R^2}}{R}.$$

We have $M_R \in (0,1)$.

Proposition 42. *Let $N \subset \tilde{\Gamma}_1(u)$ be an s-net of radius $d < 1$ such that $R_N < 1$ and $\vartheta_N \geq \vartheta_k$. The inequality $\vartheta_k \geq \vartheta_N$ implies that the function $\ell_N^{\pm}$ is defined on $[-\vartheta_k, \vartheta_k]$. We assume that $\ell_N^+(\vartheta_k) \geq 1$ (follows from $R_N < 1$) and $\ell_N^-(\vartheta_k) \leq M_R$ for some $R \geq R_N$. Then $\pi(Q_x) \cap \pi(F_N) = \varnothing$ for all $x \in \tilde{\Gamma}_1(u)$ such that $|x| \geq R$.*

Proof. Let x be a point in $\tilde{\Gamma}_1(u) \backslash \{u\}$ such that $|x| \geq R$. Let $w = 1 + i \tan \vartheta$ such that $|\vartheta| \leq \vartheta_k$. We have $|w| = \frac{1}{\cos\vartheta}$ and $\ell_N^-(\vartheta_k) \leq M_R < 1$. So, all the conditions of Theorem 38 are satisfied, hence for all $(w,z) \in \pi(F_N)$ we have

$$|z| < \ell_N^-(\vartheta) \cdot |w|.$$

On the other hand, Proposition 41 implies that for all $(w,z) \in \pi(Q_x)$ we have

$$\begin{aligned} |z| \quad &\geq \quad \left(\frac{1}{R} - \frac{\sqrt{1-R^2}}{R \cdot |w| \cos\vartheta_k} \right) \cdot |w| \\ &= \quad \left(\frac{1}{R} - \frac{\cos\vartheta}{\cos\vartheta_k} \cdot \frac{\sqrt{1-R^2}}{R} \right) \cdot |w|. \end{aligned}$$

So, it is sufficient to show that the function

$$\Delta(\vartheta) = \left(\frac{1}{R} - \frac{\cos\vartheta}{\cos\vartheta_k} \cdot \frac{\sqrt{1-R^2}}{R} \right) - \ell_N^-(\vartheta)$$

is non-negative on $[0, \vartheta_k]$. The function $\Delta(\vartheta)$ can be written as

$$\Delta(\vartheta) = a(\vartheta) \cdot \frac{\cos\vartheta}{\cos\vartheta_k} + \left(\frac{1}{R} - \frac{1}{R_N} \right),$$

where

$$a(\vartheta) = \left(\frac{1}{R_N} - \ell_N^-(\vartheta)\right) \cdot \frac{\cos\vartheta_k}{\cos\vartheta} - \frac{\sqrt{1-R^2}}{R}.$$

We compute

$$\begin{aligned}
\left(\frac{1}{R_N} - \ell_N^-(\vartheta)\right) &= \left(\frac{1}{R_N} - \left(\frac{1}{R_N} - \frac{1}{2d^2}\sqrt{4d^2(1-d^2)\cos^2\vartheta - s^2}\right)\right) \\
&= \frac{1}{2d^2}\sqrt{4d^2(1-d^2)\cos^2\vartheta - s^2}.
\end{aligned}$$

We see that

$$\begin{aligned}
a(\vartheta) &= \left(\frac{1}{R_N} - \ell_N^-(\vartheta)\right) \cdot \frac{\cos\vartheta_k}{\cos\vartheta} - \frac{\sqrt{1-R^2}}{R} \\
&= \frac{\cos\vartheta_k}{2d^2} \cdot \sqrt{4d^2(1-d^2) - \frac{s^2}{\cos^2\vartheta}} - \frac{\sqrt{1-R^2}}{R}
\end{aligned}$$

is monotone increasing on $\vartheta \in [-\vartheta_k, 0]$ and monotone decreasing for $\vartheta \in [0, \vartheta_k]$. Hence we have for all $\vartheta \in [-\vartheta_k, \vartheta_k]$ that

$$a(\vartheta) \geq a(\vartheta_k).$$

We compute

$$\begin{aligned}
a(\vartheta_k) &= \left(\frac{1}{R_N} - \ell_N^-(\vartheta_k)\right) - \frac{\sqrt{1-R^2}}{R} \\
&= \frac{1}{2d^2}\sqrt{4d^2(1-d^2)\cos^2\vartheta_k - s^2} - \frac{\sqrt{1-R^2}}{R} \\
&= \left(\frac{1}{R_N} - \frac{1}{R}\right) + \frac{1}{R} - \frac{\sqrt{1-R^2}}{R} - \frac{1}{R_N} + \frac{1}{2d^2}\sqrt{4d^2(1-d^2)\cos^2\vartheta_k - s^2} \\
&= \left(\frac{1}{R_N} - \frac{1}{R}\right) + \frac{1-\sqrt{1-R^2}}{R} - \left(\frac{1}{R_N} - \frac{1}{2d^2}\sqrt{4d^2(1-d^2)\cos^2\vartheta_k - s^2}\right) \\
&= \left(\frac{1}{R_N} - \frac{1}{R}\right) + (M_R - \ell_N^-(\vartheta_k)).
\end{aligned}$$

and therefore

$$\begin{aligned}
\Delta(\vartheta) &= a(\vartheta)\cdot\frac{\cos\vartheta}{\cos\vartheta_k} + (\frac{1}{R} - \frac{1}{R_N}) \\
&\geq \frac{\cos\vartheta}{\cos\vartheta_k}\left(\left(\frac{1}{R_N} - \frac{1}{R}\right) + (M_R - \ell_N^-(\vartheta_k))\right) + \left(\frac{1}{R} - \frac{1}{R_N}\right) \\
&\geq \left(\left(\frac{1}{R_N} - \frac{1}{R}\right) + (M_R - \ell_N^-(\vartheta_k))\right) + \left(\frac{1}{R} - \frac{1}{R_N}\right) \\
&= M_R - \ell_N^-(\vartheta_k).
\end{aligned}$$

Recall that from our assumptions we have $\ell_N^-(\vartheta_k) \leq M_R \leq 1$. Hence $M_R - \ell_N^-(\vartheta_k) \geq 0$ and thereofre $\Delta(\vartheta) \geq 0$. So, we have shown that

$$\pi(Q_x) \cap \pi(F_N) = \varnothing$$

for all $x \in \tilde{\Gamma}_1(u)$ such that $|x| \geq R$. ∎

Now we can specialise Proposition 42 for the edge crown $\mathcal{E}$ as follows:

Proposition 43. *Let $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(p_1, q, r)^k \times (C_{p_2})^k$ with $p_1 \geq q \geq r$. We assume that $\vartheta_k \leq \frac{\pi}{q}$ and $\ell_{\mathcal{E}}^-(\vartheta_k) \leq M_R$ for some $R \geq R_{\mathcal{E}}$. Then*

$$\pi(Q_x) \cap \pi(F_{\mathcal{E}}) = \varnothing$$

for all $x \in \tilde{\Gamma}_1(u)$ such that $|x| \geq R$.

Theorem 44. *Let $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(p_1, q, r)^k \times (C_{p_2})^k$ with $p_1 \geq q \geq r$. We assume that $\vartheta_k \leq \frac{\pi}{q}$. Let $R \geq R_{\mathcal{E}}$ be such that $\tilde{\Gamma}_1(u) \cap U(u, R) \subset \mathcal{E} \cup \{u\}$ and $\ell_{\mathcal{E}}^-(\vartheta_k) \leq M_R$. Then*

$$F_{\tilde{\Gamma}_1(u)\setminus\{u\}} = F_{\mathcal{E}}.$$

Proof. We have $\ell_{\mathcal{E}}^-(\vartheta_k) \leq M_R < 1$. Proposition 43 implies that $\pi(Q_x) \cap \pi(F_{\mathcal{E}}) = \varnothing$ for all $x \in \tilde{\Gamma}_1(u)\backslash(\mathcal{E} \cup \{u\})$ and therefore $F_{\tilde{\Gamma}_1(u)\setminus\{u\}} = F_{\mathcal{E}}$. ∎

4.1.3 Applying the net estimate for the cases $\tilde{\Gamma}(p_1,3,3)^k \times \tilde{\Gamma}(C_3)^k$

We are going to use the net estimate for the groups $\tilde{\Gamma}(p_1,q,r)^k \times (C_{p_2})^k = \tilde{\Gamma}(p_1,3,3)^k \times (C_3)^k$ and we will see how the net estimate works for these cases. In particular, we shall use Theorem 44 in order to show that $F_{\tilde{\Gamma}_1(u)\smallsetminus\{u\}} = F_{\mathcal{E}}$. Let us recall some formulas. Recall from Proposition 34 the notation $C = \cosh \ell_v = \cosh \rho(0,v)$ and $S = \sinh \ell_v = \sinh \rho(0,v)$. Recall from subsection 3.1

$$
\begin{aligned}
C &= \cosh \ell_v = \frac{\cos\frac{\pi}{p}\cos\frac{\pi}{q} + \cos\frac{\pi}{r}}{\sin\frac{\pi}{p}\sin\frac{\pi}{q}} \\
&= \frac{\cos\frac{\pi}{p_1}\cos\frac{\pi}{3} + \cos\frac{\pi}{3}}{\sin\frac{\pi}{p_1}\sin\frac{\pi}{3}} = \frac{\cos\frac{\pi}{p_1} + 1}{\sin\frac{\pi}{p_1}} \cdot \frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} \\
&= \frac{(2\cos^2\frac{\pi}{2p_1} - 1) + 1}{2\sin\frac{\pi}{2p_1}\cos\frac{\pi}{2p_1}} \cdot \cot\frac{\pi}{3} = \frac{2\cos^2\frac{\pi}{2p_1}}{2\sin\frac{\pi}{2p_1}\cos\frac{\pi}{2p_1}} \cdot \frac{1}{\sqrt{3}} \\
&= \frac{1}{\sqrt{3}} \cdot \cot\frac{\pi}{2p_1},
\end{aligned}
$$

$$
\begin{aligned}
S &= \sqrt{\cosh^2 \ell_v - 1} = \sqrt{\left(\frac{1}{\sqrt{3}} \cdot \cot\frac{\pi}{2p_1}\right)^2 - 1} \\
&= \sqrt{\frac{\cos^2\frac{\pi}{2p_1}}{3\sin^2\frac{\pi}{2p_1}} - 1} = \sqrt{\frac{\cos^2\frac{\pi}{2p_1} - 3\sin^2\frac{\pi}{2p_1}}{3\sin^2\frac{\pi}{2p_1}}} \\
&= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{\cos^2\frac{\pi}{2p_1} - 3\sin^2\frac{\pi}{2p_1}}}{\sin\frac{\pi}{2p_1}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{\frac{1}{2}(1+\cos\frac{\pi}{p_1}) - \frac{3}{2}(1-\cos\frac{\pi}{p_1})}}{\sin\frac{\pi}{2p_1}} \\
&= \frac{1}{\sqrt{3}} \frac{\sqrt{2\cos\frac{\pi}{p_1} - 1}}{\sin\frac{\pi}{2p_1}}.
\end{aligned}
$$

Using the formula (8.5) (in appendix) which says that $\cos 3x = \cos x(2\cos 2x - 1)$ we have

$$S = \frac{1}{\sqrt{3}} \frac{\sqrt{2\cos\frac{\pi}{p_1} - 1}}{\sin\frac{\pi}{2p_1}} = \frac{1}{\sqrt{3}\sin\frac{\pi}{2p_1}} \sqrt{\frac{\cos\frac{3\pi}{2p_1}}{3\cos\frac{\pi}{2p_1}}} = \frac{1}{\sqrt{3}\sin\frac{\pi}{2p_1} D},$$

where $D := \sqrt{\frac{\cos\frac{\pi}{2p_1}}{\cos\frac{3\pi}{2p_1}}} > 1$. Recall that $R_{\mathcal{E}} = \tanh \ell_v$ and $B = \sinh \ell_v \cdot \sin\frac{\pi}{q}$ (see Proposition 39). Hence

$$R_{\mathcal{E}} = \frac{\sinh \ell_v}{\cosh \ell_v} = \frac{S}{C} = \frac{\frac{1}{\sqrt{3}\sin\frac{\pi}{2p_1} D}}{\frac{1}{\sqrt{3}} \cdot \cot\frac{\pi}{2p_1}} = \frac{1}{\cos\frac{\pi}{2p_1} D},$$

and

$$B = S \cdot \sin\frac{\pi}{3} = \frac{1}{\sqrt{3}\sin\frac{\pi}{2p_1} D} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2\sin\frac{\pi}{2p_1} D}.$$

Note that D can be rewritten as $D = \frac{1}{\sqrt{2\cos\frac{\pi}{p_1} - 1}}$, hence $S = \frac{1}{\sqrt{3}} \frac{\sqrt{2\cos\frac{\pi}{p_1} - 1}}{\sin\frac{\pi}{2p_1}}$, $R_{\mathcal{E}} = \frac{\sqrt{2\cos\frac{\pi}{p_1} - 1}}{\cos\frac{\pi}{2p_1}}$ and $B = \frac{\sqrt{2\cos\frac{\pi}{p_1} - 1}}{2\sin\frac{\pi}{2p_1}}$. Recall that $\vartheta_k = \frac{k\pi}{2\,\mathrm{lcm}(p_1, p_2)}$, hence in our case $\vartheta_k = \frac{k\pi}{6p_1}$.

Proposition 45. *We have*

$$\ell_{\mathcal{E}}^{-}(\vartheta_k) = D\left(\cos\frac{\pi}{2p_1} - \sin\frac{\pi}{2p_1}\sqrt{4\cos^2\vartheta_k - 1}\right).$$

Proof. We know that

$$\begin{aligned} \ell_{\mathcal{E}}^{-}(\vartheta_k) &= \frac{1}{R_{\mathcal{E}}} - \frac{1}{B} \cdot \sqrt{\cos^2\vartheta_k - \cos^2\frac{\pi}{q}} \\ &= \cos\frac{\pi}{2p_1} \cdot D - 2\sin\frac{\pi}{2p_1} \cdot D \cdot \sqrt{\cos^2\vartheta_k - \frac{1}{4}} \\ &= D \cdot \left(\cos\frac{\pi}{2p_1} - \sin\frac{\pi}{2p_1}\sqrt{4\cos^2\vartheta_k - 1}\right). \end{aligned}$$

∎

Now, in order to be able to apply Theorem 44 we need to find $R \geq R_{\mathcal{E}}$ such that $\tilde{\Gamma}_1(u) \cap U(u,R) \subset \mathcal{E} \cup \{u\}$.

Proposition 46. *For*

$$R = \frac{\cos\frac{\pi}{2p_1}}{\cos\frac{\pi}{p_1}} \cdot \frac{1}{D} = \frac{\cos\frac{\pi}{2p_1}}{\cos\frac{\pi}{p_1}} \cdot \sqrt{2\cos\frac{\pi}{p_1} - 1},$$

we have $R > R_{\mathcal{E}}$ and $\tilde{\Gamma}_1(u) \cap U(u,R) \subset \mathcal{E} \cup \{u\}$.

Proof. For $\tilde{\Gamma}_1 = \Gamma(p_1,3,3)^k$ we have

$$\cosh\rho(0,x) \geq \frac{2\cos^2\frac{\pi}{p_1}}{\sin^2\frac{\pi}{2p_1}} - 1$$

for any $x \in \left(\tilde{\Gamma}_1(u)\backslash\{u\}\right) - \mathcal{E}$ as a consequence of Lemma 11.2 and Formula (11.6) in [8], where the angle α_u in [8] corresponds here to the angle $\frac{\pi}{p_1}$. Using the formula

$$|x|^2 = \frac{\cosh\rho(0,x) - 1}{\cosh\rho(0,x) + 1},$$

which connects the Euclidean and hyperbolic distance in $\mathbb{D}$, we obtain

$$\begin{aligned} |x|^2 = \frac{\cosh\rho(0,x) - 1}{\cosh\rho(0,x) + 1} &= 1 - \frac{2}{\cosh\rho(0,x) + 1} \geq 1 - \frac{2}{\left(\frac{2\cos^2\frac{\pi}{p_1}}{\sin^2\frac{\pi}{2p_1}} - 1\right) + 1} \\ &= 1 - \frac{\sin^2\frac{\pi}{2p_1}}{\cos^2\frac{\pi}{p_1}} = \frac{\cos^2\frac{\pi}{p_1} - \sin^2\frac{\pi}{2p_1}}{\cos^2\frac{\pi}{p_1}}. \end{aligned}$$

Moreover

$$\begin{aligned} \cos^2\frac{\pi}{p_1} - \sin^2\frac{\pi}{2p_1} &= \left(\frac{1 + \cos\frac{2\pi}{p_1}}{2}\right) - \left(\frac{1 - \cos\frac{\pi}{p_1}}{2}\right) \\ &= \frac{\cos\frac{2\pi}{p_1} + \cos\frac{\pi}{p_1}}{2} = \cos\frac{3\pi}{2p_1}\cos\frac{\pi}{2p_1}. \end{aligned}$$

We obtain

$$|x|^2 \geq \frac{\cos\frac{\pi}{2p_1}\cos\frac{3\pi}{2p_1}}{\cos^2\frac{\pi}{p_1}} = R^2$$

for any $x \in \tilde{\Gamma}_1(u) \backslash (\{u\} - \mathcal{E})$. Moreover

$$\frac{1}{R} = \frac{\cos\frac{\pi}{p_1}}{\cos\frac{\pi}{2p_1}} \cdot D < \cos\frac{\pi}{2p_1} \cdot D = \frac{1}{R_{\mathcal{E}}},$$

hence

$$R > R_{\mathcal{E}}.$$

∎

Proposition 47. *For R as in Proposition 46 we have*

$$M_R = \frac{1 - \sqrt{1 - R^2}}{R} = D \cdot \frac{\cos\frac{\pi}{p_1} - \sin\frac{\pi}{2p_1}}{\cos\frac{\pi}{2p_1}}.$$

Proof. We know that

$$D = \sqrt{\frac{\cos\frac{\pi}{2p_1}}{\cos\frac{3\pi}{2p_1}}}, \quad R = \frac{\cos\frac{\pi}{2p_1}}{\cos\frac{\pi}{p_1} \cdot D}.$$

Then

$$\begin{aligned} M_R &= \frac{1 - \sqrt{1 - R^2}}{R} = \frac{1 - \sqrt{1 - \left(\frac{\cos\frac{\pi}{2p_1}}{\cos\frac{\pi}{p_1} \cdot D}\right)^2}}{\frac{\cos\frac{\pi}{2p_1}}{\cos\frac{\pi}{p_1} \cdot D}} \\ &= D \frac{\cos\frac{\pi}{p_1}\left(1 - \sqrt{1 - \frac{\cos^2\frac{\pi}{2p_1}\cos\frac{3\pi}{2p_1}}{\cos^2\frac{\pi}{p_1}\cos\frac{\pi}{2p_1}}}\right)}{\cos\frac{\pi}{2p_1}} \\ &= D \frac{\cos\frac{\pi}{p_1}\left(1 - \sqrt{\frac{\cos^2\frac{\pi}{p_1} - \cos\frac{\pi}{2p_1}\cos\frac{3\pi}{2p_1}}{\cos^2\frac{\pi}{p_1}}}\right)}{\cos\frac{\pi}{2p_1}}. \end{aligned}$$

Let $\alpha = \frac{\pi}{2p_1}$, then $\cos\frac{\pi}{p_1} = \cos 2\alpha = 2\cos^2\alpha - 1$, $\cos\frac{3\pi}{2p_1} = \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$. Let $x = \cos\alpha$. Then

$$\begin{aligned}\cos^2\frac{\pi}{p_1} - \cos\frac{\pi}{2p_1}\cos\frac{3\pi}{2p_1} &= \cos^2(2\alpha) - \cos\alpha\cos(3\alpha)\\ &= (2x^2-1)^2 - x(4x^3-3x) = 4x^4 - 4x^2 + 1 - 4x^4 + 3x^2\\ &= -x^2 + 1 = 1 - \cos^2\alpha = \sin^2\alpha = \sin^2\frac{\pi}{2p_1}.\end{aligned}$$

So,

$$M_R = D\cdot\frac{\cos\frac{\pi}{p_1}\left(1-\sqrt{\frac{\sin^2\frac{\pi}{2p_1}}{\cos^2\frac{\pi}{p_1}}}\right)}{\cos\frac{\pi}{2p_1}} = D\cdot\frac{\cos\frac{\pi}{p_1} - \sin\frac{\pi}{2p_1}}{\cos\frac{\pi}{2p_1}}.$$

∎

Therefore we can now apply Theorem 44 as follows:

Proposition 48. *In the case* $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(p_1,3,3)^k \times (C_3)^k$ *we have* $\ell^-_{\mathcal{E}}(\vartheta_k) \le M_R$ *and hence*

$$F_{\tilde{\Gamma}_1(u)\smallsetminus\{u\}} = F_{\mathcal{E}} = F_{\tilde{e}}.$$

Proof. We apply Theorem 44 for the radius R as defined in Proposition 46. We have to show that $M_R \ge \ell^-_{\mathcal{E}}(\vartheta_k)$. By using Propositions 45 and 47 we can show that the inequality

$$M_R \ge \ell^-_{\mathcal{E}}(\vartheta_k)$$

is equivalent to

$$\sqrt{4\cos^2\vartheta_k - 1} \ge \frac{\cos^2\frac{\pi}{2p_1} - \cos\frac{\pi}{p_1} + \sin\frac{\pi}{2p_1}}{\sin\frac{\pi}{2p_1}\cos\frac{\pi}{2p_1}},$$

as follows:

We know that

$$M_R = D\cdot\frac{\cos\frac{\pi}{p_1} - \sin\frac{\pi}{2p_1}}{\cos\frac{\pi}{2p_1}},$$

$$\ell^-_{\mathcal{E}}(\vartheta_k) = D\left(\cos\frac{\pi}{2p_1} - \sin\frac{\pi}{2p_1}\sqrt{4\cos^2\vartheta_k - 1}\right).$$

So,

$$\begin{aligned} M_R &\geq \ell_{\mathcal{E}}^{-}(\vartheta_k) \\ D\cdot\frac{\cos\frac{\pi}{p_1}-\sin\frac{\pi}{2p_1}}{\cos\frac{\pi}{2p_1}} &\geq D\cdot\left(\cos\frac{\pi}{2p_1}-\sin\frac{\pi}{2p_1}\sqrt{4\cos^2\vartheta_k-1}\right) \\ \frac{\cos\frac{\pi}{p_1}-\sin\frac{\pi}{2p_1}}{\cos\frac{\pi}{2p_1}}-\cos\frac{\pi}{2p_1} &\geq -\sin\frac{\pi}{2p_1}\sqrt{4\cos^2\vartheta_k-1} \\ \frac{\cos^2\frac{\pi}{2p_1}+\sin\frac{\pi}{2p_1}-\cos\frac{\pi}{p_1}}{\cos\frac{\pi}{2p_1}\sin\frac{\pi}{2p_1}} &\leq \sqrt{4\cos^2\vartheta_k-1} \end{aligned}$$

Using formula (8.8) and (8.9) (in appendix) we obtain

$$\begin{aligned} \frac{\cos^2\frac{\pi}{2p_1}+\sin\frac{\pi}{2p_1}-\cos\frac{\pi}{p_1}}{\cos\frac{\pi}{2p_1}\sin\frac{\pi}{2p_1}} &= \frac{\cos^2\frac{\pi}{2p_1}+\sin\frac{\pi}{2p_1}-\cos^2\frac{\pi}{2p_1}+\sin^2\frac{\pi}{2p_1}}{\cos\frac{\pi}{2p_1}\sin\frac{\pi}{2p_1}} \\ &= \frac{\sin\frac{\pi}{2p_1}+1}{\cos\frac{\pi}{2p_1}} = \cot\left(\frac{\pi}{4}-\frac{\pi}{4p_1}\right). \end{aligned}$$

Then, it remains to show that

$$\begin{aligned} \sqrt{4\cos^2\vartheta_k-1} &\geq \cot\left(\frac{\pi}{4}-\frac{\pi}{4p_1}\right) \\ 4\cos^2\vartheta_k-1 &\geq \cot^2\left(\frac{\pi}{4}-\frac{\pi}{4p_1}\right). \end{aligned}$$

We also can obtain the following:

$$\cot^2\left(\frac{\pi}{4}-\frac{\pi}{4p_1}\right)+1=\frac{1}{\sin^2\left(\frac{\pi}{4}-\frac{\pi}{4p_1}\right)}.$$

Hence

$$4\cos^2\vartheta_k \geq \frac{1}{\sin^2\left(\frac{\pi}{4}-\frac{\pi}{4p_1}\right)}$$

$$2\cos\vartheta_k\sin\left(\frac{\pi}{4}-\frac{\pi}{4p_1}\right)\geq 1$$

We know that $\vartheta_k=\frac{k\pi}{2\,\mathrm{lcm}(p_1,p_2)}$. We have $p_2=3$ and $p_1=lk+3$, where

$l \in \{1,2\}$. Then $\vartheta_k = \frac{k\pi}{6p_1} = \frac{k\pi}{6(lk+3)} = \frac{\pi\left(\frac{(lk+3)-3}{l}\right)}{6(lk+3)} = \frac{\pi}{6l} - \frac{3\pi}{6(lk+3)l} = \frac{\pi}{6l} - \frac{\pi}{2p_1 l}$. For $l = 1$ we have $\vartheta_k = \frac{\pi}{6} - \frac{\pi}{2p_1}$ and for $l = 2$ we have $\vartheta_k = \frac{\pi}{12} - \frac{\pi}{4p_1}$.

We have $0 < \vartheta_k = \frac{\pi}{6l} - \frac{\pi}{2p_1 l} \le \frac{\pi}{6} - \frac{\pi}{2p_1} \le \frac{\pi}{4} - \frac{3\pi}{4p_1} < \frac{\pi}{4}$. Hence $\cos\vartheta_k \ge \cos\left(\frac{\pi}{4} - \frac{3\pi}{4p_1}\right)$. Therefore

$$\begin{aligned} & 2\cos\vartheta_k \sin\left(\frac{\pi}{4} - \frac{\pi}{4p_1}\right) \\ \ge\ & 2\cos\left(\frac{\pi}{4} - \frac{3\pi}{4p_1}\right)\sin\left(\frac{\pi}{4} - \frac{\pi}{4p_1}\right) \\ =\ & \sin\left(\frac{\pi}{4} - \frac{3\pi}{4p_1} + \frac{\pi}{4} - \frac{\pi}{4p_1}\right) + \sin\left(\frac{\pi}{4} - \frac{\pi}{4p_1} - \frac{\pi}{4} + \frac{3\pi}{4p_1}\right) \\ =\ & \sin\left(\frac{\pi}{2} - \frac{\pi}{p_1}\right) + \sin\left(\frac{\pi}{2p_1}\right) \\ =\ & \cos\left(\frac{\pi}{p_1}\right) + \sin\left(\frac{\pi}{2p_1}\right). \end{aligned}$$

Let $\alpha = \frac{\pi}{2p_1}$. We know that $\alpha \in \left(0, \frac{\pi}{8}\right)$ since $p_1 \ge 4$. Then

$$\begin{aligned} & \cos\left(\frac{\pi}{p_1}\right) + \sin\left(\frac{\pi}{2p_1}\right) \\ =\ & \cos 2\alpha + \sin\alpha \\ =\ & 1 - 2\sin^2\alpha + \sin\alpha \\ =\ & 1 + \sin\alpha(1 - 2\sin\alpha). \end{aligned}$$

Note that $0 < \alpha < \frac{\pi}{6}$, hence, $\sin 0 = 0 < \sin\alpha < \sin\frac{\pi}{6} = \frac{1}{2}$. We obtain $\sin\alpha \in (0, \frac{1}{2}) \Rightarrow 2\sin\alpha \in (0,1) \Rightarrow 1 - 2\sin\alpha \in (0,1)$. Therefore,

$$1 + \sin\alpha(1 - 2\sin\alpha) \ge 1.$$

∎

4.2 Combinatorial criterium

This section follows [8] and [15].
In the cases where we have a polytope which we think is a fundamental domain, we will use the following combinatorial criterium (Theorem 8 in [15]) in the spirit of the Poincaré-Maskit Theorem to check that it is a fundamental domain. The Theorem describes sufficient conditions for a subset of a 3-manifold which is homeomorphic to a polyhedron to be a fundamental domain for a group action. See [15] for the proof and more details, also compare with [8].

Theorem 49. *Let M be a simply connected 3-manifold without boundary. Let $\tilde{\Gamma}$ be a discrete group which acts on M properly discontinuously via homeomorphisms. We assume that there exists a subset $\mathcal{F}$ of M such that the family $(\gamma\mathcal{F})_{\gamma\in\tilde{\Gamma}}$ covers M and is locally finite in M. Let $\mathcal{P}$ be a subset of M with the following properties:*

(1) $\mathcal{P}$ is connected, the boundary of $\mathcal{P}$ is a manifold.

(2) A homeomorphism from $\mathcal{P}$ to a compact polytope in $\mathbb{R}^3$ with a homogeneous 3-dimensional flag complex is given. Then it is clear what is the set of faces $\mathfrak{F}$ and the set of edges $\mathfrak{E}$ of $\mathcal{P}$. Let

$$\mathfrak{C} = \{(f,k) \in \mathfrak{F} \times \mathfrak{E} : k \subset f\}.$$

Suppose that there is an involution $\tau : \mathfrak{C} \mapsto \mathfrak{C}$ such that for $(g,l) = \tau(f,k)$ holds $f \cap g = k = l$.

(3) There exists an involution σ on $\mathfrak{C}$ and a family $(\gamma_f)_{f\in\mathfrak{F}}$ of elements of $\tilde{\Gamma}$ with the following properties:

- *$\sigma(f,k) = (\gamma_f f, \gamma_f k)$ for any $(f,k) \in \mathfrak{C}$.*
- *For any face $f \in \mathfrak{F}$ and any point $x \in f^\circ$ there exists a neighborhood U of x such that $\gamma_f(U \cap (\mathcal{P})^\circ) \cap (\mathcal{P})^\circ = \varnothing$.*
- *$\gamma_{\gamma_f f} = \gamma_f^{-1}$ for any face $f \in \mathfrak{F}$.*
- *The group $\tilde{\Gamma}$ is generated by the set $\{\gamma_f \mid f \in \mathfrak{F}\}$.*

(4) Let $(f,k) \in \mathfrak{C}$ *and let* m *be the length of the* $\tau\sigma$*-orbit of* (f,k)*. In the case* $m > 1$ *for* $i \in \{1,\dots,m+1\}$ *let* $(f_i,k_i) = (\tau\sigma)^{i-1}(f,k)$*,* $\gamma_i = (\gamma_{f_i}\cdot\dots\cdot\gamma_{f_1})^{-1}$*,* $\mathcal{P}_i = \gamma_i\mathcal{P}$ *and* $h_i = \gamma_i f_{i+1}$*.*

In the case $m = 1$ *let us replace* m *with the order of* γ_f *and define* $\gamma_i = (\gamma_f)^{-i}$*,* $(f_i,k_i) = \gamma_i(f,k)$*,* $\mathcal{P}_i = \gamma_i\mathcal{P}$ *and* $h_i = \gamma_i f_{i+1}$*.*

- $\gamma_m = 1$.
- *For any point* $x \in k^\circ$ *there exists a neighbourhood* U *of* x *such that*

$$U \cap \mathcal{P}_i \cap \mathcal{P}_{i+1} = U \cap h_i$$

for all $i \in \{1,\dots,m+1\}$ *and*

$$U \cap \mathcal{P}_i \cap \mathcal{P}_j = U \cap k$$

for all $i,j \in \{1,\dots,m+1\}$ *with* $i < j-1$*.*

Let $N = \tilde{\Gamma} \times \mathcal{P}/\sim$*, where* $(\delta\gamma_f, x) \sim (\delta, \gamma_f x)$ *for* $\delta \in \tilde{\Gamma}$*,* $f \in \mathfrak{F}$ *and* $x \in f$*. Let the projection* $pr : N \to M$ *be given by* $pr([(\gamma,x)]) = \gamma x$*.*

Then the map pr *is a homeomorphism and the set* $\mathcal{P}$ *is a fundamental domain for the action of the group* $\tilde{\Gamma}$ *on* M*.*

Remark 50. *The case* $m = 1$ *in part* (4) *of Theorem 49 corresponds to the action of* $\tilde{\Gamma}$ *having fixed points. This case was not included in [15], but only minor modifications of the proof are needed to include it.*

4.2.1 Application of the combinatorial criterium

We shall now apply the combinatorial criterium in the following situation: $M = \tilde{G}$ is a simply connected 3-dimensional manifold without boundary and $\tilde{\Gamma} = \tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(p_1,3,3)^k \times (C_{p_2})^k$. The group $\tilde{\Gamma} = \tilde{\Gamma}_1 \times \tilde{\Gamma}_2/\langle(c^k,c^k)\rangle$ acts on $M = \tilde{G}$ be left-right multiplication properly discontinuously via homeomorphisms. For a subset $\mathcal{F}$ of $\tilde{G}$ such that the family $(\gamma\mathcal{F})_{\gamma\in\tilde{\Gamma}_1\times\tilde{\Gamma}_2}$ covers $\tilde{G}$ and is locally finite in $\tilde{G}$ we can take $\mathcal{F} = \mathcal{F}_{\tilde{e}}$. We construct a subset $W \subset \tilde{S}_{\tilde{e}}$ for which we would like to show that $W = F_{\tilde{e}}$. If the set $\mathcal{P} = \Psi(W)$ satisfies the conditions of Theorem 49 then we can apply the Theorem and conclude that $\mathcal{P} = \mathcal{F}_{\tilde{e}}$ and $W = F_{\tilde{e}}$.

We shall now list the conditions and describe in more detail how we shall go about checking that they are satisfied. It is sufficient to check the conditions for W instead of $\mathcal{P} = \Psi(W)$ since the map $\Psi|_{\tilde{S}_{\tilde{e}}}$ is an equivariant homeomorphism onto the image.

The first two conditions, that the set W is connected and homeomorphic to a compact polytope in $\mathbb{R}^3$ with a homogeneous 3-dimensional flag complex, will follow from the precise description of the set W.

The next two conditions will be confirmed with combinatorial methods.

In all cases that we will consider for every face f of W there exists an element $(\tilde{g}_1, \tilde{g}_2) \in \tilde{\Gamma}_1 \times \tilde{\Gamma}_2 / \langle (c^k, c^k) \rangle$ such that $f \subset \tilde{E}_{\tilde{g}_1 \tilde{g}_2} \cap \tilde{E}_{\tilde{e}}$. Then let $\gamma_f := (\tilde{g}_1^{-1}, \tilde{g}_2)$.

Next we need to show that $\sigma(f,k) = (\gamma_f f, \gamma_f k)$ for every $(f,k) \in \mathfrak{C}$. For every face f of W we shall compute where f is mapped under γ_f. Since the action of γ_f is a linear map $\mathbb{C}^2 \mapsto \mathbb{C}^2$, it is sufficient to compute the images of the vertices of f. Let f be a face of W which is contained in $\tilde{E}_{\tilde{g}_1 \tilde{g}_2} \cap \tilde{E}_{\tilde{e}}$. A vertex of f can be represented as an intersection $\tilde{E}_{\tilde{e}} \cap \tilde{E}_{\tilde{g}_1 \tilde{g}_2} \cap \tilde{E}_{\tilde{h}_1} \cap \tilde{E}_{\tilde{h}_2}$ of 4 hyperplanes. Hence the image of this vertex under $\gamma_f = (\tilde{g}_1^{-1}, \tilde{g}_2)$ can be represented as

$$
\begin{aligned}
&(\tilde{g}_1^{-1}, \tilde{g}_2) \cdot (\tilde{E}_{\tilde{e}} \cap \tilde{E}_{\tilde{g}_1 \tilde{g}_2} \cap \tilde{E}_{\tilde{h}_1} \cap \tilde{E}_{\tilde{h}_2}) \\
=& \tilde{E}_{\tilde{g}_1^{-1} \tilde{g}_2^{-1}} \cap \tilde{E}_{\tilde{g}^{-1} \tilde{g}_1 \tilde{g}_2 \tilde{g}_2^{-1}} \cap \tilde{E}_{\tilde{g}_1^{-1} \tilde{h}_1 \tilde{g}_2^{-1}} \cap \tilde{E}_{\tilde{g}_1^{-1} \tilde{h}_2 \tilde{g}_2^{-1}} \\
=& \tilde{E}_{\tilde{e}} \cap \tilde{E}_{\tilde{g}_1^{-1} \tilde{g}_2^{-1}} \cap \tilde{E}_{\tilde{g}_1^{-1} \tilde{h}_1 \tilde{g}_2^{-1}} \cap \tilde{E}_{\tilde{g}_1^{-1} \tilde{h}_2 \tilde{g}_2^{-1}}.
\end{aligned}
$$

If the images of all vertices of f are again vertices of a face f' of W in the correct order, i.e. if two vertices are connected by an edge then their images are also connected by an edge, then the action of γ_f maps the face f bijectively onto the face f' and maps vertices and edges of f into the vertices and edges of f'.

Next we need to show that for every face $f \in \mathfrak{F}$ and every point $x \in f^\circ$ there exists a neighbourhood U of x such that $\gamma_f(U \cap (W)^\circ) \cap (W)^\circ = \varnothing$. Let the face f be contained in $\tilde{E}_{\tilde{g}_1 \tilde{g}_2} \cap \tilde{E}_{\tilde{e}}$. The map given by action of $\gamma_f = (\tilde{g}_1^{-1}, \tilde{g}_2)$ preserves the orientation and maps the face f into the face contained in $\tilde{E}_{\tilde{g}_1^{-1} \tilde{g}_2^{-1}} \cap \tilde{E}_{\tilde{e}}$. If the orientation of the sequence of edges of the face f is given, then one can see from the orientation of the image face induced by the action of γ_f that for every $x \in f^\circ$ there exists a neighbourhood U such that $\gamma_f \cdot (U \cap (W)^\circ) \cap (W)^\circ = \varnothing$.

Next we need to show that $\gamma_{\gamma_f f} = \gamma_f^{-1}$ for every face $f \in \mathfrak{F}$: Let the face f be contained in $\tilde{E}_{\tilde{g}_1\tilde{g}_2}$, then $\gamma_f = (\tilde{g}_1^{-1}, \tilde{g}_2)$. Since $\gamma_f f$ is contained in $(\tilde{g}_1^{-1}, \tilde{g}_2)\cdot(\tilde{E}_{\tilde{g}_1\tilde{g}_2} \cap \tilde{E}_{\tilde{e}}) = \tilde{E}_{\tilde{g}_1^{-1}\tilde{g}_2^{-1}} \cap \tilde{E}_{\tilde{e}}$, we conclude that $\gamma_{\gamma_f f} = \big((\tilde{g}_1^{-1})^{-1}, \tilde{g}_2^{-1}\big) = (\tilde{g}_1, \tilde{g}_2^{-1}) = (\tilde{g}_1^{-1}, \tilde{g}_2)^{-1} = \gamma_f^{-1}$.

Next we have to show that the group $\tilde{\Gamma}$ is generated by the set $\{\gamma_f \mid f \in \mathfrak{F}\}$. Theorem 25 implies that the group $\tilde{\Gamma}_1$ is generated by $\tilde{r}_u c^{n_u}, \tilde{r}_v c^{n_v}, c^k$, where $n_u p + 1$ and $n_v q + 1$ are divisible by k. The group $\tilde{\Gamma}_2$ is generated by $\tilde{d}_2 = \tilde{r}_u(\frac{2\pi k}{p_2}) = \tilde{d}^{p_1}$. Hence $\tilde{\Gamma} = \tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ is generated by $(\tilde{r}_u c^{n_u}, \tilde{e}), (\tilde{r}_v c^{n_v}, \tilde{e}), (c^k, \tilde{e})$, $(\tilde{e}, \tilde{d}^{p_1})$. It is sufficient to represent these generators of $\tilde{\Gamma}$ in terms of γ_f.

In all cases that we will consider explicitly we will have $\gcd(p_1, p_2) = 1$, there will be a face of W, the bottom face, such that the corresponding group element is of the form $\gamma_{\text{top}} = (\tilde{d}_1^{-x}, \tilde{d}_2^{-y})$, where x, y are such integers that $xp_2 + yp_1 = 1$. Therefore,

$$\begin{aligned}
\gamma_{\text{top}}^{p_2} &= (\tilde{d}_1^{-p_2 x}, \tilde{d}_2^{p_2 y}) = (\tilde{d}_1^{-1+p_1 y}, \tilde{d}_2^{p_2 y}) = (\tilde{d}_1^{-1} c^{ky}, c^{ky}) = (\tilde{d}_1^{-1}, \tilde{e}),\\
\gamma_{\text{top}}^{p_1} &= (\tilde{d}_1^{-xp_1}, \tilde{d}_2^{yp_1}) = (\tilde{d}_1^{-xp_1}, \tilde{d}_2^{1-p_2 x}) = (c^{-kx}, c^{-kx}\tilde{d}_2) = (\tilde{e}, \tilde{d}_2),\\
\gamma_{\text{top}}^{p_1 p_2} &= (\tilde{e}, \tilde{d}_2^{p_2}) = (\tilde{e}, c^k),\\
\gamma_{\text{top}}^{-p_1 p_2} &= (\tilde{e}, c^{-k}) = (c^k, \tilde{e}).
\end{aligned}$$

Therefore the group generated by $(\tilde{d}_1, \tilde{e}), (c^k, \tilde{e})$ and $(\tilde{e}, \tilde{d}_2)$ is contained in the group generated by γ_{top}. It remains to express $\tilde{r}_v c^{n_v}$ in terms of elements $\gamma_f, f \in \mathfrak{F}$. In all cases we will find an element γ_f of the form $(\tilde{r}_v c^{n_v}) \cdot h$, where h is generated by $(\tilde{d}_1, \tilde{e}), (c^k, \tilde{e})$ and $(\tilde{e}, \tilde{d}_2)$, hence $\tilde{r}_v c^{n_v}$ can also be expressed in terms of elements γ_f. More precisely, this element γ_f will be of the form $(f_{0,-l}\tilde{d}_1^t, \tilde{d}_2^s)$, where $f_{0,-l}$ will be defined later.
From now on we shall denote the edge which is contained in $\tilde{E}_a \cap \tilde{E}_b \cap \tilde{E}_{\tilde{e}}$ of the face which is contained in $\tilde{E}_a \cap \tilde{E}_{\tilde{e}}$ as $(a; b)$. The same edge can sometimes be decoded by several pairs (a, b), but the following statements will be true for all such edges and faces. After we have described the action

of the involution σ, we can compute for every pair $(f,k) \in \mathfrak{E}$ the edge cycle, i.e. the sequence of pairs $(f_i,k_i) = (\tau\sigma)^{i-1}(f,k)$. In all cases that we shall explicitly consider in this thesis the edge cycles will be of one of the following two types:

- **A.** The cycle consists of 3 edges of which two are convex and contained in the top resp. bottom face, while the third edge is not convex and not contained in the top or bottom face. Let the non-convex edge be $(a;b)$. Let $a = a_1a_2, b = b_1b_2$ with $a_1, b_1 \in \tilde{\Gamma}_1,\ a_2, b_2 \in \tilde{\Gamma}_2$.
  The edge cycle is of the form

$$\begin{array}{c}((a_1,a_2);(b_1,b_2))\\ \sigma\big\downarrow\\ \big((a_1^{-1},a_2^{-1});(a_1^{-1}b_1,b_2a_2^{-1})\big)\\ \tau\big\downarrow\\ \big((a_1^{-1}b_1,b_2a_2^{-1});(a_1^{-1},a_2^{-1})\big)\\ \sigma\big\downarrow\\ \big((b_1^{-1}a_1,a_2b_2^{-1});(b_1^{-1},b_2^{-1})\big)\\ \tau\big\downarrow\\ \big((b_1^{-1},b_2^{-1});(b_1^{-1}a_1,a_2b_2^{-1})\big)\\ \sigma\big\downarrow\\ ((b_1,b_2);(a_1,a_2))\\ \tau\big\downarrow\\ ((a_1,a_2);(b_1,b_2))\end{array}$$

  or, in the short notation,

$$\begin{array}{c}((a_1,a_2);(b_1,b_2))\\ \mapsto\\ \big((a_1^{-1}b_1,b_2a_2^{-1});(a_1^{-1},a_2^{-1})\big)\\ \mapsto\\ \big((b_1^{-1},b_2^{-1});(b_1^{-1}a_1,a_2b_2^{-1})\big).\end{array}$$

We have $\gamma_{f_1} = (a_1^{-1}, a_2), \gamma_{f_2} = (b_1^{-1}a_1, b_2a_2^{-1})$ and $\gamma_{f_3} = (b_1, b_2^{-1})$. Then

$$\begin{aligned}
\gamma_1 &= (\gamma_{f_1})^{-1} = (a_1^{-1}, a_2)^{-1} = (a_1, a_2^{-1}), \\
\gamma_2 &= (\gamma_{f_2}\gamma_{f_1})^{-1} = \big((b_1^{-1}a_1, b_2a_2^{-1}) \cdot (a_1^{-1}, a_2)\big)^{-1} = (b_1^{-1}, b_2)^{-1} = (b_1, b_2^{-1}), \\
\gamma_3 &= (\gamma_{f_3}\gamma_{f_2}\gamma_{f_1})^{-1} = \big((b_1, b_2^{-1}) \cdot (b_1^{-1}a_1, b_2a_2^{-1}) \cdot (a_1^{-1}, a_2)\big)^{-1} \\
&= \big((b_1, b_2^{-1}) \cdot (b_1^{-1}, b_2)\big)^{-1} = \tilde{e}^{-1} = \tilde{e}.
\end{aligned}$$

The edge $(a; b)$ is non-convex and W looks like $(\tilde{I}_a \cup \tilde{I}_b) \cap \tilde{E}_{\tilde{e}}$ near this edge. In all such cases the other two edges are convex and the elements $(a_1^{-1}b_1, b_2a_2^{-1})$ and $(b_1^{-1}a_1, a_2b_2^{-1})$ correspond to the top/bottom edges. We have that W looks like $\tilde{H}_{a_1^{-1}b_1b_2a_2^{-1}} \cap \tilde{I}_{a_1^{-1}a_2^{-1}} \cap \tilde{E}_{\tilde{e}}$ near the edge $\big((a_1^{-1}b_1, b_2a_2^{-1}); (a_1^{-1}, a_2^{-1})\big)$ and like $\tilde{I}_{b_1^{-1}b_2^{-1}} \cap \tilde{H}_{b_1^{-1}a_1a_2b_2^{-1}} \cap \tilde{E}_{\tilde{e}}$ near the edge $\big((b_1^{-1}, b_2^{-1}); (b_1^{-1}a_1, a_2b_2^{-1})\big)$.
We obtain for a sufficiently small neighbourhood U of an interior point of the edge $(a; b)$:

$$\begin{aligned}
W \cap U &= ((\tilde{I}_a \cup \tilde{I}_b) \cap \tilde{E}_{\tilde{e}}) \cap U, \\
\gamma_1 W \cap U &= \gamma_1(\tilde{H}_{a_1^{-1}b_1b_2a_2^{-1}} \cap \tilde{I}_{a_1^{-1}a_2^{-1}} \cap \tilde{E}_{\tilde{e}}) \cap U \\
&= (\tilde{H}_{b_1b_2} \cap \tilde{I}_{\tilde{e}} \cap \tilde{E}_{a_1a_2}) \cap U = (\tilde{H}_b \cap \tilde{I}_{\tilde{e}} \cap \tilde{E}_a) \cap U, \\
\gamma_2 W \cap U &= \gamma_2(\tilde{I}_{b_1^{-1}b_2^{-1}} \cap \tilde{H}_{b_1^{-1}a_1a_2b_2^{-1}} \cap \tilde{E}_{\tilde{e}}) \cap U \\
&= (\tilde{I}_{\tilde{e}} \cap \tilde{H}_{a_1a_2} \cap \tilde{E}_{b_1b_2}) \cap U = (\tilde{I}_{\tilde{e}} \cap \tilde{H}_a \cap \tilde{E}_b) \cap U.
\end{aligned}$$

Hence
$$W \cap \gamma_1 W \cap U = (\tilde{H}_b \cap \tilde{E}_a \cap \tilde{E}_{\tilde{e}}) \cap U$$
and
$$W \cap \gamma_2 W \cap U = (\tilde{H}_a \cap \tilde{E}_b \cap \tilde{E}_{\tilde{e}}) \cap U$$
are equal to the intersections of the neighbourhood U with the faces which are contained in $\tilde{E}_a \cap \tilde{E}_{\tilde{e}}$ resp. $\tilde{E}_b \cap \tilde{E}_{\tilde{e}}$. Furthermore, the set

$$\gamma_1 W \cap \gamma_2 W \cap U = (\tilde{I}_{\tilde{e}} \cap \tilde{E}_b \cap \tilde{E}_a) \cap U = \gamma_1(\tilde{I}_{a_1^{-1}a_2^{-1}} \cap \tilde{E}_{a_1^{-1}b_1b_2a_2^{-1}} \cap \tilde{E}_{\tilde{e}}) \cap U$$

is equal to the image of the face of W which is contained in $\tilde{E}_{a_1^{-1}b_1b_2a_2^{-1}} \cap \tilde{E}_{\tilde{e}}$ under the action of $\gamma_1 = (a_1, a_2^{-1})$.

- **B.** The cycle consists of one convex edge, say $(a;b)$. We can write a and b as products $a = a_1a_2, b = b_1b_2$ with $a_1, b_1 \in \tilde{\Gamma}_1,\ a_2, b_2 \in \tilde{\Gamma}_2$.
In all such cases we will be able to check by computation that $a_1^3 = c^x, a_2^3 = c^{-x}$ for some integer x and $b_1 = a_1^{-1}, b_2 = a_2^{-1}$. Consequently $a_1^3a_2^3 = c^xc^{-x} = \tilde{e}$ and $a_1^2a_2^2 = a_1^{-1}a_2^{-1} = b_1b_2 = b$.

For an edge cycle of type **B** we set $\gamma_i = (\gamma_f)^{-i}$, where f is the face contained in $\tilde{E}_a \cap \tilde{E}_{\tilde{e}}$, i.e. $\gamma_f = (a_1^{-1}, a_2)$. Thus $\gamma_i = (a_1^{-1}, a_2)^{-i} = (a_1, a_2^{-1})^i$. The edge $(a;b)$ is convex and W looks like $\tilde{I}_a \cap \tilde{I}_b \cap \tilde{E}_{\tilde{e}}$ near this edge.

For a sufficiently small neighbourhood U of an interior point of the edge $(a;b)$ we obtain

$$W \cap U = (\tilde{I}_a \cap \tilde{I}_b \cap \tilde{E}_{\tilde{e}}) \cap U = (\tilde{I}_{a_1a_2} \cap \tilde{I}_{b_1b_2} \cap \tilde{E}_{\tilde{e}}) \cap U.$$

The edge $(a;b)$ is fixed under the action of γ_i, hence

$$\begin{aligned}
\gamma_1 W \cap U = \gamma_1(W \cap U) &= (a_1, a_2^{-1}) \cdot \big((\tilde{I}_{a_1a_2} \cap \tilde{I}_{b_1b_2} \cap \tilde{E}_{\tilde{e}}) \cap U\big) \\
&= (\tilde{I}_{a_1^2a_2^2} \cap \tilde{I}_{a_1b_1b_2a_2} \cap \tilde{E}_{a_1a_2}) \cap U = (\tilde{I}_b \cap \tilde{I}_{\tilde{e}} \cap \tilde{E}_a) \cap U, \\
\gamma_2 W \cap U = \gamma_2(W \cap U) &= (a_1^2, a_2^{-2}) \cdot \big((\tilde{I}_{a_1a_2} \cap \tilde{I}_{b_1b_2} \cap \tilde{E}_{\tilde{e}}) \cap U\big) \\
&= (\tilde{I}_{a_1^3a_2^3} \cap \tilde{I}_{a_1^2b_1b_2a_2^2} \cap \tilde{E}_{a_1^2a_2^2}) \cap U = (\tilde{I}_{\tilde{e}} \cap \tilde{I}_a \cap \tilde{E}_b) \cap U.
\end{aligned}$$

Hence

$$W \cap \gamma_1 W \cap U = (\tilde{I}_b \cap \tilde{E}_a \cap \tilde{E}_{\tilde{e}}) \cap U$$

and

$$W \cap \gamma_2 W \cap U = (\tilde{I}_a \cap \tilde{E}_b \cap \tilde{E}_{\tilde{e}}) \cap U$$

are equal to the intersections of the neighbourhood U with the faces which are contained in $\tilde{E}_a \cap \tilde{E}_{\tilde{e}}$ and $\tilde{E}_b \cap \tilde{E}_{\tilde{e}}$. Furthermore, the set

$$\gamma_1 W \cap \gamma_2 W \cap U = (\tilde{I}_{\tilde{e}} \cap \tilde{E}_a \cap \tilde{E}_b) \cap U = \gamma_1 (\tilde{I}_{a_1^{-1} a_2^{-1}} \cap \tilde{E}_{a_1^{-1} b_1 b_2 a_2^{-1}} \cap \tilde{E}_{\tilde{e}}) \cap U$$
$$= \gamma_1 (\tilde{I}_b \cap \tilde{E}_a \cap \tilde{E}_{\tilde{e}}) \cap U$$

is in the image of the face of W which is contained in $\tilde{E}_{a_1^{-1} b_1 b_2 a_2^{-1}} \cap \tilde{E}_{\tilde{e}}$ under the action of γ_1.

Chapter 5

Construction of fundamental domains for the groups $\tilde{\Gamma}(p,3,3)^k \times (C_3)^k$

NOTATION: We will use p instead of p_1 from now on since $p_2 = 3$.

5.1 Generators of the group $\tilde{\Gamma}(p,3,3)^k \times (C_3)^k$

Let $\tilde{\Gamma}_1 = \tilde{\Gamma}(p,3,3)^k$, $\tilde{\Gamma}_2 = (C_3)^k$, $p \neq 0 \mod 3$, $p = kl + 3$, $l = 1,2$. Recall that: $c = \tilde{r}_u(2\pi)$, $\tilde{r}_u = \tilde{r}_u(\frac{2\pi}{p})$, $\tilde{d}_1 = \tilde{r}_u(\frac{2\pi k}{p}) = \tilde{r}_u^k$, $\tilde{d}_2 = \tilde{r}_u(\frac{2\pi k}{3})$, $\tilde{d} = \tilde{r}_u(\frac{2\pi k}{3p})$. Note that $\tilde{d}_1 = \tilde{d}^3$, $\tilde{d}_2 = \tilde{d}^p$ We know that $\tilde{\Gamma}_1 = \tilde{\Gamma}(p,3,3)^k$ can be generated by elements of the form $\tilde{d}_1^a \tilde{r}_v^b \tilde{d}_1^h c^m$, where $a, h \in \{0, \cdots, (p-1)\}$, $b = 2l$ and $m \in \mathbb{Z}$. Moreover, we know that $\tilde{\Gamma}_2 = (C_3)^k$ can be generated by the element $\tilde{d}_2$.

Now, we have found the generators of $\tilde{\Gamma}_1 = \tilde{\Gamma}(p,3,3)^k$ and the generator of $\tilde{\Gamma}_2 = (C_3)^k$. Therefore, any element of the form $\tilde{g}_1\tilde{g}_2$, where $\tilde{g}_1 \in \tilde{\Gamma}_1$ and $\tilde{g}_2 \in \tilde{\Gamma}_2$, can be written in the form

$$\tilde{g}_1\tilde{g}_2 = \tilde{d}_1^a \tilde{r}_v^b \tilde{d}_1^h \tilde{d}_2^n c^m.$$

We compute $\tilde{d}_1^h \tilde{d}_2^n$ as follows:

$$\begin{aligned}
\tilde{d}_1^h \tilde{d}_2^n &= \tilde{r}_u \left(\frac{2\pi k}{p}\right)^h \tilde{r}_u \left(\frac{2\pi k}{3}\right)^n \\
&= \tilde{r}_u \left(2\pi k \left(\frac{h}{p} + \frac{n}{3}\right)\right) \\
&= \tilde{r}_u \left(2\pi k \left(\frac{3h + np}{3p}\right)\right) \\
&= \tilde{r}_u \left(\frac{2\pi k}{3p}\right)^{(3h+np)} \\
&= \tilde{d}^{(3h+np)}.
\end{aligned}$$

Hence,

$$\tilde{g}_1 \tilde{g}_2 = \tilde{d}_1^a \tilde{r}_v^b \tilde{d}_1^h \tilde{d}_2^n c^m = \tilde{d}^{3a} \tilde{r}_v^b \tilde{d}^{(3h+np)} c^m.$$

Note that $1 = \gcd(p, 3)$ can be written as a linear combination of p and 3 with integer coefficients (Euclidean algorithm), hence $3h + np$ can take any integer value for a suitable choice of h and n.

Proposition 51. *We have* $\tilde{r}_u = c^{-t} \tilde{d}_1^{\frac{tp+1}{k}} = c^{-t} \tilde{d}^{\frac{3(tp+1)}{k}}$, *where t is an integer such that* $3t + 1 = 0 \mod k$. *In particular*

1. *For* $k = 1 \mod 3$ *we have* $t = \frac{k-1}{3}$ *and* $3(tp+1) = k(p - l)$,

2. *For* $k = 2 \mod 3$ *we have* $t = \frac{2k-1}{3}$ *and* $3(tp+1) = k(2p - l)$.

Remark 52. *Note that* $tp + 1 = t(kl + 3) + 1 = tkl + (3t + 1) = 0 \mod k$, *hence* $\frac{tp+1}{k}$ *is an integer.*

Proof.

We have

$$\tilde{d}_1^{\frac{tp+1}{k}} = \tilde{r}_u \left(\frac{2\pi k}{p} \cdot \frac{tp+1}{k}\right) = \tilde{r}_u \left(2\pi t + \frac{2\pi}{p}\right) = \tilde{r}_u (2\pi)^t \cdot \tilde{r}_u \left(\frac{2\pi}{p}\right) = c^t \tilde{r}_u,$$

hence

$$\tilde{r}_u = c^{-t} \tilde{d}_1^{\frac{tp+1}{k}} = c^{-t} \tilde{d}^{\frac{3(tp+1)}{k}}.$$

Therefore,

1. For $k = 1 \mod 3$ we have that $t = \frac{k-1}{3}$ is an integer and $3t = k - 1 \Rightarrow 3t + 1 = k \Rightarrow 3t + 1 = 0 \mod k$
   We compute
$$3(tp+1) = 3\left(\frac{k-1}{3}p + 1\right) = kp - p + 3 = kp - kl = k(p-l).$$

2. For $k = 2 \mod 3$ we have that $t = \frac{2k-1}{3}$ is an integer and $3t = 2k - 1 \Rightarrow 3t + 1 = 2k \Rightarrow 3t + 1 = 0 \mod k$
   We compute
$$3(tp+1) = 3\left(\frac{2k-1}{3}p + 1\right) = 2kp - p + 3 = 2kp - kl = k(2p-l).$$

∎

Corollary 53. *We have*
$$\tilde{r}_u = c^{\frac{1-\lambda k}{3}} \cdot \tilde{d}_1^{\frac{\lambda p - l}{3}} = c^{\frac{1-\lambda k}{3}} \cdot \tilde{d}^{\lambda p - l},$$
where $\lambda = 1$ *if* $k = 1 \mod 3$ *and* $\lambda = 2$ *if* $k = 2 \mod 3$.

Let
$$f_{0,-l} = \tilde{r}_v \cdot \tilde{r}_u^2 \cdot c^{-1}.$$
Recall that $\pi : \widetilde{\mathrm{SU}(1,1)} \to \mathrm{SU}(1,1)$ is the covering map. We have $\pi(c) = -1, \pi(\tilde{r}_u) = r_u = r_u(2\pi/p)$ and $\pi(\tilde{r}_v) = r_v = r_v(2\pi/3)$. Now let $d_1 = r_u(2\pi k/p) = r_u^k, d = r_u(2\pi k/3p), d_1 = d^3$. Note that $\pi(\tilde{d}_1) = d_1$ and $\pi(\tilde{d}) = d$.

Corollary 54. *Let t be an integer such that $3t + 1 = 0 \mod k$. Then*
$$\pi(\tilde{r}_u) = (-1)^{-t} \cdot d^{\frac{3(tp+1)}{k}},\ \pi(\tilde{r}_u^2) = d^{\frac{6(tp+1)}{k}},\ \pi(f_{0,-l}) = r_v^4 \cdot d^{\frac{6(tp+1)}{k}}.$$

Proof. We have
$$\pi(\tilde{r}_u) = \pi(c^{-t} \cdot \tilde{d}^{\frac{3(tp+1)}{k}}) = (-1)^{-t} \cdot d^{\frac{3(tp+1)}{k}},$$
$$\pi(\tilde{r}_u^2) = (-1)^{-2t} \cdot d^{\frac{6(tp+1)}{k}} = d^{\frac{6(tp+1)}{k}},$$

$$\pi(f_{0,-l}) = \pi(\tilde{r}_v \cdot \tilde{r}_u^2 \cdot c^{-1}) = \pi(\tilde{r}_v)\pi(\tilde{r}_u^2)\pi(c^{-1}) = r_v d^{\frac{6(tp+1)}{k}}(-1) = r_v^4 \cdot d^{\frac{6(tp+1)}{k}}.$$

∎

Corollary 55. *1. For $k = 1 \mod 3$ we have*

$$\pi(\tilde{r}_u) = (-1)^{k-1} \cdot d^{p-l}, \pi(\tilde{r}_u^2) = d^{2(p-l)}, \pi(f_{0,-l}) = r_v^4 \cdot d^{2(p-l)}.$$

2. For $k = 2 \mod 3$ we have

$$\pi(\tilde{r}_u) = -d^{2p-l}, \pi(\tilde{r}_u^2) = d^{2(2p-l)}, \pi(f_{0,-l}) = r_v^4 \cdot d^{2(2p-l)}.$$

Corollary 56. *1. For $p = k + 3$ (i.e. $l = 1$) and $k = 1 \mod 3$ we have*

$$\pi(\tilde{r}_u) = (-1)^{k-1} \cdot d^{k+2}, \pi(\tilde{r}_u^2) = d^{2(k+2)}, \pi(f_{0,-l}) = r_v^4 \cdot d^{2(k+2)}.$$

2. For $p = k + 3$ (i.e. $l = 1$) and $k = 2 \mod 3$ we have

$$\pi(\tilde{r}_u) = -d^{2k+5}, \pi(\tilde{r}_u^2) = d^{2(2k+5)}, \pi(f_{0,-l}) = r_v^4 \cdot d^{2(2k+5)}.$$

3. For $p = 2k + 3$ (i.e. $l = 2$) and $k = 1 \mod 3$ we have

$$\pi(\tilde{r}_u) = (-1)^{k-1} \cdot d^{2k+1}, \pi(\tilde{r}_u^2) = d^{2(2k+1)}, \pi(f_{0,-l}) = r_v^4 \cdot d^{2(2k+1)}.$$

4. For $p = 2k + 3$ (i.e. $l = 2$) and $k = 2 \mod 3$ we have

$$\pi(\tilde{r}_u) = -d^{4(k+1)}, \pi(\tilde{r}_u^2) = d^{8(k+1)}, \pi(f_{0,-l}) = r_v^4 \cdot d^{8(k+1)}.$$

5.2 Symmetries of the fundamental domain $F_{\tilde{e}}$

We call an isometry φ of $\tilde{L}$ a symmetry of the fundamental domain $F_{\tilde{e}}$ if $\varphi(\partial P) = \partial P$ and $\varphi(F_{\tilde{e}}) = F_{\tilde{e}}$. For each isometry φ of $\tilde{L}$ we have :

$$\varphi(\tilde{H}_{\tilde{g}}) = \tilde{H}_{\varphi(\tilde{g})}, \varphi(\tilde{I}_{\tilde{g}}) = \tilde{I}_{\varphi(\tilde{g})}.$$

From the definitions of ∂P and $F_{\tilde{e}}$ we obtain that an isometry φ of $\tilde{L}$ is a symmetry of $F_{\tilde{e}}$ if and only if $\varphi(\tilde{\Gamma}_1) = \tilde{\Gamma}_1, \varphi(\tilde{g}_1(\tilde{\Gamma}_1)_u \cdot \tilde{\Gamma}_2) = \varphi(\tilde{g}_1) \cdot (\tilde{\Gamma}_1)_u \cdot$

$(\tilde{\Gamma}_2)$ for all $\tilde{g}_1 \in \tilde{\Gamma}_1, \varphi(\tilde{e}) = \tilde{e}$. If an isometry φ of $\tilde{L}$ is compatible with multiplication, i.e. if $\varphi(ab) = \varphi(a)\cdot\varphi(b)$ for all $a, b \in \tilde{L}$, then φ is a symmetry of $F_{\tilde{e}}$ if and only if $\varphi(\tilde{\Gamma}_1) = \tilde{\Gamma}_1, \varphi((\tilde{\Gamma}_1)_u\cdot\tilde{\Gamma}_2) = (\tilde{\Gamma}_1)_u\cdot\tilde{\Gamma}_2, \varphi(\tilde{\Gamma}_2) = \tilde{\Gamma}_2$ and $\varphi(\tilde{e}) = \tilde{e}$. We are going to describe some isometries of $\tilde{L}$. Using the criteria above we are going to show that they are symmetries of the fundamental domain $F_{\tilde{e}}$. These symmetries will help us to reduce the amount of calculation.

We are going to look at the cases $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(p,3,3)^k \times (C_3)^k$.

Definition 57. *(Taken from [15], P. 38 and 44)*
Let the map $\rho : \tilde{L} \to \tilde{L}$ be defined by $\rho(\tilde{g}) = \tilde{r}_u\tilde{g}\tilde{r}_u^{-1}$.
Let $\eta : L \to L$ be defined by $\eta((w,z)) = (\bar{w},\bar{z})$. Let $\tilde{\eta} : \tilde{L} \to \tilde{L}$ be defined as the lift of η with $\tilde{\eta}(\tilde{e}) = \tilde{e}$. We have $\arg\circ\tilde{\eta} = -\arg$. The isometry $\tilde{\eta}|_{\tilde{G}}$ is an automorphism.
If $q = r = 3$ we can have the following map $\tilde{\rho} : \tilde{L} \to \tilde{L}$ which is defined by $\tilde{\rho}(\tilde{g}) = \tilde{r}'_u\tilde{g}\tilde{r}'^{-1}_u$ where $\tilde{r}'_u = \tilde{r}_u(\pi/p)$. Note that $(\tilde{r}'_u)^2 = \tilde{r}_u$, hence $\tilde{\rho}^2 = \rho$.

Proposition 58. *(Taken from [15], Propositions 19 and 61).*
We have

$$\rho : \tilde{r}_u \mapsto \tilde{r}_u, \tilde{r}_v \mapsto \tilde{r}_u\tilde{r}_v\tilde{r}_u^{-1}, c \mapsto c, \quad \tilde{\eta} : \tilde{r}_u \mapsto \tilde{r}_u^{-1}, \tilde{r}_v \mapsto \tilde{r}_v^{-1}, c \mapsto c^{-1}$$

$$\tilde{\rho} : \tilde{r}_u \mapsto \tilde{r}_u, \tilde{r}_v \mapsto \tilde{r}_v^2\tilde{r}_u^{-1}, c \mapsto c.$$

Corollary 59. *We have*

$$\rho : \tilde{d}_1 \mapsto \tilde{d}_1, \tilde{d}_2 \mapsto \tilde{d}_2, \tilde{d} \mapsto \tilde{d}, \quad \tilde{\eta} : \tilde{d}_1 \mapsto \tilde{d}_1^{-1}, \tilde{d}_2 \mapsto \tilde{d}_2^{-1}, \tilde{d} \mapsto \tilde{d}^{-1},$$

$$\tilde{\rho} : \tilde{d}_1 \mapsto \tilde{d}_1, \tilde{d}_2 \mapsto \tilde{d}_2, \tilde{d} \mapsto \tilde{d}.$$

Proposition 60. *For $a, b, h, n, m \in \mathbb{Z}$, in the case $\tilde{\Gamma}_1(p,3,3)^k \times (C_3)^k$, where $p = kl + 3$ and $l = 1, 2$ we have*

$$\begin{aligned}
\rho(\tilde{d}^{3a}\tilde{r}_v^b\tilde{d}^f c^m) &= \tilde{d}^{3a+\lambda p-l}\tilde{r}_v^b\tilde{d}^{l-\lambda p+f}c^m,\\
\tilde{\eta}(\tilde{d}^{3a}\tilde{r}_v^b\tilde{d}^f c^m) &= \tilde{d}^{-3a}\tilde{r}_v^{-b}\tilde{d}^{-f}c^{-m},\\
\tilde{\rho}(\tilde{d}^{3a}\tilde{r}_v^2\tilde{d}^f c^m) &= \tilde{d}^{3a+\lambda p-l}\tilde{r}_v^4\tilde{d}^f c^{m-\frac{\lambda k+2}{3}},\\
\tilde{\rho}(\tilde{d}^{3a}\tilde{r}_v^4\tilde{d}^f c^m) &= \tilde{d}^{3a}\tilde{r}_v^2\tilde{d}^{l-\lambda p+f}c^{m+\frac{\lambda k+2}{3}},
\end{aligned}$$

where $\lambda = 1$ if $k \equiv 1 \mod 3$ and $\lambda = 2$ if $k \equiv 2 \mod 3$.

Proof. Recall that $\tilde{r}_u = c^{\frac{1-\lambda k}{3}}\tilde{d}^{\lambda p-l}$ according to Corollary 53. For the case of $\rho(\tilde{d}^{3a}\tilde{r}_v^b\tilde{d}^f c^m)$ we have

$$\begin{aligned}
\rho(\tilde{d}^{3a}\tilde{r}_v^b\tilde{d}^f c^m) &= \rho(\tilde{d}^{3a})\rho(\tilde{r}_v^b)\rho(\tilde{d}^f)\rho(c^m) \\
&= \tilde{d}^{3a}\tilde{r}_u\tilde{r}_v^b\tilde{r}_u^{-1}\tilde{d}^f c^m \\
&= \tilde{d}^{3a}c^{\frac{1-\lambda k}{3}}\tilde{d}^{\lambda p-l}\tilde{r}_v^b c^{\frac{\lambda k-1}{3}}\tilde{d}^{l-\lambda p}\tilde{d}^f c^m \\
&= \tilde{d}^{3a+\lambda p-l}\tilde{r}_v^b\tilde{d}^{l-\lambda p+f}c^m.
\end{aligned}$$

For the case of $\tilde{\eta}(\tilde{d}^{3a}\tilde{r}_v^b\tilde{d}^f c^m)$ we have

$$\begin{aligned}
\tilde{\eta}(\tilde{d}^{3a}\tilde{r}_v^b\tilde{d}^f c^m) &= \tilde{\eta}(\tilde{d}^{3a})\tilde{\eta}(\tilde{r}_v^b)\tilde{\eta}(\tilde{d}^f)\tilde{\eta}(c^m) \\
&= \tilde{d}^{-3a}\tilde{r}_v^{-b}\tilde{d}^{-f}c^{-m}.
\end{aligned}$$

For the case of $\tilde{\rho}(\tilde{d}^{3a}\tilde{r}_v^2\tilde{d}^f c^m)$ we have

$$\begin{aligned}
\tilde{\rho}(\tilde{d}^{3a}\tilde{r}_v^2\tilde{d}^f c^m) &= \tilde{\rho}(\tilde{d}^{3a})\tilde{\rho}(\tilde{r}_v^2)\tilde{\rho}(\tilde{d}^f)\tilde{\rho}(c^m) \\
&= \tilde{d}^{3a}\tilde{\rho}(\tilde{r}_v)\tilde{\rho}(\tilde{r}_v)\tilde{d}^f c^m \\
&= \tilde{d}^{3a}(\tilde{r}_v^2\tilde{r}_u^{-1})(\tilde{r}_v^2\tilde{r}_u^{-1})\tilde{d}^f c^m.
\end{aligned}$$

Using Proposition 24 $(\tilde{r}_v^2\tilde{r}_u^{-1})^2 = \tilde{r}_u\tilde{r}_v^{-2}c$ we obtain

$$\begin{aligned}
\tilde{d}^{3a}(\tilde{r}_v^2\tilde{r}_u^{-1})(\tilde{r}_v^2\tilde{r}_u^{-1})\tilde{d}^f c^m &= \tilde{d}^{3a}\tilde{r}_u c\tilde{r}_v^{-2}\tilde{d}^f c^m \\
&= \tilde{d}^{3a}c^{\frac{1-\lambda k}{3}}\tilde{d}^{\lambda p-l}\tilde{r}_v^3 c^{-1}c\tilde{r}_v^{-2}\tilde{d}^f c^m \\
&= d^{3a+\lambda p-l}\tilde{r}_v\tilde{r}_v^3 c^{-1}\tilde{d}^f c^{\frac{1-\lambda k+3m}{3}} \\
&= \tilde{d}^{3a+\lambda p-l}\tilde{r}_v^4\tilde{d}^f c^{m-\frac{\lambda k+2}{3}}.
\end{aligned}$$

For the case of $\tilde{\rho}(\tilde{d}^{3a}\tilde{r}_v^4\tilde{d}^f c^m)$ we have

$$\begin{aligned}
\tilde{\rho}(\tilde{d}^{3a}\tilde{r}_v^4\tilde{d}^f c^m) &= \tilde{\rho}(\tilde{d}^{3a}\tilde{r}_v^{-3}c\tilde{r}_v^4\tilde{d}^f c^m) \\
&= \tilde{\rho}(\tilde{d}^{3a})\tilde{\rho}(\tilde{r}_v)\tilde{\rho}(\tilde{d}^f)\tilde{\rho}(c^{m+1}) \\
&= \tilde{d}^{3a}\tilde{r}_v^2\tilde{r}_u^{-1}\tilde{d}^f c^{m+1} \\
&= \tilde{d}^{3a}\tilde{r}_v^2 c^{\frac{\lambda k-1}{3}}\tilde{d}^{l-\lambda p}\tilde{d}^f c^{m+1} \\
&= \tilde{d}^{3a}\tilde{r}_v^2\tilde{d}^{l-\lambda p+f} c^{m+\frac{\lambda k+2}{3}}.
\end{aligned}$$

∎

Proposition 61. *The isometries $\tilde{\rho}$ and $\tilde{\eta}$ are symmetries of $F_{\tilde{e}}$.*

Proof. The isometries $\tilde{\rho}|_{\tilde{G}}$ and $\tilde{\eta}|_{\tilde{G}}$ are automorphisms, thus it remains to show that $\tilde{\Gamma}_1, \tilde{\Gamma}_2$ and $(\tilde{\Gamma}_1)_u$ are invariant under $\tilde{\rho}$ and $\tilde{\eta}$. Recall that the group $\tilde{\Gamma}_1$ is generated by $\tilde{r}_v c^n, \tilde{d}_1$ and the group $\tilde{\Gamma}_2$ is generated by $\tilde{d}_2$. The stabiliser $(\tilde{\Gamma}_1)_u \cdot \tilde{\Gamma}_2$ is generated by $\tilde{d}$. Proposition 58 shows that $\tilde{\rho}(\tilde{d}) = \tilde{d}, \tilde{\rho}(\tilde{d}_1) = \tilde{d}_1, \tilde{\rho}(\tilde{d}_2) = \tilde{d}_2, \tilde{\rho}(\tilde{r}_v) = \tilde{r}_v^2\tilde{r}_u^{-1}, \tilde{\rho}(c) = c$ and $\tilde{\rho}(\tilde{r}_v c^n) = \tilde{r}_v^2\tilde{r}_u^{-1}c^n$. Moreover, $\tilde{\eta}(\tilde{d}) = \tilde{d}^{-1}, \tilde{\eta}(\tilde{d}_1) = \tilde{d}_1^{-1}, \tilde{\eta}(\tilde{d}_2) = \tilde{d}_2^{-1}, \tilde{\eta}(\tilde{r}_v) = \tilde{r}_v^{-1}, \tilde{\eta}(c) = c^{-1}$ and $\tilde{\eta}(\tilde{r}_v c^n) = \tilde{r}_v^{-1}c^{-1}$. ∎

Proposition 62. *We have*

$$\begin{aligned}
\tilde{\rho}^{-1}(\tilde{d}^{3a}\tilde{r}_v^2\tilde{d}^f c^m) &= \tilde{d}^{3a}\tilde{r}_v^4\tilde{d}^{\lambda p-l+f} c^{m-\frac{\lambda k+2}{3}}, \\
\tilde{\rho}^{-1}(\tilde{d}^{3a}\tilde{r}_v^4\tilde{d}^f c^m) &= \tilde{d}^{3a+l-\lambda p}\tilde{r}_v^2\tilde{d}^f c^{m+\frac{\lambda k+2}{3}}.
\end{aligned}$$

5.3 The intersection of $E_g \cap E_e$ and the intersection of $\tilde{I}_{\tilde{g}} \cap \tilde{E}_{\tilde{e}}$

The section taken from sections 10 and 11 in [15].

Proposition 63. *Let $g = (a, b)$ be an elements in G with $b \neq 0$ and therefore $|a|^2 = 1 + |b|^2 > 1$. In the case $Re(a) > -1$ the set $E_g \cap E_e$ is connected. In the case $Re(a) < -1$ the set $E_g \cap E_e$ consists of two connected components whose image under the projection $(1 + i\omega, z) \mapsto \omega$ is a complement of a compact interval.*

Notation. For $M \subset \mathbb{C}^2$, $\omega \in \mathbb{R}$ we have $M(\omega) = \{z \in \mathbb{C} \mid (1+i\omega, z) \in M\}$.

Lemma 64. *Let $g = (a,b)$ be an element in G.*

1. *If $Re(a) \geq 0$ then there exists an element $\tilde{g}$ in $\tilde{G}$ such that $|\arg(\tilde{g})| \leq \frac{\pi}{2}$ and $\pi(\tilde{g}) = g$. Moreover $I_g \cap E_e = \pi(\tilde{I}_{\tilde{g}} \cap \tilde{E}_{\tilde{e}})$.*

2. *If $Re(a) < 0$ then there exist elements $\tilde{g}'$ and $\tilde{g}''$ in $\tilde{G}$ such that $\pi(\tilde{g}') = \pi(\tilde{g}'') = g$, $\arg(\tilde{g}') \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\arg(\tilde{g}'') \in (-\frac{3\pi}{2}, -\frac{\pi}{2})$. Let $t = \tan(\arg(\tilde{g}')) = \tan(\arg(\tilde{g}''))$. For $p = (1+i\omega, z) \in I_g \cap E_e$ we have $p \in \pi(\tilde{I}_{\tilde{g}'} \cap \tilde{E}_{\tilde{e}})$ in the case $\omega > t$ and $p \in \pi(\tilde{I}_{\tilde{g}''} \cap \tilde{E}_{\tilde{e}})$ in the case $\omega < t$, while the case $\omega = t$ does not occur.*

Notation. For $M \subset E_e$ and $t \in \mathbb{R}$ let $M[< t] = \{(1+i\omega, z) \in M \mid \omega < t\}$ and $M[> t] = \{(1+i\omega, z) \in M \mid \omega > t\}$.

Proposition 65. *Let $\tilde{g} \in \tilde{G}$ and $\varphi := \arg(\tilde{g}) \in \mathbb{R}$. In the case $|\varphi| \geq \frac{3\pi}{2}$ we have $\tilde{I}_{\tilde{g}} \cap \tilde{E}_{\tilde{e}} = \varnothing$ and $\pi(\tilde{I}_{\tilde{g}} \cap \tilde{E}_{\tilde{e}}) = \varnothing$. In the case $|\varphi| < \frac{3\pi}{2}$ the map $\pi|_{\tilde{I}_{\tilde{g}} \cap \tilde{E}_{\tilde{e}}} : \tilde{I}_{\tilde{g}} \cap \tilde{E}_{\tilde{e}} \to \pi(\tilde{I}_{\tilde{g}} \cap \tilde{E}_{\tilde{e}})$ is a homeomorphism and*

$$\begin{aligned}
\pi(\tilde{I}_{\tilde{g}} \cap \tilde{E}_{\tilde{e}}) &= I_{\pi(\tilde{g})} \cap E_e, \ \text{if } |\varphi| \leq \frac{\pi}{2}, \\
\pi(\tilde{I}_{\tilde{g}} \cap \tilde{E}_{\tilde{e}}) &= (I_{\pi(\tilde{g})} \cap E_e)[> \tan\varphi], \ \text{if } \varphi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \\
\pi(\tilde{I}_{\tilde{g}} \cap \tilde{E}_{\tilde{e}}) &= (I_{\pi(\tilde{g})} \cap E_e)[< \tan\varphi], \ \text{if } \varphi \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right).
\end{aligned}$$

5.4 The arguments of the group elements in $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$

In this section we are going to look at the arguments of the elements of the group $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(p,3,3)^k \times (C_3)^k$ where, $p = lk+3$ and $l = 1,2$. In particularly, we are interested in finding the elements with arguments between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ because of Proposition 65.

Proposition 66. *(Taken from [15], Proposition 60 and the proof of Proposition 18).*
We have

$$\arg(\tilde{r}_u) = \frac{\pi}{p}, \quad \arg(\tilde{r}_v) = \frac{\pi}{2} - \frac{\pi}{2p}, \quad \arg(\tilde{r}_v^2) = \frac{\pi}{2} + \frac{\pi}{2p}, \quad \arg(c) = \pi.$$

Lemma 67. *Consider the element $f_{0,-l} = \tilde{r}_v \tilde{r}_u^2 c^{-1}$ in $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$. Let $\vartheta_k = \frac{\pi k}{6p}$ we have*

$$\arg(f_{0,-l}\tilde{d}^j) = (2j - 3l)\vartheta_k,$$

and

$$|\arg(f_{0,-l}\tilde{d}^j)| < \frac{\pi}{2}$$

for $j = 0, \cdots, 3l$.

Proof. First we are going to look at the argument of $f_{0,-l}$. Using Propositions 12 and 66 we have

$$\begin{aligned}
\arg(f_{0,-l}) &= \arg(\tilde{r}_v \tilde{r}_u^2 c^{-1}) \\
&= \arg(\tilde{r}_v) + 2\arg(\tilde{r}_u) - \arg(c^{-1}) = \left(\frac{\pi}{2} - \frac{\pi}{2p}\right) + \frac{4\pi}{2p} - \pi \\
&= \frac{3\pi}{2p} - \frac{\pi}{2}.
\end{aligned}$$

We know that $p = lk + 3$. So,

$$\begin{aligned}
\arg(f_{0,-l}) &= \frac{3\pi}{2p} - \frac{\pi}{2} \\
&= \frac{3\pi}{2(lk+3)} - \frac{\pi(lk+3)}{2(lk+3)} = \frac{3\pi - \pi lk - 3\pi}{2(lk+3)} = \frac{-\pi lk}{2(lk+3)} \\
&= \frac{-\pi lk}{2p} = -3l \cdot \frac{\pi k}{6p} = -3l\vartheta_k.
\end{aligned}$$

Now, we look at the argument of $\tilde{d}$. According to Proposition 66 we have

$$\arg(\tilde{d}) = \arg\left(\tilde{r}_u\left(\frac{2\pi k}{3p}\right)\right) = \frac{\pi k}{3p} = 2\vartheta_k.$$

Hence, $\arg(\tilde{d}^j) = \frac{\pi k j}{3p} = 2j \cdot \vartheta_k$. So, the argument for the element $f_{0,-l}\tilde{d}^j$ is as follows:

$$\begin{aligned}\arg(f_{0,-l}\tilde{d}^j) &= \arg(f_{0,-l}) + \arg(\tilde{d}^j)\\ &= -3l\vartheta_k + j2\vartheta_k = (2j-3l)\vartheta_k.\end{aligned}$$

Since $0 \le j \le 3l \Rightarrow -3l \le 2j - 3l \le 3l$, we obtain that $-3l\vartheta_k \le \arg(f_{0,-l}\tilde{d}^j) \le 3l\vartheta_k$. Note that

$$3l\vartheta_k = 3l\frac{\pi k}{6p} = \frac{(p-3)\pi}{2p} = \frac{\pi}{2} - \frac{3\pi}{2p} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

So, the elements
$f_{0,-l}, f_{0,-l}d^1, f_{0,-l}d^2, ..., f_{0,-l}d^{3l}$ have arguments in $(-\frac{\pi}{2}, \frac{\pi}{2})$. ∎

5.5 Projections of elements with small argument

In section 5.4 we obtained the elements which have small arguments and we are going to look at the projection of these elements in SU(1,1).

Let $\epsilon^m d^a r_v^b d^t$ be an element in SU(1,1), where $\epsilon^m = (-1)^m, m \in \mathbb{Z}$. We are going to write the element as a matrix since it is an element in SU(1,1). Recall that from section 3.3 we have $r_u = r_0(\frac{2k\pi}{p}) = (e^{i\frac{k\pi}{p}}, 0)$ and $d = r_u(\frac{2\pi k}{3p}) = r_0(\frac{2\pi k}{3p}) = (e^{i\frac{\pi k}{3p}}, 0)$. Proposition 34 tells us that $r_v^l = (\cos\frac{l\pi}{q} + iC\sin\frac{l\pi}{q}, -iB(l))$, where $B(l) = S\sin\frac{l\pi}{q}, C = \cosh\rho(0,v), S = \sinh\rho(0,v)$. In our case we have $q = r = 3$. The element $\epsilon^m d^a r_v^b d^t$ is equal to

$$\epsilon^m d^a r_v^b d^t$$

$$= \epsilon^m \left(e^{i\frac{k\pi}{3p}}, 0 \right)^a \left(\cos\frac{\pi}{3} + iC\sin\frac{\pi}{3}, -iS\sin\frac{\pi}{3} \right)^b \left(e^{i\frac{k\pi}{3p}}, 0 \right)^t$$

$$= \epsilon^m \left(e^{i\frac{ak\pi}{3p}}, 0 \right) \left(\cos\frac{b\pi}{3} + iC\sin\frac{b\pi}{3}, -iS\sin\frac{b\pi}{3} \right) \left(e^{i\frac{tk\pi}{3p}} \right)$$

$$= \epsilon^m \begin{pmatrix} e^{i\frac{ak\pi}{3p}} & 0 \\ 0 & e^{-i\frac{ak\pi}{3p}} \end{pmatrix} \begin{pmatrix} \cos\frac{b\pi}{3} + iC\sin\frac{b\pi}{3} & -iS\sin\frac{b\pi}{3} \\ iS\sin\frac{b\pi}{3} & \cos\frac{b\pi}{3} - iC\sin\frac{b\pi}{3} \end{pmatrix}$$

$$\cdot \begin{pmatrix} e^{i\frac{tk\pi}{3p}} & 0 \\ 0 & e^{-i\frac{tk\pi}{3p}} \end{pmatrix}$$

$$= \epsilon^m \begin{pmatrix} e^{i\frac{ak\pi}{3p}}(\cos\frac{b\pi}{3} + iC\sin\frac{b\pi}{3}) & e^{i\frac{ak\pi}{3p}}(-iS\sin\frac{b\pi}{3}) \\ e^{-i\frac{ak\pi}{3p}}(iS\sin\frac{b\pi}{3}) & e^{-i\frac{ak\pi}{3p}}(\cos\frac{b\pi}{3} - iC\sin\frac{b\pi}{3}) \end{pmatrix}$$

$$\cdot \begin{pmatrix} e^{i\frac{tk\pi}{3p}} & 0 \\ 0 & e^{-i\frac{tk\pi}{3p}} \end{pmatrix}$$

$$= \epsilon^m \begin{pmatrix} e^{i\frac{ak\pi}{3p}} e^{i\frac{tk\pi}{3p}}(\cos\frac{b\pi}{3} + iC\sin\frac{b\pi}{3}) & e^{i\frac{ak\pi}{3p}} e^{-i\frac{tk\pi}{3p}}(-iS\sin\frac{b\pi}{3}) \\ e^{-i\frac{ak\pi}{3p}} e^{i\frac{tk\pi}{3p}}(iS\sin\frac{b\pi}{3}) & e^{-i\frac{ak\pi}{3p}} e^{-i\frac{tk\pi}{3p}}(\cos\frac{b\pi}{3} - iC\sin\frac{b\pi}{3}) \end{pmatrix}$$

$$= \epsilon^m \begin{pmatrix} e^{i\frac{(a+t)k\pi}{3p}}(\cos\frac{b\pi}{3} + iC\sin\frac{b\pi}{3}) & e^{-i\frac{(t-a)k\pi}{3p}}(-iS\sin\frac{b\pi}{3}) \\ e^{i\frac{(t-a)k\pi}{3p}}(iS\sin\frac{b\pi}{3}) & e^{-i\frac{(a+t)k\pi}{3p}}(\cos\frac{b\pi}{3} - iC\sin\frac{b\pi}{3}) \end{pmatrix}$$

$$= \epsilon^m \left(e^{i\frac{(a+t)k\pi}{3p}}(\cos\frac{b\pi}{3} + iC\sin\frac{b\pi}{3}), e^{-i\frac{(t-a)k\pi}{3p}}(-iS\sin\frac{b\pi}{3}) \right)$$

Let

$$\beta = \frac{b\pi}{3}, \quad \alpha = \frac{(a+t)k\pi}{3p} \quad \text{and} \quad \sigma = \frac{(t-a)k\pi}{3p}.$$

Hence,

$$\epsilon^m d^a r_v^b d^t \quad = \quad \epsilon^m \Big(e^{i\alpha}(\cos\beta + iC\sin\beta), -ie^{-i\sigma}B(b) \Big).$$

5.6 The equations of the embedded tangent spaces E_g and half-spaces H_g and I_g

We are going to find the equations of the half spaces $H_{\epsilon^m d^a r_v^b d^t}, I_{\epsilon^m d^a r_v^b d^t}$ and the embedded tangent space $E_{\epsilon^m d^a r_v^b d^t}$. First, we are going to find $E_{\epsilon^m d^a r_v^b d^t}$. So, from the definition of E_g we need to work out $(g, y) = 1$. The element $g = \epsilon^m d^a r_v^b d^t$ is equal to

$$
\begin{aligned}
&\epsilon^m\Big(e^{i\alpha}(\cos\beta + iC\sin\beta), -ie^{-i\sigma}B(b)\Big)\\
=&\epsilon^m\Big((\cos\alpha + i\sin\alpha)(\cos\beta + iC\sin\beta), -i(\cos\sigma - i\sin\sigma)B(b)\Big)\\
=&\epsilon^m\Big(\cos\alpha\cos\beta + i\cos\alpha\sin\beta C + i\sin\alpha\cos\beta - \sin\alpha\sin\beta C,\\
&-i\cos\sigma B(b) - B(b)\sin\sigma\Big)\\
=&\epsilon^m\Big((\cos\alpha\cos\beta - \sin\alpha\sin\beta C) + i(\cos\alpha\sin\beta C + \sin\alpha\cos\beta),\\
&-\sin\sigma B(b) - i\cos\sigma B(b)\Big).
\end{aligned}
$$

Let $M = \cos\alpha\cos\beta - \sin\alpha\sin\beta C$, $N = \cos\alpha\sin\beta C + \sin\alpha\cos\beta$, $T = -\sin\sigma B(b)$ and $R = -\cos\sigma B(b)$. Then we have $g = \epsilon^m d^a r_v^b d^t = \epsilon^m(M + iN, T + iR)$. For $y = (w_1 + iw_2, z_1 + iz_2)$. We have $(g, y) = (\epsilon^m(M + iN, T + iR), (w_1 + iw_2, z_1 + iz_2))$ is equal to

$$
\begin{aligned}
&= \epsilon^m(Mw_1 + Nw_2 - Tz_1 - Rz_2)\\
&= \epsilon^m((\cos\alpha\cos\beta - \sin\alpha\sin\beta C)w_1 + (\cos\alpha\sin\beta C + \sin\alpha\cos\beta)w_2 -\\
&\quad (-\sin\sigma B(b))z_1 - (-\cos\sigma B(b))z_2)\\
&= \epsilon^m((\cos\alpha\cos\beta - \sin\alpha\sin\beta C)w_1 + (\cos\alpha\sin\beta C + \sin\alpha\cos\beta)w_2 +\\
&\quad \sin\sigma B(b)z_1 + \cos\sigma B(b)z_2).
\end{aligned}
$$

For $y = (w_1, +iw_2, z_1 + iz_2) \in E_e$ we have $w_1 = 1$. Let $w_2 = \omega$. Then $(g, y) = 1$ means

$$
\begin{aligned}
\epsilon^m\sin\sigma B(b)z_1 + \epsilon^m\cos\sigma B(b)z_2 &= 1 - \epsilon^m(\cos\alpha\cos\beta - \sin\alpha\sin\beta C)\\
&\quad -\epsilon^m(\cos\alpha\sin\beta C + \sin\alpha\cos\beta)\omega.
\end{aligned}
$$

Recall: For $M \subset \mathbb{C}^2$, $\omega \in \mathbb{R}$ we have $M(\omega) = \{z \in \mathbb{C} \mid (1+i\omega, z) \in M\}$.
We have $b \in \{2,4\}$. Therefore, we have either $B(2) = \sinh\rho(0,v)\sin(\frac{2\pi}{3}) > 0$ or $B(4) = \sinh\rho(0,v)\sin(\frac{4\pi}{3}) < 0$. We know that $C = \cosh\rho(0,v) = \frac{1}{\sqrt{3}}\cot\frac{\pi}{2p}$.
Hence, we need to study the following two cases:

(1) The case $b = 2$:

We have $\beta = \frac{2\pi}{3}$. Therefore, $\epsilon^m \sin\sigma B(2) z_1 + \epsilon^m \cos\sigma B(2) z_2$ is equal to

$$\begin{aligned}
&= 1 - \epsilon^m\left(\left(-\frac{1}{2}\right)\cos\alpha - \left(\frac{\sqrt{3}}{2}\right)\sin\alpha\left(\frac{1}{\sqrt{3}}\cot\frac{\pi}{2p}\right)\right) \\
&\quad -\epsilon^m\left(\left(\frac{\sqrt{3}}{2}\right)\cos\alpha\left(\frac{1}{\sqrt{3}}\cot\frac{\pi}{2p}\right) + \left(-\frac{1}{2}\right)\sin\alpha\right)\omega \\
&= 1 - \epsilon^m\left(-\frac{1}{2}\left(\cos\alpha + \sin\alpha\cot\frac{\pi}{2p}\right)\right) - \epsilon^m\left(\frac{1}{2}\left(\cos\alpha\cot\frac{\pi}{2p} - \sin\alpha\right)\right)\omega \\
&= 1 + \epsilon^m\left(\frac{\cos\alpha\sin\frac{\pi}{2p} + \sin\alpha\cos\frac{\pi}{2p}}{2\sin\frac{\pi}{2p}}\right) - \epsilon^m\left(\frac{\cos\alpha\cos\frac{\pi}{2p} - \sin\alpha\sin\frac{\pi}{2p}}{2\sin\frac{\pi}{2p}}\right)\omega \\
&= 1 + \frac{\epsilon^m\sin(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}} - \frac{\epsilon^m\cos(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}}\omega.
\end{aligned}$$

Hence we have for $z = z_1 + iz_2$:

- $z \in E_{\epsilon^m d^a r_v^2 d^t}(\omega)$ if and only if

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 = \frac{1}{B(2)}\left(1 + \frac{\epsilon^m\sin(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}} - \frac{\epsilon^m\cos(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}}\omega\right).$$

- $z \in H_{\epsilon^m d^a r_v^2 d^t}(\omega)$ if and only if

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 \leq \frac{1}{B(2)}\left(1 + \frac{\epsilon^m\sin(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}} - \frac{\epsilon^m\cos(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}}\omega\right).$$

- $z \in I_{\epsilon^m d^a r_v^2 d^t}(\omega)$ if and only if

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 \geq \frac{1}{B(2)}\left(1 + \frac{\epsilon^m\sin(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}} - \frac{\epsilon^m\cos(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}}\omega\right).$$

(2) The case $b = 4$:
We have $\beta = \frac{4\pi}{3}$. Therefore, $\epsilon^m \sin\sigma B(4) z_1 + \epsilon^m \cos\sigma B(4) z_2$ is equal to

$$\begin{aligned}
&= 1 - \epsilon^m \left(\left(-\frac{1}{2}\right) \cos\alpha - \left(-\frac{\sqrt{3}}{2}\right) \sin\alpha \left(\frac{1}{\sqrt{3}} \cot\frac{\pi}{2p}\right)\right) \\
&\quad -\epsilon^m \left(\left(-\frac{\sqrt{3}}{2}\right) \cos\alpha \left(\frac{1}{\sqrt{3}} \cot\frac{\pi}{2p}\right) + \left(-\frac{1}{2}\right) \sin\alpha \right) \omega \\
&= 1 - \epsilon^m \left(-\frac{1}{2}(\cos\alpha - \sin\alpha \cot\frac{\pi}{2p})\right) - \epsilon^m \left(-\frac{1}{2}(\cos\alpha \cot\frac{\pi}{2p} + \sin\alpha)\right)\omega \\
&= 1 + \epsilon^m \left(\frac{\cos\alpha \sin\frac{\pi}{2p} - \sin\alpha \cos\frac{\pi}{2p}}{2\sin\frac{\pi}{2p}} \right) + \epsilon^m \left(\frac{\cos\alpha \cos\frac{\pi}{2p} + \sin\alpha \sin\frac{\pi}{2p}}{2\sin\frac{\pi}{2p}} \right)\omega \\
&= 1 + \frac{\epsilon^m \sin(\frac{\pi}{2p} - \alpha)}{2\sin\frac{\pi}{2p}} + \frac{\epsilon^m \cos(\frac{\pi}{2p} - \alpha)}{2\sin\frac{\pi}{2p}}\omega.
\end{aligned}$$

Hence we have for $z = z_1 + iz_2$:

- $z \in E_{\epsilon^m d^a r_v^4 d^t}(\omega)$ if and only if

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 = \frac{1}{B(4)} \left(1 + \frac{\epsilon^m \sin(\frac{\pi}{2p} - \alpha)}{2\sin\frac{\pi}{2p}} + \frac{\epsilon^m \cos(\frac{\pi}{2p} - \alpha)}{2\sin\frac{\pi}{2p}}\omega \right).$$

- $z \in H_{\epsilon^m d^a r_v^4 d^t}(\omega)$ if and only if

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 \geq \frac{1}{B(4)} \left(1 + \frac{\epsilon^m \sin(\frac{\pi}{2p} - \alpha)}{2\sin\frac{\pi}{2p}} + \frac{\epsilon^m \cos(\frac{\pi}{2p} - \alpha)}{2\sin\frac{\pi}{2p}}\omega \right).$$

- $z \in I_{\epsilon^m d^a r_v^4 d^t}(\omega)$ if and only if

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 \leq \frac{1}{B(4)} \left(1 + \frac{\epsilon^m \sin(\frac{\pi}{2p} - \alpha)}{2\sin\frac{\pi}{2p}} + \frac{\epsilon^m \cos(\frac{\pi}{2p} - \alpha)}{2\sin\frac{\pi}{2p}}\omega \right).$$

Proposition 68. *For a complex number $z = z_1 + iz_2$ and a real number ω we have*

- $z \in E_{\epsilon^m d^a r_v^2 d^t}(\omega)$ *if and only if*

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 = \frac{1}{B(2)}\left(1 + \frac{\epsilon^m \sin(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}} - \frac{\epsilon^m \cos(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}}\omega\right).$$

- $z \in H_{\epsilon^m d^a r_v^2 d^t}(\omega)$ *if and only if*

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 \leq \frac{1}{B(2)}\left(1 + \frac{\epsilon^m \sin(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}} - \frac{\epsilon^m \cos(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}}\omega\right).$$

- $z \in I_{\epsilon^m d^a r_v^2 d^t}(\omega)$ *if and only if*

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 \geq \frac{1}{B(2)}\left(1 + \frac{\epsilon^m \sin(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}} - \frac{\epsilon^m \cos(\frac{\pi}{2p}+\alpha)}{2\sin\frac{\pi}{2p}}\omega\right).$$

- $z \in E_{\epsilon^m d^a r_v^4 d^t}(\omega)$ *if and only if*

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 = \frac{1}{B(4)}\left(1 + \frac{\epsilon^m \sin(\frac{\pi}{2p}-\alpha)}{2\sin\frac{\pi}{2p}} + \frac{\epsilon^m \cos(\frac{\pi}{2p}-\alpha)}{2\sin\frac{\pi}{2p}}\omega\right).$$

- $z \in H_{\epsilon^m d^a r_v^4 d^t}(\omega)$ *if and only if*

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 \geq \frac{1}{B(4)}\left(1 + \frac{\epsilon^m \sin(\frac{\pi}{2p}-\alpha)}{2\sin\frac{\pi}{2p}} + \frac{\epsilon^m \cos(\frac{\pi}{2p}-\alpha)}{2\sin\frac{\pi}{2p}}\omega\right).$$

- $z \in I_{\epsilon^m d^a r_v^4 d^t}(\omega)$ *if and only if*

$$\epsilon^m \sin\sigma z_1 + \epsilon^m \cos\sigma z_2 \leq \frac{1}{B(4)}\left(1 + \frac{\epsilon^m \sin(\frac{\pi}{2p}-\alpha)}{2\sin\frac{\pi}{2p}} + \frac{\epsilon^m \cos(\frac{\pi}{2p}-\alpha)}{2\sin\frac{\pi}{2p}}\omega\right).$$

Here,

$$\alpha = \frac{(a+t)k\pi}{3p} \quad \sigma = \frac{(t-a)k\pi}{3p} \quad \textit{and} \quad B(b) = \sinh\rho(0,v)\cdot\sin\frac{\pi b}{3}.$$

5.7 Fundamental domains for the case $\tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$

In this section we are going to compute fundamental domains for the case $\tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$ explicitly. Recall that according to Proposition 20 the fundamental domain $F_{\tilde{e}}$ can be described as

$$F_{\tilde{e}} = \text{CI Int}\left(F_{\tilde{\Gamma}_1(u)\smallsetminus\{u\}}\right),$$

where $F_N = \tilde{S}_{\tilde{e}} \cap \bigcap_{x\in N} R_x = \tilde{E}_{\tilde{e}} \cap \partial Q_u \cap \bigcap_{x\in N} R_x$ for $N \subset \tilde{\Gamma}_1(u) \smallsetminus \{u\}$.

First we will construct a polyhedron $P \subset \tilde{S}_{\tilde{e}}$ such that $P \subset F_{\mathcal{E}}$, where $\mathcal{E}$ is the edge crown of $\tilde{\Gamma}_1$. Proposition 48 shows that $F_{\tilde{\Gamma}_1(u)\smallsetminus\{u\}} = F_{\mathcal{E}} = F_{\tilde{e}}$. Hence, we have $P \subset F_{\tilde{\Gamma}_1(u)\smallsetminus\{u\}} = F_{\tilde{e}}$. Therefore, if we show that P is a fundamental domain then we obtain $P \subseteq F_{\tilde{e}}$ and they are both fundamental domains for $\tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$. Thus, we obtain $\text{Int}(P) = \text{Int}(F_{\tilde{e}})$ since if a fundamental domain is inside another fundamental domain then their interior is exactly the same. Then $P = \text{CI Int}(P) = \text{CI Int}(F_{\tilde{e}}) = F_{\tilde{e}}$.
To be more clear, we describe the image $\pi(P)$ in S_e and construct a polyhedron $\hat{P}$ in $\hat{S}_e$ such that $\pi(P) = \hat{P} \cap L$. After that, we study the polyhedron $\hat{P}$ to determine the combinatorial structure of its faces and we show the picture of some faces of $\hat{P}$ as in figure 5.1. Then, we obtain $\pi(P) = \hat{P} \cap L = \hat{P}$ since $\hat{P} \subset L$. Now, we need to apply Theorem 49 to $\Psi(P)$ as we described in subsection 4.2.1. After checking all the conditions of Theorem 49 we obtain that $\Psi(P)$ is a fundamental domain for the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$. Finally, as we mentioned in the beginning of this section, we have from Proposition 48 that $\Psi(P) \subset \Psi(F_{\tilde{\Gamma}_1(u)\smallsetminus\{u\}}) = \Psi(F_{\tilde{e}})$ and they are both fundamental domains of the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$. Hence $P = F_{\tilde{\Gamma}_1(u)\smallsetminus\{u\}} = F_{\tilde{e}}$ (up to the boundary points).

Recall: The elements with small argument which we obtained in Lemma 67 are:

$$f_{0,-1}, f_{0,-1}\tilde{d}, f_{0,-1}\tilde{d}^2, f_{0,-1}\tilde{d}^3,$$

where $f_{0,-1} = \tilde{r}_v \tilde{r}_u^2 c^{-1}$.

Using Corollary 53 we obtain

$$\begin{aligned} f_{0,-1} &= \tilde{r}_v \tilde{r}_u^2 c^{-1} \\ &= \tilde{r}_v \left(c^{\frac{1-\lambda k}{3}} \tilde{d}^{\lambda p-1} \right)^2 c^{-1} = \tilde{r}_v \tilde{d}^{2(\lambda p-1)} c^{\frac{-(2\lambda k+1)}{3}}, \end{aligned}$$

where $\lambda = 1$ if $k \equiv 1 \mod 3$ and $\lambda = 2$ if $k \equiv 2 \mod 3$.

We know that $\tilde{r}_v^3 = c$. So, we can write $f_{0,-1}$ as

$$f_{0,-1} = \tilde{r}_v^4 \tilde{d}^{2(\lambda p-1)} c^{\frac{-2(\lambda k+2)}{3}}.$$

Let

$$\begin{aligned} a_0 = f_{0,-1}\tilde{d} &= \tilde{r}_v^4 \tilde{d}^{2\lambda p-1} c^{\frac{-2(\lambda k+2)}{3}}, \\ b_0 = f_{0,-1}\tilde{d}^2 &= \tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}}. \end{aligned}$$

Using the symmetries in Section 5.2 we can define $a_m = \tilde{\rho}^m(a_0)$ and $b_m = \tilde{\rho}^m(b_0)$. Using Proposition 62 we obtain

$$\begin{aligned} b_{-1} &= \tilde{\rho}^{-1}(b_0) = \tilde{\rho}^{-1}\left(\tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}} \right) \\ &= \tilde{d}^{1-\lambda p} \tilde{r}_v^2 \tilde{d}^{2\lambda p} c^{\frac{-(\lambda k+2)}{3}}. \end{aligned}$$

Now, we are going to look at the projection of these elements. Let $A_m = \pi(a_m)$ and $B_m = \pi(b_m)$. Using Corollary 56 we obtain

$$\begin{aligned} A_0 &= \pi(a_0) = r_v^4 d^{2\lambda p-1}, \\ B_0 &= \pi(b_0) = r_v^4 d^{2\lambda p}, \\ B_{-1} &= \pi(b_{-1}) = d^{1-\lambda p} r_v^2 d^{2\lambda p} \epsilon^{\frac{\lambda k+2}{3}}. \end{aligned}$$

see figure 5.1.

Notation: From now on for the ease of notation we will write $\tilde{E}a_m$ instead of $\tilde{E}_{a_m}$, EA_m instead of E_{A_m} and similar for b_m, B_m.

Definition 69. *Let* $\omega^{\pm} = \pm \tan \frac{\pi k}{6p}$. *Let*

$$P = \bigcap_{m \in \mathbb{Z}} (\tilde{I} A_m \cup \tilde{I} B_m)[\omega^-, \omega^+].$$

Note that $(\tilde{I} A_m \cup \tilde{I} B_m)$ *is contained in the complement of* Q_{x_m}*, hence* P *is contained in the complement of* $\bigcup_{m \in \mathbb{Z}} Q_{x_m} = \bigcup_{x \in \mathcal{E}} Q_x$. *Therefore,* $P \subset F_{\mathcal{E}}$. *Proposition 65 and Lemma 67 imply that*

$$\pi(P) = \bigcap_{m \in \mathbb{Z}} (I A_m \cup I B_m)[\omega^-, \omega^+]$$

and that $\pi|_P : P \mapsto \pi(P)$ *is a homeomorphism. Let*

$$\hat{P} = \bigcap_{m \in \mathbb{Z}} (\hat{I} A_m \cup \hat{I} B_m)[\omega^-, \omega^+].$$

We will use some of the net estimate results to show that $\hat{P} = \pi(P)$. We shall study the combinatorial structure of the boundary of $\hat{P}$ as illustrated in figure 5.1. To do this we are going to use the analytic geometry in the appendix 8.6.

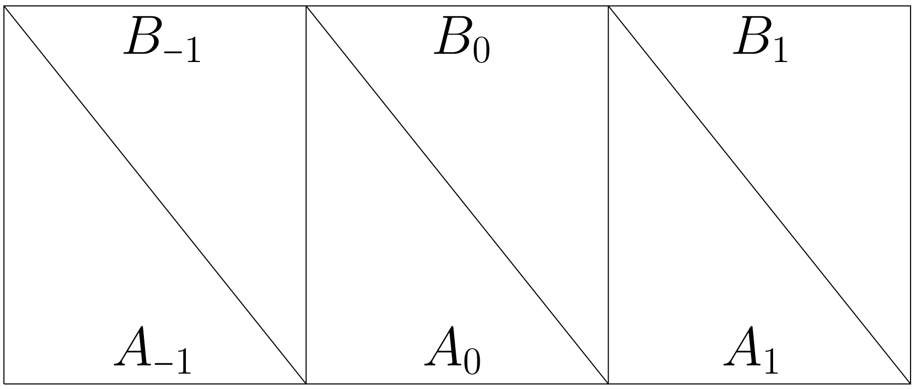

Figure 5.1: The surface of $\hat{P}$ for $\tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$.

Proposition 70. *We have*

- $z \in HA_0(\omega)$ *if and only if*

$$\sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2 \geq \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}+\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right),$$

- $z \in HB_0(\omega)$ *if and only if*

$$\sin\frac{2\pi}{3}z_1+\cos\frac{2\pi}{3}z_2 \quad \geq \quad \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}-\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right),$$

- $z \in HB_{-1}(\omega)$ *if and only if*

$$-\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_1+\cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_2 \leq \frac{1}{B(2)}\left(1-\frac{\sin(\frac{\pi}{2p}+\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}-\frac{\cos(\frac{\pi}{2p}+\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}\omega\right).$$

Proof. We will use Proposition 68.

- *The case HA_0*: We have $A_0 = r_v^4 d^{2\lambda p-1}$, so from the definition of σ and α we obtain that

$$\sigma_1 = \alpha_1 \quad = \quad \frac{(2\lambda p-1)k\pi}{3p} = \frac{(2\lambda p-1)(p-3)\pi}{3p} = \frac{2\lambda p\pi}{3} - \frac{(6\lambda+1)\pi}{3} + \frac{\pi}{p}.$$

If $k = 1 \mod 3$ then $\lambda = 1$ and $p = k+3 \equiv 1 \mod 3 \Rightarrow p = 3x+1$ for some $x \in \mathbb{Z}$. Therefore,

$$\frac{2\lambda p\pi}{3}-\frac{(6\lambda+1)\pi}{3}+\frac{\pi}{p} = \frac{2}{3}(3x+1)\pi-\frac{7\pi}{3}+\frac{\pi}{p} = 2x\pi-\frac{5\pi}{3}+\frac{\pi}{p} \equiv \frac{\pi}{p}+\frac{\pi}{3} \mod 2\pi.$$

If $k = 2 \mod 3$ then $\lambda = 2$ and $p = k+3 \equiv 2 \mod 3 \Rightarrow p = 3x+2$ for some $x \in \mathbb{Z}$. Therefore,

$$\frac{2\lambda p\pi}{3}-\frac{(6\lambda+1)\pi}{3}+\frac{\pi}{p} = \frac{4}{3}(3x+2)\pi-\frac{13\pi}{3}+\frac{\pi}{p} = 4x\pi-\frac{5\pi}{3}+\frac{\pi}{p} \equiv \frac{\pi}{p}+\frac{\pi}{3} \mod 2\pi.$$

Hence for the case HA_0 we obtain the following inequality: $z \in HA_0(\omega)$ if and only if

$$\sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2$$
$$\geq \frac{1}{B(4)}\left(1+\frac{\sin(-\frac{\pi}{2}+(\frac{\pi}{6}-\frac{\pi}{2p}))}{2\sin\frac{\pi}{2p}}+\frac{\cos(-\frac{\pi}{2}+(\frac{\pi}{6}-\frac{\pi}{2p}))}{2\sin\frac{\pi}{2p}}\omega\right).$$

Also, we can write this inequality as $z \in HA_0(\omega)$ if and only if

$$\sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2$$
$$\geq \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}+\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).$$

- *The case HB_0*: We have $B_0 = r_v^4 d^{2\lambda p}$, so from the definition of σ and α we obtain that

$$\sigma_2=\alpha_2 \quad = \quad \frac{(2\lambda p)k\pi}{3p}=\alpha_1+\frac{k\pi}{3p}=\frac{\pi}{p}+\frac{\pi}{3}+\frac{\pi(p-3)}{3p}\equiv\frac{2\pi}{3} \mod 2\pi.$$

Note that $\frac{\pi}{2p}-\alpha_2 \equiv -\frac{\pi}{2}-(\frac{\pi}{6}-\frac{\pi}{2p}) \mod 2\pi$. Thus for the case HB_0 we obtain the following inequality: $z \in HB_0(\omega)$ if and only if

$$\sin\frac{2\pi}{3}z_1+\cos\frac{2\pi}{3}z_2$$
$$\geq \frac{1}{B(4)}\left(1+\frac{\sin(-\frac{\pi}{2}-(\frac{\pi}{6}-\frac{\pi}{2p}))}{2\sin\frac{\pi}{2p}}+\frac{\cos(-\frac{\pi}{2}-(\frac{\pi}{6}-\frac{\pi}{2p}))}{2\sin\frac{\pi}{2p}}\omega\right).$$

We can write this conditions as $z \in HB_0(\omega)$ if and only if

$$\sin\frac{2\pi}{3}z_1+\cos\frac{2\pi}{3}z_2 \quad \geq \quad \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}-\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).$$

- *The case HB_{-1}*: We have $B_{-1} = d^{1-\lambda p} r_v^2 d^{2\lambda p}\epsilon^{\frac{\lambda k+2}{3}}$, so from the definition of σ and α we obtain that

$$\sigma_3 = \frac{(3\lambda p - 1)k\pi}{3p}, \quad \alpha_3 = \frac{(1+\lambda p)k\pi}{3p}, \quad \epsilon^{\frac{\lambda k+2}{3}} = (-1)^{\frac{\lambda k+2}{3}}.$$

If $k = 1 \mod 3$ then $\lambda = 1$,

$$\sigma_3 = \pi p - 3\pi - \frac{\pi}{3} + \frac{\pi}{p}, \quad \alpha_3 = \frac{(p-2)\pi}{3} - \frac{\pi}{p}, \quad \epsilon^{\frac{k+2}{3}} = (-1)^{\frac{k+2}{3}}.$$

If $k = 1 \mod 3$ and k is even, then $k = 4 \mod 6$, i.e. $k = 6x+4$ for some $x \in \mathbb{Z}$.

Then $\epsilon^{\frac{k+2}{3}} = 1$ and we obtain

$$\sigma_3 = \pi(p-3) - \left(\frac{\pi}{3} - \frac{\pi}{p}\right) = \pi k - \left(\frac{\pi}{3} - \frac{\pi}{p}\right) \equiv -\left(\frac{\pi}{3} - \frac{\pi}{p}\right) \mod 2\pi,$$

and

$$\frac{\pi}{2p} + \alpha_3 = -\left(\frac{\pi}{2p} + \frac{\pi}{3}\right) + \frac{(p-1)\pi}{3} \equiv -\left(\frac{\pi}{2p} + \frac{\pi}{3}\right) \mod 2\pi$$

since $\frac{(p-1)\pi}{3} = \frac{(k+2)\pi}{3} = \frac{(6x+6)\pi}{3} = 2(x+1)\pi \equiv 0 \mod 2\pi$.

If $k = 1 \mod 3$ and k is odd then $k = 1 \mod 6$, i.e. $k = 6x+1$ for some $x \in \mathbb{Z}$. Then $\epsilon^{\frac{k+2}{3}} = -1$ and we obtain

$$\sigma_3 = \pi(p-3) - \left(\frac{\pi}{3} - \frac{\pi}{p}\right) = \pi k - \left(\frac{\pi}{3} - \frac{\pi}{p}\right) \equiv \pi - \left(\frac{\pi}{3} - \frac{\pi}{p}\right) \mod 2\pi$$

since k is odd and

$$\frac{\pi}{2p} + \alpha_3 = -\left(\frac{\pi}{2p} + \frac{\pi}{3}\right) + \frac{(p-1)\pi}{3} \equiv \pi - \left(\frac{\pi}{2p} + \frac{\pi}{3}\right) \mod 2\pi$$

since $\frac{(p-1)\pi}{3} = \frac{(k+2)\pi}{3} = \frac{(6x+3)\pi}{3} = (2x+1)\pi \equiv \pi \mod 2\pi$. However, we know that $\cos(\phi+\pi) = -\cos\varphi$ and $\sin(\varphi+\pi) = -\sin\varphi$ and we know that for every sin and cos there is ϵ and in the case where k is odd we have $\epsilon = -1$. Hence, the sign will cancel each other and we will have the following inequality for HB_0 for $\lambda = 1$: $z \in HB_{-1}(\omega)$ if and only if

$$-\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_1+\cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_2 \\ \leq \frac{1}{B(2)}\left(1-\frac{\sin(\frac{\pi}{2p}+\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}-\frac{\cos(\frac{\pi}{2p}+\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}\omega\right).$$

If $k = 2 \mod 3$ then $\lambda = 2$,

$$\sigma_3 = 2\pi p - 6\pi - \frac{\pi}{3} + \frac{\pi}{p}, \quad \alpha_3 = \frac{(2p-5)\pi}{3} - \frac{\pi}{p}, \quad \epsilon^{\frac{2(k+1)}{3}} = (-1)^{\frac{2(k+1)}{3}} = 1.$$

We have

$$\sigma_3 = 2\pi(p-3) - \left(\frac{\pi}{3}-\frac{\pi}{p}\right) \equiv -\left(\frac{\pi}{3}-\frac{\pi}{p}\right) \mod 2\pi$$

and

$$\frac{\pi}{2p} + \alpha_3 = -\left(\frac{\pi}{2p}+\frac{\pi}{3}\right) + 2\pi\left(\frac{(p-2)}{3}\right) \equiv -\left(\frac{\pi}{2p}+\frac{\pi}{3}\right) \mod 2\pi$$

since $p-2 = k+1 \equiv 0 \mod 3$, i.e. $\frac{p-2}{3}$ is an integer.

We can see that there is no difference between the cases $\lambda = 1$ and $\lambda = 2$. Therefore, we have $z \in HB_{-1}(\omega)$ if and only if

$$-\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_1+\cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_2 \\ \leq \frac{1}{B(2)}\left(1-\frac{\sin(\frac{\pi}{2p}+\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}-\frac{\cos(\frac{\pi}{2p}+\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}\omega\right).$$

∎

Proposition 71. *We have* $0 \notin (HA_0 \cap HB_0)(\omega)$.

Proof. We have

- $z \in HA_0(\omega)$ if and only if

$$\sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2 \geq \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}+\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right),$$

- $z \in HB_0(\omega)$ if and only if

$$\sin\frac{2\pi}{3}z_1+\cos\frac{2\pi}{3}z_2 \quad \geq \quad \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}-\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).$$

Thus $0 \in (HA_0 \cap HB_0)(\omega)$ would imply that

$$\frac{1}{B(4)}\left(1-\frac{1}{2\sin\frac{\pi}{2p}}\left(\cos\left(\frac{\pi}{6}-\frac{\pi}{2p}\right)\pm\sin\left(\frac{\pi}{6}-\frac{\pi}{2p}\right)\omega\right)\right)\leq 0.$$

We know that $B(4)<0$, hence $1-\frac{1}{2\sin\frac{\pi}{2p}}\left(\cos(\frac{\pi}{6}-\frac{\pi}{2p})\pm\sin(\frac{\pi}{6}-\frac{\pi}{2p})\omega\right)\geq 0$ and therefore $\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\leq 1$. However, we have $\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}>\frac{\cos(\frac{\pi}{6})}{2\sin\frac{\pi}{8}}>1$ since $p\geq 4$. Hence $0 \notin (HA_0 \cap HB_0)(\omega)$. ∎

Proposition 72. *The bisector of the sector $(HA_0 \cap HB_0)(0)$ contains the origin.*

Proof. We apply Proposition 92 (in the appendix) for $H^- = HA_0(0)$ and $H^+ = HB_0(0)$. The inequalities of the half planes are

- $z \in HA_0(\omega)$ if and only if

$$\sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2 \quad \geq \quad \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right),$$

- $z \in HB_0(\omega)$ if and only if

$$\sin\frac{2\pi}{3}z_1+\cos\frac{2\pi}{3}z_2 \quad \geq \quad \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right).$$

∎

Proposition 73. *The lines $EA_0(\omega^+), EB_0(\omega^+)$ and $EB_{-1}(\omega^+)$ meet at one point. Here $\omega^+ = \tan(\frac{\pi k}{6p})$*

Proof. The equations of the lines are

- $z \in EA_0(\omega)$ if and only if

$$\sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2$$
$$=\frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}+\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right),$$

- $z \in EB_0(\omega)$ if and only if

$$\sin\frac{2\pi}{3}z_1+\cos\frac{2\pi}{3}z_2 \quad = \quad \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}-\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right),$$

- $z \in EB_{-1}(\omega)$ if and only if

$$-\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_1+\cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_2$$
$$=\frac{1}{B(2)}\left(1-\frac{\sin(\frac{\pi}{2p}+\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}-\frac{\cos(\frac{\pi}{2p}+\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}\omega\right).$$

For the equation of the line EA_0 we have the right hand side

$$\begin{aligned}
&\frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}+\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right)\\
=\ &\frac{1}{B(4)}\left(\frac{2\sin\frac{\pi}{2p}-\cos\frac{\pi}{6}\cos\frac{\pi}{2p}-\sin\frac{\pi}{6}\sin\frac{\pi}{2p}+\omega\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right)\\
=\ &\frac{1}{B(4)}\left(\frac{\sqrt{3}(\frac{\sqrt{3}}{2}\sin\frac{\pi}{2p}-\frac{1}{2}\cos\frac{\pi}{2p})+\omega\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right)\\
=\ &\frac{1}{B(4)}\left(\frac{-\sqrt{3}\sin(\frac{\pi}{6}-\frac{\pi}{2p})+\omega\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right).
\end{aligned}$$

Thus,

$$z\in EA_0(\omega)\iff \sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2=\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{B(4)2\sin\frac{\pi}{2p}}\left(\omega-\sqrt{3}\right).$$

Similarly, for the equation of $EB_0(\omega)$ we have

$$z\in EB_0(\omega)\iff \sin\frac{2\pi}{3}z_1+\cos\frac{2\pi}{3}z_2=\frac{-\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{B(4)2\sin\frac{\pi}{2p}}\left(\omega+\sqrt{3}\right).$$

For the equation of HB_{-1} we have the right hand side

$$\begin{aligned}
&\frac{1}{B(2)}\left(1-\frac{\sin(\frac{\pi}{2p}+\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}-\frac{\cos(\frac{\pi}{2p}+\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}\omega\right)\\
&=\frac{1}{B(2)}\left(\frac{2\sin\frac{\pi}{2p}-\sin(\frac{\pi}{2p}+\frac{\pi}{3})-\omega\cos(\frac{\pi}{2p}+\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}\right)\\
&=\frac{1}{B(2)}\left(\frac{2\sin\frac{\pi}{2p}-\sin\frac{\pi}{2p}\cos\frac{\pi}{3}-\sin\frac{\pi}{3}\cos\frac{\pi}{2p}-\omega(\cos\frac{\pi}{2p}\cos\frac{\pi}{3}-\sin\frac{\pi}{2p}\sin\frac{\pi}{3})}{2\sin\frac{\pi}{2p}}\right)\\
&=\frac{1}{B(2)}\left(\frac{-\sqrt{3}\sin(\frac{\pi}{6}-\frac{\pi}{2p})-\omega\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right).
\end{aligned}$$

Therefore, $z \in EB_{-1}(\omega)$ if and only if

$$-\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_1+\cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_2 \quad = \quad \frac{-\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{B(2)2\sin\frac{\pi}{2p}}\left(\sqrt{3}+\omega\right).$$

Recall that $B(2)=S\cdot\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}\cdot S$ and $B(4)=S\cdot\sin\frac{4\pi}{3}=-\frac{\sqrt{3}}{2}\cdot S$. We can now apply Corollary 91 (in appendix). We have

$$\begin{aligned}
\Delta(\omega) &= \begin{vmatrix} \sin(\frac{\pi}{p}+\frac{\pi}{3}) & \cos(\frac{\pi}{p}+\frac{\pi}{3}) & -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\left(\omega-\sqrt{3}\right) \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}-\left(\omega+\sqrt{3}\right) \\ -\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right) & \cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right) & -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\left(\omega+\sqrt{3}\right) \end{vmatrix} \\
&= -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\begin{vmatrix} \sin(\frac{\pi}{p}+\frac{\pi}{3}) & \cos(\frac{\pi}{p}+\frac{\pi}{3}) & \left(\omega-\sqrt{3}\right) \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & -\left(\omega+\sqrt{3}\right) \\ -\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right) & \cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right) & \left(\omega+\sqrt{3}\right) \end{vmatrix}. \\
&= -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\ \Delta_1(\omega).
\end{aligned}$$

First we need to check if $\delta \neq 0$, where $\delta = \begin{vmatrix} \sin(\frac{\pi}{p}+\frac{\pi}{3}) & \cos(\frac{\pi}{p}+\frac{\pi}{3}) \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{vmatrix} = \sin(\frac{\pi}{p}-\frac{\pi}{3})$. We have $-\frac{\pi}{3}<\frac{\pi}{p}-\frac{\pi}{3}<0$ since $p>3$. Thus, $\sin(\frac{\pi}{p}-\frac{\pi}{3})\neq 0$, i.e. $\delta\neq 0$.

We compute

$$\begin{aligned}
\Delta_1(\omega) &= (\omega-\sqrt{3})\sin(\frac{2\pi}{3}+(\frac{\pi}{3}-\frac{\pi}{p})) \\
&\quad -(-(\omega+\sqrt{3}))\sin((\frac{\pi}{p}+\frac{\pi}{3})+(\frac{\pi}{3}-\frac{\pi}{p})) \\
&\quad +(\omega+\sqrt{3})\sin((\frac{\pi}{p}+\frac{\pi}{3})-\frac{2\pi}{3}) \\
&= (\omega-\sqrt{3})\sin(\pi-\frac{\pi}{p})+(\omega+\sqrt{3})(\sin(\frac{2\pi}{3})+\sin(\frac{\pi}{p}-\frac{\pi}{3})) \\
&= (\omega-\sqrt{3})\sin(\frac{\pi}{p})+(\omega+\sqrt{3})(\frac{\sqrt{3}}{2}(1-\cos\frac{\pi}{p})+\frac{1}{2}\sin\frac{\pi}{p}) \\
&= (\omega-\sqrt{3})2\sin\frac{\pi}{2p}\cos\frac{\pi}{2p}+(\omega+\sqrt{3})\sin\frac{\pi}{2p}\cos\frac{\pi}{2p} \\
&\quad +(\omega+\sqrt{3})(\frac{\sqrt{3}}{2}(2\sin^2\frac{\pi}{2p})) \\
&= \sin\frac{\pi}{2p}\left((3\omega-\sqrt{3})\cos\frac{\pi}{2p}+\sin\frac{\pi}{2p}(\sqrt{3}\omega+3)\right) \\
&= 2\sqrt{3}\sin\frac{\pi}{2p}\left(\omega\left(\frac{\sqrt{3}}{2}\cos\frac{\pi}{2p}+\frac{1}{2}\sin\frac{\pi}{2p}\right)+\left(\frac{\sqrt{3}}{2}\sin\frac{\pi}{2p}-\frac{1}{2}\cos\frac{\pi}{2p}\right)\right) \\
&= 2\sqrt{3}\sin\frac{\pi}{2p}\left(\omega\cos(\frac{\pi}{6}-\frac{\pi}{2p})-\sin(\frac{\pi}{6}-\frac{\pi}{2p})\right).
\end{aligned}$$

We know that $\omega^+ = \tan\frac{\pi k}{6p} = \frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}$. Thus,

$$\Delta_1(\omega^+) = 2\sqrt{3}\sin\frac{\pi}{2p}\left(\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}-\sin(\frac{\pi}{6}-\frac{\pi}{2p})\right) = 0.$$

$\therefore$ The lines $EA_0(\omega^+), EB_0(\omega^+)$ and $EB_{-1}(\omega^+)$ meet at one point. ∎

Using Propositions 71, 72 and 73 we can follow the mutual position of the lines EA_m, EB_m as ω changes from $\omega=\omega^-$ to $\omega=\omega^+$, see figures 5.2, 5.3 and 5.4 for the case $k=2, p=5$.

Now, we understand the combinatorial structure of the faces of $\hat{P}$, compare with figure 5.1.

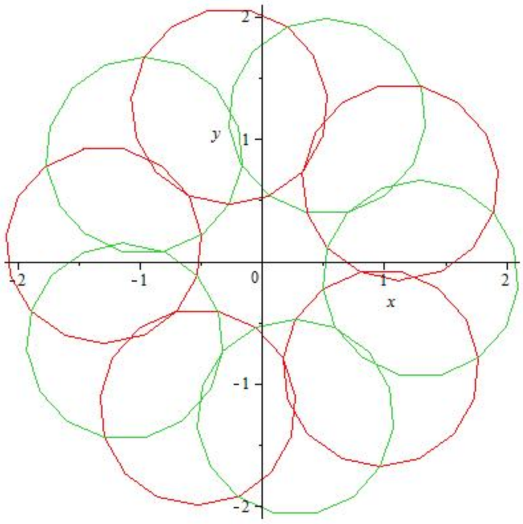

Figure 5.2: The intersections of the prisms Q_x for $\tilde{\Gamma}_1(5,3,3)^2 \times (C_3)^2$ with the plane $\omega = \omega^-$.

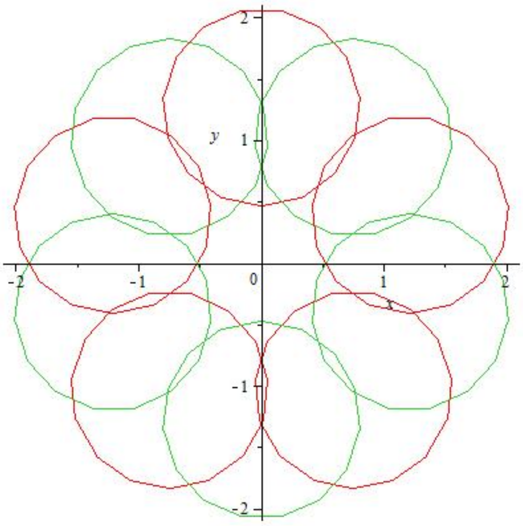

Figure 5.3: The intersections of the prisms Q_x for $\tilde{\Gamma}_1(5,3,3)^2 \times (C_3)^2$ with the plane $\omega = 0$.

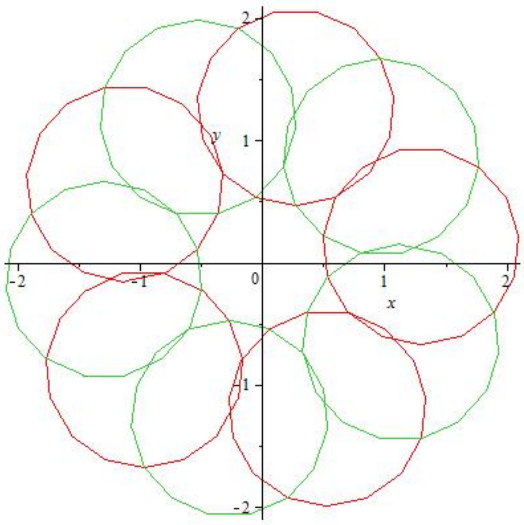

Figure 5.4: The intersections of the prisms Q_x for $\tilde{\Gamma}_1(5,3,3)^2 \times (C_3)^2$ with the plane $\omega = \omega^+$.

Proposition 74. *The identification scheme for the faces of the boundary of P under the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ is:*

$$
\begin{array}{rcl}
(\tilde{d}; b_m) & \mapsto & (\tilde{d}^{-1}; a_{m+2}), \\
(b_m; \tilde{d}, a_m, a_{m+1}) & \mapsto & (a_{m+1}; b_{m+1}, \tilde{d}^{-1}, b_m), \\
(a_m; \tilde{d}^{-1}, b_m, b_{m-1}) & \mapsto & (b_{m-1}; a_{m-1}, \tilde{d}, a_m)
\end{array}
$$

and the edge cycles

$$(b_m; \tilde{d}) \mapsto (b_{m+1}; a_{m+1}) \mapsto (\tilde{d}^{-1}; a_{m+2}),$$

$$(b_m; a_{m+1}) \circlearrowleft .$$

Proof.

In the beginning of this section we studied the polyhedron $\hat{P}$. The faces of P correspond to the elements

$$
\begin{array}{rcl}
a_0 = f_{0,-1}\tilde{d} & = & \tilde{r}_v^4 \tilde{d}^{2\lambda p - 1} c^{\frac{-2(\lambda k+2)}{3}}, \\
b_0 = f_{0,-1}\tilde{d}^2 & = & \tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}},
\end{array}
$$

and their images $a_m = \tilde{\rho}^m(a_0), b_m = \tilde{\rho}^m(b_0)$. Here $\lambda = 1$ if $k = 1 \mod 3$ and

$\lambda = 2$ if $k = 2 \mod 3$. We will look at the identification schemes for this case. We know that the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$ is

$$(\tilde{g}_1, \tilde{g}_2) \cdot x = \tilde{g}_1 x \tilde{g}_2^{-1}.$$

So,

$$(\tilde{g}_1^{-1}, \tilde{g}_2) \cdot x = \tilde{g}_1^{-1} x \tilde{g}_2^{-1}.$$

Hence, $x \mapsto \tilde{g}_1^{-1} x \tilde{g}_2^{-1}$. Therefore, $\tilde{E}_{a_m} \mapsto \tilde{E}_{\tilde{g}_1^{-1} a_m \tilde{g}_2^{-1}}$ and $\tilde{E}_{b_m} \mapsto \tilde{E}_{\tilde{g}_1^{-1} b_m \tilde{g}_2^{-1}}$. Let

$$a_m = \tilde{\rho}^m(a_0), \quad b_m = \tilde{\rho}^m(b_0).$$

We can write a_0 and b_0 as

$$\begin{aligned} a_0 &= \tilde{r}_v^4 \tilde{d}^{2\lambda p-1} c^{\frac{-2(\lambda k+2)}{3}} = \tilde{r}_v \tilde{d}^{2\lambda p-1} c^{\frac{-(2\lambda k+1)}{3}}, \\ b_0 &= \tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}} = \tilde{r}_v \tilde{d}^{2\lambda p} c^{\frac{-(2\lambda k+1)}{3}} \end{aligned}$$

Hence,

$$\begin{aligned} a_1 &= \tilde{\rho}(a_0) = \tilde{\rho}(\tilde{r}_v \tilde{d}^{2\lambda p-1} c^{\frac{-(2\lambda k+1)}{3}}) \\ &= \tilde{\rho}(\tilde{r}_v) \tilde{\rho}(\tilde{d}^{2\lambda p-1}) \tilde{\rho}(c^{\frac{-(2\lambda k+1)}{3}}) \\ &= \tilde{r}_v^2 \tilde{r}_u^{-1} \tilde{d}^{2\lambda p-1} c^{\frac{-(2\lambda k+1)}{3}} = \tilde{r}_v^2 \tilde{d}^{\lambda p} c^{\frac{-(\lambda k+2)}{3}}, \end{aligned}$$

since $\tilde{r}_u^{-1} = c^{\frac{\lambda k-1}{3}} d^{1-\lambda p}$ according to Corollary 53. Similarly, $b_1 = \tilde{\rho}(b_0) = \tilde{r}_v^2 \tilde{d}^{\lambda p+1} c^{\frac{-(\lambda k+2)}{3}}$. Note that $b_m = a_m \tilde{d}$.

1. The face $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}}$:

   We have $b_0 = \tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}}$. We need to write b_0 as a product $\tilde{g}_1 \cdot \tilde{g}_2$ with $\tilde{g}_1 \in \tilde{\Gamma}_1$ and $\tilde{g}_2 \in \tilde{\Gamma}_2$:

   $$b_0 = \tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}} = \left(\tilde{r}_v^4 c^{\frac{-2(\lambda k+2)}{3}}\right)\left(\tilde{d}_2^{2\lambda}\right) = \tilde{g}_1 \cdot \tilde{g}_2.$$

   We consider the action of $(\tilde{g}_1^{-1}, \tilde{g}_2) \in \tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$ by $x \mapsto \tilde{g}_1^{-1} x \tilde{g}_2^{-1}$. Then, we need to calculate $\tilde{g}_1^{-1} \tilde{g}_2^{-1}$ in order to find where $\tilde{E}_{\tilde{e}}$ is mapped

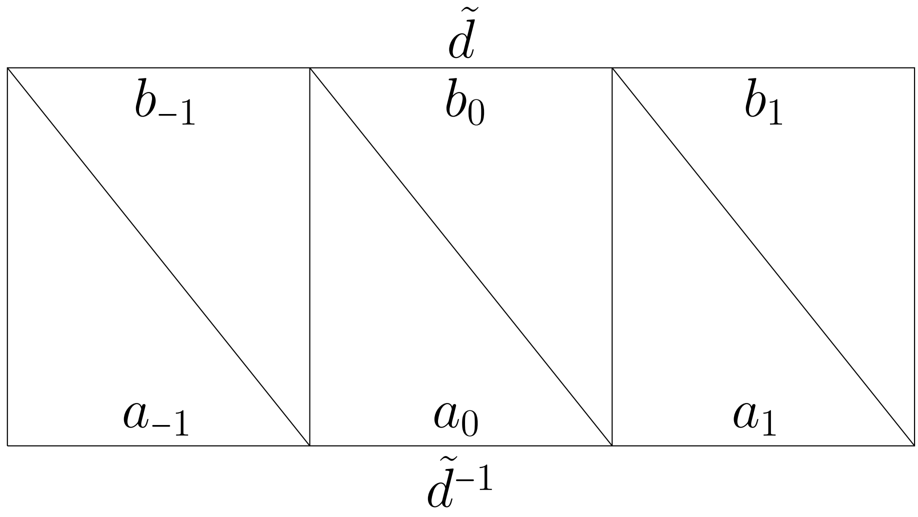

Figure 5.5: The surface of P for $\tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$.

to:

$$\begin{aligned}\tilde{g}_1^{-1}\tilde{g}_2^{-1} &= \left(\tilde{r}_v^4 c^{\frac{-2(\lambda k+2)}{3}}\right)^{-1}\left(\tilde{d}_2^{2\lambda}\right)^{-1} \\ &= \tilde{r}_v^{-4}\tilde{d}^{-2\lambda p} c^{\frac{2(\lambda k+2)}{3}} = \tilde{r}_v^2 \tilde{d}^{\lambda p} c^{\frac{-(\lambda k+2)}{3}} = a_1.\end{aligned}$$

Here we used $\tilde{r}_v^3 = c$ and $\tilde{d}^{3p} = c^k$. Hence

$$\tilde{E}_{\tilde{e}} \mapsto \tilde{E}_{a_1}.$$

Now, let us look at the image of $\tilde{E}_{b_0}$:

$$\begin{aligned}b_0 &\mapsto \tilde{g}_1^{-1} b_0 \tilde{g}_2^{-1} \\ &= \left(\tilde{r}_v^{-4} c^{\frac{2(\lambda k+2)}{3}}\right)\left(\tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}}\right)\left(\tilde{d}^{-2\lambda p}\right) = \tilde{e},\end{aligned}$$

hence

$$\tilde{E}_{b_0} \mapsto \tilde{E}_{\tilde{e}}.$$

So,

$$\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1},$$

$$\textit{i.e. the face } b_0 \textit{ is glued to the face } a_1.$$

Note that b_0 and a_1 have the following properties: $b_0 = \tilde{g}_1\tilde{g}_2, a_1 = \tilde{g}_1^{-1}\tilde{g}_2^{-1}$, $\tilde{g}_1^3 = \left(\tilde{r}_v^4 c^{\frac{-2(\lambda k+2)}{3}}\right)^3 = \tilde{r}_v^{12} c^{-2(\lambda k+2)} = c^4 c^{-2\lambda k-4} = c^{-2\lambda k}$, $\tilde{g}_2^3 = (\tilde{d}_2^{2\lambda})^3 = \tilde{d}^{6\lambda p} = (\tilde{d}^{3p})^{2\lambda} = c^{2\lambda k}$. We will see later that the edge $(b_0; a_1)$ is in an edge cycle of length 1 and the properties of b_0 and a_0 discussed above will mean that the conditions of Theorem 49 are satisfied.

Now, we still need to look at the edges of the face $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}}$:

(a) $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{\tilde{d}}$:

We have

$$\begin{aligned} \tilde{d} &\mapsto \tilde{g}_1^{-1}\tilde{d}\tilde{g}_2^{-1} \\ &= \left(\tilde{r}_v^{-4} c^{\frac{2(\lambda k+2)}{3}}\right)\tilde{d}\left(\tilde{d}^{-2\lambda p}\right) \\ &= \tilde{r}_v^2 \tilde{d}^{\lambda p+1} c^{\frac{-(\lambda k+2)}{3}} = b_1. \end{aligned}$$

So,

$$\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{\tilde{d}} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1} \cap \tilde{E}_{b_1}.$$

Hence,

The edge $(b_0; \tilde{d})$ is glued to the edge $(a_1; b_1)$.

(b) $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_0}$:

We have

$$\begin{aligned} a_0 &\mapsto \tilde{g}_1^{-1} a_0 \tilde{g}_2^{-1} \\ &= \left(\tilde{r}_v^{-4} c^{\frac{2(\lambda k+2)}{3}}\right)\left(\tilde{r}_v^4 \tilde{d}^{2\lambda p-1} c^{\frac{-2(\lambda k+2)}{3}}\right)\left(\tilde{d}^{-2\lambda p}\right) \\ &= \tilde{d}^{-1}. \end{aligned}$$

So,

$$\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_0} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1} \cap \tilde{E}_{\tilde{d}^{-1}}$$

Hence,

The edge $(b_0; a_0)$ is glued to the edge $(a_1; \tilde{d}^{-1})$.

(c) $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1}$:
We have

$$\begin{aligned} a_1 &\mapsto \tilde{g}_1^{-1} a_1 \tilde{g}_2^{-1} \\ &= \left(\tilde{r}_v^{-4} c^{\frac{2(\lambda k+2)}{3}}\right)\left(\tilde{r}_v^{2} \tilde{d}^{\lambda p} c^{\frac{-(\lambda k+2)}{3}}\right)\left(\tilde{d}^{-2\lambda p}\right) \\ &= \tilde{r}_v^{4} \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}} = b_0. \end{aligned}$$

So,

$$\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1} \cap \tilde{E}_{b_0}$$

Hence,

The edge $(b_0; a_1)$ *is glued with to the edge* $(a_1; b_0)$.

2. The face $\tilde{E}_{a_0} \cap \tilde{E}_{\tilde{e}}$:

We have shown that the edge gluings of the face $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}}$ are the following:

(a) *The edge* $(b_0; \tilde{d})$ *is glued with to the edge* $(a_1; b_1)$,

(b) *The edge* $(b_0; a_0)$ *is glued with to the edge* $(a_1; \tilde{d}^{-1})$,

(c) *The edge* $(b_0; a_1)$ *is glued with to the edge* $(a_1; b_0)$.

Now, by changing the directions and shifting m by 1 (i.e. applying the symmetrie $\tilde{\rho}$) we obtain the edge gluings of the face $\tilde{E}_{a_0} \cap \tilde{E}_{\tilde{e}}$:

(a) *The edge* $(a_0; b_0)$ *is glued with to the edge* $(b_{-1}; \tilde{d})$,

(b) *The edge* $(a_0; \tilde{d}^{-1})$ *is glued with to the edge* $(b_{-1}; a_{-1})$,

(c) *The edge* $(a_0; b_{-1})$ *is glued with to the edge* $(b_{-1}; a_0)$.

3. The face $\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}}$:

Now, we still need to look at the top face $\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}}$. We need to write $\tilde{d}$ as a product $\tilde{g}_1 \cdot \tilde{g}_2$ with $\tilde{g}_1 \in \tilde{\Gamma}_1$ and $\tilde{g}_2 \in \tilde{\Gamma}_2$:
We know that $\tilde{d}^{3h+np} = \tilde{d}_1^h \tilde{d}_2^n$ and what we need to find is $\tilde{d}$. Therefore, we need to solve the equation $1 = 3h + np$. If $\lambda = 1 \mod 3$, we have $\tilde{d} = \tilde{d}_1^{\frac{-(k+1)-1}{3}} \tilde{d}_2$. If $\lambda = 2 \mod 3$, we have $\tilde{d} = \tilde{d}_1^{\frac{k+4}{3}} \tilde{d}_2^{-1}$. We know that $\tilde{d}_2^3 = c^k, \tilde{d}_1^p = c^k$, therefore $\tilde{d}_2^{-1} = c^{-k}\tilde{d}_2^2 = \tilde{d}_1^{-p}\tilde{d}_2^2 = \tilde{d}_1^{-k-3}\tilde{d}_2^2$. Hence, $\tilde{d} = \tilde{d}_1^{\frac{-2(k+2)-1}{3}} \tilde{d}_2^2$. Thus,

$$\tilde{d} = (\tilde{d}_1)^{\frac{-\lambda(k+\lambda)-1}{3}} (\tilde{d}_2)^{\lambda} = \tilde{g}_1 \cdot \tilde{g}_2.$$

We consider the action of $(\tilde{g}_1^{-1}, \tilde{g}_2) \in \tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$ by $x \mapsto \tilde{g}_1^{-1} x \tilde{g}_2^{-1}$. Therefore, in order to obtain the face which is glued with the top face $\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}}$, we do the following:

$$\tilde{d} \mapsto \tilde{d}_1^{\frac{\lambda(k+\lambda)+1}{3}} \tilde{d}\tilde{d}_2^{-\lambda} = \tilde{d}^{\lambda(k+\lambda)+1}\tilde{d}\tilde{d}^{-\lambda p} = \tilde{d}^{\lambda^2-3\lambda+2} = \tilde{d}^{(\lambda-1)(\lambda-2)} = \tilde{d}^0 = \tilde{e}.$$

Here we obtained that $\tilde{d}^{(\lambda-1)(\lambda-2)} = \tilde{d}^0$ since $\lambda \in \{1,2\}$.

So, $\tilde{E}_{\tilde{d}} \mapsto \tilde{E}_{\tilde{e}}$. We still need to look at where $\tilde{E}_{\tilde{e}}$ is mapped to.

$$\tilde{e} \mapsto \tilde{d}_1^{\frac{\lambda(k+\lambda)+1}{3}} \tilde{e}\tilde{d}_2^{-\lambda} = \tilde{d}^{\lambda(k+\lambda)+1}\tilde{d}^{-\lambda p} = \tilde{d}^{\lambda^2-3\lambda+1} = \tilde{d}^{-1}.$$

Here we obtained that $\tilde{d}^{\lambda^2-3\lambda+1} = \tilde{d}^{-1}$ since $\lambda \in \{1,2\}$.

So $\tilde{E}_{\tilde{e}} \mapsto \tilde{E}_{\tilde{d}^{-1}}$ and

$$\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}} \mapsto \tilde{E}_{\tilde{d}^{-1}} \cap \tilde{E}_{\tilde{e}}.$$

Hence,

The face in $\tilde{E}_{\tilde{d}}$ is glued to the face in $\tilde{E}_{\tilde{d}^{-1}}$.

We are going to looked at the edges of the top face $\tilde{E}_{\tilde{d}}$:

(a) $\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_0}$:

Let us look at $\tilde{E}_{b_0}$:

$$\begin{aligned} b_0 &\mapsto \tilde{d}_1^{\frac{\lambda(k+\lambda)+1}{3}} b_0 \tilde{d}_2^{-\lambda} = \tilde{d}^{\lambda(k+\lambda)+1}\left(\tilde{r}_v \tilde{d}^{2\lambda p} c^{\frac{-(2\lambda k+1)}{3}}\right)\tilde{d}^{-\lambda p} \\ &= \tilde{d}^{\lambda(k+\lambda)+1}\tilde{r}_v \tilde{d}^{\lambda p} c^{\frac{-(2\lambda k+1)}{3}}. \end{aligned}$$

On the other hand,

$$a_2 = \tilde{\rho}^2(a_0) = \tilde{r}_u a_0 \tilde{r}_u^{-1} = \tilde{r}_u\left(\tilde{r}_v \tilde{d}^{2\lambda p-1} c^{\frac{-(2\lambda k+1)}{3}}\right)\tilde{r}_u^{-1}.$$

Using $\tilde{r}_u = \tilde{d}^{\lambda p-1} c^{\frac{(1-\lambda k)}{3}}$ and $\tilde{r}_u^{-1} = c^{\frac{(\lambda k-1)}{3}} \tilde{d}^{1-\lambda p}$, we can see that

$$\begin{aligned} a_2 &= \left(\tilde{d}^{\lambda p-1} c^{\frac{1-\lambda k}{3}}\right)\left(\tilde{r}_v \tilde{d}^{2\lambda p-1} c^{\frac{-(2\lambda k+1)}{3}}\right)\left(\tilde{d}^{\lambda p-1} c^{\frac{1-\lambda k}{3}}\right)^{-1} \\ &= \tilde{d}^{\lambda p-1}\tilde{r}_v \tilde{d}^{2\lambda p-1}\tilde{d}^{1-\lambda p} c^{\frac{1-\lambda k}{3}} c^{\frac{-(2\lambda k+1)}{3}} c^{\frac{-1+\lambda k}{3}} = \tilde{d}^{\lambda p-1}\tilde{r}_v \tilde{d}^{\lambda p} c^{\frac{-(2\lambda k+1)}{3}}. \end{aligned}$$

Hence

$$\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_0} \mapsto \tilde{E}_{\tilde{d}^{-1}} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_2},$$

i.e. the edge $(\tilde{d}; b_0)$ *is glued to the edge* $(\tilde{d}^{-1}; a_2)$.

Shifting m (i.e. applying $\tilde{\rho}$) we obtain:

The edge $(\tilde{d}; b_m)$ *is glued to the edge*$(\tilde{d}^{-1}; a_{m+2})$.

4. The face $\tilde{E}_{\tilde{d}^{-1}} \cap \tilde{E}_{\tilde{e}}$:

We obtained that the gluing of the edges of the face $\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}}$ is as follows:

The edge $(\tilde{d}; b_m)$ *is glued to the edge* $(\tilde{d}^{-1}; a_{m+2})$.

Now, by changing the direction and shifting m by 2 we will obtain the gluings of the edges of the bottom face $\tilde{E}_{\tilde{d}^{-1}} \cap \tilde{E}_{\tilde{e}}$:

The edge $(\tilde{d}^{-1}; a_m)$ *is glued to the edge* $(\tilde{d}; b_{m-2})$.

Now, we are going to find the edge cycles for P. We have determined the identifications of the faces and edges of P:

$$\begin{aligned} (\tilde{d}; b_0) &\mapsto (\tilde{d}^{-1}; a_2), \\ (b_0; \tilde{d}, a_0, a_1) &\mapsto (a_1; b_1, \tilde{d}^{-1}, b_0), \\ (a_0; \tilde{d}^{-1}, b_0, b_{-1}) &\mapsto (b_{-1}; a_{-1}, \tilde{d}, a_0). \end{aligned}$$

So, under the symmetry $\tilde{\rho}$ we have:

$$\tilde{\rho}(a_m) = a_{m+1}, \quad \tilde{\rho}(b_m) = b_{m+1}, \quad \tilde{\rho}(\tilde{d}) = \tilde{d}.$$

We obtain the following face and edge identifications:

$$\begin{aligned} (\tilde{d}; b_m) &\mapsto (\tilde{d}^{-1}; a_{m+2}), \\ (b_m; \tilde{d}, a_m, a_{m+1}) &\mapsto (a_{m+1}; b_{m+1}, \tilde{d}^{-1}, b_m), \\ (a_m; \tilde{d}^{-1}, b_m, b_{m-1}) &\mapsto (b_{m-1}; a_{m-1}, \tilde{d}, a_m). \end{aligned}$$

We work out the edge cycles using the identifications and the combinatorics of the faces:

$$(b_m; \tilde{d}) \mapsto (b_{m+1}; a_{m+1}) \mapsto (\tilde{d}^{-1}; a_{m+2}),$$

$$(b_m; a_{m+1}) \circlearrowleft,$$

see figure 5.6.

∎

Now, we can apply Theorem 49 and we obtain the following result:

Theorem 75. *We have*

$$F_{\tilde{e}} = P.$$

Proof. Theorem 49 implies that $\Psi(P)$ is a fundamental domain for $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$. Moreover, we will show that the fundamental domains $\Psi(P)$ and $F_{\tilde{e}}$ coincide. Proposition 48 implies that $F_{\tilde{\Gamma}_1(u) \setminus \{u\}} = F_{\mathcal{E}} = F_{\tilde{e}}$. We know that $P = \mathrm{Cl}\,\mathrm{Int}(P)$ and $F_{\tilde{e}} = \mathrm{Cl}\,\mathrm{Int}(F_{\mathcal{E}})$. We also know that $P \subset F_{\mathcal{E}}$. This all implies that $P \subset F_{\tilde{e}}$ and hence

$$F_{\tilde{e}} = P$$

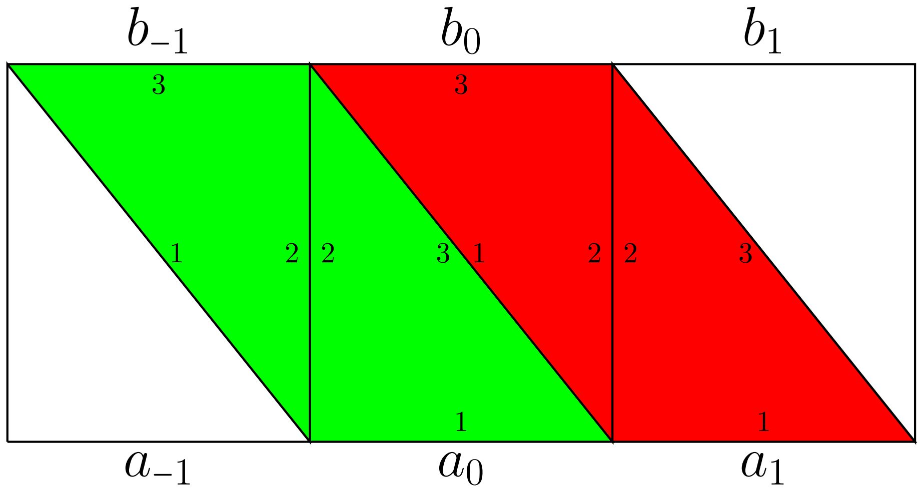

Figure 5.6: Identification scheme for $\tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$.

Figures of Fundamental Domains for $\tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$

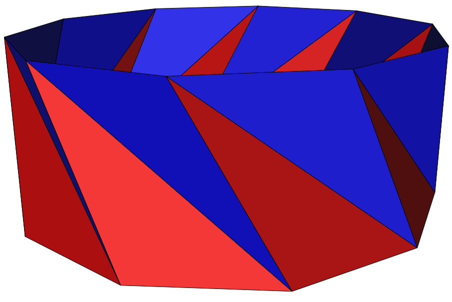

Figure 5.7: Fundamental Domain for $\tilde{\Gamma}(5,3,3)^2 \times (C_3)^2$.

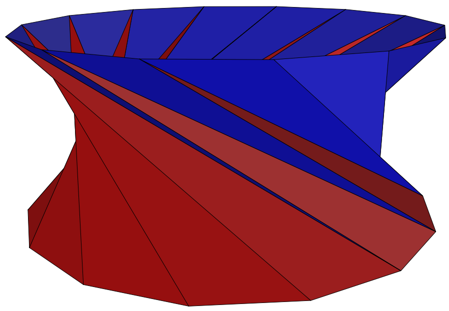

Figure 5.8: Fundamental Domain for $\tilde{\Gamma}(7,3,3)^4 \times (C_3)^4$.

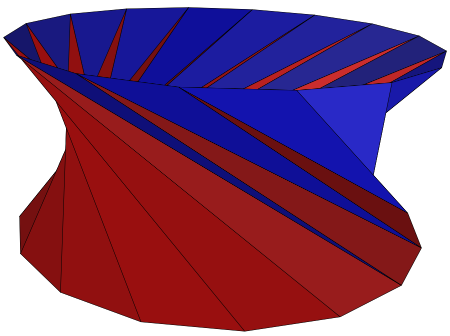

Figure 5.9: Fundamental Domain for $\tilde{\Gamma}(8,3,3)^5 \times (C_3)^5$.

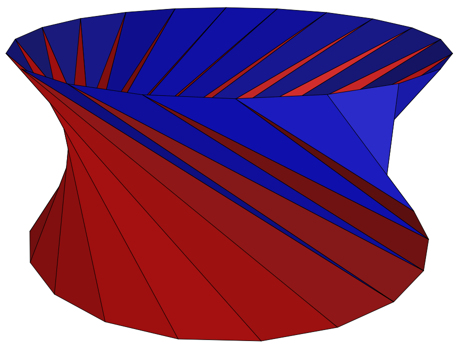

Figure 5.10: Fundamental Domain for $\tilde{\Gamma}(10,3,3)^7 \times (C_3)^7$.

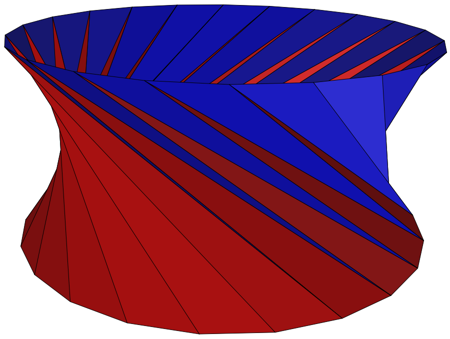

Figure 5.11: Fundamental Domain for $\tilde{\Gamma}(11,3,3)^8 \times (C_3)^8$.

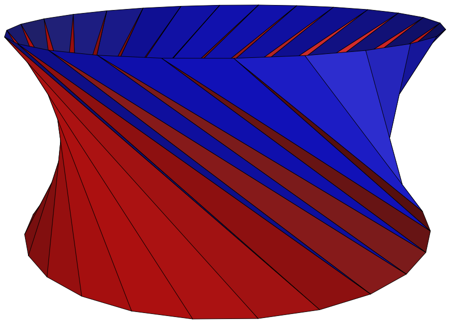

Figure 5.12: Fundamental Domain for $\tilde{\Gamma}(13,3,3)^{10} \times (C_3)^{10}$.

5.8 Fundamental domains for the case $\tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$

In this section we are going to compute fundamental domains for the case $\tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$ explicitly. Recall that according to Proposition 20 the fundamental domain $F_{\tilde{e}}$ can be described as

$$F_{\tilde{e}} = \text{CI Int}\left(F_{\tilde{\Gamma}_1(u)\setminus\{u\}}\right),$$

where $F_N = \tilde{S}_{\tilde{e}} \cap \bigcap_{x\in N} R_x = \tilde{E}_{\tilde{e}} \cap \partial Q_u \cap \bigcap_{x\in N} R_x$ for $N \subset \tilde{\Gamma}_1(u) \setminus \{u\}$.

First we will construct a polyhedron $P \subset \tilde{S}_{\tilde{e}}$ such that $P \subset F_{\mathcal{E}}$, where $\mathcal{E}$ is the edge crown of $\tilde{\Gamma}_1$. Proposition 48 shows that $F_{\tilde{\Gamma}_1(u)\setminus\{u\}} = F_{\mathcal{E}} = F_{\tilde{e}}$. Hence, we have $P \subset F_{\tilde{e}}$. Therefore, if we show that P is a fundamental domain then we obtain $\text{Int}(P) \subseteq \text{Int}(F_{\tilde{e}})$ and they are both fundamental domains for $\tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$. Thus, we obtain $P = F_{\tilde{\Gamma}_1(u)\setminus\{u\}} = F_{\tilde{e}}$ since if a fundamental domain is inside another fundamental domain then their interior is exactly the same. Then $P = \text{Cl Int}(P) = \text{Cl Int}(F_{\tilde{e}}) = F_{\tilde{e}}$.
To be more clear, we describe the image $\pi(P)$ in S_e and construct a polyhedron $\hat{P}$ in $\hat{S}_e$ such that $\pi(P) = \hat{P} \cap L$. After that, we study the polyhedron $\hat{P}$ to determine the combinatorial structure of its faces and we show the picture of some faces of $\hat{P}$ as in figure 5.13. Then, we obtain $\pi(P) = \hat{P} \cap L = \hat{P}$

since $\hat{P} \subset L$. Now, we need to apply Theorem 49 to $\Psi(P)$ as we described in subsection 4.2.1. After checking all the conditions of Theorem 49 we obtain that $\Psi(P)$ is a fundamental domain for the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$. Finally, as we mentioned in the beginning of this section, we have from Proposition 48 that $\Psi(P) \subset \Psi(F_{\tilde{\Gamma}_1(u)\setminus\{u\}}) = \Psi(F_{\tilde{e}})$ and they are both fundamental domains of the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$. Hence $P = F_{\tilde{\Gamma}_1(u)\setminus\{u\}} = F_{\tilde{e}}$ (up to the boundary points).

Recall: Lemma 67 shows that the elements with small argument are:

$$f_{0,-2}, f_{0,-2}\tilde{d}, f_{0,-2}\tilde{d}^2, f_{0,-2}\tilde{d}^3, f_{0,-2}\tilde{d}^4, f_{0,-2}\tilde{d}^5, f_{0,-2}\tilde{d}^6,$$

where $f_{0,-2} = \tilde{r}_v \tilde{r}_u^2 c^{-1}$.
Using Corollary 53 we obtain

$$\begin{aligned} f_{0,-2} &= \tilde{r}_v \tilde{r}_u^2 c^{-1} \\ &= \tilde{r}_v (c^{\frac{1-\lambda k}{3}} \tilde{d}^{\lambda p-2})^2 c^{-1} = \tilde{r}_v \tilde{d}^{2(\lambda p-2)} c^{\frac{-(2\lambda k+1)}{3}}, \end{aligned}$$

where $\lambda = 1$ if $k \equiv 1 \mod 3$ and $\lambda = 2$ if $k \equiv 2 \mod 3$.
We know that $\tilde{r}_v^3 = c$. So, we can write $f_{0,-2}$ as

$$f_{0,-2} = \tilde{r}_v^4 \tilde{d}^{2(\lambda p-2)} c^{\frac{-2(\lambda k+2)}{3}}.$$

Let

$$\begin{aligned} a_0 = f_{0,-2}\tilde{d}^2 &= \tilde{r}_v^4 \tilde{d}^{2(\lambda p-2)} \tilde{d}^2 c^{\frac{-2(\lambda k+2)}{3}} = \tilde{r}_v^4 \tilde{d}^{2\lambda p-2} c^{\frac{-2(\lambda k+2)}{3}}, \\ b_0 = f_{0,-2}\tilde{d}^3 &= \tilde{r}_v^4 \tilde{d}^{2(\lambda p-2)} \tilde{d}^3 c^{\frac{-2(\lambda k+2)}{3}} = \tilde{r}_v^4 \tilde{d}^{2\lambda p-1} c^{\frac{-2(\lambda k+2)}{3}}, \\ c_0 = f_{0,-2}\tilde{d}^4 &= \tilde{r}_v^4 \tilde{d}^{2(\lambda p-2)} \tilde{d}^4 c^{\frac{-2(\lambda k+2)}{3}} = \tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}}. \end{aligned}$$

Using the symmetries in Section 5.2 we define $a_m = \tilde{\rho}^m(a_0), b_m = \tilde{\rho}^m(b_0)$

and $c_m = \tilde{\rho}^m(c_0)$. Using Proposition 62 we obtain

$$\begin{aligned} c_{-1} &= \tilde{\rho}^{-1}(c_0) = \tilde{\rho}^{-1}\left(\tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}}\right) \\ &= \tilde{d}^{2-\lambda p}\tilde{r}_v^2 \tilde{d}^{2\lambda p} c^{\frac{-\lambda k-2}{3}} = \tilde{d}^{2-\lambda p}\tilde{d}^{3\lambda p} c^{-\lambda k}\tilde{r}_v^2 \tilde{d}^{2\lambda p} c^{\frac{-\lambda k-2}{3}} \\ &= \tilde{d}^{2\lambda p+2}\tilde{r}_v^2 \tilde{d}^{2\lambda p} c^{\frac{-4\lambda k-2}{3}} \end{aligned}$$

since we know that $\tilde{d}^{3\lambda p} = c^{\lambda k}$. Now, we are going to look at the projections of these elements. Let $A_m = \pi(a_m), B_m = \pi(b_m)$ and $C_m = \pi(c_m)$. Using Corollary 56 we obtain

$$\begin{aligned} A_0 &= \pi(a_0) = r_v^4 d^{2\lambda p-2}, \\ B_0 &= \pi(b_0) = r_v^4 d^{2\lambda p-1}, \\ C_0 &= \pi(c_0) = r_v^4 d^{2\lambda p}, \\ C_{-1} &= \pi(c_{-1}) = d^{2\lambda p+2} r_v^2 d^{2\lambda p}. \end{aligned}$$

Notation: From now on for the ease of notation we will write $\tilde{E}a_m$ instead of $\tilde{E}_{a_m}$, EA_m instead of E_{A_m} and similar for b_m, B_m, c_m and C_m.

Definition 76. *Let* $\omega^{\pm} = \pm\tan(\frac{\pi k}{6p})$. *Let*

$$P = \bigcap_{m\in\mathbb{Z}} (\tilde{I}A_m \cup \tilde{I}B_m \cup \tilde{I}C_m)[\omega^-, \omega^+].$$

Note that $(\tilde{I}A_m \cup \tilde{I}B_m \cup \tilde{I}C_m)$ *is contained in the complement of* Q_{x_m}, *hence* P *is contained in the complement of* $\bigcup_{m\in\mathbb{Z}} Q_{x_m} = \bigcup_{x\in\mathcal{E}} Q_x$. *Therefore,* $P \subset F_{\mathcal{E}}$. *Proposition 65 and Lemma 67 imply that*

$$\pi(P) = \bigcap_{m\in\mathbb{Z}} (IA_m \cup IB_m \cup IC_m)[\omega^-, \omega^+]$$

and that $\pi|_P : P \mapsto \pi(P)$ *is a homeomorphism. Let*

$$\hat{P} = \bigcap_{m\in\mathbb{Z}} (\hat{I}A_m \cup \hat{I}B_m \cup \hat{I}C_m)[\omega^-, \omega^+].$$

We will use some of the net estimate results to show that $\hat{P} = \pi(P)$. We shall study the combinatorial structure of the boundary of $\hat{P}$ as illustrated in figure 5.13. To do this we are going to use the analytic geometry in the appendix 8.6.

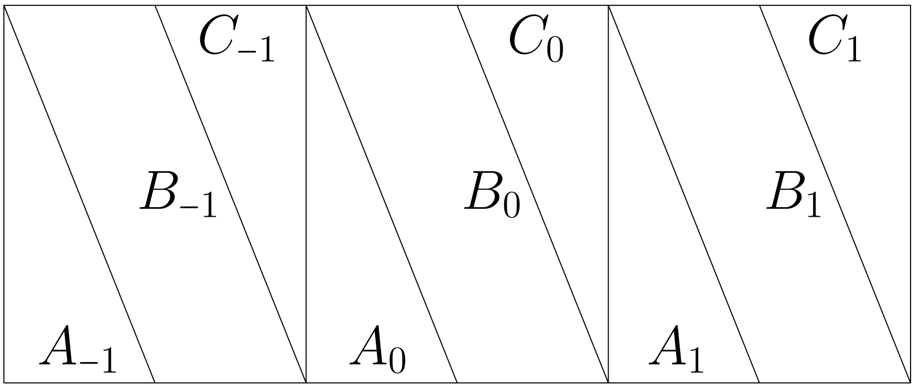

Figure 5.13: The surface of $\hat{P}$ for $\tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$.

Proposition 77. *We have*

- $z \in HA_0(\omega)$ *if and only if*

$$\sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2$$
$$\geq \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}+\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).$$

- $z \in HB_0(\omega)$ *if and only if*

$$\sin\left(\frac{\pi}{2p}+\frac{\pi}{2}\right)z_1+\cos\left(\frac{\pi}{2p}+\frac{\pi}{2}\right)z_2 \quad \geq \quad \frac{1}{B(4)}\left(1-\frac{1}{2\sin\frac{\pi}{2p}}\right).$$

- $z \in HC_0(\omega)$ *if and only if*

$$\sin(\frac{2\pi}{3})z_1+\cos(\frac{2\pi}{3})z_2 \quad \geq \quad \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}-\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).$$

- $z \in HC_{-1}(\omega)$ *if and only if*

$$-\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_1+\cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_2$$
$$\le \frac{1}{B(2)}\left(1+\frac{\sin(\frac{5\pi}{3}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}-\frac{\cos(\frac{5\pi}{3}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).$$

Proof.

We will use Proposition 68.

- *The case HA_0*: We have $A_0 = r_v^4 d^{2\lambda p-2}$, so from the definition of σ and α we obtain that

$$\sigma_1=\alpha_1 = \frac{(2\lambda p-2)k\pi}{3p}=\frac{(2\lambda p-2)(p-3)\pi}{6p}=\frac{2\lambda p\pi}{6}+\frac{(-6\lambda-2)\pi}{6}+\frac{\pi}{p}.$$

If $k=1 \mod 3$ then $\lambda=1$ and $p=2k+3\equiv 5 \mod 6$, hence $p=6x+5$ for some $x\in\mathbb{Z}$. Therefore,

$$\frac{2\lambda p\pi}{6}-\frac{(6\lambda+2)\pi}{6}+\frac{\pi}{p}=\frac{2}{6}(6x+5)\pi-\frac{8\pi}{6}+\frac{\pi}{p}=2x\pi+\frac{2\pi}{6}+\frac{\pi}{p}\equiv\frac{\pi}{p}+\frac{\pi}{3} \mod 2\pi.$$

If $k=2 \mod 3$ then $\lambda=2$ and $p=2k+3\equiv 7 \mod 6$, hence $p=6x+7$ for some $x\in\mathbb{Z}$. Therefore,

$$\frac{2\lambda p\pi}{6}+\frac{(-6\lambda-2)\pi}{6}+\frac{\pi}{p}=\frac{4}{6}(6x+7)\pi-\frac{14\pi}{6}+\frac{\pi}{p}=4x\pi+\frac{7\pi}{3}+\frac{\pi}{p}\equiv\frac{\pi}{p}+\frac{\pi}{3} \mod 2\pi.$$

Note that $\frac{\pi}{2p}-\alpha_1=-\frac{\pi}{2}+(\frac{\pi}{6}-\frac{\pi}{2p}) \mod 2\pi$. Hence for the case HA_0 we obtain the following inequality:

$$z\in HA_0(\omega) \iff \sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2$$
$$\ge\frac{1}{B(4)}\left(1+\frac{\sin(-\frac{\pi}{2}+(\frac{\pi}{6}-\frac{\pi}{2p}))}{2\sin\frac{\pi}{2p}}+\frac{\cos(-\frac{\pi}{2}+(\frac{\pi}{6}-\frac{\pi}{2p}))}{2\sin\frac{\pi}{2p}}\omega\right).$$

We can write this condition as

$$z \in HA_0(\omega) \iff \sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2$$
$$\geq \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}+\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).$$

- *The case* HB_0: We have $B_0 = r_v^4 d^{2\lambda p-1}$, so from the definition of σ and α we obtain that

$$\sigma_2 = \alpha_2 = \frac{(2\lambda p-1)k\pi}{3p} = \alpha_1+\frac{k\pi}{3p} \equiv \frac{\pi}{p}+\frac{\pi}{3}+\frac{\pi(p-3)}{6p} \equiv \frac{\pi}{2p}+\frac{\pi}{2} \mod 2\pi.$$

Hence for the case HB_0 we obtain the following inequality: $z \in HB_0(\omega)$ if and only if

$$\sin(\frac{\pi}{2p}+\frac{\pi}{2})z_1+\cos(\frac{\pi}{2p}+\frac{\pi}{2})z_2 \geq \frac{1}{B(4)}\left(1-\frac{\sin\frac{\pi}{2}}{2\sin\frac{\pi}{2p}}+\frac{\cos\frac{\pi}{2}}{2\sin\frac{\pi}{2p}}\omega\right).$$

Moreover, we obtain $z \in HB_0(\omega)$ if and only if

$$\sin\left(\frac{\pi}{2p}+\frac{\pi}{2}\right)z_1+\cos\left(\frac{\pi}{2p}+\frac{\pi}{2}\right)z_2 \geq \frac{1}{B(4)}\left(1-\frac{1}{2\sin\frac{\pi}{2p}}\right).$$

- *The case* HC_0: We have $C_0 = r_v^4 d^{2\lambda p}$, so from the definition of σ and α we obtain that

$$\sigma_3 = \alpha_3 = \frac{(2\lambda p)k\pi}{3p} = \alpha_2+\frac{k\pi}{3p} = \frac{\pi}{2p}+\frac{\pi}{2}+\frac{\pi(p-3)}{6p} \equiv \frac{2\pi}{3} \mod 2\pi.$$

Hence for the case HC_0 we obtain the following inequality:

$$z \in HC_0(\omega) \iff \sin(\frac{2\pi}{3})z_1+\cos(\frac{2\pi}{3})z_2$$
$$\geq \frac{1}{B(4)}\left(1+\frac{\sin(-\frac{\pi}{2}-(\frac{\pi}{6}-\frac{\pi}{2p}))}{2\sin\frac{\pi}{2p}}+\frac{\cos(-\frac{\pi}{2}-(\frac{\pi}{6}-\frac{\pi}{2p}))}{2\sin\frac{\pi}{2p}}\omega\right).$$

Moreover, we obtain $z \in HC_0(\omega)$ if and only if

$$\sin(\frac{2\pi}{3})z_1 + \cos(\frac{2\pi}{3})z_2 \quad \geq \quad \frac{1}{B(4)}\left(1 - \frac{\cos(\frac{\pi}{6} - \frac{\pi}{2p})}{2\sin\frac{\pi}{2p}} - \frac{\sin(\frac{\pi}{6} - \frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).$$

- *The case HC_{-1}*: We have $C_{-1} = d^{2\lambda p+2}r_v^2 d^{2\lambda p}$, so from the definition of σ and α we obtain that

$$\sigma_4 = \frac{-2k\pi}{3p}, \quad \alpha_4 = \frac{(4\lambda p + 2)k\pi}{3p} = \frac{4\lambda p\pi}{6} + \frac{(-12\lambda + 2)\pi}{6} - \frac{\pi}{p}.$$

If $k = 1 \mod 3$ then $\lambda = 1$ and $p = 2k + 3 \equiv 5 \mod 6$, hence $p = 6x + 5$ for some $x \in \mathbb{Z}$. Therefore,

$$\sigma_4 = -\left(\frac{\pi}{3} - \frac{\pi}{p}\right).$$

$$\alpha_4 = \frac{4\pi}{6}(6x + 5) + \frac{-10\pi}{6} - \frac{\pi}{p} = 4x\pi + \frac{10\pi}{6} - \frac{\pi}{p} \equiv \frac{5\pi}{3} - \frac{\pi}{p} \mod 2\pi.$$

If $k = 2 \mod 3$ then $\lambda = 2$ and $p = 2k + 3 \equiv 7 \mod 6$, hence $p = 6x + 7$ for some $x \in \mathbb{Z}$. Therefore,

$$\sigma_4 = -\left(\frac{\pi}{3} - \frac{\pi}{p}\right).$$

$$\alpha_4 = \frac{8\pi}{6}(6x + 7) + \frac{-22\pi}{6} - \frac{\pi}{p} = 8x\pi + \frac{34\pi}{6} - \frac{\pi}{p} \equiv \frac{5\pi}{3} - \frac{\pi}{p} \mod 2\pi.$$

Hence for the case HC_{-1} we obtain the following equation:

$$z \in HC_{-1}(\omega) \iff -\sin\left(\frac{\pi}{3} - \frac{\pi}{p}\right)z_1 + \cos\left(\frac{\pi}{3} - \frac{\pi}{p}\right)z_2$$
$$\leq \frac{1}{B(2)}\left(1 + \frac{\sin(\frac{5\pi}{3} - \frac{\pi}{2p})}{2\sin\frac{\pi}{2p}} - \frac{\cos(\frac{5\pi}{3} - \frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).$$

∎

Proposition 78. *We have $0 \notin (HA_0 \cap HC_0)(\omega)$.*

Proof. We have

- $z \in HA_0(\omega)$ if and only if

$$\begin{aligned}&\sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2\\ &\geq \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}+\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right),\end{aligned}$$

- $z \in HC_0(\omega)$ if and only if

$$\sin(\frac{2\pi}{3})z_1+\cos(\frac{2\pi}{3})z_2 \quad\geq\quad \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}-\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).$$

Thus $0 \in (HA_0 \cap HC_0)(\omega)$ would imply that

$$\frac{1}{B(4)}\left(1-\frac{1}{2\sin\frac{\pi}{2p}}\left(\cos(\frac{\pi}{6}-\frac{\pi}{2p})\pm\sin(\frac{\pi}{6}-\frac{\pi}{2p})\omega\right)\right)\leq 0.$$

We know that $B(4)<0$, hence $1-\frac{1}{2\sin\frac{\pi}{2p}}\left(\cos(\frac{\pi}{6}-\frac{\pi}{2p})\pm\sin(\frac{\pi}{6}-\frac{\pi}{2p})\omega\right)\geq 0$ and therefore $\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\leq 1$. However, we have $\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}>\frac{\cos(\frac{\pi}{6})}{2\sin\frac{\pi}{10}}>1$ since $p\geq 5$. Hence $0 \notin (HA_0 \cap HC_0)(\omega)$. ∎

Proposition 79. *The bisector of the sector $(HA_0 \cap HC_0)(0)$ contains the origin.*

Proof. We apply Proposition 92 (in the appendix) for $H^- = HA_0(0)$ and $H^+ = HC_0(0)$. The inequalities of the half planes are

$$\begin{aligned}z\in HA_0(\omega) &\iff \sin\left(\tfrac{\pi}{p}+\tfrac{\pi}{3}\right)z_1+\cos\left(\tfrac{\pi}{p}+\tfrac{\pi}{3}\right)z_2\\ &\qquad\geq \tfrac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right).\\ z\in HC_0(\omega) &\iff \sin(\tfrac{2\pi}{3})z_1+\cos(\tfrac{2\pi}{3})z_2\geq\tfrac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right).\end{aligned}$$

∎

Proposition 80. *The lines* $EA_0\left(\tan\left(\frac{\pi}{6}-\frac{\pi}{2p}\right)\right)$, $EC_0\left(\tan\left(\frac{\pi}{6}-\frac{\pi}{2p}\right)\right)$ *and* EC_{-1} $\left(\tan\left(\frac{\pi}{6}-\frac{\pi}{2p}\right)\right)$ *meet at one point. Note that* $\tan\left(\frac{\pi}{6}-\frac{\pi}{2p}\right) > \tan\left(\frac{\pi}{12}-\frac{\pi}{4p}\right) = \omega^+$.

Proof. The equations of the lines are

- $z \in EA_0(\omega)$ if and only if

$$\begin{aligned}&\sin(\frac{\pi}{p}+\frac{\pi}{3})z_1+\cos(\frac{\pi}{p}+\frac{\pi}{3})z_2\\&=\frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}+\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right),\end{aligned}$$

- $z \in EC_0(\omega)$ if and only if

$$\sin(\frac{2\pi}{3})z_1+\cos(\frac{2\pi}{3})z_2 = \frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}-\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right),$$

- $z \in EC_{-1}(\omega)$ if and only if

$$\begin{aligned}&-\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_1+\cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_2\\&=\frac{1}{B(2)}\left(1+\frac{\sin(\frac{5\pi}{3}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}-\frac{\cos(\frac{5\pi}{3}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).\end{aligned}$$

For the equation of the line EA_0 we have the right hand side

$$\begin{aligned}&\frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}+\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right)\\
=\;&\frac{1}{B(4)}\left(\frac{2\sin\frac{\pi}{2p}-\cos\frac{\pi}{6}\cos\frac{\pi}{2p}-\sin\frac{\pi}{6}\sin\frac{\pi}{2p}+\omega\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right)\\
=\;&\frac{1}{B(4)}\left(\frac{\sqrt{3}(\frac{\sqrt{3}}{2}\sin\frac{\pi}{2p}-\frac{1}{2}\cos\frac{\pi}{2p})+\omega\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right)\\
=\;&\frac{1}{B(4)}\left(\frac{-\sqrt{3}\sin(\frac{\pi}{6}-\frac{\pi}{2p})+\omega\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right).\end{aligned}$$

Thus,

$$z \in EA_0(\omega) \iff \sin(\frac{\pi}{p} + \frac{\pi}{3})z_1 + \cos(\frac{\pi}{p} + \frac{\pi}{3})z_2 = \frac{\sin(\frac{\pi}{6} - \frac{\pi}{2p})}{B(4)2\sin\frac{\pi}{2p}}\left(\omega - \sqrt{3}\right).$$

Similarly

$$z \in EC_0(\omega) \iff \sin\frac{2\pi}{3}z_1 + \cos\frac{2\pi}{3}z_2 = -\frac{\sin(\frac{\pi}{6} - \frac{\pi}{2p})}{B(4)2\sin\frac{\pi}{2p}}\left(\omega + \sqrt{3}\right).$$

For the equation of HC_{-1} we have the right hand side

$$\begin{aligned}
&\frac{1}{B(2)}\left(1 + \frac{\sin(\frac{5\pi}{3} - \frac{\pi}{2p})}{2\sin\frac{\pi}{2p}} - \frac{\cos(\frac{5\pi}{3} - \frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right) \\
&= \frac{1}{B(2)}\left(\frac{2\sin\frac{\pi}{2p} + \sin(\frac{5\pi}{3} - \frac{\pi}{2p}) - \omega\cos(\frac{5\pi}{3} - \frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right) \\
&= \frac{1}{B(2)}\left(\frac{2\sin\frac{\pi}{2p} + \sin\frac{5\pi}{3}\cos\frac{\pi}{2p} - \cos\frac{5\pi}{3}\sin\frac{\pi}{2p} - \omega(\cos\frac{\pi}{2p}\cos\frac{5\pi}{3} + \sin\frac{\pi}{2p}\sin\frac{5\pi}{3})}{2\sin\frac{\pi}{2p}}\right) \\
&= \frac{1}{B(2)}\left(\frac{-\sqrt{3}\sin(\frac{\pi}{6} - \frac{\pi}{2p}) - \omega\sin(\frac{\pi}{6} - \frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\right).
\end{aligned}$$

Therefore, $z \in EC_{-1}(\omega)$ if and only if

$$-\sin\left(\frac{\pi}{3} - \frac{\pi}{p}\right)z_1 + \cos\left(\frac{\pi}{3} - \frac{\pi}{p}\right)z_2 \quad = \quad \frac{-\sin(\frac{\pi}{6} - \frac{\pi}{2p})}{B(2)2\sin\frac{\pi}{2p}}\left(\sqrt{3} + \omega\right).$$

Recall that $B(2) = S \cdot \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \cdot S$ and $B(4) = S \cdot \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2} \cdot S$. We can now apply Corollary 91 (in appendix). We have

$$\Delta(\omega) = \begin{vmatrix} \sin(\frac{\pi}{p}+\frac{\pi}{3}) & \cos(\frac{\pi}{p}+\frac{\pi}{3}) & -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\left(\omega-\sqrt{3}\right) \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & \frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\left(\omega+\sqrt{3}\right) \\ -\sin(\frac{\pi}{3}-\frac{\pi}{p}) & \cos(\frac{\pi}{3}-\frac{\pi}{p}) & -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\left(\omega+\sqrt{3}\right) \end{vmatrix}$$

$$= -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\begin{vmatrix} \sin(\frac{\pi}{p}+\frac{\pi}{3}) & \cos(\frac{\pi}{p}+\frac{\pi}{3}) & \left(\omega-\sqrt{3}\right) \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & -\left(\omega+\sqrt{3}\right) \\ -\sin(\frac{\pi}{3}-\frac{\pi}{p}) & \cos(\frac{\pi}{3}-\frac{\pi}{p}) & \left(\omega+\sqrt{3}\right) \end{vmatrix}.$$

$$= -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\cdot\Delta_1(\omega).$$

First we need to check if $\delta \neq 0$, where $\delta = \begin{vmatrix} \sin(\frac{\pi}{p}+\frac{\pi}{3}) & \cos(\frac{\pi}{p}+\frac{\pi}{3}) \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{vmatrix} = \sin(\frac{\pi}{p}-\frac{\pi}{3})$. We have $-\frac{\pi}{3} < \frac{\pi}{p}-\frac{\pi}{3} < 0$ since $p > 3$. Thus, $\sin(\frac{\pi}{p}-\frac{\pi}{3}) \neq 0$, i.e. $\delta \neq 0$.

Note that $\Delta_1(\omega)$ coincides with the determinate computed in the proof of Proposition 73, i.e.

$$\Delta_1(\omega) = 2\sqrt{3}\sin\frac{\pi}{2p}\left(\omega\cos(\frac{\pi}{6}-\frac{\pi}{2p})-\sin(\frac{\pi}{6}-\frac{\pi}{2p})\right).$$

We know that $\omega^+ = \tan(\frac{\pi}{12}-\frac{\pi}{4p})$. Thus, $\Delta_1(\omega)=0$ if and only if $\omega = \tan(\frac{\pi}{6}-\frac{\pi}{2p})$.

Therefore, the lines $EA_0\left(\tan\left(\frac{\pi}{6}-\frac{\pi}{2p}\right)\right)$, $EC_0\left(\tan\left(\frac{\pi}{6}-\frac{\pi}{2p}\right)\right)$ and $EC_{-1}\left(\tan\left(\frac{\pi}{6}-\frac{\pi}{2p}\right)\right)$ meet at one point. ∎

Proposition 81. *The lines $EA_0(\omega^+), EB_0(\omega^+)$ and $EC_{-1}(\omega^+)$ meet at one point. Here $\omega^+ = \tan\left(\frac{\pi}{12}-\frac{\pi}{4p}\right)$.*

Proof. The equations of the lines are

- $z \in EA_0(\omega)$ if and only if

$$\begin{aligned}
&\sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2\\
&=\frac{1}{B(4)}\left(1-\frac{\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}+\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right),
\end{aligned}$$

- $z \in EB_0(\omega)$ if and only if

$$\sin\left(\frac{\pi}{2p}+\frac{\pi}{2}\right)z_1+\cos\left(\frac{\pi}{2p}+\frac{\pi}{2}\right)z_2 = \frac{1}{B(4)}\left(\frac{2\sin\frac{\pi}{2p}-1}{2\sin\frac{\pi}{2p}}\right),$$

- $z \in EC_{-1}(\omega)$ if and only if

$$\begin{aligned}
&-\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_1+\cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_2\\
&=\frac{1}{B(2)}\left(1+\frac{\sin(\frac{5\pi}{3}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}-\frac{\cos(\frac{5\pi}{3}-\frac{\pi}{2p})}{2\sin\frac{\pi}{2p}}\omega\right).
\end{aligned}$$

We know from the proof of Proposition 80 that

$$z \in EA_0(\omega) \iff \sin\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_1+\cos\left(\frac{\pi}{p}+\frac{\pi}{3}\right)z_2=\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{B(4)2\sin\frac{\pi}{2p}}\left(\omega-\sqrt{3}\right)$$

and

$$\begin{aligned}
z \in EC_{-1}(\omega) \iff &-\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_1+\cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right)z_2\\
&=\frac{-\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{B(2)2\sin\frac{\pi}{2p}}\left(\sqrt{3}+\omega\right).
\end{aligned}$$

Recall that $B(2)=S\cdot\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}\cdot S$ and $B(4)=S\cdot\sin\frac{4\pi}{3}=-\frac{\sqrt{3}}{2}\cdot S$. We can now apply Corollary 91 (in appendix). We have

$$
\begin{aligned}
\Delta(\omega) &= \begin{vmatrix} \sin(\frac{\pi}{p}+\frac{\pi}{3}) & \cos(\frac{\pi}{p}+\frac{\pi}{3}) & -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\left(\omega-\sqrt{3}\right) \\ \sin\left(\frac{\pi}{2p}+\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2p}+\frac{\pi}{2}\right) & -\frac{2\sin\frac{\pi}{2p}-1}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}} \\ -\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right) & \cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right) & -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\left(\omega+\sqrt{3}\right) \end{vmatrix} \\
&= -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}} \begin{vmatrix} \sin(\frac{\pi}{p}+\frac{\pi}{3}) & \cos(\frac{\pi}{p}+\frac{\pi}{3}) & \left(\omega-\sqrt{3}\right) \\ \sin\left(\frac{\pi}{2p}+\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2p}+\frac{\pi}{2}\right) & \frac{2\sin\frac{\pi}{2p}-1}{\sin(\frac{\pi}{6}-\frac{\pi}{2p})} \\ -\sin\left(\frac{\pi}{3}-\frac{\pi}{p}\right) & \cos\left(\frac{\pi}{3}-\frac{\pi}{p}\right) & \left(\omega+\sqrt{3}\right) \end{vmatrix}. \\
&= -\frac{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}{\sqrt{3}\cdot S\sin\frac{\pi}{2p}}\,\Delta_1(\omega).
\end{aligned}
$$

First we need to check if $\delta \neq 0$, where $\delta = \begin{vmatrix} \sin(\frac{\pi}{p}+\frac{\pi}{3}) & \cos(\frac{\pi}{p}+\frac{\pi}{3}) \\ \sin\left(\frac{\pi}{2p}+\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2p}+\frac{\pi}{2}\right) \end{vmatrix} = \sin(\frac{\pi}{2p}-\frac{\pi}{6})$. We have $-\frac{\pi}{6} < \frac{\pi}{2p}-\frac{\pi}{6} < 0$ since $p > 3$. Thus, $\sin(\frac{\pi}{2p}-\frac{\pi}{6}) \neq 0$, i.e. $\delta \neq 0$.

$$
\begin{aligned}
\Delta_1(\omega) &= (\omega-\sqrt{3})\sin\left((\frac{\pi}{2p}+\frac{\pi}{2})+(\frac{\pi}{3}-\frac{\pi}{p})\right)\\
&\quad -\left(\frac{2\sin\frac{\pi}{2p}-1}{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}\right)\sin\left((\frac{\pi}{p}+\frac{\pi}{3})+(\frac{\pi}{3}-\frac{\pi}{p})\right)\\
&\quad +(\omega+\sqrt{3})\sin\left((\frac{\pi}{p}+\frac{\pi}{3})-(\frac{\pi}{2p}+\frac{\pi}{2})\right)\\
&= (\omega-\sqrt{3})\sin\left(\frac{5\pi}{6}-\frac{\pi}{2p}\right)\\
&\quad -\left(\frac{2\sin\frac{\pi}{2p}-1}{\sin(\frac{\pi}{6}-\frac{\pi}{2p})}\right)\sin(\frac{2\pi}{3})\\
&\quad +(\omega+\sqrt{3})\sin\left(\frac{\pi}{2p}-\frac{\pi}{6}\right)\\
&= (\omega-\sqrt{3})\left(\frac{1}{2}\cos\frac{\pi}{2p}+\frac{\sqrt{3}}{2}\sin\frac{\pi}{2p}\right)\\
&\quad -\left(\frac{\sqrt{3}}{2}\frac{2\sin\frac{\pi}{2p}-1}{\frac{1}{2}\cos\frac{\pi}{2p}-\frac{\sqrt{3}}{2}\sin\frac{\pi}{2p}}\right)\\
&\quad +(\omega+\sqrt{3})\left(\frac{\sqrt{3}}{2}\sin\frac{\pi}{2p}-\frac{1}{2}\cos\frac{\pi}{2p}\right)\\
&= \omega(\sqrt{3}\sin\frac{\pi}{2p})-\sqrt{3}\cos\frac{\pi}{2p}-\sqrt{3}\left(\frac{2\sin\frac{\pi}{2p}-1}{\cos\frac{\pi}{2p}-\sqrt{3}\sin\frac{\pi}{2p}}\right)\\
&= \sqrt{3}\left(\omega\sin\frac{\pi}{2p}-\left(\frac{\cos^2\frac{\pi}{2p}-\sqrt{3}\cos\frac{\pi}{2p}\sin\frac{\pi}{2p}+2\sin\frac{\pi}{2p}-1}{\cos\frac{\pi}{2p}-\sqrt{3}\sin\frac{\pi}{2p}}\right)\right)\\
&= \sqrt{3}\left(\omega\sin\frac{\pi}{2p}-\left(\frac{-\sin^2\frac{\pi}{2p}-\sqrt{3}\cos\frac{\pi}{2p}\sin\frac{\pi}{2p}+2\sin\frac{\pi}{2p}}{\cos\frac{\pi}{2p}-\sqrt{3}\sin\frac{\pi}{2p}}\right)\right).
\end{aligned}
$$

We know that

$$
\begin{aligned}
\omega^+ &= \tan(\frac{\pi}{12}-\frac{\pi}{4p}) = \tan\frac{1}{2}(\frac{\pi}{6}-\frac{\pi}{2p})\\
&= \frac{1-\cos(\frac{\pi}{6}-\frac{\pi}{2p})}{\sin(\frac{\pi}{6}-\frac{\pi}{2p})} = \frac{1-\frac{1}{2}(\sqrt{3}\cos\frac{\pi}{2p}+\sin\frac{\pi}{2p})}{\frac{1}{2}(\cos\frac{\pi}{2p}-\sqrt{3}\sin\frac{\pi}{2p})}.
\end{aligned}
$$

Hence, we have

$$\omega^{+}\sin\frac{\pi}{2p} = \frac{2\sin\frac{\pi}{2p}-\sqrt{3}\cos\frac{\pi}{2p}\sin\frac{\pi}{2p}-\sin^{2}\frac{\pi}{2p}}{\cos\frac{\pi}{2p}-\sqrt{3}\sin\frac{\pi}{2p}}.$$

Thus,

$$\begin{aligned}\Delta_1(\omega^{+}) &= \sqrt{3}\left(\omega^{+}\sin\frac{\pi}{2p}-\left(\frac{-\sin^{2}\frac{\pi}{2p}-\sqrt{3}\cos\frac{\pi}{2p}\sin\frac{\pi}{2p}+2\sin\frac{\pi}{2p}}{\cos\frac{\pi}{2p}-\sqrt{3}\sin\frac{\pi}{2p}}\right)\right)\\ &= \sqrt{3}\left(\frac{2\sin\frac{\pi}{2p}-\sqrt{3}\cos\frac{\pi}{2p}\sin\frac{\pi}{2p}-\sin^{2}\frac{\pi}{2p}}{\cos\frac{\pi}{2p}-\sqrt{3}\sin\frac{\pi}{2p}}\right)\\ &\quad -\sqrt{3}\left(\frac{-\sin^{2}\frac{\pi}{2p}-\sqrt{3}\cos\frac{\pi}{2p}\sin\frac{\pi}{2p}+2\sin\frac{\pi}{2p}}{\cos\frac{\pi}{2p}-\sqrt{3}\sin\frac{\pi}{2p}}\right)\\ &= 0\end{aligned}$$

$\therefore$ The lines $EA_0(\omega^{+}), EB_0(\omega^{+})$ and $EC_{-1}(\omega^{+})$ meet at one point. ∎

Using Propositions 78, 79, 80 and 81 we can follow the mutual position of the lines EA_m, EB_m, EC_m as ω changes from $\omega=\omega^{-}$ to $\omega=\omega^{+}$, see figures 5.14, 5.15 and 5.16 for the case $k=2, p=7$.

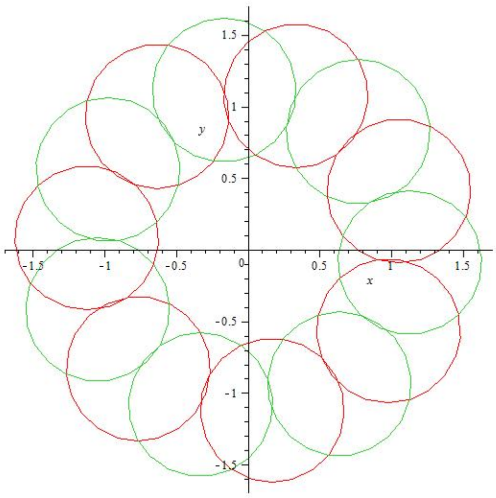

Figure 5.14: The intersections of the prisms Q_x for $\tilde{\Gamma}_1(7,3,3)^2\times(C_3)^2$ with the plane $\omega=\omega^{-}$.

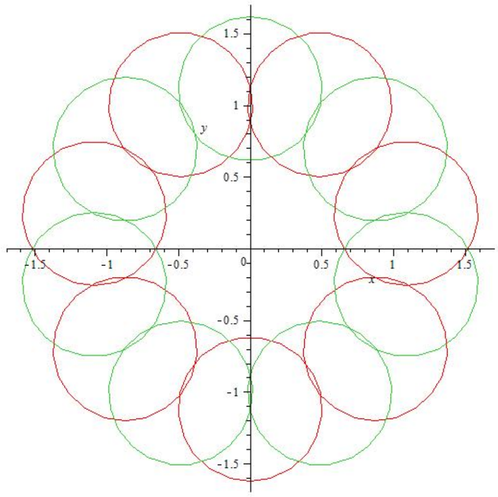

Figure 5.15: The intersections of the prisms Q_x for $\tilde{\Gamma}_1(7,3,3)^2 \times (C_3)^2$ with the plane $\omega = 0$.

Now, we understand the combinatorial structure of the faces of $\hat{P}$, compare with figure 5.13.

Proposition 82. *The identification scheme for the faces of the boundary of P under the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ is:*

$$\begin{aligned}
(\tilde{d}; c_m) &\mapsto (\tilde{d}^{-1}; b_{m+p+1}),\\
(\tilde{d}; b_m) &\mapsto (\tilde{d}^{-1}; a_{m+p+1}),\\
(c_m; \tilde{d}, a_{m+1}, b_m) &\mapsto (a_{m+1}; b_{m+1}, c_m, \tilde{d}^{-1}),\\
(b_m; \tilde{d}, c_m, \tilde{d}^{-1}, a_m) &\mapsto (b_{m+p}; c_{m+p}, \tilde{d}, a_{m+p}, \tilde{d}^{-1}),\\
(a_m; b_m, \tilde{d}^{-1}, c_{m-1}) &\mapsto (c_{m-1}; \tilde{d}, b_{m-1}, a_m),
\end{aligned}$$

and the edge cycles

$$(c_m; \tilde{d}) \mapsto (b_{m+1}; a_{m+1}) \mapsto (\tilde{d}^{-1}; b_{m+p+1}),$$

$$(c_m; b_m) \mapsto (\tilde{d}^{-1}; a_{m+1}) \mapsto (b_{m-p}; \tilde{d}),$$

$$(c_m; a_{m+1}) \circlearrowleft .$$

Proof.

In the beginning of this section we study the polyhedron $\hat{P}$. The faces of

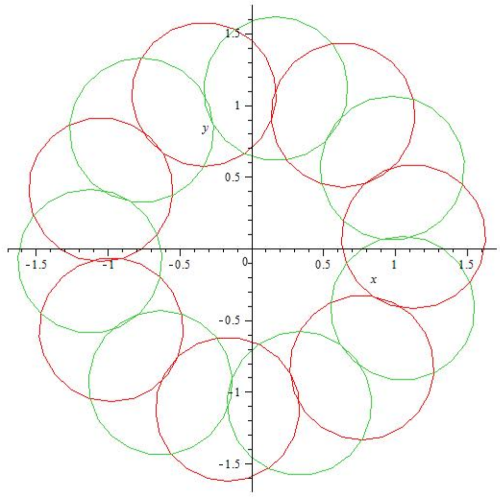

Figure 5.16: The intersections of the prisms Q_x for $\tilde{\Gamma}_1(7,3,3)^2 \times (C_3)^2$ with the plane $\omega = \omega^+$.

$\hat{P}$ correspond to the elements

$$\begin{aligned} a_0 &= \tilde{r}_v^4 \tilde{d}^{2\lambda p-2} c^{\frac{-2(\lambda k+2)}{3}}, \\ b_0 &= \tilde{r}_v^4 \tilde{d}^{2\lambda p-1} c^{\frac{-2(\lambda k+2)}{3}}, \\ c_0 &= \tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}}, \end{aligned}$$

and their images $a_m = \tilde{\rho}^m(a_0), b_m = \tilde{\rho}^m(b_0), c_m = \tilde{\rho}^m(c_0)$ under $\tilde{\rho}$. Here $\lambda = 1$ if $k = 1 \mod 3$ and $\lambda = 2$ if $k = 2 \mod 3$. We will look at the identification schemes for this case. We know that the action of $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$ is

$$(\tilde{g}_1, \tilde{g}_2) \cdot x = \tilde{g}_1 x \tilde{g}_2^{-1}.$$

So,

$$(\tilde{g}_1^{-1}, \tilde{g}_2) \cdot x = \tilde{g}_1^{-1} x \tilde{g}_2^{-1}.$$

Hence, $x \mapsto \tilde{g}_1^{-1} x \tilde{g}_2^{-1}$. Therefore, $\tilde{E}_{a_m} \mapsto \tilde{E}_{\tilde{g}_1^{-1} a_m \tilde{g}_2^{-1}}$ and similar for $\tilde{E}_{b_m}, \tilde{E}_{c_m}$. We can write a_0, b_0 and c_0 as

$$
\begin{aligned}
a_0 &= \tilde{r}_v^4 \tilde{d}^{2\lambda p-2} c^{\frac{-2(\lambda k+2)}{3}} = \tilde{r}_v \tilde{d}^{2\lambda p-2} c^{\frac{-(2\lambda k+1)}{3}}, \\
b_0 &= \tilde{r}_v^4 \tilde{d}^{2\lambda p-1} c^{\frac{-2(\lambda k+2)}{3}} = \tilde{r}_v \tilde{d}^{2\lambda p-1} c^{\frac{-(2\lambda k+1)}{3}}, \\
c_0 &= \tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}} = \tilde{r}_v \tilde{d}^{2\lambda p} c^{\frac{-(2\lambda k+1)}{3}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
a_1 &= \tilde{\rho}(a_0) = \tilde{\rho}(\tilde{r}_v \tilde{d}^{2(\lambda p-1)} c^{\frac{-(2\lambda k+1)}{3}}) \\
&= \tilde{\rho}(\tilde{r}_v)\tilde{\rho}(\tilde{d}^{2(\lambda p-1)})\tilde{\rho}(c^{\frac{-(2\lambda k+1)}{3}}) \\
&= \tilde{r}_v^2 \tilde{r}_u^{-1} \tilde{d}^{2(\lambda p-1)} c^{\frac{-(2\lambda k+1)}{3}} = \tilde{r}_v^2 \tilde{d}^{\lambda p} c^{\frac{-(\lambda k+2)}{3}},
\end{aligned}
$$

since $\tilde{r}_u^{-1} = c^{\frac{\lambda k-1}{3}} d^{-\lambda p+2}$ according to Corollary 53. Similarly, $b_1 = \tilde{\rho}(b_0) = \tilde{r}_v^2 \tilde{d}^{\lambda p+1} c^{\frac{-(\lambda k+2)}{3}}$ and $c_1 = \tilde{\rho}(c_0) = \tilde{r}_v^2 \tilde{d}^{\lambda p+2} c^{\frac{-(\lambda k+2)}{3}}$. Note that $b_m = a_m \tilde{d}$ and $c_m = b_m \tilde{d}$.

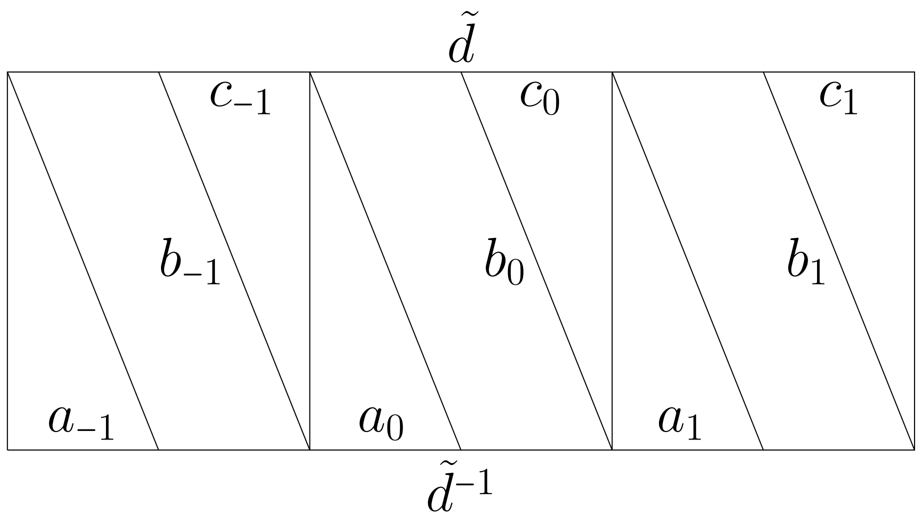

Figure 5.17: The surface of P for $\tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$.

1. The face $\tilde{E}_{c_0} \cap \tilde{E}_{\tilde{e}}$:

   We have $c_0 = \tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}}$. We need to write c_0 as a product $\tilde{g}_1 \cdot \tilde{g}_2$ with $\tilde{g}_1 \in \tilde{\Gamma}_1$, $\tilde{g}_2 \in \tilde{\Gamma}_2$. We have

$$c_0 = \tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}} = \left(\tilde{r}_v^4 c^{\frac{-2(\lambda k+2)}{3}}\right)\left(\tilde{d}_2^{2\lambda}\right) = \tilde{g}_1 \tilde{g}_2.$$

We consider the action of $(\tilde{g}_1^{-1}, \tilde{g}_2) \in \tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$ by $x \mapsto \tilde{g}_1^{-1} x \tilde{g}_2^{-1}$. Then, we need to calculate $\tilde{g}_1^{-1}\tilde{g}_2^{-1}$ in order to find the image of $\tilde{E}_{\tilde{e}}$.

$$\begin{aligned}
\tilde{g}_1^{-1}\tilde{g}_2^{-1} &= \left(\tilde{r}_v^4 c^{\frac{-2(\lambda k+2)}{3}}\right)^{-1}\left(\tilde{d}_2^{2\lambda}\right)^{-1} \\
&= \tilde{r}_v^{-4}\tilde{d}^{-2\lambda p} c^{\frac{2(\lambda k+2)}{3}} = \tilde{r}_v^6 c^{-2} \tilde{r}_v^{-4} \tilde{d}^{\lambda p} c^{-\lambda k} c^{\frac{2(\lambda k+2)}{3}} \\
&= \tilde{r}_v^2 \tilde{d}^{\lambda p} c^{\frac{-(\lambda k+2)}{3}} = a_1.
\end{aligned}$$

Here we used $\tilde{r}_v^3 = c$ and $\tilde{d}^{3p} = c^k$. Hence

$$\tilde{E}_{\tilde{e}} \mapsto \tilde{E}_{a_1}.$$

Now, let us look at the image of $\tilde{E}_{c_0}$:

$$\begin{aligned}
c_0 &\mapsto \tilde{g}_1^{-1} c_0 \tilde{g}_2^{-1} \\
&= \left(\tilde{r}_v^{-4} c^{\frac{2(\lambda k+2)}{3}}\right)\left(\tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}}\right)\left(\tilde{d}^{-2\lambda p}\right) = \tilde{e},
\end{aligned}$$

hence

$$\tilde{E}_{c_0} \mapsto \tilde{E}_{\tilde{e}}.$$

So,

$$\tilde{E}_{c_0} \cap \tilde{E}_{\tilde{e}} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1},$$

$$i.e.\ the\ face\ c_0\ is\ glued\ to\ the\ face\ a_1.$$

Note that c_0 and a_1 have the following properties: $c_0 = \tilde{g}_1\tilde{g}_2, a_1 = \tilde{g}_1^{-1}\tilde{g}_2^{-1}$, $\tilde{g}_1^3 = \left(\tilde{r}_v^4 c^{\frac{-2(\lambda k+2)}{3}}\right)^3 = \tilde{r}_v^{12} c^{-2(\lambda k+2)} = c^4 c^{-2\lambda k-4} = c^{-2\lambda k}$, $\tilde{g}_2^3 = (\tilde{d}_2^{2\lambda})^3 = \tilde{d}^{6\lambda p} = (\tilde{d}^{3p})^{2\lambda} = c^{2\lambda k}$. We will see later that the edge $(c_0; a_1)$ is in an edge cycle of length 1 and the properties of c_0 and a_0 discussed above will mean that the conditions of Theorem 49 are satisfied.

Now, we still need to look at the edges of the face $\tilde{E}_{c_0} \cap \tilde{E}_{\tilde{e}}$:

(a) $\tilde{E}_{c_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{\tilde{d}}$:
We have

$$\begin{aligned}
\tilde{d} &\mapsto \tilde{g}_1^{-1}\tilde{d}\tilde{g}_2^{-1} \\
&= \left(\tilde{r}_v^{-4}c^{\frac{2(\lambda k+2)}{3}}\right)\tilde{d}\left(\tilde{d}^{-2\lambda p}\right) \\
&= \tilde{r}_v^{6}\tilde{r}_v^{-4}\tilde{d}\tilde{d}^{\lambda p}c^{-2}c^{-\lambda k}c^{\frac{2(\lambda k+2)}{3}} \\
&= \tilde{r}_v^{2}\tilde{d}^{\lambda p+1}c^{\frac{-(\lambda k+2)}{3}} = b_1.
\end{aligned}$$

So,

$$\tilde{E}_{c_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{\tilde{d}} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1} \cap \tilde{E}_{b_1}$$

Hence,

The edge $(c_0; \tilde{d})$ *is glued to the edge* $(a_1; b_1)$.

(b) $\tilde{E}_{c_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1}$:
We have

$$\begin{aligned}
a_1 &\mapsto \tilde{g}_1^{-1}a_1\tilde{g}_2^{-1} \\
&= \left(\tilde{r}_v^{-4}c^{\frac{2(\lambda k+2)}{3}}\right)\left(\tilde{r}_v^{2}\tilde{d}^{\lambda p}c^{\frac{-(\lambda k+2)}{3}}\right)\left(\tilde{d}^{-2\lambda p}\right) \\
&= \tilde{r}_v^{-4}\tilde{r}_v^{6}\tilde{r}_v^{2}\tilde{d}^{\lambda p}\tilde{d}^{\lambda p}c^{-2}c^{-\lambda k}c^{\frac{2(\lambda k+2)}{3}}c^{\frac{-(\lambda k+2)}{3}} \\
&= \tilde{r}_v^{4}\tilde{d}^{2\lambda p}c^{\frac{-2(\lambda k+2)}{3}} = c_0.
\end{aligned}$$

So,

$$\tilde{E}_{c_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1} \cap \tilde{E}_{c_0}.$$

Hence,

The edge $(c_0; a_1)$ *is glued to the edge* $(a_1; c_0)$.

(c) $\tilde{E}_{c_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_0}$:

We have

$$\begin{aligned} b_0 &\mapsto \tilde{g}_1^{-1} b_0 \tilde{g}_2^{-1} \\ &= \left(\tilde{r}_v^{-4} c^{\frac{2(\lambda k+2)}{3}}\right)\left(\tilde{r}_v^4 \tilde{d}^{2\lambda p-1} c^{\frac{-2(\lambda k+2)}{3}}\right)\left(\tilde{d}^{-2\lambda p}\right) \\ &= \tilde{d}^{2\lambda p-1} \tilde{d}^{-2\lambda p} = \tilde{d}^{-1}. \end{aligned}$$

So,

$$\tilde{E}_{c_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_0} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_1} \cap \tilde{E}_{\tilde{d}^{-1}}.$$

Hence,

The edge $(c_0; b_0)$ is glued to the edge $(a_1; \tilde{d}^{-1})$.

2. The face $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}}$:

   We know that

   $$b_0 = \tilde{r}_v^4 \tilde{d}^{2\lambda p-1} c^{\frac{-2(\lambda k+2)}{3}}.$$

   We need to write b_0 as a product $\tilde{g}_1 \cdot \tilde{g}_2$ with $\tilde{g}_1 \in \tilde{\Gamma}_1$, $\tilde{g}_2 \in \tilde{\Gamma}_2$. We have

   $$\begin{aligned} b_0 &= \tilde{r}_v^4 \tilde{d}^{2\lambda p-1} c^{\frac{-2(\lambda k+2)}{3}} \\ &= \left(\tilde{r}_v^4 \tilde{d}^{-(\lambda p+1)} c^{\frac{-2(\lambda k+2)}{3}}\right)\left(\tilde{d}^{3\lambda p}\right) \\ &= \left(\tilde{r}_v^4 \tilde{d}_1^{\frac{-(\lambda(2k+3)+1)}{3}} c^{\frac{-2(\lambda k+2)}{3}}\right)\left(\tilde{d}_2^{3\lambda}\right) = \tilde{g}_1 \tilde{g}_2. \end{aligned}$$

   Here we used the fact that for $k = 1 \mod 3$ we have $\lambda = 1$ and $\lambda(2k + 3) + 1 = 2k + 4 = 0 \mod 3$ and for $k = 2 \mod 3$ we have $\lambda = 2$ and $\lambda(2k + 3) + 1 = 4k + 7 = 0 \mod 3$. So, $\frac{\lambda(2k+3)+1}{3} \in \mathbb{Z}$ in both cases.
   We consider the action of $(\tilde{g}_1^{-1}, \tilde{g}_2) \in \tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$ by $x \mapsto \tilde{g}_1^{-1} x \tilde{g}_2^{-1}$. We look at the image of $\tilde{E}_{\tilde{e}}$:

$$
\begin{aligned}
\tilde{e} &\mapsto \tilde{g}_1^{-1}\tilde{e}\tilde{g}_2^{-1} \\
&= \left(\tilde{r}_v^4\tilde{d}^{-(\lambda p+1)}c^{\frac{-2(\lambda k+2)}{3}}\right)^{-1}\tilde{e}\left(\tilde{d}^{3\lambda p}\right)^{-1} \\
&= \tilde{d}^{(\lambda p+1)}\tilde{r}_v^{-4}c^{\frac{2(\lambda k+2)}{3}}\tilde{d}^{-3\lambda p} = \tilde{d}^{(\lambda p+1)}\tilde{r}_v^{6}\tilde{r}_v^{-4}c^{-2}c^{-\lambda k}c^{\frac{2(\lambda k+2)}{3}} \\
&= \tilde{d}^{(\lambda p+1)}\tilde{r}_v^{2}c^{\frac{-(\lambda k+2)}{3}}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
b_p &= b_{2k+3} = \tilde{\rho}^{2k+3}(b_0) = \tilde{\rho}^{2k+2}\left(\tilde{\rho}(b_0)\right) \\
&= \tilde{\rho}^{2(k+1)}(b_1) = \rho^{k+1}(b_1) = \tilde{r}_u b_1 \tilde{r}_u^{-1}
\end{aligned}
$$

since $\tilde{\rho}^2 = \rho$. We also know from Corollary 53 that $\tilde{r}_u = c^{\frac{1-\lambda k}{3}}\tilde{d}^{\lambda p-2}$ and $\tilde{r}_u^{-1} = c^{\frac{\lambda k-1}{3}}\tilde{d}^{2-\lambda p}$. Hence

$$
\begin{aligned}
b_p &= \rho^{k+1}(b_1) = \tilde{r}_u^{k+1}b_1\tilde{r}_u^{-(k+1)} \\
&= \tilde{r}_u^{k+1}\left(\tilde{r}_v^2\tilde{d}^{\lambda p+1}c^{\frac{-(\lambda k+2)}{3}}\right)\tilde{r}_u^{-(k+1)} \\
&= c^{\frac{(1-\lambda k)(k+1)}{3}}\tilde{d}^{(\lambda p-2)(k+1)}\left(\tilde{r}_v^2\tilde{d}^{\lambda p+1}c^{\frac{-(\lambda k+2)}{3}}\right)c^{\frac{(\lambda k-1)(k+1)}{3}}\tilde{d}^{(2-\lambda p)(k+1)} \\
&= \tilde{d}^{(\lambda(2k+3)-2)(k+1)}\tilde{r}_v^2\tilde{d}^{\lambda(2k+3)+1}\tilde{d}^{(2-\lambda(2k+3))(k+1)}c^{\frac{-(\lambda k+2)}{3}} \\
&= \tilde{d}^{2\lambda k^2+k(5\lambda-2)+3\lambda-2}\tilde{r}_v^2\tilde{d}^{-2\lambda k^2+k(2-3\lambda)+3}c^{\frac{-(\lambda k+2)}{3}}.
\end{aligned}
$$

We have

$$
-(2\lambda k^2 + k(3\lambda-2) - 3) = -(\lambda k - 1)(2k+3) = -p(\lambda k - 1).
$$

Therefore,

$$
\tilde{d}^{-2\lambda k^2+k(2-3\lambda)+3} = \tilde{d}^{-p(\lambda k-1)} = \tilde{d}^{3p\frac{(1-\lambda k)}{3}} = \left(\tilde{d}^{3p}\right)^{\frac{1-\lambda k}{3}} = \left(c^k\right)^{\frac{1-\lambda k}{3}} = c^{\frac{k(1-\lambda k)}{3}}
$$

since we know that $1 - \lambda k \equiv 0 \mod 3$, so $\frac{1-\lambda k}{3} \in \mathbb{Z}$. Thus,

$$\begin{aligned}
b_p &= \tilde{d}^{2\lambda k^2+k(5\lambda-2)+3\lambda-2}\tilde{r}_v^2\tilde{d}^{-2\lambda k^2+k(2-3\lambda)+3}c^{\frac{-(\lambda k+2)}{3}} \\
&= \tilde{d}^{2\lambda k^2+k(5\lambda-2)+3\lambda-2}\tilde{r}_v^2\tilde{d}^{3p\frac{1-\lambda k}{3}}c^{\frac{-(\lambda k+2)}{3}} \\
&= \tilde{d}^{2\lambda k^2+k(5\lambda-2)+3\lambda-2}\tilde{r}_v^2 c^{\frac{k(1-\lambda k)}{3}}c^{\frac{-(\lambda k+2)}{3}} \\
&= \tilde{d}^{2\lambda k^2+k(5\lambda-2)+3\lambda-2}c^{\frac{k(1-\lambda k)}{3}}\tilde{r}_v^2 c^{\frac{-(\lambda k+2)}{3}} \\
&= \tilde{d}^{2\lambda k^2+k(5\lambda-2)+3\lambda-2}\tilde{d}^{-p(\lambda k-1)}\tilde{r}_v^2 c^{\frac{-(\lambda k+2)}{3}} \\
&= \tilde{d}^{2\lambda k^2+k(5\lambda-2)+3\lambda-2}\tilde{d}^{(2k+3)(1-\lambda k)}\tilde{r}_v^2 c^{\frac{-(\lambda k+2)}{3}} \\
&= \tilde{d}^{2\lambda k+3\lambda+1}\tilde{r}_v^2 c^{\frac{-(\lambda k+2)}{3}} \\
&= \tilde{d}^{\lambda p+1}\tilde{r}_v^2 c^{\frac{-(\lambda k+2)}{3}}.
\end{aligned}$$

So, we have shown that $\tilde{e} \mapsto b_p$, hence

$$\tilde{E}_{\tilde{e}} \mapsto \tilde{E}_{b_p}.$$

Note that similarly

$$\begin{aligned}
c_p &= \tilde{\rho}^p(c_0) = \tilde{d}^{\lambda p+1}\tilde{r}_v^2\tilde{d}c^{\frac{-(\lambda k+2)}{3}}, \\
a_p &= \tilde{\rho}^p(a_0) = \tilde{d}^{\lambda p+1}\tilde{r}_v^2\tilde{d}^{-1}c^{\frac{-(\lambda k+2)}{3}} = \tilde{d}^{\lambda p+1}\tilde{r}_v^2\tilde{d}^{3p-1}c^{\frac{-3k-(\lambda k+2)}{3}}.
\end{aligned}$$

Now, we need to look at the image of $\tilde{E}_{b_0}$. We have

$$\begin{aligned}
b_0 &\mapsto \tilde{g}_1^{-1}b_0\tilde{g}_2^{-1} \\
&= \left(\tilde{r}_v^4\tilde{d}^{-(\lambda p+1)}c^{\frac{-2(\lambda k+2)}{3}}\right)^{-1}\left(\tilde{r}_v^4\tilde{d}^{-(\lambda p+1)}c^{\frac{-2(\lambda k+2)}{3}}\tilde{d}^{3\lambda p}\right)\left(\tilde{d}^{3\lambda p}\right)^{-1} = \tilde{e},
\end{aligned}$$

hence,

$$\tilde{E}_{b_0} \mapsto \tilde{E}_{\tilde{e}}.$$

Therefore,

$$\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_p}.$$

Now, we have to study the edges of the face $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}}$:

(a) $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{\tilde{d}}$:
We have

$$\begin{aligned} \tilde{d} &\mapsto \tilde{g}_1^{-1}\tilde{d}\tilde{g}_2^{-1} \\ &= \left(\tilde{d}^{\lambda p+1}\tilde{r}_v^{-4}c^{\frac{2(\lambda k+2)}{3}}\right)\tilde{d}\left(\tilde{d}^{-3\lambda p}\right) \\ &= \tilde{d}^{\lambda p+1}\tilde{r}_v^{6}\tilde{r}_v^{-4}\tilde{d}^{-3\lambda p+1}c^{-2}c^{\frac{2(\lambda k+2)}{3}} \\ &= \tilde{d}^{\lambda p+1}\tilde{r}_v^{2}\tilde{d}c^{\frac{-(\lambda k+2)}{3}} = c_p. \end{aligned}$$

So,

$$\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{\tilde{d}} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_p} \cap \tilde{E}_{c_p}.$$

Hence,

The edge $(b_0; \tilde{d})$ *is glued to the edge* $(b_p; c_p)$.

(b) $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{c_0}$:
We have

$$\begin{aligned} c_0 &\mapsto \tilde{g}_1^{-1}c_0\tilde{g}_2^{-1} \\ &= \left(\tilde{d}^{\lambda p+1}\tilde{r}_v^{-4}c^{\frac{2(\lambda k+2)}{3}}\right)\left(\tilde{r}_v^{4}\tilde{d}^{2\lambda p}c^{\frac{-2(\lambda k+2)}{3}}\right)\left(\tilde{d}^{-3\lambda p}\right) \\ &= \tilde{d}^{\lambda p+1}\tilde{d}^{2\lambda p}\tilde{d}^{-3\lambda p} = \tilde{d}. \end{aligned}$$

So,

$$\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{c_0} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_p} \cap \tilde{E}_{\tilde{d}}.$$

Hence,

The edge $(b_0; c_0)$ *is glued to the edge* $(b_p; \tilde{d})$.

(c) $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{\tilde{d}^{-1}}$:

We have

$$\begin{aligned}
\tilde{d}^{-1} &\mapsto \tilde{g}_1^{-1}\tilde{d}^{-1}\tilde{g}_2^{-1} \\
&= \big(\tilde{d}^{\lambda p+1}\tilde{r}_v^{-4}c^{\frac{2(\lambda k+2)}{3}}\big)\tilde{d}^{-1}\big(\tilde{d}^{-3\lambda p}\big) \\
&= \tilde{d}^{\lambda p+1}\tilde{r}_v^{6}\tilde{r}_v^{-4}\tilde{d}^{-1}c^{-2}c^{-\lambda k}c^{\frac{2(\lambda k+2)}{3}} \\
&= \tilde{d}^{\lambda p+1}\tilde{r}_v^{2}\tilde{d}^{-1}c^{\frac{-(\lambda k+2)}{3}} = a_p.
\end{aligned}$$

So,

$$\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{\tilde{d}^{-1}} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_p} \cap \tilde{E}_{a_p}.$$

Hence,

The edge $(b_0; \tilde{d}^{-1})$ *is glued to the edge* $(b_p; a_p)$.

(d) $\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_0}$:
We have

$$\begin{aligned}
a_0 &\mapsto \tilde{g}_1^{-1}a_0\tilde{g}_2^{-1} \\
&= \big(\tilde{d}^{\lambda p+1}\tilde{r}_v^{-4}c^{\frac{2(\lambda k+2)}{3}}\big)\big(\tilde{r}_v^{4}\tilde{d}^{2(\lambda p-1)}c^{\frac{-2(\lambda k+2)}{3}}\big)\big(\tilde{d}^{-3\lambda p}\big) \\
&= \tilde{d}^{\lambda p+1}\tilde{d}^{2\lambda p-2}\tilde{d}^{-3\lambda p} = \tilde{d}^{-1}.
\end{aligned}$$

So,

$$\tilde{E}_{b_0} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_0} \mapsto \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_p} \cap \tilde{E}_{\tilde{d}^{-1}}.$$

Hence,

The edge $(b_0; a_0)$ *is glued to the edge* $(b_p; \tilde{d}^{-1})$.

3. The face $\tilde{E}_{a_0} \cap \tilde{E}_{\tilde{e}}$:
We know the gluing rules for the face $\tilde{E}_{c_0} \cap \tilde{E}_{\tilde{e}}$:

$$(c_0; \tilde{d}, a_1, b_0) \mapsto (a_1; b_1, c_0, \tilde{d}^{-1}).$$

Inverting the identification and shifting m by 1 (i.e. applying the symmetry $\tilde{\rho}^{-1}$) we obtain the following gluings for the face $\tilde{E}_{a_0} \cap \tilde{E}_{\tilde{e}}$:

$$(a_0; b_0, c_{-1}, \tilde{d}^{-1}) \mapsto (c_{-1}; \tilde{d}, a_0, b_{-1}).$$

4. The face $\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}}$:

We still need to look at the top face $\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}}$. We are going to write $\tilde{d}$ as a product $\tilde{g}_1 \cdot \tilde{g}_2$ with $\tilde{g}_1^{-1} \in \tilde{\Gamma}_1, \tilde{g}_2 \in \tilde{\Gamma}_2$:

$$\tilde{d} = (\tilde{d}_1)^{\frac{-2\lambda p+1}{3}} (\tilde{d}_2)^{2\lambda} = \tilde{g}_1 \tilde{g}_2.$$

We consider the action of $(\tilde{g}_1^{-1}, \tilde{g}_2) \in \tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ on $\tilde{G}$ by $x \mapsto \tilde{g}_1^{-1} x \tilde{g}_2$ on $\tilde{G}$ by $x \mapsto \tilde{g}_1^{-1} x \tilde{g}_2^{-1}$. Therefore, in order to obtain the image of the top face $\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}}$, we compute

$$\tilde{d} \mapsto \tilde{g}_1^{-1} \tilde{d} \tilde{g}_2^{-1} = \tilde{d}_1^{\frac{2\lambda p-1}{3}} \tilde{d} \tilde{d}_2^{-2\lambda} = \tilde{d}^{2\lambda p-1} \tilde{d} \tilde{d}^{-2\lambda p} = \tilde{e}.$$

So, $\tilde{E}_{\tilde{d}} \mapsto \tilde{E}_{\tilde{e}}$. We now need to look at the image of $\tilde{e}$:

$$\tilde{e} \mapsto \tilde{g}_1^{-1} \tilde{e} \tilde{g}_2^{-1} = \tilde{d}_1^{\frac{2\lambda p-1}{3}} \tilde{e} \tilde{d}_2^{-2\lambda} = \tilde{d}^{2\lambda p-1} \tilde{d}^{-2\lambda p} = \tilde{d}^{-1}.$$

So, $\tilde{E}_{\tilde{e}} \mapsto \tilde{E}_{\tilde{d}^{-1}}$. Hence,

The face in $\tilde{E}_{\tilde{d}}$ is glued to the face in $\tilde{E}_{\tilde{d}^{-1}}$.

We will now consider the edge gluings:

(a) $\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{c_0}$:

$$\begin{aligned} c_0 \quad \mapsto \quad & \tilde{g}_1^{-1} c_0 \tilde{g}_2^{-1} = \tilde{d}_1^{\frac{2\lambda p-1}{3}} \left(\tilde{r}_v^4 \tilde{d}^{2\lambda p} c^{\frac{-2(\lambda k+2)}{3}} \right) \tilde{d}_2^{-2\lambda} \\ = \quad & \tilde{d}^{2\lambda p-1} \tilde{r}_v^4 c^{\frac{-2(\lambda k+2)}{3}}. \end{aligned}$$

On the other hand we obtain using Proposition 60 that

$$\begin{aligned} b_{p+1} &= \tilde{\rho}(b_p) = \tilde{\rho}(\tilde{d}^{\lambda p+1} \tilde{r}_v^2 c^{\frac{-\lambda k+2}{3}}) \\ &= \tilde{d}^{(\lambda p+1)+(\lambda p-2)} \tilde{r}_v^4 c^{\frac{-\lambda k+2}{3} \frac{-\lambda k+2}{3}} \\ &= \tilde{d}^{2\lambda p-1} \tilde{r}_v^4 c^{\frac{-2(\lambda k+2)}{3}}. \end{aligned}$$

So,

$$\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{c_0} \mapsto \tilde{E}_{\tilde{d}^{-1}} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_{p+1}}.$$

Hence,

The edge $(\tilde{d}; c_0)$ *is glued to the edge* $(\tilde{d}^{-1}; b_{p+1})$.

Note that we can compute similarly that

$$a_{p+1} = \tilde{d}^{2\lambda p-1}\tilde{r}_v^4\tilde{d}^{-1}c^{\frac{-2(\lambda k+2)}{3}},$$
$$c_{p+1} = \tilde{d}^{2\lambda p-1}\tilde{r}_v^4\tilde{d}c^{\frac{-2(\lambda k+2)}{3}}.$$

(b) $\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_0}$:

$$\begin{aligned} b_0 &\mapsto \tilde{g}_1^{-1} b_0 \tilde{g}_2^{-1} = \tilde{d}_1^{\frac{2\lambda p-1}{3}}\left(\tilde{r}_v^4\tilde{d}^{2\lambda p-1}c^{\frac{-2(\lambda k+2)}{3}}\right)\tilde{d}_2^{-2\lambda} \\ &= \tilde{d}^{2\lambda p-1}\tilde{r}_v^4\tilde{d}^{-1}c^{\frac{-2(\lambda k+2)}{3}} = a_{p+1}. \end{aligned}$$

So,

$$\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{b_0} \mapsto \tilde{E}_{\tilde{d}^{-1}} \cap \tilde{E}_{\tilde{e}} \cap \tilde{E}_{a_{p+1}}.$$

Hence,

The edge $(\tilde{d}; b_0)$ *is glued to the edge* $(\tilde{d}^{-1}; a_{p+1})$.

Applying the symmetry $\tilde{\rho}$ we obtain that:

$$\begin{aligned} (\tilde{d}; b_m) &\mapsto (\tilde{d}^{-1}; a_{m+p+1}), \\ (\tilde{d}; c_m) &\mapsto (\tilde{d}^{-1}; b_{m+p+1}). \end{aligned}$$

5. The face $\tilde{E}_{\tilde{d}^{-1}} \cap \tilde{E}_{\tilde{e}}$:

   We obtained the gluing of the edges of $\tilde{E}_{\tilde{d}} \cap \tilde{E}_{\tilde{e}}$.

   Hence, by inverting the identifications and shifting m by $p+1$ we obtain:

   (a) *The edge* $(\tilde{d}^{-1}; b_m)$ *is glued with the edge* $(\tilde{d}^{-1}; c_{m-p-1})$,

(b) *The edge* $(\tilde{d}^{-1}; a_m)$ *is glued with the edge* $(\tilde{d}^{-1}; b_{m-p-1})$.

Now, we are going to find the edge cycles for P. We have determined the identifications of the faces and edges of P:

$$\begin{aligned}
(\tilde{d}; c_0) &\mapsto (\tilde{d}^{-1}; b_{p+1}),\\
(\tilde{d}; b_0) &\mapsto (\tilde{d}^{-1}; a_{p+1}),\\
(c_0; \tilde{d}, a_1, b_0) &\mapsto (a_1; b_1, c_0, \tilde{d}^{-1}),\\
(b_0; \tilde{d}, c_0, \tilde{d}^{-1}, a_0) &\mapsto (b_p; c_p, \tilde{d}, a_p, \tilde{d}^{-1}),\\
(a_0; b_0, \tilde{d}^{-1}, c_{-1}) &\mapsto (c_{-1}; \tilde{d}, b_{-1}, a_0).
\end{aligned}$$

So, under the symmetrie $\tilde{\rho}$ we have:

$$\tilde{\rho}(a_m) = a_{m+1}, \quad \tilde{\rho}(b_m) = b_{m+1}, \quad \tilde{\rho}(\tilde{d}) = \tilde{d}.$$

We obtain the following face and edge identifications:

$$\begin{aligned}
(\tilde{d}; c_m) &\mapsto (\tilde{d}^{-1}; b_{m+p+1}),\\
(\tilde{d}; b_m) &\mapsto (\tilde{d}^{-1}; a_{m+p+1}),\\
(c_m; \tilde{d}, a_{m+1}, b_m) &\mapsto (a_{m+1}; b_{m+1}, c_m, \tilde{d}^{-1}),\\
(b_m; \tilde{d}, c_m, \tilde{d}^{-1}, a_m) &\mapsto (b_{m+p}; c_{m+p}, \tilde{d}, a_{m+p}, \tilde{d}^{-1}),\\
(a_m; b_m, \tilde{d}^{-1}, c_{m-1}) &\mapsto (c_{m-1}; \tilde{d}, b_{m-1}, a_m).
\end{aligned}$$

We work out the edge cycles using the identifications and the combinatorics of the faces:

$$(c_m; \tilde{d}) \mapsto (b_{m+1}; a_{m+1}) \mapsto (\tilde{d}^{-1}; b_{m+p+1}),$$

$$(c_m; b_m) \mapsto (\tilde{d}^{-1}; a_{m+1}) \mapsto (b_{m-p}; \tilde{d}),$$

$$(c_m; a_{m+1}) \circlearrowleft,$$

compare with figure 5.18.

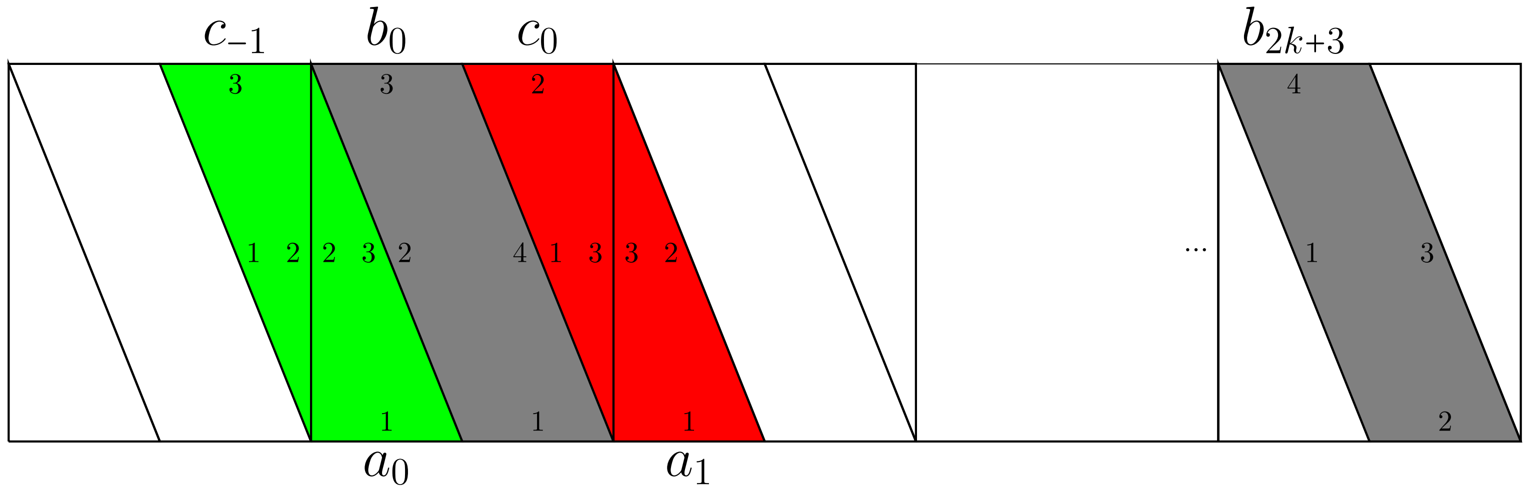

Figure 5.18: The surface of P for $\tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$.

∎

Now, we can apply Theorem 49 and we obtain the following result:

Theorem 83. *We have*

$$F_{\tilde{e}} = P.$$

Proof. Theorem 49 implies that $\Psi(P)$ is a fundamental domain for $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$. Moreover, we will show that the fundamental domains $\Psi(P)$ and $F_{\tilde{e}}$ coincide. Proposition 48 implies that $F_{\tilde{\Gamma}_1(u)\setminus\{u\}} = F_{\mathcal{E}} = F_{\tilde{e}}$. We know that $P = \mathrm{Cl}\,\mathrm{Int}(P)$ and $F_{\tilde{e}} = \mathrm{Cl}\,\mathrm{Int}(F_{\mathcal{E}})$. We also know that $P \subset F_{\mathcal{E}}$. This all implies that $P \subset F_{\tilde{e}}$ and hence

$$F_{\tilde{e}} = P$$

∎

Figures of Fundamental Domains for $\tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$

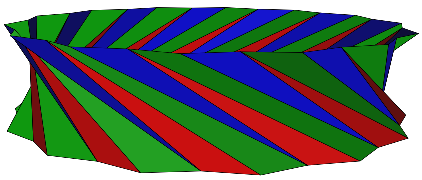

Figure 5.19: Fundamental Domain for $\tilde{\Gamma}(7,3,3)^2 \times (C_3)^2$.

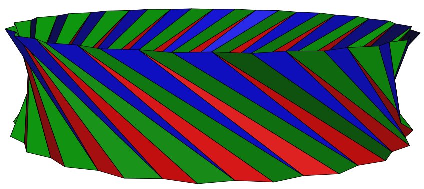

Figure 5.20: Fundamental Domain for $\tilde{\Gamma}(11,3,3)^4 \times (C_3)^4$.

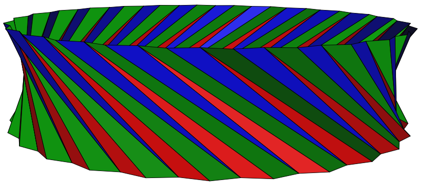

Figure 5.21: Fundamental Domain for $\tilde{\Gamma}(13, 3, 3)^5 \times (C_3)^5$.

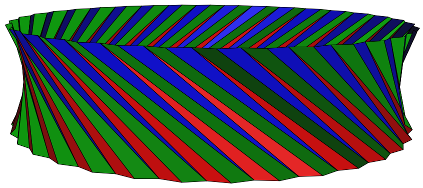

Figure 5.22: Fundamental Domain for $\tilde{\Gamma}(17, 3, 3)^7 \times (C_3)^7$.

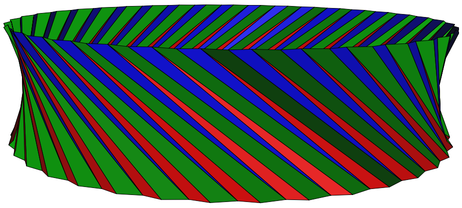

Figure 5.23: Fundamental Domain for $\tilde{\Gamma}(19, 3, 3)^8 \times (C_3)^8$.

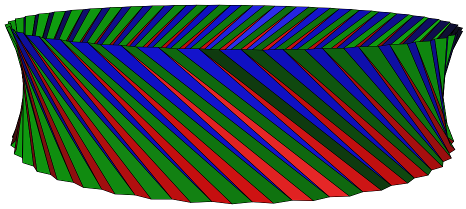

Figure 5.24: Fundamental Domain for $\tilde{\Gamma}(23, 3, 3)^{10} \times (C_3)^{10}$.

Chapter 6

On the link space of a $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularity

In this chapter we are going to look at our motivation for computing the fundamental domains for the groups $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$. Milnor [10] studied the link spaces of isolated singularities. Dolgachev [6] proved that the link space of a Gorenstein quasihomogeneous surface singularity is diffeomorphic to a quotient $\tilde{\Gamma}_1 \backslash \tilde{G}$. After that, Pratoussevitch [16] generalised the result of Dolgachev and proved the link space of a $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularity is diffeomorphic to a biquotient $\tilde{\Gamma}_1 \diagdown \tilde{G} \diagup \tilde{\Gamma}_2$.

6.1 Basic concepts

We are going to recall the basic concepts of singularity theory.

Definition 84. *We consider the space $\mathbb{C}^n$ with fixed coordinates $x_1, \cdots, x_n$. A holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is said to be quasihomogeneous of degree d with indices $\alpha_1, \cdots, \alpha_n$, if for any $\lambda > 0$ we have*

$$f(\lambda^{\alpha_1} x_1, \cdots, \lambda^{\alpha_n} x_n) = \lambda^d f(x_1, \cdots, x_n).$$

The index α_s is also called the weight of the variable x_s.

In terms of the Taylor series $f = \sum f_k x^k$ the condition of quasihomogeneity of degree 1 means that all the indices $(k_1, \cdots, k_n)$ of the non-zero terms $f(k_1, \cdots, k_n) x_a^{k_1} \cdots x_n^{k_n}$ of the series lie on a hyperplane

$$\gamma = \{K : \alpha_1 k_1 + \cdots + \alpha_n k_n = 1\}.$$

Recall the definitions of a Gorenstein singularity and a $\mathbb{Q}$-Gorenstein singularity:

Definition 85. *A normal isolated singularity of dimension n is Gorenstein if and only if there exists a nowhere vanishing $n-$form on a punctured neighbourhood of the singular point. For example all isolated singularities of complete intersections are Gorenstein.*

A natural generalisation of Gorenstein singularities are the $\mathbb{Q}$-Gorenstein singularities.

Definition 86. *A normal isolated singularity of dimension at least 2 is $\mathbb{Q}$-Gorenstein if there is a natural number r such that the divisor $r \cdot \mathcal{K}_X$ is defined on a punctured neighbourhood of the singular point by a function. Here $\mathcal{K}_X$ is the canonical divisor of X.*

The smallest such number r is called the *index* of the singularity. A normal isolated singularity is Gorenstein if and only if it is $\mathbb{Q}$-Gorenstein of index 1.

6.2 On the link spaces of Gorenstein and $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities

Now we are going to look at the definition of the link space of a singularity. As we mentioned, Milnor studied the topology of the link spaces of isolated singularities. We consider the following:
Let $f(z_1, \cdots, z_{n+1})$ be a non-constant polynomial in $n+1$ complex variables, and let V be the algebraic set consisting of all $(n+1)-$tuples $Z = (z_1, \cdots, z_{n+1})$ of complex number with $f(Z) = 0$. (Such a set called a complex hypersurface.) Let $K = V \cap S_\epsilon$ be the intersection of the hypersurface V with the small sphere S_ϵ centered at the given point Z_0. Now we are going to state the result introduced by Milnor [10].

Theorem 87. *For a small $\epsilon > 0$ the intersection of V with the ball D_ϵ of radius ϵ is homeomorphic to the cone over $K = V \cap S_\epsilon$. In fact the pair $(D_\epsilon, V \cap D_\epsilon)$ is homeomorphic to the pair consisting of the cone over S_ϵ and the cone over K. The space $K = V \cap S_\epsilon$ for sufficiently small ϵ is the link space of V.*
Here, by the cone over K, denoted Cone (K), we mean the union of all line segments $tk + (1-t)x^0$, $0 \le t \le 1$, joining a point $k \in K$ to the base point x^0. The set Cone (S_ϵ), defined similarly, is of course precisely equal to D_ϵ.

After we looked at the topology of the link space of singularities, we recall the result of Dolgachev [6].

Theorem 88. *Let $(X, 0)$ be a hyperbolic Gorenstein quasi-homogeneous surface singularity and M be its link space. Then M is diffeomorphic to a quotient $\tilde{\Gamma}_1 \backslash \tilde{G}$, where $\tilde{G}$ is the universal cover $\widetilde{PSU(1,1)}$ of the 3-dimensional Lie group $PSU(1,1)$, while $\tilde{\Gamma}_1$ is a discrete subgroup of finite level in $\tilde{G}$. Conversely, if M is the link space of a normal quasi-homogeneous surface singularity and M is diffeomorphic to a quotient as above, then M is the link space of a hyperbolic Gorenstein quasi-homogeneous surface singularity.*

Pratoussevitch [16] generalised the result of Dolgachev as follows:

Theorem 89. *The link space of a hyperbolic $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularity of index r is diffeomorphic to a biquotient*

$$\tilde{\Gamma}_1 \backslash \tilde{G} / \tilde{\Gamma}_2,$$

where $\tilde{G}$ is the universal cover $\widetilde{PSU(1,1)}$ of the 3-dimensional Lie group $PSU(1,1)$, while $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are discrete subgroups of the same finite level m in $\tilde{G}$, $\tilde{\Gamma}_1$ is co-compact, and the image of $\tilde{\Gamma}_2$ is a cyclic subgroup of order r. Conversely, any biquotient as above is diffeomorphic to the link space of a quasi-homogeneous hyperbolic $\mathbb{Q}$-Gorenstein singularity.

Our motivation for choosing the discrete subgroup $\tilde{\Gamma}_1$ is that it corresponds to some singularities in the series E and Z according to the classification by V.I.Arnold. We listed in the table below the normal form, level and signature of the image in $\mathrm{PSU}(1,1)$ for those subgroup of $\widetilde{\mathrm{PSU}(1,1)}$. For more details see [1] ,[4], [5] and [15].

Type	normal form	Restriction	Level k	Signature
E_{6j+2}	$x^3 + y^{3j+2} + axy^{2j+2}$	$j \geq 2$, j is even	$((6j+2)-10)/4$	(0,k+3,3,3)
E_{6j}	$x^3 + y^{3j+1} + axy^{2j+1}$	$j \geq 3$, j is odd	$((6j)-10)/4$	(0,k+3,3,3)
$Z_{12j+6i+1}$	$x^3 + bxy^{2j+2i+2} + y^{3j+3i+2}$	$j \geq 1$, $i \geq 0$, i is even	$((12j+6i+1)-9)/4$	(0,2k+3,3,3)
$Z_{12j+6i-1}$	$x^3 + bxy^{2j+2i+1} + y^{3j+3i+1}$	$j \geq 1$, $i \geq 1$, i is odd	$((12j+6i-1)-9)/4$	(0,2k+3,3,3)

Table 6.1: The normal form for $\tilde{\Gamma}_1$.

Chapter 7

Conclusion

In this thesis we have computed the fundamental domains for two series of groups of the form $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2$ acting on $\tilde{G}$ by left-right multiplication, i.e. $(g,h) \cdot x = gxh^{-1}$, where $\tilde{\Gamma}_1$ is a discrete subgroup of $\tilde{G}$ of the form $\tilde{\Gamma}(p,3,3)^k, p = kl+3, l = 1,2$ and $\tilde{\Gamma}_2$ is a cyclic discrete subgroup of $\tilde{G}$ of order 3. We can see clearly from figures 5.7, 5.8, 5.9, 5.10, 5.11 and 5.12 that the series of the form $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(k+3,3,3)^k \times (C_3)^k$ has similar fundamental domains. Similarly, from figures 5.19, 5.20, 5.21, 5.22, 5.23 and 5.24 we see that the series of the form $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(2k+3,3,3)^k \times (C_3)^k$ has similar fundamental domains. Note that while $\tilde{\Gamma}_1$ in all these cases are triangle groups $\tilde{\Gamma}(p,q,r)^k$, the combinatorics of fundamental domains for $\tilde{\Gamma}_1 = \tilde{\Gamma}(k+3,3,3)^k$ and $\tilde{\Gamma}_1 = \tilde{\Gamma}(2k+3,3,3)^k$ is different. Hence the combinatorics of fundamental domains computed here shows the structure of these groups which does not show in their fundamental domains in the hyperbolic plane. Clearly there is plenty of scope for future research. There are other infinite series of possible cases to study, for example the fundamental domains for the series of the form $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(p,3,2)^k \times (C_3)^k$ and $\tilde{\Gamma}_1 \times \tilde{\Gamma}_2 = \tilde{\Gamma}(p,4,2)^k \times (C_3)^k$. We also can consider the case where $\tilde{\Gamma}_2$ is a cyclic group of order 5 or 7.

Chapter 8

Appendix

This appendix is taken from [15].
In this appendix we shall collect all the formulae and facts of the Euclidean and hyperbolic trigonometry, linear algebra and analytic geometry which we often use in computations.

8.1 Trigonometry

$$\sin^2 x + \cos^2 x = 1 \tag{8.1}$$

$$\begin{aligned} \sin(x \pm y) &= \sin x \cos y \pm \sin y \cos x \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\ \tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \end{aligned} \tag{8.2}$$

$$\begin{aligned} \sin x \pm \tan y \cos x &= \frac{\sin(x \pm y)}{\cos y} \\ \cos x \pm \cot y \sin x &= \frac{\sin(y \pm x)}{\sin y} \end{aligned} \tag{8.3}$$

$$
\begin{aligned}
\sin 2x &= 2\sin x \cos x \\
\cos 2x &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \\
\tan 2x &= \frac{2\tan x}{1 - \tan^2 x} \\
\tan \frac{x}{2} &= \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}
\end{aligned} \tag{8.4}
$$

$$
\begin{aligned}
\sin 3x &= 3\sin x - 4\sin^3 x = \sin x(2\cos 2x + 1) \\
&= \sin x(3 - 4\sin^2 x) = \sin x(4\cos^2 x - 1) \\
\cos 3x &= 4\cos^3 x - 3\cos x = \cos x(2\cos 2x - 1) \\
&= \cos x(4\cos^2 x - 3) = \cos x(1 - 4\sin^2 x)
\end{aligned} \tag{8.5}
$$

$$
\begin{aligned}
\sin x \sin y &= \frac{1}{2}(\cos(x - y) - \cos(x + y)) \\
\cos x \cos y &= \frac{1}{2}(\cos(x - y) + \cos(x + y)) \\
\sin x \cos y &= \frac{1}{2}(\sin(x - y) + \sin(x + y))
\end{aligned} \tag{8.6}
$$

$$
\begin{aligned}
\sin x + \sin y &= 2\sin \frac{x + y}{2} \cos \frac{x - y}{2} \\
\sin x - \sin y &= 2\sin \frac{x - y}{2} \cos \frac{x + y}{2} \\
\cos x + \cos y &= 2\cos \frac{x - y}{2} \cos \frac{x + y}{2} \\
\cos x - \cos y &= -2\sin \frac{x + y}{2} \sin \frac{x - y}{2}
\end{aligned} \tag{8.7}
$$

$$
\begin{aligned}
\sin\left(\frac{\pi}{4} - x\right) &= \frac{1}{\sqrt{2}} \cdot (\cos x - \sin x) \\
\cos\left(\frac{\pi}{4} - x\right) &= \frac{1}{\sqrt{2}} \cdot (\sin x + \cos x) \\
\sin^2\left(\frac{\pi}{4} - x\right) &= \frac{1}{2} \cdot (1 - \sin 2x) \\
\cos^2\left(\frac{\pi}{4} - x\right) &= \frac{1}{2} \cdot (1 + \sin 2x)
\end{aligned} \tag{8.8}
$$

$$\tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x} = \frac{1 - \sin 2x}{\cos 2x} = \frac{\cos 2x}{1 + \sin 2x} \tag{8.9}$$

8.2 Hyperbolic trigonometry

$$\begin{aligned} &\cosh^2 x - \sinh^2 x = 1 \\ &\sinh x = \sqrt{\cosh^2 x - 1} \\ &\cosh x = \sqrt{\sinh^2 x + 1} \quad \text{for } x > 0 \end{aligned} \tag{8.10}$$

$$\begin{aligned} &\sinh 2x = 2 \sinh x \cosh x \\ &\cosh 2x = \sinh^2 x + \cosh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1 \end{aligned} \tag{8.11}$$

$$\begin{aligned} \sinh \frac{x}{2} &= \sqrt{\frac{\cosh x - 1}{2}} \\ \cosh \frac{x}{2} &= \sqrt{\frac{\cosh x + 1}{2}} \\ \tanh \frac{x}{2} &= \sqrt{\frac{\cosh x - 1}{\cosh x + 1}} \quad \text{for } x > 0 \end{aligned} \tag{8.12}$$

$$\begin{aligned} \sinh^2 x &= \frac{\tanh^2 x}{1 - \tanh^2 x} \\ \cosh^2 x &= \frac{1}{1 - \tanh^2 x} \\ \tanh^2 x &= 1 - \frac{1}{\cosh^2 x} \\ \coth^2 x &= \frac{\cosh^2 x}{\cosh^2 x - 1} \end{aligned} \tag{8.13}$$

8.3 Formulae for hyperbolic triangles

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma} \tag{8.14}$$

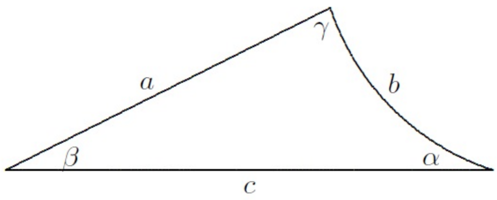

Figure 8.1: Hyperbolic triangle.

$$\cosh c = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta} \tag{8.15}$$

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma \tag{8.16}$$

8.4 Data of the triangle $\Delta(p_1, q, r)$

$\Delta(p_1, q, r)$ is the hyperbolic triangle with vertices $u, v, w \in \mathbb{D}$ and angles

$$\alpha_u = \frac{\pi}{p_1}, \quad \alpha_v = \frac{\pi}{q} \quad \text{and} \quad \alpha_w = \frac{\pi}{r}.$$

The lengths of the sides $\ell_v = \rho(u, v)$, $\ell_w = \rho(u, w)$ and $\ell_{vw} = \rho(v, w)$ can be computed from the angles using Formula (8.15).

In the case $\tilde{\Gamma}_1 = \tilde{\Gamma}(p_1, 3, 3)$ we have

$$\cosh \ell_v = \frac{1}{\sqrt{3}} \cdot \cot\alpha, \quad \sinh \ell_v = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{\cos 3\alpha}}{\sin\alpha\sqrt{\cos\alpha}}, \quad \tanh \ell_v = \frac{\sqrt{\cos 3\alpha}}{\cos\alpha\sqrt{\cos\alpha}},$$

where $\alpha = \frac{\pi}{p_1}$.

8.5 Hyperbolic geometry

Let ρ be the hyperbolic and d the Euclidean metric on $\mathbb{D}$. Let $a, b \in \mathbb{D}$, then

$$d^2(a,b) = \frac{\cosh\rho(0,a)-1}{\cosh\rho(0,a)+1} + \frac{\cosh\rho(0,b)-1}{\cosh\rho(0,b)+1} \\ -2\cdot\frac{\cosh\rho(0,a)\cosh\rho(0,b)-\cosh\rho(a,b)}{\sinh\rho(0,a)\sinh\rho(0,b)}\cdot\sqrt{\frac{(\cosh\rho(0,a)-1)(\cosh\rho(0,b)-1)}{(\cosh\rho(0,a)+1)(\cosh\rho(0,b)+1)}} \tag{8.17}$$

$$d(a,b) = \frac{\sqrt{2(\cosh\rho(a,b)-1)}}{\cosh\rho(0,a)+1}, \quad \text{if } \rho(0,a)=\rho(0,b) \tag{8.18}$$

$$\cosh\rho(0,a) = \frac{1+|a|^2}{1-|a|^2}, \quad |\sinh\rho(0,a)| = \frac{2|a|}{1-|a|^2}, \quad |a|^2 = \frac{\cosh\rho(0,a)-1}{\cosh\rho(0,a)+1} \tag{8.19}$$

8.6 Analytic geometry

Proposition 90. *Let us consider two lines and a closed halfplane H in $\mathbb{R}^2$ given by the equations/inequalities*

$$a_1x + b_1y = c_1,$$

$$a_2x + b_2y = c_2,$$

$$a_3x + b_3y \le c_3.$$

Let

$$\delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \qquad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

We assume that $\delta \neq 0$, i.e. the two lines intersect each other in exactly one point, say p. Then

$$p \in IntH \iff \delta \cdot \Delta > 0,$$
$$p \notin H \iff \delta \cdot \Delta < 0,$$
$$p \in \partial H \iff \Delta = 0.$$

Corollary 91. *We consider three lines in $\mathbb{R}^2$ given by the equations*

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$
$$a_3x + b_3y = c_3.$$

Let

$$\delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \qquad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

We assume that $\delta \neq 0$, i.e. the first two lines intersect each other in exactly one point, say p. Then all three lines meet at one point if and only if $\Delta = 0$.

Proposition 92. *Let $z = (z_1, z_2)$ be the coordinates in $\mathbb{C} \cong \mathbb{R}^2$. We consider the halfplanes H^+ and H^- in $\mathbb{C}$ given by the inequalities*

$$z \in H^- \Leftrightarrow z_1 \sin x + z_2 \cos x \geq c,$$
$$z \in H^+ \Leftrightarrow z_1 \sin y + z_2 \cos y \geq c,$$

then the bisector of the sector $H^- \cap H^+$ is given by the equation

$$(\sin x - \sin y)z_1 + (\cos x - \cos y)z_2 = 0$$

and hence contains the origin 0.

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