ON THE METRIC THEORY OF NUMBERS IN NON-ARCHIMEDEAN SETTINGS

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Abstract

This thesis is a contribution to some fields of the metrical theory of numbers in non-Archimedean settings. This is a branch of number theory that studies and characterizes sets of numbers, which occur in a locally compact topological field endowed with a non-Archimedean absolute value. This is done from a probabilistic or measure-theoretic point of view. In particular, we develop new formulations of ergodicity and unique ergodicity based on certain subsequences of the natural numbers, called Hartman uniformly distributed sequences. We use subsequence ergodic theory to establish a generalised metrical theory of continued fractions in both the settings of the p-adic numbers and the formal Laurent series over a finite field. We introduce the a-adic van der Corput sequence which significantly generalises the classical van der Corput sequence. We show that it provides a wealth of examples of low-discrepancy sequences which are very useful in the quasi-Monte Carlo method. We use our subsequential characterization of unique ergodicity to solve the generalised version of an open problem asked by O. Strauch on the distribution of the sequence of consecutive van der Corput sequences. In addition to these problems in ergodic methods and number theory, we employ some geometric measure theory to settle the positive characteristic analogue of an open problem asked by R.D. Mauldin on the complexity of the Liouville numbers in the field of formal Laurent series over a finite field by giving a complete characterization of all Hausdorff measures of the set of Liouville numbers.

Contents

Introduction

Metric number theory is a branch of number theory which studies and characterizes sets of numbers with fixed arithmetic properties from a probabilistic or measure-theoretic point of view. The central theme of this theory is to determine whether or not a given property holds everywhere except on an exceptional set of measure zero. In addition, the metric theory of numbers includes the study of the complexity of those exceptional sets in terms of Hausdorff dimension. Nowadays, the theory is deeply intertwined with measure theory, ergodic theory, dynamical systems, fractal geometry and other areas of mathematics. This thesis emphasizes the study of non-Archimedean settings, such as the a-adic integers, the p-adic numbers and the formal Laurent series over a finite field, which possess a different geometric nature from the classical real numbers. It also aims to make contributions to other fields, including subsequence ergodic theory, continued fractions, diophantine approximation and uniform distribution theory, of the non-standard metric number theory.

To begin with, we consider a locally compact topological field X endowed with a non-Archimedean absolute value $|\cdot|$ defined on it. This is a map of X into the nonnegative real numbers with the following three properties:

(1) $|x| = 0$ if and only if $x = 0$;

(2) $|xy| = |x||y|$;

(3) $|x + y| \leq \max(|x|, |y|)$.

The a-adic integers, the p-adic numbers and the formal Laurent series over a finite field are some examples of a non-Archimedean space.

The familiar triangle inequality $|x+y| \leq |x|+|y|$ can be thought of as an expression of the Euclidean axiom that the shortest distance between two points is a straight line. In our non-Archimedean setting, the absolute value however satisfies a stronger inequality, called the ultrametric triangle inequality: $|x+y| \leq \max(|x|, |y|)$. It produces a geometry that is quite different from that produced by the ordinary triangle inequality. Some of the ways in which it differs is relevant to this thesis.

Any locally compact topological group X is endowed with a translation invariant Haar measure. This makes the space interesting for the study of the metrical theory of

numbers. We refer to elements of the field X as numbers because their elements in a very precise sense are like the integers, the rational numbers and the irrational numbers in the real numbers.

In spite of their strange structure, the non-Archimedean spaces turn out to be of particular interest and importance in number theory and other areas of mathematics. We shall see some motivation and applications to several topics in the beginning part of each chapter. We omit them here to make the thesis a reasonable size.

This thesis is a product of the following publications:

[A] J. Hančl, A. Jaššová, P. Lertchoosakul and R. Nair. *On the metric theory of p-adic continued fractions.* Indag. Math. (N.S.), 24(1):42–56, 2013.

[B] J. Hančl, A. Jaššová, P. Lertchoosakul and R. Nair. *Polynomial actions in positive characteristic.* Proc. Steklov Inst. Math., 280(suppl.2):37–42, 2013.

[C] J. Hančl, A. Jaššová, P. Lertchoosakul and R. Nair. *On the quantitative metric theory of continued fractions.* Preprint 2014.

[D] J. Hančl, A. Jaššová, P. Lertchoosakul and R. Nair. *Polynomial actions in positive characteristic II.* Preprint 2014.

[E] A. Jaššová, S. Kristensen, P. Lertchoosakul and R. Nair. *On recurrence in positive characteristic.* Preprint 2014.

[F] A. Jaššová, P. Lertchoosakul and R. Nair. *On variants of the Halton sequence.* Preprint 2014.

[G] P. Lertchoosakul and R. Nair. *Distribution functions for subsequences of the van der Corput sequence.* Indag. Math. (N.S.), 24(3):593–601, 2013.

[H] P. Lertchoosakul and R. Nair. *On the complexity of the Liouville numbers in positive characteristic.* Q.J. Math., 65(2):439–457, 2014.

[I] P. Lertchoosakul and R. Nair. *On the metric theory of continued fractions in positive characteristic.* Mathematika, 60(2):307–320, 2014.

[J] P. Lertchoosakul an R. Nair. *Quantitative metric theory of continued fractions in positive characteristic.* Preprint, 2014.

Of necessity only a part of this work appears in this thesis. A summary of the excluded work can be found in the appendix.

In this thesis, we refer to work of the author by a capital letter in brackets [A]–[J]. The results without any reference are new and have not appeared in other publications.

To make the thesis relatively self-contained, we put in Chapter 1 a concise review of background material necessary for understanding the metrical theory of numbers in non-Archimedean settings. This preliminary chapter is also meant to collect all the definitions and results which will be referred to in the sequel. It includes basic notions and well-known results from measure theory, ergodic theory, uniform distribution theory of sequences and non-Archimedean spaces.

In Chapter 2, we learn some subsequence ergodic theory. In particular, we develop new formulations of ergodicity and unique ergodicity based on certain subsequences of the natural numbers, called Hartman uniformly distributed sequences. These new ideas appear in Theorem 2.3.2 and Theorem 2.5.1. They will prove fruitful later in our metrical studies of continued fractions in the positive characteristic setting and the p-adic setting and of the distribution of some low-discrepancy sequences.

In Chapter 3, we introduce the Liouville numbers in positive characteristic which play an important role in Diophantine approximation in this setting. We investigate the complexity of the Liouville numbers in terms of measure and dimension. Particularly, we show that the set of Liouville numbers is large in the sense that it is uncountable and dense in the field of formal Laurent series; nevertheless, we show further that the set of Liouville numbers is small in the sense that it has Haar measure zero and Hausdorff dimension zero. By using some geometric measure theory, we prove Theorem 3.5.1 which gives a complete characterization of all Hausdorff measures of the set of Liouville numbers in positive characteristic. This settles the positive characteristic analogue of an open problem raised by R.D. Mauldin.

In Chapter 4, we introduce the theory of continued fractions in the field of formal Laurent series. In Theorem 4.4.1, we prove that the continued fraction map is exact with respect to Haar measure. This fact of exactness implies a number of strictly weaker properties. In particular, we then use the weak-mixing property and ergodicity, together with some subsequence and moving average ergodic theory, to establish Theorem 4.6.1 and Theorem 4.7.1 which provide a generalised metric theory of continued fractions in positive characteristic. We add further to the theory by employing Gál and Koksma's method to establish Theorem 4.9.1 which gives a quantitative metric theory of continued fractions in positive characteristic.

In Chapter 5, we study an analogue of the regular continued fraction expansion for the p-adic numbers which was given by T. Schneider. In Theorem 5.4.1, we prove that Schneider's continued fraction map is exact with respect to Haar measure. This fact of exactness implies a number of strictly weaker properties. Again, we use the weak-mixing property and ergodicity, together with some machinery from subsequence ergodic theory, to establish Theorem 5.6.1 and Theorem 5.7.1 which give a generalised metric theory of p-adic continued fractions.

In Chapter 6, we generalise the classical notion of van der Corput sequences to the a-adic van der Corput sequence. In Theorem 6.3.2, we prove that this newly defined sequence provides a wealth of low-discrepancy sequences which are very useful in numerical integration. Then we give another construction of the a-adic van der Corput sequence using a generalization of the Kakutani-von Neumann odometer. Moreover, we show that the generalised Kakutani-von Neumann odometer is uniquely ergodic. Finally, we use our subsequential characterization of unique ergodicity to establish Theorem 6.4.1 which solves the generalised version of an open problem asked by O. Strauch on the distribution of the sequence of consecutive van der Corput sequences.

Chapter 1

Preliminaries

This chapter is meant to provide a concise review of background materials necessary for understanding the metrical theory of numbers in non-Archimedean settings. More importantly, it aims to collect together in one place all the definitions and results which will be referred to in the sequel. The first four Sections 1.1–1.4 develop the basic notions of measure theory, which include measure spaces, examples of measures, integration and function spaces. They are what is needed to introduce the concepts of measure-preserving dynamical system and ergodicity in Section 1.5. Then we investigate two further specialized subfields of ergodic theory in the next two sections. In fact, Section 1.6 introduces the spectral study of the Koopman operator induced by a measure-preserving transformation. Section 1.7 specializes the study of invariant measures to continuous transformations. In Section 1.8, we summarize the basic ideas of the classical theory of uniform distribution. In the last two sections, we discuss the non-Archimedean spaces on which this thesis is based. In particular, Section 1.9 introduces the a-adic integers, and Section 1.10 introduces the p-adic numbers and the fields of formal Laurent series.

We shall refer to the following textbooks [10], [14], [20], [27], [54] and [58] as standard references. All well-known results we quote can be found in these references.

We now list some notation which will be used repeatedly throughout the thesis:

- We use the notation $\mathbb{N}_0$ and $\mathbb{N}_{>1}$ to denote the set of non-negative integers and the set of natural numbers greater than 1, respectively.
- Given a set E, we denote by $\#E$ its cardinality, and we define the characteristic function of E by $\mathbb{1}_E(x) = 1$ if $x \in E$ and $\mathbb{1}_E(x) = 0$ if $x \notin E$.
- For any real number α, we define $e(\alpha) = e^{2\pi i\alpha}$.
- Given two real-valued functions f and g, we write $f = O(g)$ or $f \ll g$ if $|f| < c|g|$ for some positive constant c. We write $f = o(g)$ if $\lim_{x\to\infty} f(x)/g(x) = 0$, and we say that f and g are comparable if $f = O(g)$ and $g = O(f)$.

1.1 Measure spaces

Definition 1.1.1. *Let X be a non-empty set. A collection $\mathcal{B}$ of subsets of X is called a* σ-algebra *if it has the following three properties:*

(1) $\emptyset \in \mathcal{B}$;

(2) for any $E \in \mathcal{B}$, we have $X \setminus E \in \mathcal{B}$;

(3) for any countable collection $\{E_n\}_{n=1}^{\infty}$ of sets in $\mathcal{B}$, we have $\bigcup_{n=1}^{\infty} E_n \in \mathcal{B}$.

We call $(X, \mathcal{B})$ a measurable space.

Clearly, if $\mathcal{B}$ is a σ-algebra of subsets of X, then it is easy to see that $X \in \mathcal{B}$ and that $\mathcal{B}$ is also closed under taking countable intersections. If X is a compact metric space, the *Borel σ-algebra* of subsets of X is the smallest σ-algebra that contains every open subset of X. An element of the Borel σ-algebra is called a *Borel set*.

Definition 1.1.2. *Let $(X, \mathcal{B})$ be a measurable space. A function $\mu : \mathcal{B} \to [0, \infty]$ is said to be a* measure *if it satisfies the following two conditions:*

(1) $\mu(\emptyset) = 0$;

(2) for any countable collection $\{E_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{B}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

We call $(X, \mathcal{B}, \mu)$ a measure space.

It is an immediate consequence of the condition (2) in Definition 1.1.2 that, for any $E, F \in \mathcal{B}$ such that $E \subseteq F$, we have $\mu(E) \leq \mu(F)$.

Definition 1.1.3. *Let $(X, \mathcal{B}, \mu)$ be a measure space. Whenever $\mu(X) = 1$, we call μ a* probability measure *and refer to $(X, \mathcal{B}, \mu)$ as a* probability space.

Definition 1.1.4. *Let $(X, \mathcal{B}, \mu)$ be a measure space. The space X is said to be* σ-finite *if there exist countably many sets $E_n \in \mathcal{B}$ such that $X = \bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n) < \infty$.*

Definition 1.1.5. *We shall say that a property holds* almost everywhere *if the set of points on which the property fails to hold has measure zero.*

In order to define a measure, it is necessary to define the measure of every set in the σ-algebra under consideration. This is usually impractical; instead, we seek a method that allows us to define a measure on an easily managed subcollection of subsets and then extend it to the required σ-algebra.

Definition 1.1.6. *Let X be a nonempty set. A collection $\mathcal{A}$ of subsets of X is called an* algebra *if it has the following three properties:*

(1) $\emptyset \in \mathcal{A}$;

(2) for any $E \in \mathcal{A}$, we have $X \setminus E \in \mathcal{A}$;

(3) for any two sets E_1 and E_2 in $\mathcal{A}$, we have $E_1 \cup E_2 \in \mathcal{A}$.

We define the σ-algebra generated by $\mathcal{A}$ *to be the smallest σ-algebra containing $\mathcal{A}$ and denote it by $\mathcal{B}(\mathcal{A})$.*

Theorem 1.1.7 (Kolmogorov Extension Theorem)**.** *Let X be a set, and let $\mathcal{A}$ be an algebra of subsets of X. Suppose $\mu^* : \mathcal{A} \to [0, \infty]$ satisfies the following three properties:*

(1) $\mu^*(\emptyset) = 0$;

(2) for any countable collection $\{E_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$, we have

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu^*(E_n);$$

(3) there are countably many sets $E_n \in \mathcal{A}$ such that $X = \bigcup_{n=1}^{\infty} E_n$ and $\mu^*(E_n) < \infty$.

Then there exists a unique measure $\mu : \mathcal{B}(\mathcal{A}) \to [0, \infty]$ which is an extension of μ^*.

The Kolmogorov extension theorem says that if we have a function that looks like a measure on an algebra $\mathcal{A}$, then it is indeed a measure when it is extended to $\mathcal{B}(\mathcal{A})$. The important hypotheses are (1) and (2), while the hypothesis (3) is a technical assumption saying the space X is σ-finite. Note that we shall often use the Kolmogorov extension theorem with the algebra of finite unions of cylinder sets which we define presently.

For each $i \in \mathbb{N}$, let $(X_i, \mathcal{B}_i, \mu_i)$ be a probability space. Let $X = \prod_{i=1}^{\infty} X_i$, so that a point of X is a sequence $(x_i)_{i=1}^{\infty}$ with $x_i \in X_i$ for each i. We now define a σ-algebra $\mathcal{B}$ of subsets of X as follows. Let $n \in \mathbb{N}$, let $E_j \in \mathcal{B}_j$ $(1 \leq j \leq n)$, and consider the set

$$\prod_{j=1}^{n} E_j \times \prod_{i=n+1}^{\infty} X_i = \{(x_i)_{i=1}^{\infty} \in X : x_j \in E_j,\ 1 \leq j \leq n\}.$$

Let $\mathcal{A}$ denote the algebra of finite unions of all such subsets of X. The σ-algebra $\mathcal{B}$ is the σ-algebra generated by $\mathcal{A}$. We write $(X, \mathcal{B}) = \prod_{i=1}^{\infty}(X_i, \mathcal{B}_i)$. If we define $\mu^* : \mathcal{A} \to [0, \infty]$ by giving the above rectangle the value $\prod_{j=1}^{n} \mu_j(E_j)$, then we can use the Kolmogorov extension theorem to extend μ^* to a probability measure μ, called the *product measure*, on $(X, \mathcal{B})$. The probability space $(X, \mathcal{B}, \mu)$ is called the *product space* of the $(X_i, \mathcal{B}_i, \mu_i)$.

A special type of product space will be important for us. Here each space $(X_i, \mathcal{B}_i, \mu_i)$ is the same space $(Y, \mathcal{C}, \nu)$, where Y is a countable set $\{y_k\}_{k=1}^{\infty}$ and ν is given by a probability sequence $(p_k)_{k=1}^{\infty}$ with $p_k = \nu(\{y_k\})$. We now take elementary rectangles where each E_j, in the description above, is taken to be one point of Y. That is, for each $n \in \mathbb{N}$ and $a_j \in Y$ $(1 \leq j \leq n)$, we consider the set

$$\Delta_{a_1,\ldots,a_n} = \{(x_i)_{i=1}^{\infty} \in X \colon x_j = a_j,\ 1 \leq j \leq n\}$$

and call it a *cylinder set* of length n. The algebra of finite unions of all cylinder sets generates the product σ-algebra $\mathcal{B}$. We have $\mu(\Delta_{a_1,\ldots,a_n}) = \prod_{j=1}^{n} p_j$.

1.2 Examples of measures

Lebesgue measure: Take $X = [0,1]$, and let $\mathcal{A}$ denote the algebra of all finite unions of subintervals of $[0,1]$. For an interval $[a,b]$, define $\lambda^*([a,b]) = b - a$ and extend this to $\mathcal{A}$. Then this satisfies the hypotheses of the Kolmogorov extension theorem, and so it defines a Borel probability measure. This is the *Lebesgue measure* on $[0,1]$.

In a similar way, we can define Lebesgue measure on $[0,1]^n$. An n-dimensional cube is a set of the form $[a_1, b_1] \times \cdots \times [a_n, b_n]$, where $0 \leq a_j \leq b_j \leq 1$ for each $1 \leq j \leq n$. Let $\mathcal{A}$ denote the algebra of all finite unions of n-dimensional cubes. Define

$$\lambda_n^*([a_1, b_1] \times \cdots \times [a_n, b_n]) = \prod_{j=1}^{n} (b_j - a_j)$$

and extend this to $\mathcal{A}$. Again, this satisfies the hypotheses of the Kolmogorov extension theorem and defines the n-dimensional *Lebesgue measure* on $[0,1]^n$.

Throughout this thesis, we shall reserve λ_n for the n-dimensional Lebesgue measure.

Stieltjes measure: Take $X = [0,1]$, let $\mathcal{A}$ denote the algebra of finite unions of subintervals of $[0,1]$, and let $\nu : [0,1] \to [0,\infty]$ be a non-decreasing function. For an interval $[a,b] \subseteq [0,1]$, define $\mu_\nu^*([a,b]) = \nu(b) - \nu(a)$ and extend this to $\mathcal{A}$. Then the Kolmogorov extension theorem extends μ_ν^* to a Borel measure.

A wide range of measures can be constructed using this method. Lebesgue measure can also be viewed as a special example of this construction, that is, by taking $\nu(x) = x$. A more interesting example that will prove useful when we study continued fractions is given by taking

$$\nu(x) = \frac{1}{\log 2} \int_0^x \frac{dx}{1+x}.$$

This measure μ_ν is called the *Gauss measure*. It is worth noting that Gauss measure and Lebesgue measure are comparable in the sense that they have the same null sets.

Dirac measure: Let $(X, \mathcal{B})$ be any measurable space, and let $x \in X$. For any $E \in \mathcal{B}$, we define the measure

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We call δ_x the *Dirac δ-measure* supported at x.

Haar measure: Let X be a locally compact topological group. There exists a way to assign a probability measure to subsets of X, which ties in with its group structure, and subsequently to define an integral for functions on X.

Theorem 1.2.1 (Haar Theorem)**.** *Let X be a locally compact topological group. There is a unique, up to a positive multiplicative constant, measure μ on the Borel σ-algebra $\mathcal{B}$ of subsets of X satisfying the following four properties:*

(1) for any $x \in X$ and $E \in \mathcal{B}$, we have $\mu(x + E) = \mu(E)$;

(2) for any compact set $K \subseteq X$, we have $\mu(K) < \infty$;

(3) for any $E \in \mathcal{B}$, we have $\mu(E) = \inf\{\mu(U) \colon E \subseteq U,\ U \text{ open}\}$;

(4) for any open set $E \subseteq X$, we have $\mu(E) = \sup\{\mu(K) \colon K \subseteq E,\ K \text{ compact}\}$.

This unique measure μ is called *Haar measure* on X.

Hausdorff measure and dimension: Let (X, d) be a metric space, and let $s \geq 0$. The *s-dimensional Hausdorff measure* of any set $E \subseteq X$ is defined by

$$\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \left(\inf \left\{ \sum_{j=1}^{\infty} (\mathrm{diam} B_j)^s \colon E \subseteq \bigcup_{j=1}^{\infty} B_j,\ \mathrm{diam} B_j < \delta \right\} \right) \in [0, \infty],$$

where the infimum is taken over all countable covers of E by balls with diameter less than or equal to δ. Here, $\mathrm{diam} B_j = \sup\{d(x, y) \colon x, y \in B_j\}$. It is worth noting that if $X = \mathbb{R}^n$, then λ_n and $\mathcal{H}^n$ are comparable, and noting that if X is a compact topological group with Haar measure μ, then μ and $\mathcal{H}^1$ are comparable.

We define here the *Hausdorff dimension* of any set $E \subseteq X$ to be

$$\dim_{\mathrm{H}} E = \inf\{s \geq 0 \colon \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 \colon \mathcal{H}^s(E) > 0\}.$$

The notion of Hausdorff dimension will be useful for the study of fractal sets; i.e., it is used to investigate the complexity of sets with non-integer dimension in the sense that the more complicated sets have bigger Hausdorff dimension. For an introduction to the subject of Hausdorff measure and dimension, the reader is referred to [12] and [47].

1.3 Integration

Let $(X, \mathcal{B}, \mu)$ be a measure space. We give a brief introduction to integration on X.

Definition 1.3.1. *A function $f : X \to \mathbb{R}$ is said to be* measurable *if $f^{-1}(E) \in \mathcal{B}$ for every Borel subset E of $\mathbb{R}$, or equivalently, if $f^{-1}((c, \infty)) \in \mathcal{B}$ for all $c \in \mathbb{R}$. A function $f : X \to \mathbb{C}$ is said to be* measurable *if both its real and imaginary parts are measurable.*

We now define integration via simple functions.

Definition 1.3.2. *A function $f : X \to \mathbb{R}$ is said to be* simple *if it can be written as a linear combination of characteristic functions of sets in $\mathcal{B}$:*

$$f = \sum_{i=1}^{r} a_i \mathbb{1}_{E_i}$$

for some $a_i \in \mathbb{R}$ and $E_i \in \mathcal{B}$, where E_i are pairwise disjoint, and $\mathbb{1}_{E_i}$ denotes the characteristic function of E_i.

Simple functions are measurable. We define the integral for simple functions by

$$\int_X f \, d\mu = \sum_{i=1}^{r} a_i \mu(E_i).$$

This value can be shown to be independent of the representation of f as a simple function. Thus, for simple functions, the integral can be thought of as being defined to be the area underneath the graph.

Suppose that $f : X \to \mathbb{R}$ is measurable and $f \geq 0$. Then there exists an increasing sequence of simple functions $(f_n)_{n=1}^{\infty}$ which converges pointwise to f. We define

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.$$

Note that this definition is independent of the choice of sequence $(f_n)_{n=1}^{\infty}$.

For an arbitrary measurable function $f : X \to \mathbb{R}$, we write $f = f^+ - f^-$, where $f^+ = \max(f, 0) \geq 0$ and $f^- = \max(-f, 0) \geq 0$. Define

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

Finally, for a measurable complex-valued function $f : X \to \mathbb{C}$, we define

$$\int_X f \, d\mu = \int_X \mathrm{Re}(f) \, d\mu + i \int_X \mathrm{Im}(f) \, d\mu,$$

where $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ denote the real part and the imaginary part of f, respectively.

Definition 1.3.3. *A real-valued or complex-valued function f defined on X is said to be* integrable *if $\int_X |f| \, d\mu < \infty$.*

Observe that f is integrable if and only if $|f|$ is integrable. If $f = g$ μ-almost everywhere, then one is integrable if the other is and $\int_X f\,d\mu = \int_X g\,d\mu$. The space of integrable functions is defined to be

$$L^1(\mu) = \left\{ f : X \to \mathbb{C} \colon f \text{ measurable}, \int_X |f|\,d\mu < \infty \right\},$$

where two integrable functions are identified if they are equal μ-almost everywhere.

The following three basic theorems will be useful later. The first result establishes an inequality relating the integral of the limit inferior of a sequence of functions to the limit inferior of integrals of these functions, the second one is a generalised version of the dominated convergence theorem, which basically justifies the passage of the limit under the integral sign when $p = 1$, and the last one allows the order of integration to be changed in iterated integrals.

Theorem 1.3.4 (Fatou's Lemma)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable real-valued functions on X which is bounded below by an integrable function. If $\liminf_{n\to\infty} \int_X f_n\,d\mu < \infty$, then $\liminf_{n\to\infty} f_n$ is integrable and $\int_X \liminf_{n\to\infty} f_n\,d\mu \leq \liminf_{n\to\infty} \int_X f_n\,d\mu$.*

Theorem 1.3.5 (Dominated Convergence Theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space, let $1 \leq p < \infty$ be a real number, and let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable real-valued functions on X converging μ-almost everywhere to a measurable function $f : X \to \mathbb{R}$. Suppose there is a function $g : X \to \mathbb{R}$ in $L^p(\mu)$ such that $|f_n(x)| \leq g(x)$ for μ-almost everywhere $x \in X$. Then $f \in L^p(\mu)$ and $\lim_{n\to\infty} \|f_n - f\|_p = 0$.*

Theorem 1.3.6 (Fubini's Theorem)**.** *Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be two σ-finite measure spaces. Suppose that $f : X \times Y \to \mathbb{R}$ is an integrable function with respect to the product measure $\mu_X \times \mu_Y$. Then we have*

$$\int_{X\times Y} f\,d(\mu_X \times \mu_Y) = \int_X \int_Y f(x,y)\,d\mu_Y(y)\,d\mu_X(x) = \int_Y \int_X f(x,y)\,d\mu_X(x)\,d\mu_Y(y).$$

1.4 Function Spaces

Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $p \in \mathbb{R}$ with $p \geq 1$. Consider the set of all measurable functions $f : X \to \mathbb{C}$ with $|f|^p$ integrable. This space is a vector space under the usual addition and scalar multiplication of functions. If we define an equivalence relation on this set by identifying two such functions when they are equal μ-almost everywhere, then the space of equivalence classes is also a vector space. Let $L^p(\mu)$ denote the space of these equivalence classes of functions f such that $|f|^p$ is integrable. This is called the L^p *space*. We write $f \in L^p(\mu)$ to mean that the function $f : X \to \mathbb{C}$ has $|f|^p$ integrable. The expression $\|f\|_p = (\int_X |f|^p\,d\mu)^{1/p}$ defines a norm on $L^p(\mu)$, and

this norm is complete. Therefore, $L^p(\mu)$ is a Banach space. The bounded measurable functions are dense in $L^p(\mu)$. It is worth mentioning that if $\mu(X) < \infty$ and $1 \leq p < q$, then we have $L^q(\mu) \subseteq L^p(\mu)$.

The Banach space $L^p(\mu)$ is a Hilbert space if and only if $p = 2$. The inner product in $L^2(\mu)$ is given by $\langle f, g \rangle = \int_X f\overline{g}\, d\mu$. We note the *Schwarz inequality*, which says that

$$|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2$$

for all $f, g \in L^2(\mu)$.

1.5 Ergodic theory

Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $T : X \to X$ be a transformation on X. We shall call $(X, \mathcal{B}, \mu, T)$ a *dynamical system*.

Definition 1.5.1. *The dynamical system $(X, \mathcal{B}, \mu, T)$ is said to be* measure-preserving *if, for every $E \in \mathcal{B}$, we have $\mu(T^{-1}E) = \mu(E)$. If the transformation $T : X \to X$ has the further property that, whenever $E \in \mathcal{B}$ satisfies $T^{-1}E = E$, we have $\mu(E) = 0$ or $\mu(E) = 1$, then T is called* ergodic.

Ergodicity can be viewed as an indecomposability condition. If ergodicity does not hold, then there exists a set $E \in \mathcal{B}$ such that $T^{-1}E = E$ and $0 < \mu(E) < 1$. We can then split $T : X \to X$ into $T : E \to E$ and $T : X \setminus E \to X \setminus E$ with invariant probability measures $\frac{1}{\mu(E)}\mu(A_1 \cap E)$ and $\frac{1}{1-\mu(E)}\mu(A_2 \cap (X \setminus E))$ for $A_1 \in \mathcal{B}(E)$ and $A_2 \in \mathcal{B}(X \setminus E)$, respectively.

There are several ways of stating the ergodicity condition, the reader can consult [10], [53] and [58]. The following result from [58, Theorem 1.6] characterizes ergodicity in terms of an operator $U_T f = f \circ T$.

Lemma 1.5.2. [58] *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system. Then the following statements are equivalent:*

(1) T is ergodic.

(2) Whenever f is measurable and $(f \circ T)(x) = f(x)$ for all $x \in X$, we have f is constant μ-almost everywhere.

(3) Whenever f is measurable and $(f \circ T)(x) = f(x)$ for μ-almost everywhere $x \in X$, we have f is constant μ-almost everywhere.

(4) Whenever f is in $L^2(\mu)$ and $(f \circ T)(x) = f(x)$ for all $x \in X$, we have f is constant μ-almost everywhere.

(5) *Whenever f is in $L^2(\mu)$ and $(f \circ T)(x) = f(x)$ for μ-almost everywhere $x \in X$, we have f is constant μ-almost everywhere.*

A similar characterization in terms of $L^p(\mu)$ functions is true for any $p \geq 1$.

We now introduce the following famous ergodic theorem which is a consequence of measure preservation and ergodicity.

Theorem 1.5.3 (Birkhoff Ergodic Theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system, and let $f \in L^1(\mu)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n \alpha)$$

exists μ-almost everywhere $\alpha \in X$. If $(X, \mathcal{B}, \mu, T)$ is an ergodic dynamical system, then the limit equals $\int_X f \, d\mu$ for μ-almost everywhere $\alpha \in X$.

It is worth mentioning a way to further analyze the properties of a dynamical system is to find an isomorphism with a well-understood ergodic or uniquely ergodic system.

Definition 1.5.4. *Let $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ be two measure-preserving dynamical systems. We say that $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ are* metrically isomorphic *if there exist two sets $E_1 \in \mathcal{B}_1$ and $E_2 \in \mathcal{B}_2$, with $\mu_1(E_1) = 1$, $\mu_2(E_2) = 1$, $T_1(E_1) \subseteq E_1$ and $T_2(E_2) \subseteq E_2$, and if there exists an invertible measure-preserving transformation $\varphi : (E_1, \mathcal{B}_1, \mu_1) \to (E_2, \mathcal{B}_2, \mu_2)$ such that $(T_2 \circ \varphi)(x) = (\varphi \circ T_1)(x)$ for all $x \in E_1$.*

The ergodicity, all mixing properties and unique ergodicity are preserved under a metrical isomorphism. See [10, p 9–11] and [58, p 57–67] for more details.

1.6 Associated isometries and spectral theory

Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system. For each $p \geq 1$, $T : X \to X$ induces the *Koopman operator* $U_T : L^p(\mu) \to L^p(\mu)$ defined by

$$(U_T f)(x) = f(Tx)$$

for $f \in L^p(\mu)$ and $x \in X$. Because T is measure-preserving, for each function $f \in L^1(\mu)$, we have $\int_X f \circ T \, d\mu = \int_X f \, d\mu$ and so in particular $\|U_T f\|_p = \|f\|_p$. This means that U_T is a linear isometry[1] on the Hilbert space $L^2(\mu)$. The study of U_T is usually called the spectral study of T, and we shall see later how this is useful in proving some results relating to ergodicity.

[1] An *isometry* of a Hilbert space H is a linear operator U such that $\langle Uv, Uw \rangle = \langle v, w \rangle$ for all $v, w \in H$. We say that U is *unitary* if, in addition, it is invertible. Recall that $L^2(\mu)$ is a Hilbert space with respect to the inner product $\langle f, g \rangle = \int_X f\overline{g} \, d\mu$.

Definition 1.6.1. *A sequence of complex numbers* $(a_n)_{n=-\infty}^{\infty}$ *is called* positive definite *if, for any sequence* $(z_n)_{n=-\infty}^{\infty}$ *in* $\mathbb{C}$ *with only a finite number of non-zero terms,*

$$\sum_{n,\,m} a_{n-m} z_n \overline{z_m} \geq 0.$$

Theorem 1.6.2 (Bochner-Herglotz Theorem)**.** *A sequence* $(a_n)_{n=-\infty}^{\infty}$ *in* $\mathbb{C}$ *is positive definite if and only if there exists a finite measure* ω *on the torus* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ *such that*

$$a_n = \int_{\mathbb{T}} z^n \, d\omega(z).$$

If we consider $U_T : L^2(\mu) \to L^2(\mu)$, and if we denote the adjoint of U_T by U_T^{-1}, then the sequence $(\langle U_T^n f, f\rangle)_{n=-\infty}^{\infty}$ is positive definite, see e.g. [10, p 26–29]. Therefore, by the Bochner-Herglotz theorem, there is a measure ω_f satisfying

$$\langle U_T^n f, f\rangle = \int_{\mathbb{T}} z^n \, d\omega_f(z).$$

The measure ω_f is called the *spectral measure*. It is worth noting another consequence of measure preservation of T that $\langle U_T^n f, U_T^m f\rangle = \langle U_T^{n-m} f, f\rangle$ for all $n, m \in \mathbb{Z}$.

1.7 Invariant measures for continuous transformations

Let X be a compact metric space equipped with the Borel σ-algebra $\mathcal{B}$. We denote by $M(X)$ the collection of all probability measures defined on X. It is well-known that $M(X)$ is convex[2] and compact in the weak* topology[3], see [58, p 146–150]. Let $T : X \to X$ be a continuous transformation. It is clear that T is measurable. Recall that a probability measure $\mu \in M(X)$ is preserved by T if, for each $E \in \mathcal{B}$, $\mu(T^{-1}E) = \mu(E)$. Let $M(X,T)$ denote the set of probability measures defined on X and preserved by T. Again, $M(X,T)$ is a convex non-empty compact subset of $M(X)$ in the weak* topology, see [58, p 152]. Furthermore, the set of all probability measures on which T is ergodic is precisely the set of extremal points of $M(X,T)$.

Definition 1.7.1. *Let* $M(X,T)$ *be the set of probability measures defined on* X *and preserved by* T*. We shall call* T uniquely ergodic *if* $M(X,T)$ *is a singleton.*

In other words, T is uniquely ergodic if there exists only one probability measure preserved by T.

The following theorem relates elements of $M(X)$ to linear functionals on the space $C(X)$ of all complex-valued continuous functions defined on X. Indeed, if we have a map $J : C(X) \to \mathbb{C}$ that is continuous, linear, positive and normalized, then J must be given by integrating with respect to a Borel probability measure.

[2] If $\mu_1, \mu_2 \in M(X)$ and $0 \leq c \leq 1$, then $c\mu_1 + (1-c)\mu_2 \in M(X)$.

[3] The *weak* topology* on $M(X)$ is the smallest topology such that, for every continuous function $f : X \to \mathbb{C}$, the map $\mu \mapsto \int_X f \, d\mu$ is continuous.

Theorem 1.7.2 (Riesz Representation Theorem)**.** *Let X be a compact metric space. Suppose that $J : C(X) \to \mathbb{C}$ is a continuous linear map satisfying two conditions:*

(1) $J(1) = 1$, where $1 : X \to \mathbb{C}$ is defined by $1(x) = 1$ for all x;

(2) $f \geq 0$ implies $J(f) \geq 0$.

Then there exists $\mu \in M(X)$ such that $J(f) = \int_X f \, d\mu$ for all $f \in C(X)$.

The following result from [58, Theorem 6.2] says that each $\mu \in M(X)$ is determined by how it integrates continuous functions.

Lemma 1.7.3. [58] *Let X be a compact metric space, and let μ and ν be two Borel probability measures on X. If $\int_X f \, d\mu = \int_X f \, d\nu$ for every $f \in C(X)$, then $\mu = \nu$.*

The following result from [58, Theorem 6.8] gives a useful criterion for checking whether a measure is preserved by the transformation T.

Lemma 1.7.4. [58] *Let X be a compact metric space, and let $\mu \in M(X)$. Suppose $T : X \to X$ is a continuous transformation of X. Then $\mu \in M(X, T)$ if and only if $\int_X f \circ T \, d\mu = \int_X f \, d\mu$ for every $f \in C(X)$.*

The last result in this section is a well-known characterization of unique ergodicity.

Theorem 1.7.5 (Characterization Theorem of Unique Ergodicity)**.** *Let X be a compact metric space, and let $T : X \to X$ be a continuous transformation. Then the following statements are equivalent:*

(1) For every $f \in C(X)$, $(1/N) \sum_{n=1}^{N} f(T^n \alpha)$ converges uniformly to a constant.

(2) For every $f \in C(X)$, $(1/N) \sum_{n=1}^{N} f(T^n \alpha)$ converges pointwise to a constant.

(3) There exists $\mu \in M(X, T)$ such that, for all $f \in C(X)$ and all $\alpha \in X$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n \alpha) = \int_X f \, d\mu.$$

(4) T is uniquely ergodic.

1.8 Uniform distribution

A sequence of real numbers is uniformly distributed if the proportion of terms falling in a subinterval is proportional to the length of that interval. Such sequences are studied in Diophantine approximation and dynamical systems and have applications to Monte Carlo integration. For a general reference on this subject, we refer the reader to [27].

Definition 1.8.1. *Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that the sequence $(x_n)_{n=1}^{\infty}$ is* uniformly distributed *on an interval $I \subseteq \mathbb{R}$ if, for any subinterval $[a,b] \subseteq I$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon x_n \in [a,b]\} = \frac{b-a}{|I|}.$$

In this thesis, we shall restrict our attention to the interval $[0,1]$, and we shall denote by $\{x_n\}$ the fractional part of x_n.

Definition 1.8.2. *Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that the sequence $(x_n)_{n=1}^{\infty}$ is* uniformly distributed mod 1 *if*[4]*, for any Jordan-measurable subset $B \subseteq [0,1]$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon \{x_n\} \in B\} = \lambda(B),$$

where λ denotes the Lebesgue measure on $[0,1]$.

We note that the requirement of any Jordan-measurable subset B in Definition 1.8.2 can be replaced by any subinterval $[a,b]$, $[a,b)$, $(a,b]$ or (a,b) of $[0,1]$ without changing the definition. The condition is saying that the frequency with which the sequence $\{x_n\}$ lies in $[a,b]$ converges to $b-a$, the length of the subinterval. In a multi-dimensional space $[0,1]^s$, the definition of uniform distribution can be established in a similar fashion. The following famous result gives a necessary and sufficient condition for $(x_n)_{n=1}^{\infty}$ to be uniformly distributed mod 1.

Theorem 1.8.3 (Weyl Criterion)**.** *The following statements are equivalent:*

(1) The sequence $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1.

(2) For any continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\,dx.$$

(3) For each $\alpha \in \mathbb{Z} \setminus \{0\}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \alpha x_n} = 0.$$

From the Weyl criterion, we can extend the definition of uniform distribution to a more general setting.

[4] A set in the Euclidean space is *Jordan-measurable* if it can be well approximated by polyrectangles or a finite unions of rectangles. Such sets include all rectangles, balls, and simplexes. Also, any finite union and intersection of Jordan-measurable sets is Jordan-measurable, and so is the set difference of any two Jordan-measurable sets. For a precise definition of Jordan-measurable sets, see [2, p 396–397].

Definition 1.8.4. *Let X be a locally compact topological group equipped with the Haar measure μ. Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in X. We say that the sequence $(x_n)_{n=1}^{\infty}$ is* uniformly distributed *on X if, for any $f \in C(X)$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_X f \, d\mu.$$

Another related concept of uniform distribution is discrepancy. Let $\omega = (x_n)_{n=1}^{\infty}$ be a sequence in $[0,1]^s$.

Definition 1.8.5. *For each $N \in \mathbb{N}$, define the* discrepancy *$D_N(\omega)$ of $x_1, \ldots, x_N$ by*

$$D_N(\omega) = \sup_{B\in I} \left| \frac{1}{N} \cdot \#\{1 \le n \le N \colon x_n \in B\} - \lambda_s(B) \right|,$$

where $I = \{\prod_{i=1}^{s}[0,u_i) \colon 0 < u_i \le 1\}$, and λ_s denotes the s-dimensional Lebesgue measure on $[0,1]^s$.

The discrepancy is nothing other than a quantitative measure of uniformity of distribution. In particular, the sequence ω is uniformly distributed on $[0,1]^s$ if and only if $D_N(\omega) \to 0$ as $N \to \infty$. In a sense, the faster $D_N(\omega)$ decays as a function of N, the better uniformly distributed the sequence ω is. One of the fundamental obstructions in nature in this subject is that there is a limit to how well distributed any sequence can be. Precisely, it is one of the most famous conjectures in the theory of uniform distribution that, for any sequence ω in $[0,1]^s$, the inequality $D_N(\omega) > N^{-1}(\log N)^s$ holds for infinitely many $N \in \mathbb{N}$.

Definition 1.8.6. *We call ω a* low-discrepancy sequence *if $D_N(\omega) = O(N^{-1}(\log N)^s)$, where the constant depends only on the dimension s.*

The following theorem illustrates the importance of low discrepancy in the quasi-Monte Carlo method.

Theorem 1.8.7 (Koksma-Hlawka Inequality)**.** *Let $\omega = (x_n)_{n=1}^{\infty}$ be a sequence in $[0,1]^s$. Suppose $f : [0,1]^s \to \mathbb{R}$ is a function of bounded variation $V(f)$. Then, for each $N \in \mathbb{N}$,*

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_{[0,1]^s} f(x)\, dx \right| \le V(f) D_N(\omega),$$

where $D_N(\omega)$ is the discrepancy of $x_1, \ldots, x_N$.

In the theory of uniform distribution, we are usually interested in the distribution of a given sequence. Let $\omega = (x_n)_{n=1}^{\infty}$ be a sequence in $[0,1]^s$, and let J denote the set of all Jordan-measurable subsets of $[0,1]^s$.

Definition 1.8.8. *A function $\nu : J \to [0,1]$ is said to be a* distribution function *of the sequence ω if there is an increasing sequence $(N_k)_{k=1}^{\infty}$ of natural numbers such that, for every $B \in J$, we have*

$$\nu(B) = \lim_{k\to\infty} \frac{1}{N_k} \cdot \#\{1 \le n \le N_k \colon x_n \in B\}.$$

Let $G(\omega)$ be the set of all distribution functions of ω. If $G(\omega) = \{\nu\}$ is a singleton, then ν is called an asymptotic distribution function *of the sequence ω.*

For each $1 \le i \le s$, let $x_{n,i}$ denote the ith coordinate of x_n. If $(x_{n,i})_{n=1}^{\infty}$ is uniformly distributed mod 1 for every $i = 1, \dots, s$, then every distribution function ν of ω satisfies two further properties:

(1) for every $B = I_1 \times I_2 \times \cdots \times [0,0] \times \cdots \times I_s \in J$, we have $\nu(B) = 0$;

(2) for every $B = [0,1] \times [0,1] \times \cdots \times [0,x] \times \cdots \times [0,1] \in J$, we have $\nu(B) = x$.

The distribution function which satisfies the conditions (1) and (2) is called a *copula*. For a full treatment of the theory of copulas, we refer the reader to [39].

1.9 a-adic integers

If $b \ge 2$ is an integer, then every nonnegative integer has a b-adic representation of the form $n_0 + n_1 b + n_2 b^2 + \cdots + n_k b^k$ for some $k \in \mathbb{N}_0$ with $n_i \in \{0, 1, \dots, b-1\}$ $(0 \le i \le k)$. This idea of expressing an integer is an extension of the decimal numeral system. We now describe a class of locally compact groups, called the a-adic integers, which can be seen as a general framework of a numeral system. For more details, the reader is referred to [20, p 106–117].

Let $a = (a_n)_{n=1}^{\infty}$ be a sequence of natural numbers greater than 1. We define the *a-adic integers* Δ_a to be the set of infinite sequences in $\prod_{n=1}^{\infty}\{0, 1, \dots, a_n - 1\}$.

For two a-adic integers $x = (x_n)_{n=1}^{\infty}$ and $y = (y_n)_{n=1}^{\infty}$, let $z = (z_n)_{n=1}^{\infty}$ be defined as follows. Write $x_1 + y_1 = t_1 a_1 + z_1$, where $z_1 \in \{0, 1, \dots, a_1 - 1\}$ and $t_1 \in \mathbb{N}_0$. Write $x_2 + y_2 + t_1 = t_2 a_2 + z_2$, where $z_2 \in \{0, 1, \dots, a_2 - 1\}$ and $t_2 \in \mathbb{N}_0$. Suppose $z_0, \dots, z_k$ and $t_0, \dots, t_k$ have been defined. Then write $x_{k+1} + y_{k+1} + t_k = t_{k+1} a_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0, 1, \dots, a_{k+1} - 1\}$ and $t_{k+1} \in \mathbb{N}_0$. We have thus inductively defined the sequence $z = (z_n)_{n=1}^{\infty}$, which we deem to be $x + y$. The binary operation $+$ which we call addition makes Δ_a an Abelian group.

For each nonnegative integer k, let

$$\Lambda_k = \big\{(x_n)_{n=1}^{\infty} \in \Delta_a \colon x_j = 0,\ j \le k\big\}.$$

These sets Λ_k form a basis at $0 = (0)_{n=1}^{\infty}$ for a topology on Δ_a. With respect to this topology, Δ_a is compact and the group operations are continuous, making Δ_a a compact

Abelian topological group. A second binary operation called multiplication, denoted by $\times$ and compatible with addition, is defined as follows. Let $u = (1, 0, 0, 0, \dots)$. Note that $\{nu\}_{n=0}^{\infty}$ is dense in Δ_a. First on $\{nu\}_{n=0}^{\infty}$, define $n_1 u \times n_2 u$ to be $(n_1 n_2)u$. Deeming multiplication to be continuous on Δ_a defines it off $\{nu\}_{n=0}^{\infty}$. The two binary operations, addition and multiplication, make Δ_a a topological ring.

It is worth mentioning the non-Archimedean structure of Δ_a which can be derived from the topology. Let $x = (x_n)_{n=1}^{\infty}$ and $y = (y_n)_{n=1}^{\infty}$ be two arbitrary elements in Δ_a. We can define an absolute value on Δ_a by $|x| = 2^{-m}$, where m is the least integer for which $x_m \neq 0$, and $|0| = 0$. Then it is not hard to check that the absolute value satisfies the ultrametric triangle inequality $|x + y| \leq \max(|x|, |y|)$.

For each $n \in \mathbb{N}$ and $E \subseteq \{0, 1, \dots, a_n - 1\}$, let $\mu_n(E)$ denote the measure on the finite set $\{0, 1, \dots, a_n - 1\}$ given by

$$\mu_n(E) = \frac{1}{a_n} \cdot \#E.$$

The Haar measure μ is the corresponding product measure on Δ_a.

The dual group[5] of Δ_a, which we denote as $Z(a^{\infty})$, consists of all rational numbers $t = \ell/A_r$, where $A_r = a_1 \cdots a_r$ and $0 \leq \ell \leq A_r$ for some natural number r. To evaluate a character χ_t at $x = (x_n)_{n=1}^{\infty}$ in Δ_a, we write

$$\chi_t(x) = e\left(\frac{\ell}{A_r}(x_1 + a_1 x_2 + a_1 a_2 x_3 + \cdots + a_1 \cdots a_{r-1} x_r)\right),$$

where $e(\alpha)$ denotes $e^{2\pi i \alpha}$ for a real number α.

1.10 Non-Archimedean local fields

A *local field* is a locally compact topological field with respect to a non-discrete topology. Given such a field, an absolute value can be defined on it. There are two basic types of local fields: those local fields with Archimedean absolute values and those local fields with non-Archimedean absolute values. Every local field is isomorphic, as a topological field, to one of the following fields: the real numbers, the complex numbers, the finite extension of the p-adic numbers and the field of formal Laurent series over a finite field. For a full account of local fields, we refer to [52]. We now deal with the non-Archimedean local fields: the p-adic numbers and the field of formal Laurent series.

[5]For a locally compact Abelian group, a *character* of G is a continuous group homomorphism from G with values in the torus $\mathbb{T}$. The set of all characters on G can be made into a locally compact Abelian group, called the *dual group* of G. The group operation on the dual group is given by pointwise multiplication of characters, the inverse of a character is its complex conjugate, and the topology on the space of characters is that of uniform convergence on compact sets.

p-adic numbers: The p-adic number system, for any prime number p, extends the ordinary arithmetic of the rational numbers in a way different from the extension of the rational number system to the real number system. The extension is achieved by an alternative interpretation of the concept of closeness or absolute value. It turns out that the extended fields are of particular interest and importance in number theory.

Let p be a prime number. For any rational number r, we can write

$$r = p^{v(r)}\frac{u}{v}$$

with u and v coprime to p and to each other. Let $|r|_p = p^{-v(r)}$. Then $d_p(r, r') = |r - r'|_p$ defines a metric on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric d_p is denoted by $\mathbb{Q}_p$ and referred to as the *p-adic numbers*. We use $\mathbb{Z}_p$ to denote $\{x \in \mathbb{Q}_p \colon |x|_p \leq 1\}$ – the *ring of p-adic integers*. We shall be interested in the open unit ball

$$p\mathbb{Z}_p = \{px \colon x \in \mathbb{Z}_p\} = \{x \in \mathbb{Q}_p \colon |x|_p < 1\}.$$

It is worth keeping in mind that the metric d_p has the ultrametric property, i.e., $d_p(r, r'') \leq \max(d_p(r, r'), d_p(r', r''))$ for all $r, r', r'' \in \mathbb{Q}_p$. A basic and easily-verified property of $\mathbb{Q}_p$ is that each element α of $\mathbb{Q}_p$ has a unique p-adic expansion of the form $\alpha = \sum_{n=n_0}^{\infty} k_n p^n$, where $n_0 \in \mathbb{Z}$, $k_n \in \{0, 1, \ldots, p-1\}$ for all n, and $a_{k_0} \neq 0$. From this, we note that $|\alpha|_p = p^{-n_0}$. The main characteristics of $\mathbb{Q}_p$ that distinguish it from $\mathbb{R}$ stem from the non-Archimedean property. It turns out that $\mathbb{Q}_p$ is a locally compact field, and hence it is endowed with the Haar measure μ characterized by $\mu(p\mathbb{Z}_p) = 1$. See [9, Ch 4] or [26, Ch 1] for a clear and succinct introduction to the p-adic numbers.

Fields of formal Laurent series: The field of formal Laurent series over a finite field is considered the positive characteristic analogue of the real numbers. Many issues in number theory which have been studied in the setting of the real numbers can be addressed in the setting of the formal Laurent series. We refer the reader to [54, Part II] and [29] for the construction and structure of this field and for a survey of Diophantine approximation in this setting, respectively.

Let $\mathbb{F}_q$ be the finite field of q elements, where q is a power of a prime p. If Z is an indeterminate, we denote by $\mathbb{F}_q[Z]$ and $\mathbb{F}_q(Z)$ the ring of polynomials in Z with coefficients in $\mathbb{F}_q$ and the quotient field of $\mathbb{F}_q[Z]$, respectively. For each $P, Q \in \mathbb{F}_q[Z]$ with $Q \neq 0$, define $|P/Q| = q^{\deg(P)-\deg(Q)}$ and $|0| = 0$. The *field $\mathbb{F}_q((Z^{-1}))$ of formal Laurent series*[6] is the completion of $\mathbb{F}_q(Z)$ with respect to the valuation $|\cdot|$. That is,

$$\mathbb{F}_q((Z^{-1})) = \left\{a_n Z^n + a_{n-1}Z^{n-1} + \cdots + a_0 + a_{-1}Z^{-1} + \cdots : n \in \mathbb{Z},\ a_i \in \mathbb{F}_q\right\}$$

[6]The field of formal Laurent series over a finite field, or the non-Archimedean local field of positive characteristic, is usually referred to as "positive characteristic". This distinguishes it from the p-adic numbers which is the non-Archimedean local field of characteristic zero.

and we have $|a_n Z^n + a_{n-1} Z^{n-1} + \cdots| = q^n$ $(a_n \neq 0)$ and $|0| = 0$, where q is the number of elements of $\mathbb{F}_q$. It is worth keeping in mind that $|\cdot|$ is a non-Archimedean norm, since $|\alpha + \beta| \leq \max(|\alpha|, |\beta|)$. Moreover, every ball in the field of formal Laurent series of finite radius is compact. In fact, $\mathbb{F}_q((Z^{-1}))$ is the non-Archimedean local field of positive characteristic p. As a result, up to a positive multiplicative constant, there exists a unique countably additive Haar measure μ on the Borel subsets of $\mathbb{F}_q((Z^{-1}))$. In [54, p 65–70], Sprindžuk finds a characterization of Haar measure on $\mathbb{F}_q((Z^{-1}))$ by its value on the balls $B(\alpha; q^n) = \{\beta \in \mathbb{F}_q((Z^{-1})) \colon |\alpha - \beta| < q^n\}$. Indeed, it was shown that the equation $\mu(B(\alpha; q^n)) = q^n$ completely characterizes Haar measure.

It is worth noting that the sequence of the coefficients of a formal Laurent series can be considered analogous to the sequence of the digits in the decimal expansion of a real number. In [29], Lasjaunias pointed out that if $\alpha = \sum_{n=-n_0}^{\infty} a_{-n} Z^{-n}$ $(n_0 \in \mathbb{Z})$ is an element of $\mathbb{F}_q((Z^{-1}))$, then we have $\alpha \in \mathbb{F}_q(Z)$ if and only if the sequence $(a_{-n})_{n=1}^{\infty}$ is ultimately periodic.

Chapter 2

On subsequence ergodic theory

In this chapter, we make some contributions to subsequence ergodic theory and we describe some ideas which will be employed in succeeding chapters. Given a Hartman uniformly distributed sequence $(a_n)_{n=1}^{\infty}$ of natural numbers, we are able to show that if, for each $p > 1$ and $f \in L^p(\mu)$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n}\alpha)$$

exists μ-almost everywhere $\alpha \in X$, then only the ergodicity of the dynamical system $(X, \mathcal{B}, \mu, T)$ implies that the limit is equal to $\int_X f\, d\mu$ for μ-almost everywhere. This is not only interesting in and of itself but also useful in applications. Moreover, we can give a new formulation of unique ergodicity, which generalises the classical characterization theorem of unique ergodicity, using this property of $(a_n)_{n=1}^{\infty}$ being Hartman uniformly distributed. Finally, we give a remark on moving average ergodic theory which will be useful with regard to our applications.

2.1 Introduction

A topic of classical interest in ergodic theory is extending the Birkhoff ergodic theorem to various classes of subsequential ergodic averages. Indeed, we are interested in the conditions on a dynamical system $(X, \mathcal{B}, \mu, T)$ and a subsequence $(a_n)_{n=1}^{\infty}$ of the natural numbers such that, for each $f \in L^p(\mu)$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n}\alpha)$$

exists and converges to the expectation $\int_X f\, d\mu$ for μ-almost everywhere $\alpha \in X$. When the dynamical system $(X, \mathcal{B}, \mu, T)$ is ergodic, with $(a_n)_{n=1}^{\infty} = (n)_{n=1}^{\infty}$ and $p = 1$, this is the Birkhoff ergodic theorem.

One of the first fundamental achievements in subsequence ergodic theory appear in a series of papers of J. Bourgain, e.g. [6] and [7]. These showed that, for each ergodic dynamical system $(X, \mathcal{B}, \mu, T)$ and $f \in L^2(\mu)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{n^2}\alpha)$$

exists μ-almost everywhere $\alpha \in X$. More generally, he showed the existence of the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{P(n)}\alpha),$$

where $P(n)$ is a polynomial with integer coefficients, for an L^2 function f.

Another profound advance in the theory, due to Nair e.g. [33]–[37], was an elaborate extension of the work of Bourgain to a broad class of subsequences of the natural numbers. These include the cases $a_n = p_n$ and $a_n = P(p_n)$, where P is a polynomial with integer coefficients, and p_n denotes the nth prime number.

The purpose of this chapter is to make some contributions to subsequence ergodic theory by considering a class of Hartman uniformly distributed sequences of natural numbers. In addition, this chapter is meant to provide some machinery which makes it possible for the calculations in Chapters 4, 5 and 6.

The outline of this chapter is as follows. We first introduce in Section 2.2 the arithmetic framework on which our subsequence ergodic theory is based. This includes the notions of L^p-good universality and Hartman uniformly distribution. In Section 2.3, we provide two subsequence ergodic theorems. In Section 2.4, we supply a wealth of examples of L^p-good universal sequences and Hartman uniformly distributed sequences. In Section 2.5, we give a new formulation of unique ergodicity in the framework of subsequence ergodic theory. In Section 2.6, we give a moving average ergodic theorem.

2.2 Preliminary arithmetic context

In this section, we describe the arithmetic and analytic framework on which our subsequence ergodic theory is based.

Definition 2.2.1. *A sequence of natural numbers $(a_n)_{n=1}^{\infty}$ is called L^p-good universal if, for each dynamical system $(X, \mathcal{B}, \mu, T)$ and $f \in L^p(\mu)$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n}\alpha)$$

exists μ-almost everywhere $\alpha \in X$.

It is an immediate consequence of the Birkhoff ergodic theorem that the sequence of natural numbers, $a_n = n$, is L^1-good universal.

Recall that a sequence of real numbers $(x_n)_{n=1}^{\infty}$ is *uniformly distributed mod* 1 if, for each interval $I \subseteq [0,1)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon \{x_n\} \in I\} = |I|,$$

where $|I|$ denotes the length of I, and $\{x_n\}$ denotes the fractional part of x_n. In addition, we say that a sequence of natural numbers $(a_n)_{n=1}^{\infty}$ is *uniformly distributed on* $\mathbb{Z}$ if, for each $m \in \mathbb{N}_{>1}$ and $k \in \{0, 1, \ldots, m-1\}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon a_n \equiv k \bmod m\} = \frac{1}{m}.$$

That is, a sequence is uniformly distributed on $\mathbb{Z}$ if and only if it is uniformly distributed among the residue classes mod m for every natural number $m > 1$.

Definition 2.2.2. *A sequence of natural numbers* $(a_n)_{n=1}^{\infty}$ *is called* Hartman uniformly distributed *if it is uniformly distributed on* $\mathbb{Z}$ *and if* $(\gamma a_n)_{n=1}^{\infty}$ *is uniformly distributed mod* 1 *for each irrational number* γ.

A sequence being Hartman uniformly distributed has a couple of equivalent formulations. Firstly, $(a_n)_{n=1}^{\infty}$ is a Hartman uniformly distributed sequence if and only if, for each $\theta \notin \mathbb{Z}$, we have $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i a_n \theta} = 0$. In other words, $(a_n)_{n=1}^{\infty}$ is a Hartman uniformly distributed sequence if and only if, for each $z \in \mathbb{T} \setminus \{1\}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} z^{a_n} = 0.$$

Secondly, a sequence of natural numbers is Hartman uniformly distributed if and only if it is uniformly distributed on the Bohr compactification of the integers. For further background on Hartman uniform distribution, see [19] and [27].

2.3 Some subsequence ergodic theorems

We now introduce the following two pointwise subsequence ergodic theorems. The first result, due to Nair [34], enables us to calculate the limit of the ergodic averages for an L^p-good universal sequence with the requirement of weak mixing. For Hartman uniformly distributed sequences of natural numbers, it is nonetheless possible to prove a second version of subsequence ergodic theorem using only ergodicity. Note that a sketch proof of the second theorem can be found in [36].

Theorem 2.3.1. [34] *Let $(X, \mathcal{B}, \mu, T)$ be a weak-mixing dynamical system, and let $(a_n)_{n=1}^{\infty}$ be an L^2-good universal sequence. Suppose that, for any irrational number γ, the sequence $(\gamma a_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. Then, for any $f \in L^2(\mu)$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n}\alpha) = \int_X f \, d\mu$$

μ-almost everywhere $\alpha \in X$.

We note that L^2 is dense in L^p for every p in $(1, 2]$. Hence, this theorem extends readily to the case $p > 1$ by approximation by L^2 functions, and so do other results in this chapter. However, it is still an important open problem in subsequence ergodic theory for the case $p = 1$. We forego the details as we do not need this degree of generality in our applications.

Theorem 2.3.2. [A] *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system, and let $(a_n)_{n=1}^{\infty}$ be a Hartman uniformly distributed sequence which is also L^2-good universal. Then, for any $f \in L^2(\mu)$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n}\alpha) = \int_X f \, d\mu$$

μ-almost everywhere $\alpha \in X$.

Proof of Theorem 2.3.2. Let $f \in L^2(\mu)$, and set

$$Mf(\alpha) = \sup_{N\geq 1} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n}\alpha) \right|.$$

Plainly, we have $|\frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}\alpha)| \leq Mf(\alpha)$ $(N \in \mathbb{N})$ and $Mf \in L^2(\mu)$. It follows from the dominated convergence theorem that the limit

$$g(\alpha) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n}\alpha)$$

exists in L^2 norm. Our next order of business is to show that $g(T\alpha) = g(\alpha)$. We have

$$\begin{aligned}
&\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f \circ T^{a_n+1} - \frac{1}{N} \sum_{n=1}^{N} f \circ T^{a_n} \right\|_2^2 \\
&= \lim_{N\to\infty} \frac{1}{N^2} \left(\int_X \left| \sum_{n=1}^{N} (f \circ T^{a_n+1} - f \circ T^{a_n}) \right|^2 d\mu \right) \\
&= \lim_{N\to\infty} \frac{1}{N^2} \left(\sum_{1\leq n,m\leq N} \int_X (f \circ T^{a_n+1} - f \circ T^{a_n})(f \circ T^{a_m+1} - f \circ T^{a_m}) \, d\mu \right).
\end{aligned}$$

By arranging the terms using the Koopman operator, we have

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N} f\circ T^{a_n+1}-\frac{1}{N}\sum_{n=1}^{N} f\circ T^{a_n}\right\|_2^2$$
$$=\lim_{N\to\infty}\frac{1}{N^2}\Bigg(\sum_{1\le n,m\le N}\langle U_T^{a_n-a_m}f,f\rangle-\langle U_T^{a_n+1-a_m}f,f\rangle-\langle U_T^{a_n-am-1}f,f\rangle+\langle U_T^{a_n-a_m}f,f\rangle\Bigg).$$

It now follows from the Bochner-Herglotz theorem that there is a spectral measure ω_f attached to the function f such that $\langle U_T^n f,f\rangle=\int_{\mathbb{T}} z^n\,d\omega_f(z)$. This is

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N} f\circ T^{a_n+1}-\frac{1}{N}\sum_{n=1}^{N} f\circ T^{a_n}\right\|_2^2$$
$$=\lim_{N\to\infty}\frac{1}{N^2}\Bigg(\sum_{1\le n,m\le N}\int_{\mathbb{T}}(2z^{a_n-a_m}-z^{a_n-a_m+1}-z^{a_n-a_m-1})\,d\omega_f(z)\Bigg)$$
$$=\lim_{N\to\infty}\frac{1}{N^2}\Bigg(\sum_{1\le n,m\le N}\int_{\mathbb{T}}(2-z-z^{-1})z^{a_n-a_m}\,d\omega_f(z)\Bigg).$$

We now have

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N} f\circ T^{a_n+1}-\frac{1}{N}\sum_{n=1}^{N} f\circ T^{a_n}\right\|_2^2$$
$$=\lim_{N\to\infty}\int_{\mathbb{T}}(2-z-z^{-1})\left(\frac{1}{N}\sum_{n=1}^{N} z^{a_n}\right)\left(\frac{1}{N}\sum_{m=1}^{N} z^{-a_m}\right)d\omega_f(z).$$

If $z=1$, the right hand side vanishes. In addition, when $z\neq 1$, this tends to zero as $j\to\infty$ because (a_n) is Hartman uniformly distributed. This shows that $g\circ T=g$. Now it follows from Lemma 1.5.2 that if T is ergodic and $g(T\alpha)=g(\alpha)$ for a measurable function g, then $g(\alpha)$ must be $\int_X f\,d\mu$.

All we have to do now is to show the pointwise limit is the same as the norm limit. We consider an increasing sequence of natural numbers $(N_t)_{t=1}^{\infty}$ such that

$$\left\|\frac{1}{N_t}\sum_{n=1}^{N_t} f(T^{a_n}\alpha)-\int_X f\,d\mu\right\|_2\le\frac{1}{t}.$$

We have

$$\sum_{t=1}^{\infty}\left(\int_X\left|\frac{1}{N_t}\sum_{n=1}^{N_t} f(T^{a_n}\alpha)-\int_X f\,d\mu\right|^2 d\mu(\alpha)\right)\le\sum_{t=1}^{\infty}\frac{1}{t^2}<\infty.$$

The Fatou's lemma tells us that

$$\int_X\left(\sum_{t=1}^{\infty}\left|\frac{1}{N_t}\sum_{n=1}^{N_t} f(T^{a_n}\alpha)-\int_X f\,d\mu\right|^2\right)d\mu(\alpha)<\infty,$$

which implies that

$$\sum_{t=1}^{\infty}\left|\frac{1}{N_t}\sum_{n=1}^{N_t} f(T^{a_n}\alpha)-\int_X f\,d\mu\right|^2<\infty$$

μ-almost everywhere $\alpha\in X$. This means that

$$\left|\frac{1}{N_t}\sum_{n=1}^{N_t} f(T^{a_n}\alpha)-\int_X f\,d\mu\right|=o(1)$$

μ-almost everywhere $\alpha\in X$. As $(a_n)_{n=1}^{\infty}$ is L^2-good universal, we must have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}\alpha)=\int_X f\,d\mu$$

μ-almost everywhere $\alpha\in X$. This completes the proof of Theorem 2.3.2. ■

2.4 Examples of Hartman uniformly distributed and good universal sequences

The following is a list of constructions of Hartman uniformly distributed sequences. The first six are also examples of L^p-good universal sequences for some $p\geq 1$. The other examples appear in [35]. Note that the second example is not in general Hartman uniformly distributed but it provides a wealth of sequences satisfying Theorem 2.3.1.

(1) The sequence $(n)_{n=1}^{\infty}$ is L^1-good universal. This is Birkhoff ergodic theorem.

(2) *Polynomial like sequences:* Let $P(x)$ be a polynomial mapping $\mathbb{N}$ into itself, and let $(p_n)_{n=1}^{\infty}$ denote the sequence of prime numbers. Then the sequences $(P(n))_{n=1}^{\infty}$ and $(P(p_n))_{n=1}^{\infty}$ are L^p-good universal sequences ($p>1$). See [6], [7] and [33].

Note that if $n\in\mathbb{N}$, then $n^2\not\equiv 3 \bmod 4$, so in general the sequences $(P(n))_{n=1}^{\infty}$ and $(P(p_n))_{n=1}^{\infty}$ are not Hartman uniformly distributed. We do, however, know from [59] that if γ is an irrational number, then both $(\gamma P(n))_{n=1}^{\infty}$ and $(\gamma P(p_n))_{n=1}^{\infty}$ are uniformly distributed mod 1.

(3) *Condition* H*:* Sequences $(a_n)_{n=1}^{\infty}$ that are both L^p-good universal and Hartman uniformly distributed can be constructed as follows. Denote by $[x]$ and $\{x\}$ the integer part and the fractional part of real number x, respectively. Set $a_n=[g(n)]$ $(n=1,2,\dots)$ where $g:[1,\infty)\to[1,\infty)$ is a differentiable function whose derivative increases with its argument. Let A_N denote the cardinality of the set $\{n\colon a_n\leq N\}$, and suppose, for some function $a:[1,\infty)\to[1,\infty)$ increasing to infinity as its argument does, that we set

$$b(N)=\sup_{\{z\}\in\left[\frac{1}{a(N)},\frac{1}{2}\right)}\left|\sum_{n\colon a_n\leq N} e^{2\pi i z a_n}\right|.$$

Suppose also, for some decreasing function $c : [1, \infty) \to [1, \infty)$ and some positive constant $C > 0$, that

$$\frac{b(N) + A_{[a(N)]} + \frac{N}{a(N)}}{A_N} \leq Cc(N).$$

Then if we have

$$\sum_{n=1}^{\infty} c(\theta^n) < \infty$$

for all $\theta > 1$, we say that $(a_n)_{n=1}^{\infty}$ satisfies condition H, see [37].

Sequences which satisfy condition H are both Hartman uniformly distributed and L^p-good universal ($p > 1$). Specific sequences of integers satisfying condition H include $a_n = [g(n)]$ ($n = 1, 2, \dots$) where:

(a) $g(n) = n^{\omega}$ if $\omega > 1$ and $\omega \notin \mathbb{N}$.

(b) $g(n) = e^{\log^{\gamma} n}$ for $\gamma \in (1, \frac{3}{2})$.

(c) $g(n) = b_k n^k + \cdots + b_1 n + b_0$, where $b_k, \dots, b_1$ are not all rational multiples of the same real number.

(d) *Hardy fields:* By a Hardy field, we mean a closed subfield (under differentiation) of the ring of germs at $+\infty$ of continuous real-valued functions with addition and multiplication taken to be pointwise. Let $\mathcal{H}$ denote the union of all Hardy fields. If $(a_n)_{n=1}^{\infty} = ([h(n)])_{n=1}^{\infty}$, where $h \in \mathcal{H}$ satisfies the conditions that, for some $k \in \mathbb{N}_{>1}$,

$$\lim_{x \to \infty} \frac{h(x)}{x^{k-1}} = \infty \qquad \text{and} \qquad \lim_{x \to \infty} \frac{h(x)}{x^k} = 0,$$

then $(a_n)_{n=1}^{\infty}$ satisfies condition H. This example is observed in [5].

(4) *A random example:* Suppose that $(b_n)_{n=1}^{\infty}$ is a strictly increasing sequence in $\mathbb{N}$, and let $B = \{b_n\}_{n=1}^{\infty}$. By identifying B with its characteristic function $\mathbb{1}_B$, we may view it as a point in $\Lambda = \{0, 1\}^{\mathbb{N}}$, the set of maps from $\mathbb{N}$ to $\{0, 1\}$. We may endow Λ with a probability measure by viewing it as a Cartesian product $\Lambda = \prod_{n=1}^{\infty} X_n$, where, for each natural number n, we have $X_n = \{0, 1\}$ and specify the probability measure ν_n on X_n by $\nu_n(\{1\}) = \omega_n$ with $0 \leq \omega_n \leq 1$ and $\nu_n(\{0\}) = 1 - \omega_n$ such that $\lim_{n \to \infty} \omega_n n = \infty$. The desired probability measure on Λ is the corresponding product measure $\nu = \prod_{n=1}^{\infty} \nu_n$. The underlying σ-algebra $\mathcal{A}$ is that generated by the cylinder sets

$$\{(\Delta_n)_{n=1}^{\infty} \in \Lambda \colon \Delta_{n_1} = \alpha_{n_1}, \dots, \Delta_{n_k} = \alpha_{n_k}\}$$

for all possible choices of $n_1, \dots, n_k$ and $\alpha_{n_1}, \dots, \alpha_{n_k}$. Then almost every point $(a_n)_{n=1}^{\infty}$ in Λ, with respect to the measure ν, is Hartman uniformly distributed, [6].

(5) *Block sequences:* Suppose that $(a_n)_{n=1}^{\infty} = \bigcup_{n=1}^{\infty}[d_n, e_n]$ is ordered by absolute value for disjoint $([d_n, e_n])_{n=1}^{\infty}$ with $d_{n-1} = O(e_n)$ as n tends to infinity. Note that this allows the possibility that $(a_n)_{n=1}^{\infty}$ is zero density. This example is an immediate consequence of Tempelman's semigroup ergodic theorem, [56, p 218].

(6) *Random perturbation of good sequences:* Suppose $(a_n)_{n=1}^{\infty}$ is an L^p-good universal sequence which is also Hartman uniformly distributed. Let $\theta = (\theta_n)_{n=1}^{\infty}$ be a sequence of $\mathbb{N}$-valued independent, identically distributed random variables with basic probability space $(Y, \mathcal{A}, \mathcal{P})$, and a $\mathcal{P}$-complete σ-field $\mathcal{A}$. Let $\mathbb{E}$ denote the expectation with respect to the basic probability space $(Y, \mathcal{A}, \mathcal{P})$. Assume that there exist $0 < \alpha < 1$ and $\beta > 1/\alpha$ such that

$$a_n = O\left(e^{n^{\alpha}}\right) \qquad \text{and} \qquad \mathbb{E}\log_+^{\beta}|\theta_1| < \infty.$$

Then $(a_n + \theta_n(\omega))_{n=1}^{\infty}$ is also an L^p-good universal sequence which is Hartman uniformly distributed, [38].

(7) $a_n = [P(n)]$ $(n = 1, 2, \dots)$, where $P(x) = b_k x^k + \cdots + b_1 x + b_0$ and $b_k, \dots, b_1$ are not all rational multiples of the same real number.

(8) $a_n = [P(p_n)]$ $(n = 1, 2, \dots)$, where $(p_n)_{n=1}^{\infty}$ denotes the sequence of prime numbers and $P(x)$ is as in (7).

(9) $a_n = [f(n)]$ $(n = 1, 2, \dots)$, where $f(z)$ denotes a non-polynomial entire function which is real on the real numbers and such that $|f(z)| \ll e^{(\log z)^{\alpha}}$ with $\alpha < \frac{4}{3}$.

(10) $a_n = [f(p_n)]$ $(n = 1, 2, \dots)$, where $(p_n)_{n=1}^{\infty}$ denotes the sequence of prime numbers and $f(z)$ is as in (9).

(11) $a_n = [b_n \cos(b_n x)]$ $(n = 1, 2, \dots)$ for a strictly increasing sequence of integers $(b_n)_{n=1}^{\infty}$ and almost all x with respect to Lebesgue measure.

(12) $a_n = [b_n \cos(b_n x)]$ $(n = 1, 2, \dots)$ for a strictly increasing sequence of integers $(b_n)_{n=1}^{\infty}$ such that $a_n \ll n^p$, for some $p > 1$, and all x outside a set of Hausdorff dimension not greater than $1 - \frac{1}{4p+\frac{1}{2}}$.

(13) $a_n = [g_n(x)]$ $(n = 1, 2, \dots)$ for almost all $x \in [a, b]$ with respect to Lebesgue measure, where $(g_n(x))_{n=1}^{\infty}$ is a sequence of continuously differentiable functions defined on $[a, b]$ satisfying the following hypotheses. For each pair of distinct natural numbers n and m, we have:

(a) $g_n'(x) - g_m'(x)$ is monotonic on $[a, b]$.

(b) There is an absolute constant λ such that $|g_n'(x) - g_m'(x)| \geq \lambda > 0$.

(14) $a_n = [g_n(x)]$ $(n = 1, 2, \dots)$ for all x lying outside a set of Hausdorff dimension at most $1 - \frac{1}{p}$ in $[a, b]$, where $(g_n(x))_{n=1}^{\infty}$ is a sequence of continuously differentiable functions defined on $[a, b]$ satisfying the hypotheses (a) and (b) of (13) and also meeting the further two requirements:

(c) For all $x \in [a, b]$, we have

$$\sup_{x \in [a,b]} |g_n'(x)| \ll n^p$$

for some $p > 1$ and with an implied constant independent of x.

(d) For each pair of distinct natural numbers n and m, the function

$$\frac{g_n'(x) g_m'(x)}{g_n'(x) - g_m'(x)}$$

is monotonic on $[a, b]$.

2.5 Hartman uniform distribution and unique ergodicity

In this section, we give a new and much more powerful formulation of unique ergodicity. Indeed, we consider some kind of subsequential ergodic average rather than the normal ergodic average over the natural numbers as appeared in the classical characterization theorem of unique ergodicity. The reader may find some partial proof from [36].

Theorem 2.5.1. [F] *Let $T : X \to X$ be a continuous transformation of a compact metric space X, and let $(a_n)_{n=1}^{\infty}$ be a Hartman uniformly distributed sequence which is also L^2-good universal. Then the following are equivalent:*

(1) For every $f \in C(X)$, $(1/N) \sum_{n=1}^{N} f(T^{a_n} \alpha)$ converges uniformly to a constant.

(2) For every $f \in C(X)$, $(1/N) \sum_{n=1}^{N} f(T^{a_n} \alpha)$ converges pointwise to a constant.

(3) There exists $\mu \in M(X, T)$ such that, for all $f \in C(X)$ and all $\alpha \in X$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n} \alpha) = \int_X f \, d\mu.$$

(4) T is uniquely ergodic.

Proof of Theorem 2.5.1.

(1) $\Rightarrow$ (2): This holds trivially. ■

(2) $\Rightarrow$ (3): Define a linear operator $J : C(X) \to \mathbb{C}$ by

$$J(f) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n} \alpha).$$

Observe that J is continuous since

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}\alpha)\right| \leq \sup_{\alpha\in X}|f(\alpha)|.$$

In addition, we have $J(1) = 1$ and the fact that $f \geq 0$ implies $J(f) \geq 0$. Hence, by the Riesz representation theorem, there exists $\mu \in M(X)$ such that $J(f) = \int_X f\,d\mu$. Also, we note that $J(f \circ T) = J(f)$, so $\int_X f \circ T\,d\mu = \int_X f\,d\mu$. Thus, we have $\mu \in M(X,T)$ by Lemma 1.7.4. ■

(3) ⇒ (4): Suppose that $\nu \in M(X,T)$. We have, for all $\alpha \in X$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}\alpha) = \int_X f\,d\mu.$$

Integrating with respect to ν and using the dominated convergence theorem, we obtain

$$\begin{aligned}\int_X f\,d\mu &= \int_X\left(\int_X f\,d\mu\right)d\nu = \lim_{N\to\infty}\int_X \frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}\alpha)\,d\nu(\alpha)\\ &= \lim_{N\to\infty}\int_X f\,d\nu = \int_X f\,d\nu\end{aligned}$$

for all $f \in C(X)$. We have $\nu = \mu$ by Lemma 1.7.3, and hence T is uniquely ergodic. ■

(4) ⇒ (1): By Theorem 2.3.2, if $(1/N)\sum_{n=1}^{\infty} f(T^{a_n}\alpha)$ converges uniformly to a constant, then this constant must be $\int_X f\,d\mu$, where $M(X,T) = \{\mu\}$. Suppose to the contrary that (1) does not hold. Then there exist a function $g \in C(X)$, an $\epsilon > 0$ and a sequence $(\alpha_N)_{N=1}^{\infty}$ in X such that

$$\left|\frac{1}{N}\sum_{n=1}^{N} g(T^{a_n}\alpha_N) - \int_X g\,d\mu\right| \geq \epsilon.$$

For each $N \in \mathbb{N}$, set

$$\mu_N = \frac{1}{N}\sum_{n=1}^{N}\delta_{T^{a_n}\alpha_N},$$

where δ_x denotes the Dirac δ-measure supported at x. This means that

$$\left|\int_X g\,d\mu_N - \int_X g\,d\mu\right| \geq \epsilon. \tag{2.5.1}$$

Since $M(X)$ is compact, we can pick a convergent subsequence $(\mu_{N_j})_{j=1}^{\infty}$ of $(\mu_N)_{N=1}^{\infty}$, say to μ_∞. By the inequality (2.5.1), we see that $\mu_\infty \neq \mu$. To show the contradiction,

it suffices to prove that $\mu_\infty \in M(X,T)$. Let $f \in C(X)$. Then we have

$$\begin{aligned}\left|\int_X f\,d\mu_\infty - \int_X f\circ T\,d\mu_\infty\right| &= \lim_{j\to\infty}\left|\int_X (f - f\circ T)\,d\Big(\frac{1}{N_j}\sum_{n=1}^{N_j}\delta_{T^{a_n}\alpha_{N_j}}\Big)\right| \\ &= \lim_{j\to\infty}\left|\frac{1}{N_j}\int_X \sum_{n=1}^{N_j}(f\circ T^{a_n} - f\circ T^{a_n+1})\,d\delta_{\alpha_{N_j}}\right| \\ &\le \lim_{j\to\infty}\frac{1}{N_j}\int_X\left|\sum_{n=1}^{N_j}(f\circ T^{a_n} - f\circ T^{a_n+1})\right| d\delta_{\alpha_{N_j}}\end{aligned}$$

Integrating both sides of this inequality with respect to μ and noting that the left hand side is a constant, we have

$$\left|\int_X f\,d\mu_\infty - \int_X f\circ T\,d\mu_\infty\right| \le \int_X\left(\lim_{j\to\infty}\frac{1}{N_j}\int_X\left|\sum_{n=1}^{N_j}(f\circ T^{a_n} - f\circ T^{a_n+1})\right| d\delta_{\alpha_{N_j}}\right)d\mu$$

By using the dominated convergence theorem, the Fubini's theorem, and the fact that $\int_X d\delta_{\alpha_{N_j}} = 1$, we obtain

$$\left|\int_X f\,d\mu_\infty - \int_X f\circ T\,d\mu_\infty\right| \le \lim_{j\to\infty}\frac{1}{N_j}\int_X\left|\sum_{n=1}^{N_j}(f\circ T^{a_n} - f\circ T^{a_n+1})\right| d\mu$$

By using the Schwarz inequality, this is

$$\left|\int_X f\,d\mu_\infty - \int_X f\circ T\,d\mu_\infty\right| \le \lim_{j\to\infty}\frac{1}{N_j}\left\|\sum_{n=1}^{N_j}(f\circ T^{a_n} - f\circ T^{a_n+1})\right\|_2. \tag{2.5.2}$$

Next we can calculate the right hand side of (2.5.2) in terms of the Koopman operator.

$$\begin{aligned}&\lim_{j\to\infty}\frac{1}{N_j}\left\|\sum_{n=1}^{N_j}(f\circ T^{a_n} - f\circ T^{a_n+1})\right\|_2 = \lim_{j\to\infty}\frac{1}{N_j}\left(\int_X\left|\sum_{n=1}^{N_j}(f\circ T^{a_n} - f\circ T^{a_n+1})\right|^2 d\mu\right)^{\frac{1}{2}} \\ &= \lim_{j\to\infty}\frac{1}{N_j}\left(\sum_{1\le n,m\le N_j}\int_X (f\circ T^{a_n} - f\circ T^{a_n+1})(f\circ T^{a_m} - f\circ T^{a_m+1})\,d\mu\right)^{\frac{1}{2}} \\ &= \lim_{j\to\infty}\frac{1}{N_j}\left(\sum_{1\le n,m\le N_j}\langle U_T^{a_n-a_m}f, f\rangle - \langle U_T^{a_n+1-a_m}f, f\rangle - \langle U_T^{a_n-a_m-1}f, f\rangle + \langle U_T^{a_n-a_m}f, f\rangle\right)^{\frac{1}{2}}.\end{aligned}$$

By the Bochner-Herglotz theorem, there exists a spectral measure ω_f attached to the function f such that $\langle U_T^n f, f\rangle = \int_{\mathbb{T}} z^n\,d\omega_f(z)$. This is

$$\begin{aligned}&\lim_{j\to\infty}\frac{1}{N_j}\left\|\sum_{n=1}^{N_j}(f\circ T^{a_n} - f\circ T^{a_n+1})\right\|_2 \\ &= \lim_{j\to\infty}\frac{1}{N_j}\left(\sum_{1\le n,m\le N_j}\int_{\mathbb{T}}(2z^{a_n-a_m} - z^{a_n-a_m+1} - z^{a_n-a_m-1})\,d\omega_f(z)\right)^{\frac{1}{2}} \\ &= \lim_{j\to\infty}\frac{1}{N_j}\left(\sum_{1\le n,m\le N_j}\int_{\mathbb{T}}(2 - z - z^{-1})z^{a_n-a_m}\,d\omega_f(z)\right)^{\frac{1}{2}}.\end{aligned}$$

Finally, we have

$$\lim_{j\to\infty}\frac{1}{N_j}\left\|\sum_{n=1}^{N_j}(f\circ T^{a_n}-f\circ T^{a_n+1})\right\|_2$$
$$=\lim_{j\to\infty}\left(\int_{\mathbb{T}}(2-z-z^{-1})\left(\frac{1}{N_j}\sum_{n=1}^{N_j}z^{a_n}\right)\left(\frac{1}{N_j}\sum_{m=1}^{N_j}z^{-a_m}\right)d\omega_f(z)\right)^{\frac{1}{2}}.$$

If $z=1$, the right hand side vanishes. In addition, when $z\neq 1$, this tends to zero as $j\to\infty$ because $(a_n)_{n=1}^{\infty}$ is Hartman uniformly distributed. From the inequality (2.5.2), we see that $\int_X f\,d\mu_\infty=\int_X f\circ T\,d\mu_\infty$ for every $f\in C(X)$. By Lemma 1.7.4, we must have $\mu_\infty\in M(X,T)$, and this contradicts the unique ergodicity of T. ■

This completes the proof of Theorem 2.5.1. ■

2.6 Moving ergodic averages

We begin by introducing some notation. Let Z be a collection of points in $\mathbb{Z}\times\mathbb{N}$, and let

$$Z^h=\{(m,n)\in Z\colon n\geq h\},$$
$$Z_c^h=\{(x,y)\in\mathbb{Z}^2\colon |x-m|<c(y-n)\text{ for some }(m,n)\in Z^h\},$$
$$Z_c^h(k)=\{x\colon (x,k)\in Z_c^h\}\qquad (k\in\mathbb{N}).$$

Geometrically, we can think of Z_c^1 as the lattice points contained in the union of all solid cones with aperture c and vertex contained in $Z^1=Z$.

Definition 2.6.1. *A sequence of pairs of natural numbers $(a_n,b_n)_{n=1}^{\infty}$ is* Stoltz *if there exist a collection of points Z in $\mathbb{Z}\times\mathbb{N}$ and a function $h=h(t)$ tending to infinity with t such that $(a_n,b_n)_{n=t}^{\infty}\in Z^{h(t)}$, and if there exist h_0, c_0 and $d>0$ such that, for all $k\in\mathbb{N}$, we have the cardinality $\#Z_{c_0}^{h_0}(k)\leq dk$.*

This technical condition is interesting because of the following lemma.

Lemma 2.6.2. [3] *Let $(X,\mathcal{B},\mu,T)$ be an ergodic dynamical system, and let $(a_n,b_n)_{n=1}^{\infty}$ be a Stoltz sequence. Then, for any $f\in L^1(\mu)$, the limit*

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{j=1}^{b_n}f(T^{a_n+j}\alpha)$$

exists μ-almost everywhere $\alpha\in X$.

Note that if we set

$$M_f(\alpha)=\lim_{n\to\infty}\frac{1}{b_n}\sum_{j=1}^{b_n}f(T^{a_n+j}\alpha)\qquad\text{and}\qquad M_{n,f}(\alpha)=\frac{1}{b_n}\sum_{j=1}^{b_n}f(T^{a_n+j}\alpha)$$

and observe that

$$M_{n,f}(T\alpha) - M_{n,f}(\alpha) = \frac{1}{b_n}\Big(f(T^{a_n+b_n+1}\alpha) - f(T^{a_n+1}\alpha)\Big),$$

then we can see that $M_f(T\alpha) = M_f(\alpha)$ for μ-almost everywhere $\alpha \in X$. A standard fact in ergodic theory is that if $(X, \mathcal{B}, \mu, T)$ is ergodic and if $M_f(T\alpha) = M_f(\alpha)$ for μ-almost everywhere $\alpha \in X$, then $M_f(\alpha) = \int_X f\, d\mu$ for μ-almost everywhere $\alpha \in X$. Therefore, we have the following result.

Theorem 2.6.3. [A] *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system, and let $(a_n, b_n)_{n=1}^{\infty}$ be a Stoltz sequence. Then, for any $f \in L^1(\mu)$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{b_n} f(T^{a_n+j}\alpha) = \int_X f\, d\mu$$

μ-almost everywhere $\alpha \in X$.

We note that the term "Stoltz" is used here because the condition on $(a_n, b_n)_{n=1}^{\infty}$ is analogous to the condition required in the classical non-radical limit theorem for harmonic functions, also called a Stoltz condition, which suggested Lemma 2.6.2 to the authors of [3]. Averages where $a_n = 1$ for all n will be called *non-moving*. This is as opposed the more general *moving* averages which are averages along intervals whose initial element, i.e. a_n, may not be 1. Moving averages satisfying the above hypothesis can be constructed by taking, for instance, $a_n = 2^{2^n}$ and $b_n = 2^{2^{n-1}}$.

Chapter 3

Complexity of the Liouville numbers in positive characteristic

A Liouville number in the field of formal Laurent series over the finite field $\mathbb{F}_q$ can be defined analogously as in the real case. That is, the set $\mathscr{L}$ of Liouville numbers consists of irrational elements of $\mathbb{F}_q((Z^{-1}))$ that can be very well approximated by a sequence of rational functions from $\mathbb{F}_q(Z)$. Our main aim is to investigate the complexity of $\mathscr{L}$ in terms of size and dimension. Indeed, we show that $\mathscr{L}$ has Haar measure zero and Hausdorff dimension zero. We locate the exact cut-point at which the h-dimensional Hausdorff measure of $\mathscr{L}$ drops from infinity to zero for any dimension function h. Our results also include the fact that if $\mathscr{L}$ has infinite h-dimensional Hausdorff measure, then it does not have σ-finite h-dimensional Hausdorff measure.

3.1 Introduction

A *Liouville number* is an irrational number $\alpha \in \mathbb{R}$ with the property that, for every natural number n, there exist integers p and q with $q > 1$ and such that

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}.$$

Therefore, Liouville numbers are real numbers that can be very well approximated by a sequence of rational numbers. They were of significance in establishing the first proof of the existence of transcendental numbers, and they initiated the study of Diophantine approximation of algebraic real numbers.

From the point of view of measure theory, almost all real numbers are transcendental. Also, the Liouville numbers are uncountable. However, the set of Liouville numbers is quite small. More precisely, Oxtoby showed that its Lebesgue measure and Hausdorff dimension are both zero, [45, p 8–9]. It had been asked further by R.D. Mauldin what the exact cut-point at which the Hausdorff measure of the Liouville numbers drops from infinity to zero is. Recently, Olsen and Renfro solved this 20 plus year open question by

giving a complete characterization of all Hausdorff measures of the Liouville numbers, [42] and [43].

This chapter aims to extend this study of the complexity of the Liouville numbers to the setting of the fields of formal Laurent series. We now summarize the contents of this chapter. First of all, we introduce in Section 3.2 the Liouville numbers in the positive characteristic setting. We show that the set of Liouville numbers is dense in this setting and that there are uncountably many of them. In Section 3.3, we look into the complexity of the Liouville numbers in terms of size and dimension, and we arrive at the conclusion that the set of Liouville numbers is so small that it has both Haar measure and Hausdorff dimension zero. We then introduce the concept of exact Hausdorff dimension in Section 3.4 which is used to provide a further indication of the complexity of the Liouville numbers. Finally, we give a complete characterization of all Hausdorff measures of the Liouville numbers in Section 3.5; however, the proof appears in Section 3.8 as we need several preliminary results which come along in Section 3.6 and Section 3.7.

3.2 Liouville numbers

Definition 3.2.1. *A* Liouville number *in positive characteristic can be defined to be an irrational element $\alpha \in \mathbb{F}_q((Z^{-1})) \setminus \mathbb{F}_q(Z)$ with the property that, for every natural numbers n, there exist polynomials $P, Q \in \mathbb{F}_q[Z]$ with $|Q| > 1$ and such that*

$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{|Q|^n}.$$

The set $\mathscr{L}$ of Liouville numbers is

$$\mathscr{L} = \left\{\alpha \in \mathbb{F}_q((Z^{-1})) \setminus \mathbb{F}_q(Z) \colon \forall n \in \mathbb{N}, \exists P, Q \in \mathbb{F}_q[Z] \text{ with } |Q| > 1,\ \left|\alpha - \frac{P}{Q}\right| < \frac{1}{|Q|^n}\right\}.$$

It is worth noting that the set of Liouville numbers can also be expressed as

$$\mathscr{L} = \big(\mathbb{F}_q((Z^{-1})) \setminus \mathbb{F}_q(Z)\big) \cap \left(\bigcap_{n=1}^{\infty} \bigcup_{\substack{Q \in \mathbb{F}_q[Z] \\ |Q|>1}} \bigcup_{P \in \mathbb{F}_q[Z]} B\left(\frac{P}{Q}; \frac{1}{|Q|^n}\right)\right). \tag{3.2.1}$$

To describe the importance of the Liouville numbers in positive characteristic, we begin with some definitions.

Definition 3.2.2. *An element $\alpha \in \mathbb{F}_q((Z^{-1}))$ is called an* algebraic number *if there exists a polynomial*

$$f(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0,$$

where $A_n, \ldots, A_0 \in \mathbb{F}_q[Z]$ are not all zero, such that $f(\alpha) = 0$. An element in $\mathbb{F}_q((Z^{-1}))$ which is not algebraic is said to be a transcendental number.

It follows easily from the definition of algebraic numbers that almost all elements in $\mathbb{F}_q((Z^{-1}))$ are transcendental. The following theorem characterizes transcendence in this space.

Theorem 3.2.3 (Liouville Theorem in Positive Characteristic)**.** [31] *Let $\alpha \in \mathbb{F}_q((Z^{-1}))$ be an algebraic number of degree $n > 1$. Then there exists a constant $c(\alpha) > 0$ such that*

$$\left|\alpha - \frac{P}{Q}\right| > \frac{c(\alpha)}{|Q|^n}$$

for all $P, Q \in \mathbb{F}_q[Z]$ with $Q \neq 0$.

It follows immediately from the Liouville theorem that all Liouville numbers are transcendental; that is, Liouville numbers are examples of elements in $\mathbb{F}_q((Z^{-1}))$ which are not a root of any nonzero polynomial equation with coefficients from $\mathbb{F}_q[Z]$. Typical examples of Liouville numbers are $\sum_{n=1}^{\infty} a_{-n} Z^{-n!}$, where $a_{-n} \in \mathbb{F}_q$ with infinitely many $a_{-n} \neq 0$, and it follows easily from these examples that the set of Liouville numbers is uncountable. Therefore, we have the following results.

Lemma 3.2.4. *Let $(a_{-n})_{n=1}^{\infty}$ be a sequence in $\mathbb{F}_q$ with infinitely many $a_{-n} \neq 0$. Then $\sum_{n=1}^{\infty} a_{-n} Z^{-n!}$ is a Liouville number.*

Proof of Lemma 3.2.4. Let $\alpha = \sum_{n=1}^{\infty} a_{-n} Z^{-n!}$. It is clear that the sequence of the coefficients of the formal Laurent series representing α is not ultimately periodic, so it follows that $\alpha \notin \mathbb{F}_q(Z)$.

Now, for any $n \in \mathbb{N}$, define P_n and Q_n in $\mathbb{F}_q[Z]$ with $|Q_n| > 1$ as follows:

$$Q_n = Z^{n!} \qquad \text{and} \qquad P_n = Q_n \sum_{j=1}^{n} a_{-j} Z^{-j!}.$$

Then we have

$$\left|\alpha - \frac{P_n}{Q_n}\right| = \left|\sum_{j=n+1}^{\infty} a_{-j} Z^{-j!}\right| = q^{-(n+1)!} = |Q_n|^{-(n+1)} < |Q_n|^{-n}.$$

Therefore, we conclude that any such α is a Liouville number. ■

Theorem 3.2.5. *The set of Liouville numbers $\mathscr{L}$ is uncountable.*

Proof of Theorem 3.2.5. It is an immediate consequence of the fact that there are uncountably many $\sum_{n=1}^{\infty} a_{-n} Z^{-n!}$, where $a_{-n} \in \mathbb{F}_q$ with infinitely many $a_{-n} \neq 0$. ■

We can investigate a little further the complexity of the Liouville numbers, and we end this section by showing that the set of Liouville numbers is dense in $\mathbb{F}_q((Z^{-1}))$. This unsurprising fact is a consequence of the Baire category theorem and the observation in (3.2.1). A *Baire space* is a topological space X with the property that the intersection of any countable collection of open dense sets in X is dense in X.

Lemma 3.2.6 (Baire Category Theorem)**.** *A complete metric space is a Baire space.*

Theorem 3.2.7. *The set of Liouville numbers $\mathscr{L}$ is dense in $\mathbb{F}_q((Z^{-1}))$.*

Proof of Theorem 3.2.7. Note that the field of formal Laurent series is a complete metric space, so, by the Baire category theorem, $\mathbb{F}_q((Z^{-1}))$ is a Baire space. Then we consider the definition of Liouville numbers in (3.2.1) and observe that, for each $n \in \mathbb{N}$, the countable union of balls containing all the rational functions in $\mathbb{F}_q(Z)$

$$\bigcup_{\substack{Q\in\mathbb{F}_q[Z]\\ |Q|>1}} \bigcup_{P\in\mathbb{F}_q[Z]} B\left(\frac{P}{Q}; \frac{1}{|Q|^n}\right)$$

is dense in $\mathbb{F}_q((Z^{-1}))$. Since $\mathbb{F}_q((Z^{-1}))$ is a Baire space, it follows immediately that $\mathscr{L}$ is dense in $\mathbb{F}_q((Z^{-1}))$. ■

Before proceeding to the next section, we give here some comment on the quality of rational approximation relating to Theorem 3.2.3. Let $\alpha \in \mathbb{F}_q((Z^{-1})) \setminus \mathbb{F}_q(Z)$.

For each positive real number $\varepsilon > 0$, define

$$\kappa(\alpha, \varepsilon) = \liminf_{|Q|\to\infty} |Q|^{\varepsilon} \left|\alpha - \frac{P}{Q}\right|,$$

where P and Q run over polynomials in $\mathbb{F}_q[Z]$ with $Q \neq 0$. The *approximation exponent* of α is defined by

$$\eta(\alpha) = \sup\{\varepsilon > 0 \colon \kappa(\alpha, \varepsilon) < \infty\}.$$

It is clear that $\kappa(\alpha, \varepsilon) = \infty$ if $\varepsilon > \eta(\alpha)$, that $\kappa(\alpha, \varepsilon) = 0$ if $\varepsilon < \eta(\alpha)$, and that $0 \leq \kappa(\alpha, \eta(\alpha)) \leq \infty$. By using some knowledge of the continued fraction algorithm in positive characteristic, which will be developed in Chapter 4, it is true as in the real case that, for every $\varepsilon \in [2, \infty]$, there exists $\alpha \in \mathbb{F}_q((Z^{-1})) \setminus \mathbb{F}_q(Z)$ such that $\eta(\alpha) = \varepsilon$. Moreover, the Liouville theorem in positive characteristic says that if α is an algebraic number of degree $n > 1$, then $\kappa(\alpha, n) > 0$, and therefore $\eta(\alpha) \in [2, n]$. For a detailed survey on this subject, the reader is referred to [29].

Definition 3.2.8. *Let $\alpha \in \mathbb{F}_q((Z^{-1})) \setminus \mathbb{F}_q(Z)$ be an irrational element. We call α a* badly approximable number *if $\eta(\alpha) = 2$ and $\kappa(\alpha, 2) > 0$.*

The definition of a badly approximable number α is equivalent to saying that the partial quotients in the continued fraction expansion for α are bounded. Clearly, by the Liouville theorem in positive characteristic, all quadratic power series are badly approximable. This fact is also a consequence of their particular continued fraction expansion. It is worth underlining that Lasjaunias successfully described in [28] the explicit continued fraction expansion for each algebraic number of degree greater than 2 that is known to be badly approximable. This is still a very important open question in the classical real case.

When α is badly approximable, we are usually interested in the best possible value of $c(\alpha)$ in Theorem 3.2.3. In the field of formal Laurent series, Fuchs proved in [16] an analogous result of the Hurwitz theorem.

Theorem 3.2.9 (Hurwitz Theorem in Positive Characteristic)**.** [16] *Let $0 < q' < q$. Then, for all irrational element $\alpha \in \mathbb{F}_q((Z^{-1})) \setminus \mathbb{F}_q(Z)$, the inequality*

$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{q'|Q|^2}$$

has infinitely many solutions $P, Q \in \mathbb{F}_q[Z]$ with $Q \neq 0$.

Note that, in the Hurwitz theorem in positive characteristic, if $q' \geq q$, then the inequality is not true in general. Consider for example $\alpha = [0; Z, Z, Z, \dots]$. For more references of the research in this direction, the reader should consult [16].

3.3 On the complexity of the Liouville numbers

In the previous section, we see that there are uncountably many Liouville numbers and that they are dense in $\mathbb{F}_q((Z^{-1}))$. However, we shall show in this section that the set of Liouville numbers is very small in the sense of measure and dimension. In fact, we employ and adapt Oxtoby's method to prove that it has Hausdorff dimension zero and Haar measure zero.

Theorem 3.3.1. [H] *For all $s > 0$, $\mathcal{H}^s(\mathscr{L}) = 0$. In particular,* $\dim_{\mathrm{H}} \mathscr{L} = 0$.

Proof of Theorem 3.3.1. Note that we can write the field of formal Laurent series as a countable disjoint union of balls; that is, $\mathbb{F}_q((Z^{-1})) = \bigcup_{A \in \mathbb{F}_q[Z]} B(A; 1)$. Since Hausdorff measure satisfies the countably additive property, it suffices to show, for each $s > 0$, that we have $\mathcal{H}^s(\mathscr{L} \cap B(A; 1)) = 0$ for all $A \in \mathbb{F}_q[Z]$.

Let $s > 0$, and let $A \in \mathbb{F}_q[Z]$. To prove that $\mathcal{H}^s(\mathscr{L} \cap B(A; 1)) = 0$, we need to find, for each $\delta > 0$, a countable cover $\{B_j\}_{j=1}^{\infty}$ of $\mathscr{L} \cap B(A; 1)$ such that $\mathrm{diam} B_j < \delta$ and $\sum_{j=1}^{\infty} (\mathrm{diam} B_j)^s < \delta$. From identity (3.2.1), it follows that

$$\mathscr{L} \cap B(A; 1) \subseteq \bigcap_{n=1}^{\infty} \bigcup_{\substack{Q \in \mathbb{F}_q[Z] \\ |Q|>1}} \bigcup_{\substack{P \in \mathbb{F}_q[Z] \\ |P|<|Q|}} B\left(A + \frac{P}{Q}; \frac{1}{|Q|^n}\right).$$

That is, for each $n \in \mathbb{N}$,

$$\bigcup_{\substack{Q \in \mathbb{F}_q[Z] \\ |Q|>1}} \bigcup_{\substack{P \in \mathbb{F}_q[Z] \\ |P|<|Q|}} B\left(A + \frac{P}{Q}; \frac{1}{|Q|^n}\right) \tag{3.3.1}$$

is a countable cover of $\mathscr{L} \cap B(A; 1)$ by balls. We can choose n so large that it satisfies simultaneously the following three conditions:

$$\frac{1}{q^n} < \delta, \qquad ns > 2, \qquad \frac{q-1}{(ns-2)q^{ns-2} \log q} < \delta.$$

Then each of the balls in (3.3.1) has diameter

$$\mathrm{diam} B\left(A+\frac{P}{Q};\frac{1}{|Q|^n}\right)=\sup\left\{|\alpha-\beta|\colon \alpha,\beta\in B\left(A+\frac{P}{Q};\frac{1}{|Q|^n}\right)\right\}<\frac{1}{|Q|^n}\leq\frac{1}{q^n}<\delta.$$

We note two basic combinatorial results:

$$\#\big\{P\in\mathbb{F}_q[Z]\colon |P|<|Q|\big\}=|Q| \qquad \text{and} \qquad \#\big\{Q\in\mathbb{F}_q[Z]\colon |Q|=q^r\big\}=(q-1)q^r.$$

We now have

$$\begin{aligned}\sum_{\substack{Q\in\mathbb{F}_q[Z]\\|Q|>1}}\sum_{\substack{P\in\mathbb{F}_q[Z]\\|P|<|Q|}}\left(\frac{1}{|Q|^n}\right)^s &= \sum_{\substack{Q\in\mathbb{F}_q[Z]\\|Q|>1}}\frac{|Q|}{|Q|^{ns}}=\sum_{\substack{Q\in\mathbb{F}_q[Z]\\|Q|>1}}\frac{1}{|Q|^{ns-1}}\\ &=\sum_{r=1}^{\infty}\frac{(q-1)q^r}{q^{r(ns-1)}}=(q-1)\sum_{r=1}^{\infty}\frac{1}{q^{r(ns-2)}}\\ &\leq (q-1)\int_1^{\infty}\frac{1}{q^{r(ns-2)}}\,dr\\ &=(q-1)\left(\frac{1}{(ns-2)\log q}\right)\left(\frac{1}{q^{ns-2}}\right)<\delta.\end{aligned}$$

This completes the proof of Theorem 3.3.1. ■

Corollary 3.3.2. [H] $\mu(\mathscr{L})=\mathcal{H}^1(\mathscr{L})=0$.

Proof of Corollary 3.3.2. This follows from the fact that μ and $\mathcal{H}^1$ are comparable. ■

3.4 Exact Hausdorff dimension

In order to investigate sets with Hausdorff dimension zero, it is useful to introduce a finer notion of dimension that allows more discrimination than power functions.

Definition 3.4.1. *A* dimension function*, or a* gauge function*, is a non-decreasing and right continuous function* $h:[0,\infty)\to[0,\infty)$ *with* $h(0)=0$.

For a dimension function h, define the *h-dimensional Hausdorff measure* $\mathcal{H}^h(E)$ of $E\subseteq\mathbb{F}_q((Z^{-1}))$ by

$$\mathcal{H}^h(E)=\lim_{\delta\to0^+}\left(\inf\left\{\sum_{j=1}^{\infty}h(\mathrm{diam}B_j)\colon E\subseteq\bigcup_{j=1}^{\infty}B_j,\ \mathrm{diam}B_j<\delta\right\}\right)\in[0,\infty],$$

where the infimum is taken over all countable covers of E by balls with diameter less than δ. Note that if $s\geq0$ and $h(r)=r^s$, then, by a transparent abuse of notation $\mathcal{H}^s=\mathcal{H}^h$. For a comprehensive treatment of the Hausdorff measure $\mathcal{H}^h$, the reader should consult [47]. We shall say that h is an *exact Hausdorff dimension* for E if $0<\mathcal{H}^h(E)<\infty$.

It follows from Theorem 3.3.1 that no power function $h(r) = r^s$ gives the set of Liouville numbers finite positive h-dimensional Hausdorff measure. In this view, it is natural to ask whether the set of Liouville numbers has an exact Hausdorff dimension or not. This question can be answered by using the translation invariance property of $\mathscr{L}$; for related results, see [8], [41] and [42].

Lemma 3.4.2. [H] *Let $E \subseteq \mathbb{F}_q((Z^{-1}))$ be a set such that $E + A = E$ for all $A \in \mathbb{F}_q[Z]$, and let λ be a translation invariant measure on $\mathbb{F}_q((Z^{-1}))$. Then $\lambda(E) \in \{0, \infty\}$.*

Proof of Lemma 3.4.2. Suppose that $\lambda(E) > 0$. We claim that $\lambda(E) = \infty$. Since we have $\sum_{A \in \mathbb{F}_q[Z]} \lambda(E \cap B(A;1)) = \lambda(E) > 0$, there exists $A_0 \in \mathbb{F}_q[Z]$ such that $\lambda(E \cap B(A_0;1)) > 0$. Then, for all $A \in \mathbb{F}_q[Z]$, we obtain

$$\begin{aligned}\lambda(E \cap B(A;1)) &= \lambda((E \cap B(A;1)) + (A_0 - A)) \\ &= \lambda((E + (A_0 - A)) \cap (B(A;1) + (A_0 - A))) \\ &= \lambda(E \cap B(A_0;1)).\end{aligned}$$

Therefore, $\lambda(E) = \sum_{A \in \mathbb{F}_q[Z]} \lambda(E \cap B(A;1)) = \sum_{A \in \mathbb{F}_q[Z]} \lambda(E \cap B(A_0;1)) = \infty$. This completes the proof of Lemma 3.4.2. ■

Theorem 3.4.3. [H] *For any dimension function h, we have $\mathcal{H}^h(\mathscr{L}) \in \{0, \infty\}$.*

Proof of Theorem 3.4.3. It is clear that $\mathcal{H}^h$ is a translation invariant measure for any dimension function h. Moreover, the set $\mathscr{L}$ of Liouville numbers satisfies the condition that $\mathscr{L} + A = \mathscr{L}$ for all $A \in \mathbb{F}_q[Z]$. It is an immediate consequence of Lemma 3.4.2 that $\mathcal{H}^h(\mathscr{L}) \in \{0, \infty\}$. ■

3.5 On the exact Hausdorff dimension of the Liouville numbers

Due to Theorem 3.4.3, it is clearly of interest to ask two further questions about the complexity of the Liouville numbers:

(1) For which dimension function h is $\mathcal{H}^h(\mathscr{L}) = 0$, and for which dimension function h is $\mathcal{H}^h(\mathscr{L}) = \infty$?

(2) Does there exist a dimension function h such that $\mathcal{H}^h(\mathscr{L}) > 0$ and the set $\mathscr{L}$ of Liouville numbers has σ-finite $\mathcal{H}^h$ measure?

The first question asks one to locate the exact cut-point at which the Hausdorff measure of $\mathscr{L}$ drops from infinity to zero. The second question asks, in the case $\mathcal{H}^h(\mathscr{L}) = \infty$, whether the h-dimensional Hausdorff measure is truly infinite, i.e. $\mathscr{L}$ is a countable union of sets with finite h-dimension Hausdorff measure or not.

To answer these two questions, we begin with a definition. For a dimension function h, define the function Φ_h by

$$\Phi_h(r) = \inf_{0<t\leq r} r\frac{h(t)}{r}.$$

Theorem 3.5.1. [H] *Let h be an arbitrary dimension function.*

(1) If $\limsup_{q^\omega\to 0,\, \omega\in\mathbb{Z}} \Phi_h(q^\omega)/q^{\omega s} = 0$ for some $s > 0$, then we have $\mathcal{H}^h(\mathscr{L}) = 0$.

(2) If $\limsup_{q^\omega\to 0,\, \omega\in\mathbb{Z}} \Phi_h(q^\omega)/q^{\omega s} > 0$ for all $s > 0$, then we have $\mathcal{H}^h(\mathscr{L}) = \infty$ and the set $\mathscr{L}$ does not have σ-finite $\mathcal{H}^h$ measure.

This result gives a complete characterization of all Hausdorff measures $\mathcal{H}^h(\mathscr{L})$ on the set of Liouville numbers, and thus answers the last two questions. Indeed, firstly, it locates the exact cut-point at which the Hausdorff measure of $\mathscr{L}$ drops from infinity to zero. If h is a dimension function for which the function $\Phi_h(q^\omega)$ increases more slowly than a particular power function of q^ω near 0, then $\mathcal{H}^h(\mathscr{L}) = 0$, and if h is a dimension function for which the function $\Phi_h(q^\omega)$ increases faster than any power function of q^ω near 0, then $\mathcal{H}^h(\mathscr{L}) = \infty$. Also, it shows that if the h-dimensional Hausdorff measure of $\mathscr{L}$ is infinite, then $\mathscr{L}$ does not have σ-finite h-dimensional Hausdorff measure. Note that Theorem 3.5.1 considers only $r = q^\omega$ ($\omega \in \mathbb{Z}$) rather than arbitrary nonnegative $r \in \mathbb{R}$ as in the real case, and this theorem is an analogue of Olsen and Renfro's result in [43]. To prove this theorem, we need several lemmas, so we postpone the proof of Theorem 3.5.1 until the last section of this chapter.

3.6 Preliminary results I

To prove Theorem 3.5.1, we need some preliminary results, which are divided into two sections. In Preliminary results I, we prove a special case about the exact cut-point at which the Hausdorff measure of $\mathscr{L}$ drops from infinity to zero. In fact, we shall make a requirement in Lemma 3.6.1 that the function $r \mapsto h(r)/r$ is decreasing in a neighborhood of 0. In Preliminary results II, we collect some properties of the function Φ_h. Then we use these properties to extend the results in Lemma 3.6.1 to a complete characterization of all Hausdorff measures $\mathcal{H}^h(\mathscr{L})$ of the set of Liouville numbers.

Lemma 3.6.1. [H] *Let h be a dimension function.*

(1) If $\limsup_{q^\omega\to 0,\, \omega\in\mathbb{Z}} h(q^\omega)/q^{\omega s} = 0$ for some $s > 0$, then we have $\mathcal{H}^h(\mathscr{L}) = 0$.

(2) If $\limsup_{q^\omega\to 0,\, \omega\in\mathbb{Z}} h(q^\omega)/q^{\omega s} > 0$ for all $s > 0$, and if $r \mapsto h(r)/r$ is a decreasing function in a neighborhood of 0, then we have $\mathcal{H}^h(\mathscr{L}) = \infty$.

Proof of Lemma 3.6.1(1). This result is simply a rephrasing of Theorem 3.3.1 that the Hausdorff dimension of $\mathscr{L}$ equals 0. ■

Proof of Lemma 3.6.1(2). Let h be a dimension function such that, for all $s > 0$,

$$\limsup_{\substack{q^\omega \to 0 \\ \omega \in \mathbb{Z}}} \frac{h(q^\omega)}{q^{\omega s}} > 0,$$

and suppose that the function $r \mapsto h(r)/r$ is decreasing in a neighborhood of 0. Observe that, for all $s > 0$, we have

$$\limsup_{\substack{q^\omega \to 0 \\ \omega \in \mathbb{Z}}} \frac{h(q^\omega)}{q^{\omega s}} = \limsup_{\substack{q^\omega \to 0 \\ \omega \in \mathbb{Z}}} \frac{1}{(q^\omega)^{s/2}} \frac{h(q^\omega)}{(q^\omega)^{s/2}} = \infty. \tag{3.6.1}$$

To prove the result, we suppose to the contrary that $\mathcal{H}^h(\mathscr{L}) < \infty$. Since h is continuous from the right with $h(0) = 0$, it follows easily that $\mathcal{H}^h(\mathbb{F}_q(Z)) = 0$, and thus we have $\mathcal{H}^h(\mathscr{L} \cup \mathbb{F}_q(Z)) < \infty$. Then there is a real number $c_0 > 1$ such that $\mathcal{H}^h(\mathscr{L} \cup \mathbb{F}_q(Z)) < c_0$. Moreover, it follows from (3.6.1) that there exists a natural number ω_0 such that

$$\frac{h(q^{-\omega_0})}{q^{-\omega_0}} \geq q^2 c_0. \tag{3.6.2}$$

Since $\mathcal{H}^h(\mathscr{L} \cup \mathbb{F}_q(Z)) < c_0$, there exists a countable cover $\{B_j\}_{j=1}^\infty$ of $\mathscr{L} \cup \mathbb{F}_q(Z)$ by balls with $\mathrm{diam} B_j < q^{-\omega_0}$ such that $\sum_{j=1}^\infty h(\mathrm{diam} B_j) \leq c_0$.

We shall now construct a sequence $(E_n)_{n=1}^\infty$ of sets such that

$$\bigcap_{n=1}^\infty E_n \neq \emptyset; \tag{3.6.3}$$

$$\bigcap_{n=1}^\infty E_n \subseteq \mathscr{L} \cup \mathbb{F}_q(Z); \tag{3.6.4}$$

$$\left(\bigcap_{n=1}^\infty E_n \right) \cap \left(\bigcup_{j=1}^\infty B_j \right) = \emptyset. \tag{3.6.5}$$

Since $\mathscr{L} \cup \mathbb{F}_q(Z) \subseteq \bigcup_{j=1}^\infty B_j$, this will give the desired contradiction.

The construction of the sets E_n is divided into four steps. We begin by giving some notation. For a natural number n, we write $\mathbb{F}_q[Z]^n$ for the family of strings $P_1 \dots P_n$ of length n with entries $P_i \in \mathbb{F}_q[Z]$, i.e.

$$\mathbb{F}_q[Z]^n = \{P_1 \dots P_n \colon P_i \in \mathbb{F}_q[Z]\}.$$

First, we construct an auxiliary sequence $(Q_n)_{n=1}^\infty$ in $\mathbb{F}_q[Z]$. Next, by using this sequence $(Q_n)_{n=1}^\infty$, we construct a sequence of sets $(\Pi_n)_{n=1}^\infty$ and $I_{\mathbf{P}}$ for $\mathbf{P} \in \Pi_n$ with $\Pi_n \subseteq \mathbb{F}_q[Z]^n$ and $I_{\mathbf{P}} \subseteq \mathscr{L}$. Then we construct a sequence of sets $(\Gamma_n)_{n=1}^\infty$ with $\Gamma_n \subseteq \Pi_n$. Finally, we construct the sequence $(E_n)_{n=1}^\infty$. ■

Construction of $(Q_n)_{n=1}^{\infty}$: Let $c = q^8 c_0$. It follows from (3.6.1) that there exists an increasing sequence $(\omega_n)_{n=1}^{\infty}$ of natural numbers such that

$$\frac{h(q^{-\omega_n})}{(q^{-\omega_n})^{1/2n}} \geq 1; \tag{3.6.6}$$

$$q^{-\omega_n} \leq \frac{1}{q^n}\left(\frac{q^{-\omega_n-1}}{cq^{n-1}}\right)^{2n}. \tag{3.6.7}$$

Now we can construct inductively a sequence $(Q_n)_{n=1}^{\infty}$ in $\mathbb{F}_q[Z]$ so that the following two conditions are satisfied for all $n \geq 2$:

(I.1) we have

$$|Q_n| \geq (cq^{n-1}|Q_{n-1}|^{n-1})^2;$$

(I.2) we have

$$\frac{1}{q^n|Q_n|^n} \leq q^{-\omega_n} \leq \frac{1}{|Q_n|^n}.$$

The start of the induction. We simply put

$$Q_1 = 1.$$

The inductive step. Assume that $n \geq 2$ and that $Q_1, Q_2, \ldots, Q_{n-1}$ have been constructed such that (I.1) and (I.2) are satisfied. Observe that

$$(q^{\omega_n})^{1/n} - \left(\frac{q^{\omega_n}}{q^n}\right)^{1/n} = q^{(\omega_n/n)-1}(q-1).$$

This implies without difficulty that there exists a Q_n in $\mathbb{F}_q[Z]$ with

$$\left(\frac{q^{\omega_n}}{q^n}\right)^{1/n} \leq |Q_n| \leq (q^{\omega_n})^{1/n}. \tag{3.6.8}$$

Then it follows immediately that Q_n satisfies condition (I.2). To see that Q_n satisfies condition (I.1), we use (3.6.7) and (3.6.8) to obtain

$$|Q_n| \geq \left(\frac{q^{\omega_n}}{q^n}\right)^{1/n} \geq \left(\frac{cq^{n-1}}{q^{-\omega_n-1}}\right)^2 \geq (cq^{n-1}|Q_{n-1}|^{n-1})^2.$$

This completes the inductive step in the construction of the sequence $(Q_n)_{n=1}^{\infty}$.

Construction of $(\Pi_n)_{n=1}^{\infty}$ and $I_{\mathbf{P}}$: For each $n \in \mathbb{N}$ and $P \in \mathbb{F}_q[Z]$, we define sets $\Sigma_{n,P} \subseteq \mathbb{F}_q[Z]$, $\Sigma_n \subseteq \mathbb{F}_q[Z]$, $\Pi_n \subseteq \mathbb{F}_q[Z]^n$ and $I_{\mathbf{P}} \subseteq \mathscr{L}$ for $\mathbf{P} \in \Pi_n$ inductively as follows.

Put

$$\begin{aligned}\Sigma_{1,1} &= \{1\},\\ \Sigma_1 &= \Sigma_{1,1},\\ \Pi_1 &= \Sigma_1,\\ I_1 &= B\left(\frac{1}{Q_1}; \frac{1}{|Q_1|^1}\right).\end{aligned}$$

Next, assume that $n \geq 2$ and that the sets $\Sigma_{n-1,P}$, Σ_{n-1}, Π_{n-1} and $I_{\mathbf{P}}$ for $\mathbf{P} \in \Pi_{n-1}$ have been defined. Now put

$$\begin{aligned}\Sigma_{n,P} &= \mathbf{F}_q[Z] \cap B\left(Q_n \frac{P}{Q_{n-1}}; \frac{|Q_n|}{|Q_{n-1}|^{n-1}}\right) \qquad \text{for } P \in \Sigma_{n-1},\\ \Sigma_n &= \bigcup_{P \in \Sigma_{n-1}} \Sigma_{n,P},\\ \Pi_n &= \left\{P_1 P_2 \dots P_n \in \mathbb{F}_q[Z]^n \colon P_1 \in \Sigma_{1,1}, P_2 \in \Sigma_{2,P_1}, \dots, P_n \in \Sigma_{n,P_{n-1}}\right\},\\ I_{\mathbf{P}} &= B\left(\frac{P_n}{Q_n}; \frac{1}{|Q_n|^n}\right) \qquad \text{for } \mathbf{P} = P_1 P_2 \dots P_n \in \Pi_n.\end{aligned}$$

Below we collect some of the main properties of the balls $I_{\mathbf{P}}$. We shall denote the distance between two sets $X, Y \subseteq \mathbb{F}_q((Z^{-1}))$ by $\operatorname{dist}(X, Y) = \inf_{x \in X,\, y \in Y} |x - y|$.

Proposition 3.6.2. [H] *We have the following.*

(1) If $\mathbf{P}P \in \Pi_{n+1}$ with $\mathbf{P} \in \Pi_n$, then we have

$$I_{\mathbf{P}P} \subseteq I_{\mathbf{P}}.$$

(2) If $\mathbf{P}_1, \mathbf{P}_2 \in \Pi_n$ are distinct, then we have

$$\operatorname{dist}(I_{\mathbf{P}_1}, I_{\mathbf{P}_2}) \geq \frac{1}{|Q_n|} > 0.$$

(3) We have

$$\bigcap_{n=1}^{\infty} \bigcup_{\mathbf{P} \in \Pi_n} I_{\mathbf{P}} \subseteq \mathscr{L} \cup \mathbb{F}_q(Z).$$

Proof of Proposition 3.6.2. Statement (1) is obvious by an easy calculation. Next, we observe the fact that if $\mathbf{P}_1, \mathbf{P}_2 \in \Pi_n$ are distinct, and if $\mathbf{P}_1 = P_{1,1} P_{1,2} \dots P_{1,n}$ and $\mathbf{P}_2 = P_{2,1} P_{2,2} \dots P_{2,n}$, then we must have $P_{1,n} \neq P_{2,n}$. Now statement (2) follows by an easy calculation. Finally, we prove statement (3). Let $\alpha \in \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{P} \in \Pi_n} I_{\mathbf{P}}$. For each $n \in \mathbb{N}$, we have $\alpha \in \bigcup_{\mathbf{P} \in \Pi_n} I_{\mathbf{P}}$, so there exists $\mathbf{P} = P_1 P_2 \dots P_n \in \Pi_n$ such that $\alpha \in I_{\mathbf{P}} = B(P_n/Q_n; |Q_n|^{-n})$, whence $|\alpha - P_n/Q_n| < |Q_n|^{-n}$. It now follows from the definition of a Liouville number that $\alpha \in \mathscr{L} \cup \mathbb{F}_q(Z)$. ∎

Construction of $(\Gamma_n)_{n=1}^{\infty}$: We now construct sets $\Gamma_1 \subseteq \Pi_1$, $\Gamma_2 \subseteq \Pi_2$, ... inductively such that the following three conditions are satisfied for all $n \geq 2$:

(II.1) if $\mathrm{diam} B_j \geq 1/(q|Q_n|)^n$, then we have

$$B_j \cap I_{\mathbf{P}} = \emptyset$$

for all $\mathbf{P} \in \Gamma_n$;

(II.2) we have

$$\#\Gamma_n \geq \frac{|Q_n|}{q^2|Q_{n-1}|^{n-1}} \cdot \#\Gamma_{n-1};$$

(II.3) we have

$$\Gamma_n \subseteq \big\{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1}\big\}.$$

The start of the induction. We simply put

$$\Gamma_1 = \Pi_1.$$

The inductive step. Assume that $n \geq 2$ and that the sets $\Gamma_1 \subseteq \Pi_1$, $\Gamma_2 \subseteq \Pi_2$, ..., $\Gamma_{n-1} \subseteq \Pi_{n-1}$ have been constructed such that conditions (II.1)–(II.3) are satisfied. Put

$$\Gamma_n = \left\{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1},\ \forall j \in \mathbb{N},\ \mathrm{diam}(B_j \cap I_{\mathbf{P}P}) < \frac{1}{q^{n-1}}(\mathrm{diam} I_{\mathbf{P}P})\right\}.$$

We shall now prove that the set Γ_n satisfies all the conditions (II.1)–(II.3). Write $X_n = \{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n=1}\} \setminus \Gamma_n$, i.e.

$$X_n = \left\{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1},\ \exists j \in \mathbb{N},\ \mathrm{diam}(B_j \cap I_{\mathbf{P}P}) \geq \frac{1}{q^{n-1}}(\mathrm{diam} I_{\mathbf{P}P})\right\},$$

and write, for each $j \in \mathbb{N}$,

$$X_{n,j} = \left\{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1},\ \mathrm{diam}(B_j \cap I_{\mathbf{P}P}) \geq \frac{1}{q^{n-1}}(\mathrm{diam} I_{\mathbf{P}P})\right\}.$$

We clearly have $X_n = \bigcup_{j=1}^{\infty} X_{n,j}$, and we note that this union may not be disjoint. We now prove the following four propositions. To avoid confusion, we emphasize that these propositions are part of the inductive step. In particular, the variable n should be interpreted as it is used in the inductive step.

Proposition 3.6.3. [H] *We have*

$$\#\big\{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1}\big\} = \frac{|Q_n|}{|Q_{n-1}|^{n-1}} \cdot \#\Gamma_{n-1}.$$

Proposition 3.6.4. [H] *For each $j \in \mathbb{N}$, we have*

$$\#X_{n,j} \leq q|Q_n|(\mathrm{diam}B_j) + \frac{h(\mathrm{diam}B_j)}{h(1/q^n|Q_n|^n)}.$$

Proposition 3.6.5. [H] *If $2 \leq m \leq n$, then $\#\Gamma_{m-1} \geq 1$ and*

$$\frac{h(1/q^m|Q_m|^m)}{1/q^m|Q_m|^m} \geq cq^{m-1}|Q_m|^{m-1}|Q_{m-1}|^{m-1}\frac{1}{\#\Gamma_{m-1}}.$$

Proposition 3.6.6. [H] *We have*

$$\#X_n \leq \left(1 - \frac{1}{q^2}\right) \cdot \#\{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1}\}.$$

Proof of Proposition 3.6.3. By the construction of $(Q_n)_{n=1}^{\infty}$, we note from (I.1) that $|Q_n| > |Q_{n-1}|^{n-1}$. We now have

$$\begin{aligned}
\#\{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1}\} &= \sum_{\mathbf{P}=P_1\ldots P_{n-1}\in\Gamma_{n-1}} \#\left(\mathbb{F}_q[Z] \cap B\left(Q_n\frac{P_{n-1}}{Q_{n-1}}; \frac{|Q_n|}{|Q_{n-1}|^{n-1}}\right)\right) \\
&= \sum_{\mathbf{P}=P_1\ldots P_{n-1}\in\Gamma_{n-1}} \mu\left(B\left(Q_n\frac{P_{n-1}}{Q_{n-1}}; \frac{|Q_n|}{|Q_{n-1}|^{n-1}}\right)\right) \\
&= \sum_{\mathbf{P}\in\Gamma_{n-1}} \frac{|Q_n|}{|Q_{n-1}|^{n-1}} \\
&= \frac{|Q_n|}{|Q_{n-1}|^{n-1}} \cdot \#\Gamma_{n-1},
\end{aligned}$$

as required. ■

Proof of Proposition 3.6.4. We divide the proof into two cases. First, we assume that $\#X_{n,j} = 1$. In this case, $X_{n,j} = \{\mathbf{P}\}$ for some $\mathbf{P} \in \Pi_n$, whence

$$\mathrm{diam}B_j \geq \mathrm{diam}(B_j \cap I_{\mathbf{P}}) \geq \frac{1}{q^{n-1}}(\mathrm{diam}I_{\mathbf{P}}) = \frac{1}{q^n|Q_n|^n}.$$

This implies that $h(\mathrm{diam}B_j) \geq h(1/q^n|Q_n|^n)$, and so

$$\#X_{n,j} = 1 \leq \frac{h(\mathrm{diam}B_j)}{h(1/q^n|Q_n|^n)} \leq q|Q_n|(\mathrm{diam}B_j) + \frac{h(\mathrm{diam}B_j)}{h(1/q^n|Q_n|^n)}.$$

This proves Proposition 3.6.4 when $\#X_{n,j} = 1$. Next, we assume that $\#X_{n,j} \geq 2$. In this case, $X_{n,j} = \{\mathbf{P}_1, \ldots, \mathbf{P}_m\}$ for some distinct $\mathbf{P}_i \in \Pi_n$ and $m \geq 2$. Observe that $B_j \cap I_{\mathbf{P}_i} \neq \emptyset$ for all $1 \leq i \leq m$. Moreover, by Proposition 3.6.2(2), we see that the balls $I_{\mathbf{P}_i}$ are pairwise disjoint. It follows that B_j must contain every ball $I_{\mathbf{P}_i}$. Since B_j is a ball, we deduce that B_j must contain the gaps between the balls $I_{\mathbf{P}_i}$. By Proposition 3.6.2(2), we see that the length of each gap is greater than or equal to $1/|Q_n|$. Now let ω be the unique nonnegative integer such that $q^{\omega} \leq m < q^{\omega+1}$. It

follows that there exist i_1 and i_2 with $1 \le i_1 < i_2 \le m$ such that the distance between $I_{\mathbf{P}_{i_1}}$ and $I_{\mathbf{P}_{i_2}}$ is greater than or equal to $q^\omega/|Q_n|$. This implies that

$$\mathrm{diam} B_j \ge \mathrm{dist}\big(I_{\mathbf{P}_{i_1}}, I_{\mathbf{P}_{i_2}}\big) \ge \frac{q^\omega}{|Q_n|} > \frac{m}{q|Q_n|},$$

and so we obtain

$$\#X_{n,j} = m \le q|Q_n|(\mathrm{diam} B_j) \le q|Q_n|(\mathrm{diam} B_j) + \frac{h(\mathrm{diam} B_j)}{h(1/q^n|Q_n|^n)}.$$

This completes the proof of Proposition 3.6.4. ■

Proof of Proposition 3.6.5. We first show that $\#\Gamma_{m-1} \ge 1$. For $2 \le i \le n-1$, the inductive hypothesis (II.2) and the fact from (I.1) that $|Q_i| \ge (cq^{i-1}|Q_{i-1}|^{i-1})^2$ imply

$$\#\Gamma_i \ge \frac{|Q_i|}{q^2|Q_{i-1}|^{i-1}} \cdot \#\Gamma_{i-1} \ge \#\Gamma_{i-1}.$$

Repeated application of this inequality gives

$$\#\Gamma_{m-1} \ge \#\Gamma_{m-2} \ge \cdots \ge \#\Gamma_2 \ge \#\Gamma_1 = 1.$$

Next, we prove that

$$\frac{h(1/q^m|Q_m|^m)}{1/q^m|Q_m|^m} \ge cq^{m-1}|Q_m|^{m-1}|Q_{m-1}|^{m-1}\frac{1}{\#\Gamma_{m-1}}.$$

By (3.6.6), the inequality in (I.2) that $1/q^m|Q_m|^m \le q^{-\omega_m} \le 1/|Q_m|^m$ and the fact that $h(r)/r$ is non-decreasing near 0, we have

$$\frac{h(1/q^m|Q_m|^m)}{1/q^m|Q_m|^m} \ge \frac{h(q^{-\omega_m})}{q^{-\omega_m}} = \frac{1}{(q^{-\omega_m})^{1-1/2m}} \ge \frac{1}{(1/|Q_m|^m)^{1-1/2m}} = |Q_m|^{m-1/2}. \tag{3.6.9}$$

Also, since $|Q_m| \ge (cq^{m-1}|Q_{m-1}|^{m-1})^2$ and $\#\Gamma_{m-1} \ge 1$, we deduce that

$$\frac{1}{cq^{m-1}}|Q_m|^{m-1/2} \ge |Q_m|^{m-1}|Q_{m-1}|^{m-1} \ge |Q_m|^{m-1}|Q_{m-1}|^{m-1}\frac{1}{\#\Gamma_{m-1}}. \tag{3.6.10}$$

Combining (3.6.9) and (3.6.10) gives the desired result. ■

Proof of Proposition 3.6.6. We divide the proof into two cases. First, we prove the case $n = 2$. By Proposition 3.6.4, we have

$$\#X_2 \le \sum_{j=1}^{\infty} \#X_{2,j} \le q|Q_2| \sum_{j=1}^{\infty} \mathrm{diam} B_j + \frac{1}{h(1/q^2|Q_2|^2)} \sum_{j=1}^{\infty} h(\mathrm{diam} B_j). \tag{3.6.11}$$

Since $\mathrm{diam} B_j < q^{-\omega_0}$, we now use the fact that the map $r \mapsto h(r)/r$ is decreasing in a neighborhood of 0, together with (3.6.2), to obtain

$$\frac{h(\mathrm{diam} B_j)}{\mathrm{diam} B_j} > \frac{h(q^{-\omega_0})}{q^{-\omega_0}} \ge q^2 c_0.$$

We therefore conclude from (3.6.11) that

$$\begin{aligned}\#X_2 &\le \frac{q|Q_2|}{q^2c_0}\sum_{j=1}^{\infty} h(\mathrm{diam}B_j) + \frac{1}{h(1/q^2|Q_2|^2)}\sum_{j=1}^{\infty} h(\mathrm{diam}B_j)\\ &\le \frac{|Q_2|}{q} + \frac{1}{h(1/q^2|Q_2|^2)}c_0.\end{aligned}$$

By Proposition 3.6.5, we obtain

$$\frac{h(1/q^2|Q_2|^2)}{1/q^2|Q_2|^2} \ge cq|Q_2|^1|Q_1|^1\frac{1}{\#\Gamma_1} = cq|Q_2|.$$

Hence,

$$\#X_2 \le \frac{|Q_2|}{q} + \frac{q|Q_2|}{c}c_0 = \frac{|Q_2|}{q} + \frac{|Q_2|}{q^7}.$$

Also, Proposition 3.6.3 implies that

$$\#\big\{\mathbf{P}P \in \Pi_2 \colon \mathbf{P} \in \Gamma_1\big\} = \frac{|Q_2|}{|Q_1|^1}\cdot \#\Gamma_1 = |Q_2|.$$

This equation and the previous inequality give

$$\#X_2 \le \left(\frac{1}{q} + \frac{1}{q^7}\right)\cdot \#\big\{\mathbf{P}P \in \Pi_2 \colon \mathbf{P} \in \Gamma_1\big\} \le \left(1 - \frac{1}{q^2}\right)\cdot \#\big\{\mathbf{P}P \in \Pi_2 \colon \mathbf{P} \in \Gamma_1\big\}.$$

This completes the proof of Proposition 3.6.6 for $n = 2$.

Next, we prove the case $n \ge 3$. We have

$$\#X_n \le \sum_{j=1}^{\infty} \#X_{n,j}. \tag{3.6.12}$$

Now observe that

$$\text{if } j \text{ satisfies } \mathrm{diam}B_j \ge \frac{1}{q^{n-1}|Q_{n-1}|^{n-1}}, \quad \text{then } X_{n,j} = \emptyset. \tag{3.6.13}$$

We now prove (3.6.13). Indeed, if (3.6.13) were not satisfied, then there would exist $\mathbf{P}P \in \Pi_n$ with $\mathbf{P} \in \Gamma_{n-1}$ such that $\mathrm{diam}(B_j \cap I_{\mathbf{P}P}) \ge (1/q^{n-1})(\mathrm{diam}I_{\mathbf{P}P})$. Particularly, this would imply that $B_j \cap I_{\mathbf{P}P} \ne \emptyset$, and thus $B_j \cap I_{\mathbf{P}} \ne \emptyset$ by Proposition 3.6.2(1). But this would contradict the inductive hypothesis (II.1) since $\mathrm{diam}B_j \ge 1/q^{n-1}|Q_{n-1}|^{n-1}$. This proves (3.6.13).

It follows from (3.6.12), (3.6.13) and Proposition 3.6.4 that

$$\begin{aligned}\#X_n &\le \sum_{\substack{j=1\\ \mathrm{diam}B_j < 1/q^{n-1}|Q_{n-1}|^{n-1}}}^{\infty} \#X_{n,j}\\ &\le q|Q_n| \sum_{\substack{j=1\\ \mathrm{diam}B_j < 1/q^{n-1}|Q_{n-1}|^{n-1}}}^{\infty} \mathrm{diam}B_j + \frac{1}{h(1/q^n|Q_n|^n)}\sum_{j=1}^{\infty} h(\mathrm{diam}B_j).\end{aligned} \tag{3.6.14}$$

For each j with $\mathrm{diam}B_j < 1/q^{n-1}|Q_{n-1}|^{n-1}$, Proposition 3.6.5 implies that

$$\frac{h(\mathrm{diam}B_j)}{\mathrm{diam}B_j} \geq \frac{h(1/q^{n-1}|Q_{n-1}|^{n-1})}{1/q^{n-1}|Q_{n-1}|^{n-1}} \geq cq^{n-2}|Q_{n-1}|^{n-2}|Q_{n-2}|^{n-2}\frac{1}{\#\Gamma_{n-2}}.$$

We therefore conclude from (3.6.14) that

$$\begin{aligned}\#X_n &\leq \frac{q|Q_n|\cdot\#\Gamma_{n-2}}{cq^{n-2}|Q_{n-1}|^{n-2}|Q_{n-2}|^{n-2}} \sum_{\substack{j=1\\ \mathrm{diam}B_j<1/q^{n-1}|Q_{n-1}|^{n-1}}}^{\infty} h(\mathrm{diam}B_j)\\ &\quad + \frac{1}{h(1/q^n|Q_n|^n)}\sum_{j=1}^{\infty} h(\mathrm{diam}B_j)\\ &\leq \frac{|Q_n|\cdot\#\Gamma_{n-2}}{cq^{n-3}|Q_{n-1}|^{n-2}|Q_{n-2}|^{n-2}}c_0 + \frac{1}{h(1/q^n|Q_n|^n)}c_0.\end{aligned}$$

By using the inductive hypothesis (II.2), we have

$$\#\Gamma_{n-1} \geq \frac{|Q_{n-1}|}{q^2|Q_{n-2}|^{n-2}}\cdot\#\Gamma_{n-2}.$$

Hence, we obtain

$$\#X_n \leq \frac{|Q_n|\cdot\#\Gamma_{n-1}}{cq^{n-5}|Q_{n-1}|^{n-1}}c_0 + \frac{1}{h(1/q^n|Q_n|^n)}c_0.$$

By Proposition 3.6.5, we know that

$$\frac{h(1/q^n|Q_n|^n)}{1/q^n|Q_n|^n} \geq cq^{n-1}|Q_n|^{n-1}|Q_{n-1}|^{n-1}\frac{1}{\#\Gamma_{n-1}}.$$

This and the previous inequalities imply that

$$\#X_n \leq \frac{|Q_n|\cdot\#\Gamma_{n-1}}{cq^{n-5}|Q_{n-1}|^{n-1}}c_0 + \frac{q|Q_n|\cdot\#\Gamma_{n-1}}{c|Q_{n-1}|^{n-1}}c_0 = \frac{|Q_n|\cdot\#\Gamma_{n-1}}{q^{n+3}|Q_{n-1}|^{n-1}} + \frac{|Q_n|\cdot\#\Gamma_{n-1}}{q^7|Q_{n-1}|^{n-1}}.$$

Finally, using Proposition 3.6.3, we obtain

$$\begin{aligned}\#X_n &\leq \left(\frac{1}{q^{n+3}} + \frac{1}{q^7}\right)\cdot\#\{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1}\}\\ &\leq \left(1 - \frac{1}{q^2}\right)\cdot\#\{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1}\}.\end{aligned}$$

This completes the proof of Proposition 3.6.6 for $n \geq 3$. ■

We are now ready to prove that the set Γ_n satisfies (II.1)–(II.3).

Proof that Γ_n satisfies condition (II.1). It is immediate from the definition of the set Γ_n that it satisfies (II.1). Indeed, suppose to the contrary that there exist a j with $\mathrm{diam}B_j \geq 1/q^n|Q_n|^n$ and a $\mathbf{P} \in \Gamma_n$ such that $B_j \cap I_{\mathbf{P}} \neq \emptyset$. This implies that

$$\mathrm{diam}(B_j \cap I_{\mathbf{P}}) \geq \frac{1}{q^n|Q_n|^n} = \frac{1}{q^{n-1}}(\mathrm{diam}I_{\mathbf{P}}).$$

However, this inequality contradicts the fact that $\mathbf{P} \in \Gamma_n$. This proves that the set Γ_n satisfies condition (II.1). ■

Proof that Γ_n satisfies condition (II.2). Since we have $X_n = \{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1}\} \backslash \Gamma_n$, it follows from Proposition 3.6.3 and 3.6.6 that

$$\begin{aligned} \#\Gamma_n &= \#\big(\{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1}\} \setminus X_n\big) \\ &\geq \frac{1}{q^2} \cdot \#\{\mathbf{P}P \in \Pi_n \colon \mathbf{P} \in \Gamma_{n-1}\} \\ &= \frac{|Q_n|}{q^2 |Q_{n-1}|^{n-1}} \cdot \#\Gamma_{n-1}. \end{aligned}$$

This proves that the set Γ_n satisfies condition (II.2). ■

Proof that Γ_n satisfies condition (II.3). This is obvious. ■

This completes the inductive step in the construction of the sequence $(\Gamma_n)_{n=1}^{\infty}$.

Construction of $(E_n)_{n=1}^{\infty}$: We simply put

$$E_n = \bigcup_{\mathbf{P} \in \Gamma_n} I_{\mathbf{P}}.$$

We can now complete the proof of Lemma 3.6.1(2) by showing that the sequence $(E_n)_{n=1}^{\infty}$ satisfies all the conditions (3.6.3)–(3.6.5).

Proof of (3.6.3). Clearly, it follows from the induction that $E_n \neq \emptyset$ for every $n \in \mathbb{N}$. Now we observe that condition (II.3) implies that $E_{n+1} \subseteq E_n$ for each $n \in \mathbb{N}$. Using the fact that every ball in $\mathbb{F}_q((Z^{-1}))$ is compact, we can see that the sequence $(E_n)_{n=1}^{\infty}$ is a decreasing sequence of nonempty compact sets, and we thus conclude that $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$. This proves (3.6.3). ■

Proof of (3.6.4). It follows from Proposition 3.6.2(3) that

$$\bigcap_{n=1}^{\infty} E_n \subseteq \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{P} \in \Pi_n} I_{\mathbf{P}} \subseteq \mathscr{L} \cup \mathbb{F}_q(Z),$$

as required. ■

Proof of (3.6.5). It follows from (II.1) that $(\bigcap_{n=1}^{\infty} E_n) \cap (\bigcup_{j=1}^{\infty} B_j) = \emptyset$. ■

This completes the proof of Lemma 3.6.1(2). ■

3.7 Preliminary results II

In this section, we collect some properties of the function Φ_h which will be useful for extending the results in Lemma 3.6.1 to Theorem 3.5.1. The first lemma is an adapted result from [43, Lemma 2.1] which says that, for each dimension function

with some specified properties, we can always find another dimension function with the same specified properties such that it gives much less value than the original one does. The second lemma is taken from [43, Lemma 2.2]. This says that Φ_h is a dimension function with the property that $r \mapsto \Phi_h(r)/r$ is decreasing in a neighborhood of 0. Finally, our third is analogue of [43, Lemma 2.2(3)] which says that the h-dimensional Hausdorff measure and the Φ_h-dimensional Hausdorff measure of a set in $\mathbb{F}_q((Z^{-1}))$ are comparable.

Lemma 3.7.1. [H] *Let h be a dimension function such that the function $r \mapsto h(r)/r$ is decreasing in a neighborhood of* 0. *Suppose that* $\limsup_{q^\omega \to 0,\, \omega \in \mathbb{Z}} h(q^\omega)/q^{\omega s} > 0$ *for all* $s > 0$. *Then there is another dimension function g such that the function $r \mapsto g(r)/r$ is decreasing in a neighborhood of* 0 *and such that*

(1) $\limsup_{q^\omega \to 0,\, \omega \in \mathbb{Z}} g(q^\omega)/q^{\omega s} > 0$ *for all* $s > 0$;

(2) $\lim_{q^\omega \to 0,\, \omega \in \mathbb{Z}} g(q^\omega)/h(q^\omega) = 0$.

Proof of Lemma 3.7.1. First, we note that, for all $s > 0$,

$$\limsup_{\substack{q^\omega \to 0 \\ \omega \in \mathbb{Z}}} \frac{h(q^\omega)}{q^{\omega s}} = \infty. \tag{3.7.1}$$

This was proved in (3.6.1). Next, observe that

$$\lim_{\substack{q^\omega \to 0 \\ \omega \in \mathbb{Z}}} \frac{h(q^\omega)}{q^{\omega}} = \infty. \tag{3.7.2}$$

This follows from the fact that the function $r \mapsto h(r)/r$ is decreasing near 0 so that the limit $\lim_{q^\omega \to 0,\, \omega \in \mathbb{Z}} h(q^\omega)/q^\omega$ exists, and we therefore conclude from (3.7.1) that $\lim_{q^\omega \to 0,\, \omega \in \mathbb{Z}} h(q^\omega)/q^\omega = \limsup_{q^\omega \to 0,\, \omega \in \mathbb{Z}} h(q^\omega)/q^\omega = \infty$. This proves (3.7.2).

By (3.7.1) and (3.7.2), there is an increasing sequence $(\omega_n)_{n=1}^\infty$ of natural numbers such that the following two conditions are satisfied for all $n \in \mathbb{N}$:

$$\frac{h(q^{-\omega_n})}{(q^{-\omega_n})^{1/n}} \geq n^2; \tag{3.7.3}$$

$$\frac{h(q^{-\omega_{n+1}})}{q^{-\omega_{n+1}}} \geq \left(\frac{n+1}{n}\right) \frac{h(q^{-\omega_n})}{q^{-\omega_n}}. \tag{3.7.4}$$

For each $n \in \mathbb{N}$, put

$$\rho_n = q^{-\omega_n} \left(\frac{n}{n+1}\right) \frac{h(q^{-\omega_{n+1}})}{h(q^{-\omega_n})}.$$

Note that it follows from (3.7.4) that $q^{-\omega_{n+1}} \leq \rho_n \leq q^{-\omega_n}$. Now we can define the function $g : [0, \infty) \to [0, \infty)$ by

$$g(r) = \begin{cases} 0 & \text{for } r = 0, \\ \frac{h(q^{-\omega_{n+1}})}{n+1} & \text{for } q^{-\omega_{n+1}} < r \leq \rho_n, \\ r \frac{h(q^{-\omega_n})}{n q^{-\omega_n}} & \text{for } \rho_n < r \leq q^{-\omega_n}, \\ h(q^{-\omega_1}) & \text{for } r > q^{-\omega_1}. \end{cases}$$

It is not hard to check that g is increasing and right continuous with $g(0) = 0$. In particular, this says that g is a dimension function. We must now show that the function $r \mapsto g(r)/r$ is decreasing in a neighborhood of 0 and that g satisfies conditions (1) and (2).

We first prove that the function $r \mapsto g(r)/r$ is decreasing near 0. Note that

$$\frac{g(r)}{r} = \begin{cases} \frac{h(q^{-\omega_{n+1}})}{n+1}\frac{1}{r} & \text{for } q^{-\omega_{n+1}} < r \le \rho_n, \\ \frac{h(q^{-\omega_n})}{n}\frac{1}{q^{-\omega_n}} & \text{for } \rho_n < r \le q^{-\omega_n}. \end{cases} \tag{3.7.5}$$

For $q^{-\omega_{n+1}} < r \le \rho_n$, we have

$$\frac{g(r)}{r} = \frac{h(q^{-\omega_{n+1}})}{n+1}\frac{1}{r} \ge \frac{h(q^{-\omega_{n+1}})}{n+1}\frac{1}{\rho_n} = \frac{g(\rho_n)}{\rho_n}.$$

We conclude from (3.7.5) that the function $r \mapsto g(r)/r$ is decreasing on $(q^{-\omega_{n+1}}, q^{-\omega_n}]$. For $q^{-\omega_n} < r \le \rho_{n-1}$, we have

$$\frac{g(q^{-\omega_n})}{q^{-\omega_n}} = \frac{h(q^{-\omega_n})}{n}\frac{1}{q^{-\omega_n}} \ge \frac{h(q^{-\omega_n})}{n}\frac{1}{r} = \frac{g(r)}{r}.$$

Now we can also conclude from (3.7.5) that the function $r \mapsto g(r)/r$ is decreasing on $(\rho_n, \rho_{n-1}]$. Thus, we have proved that the function $r \mapsto g(r)/r$ is decreasing.

Next, we prove that g satisfies condition (1). For all $s > 0$, we use (3.7.3) to obtain

$$\begin{aligned} \limsup_{\substack{q^\omega \to 0 \\ \omega \in \mathbb{Z}}} \frac{g(q^\omega)}{q^{\omega s}} &\ge \limsup_{n\to\infty} \frac{g(q^{-\omega_n})}{(q^{-\omega_n})^s} \\ &= \limsup_{n\to\infty} \frac{h(q^{-\omega_n})}{n(q^{-\omega_n})^s} \\ &\ge \limsup_{\substack{n\to\infty \\ n\ge 1/s}} \frac{h(q^{-\omega_n})}{n(q^{-\omega_n})^{1/n}} \\ &\ge \limsup_{\substack{n\to\infty \\ n\ge 1/s}} \frac{n^2}{n} \\ &= \infty. \end{aligned}$$

This shows that condition (1) is satisfied.

Finally, we prove that g satisfies condition (2). Since the function $r \mapsto h(r)/r$ is decreasing, we conclude that $h(q^\omega)/q^\omega \ge h(q^{-\omega_n})/q^{-\omega_n}$ for all $q^\omega \in [q^{-\omega_{n+1}}, q^{-\omega_n}]$, whence we obtain

$$h(q^\omega) \ge \begin{cases} h(q^{-\omega_{n+1}}) \ge ng(q^\omega) & \text{for } q^{-\omega_{n+1}} < q^\omega \le \rho_n, \\ \frac{h(q^{-\omega_n})}{q^{-\omega_n}} q^\omega = ng(q^\omega) & \text{for } \rho_n < q^\omega \le q^{-\omega_n}. \end{cases}$$

It now follows that $g(q^\omega)/h(q^\omega) \le 1/n$ for $q^{-\omega_{n+1}} \le q^\omega \le q^{-\omega_n}$, and hence we have $\lim_{q^\omega \to 0,\, \omega\in\mathbb{Z}} g(q^\omega)/h(q^\omega) = 0$. This shows that condition (2) is satisfied. ■

Lemma 3.7.2. [43] *Let h be a dimension function.*

(1) The function Φ_h is a dimension function. In particular, the Φ_h-dimensional Hausdorff measure $\mathcal{H}^{\Phi_h}$ is well defined.

(2) The function $r \mapsto \Phi_h(r)/r$ is decreasing in a neighborhood of 0.

Lemma 3.7.3. [H] *Let h be a dimension function. We have*

$$\mathcal{H}^{\Phi_h}(E) \leq \mathcal{H}^h(E) \leq q\mathcal{H}^{\Phi_h}(E)$$

for all $E \subseteq \mathbb{F}_q((Z^{-1}))$.

Proof of Lemma 3.7.3. First we claim that $\mathcal{H}^{\Phi_h}(E) \leq \mathcal{H}^h(E)$ for all $E \subseteq \mathbb{F}_q((Z^{-1}))$. This is an easy consequence of the fact that

$$\Phi_h(r) = \inf_{0<t\leq r} r\left(\frac{h(t)}{t}\right) \leq r\left(\frac{h(r)}{r}\right) = h(r).$$

Next, we show that $\mathcal{H}^h(E) \leq q\mathcal{H}^{\Phi_h}(E)$ for all $E \subseteq \mathbb{F}_q((Z^{-1}))$. Let $\delta > 0$, and define

$$\mathcal{H}^h_\delta(E) = \inf\left\{\sum_{j=1}^{\infty} h(\mathrm{diam}B_j)\colon E \subseteq \bigcup_{j=1}^{\infty} B_j,\ \mathrm{diam}B_j < \delta\right\};$$
$$\mathcal{H}^{\Phi_h}_\delta(E) = \inf\left\{\sum_{j=1}^{\infty} \Phi_h(\mathrm{diam}B_j)\colon E \subseteq \bigcup_{j=1}^{\infty} B_j,\ \mathrm{diam}B_j < \delta\right\}.$$

Note that $\mathcal{H}^h(E) = \lim_{\delta\to 0}\mathcal{H}^h_\delta(E)$ and $\mathcal{H}^{\Phi_h}(E) = \lim_{\delta\to 0}\mathcal{H}^{\Phi_h}_\delta(E)$. Now it suffices to prove that $(1/q)\mathcal{H}^h_\delta(E) \leq \mathcal{H}^{\Phi_h}_\delta(E) + \delta$. Let $\{B_j\}_{j=1}^{\infty}$ be a countable cover of E by balls with $\mathrm{diam}B_j < \delta$. For each $j \in \mathbb{N}$, we have

$$\Phi_h(\mathrm{diam}B_j) + \frac{\delta}{2^j} > \Phi_h(\mathrm{diam}B_j) = \inf_{0<t\leq \mathrm{diam}B_j}(\mathrm{diam}B_j)\frac{h(t)}{t}.$$

It follows that, for each $j \in \mathbb{N}$, there is a t_j with $0 < t_j \leq \mathrm{diam}B_j$ such that

$$\Phi_h(\mathrm{diam}B_j) + \frac{\delta}{2^j} \geq (\mathrm{diam}B_j)\frac{h(t_j)}{t_j}. \tag{3.7.6}$$

For each $j \in \mathbb{N}$, let ω_j denote the unique nonnegative integer with

$$q^{\omega_j - 1} < \frac{\mathrm{diam}B_j}{t_j} \leq q^{\omega_j},$$

and let $q^{\omega'_j} = \max\{q^\omega \leq t_j\colon \omega \in \mathbb{Z}\}$. Then it is easy to check that $\mathrm{diam}B_j = q^{\omega'_j}q^{\omega_j}$ for all $j \in \mathbb{N}$. Observe that the set B_j can be covered by q^{ω_j} balls $B_{j,1}, \ldots, B_{j,q^{\omega_j}}$ with

$\mathrm{diam} B_{j,i} = q^{\omega'_j}$ for all $i = 1, \ldots, q^{\omega_j}$. Since $E \subseteq \bigcup_{j=1}^{\infty} B_j \subseteq \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{q^{\omega_j}} B_{j,i}$ and h is an increasing function, we conclude from (3.7.6) and (3.7) that

$$\begin{aligned}
\sum_{j=1}^{\infty} \Phi_h(\mathrm{diam} B_j) + \delta &= \sum_{j=1}^{\infty} \left(\Phi_h(\mathrm{diam} B_j) + \frac{\delta}{2^j} \right) \\
&\geq \sum_{j=1}^{\infty} \frac{\mathrm{diam} B_j}{t_j} h(t_j) \\
&\geq \frac{1}{q} \sum_{j=1}^{\infty} q^{\omega_j} h(t_j) \\
&\geq \frac{1}{q} \sum_{j=1}^{\infty} q^{\omega_j} h(q^{\omega'_j}) \\
&= \frac{1}{q} \sum_{j=1}^{\infty} \sum_{i=1}^{q^{\omega_j}} h(\mathrm{diam} B_{j,i}) \\
&\geq \frac{1}{q} \mathcal{H}_{\delta}^{h}(E).
\end{aligned}$$

This implies that $\mathcal{H}_{\delta}^{\Phi_h}(E) + \delta \geq (1/q)\mathcal{H}_{\delta}^{h}(E)$ for every $\delta > 0$. Finally, letting $\delta \to 0$ gives the desired result. ■

3.8 Proof of Theorem 3.5.1

Proof of Theorem 3.5.1(1). Let h be a dimension function such that

$$\limsup_{\substack{q^{\omega} \to 0 \\ \omega \in \mathbb{Z}}} \frac{\Phi_h(q^{\omega})}{q^{\omega s}} = 0$$

for some $s > 0$. It now follows from Lemma 3.6.1 that $\mathcal{H}^{\Phi_h}(\mathscr{L}) = 0$. In addition, by Lemma 3.7.3, we see that $\mathcal{H}^{h}(\mathscr{L})$ and $\mathcal{H}^{\Phi_h}(\mathscr{L})$ are comparable. Thus, we have $\mathcal{H}^{h}(\mathscr{L}) = 0$, and this proves Theorem 3.5.1(1). ■

Proof of Theorem 3.5.1(2). Suppose to the contrary that there exists a dimension function h, with the property that

$$\limsup_{\substack{q^{\omega} \to 0 \\ \omega \in \mathbb{Z}}} \frac{\Phi_h(q^{\omega})}{q^{\omega s}} > 0$$

for every $s > 0$, such that $\mathscr{L}$ has σ-finite h-dimensional Hausdorff measure. That is, we have $\mathscr{L} = \bigcup_{j=1}^{\infty} E_j$, where $E_j \subseteq \mathbb{F}_q((Z^{-1}))$ with $\mathcal{H}^{h}(E_j) < \infty$ for all $j \in \mathbb{N}$. Since $\limsup_{q^{\omega} \to 0,\, \omega \in \mathbb{Z}} \Phi_h(q^{\omega})/q^{\omega s} > 0$ for every $s > 0$ and, by Lemma 3.7.2, the function $r \mapsto \Phi_h(r)/r$ is decreasing in a neighborhood of 0, it now follows from Lemma 3.7.1 that we can find a further dimension function g such that the function $r \mapsto g(r)/r$ is decreasing in a neighborhood of 0 and such that

(1) $\limsup_{q^\omega \to 0,\, \omega \in \mathbb{Z}} g(q^\omega)/q^{\omega s} > 0$ for all $s > 0$;

(2) $\lim_{q^\omega \to 0,\, \omega \in \mathbb{Z}} g(q^\omega)/\Phi_h(q^\omega) = 0$.

It follows from (1) and Lemma 3.6.1 that

$$\mathcal{H}^g(\mathscr{L}) = \infty. \tag{3.8.1}$$

Since $\mathcal{H}^h(E_j) < \infty$ for all $j \in \mathbb{N}$, we conclude from Lemma 3.7.3 that $\mathcal{H}^{\Phi_h}(E_j) < \infty$ for all $j \in \mathbb{N}$. It then follows from (2) that $\mathcal{H}^g(E_j) = 0$ for all $j \in \mathbb{N}$. Hence, we obtain

$$\mathcal{H}^g(\mathscr{L}) = \mathcal{H}^g\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mathcal{H}^g(E_j) = 0. \tag{3.8.2}$$

The desired contradiction now follows from (3.8.1) and (3.8.2). This completes the proof of Theorem 3.5.1(2). ■

This completes the proof of Theorem 3.5.1. ■

Chapter 4

Metric theory of continued fractions in positive characteristic

Let $\mathbb{F}_q$ be the finite field of q elements. An analogue of the regular continued fraction expansion for an irrational element α in the field of formal Laurent series over $\mathbb{F}_q$ is given uniquely by

$$\alpha = A_0(\alpha) + \cfrac{1}{A_1(\alpha) + \cfrac{1}{A_2(\alpha) + \cfrac{1}{A_3(\alpha) + \ddots}}},$$

where $(A_n(\alpha))_{n=0}^{\infty}$ is a sequence of polynomials with coefficients in $\mathbb{F}_q$ such that, for each $n \geq 1$, $\deg(A_n(\alpha)) \geq 1$. In this chapter, we shall first prove the exactness of the continued fraction map in positive characteristic. This fact implies a number of strictly weaker properties. Particularly, we then use the weak-mixing property and ergodicity to establish various metrical results regarding the averages of partial quotients of continued fraction expansions. A sample result that we prove is that if $(p_n)_{n=1}^{\infty}$ denotes the sequence of prime numbers, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \deg(A_{p_n}(\alpha)) = \frac{q}{q-1}$$

for almost everywhere α with respect to Haar measure. Also, we prove a quantitative version of the metrical results regarding the averages of partial quotients. By using Gál and Koksma's method, we prove for instance that, given any $\epsilon > 0$, we have

$$|A_1(\alpha) \cdots A_N(\alpha)|^{\frac{1}{N}} = q^{\frac{q}{q-1}} + o\Big(N^{-\frac{1}{2}} (\log N)^{\frac{3}{2}+\epsilon}\Big)$$

for almost every α with respect to Haar measure.

4.1 Introduction

Extending the idea of the Euclidean algorithm, for a real number α, let

$$\alpha = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \ddots}}} = [c_0; c_1, c_2, c_3, \dots] \qquad (c_n \in \mathbb{N})$$

denote its regular continued fraction expansion. The terms $c_0, c_1, c_2, \dots$ are called the *partial quotients* of the continued fraction expansion, and the sequence of rational truncates

$$[c_0; c_1, \dots, c_n] = \frac{p_n}{q_n} \qquad (n = 0, 1, 2, \dots)$$

are called the *convergents* of the continued fraction expansion.

For a real number α, let $\{\alpha\}$ denote its fractional part. We now consider the particular ergodic properties of the Gauss transformation $T : [0, 1) \to [0, 1)$ defined by

$$T\alpha = \left\{\frac{1}{\alpha}\right\} \qquad \text{and} \qquad T0 = 0.$$

Notice that $c_n(\alpha) = c_1(T^{n-1}\alpha)$ for all natural numbers n. The dynamical system $([0, 1), \mathcal{B}, \nu, T)$, where $\mathcal{B}$ is the Borel σ-algebra of subsets of $[0, 1)$ and ν is the Gauss measure defined, for any $E \in \mathcal{B}$, by

$$\nu(E) = \frac{1}{\log 2} \int_E \frac{d\alpha}{\alpha + 1},$$

is ergodic. See [10, p 165–177], [24, Ch 4] or [49] for more details. This point of view can be used to prove the following result.

Suppose $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a continuous increasing function such that

$$\int_0^1 |F(c_1(x))| \, d\nu(x) < \infty.$$

Then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(c_1(T^n\alpha)) = \int_0^1 F(c_1(x)) \, d\nu(x)$$

almost everywhere α with respect to the Lebesgue measure on $[0, 1)$. Specializing for instance to the case where $F(x) = \log x$, we recover *Khinchin's famous result* that

$$\lim_{N\to\infty} (c_1(\alpha) \cdots c_N(\alpha))^{\frac{1}{N}} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n(n+2)}\right)^{\frac{\log n}{\log 2}}$$

almost everywhere $\alpha \in [0, 1)$ with respect to Lebesgue measure, [25]. Results for means other than the geometric mean can be obtained by making different choices of F. See also [24, p 230–232] for more details.

A second well-known result giving a quantitative version of the metrical theory of continued fractions, due to de Vroedt [11], is the following. Suppose $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is a nonnegative function such that $f(n) \leq Kn^{\frac{1}{2}-\omega}$ for all $n \in \mathbb{N}$, where K and ω are positive constants. Then, given any $\epsilon > 0$,

$$\frac{1}{N}\sum_{n=1}^{N} f(c_1(T^n\alpha)) = \frac{1}{\log 2}\sum_{n=1}^{\infty} f(n)\log\left(1+\frac{1}{n(n+2)}\right) + o\Big(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon}\Big),$$

as N tends to infinity, for almost everywhere $\alpha \in [0,1)$ with respect to Lebesgue measure. Again, by adopting $f(x) = \log x$, this theorem refines Khinchin's result.

A beautiful result, due to Nair [34], generalises the work of Khinchin to include a broad class of the subsequential ergodic averages of the partial quotients. For instance, let $(a_n)_{n=1}^{\infty}$ be an L^2-good universal sequence of natural numbers such that, for any irrational number γ, the sequence $(\gamma a_n)_{n=1}^{\infty}$ is uniformly distributed mod 1, and let $G : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a continuous increasing function such that $\int_0^1 |G(c_1(x))|^2\, d\nu(x) < \infty$. Then we have

$$\lim_{N\to\infty}\sum_{n=1}^{N} G(c_1(T^{a_n}\alpha)) = \int_0^1 G(c_1(x))\, d\nu(x)$$

almost everywhere α with respect to the Lebesgue measure on $[0,1)$. Note that the sequence of natural numbers satisfies the conditions on $(a_n)_{n=1}^{\infty}$. If we set $a_n = n$ and $G(x) = \log x$, then this is Khinchin's result.

The purpose of this chapter is to extend these classical studies on the metric theory of continued fractions to the setting of the fields of formal Laurent series. In Section 4.2, we introduce the continued fraction algorithm in the positive characteristic setting. Then we define a cylinder set in Section 4.3 and use it to prove in Section 4.4 the exactness of the continued fraction map in positive characteristic. In Section 4.5, we use the ergodicity of the continued fraction map to establish the metric theory of continued fractions in positive characteristic. In Section 4.6 and Section 4.7, we further investigate the metric theory of continued fractions by employing some subsequence and moving average ergodic theorems. Finally, we introduce Gál and Koksma's method in Section 4.8 and use it to establish the quantitative metric theory of continued fractions in positive characteristic in Section 4.9.

We make a final note from [30] that questions in positive characteristic are not only of mathematical interest, but can be motivated by the study of pseudorandom sequences over finite fields. Any sequence may be encoded as a formal Laurent series. The linear complexity of profile of a sequence reveals how easy or difficult it is to generate initial segments of the sequence by short linear recurrences. The profile of a given sequence may be read off from the continued fraction expansion of the formal Laurent series which encodes it. Sequences with desirable linear complexity profiles form a cryptographic point of view correspond to formal Laurent series which are difficult to approximate.

4.2 Continued fraction algorithm

As in the classical context of real numbers, we have a continued fraction algorithm for $\mathbb{F}_q((Z^{-1}))$. Note that, in the case of the field of formal Laurent series, the roles of $[0,1)$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ in the classical theory of continued fractions are played by $B(0;1)$, $\mathbb{F}_q[Z]$, $\mathbb{F}_q(Z)$ and $\mathbb{F}_q((Z^{-1}))$, respectively.

For each $\alpha \in \mathbb{F}_q((Z^{-1}))$, we can write

$$\alpha = A_0 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \ddots}}} = [A_0; A_1, A_2, A_3, \dots],$$

where $(A_n)_{n=0}^{\infty}$ is a sequence of polynomials in $\mathbb{F}_q[Z]$ with $|A_n| > 1$ for all $n \geq 1$. Here, the sequence $(A_n)_{n=0}^{\infty}$ is uniquely determined by α, and it is clear that the sequence is infinite if and only if $\alpha \notin \mathbb{F}_q(Z)$. Note that, in the context of continued fractions, we often deal with the set $\mathbb{F}_q[Z]^* = \{A \in \mathbb{F}_q[Z] \colon |A| > 1\}$. As in the classical theory, we define recursively the two sequences of polynomials $(P_n)_{n=0}^{\infty}$ and $(Q_n)_{n=0}^{\infty}$ in $\mathbb{F}_q[Z]$ by

$$P_n = A_n P_{n-1} + P_{n-2} \qquad \text{and} \qquad Q_n = A_n Q_{n-1} + Q_{n-2},$$

with the initial conditions $P_0 = A_0$, $Q_0 = 1$, $P_1 = A_1 A_0 + 1$ and $Q_1 = A_1$. Then we have $Q_n P_{n-1} - Q_{n-1} P_n = (-1)^n$, and whence P_n and Q_n are coprime. Also, we have

$$\frac{P_n}{Q_n} = [A_0; A_1, \dots, A_n].$$

In addition, it is worth noting an easily verified property that, for any $\beta \in \mathbb{F}_q((Z^{-1}))$,

$$[A_0; A_1, \dots, A_n, \beta] = \frac{\beta P_n + P_{n-1}}{\beta Q_n + Q_{n-1}}.$$

For a general reference on this subject, the reader should consult [25] and [50].

The *continued fraction map*, or the *Gauss transformation*, T on the unit ball $B(0;1) = \{a_{-1}Z^{-1} + a_{-2}Z^{-2} + \cdots : a_i \in \mathbb{F}_q\}$ is defined by

$$T\alpha = \left\{\frac{1}{\alpha}\right\} \qquad \text{and} \qquad T0 = 0,$$

where $\{a_n Z^n + \cdots + a_0 + a_{-1}Z^{-1} + \cdots\} = a_{-1}Z^{-1} + a_{-2}Z^{-2} + \cdots$ is its fraction part. We note that if $\alpha = [0; A_1(\alpha), A_2(\alpha), \dots]$, then we have

$$T^n\alpha = [0; A_{n+1}(\alpha), A_{n+2}(\alpha), \dots] \qquad \text{and} \qquad A_m(T^n\alpha) = A_{n+m}(\alpha)$$

for all $m \geq 1$ and $n \geq 0$.

4.3 Cylinder sets

When we have to prove some properties which hold for every set in a σ-algebra, it suffices to show that the properties hold on an easily managed subcollection of subsets which can be extended to the required σ-algebra by using the Kolmogorov extension theorem. Cylinder sets are those mentioned subsets which are usually employed to prove some metrical properties relating to continued fractions.

Recall that $\mathbb{F}_q[Z]^* = \{A \in \mathbb{F}_q[Z] \colon |A| > 1\}$. Let $n \in \mathbb{N}$, and let $A_1, \dots, A_n \in \mathbb{F}_q[Z]^*$. The *cylinder set* $\Delta_{A_1,\dots,A_n}$ of length n is defined to be the set of all points in $B(0;1)$ whose continued fraction expansions are of the form $[0; A_1, \dots, A_n, \dots]$. That is,

$$\Delta_{A_1,\dots,A_n} = \{[0; A_1, \dots, A_{n-1}, A_n + \beta] \colon \beta \in B(0;1)\}.$$

The relationship between a cylinder set and a ball appears in the following lemma. This is crucial for calculating the measure of each cylinder set.

Lemma 4.3.1. [I] *Let n be a natural number, and let $A_1, \dots, A_n \in \mathbb{F}_q[Z]^*$. Then*

$$\Delta_{A_1,\dots,A_n} = B\big([0; A_1, \dots, A_n]; |A_1 \cdots A_n|^{-2}\big).$$

Proof of Lemma 4.3.1. First, we show that the cylinder set $\Delta_{A_1,\dots,A_n}$ belongs to the ball $B\big([0; A_1, \dots, A_n]; |A_1 \cdots A_n|^{-2}\big)$. Let $\alpha = [0; A_1, \dots, A_{n-1}, A_n + \beta]$, where $\beta \in B(0;1)$, and let $P_n/Q_n = [0; A_1, \dots, A_n]$. Then we have

$$\begin{aligned}\left|\alpha - \frac{P_n}{Q_n}\right| &= \left|\frac{(A_n+\beta)P_{n-1} + P_{n-2}}{(A_n+\beta)Q_{n-1} + Q_{n-2}} - \frac{P_n}{Q_n}\right| = \left|\frac{\beta(P_{n-1}Q_n - P_nQ_{n-1})}{Q_n(Q_n + \beta Q_{n-1})}\right| \\ &= \frac{|\beta|}{|Q_n||Q_n + \beta Q_{n-1}|} < \frac{1}{|Q_n|^2} = \frac{1}{|A_1 \cdots A_n|^2}.\end{aligned}$$

This shows that $\alpha \in B\big([0; A_1, \dots, A_n]; |A_1 \cdots A_n|^{-2}\big)$.

To prove the converse, suppose that $\alpha \notin \Delta_{A_1,\dots,A_n}$. Then we can write α as the continued fraction $[0; B_1, \dots, B_{n-1}, B_n + \gamma]$, where $\gamma \in B(0;1)$ and $B_i \neq A_i$ for some $i = 1, \dots, n$. Let j be the first position where $B_j \neq A_j$, so that we have $\alpha = [0; A_1, \dots, A_{j-1}, B_j, \dots, B_{n-1}, B_n + \gamma]$. If $P_j/Q_j = [0; A_1, \dots, A_j]$, then

$$\begin{aligned}&\big|[0; A_1, \dots, A_{j-1}, B_j, \dots, B_{n-1}, B_n + \gamma] - [0; A_1, \dots, A_{j-1}, A_j, \dots, A_n]\big| \\ &= \left|\frac{[B_j; \dots, B_{n-1}, B_n + \gamma]P_{j-1} + P_{j-2}}{[B_j; \dots, B_{n-1}, B_n + \gamma]Q_{j-1} + Q_{j-2}} - \frac{[A_j; \dots, A_n]P_{j-1} + P_{j-2}}{[A_j; \dots, A_n]Q_{j-1} + Q_{j-2}}\right| \\ &= \frac{\big|[B_j; \dots, B_{n-1}, B_n + \gamma] - [A_j; \dots, A_n]\big|}{\big|[B_j; \dots, B_{n-1}, B_n + \gamma]Q_{j-1}\big|\big|[A_j; \dots, A_n]Q_{j-1}\big|} = \frac{|A_j - B_j|}{|A_j||B_j||Q_{j-1}|^2} \\ &= \frac{1}{\min(|A_j|, |B_j|)|Q_{j-1}|^2} \geq \frac{1}{|Q_n|^2}.\end{aligned}$$

This shows that $\alpha \notin B\big([0; A_1, \dots, A_n]; |A_1 \cdots A_n|^{-2}\big)$, as required. ■

From Lemma 4.3.1, it follows immediately that $\mu(\Delta_{A_1,\ldots,A_n}) = |A_1 \cdots A_n|^{-2}$. We note also that two cylinder sets $\Delta_{A_1,\ldots,A_n}$ and $\Delta_{B_1,\ldots,B_n}$ are disjoint if and only if $A_j \neq B_j$ for some $1 \leq j \leq n$.

Let $\mathcal{A}$ denote the algebra of finite unions of cylinder sets. Then $\mathcal{A}$ generates the Borel σ-algebra of subsets of $B(0;1)$. This follows from the fact that the cylinder sets are clearly Borel sets themselves and that they separate points, that is, if $\alpha \neq \beta$, then there exist two disjoint cylinder sets Δ_1 and Δ_2 such that $\alpha \in \Delta_1$ and $\beta \in \Delta_2$.

4.4 Exactness and weak mixing

In [23], Houndonougbo proved that the dynamical system $(B(0;1), \mathcal{B}, \mu, T)$ is measure-preserving and ergodic. Nevertheless, in order to calculate the more general averages of convergents of continued fraction expansions, we need subsequence ergodic theory which requires a stronger property of the dynamical system, called weak mixing. Indeed, we shall systematically prove that the continued fraction map in positive characteristic is exact with respect to Haar measure. This fact of exactness implies all mixing properties and ergodicity.

Let

$$\mathcal{N} = \big\{E \in \mathcal{B} \colon \mu(E) \in \{0,1\}\big\}$$

denote the trivial σ-algebra of subsets of $\mathcal{B}$ of either null or full measure. We say that a measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$ is *exact* if

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \mathcal{N},$$

where $T^{-n}\mathcal{B} = \{T^{-n}E \colon E \in \mathcal{B}\}$.

Theorem 4.4.1. [I] *The dynamical system $(B(0;1), \mathcal{B}, \mu, T)$ is exact.*

In order to prove the exactness, we need the following three lemmas. Note that the first two lemmas appear in [23] in slightly different language.

Lemma 4.4.2. [I] *The dynamical system $(B(0;1), \mathcal{B}, \mu, T)$ is measure-preserving.*

Proof of Lemma 4.4.2. By the Kolmogorov extension theorem, it suffices to show that, for any cylinder set $\Delta_{A_1,\ldots,A_n}$, we have $\mu(T^{-1}\Delta_{A_1,\ldots,A_n}) = \mu(\Delta_{A_1,\ldots,A_n})$. First, we note that $\mu(\Delta_{A_1,\ldots,A_n}) = |A_1 \cdots A_n|^{-2}$. Then we notice that

$$T^{-1}\Delta_{A_1,\ldots,A_n} = \bigcup_{A \in \mathbb{F}_q[Z]^*} \Delta_{A,A_1,\ldots,A_n}. \tag{4.4.1}$$

Note also that, for each $j \geq 1$, we have $\#\{A \in \mathbb{F}_q[Z]^* \colon |A| = q^j\} = (q-1)q^j$. Now, by the disjointness of cylinder sets, it follows from (4.4.1) that

$$\begin{aligned}\mu(T^{-1}\Delta_{A_1,\ldots,A_n}) &= \sum_{A\in\mathbb{F}_q[Z]^*} |AA_1\cdots A_n|^{-2} = |A_1\cdots A_n|^{-2} \sum_{A\in\mathbb{F}_q[Z]^*} |A|^{-2} \\ &= |A_1\cdots A_n|^{-2} \sum_{j=1}^{\infty} \frac{(q-1)q^j}{q^{2j}} = |A_1\cdots A_n|^{-2} \sum_{j=1}^{\infty} \frac{q-1}{q^j} \\ &= |A_1\cdots A_n|^{-2} = \mu(\Delta_{A_1,\ldots,A_n}).\end{aligned}$$

This shows that the continued fraction map preserves Haar measure. ■

Lemma 4.4.3. [I] *For the dynamical system* $(B(0;1), \mathcal{B}, \mu, T)$, *suppose that* $E \in \mathcal{B}$. *Then, for any natural number* n *and any cylinder set* $\Delta_{A_1,\ldots,A_n}$, *we have*

$$\mu(\Delta_{A_1,\ldots,A_n} \cap T^{-n}E) = \mu(\Delta_{A_1,\ldots,A_n})\mu(E).$$

Proof of Lemma 4.4.3. By the Kolmogorov extension theorem, we need only to prove the case that $E = \Delta_{B_1,\ldots,B_m}$ is any cylinder set. We first observe that

$$T^{-n}\Delta_{B_1,\ldots,B_m} = \bigcup_{C_1,\ldots,C_n\in\mathbb{F}_q[Z]^*} \Delta_{C_1,\ldots,C_n,B_1,\ldots,B_m}.$$

By the disjointness of cylinder sets, it follows immediately that

$$\Delta_{A_1,\ldots,A_n} \cap T^{-n}\Delta_{B_1,\ldots,B_m} = \Delta_{A_1,\ldots,A_n,B_1,\ldots,B_m}.$$

Therefore, we conclude that

$$\mu(\Delta_{A_1,\ldots,A_n} \cap T^{-n}\Delta_{B_1,\ldots,B_m}) = |A_1\cdots A_nB_1\cdots B_m|^{-2} = \mu(\Delta_{A_1,\ldots,A_n})\mu(\Delta_{B_1,\ldots,B_m}).$$

This completes the proof of Lemma 4.4.3. ■

Lemma 4.4.4. [I] *Let* $(X, \mathcal{B}, \mu)$ *be a probability space, and let* $E \in \mathcal{B}$. *Let* $\mathcal{A} \subseteq \mathcal{B}$ *be an algebra that generates* $\mathcal{B}$. *Suppose that there exists an* $\omega > 0$ *such that*

$$\mu(E \cap \Delta) \geq \omega\mu(E)\mu(\Delta)$$

for all $\Delta \in \mathcal{A}$. *Then either* $\mu(E) = 0$ *or* $\mu(E) = 1$.

Proof of Lemma 4.4.4. Let $\epsilon > 0$. As $\mathcal{A}$ generates $\mathcal{B}$, there exists a $\Delta \in \mathcal{A}$ such that $\mu\big((E^c \setminus \Delta) \cup (\Delta \setminus E^c)\big) < \epsilon$. Therefore, we have $|\mu(E^c) - \mu(\Delta)| < \epsilon$. Note that $E \cap \Delta \subseteq (E^c \setminus \Delta) \cup (\Delta \setminus E^c)$ so that $\mu(E \cap \Delta) < \epsilon$. It now follows that

$$\mu(E)\mu(E^c) < \mu(E)(\mu(\Delta) + \epsilon) \leq \mu(E)\mu(\Delta) + \epsilon \leq \frac{1}{\omega}\mu(E \cap \Delta) + \epsilon < \left(\frac{1}{\omega} + 1\right)\epsilon.$$

As $\epsilon > 0$ is arbitrary, we have $\mu(E)\mu(E^c) = 0$. Thus, either $\mu(E) = 0$ or $\mu(E) = 1$, and this completes the proof of Lemma 4.4.4. ■

We are now in a position to prove that the continued fraction map T on $B(0;1)$ is exact with respect to Haar measure.

Proof of Theorem 4.4.1. It is not hard to check that we need only to prove the inclusion $\bigcap_{n=1}^{\infty} T^{-n}\mathcal{B} \subseteq \mathcal{N}$. Let $E \in \bigcap_{n=1}^{\infty} T^{-n}\mathcal{B}$. It follows immediately that, for each $n \geq 1$, there exists an $E_n \in \mathcal{B}$ such that $E = T^{-n}E_n$ and $\mu(E_n) = \mu(E)$. Then, for each cylinder set $\Delta_{A_1,\ldots,A_n}$ of length n, we always have

$$\mu(E \cap \Delta_{A_1,\ldots,A_n}) = \mu(T^{-n}E_n \cap \Delta_{A_1,\ldots,A_n}) = \mu(E)\mu(\Delta_{A_1,\ldots,A_n}).$$

It follows that $\mu(E) \in \{0,1\}$, so $E \in \mathcal{N}$. This proves the exactness. ■

If $(X, \mathcal{B}, \mu, T)$ is exact, then a number of strictly weaker properties follow. Firstly, for any natural number n and any $E_0, E_1, \ldots, E_n \in \mathcal{B}$, we have

$$\lim_{j_1,\ldots,j_n\to\infty} \mu\Big(E_0 \cap T^{-j_1}E_1 \cap \cdots \cap T^{-(j_1+\cdots+j_n)}E_n\Big) = \mu(E_0)\mu(E_1)\cdots\mu(E_n).$$

This is called *mixing of order* n. Mixing of order $n = 1$ is

$$\lim_{j\to\infty} \mu(E_0 \cap T^{-j}E_1) = \mu(E_0)\mu(E_1),$$

and this is called *strong mixing*, which in turn implies

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \big|\mu(E_0 \cap T^{-j}E_1) - \mu(E_0)\mu(E_1)\big| = 0$$

which is called *weak mixing*. Weak-mixing property implies the condition that if $E \in \mathcal{B}$ and if $T^{-1}E = E$, then either $\mu(E) = 0$ or $\mu(E) = 1$. This last property is referred to as *ergodicity* in measurable dynamics. All these implications are known to be strict in general, see [10, p 22–26].

4.5 Metric theory of continued fractions in positive characteristic

The most basic implication of exactness is ergodicity. In this section, we shall use the fact that the continued fraction map is ergodic to give the answers to Gauss' metrical problems concerning the averages of partial quotients of continued fraction expansions. Indeed, for a typical point $\alpha = [A_0(\alpha); A_1(\alpha), A_2(\alpha), \ldots] \in \mathbb{F}_q((Z^{-1}))$, we would like to identify for instance the limits:

(1) $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \deg(A_n(\alpha))$;

(2) for each $A \in \mathbb{F}_q[Z]^*$, $\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \leq n \leq N \colon A_n(\alpha) = A\}$;

(3) for each $m \in \mathbb{N}$, $\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \leq n \leq N \colon \deg(A_n(\alpha)) = m\}$.

Theorem 4.5.1. *Let $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function such that*

$$\int_{B(0;1)} |F(|A_1(x)|)|\, d\mu(x) < \infty.$$

Then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(|A_n(\alpha)|) = (q-1) \sum_{n=1}^{\infty} \frac{F(q^n)}{q^n}$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Theorem 4.5.1. Note that $A_{n+1}(x) = A_1(T^n x)$. Apply the Birkhoff ergodic theorem with $f(x) = F(|A_1(x)|)$. By using the fact that $\mu(\Delta_{A_1}) = |A_1|^{-2}$, it is a little of combinatorial work to see that

$$\begin{aligned}\int_{B(0;1)} F(|A_1(x)|)\, d\mu(x) &= (q-1)qq^{-2}F(q) + (q-1)q^2q^{-4}F(q^2) + \cdots \\ &= (q-1)\sum_{n=1}^{\infty} \frac{F(q^n)}{q^n}.\end{aligned}$$

This completes the proof of Theorem 4.5.1. ■

Theorem 4.5.2. *Let $H : \mathbb{N}^m \to \mathbb{R}$ be a function such that*

$$\int_{B(0;1)} |H(|A_1(x)|, \ldots, |A_m(x)|)|\, d\mu(x) < \infty.$$

Then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} H(|A_n(\alpha)|, \ldots, |A_{n+m-1}(\alpha)|) = \sum_{i_1,\ldots,i_m \in \mathbb{N}} H(q^{i_1}, \ldots, q^{i_m}) \left(\frac{(q-1)^m}{q^{i_1+\cdots+i_m}} \right)$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Theorem 4.5.2. Note that $A_{n+m}(x) = A_m(T^n x)$. In view of the Birkhoff ergodic theorem, we consider $f(x) = H(|A_1(x)|, \ldots, |A_m(x)|)$. By using the fact that $\mu(\Delta_{A_1,\ldots,A_m}) = |A_1 \cdots A_m|^{-2}$, it is a little of combinatorial work to see that

$$\begin{aligned}\int_{B(0;1)} H(|A_1(x)|, \ldots, |A_m(x)|)\, d\mu(x) &= \sum_{i_1,\ldots,i_m \in \mathbb{N}} H(q^{i_1}, \ldots, q^{i_m}) \left(\frac{(q-1)^m q^{i_1+\cdots+i_m}}{(q^{i_1+\cdots+i_m})^2} \right) \\ &= \sum_{i_1,\ldots,i_m \in \mathbb{N}} H(q^{i_1}, \ldots, q^{i_m}) \left(\frac{(q-1)^m}{q^{i_1+\cdots+i_m}} \right)\end{aligned}$$

This proves Theorem 4.5.2. ■

Theorems 4.5.1 and 4.5.2 are general results for calculating means. Specializing for instance to the case $F(x) = \log_q x$, we establish the positive characteristic analogue of *Khinchin's constant*

$$\lim_{N\to\infty} |A_1(\alpha) \cdots A_N(\alpha)|^{\frac{1}{N}} = q^{\frac{q}{q-1}}$$

μ-almost everywhere $\alpha \in B(0;1)$, also proved [4] and [25]. Results for means other than the geometric mean can be obtained by making different choices of F and H, see [24, p 230–232] for more details. In addition, the following three results, which were proved in [23] and rediscovered in [4], can be viewed as corollaries of Theorem 4.5.1.

Corollary 4.5.3. [23] *We have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \deg(A_n(\alpha)) = \frac{q}{q-1}$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Corollary 4.5.3. Apply Theorem 4.5.1 with $F(x) = \log_q x$. Indeed, we have

$$\int_{B(0;1)} |\log_q(|A_1(x)|)|\, d\mu(x) = (q-1) \sum_{n=1}^{\infty} \frac{\log_q(q^n)}{q^n} = (q-1) \sum_{n=1}^{\infty} \frac{n}{q^n} = \frac{q}{q-1} < \infty.$$

This shows that the hypothesis of Theorem 4.5.1 is satisfied when we set $F(x) = \log_q x$. It now follows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \deg(A_n(\alpha)) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log_q(|A_n(\alpha)|) = (q-1) \sum_{n=1}^{\infty} \frac{\log_q(q^n)}{q^n} = \frac{q}{q-1}$$

μ-almost everywhere $\alpha \in B(0;1)$, as required. ■

Corollary 4.5.4. [23] *Given any $A \in \mathbb{F}_q[Z]^*$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon A_n(\alpha) = A\} = |A|^{-2}$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Corollary 4.5.4. Observe the fact that $\frac{1}{(q-1)|A|}\mathbb{1}_{\{|A|\}}(|A_1(x)|) = \mathbb{1}_{\{A\}}(A_1(x))$ for μ-almost everywhere $x \in B(0;1)$. Apply Theorem 4.5.1 with $F(x) = \frac{1}{(q-1)|A|}\mathbb{1}_{\{|A|\}}(x)$, where $\mathbb{1}_E(x)$ is the characteristic function of a set E. Indeed, we have

$$\int_{B(0;1)} \left| \frac{1}{(q-1)|A|} \mathbb{1}_{\{|A|\}}(|A_1(x)|) \right| d\mu(x) = (q-1) \sum_{n=1}^{\infty} \frac{\frac{1}{(q-1)|A|}\mathbb{1}_{\{|A|\}}(q^n)}{q^n} = |A|^{-2} < \infty.$$

This shows that the hypothesis of Theorem 4.5.1 is satisfied. It now follows that

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon A_n(\alpha) = A\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{A\}}(A_n(\alpha)) \\ &= \int_{B(0;1)} \mathbb{1}_{\{A\}}(A_1(x))\, d\mu(x) = \int_{B(0;1)} \frac{1}{(q-1)|A|} \mathbb{1}_{\{|A|\}}(|A_1(x)|)\, d\mu(x) = |A|^{-2} \end{aligned}$$

μ-almost everywhere $\alpha \in B(0;1)$, as required. ■

Corollary 4.5.5. [23] *Given any two natural numbers $k < l$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon \deg(A_n(\alpha)) = l\} = \frac{q-1}{q^l},$$
$$\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon \deg(A_n(\alpha)) \ge l\} = \frac{1}{q^{l-1}},$$
$$\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon k \le \deg(A_n(\alpha)) < l\} = \frac{1}{q^{k-1}}\left(1 - \frac{1}{q^{l-k}}\right)$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Corollary 4.5.5. In view of Theorem 4.5.1, we consider $F_1(x) = \mathbb{1}_{\{q^l\}}(x)$, $F_2(x) = \mathbb{1}_{[q^l,\infty)}(x)$ and $F_3(x) = \mathbb{1}_{[q^k,q^l)}(x)$, respectively. ■

Now we would like ask some further questions in two more general directions. First, given any sequence $(a_n)_{n=1}^{\infty}$ of natural numbers, we wish to identify, for a typical point $\alpha = [A_0(\alpha); A_1(\alpha), A_2(\alpha), \dots] \in \mathbb{F}_q((Z^{-1}))$, the subsequential limits:

(1) $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \deg(A_{a_n}(\alpha))$;

(2) for each $A \in \mathbb{F}_q[Z]^*$, $\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon A_{a_n}(\alpha) = A\}$;

(3) for each $m \in \mathbb{N}$, $\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon \deg(A_{a_n}(\alpha)) = m\}$;

(4) given another sequence $(b_n)_{n=1}^{\infty}$ of natural numbers, we would like to calculate the moving averages of the same quantities as in (1)–(3), for instance, the limit

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{b_n} \deg(A_{a_n+j}(\alpha)).$$

We shall answer these questions in Sections 4.6 and 4.7 for a large class of the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ by employing the subsequence and moving average ergodic theory. For the second direction, we wish to investigate some quantitative version of the metrical results regarding the ergodic averages. In particular, we shall find the error terms of the ergodic averages in Theorem 4.5.1–Corollary 4.5.5 as functions of N. For example, we shall see in Section 4.9 how the geometric mean of $|A_n(\alpha)|$ $(n = 1, \dots, N)$ deviates from the Khinchin's constant $q^{q/(q-1)}$ for μ-almost everywhere $\alpha \in B(0;1)$.

4.6 On the metric theory of continued fractions in positive characteristic I

In this section, we assume that the sequence $(a_n)_{n=1}^{\infty}$ of natural numbers is L^2-good universal. We also suppose, for any irrational number γ, that the sequence $(\gamma a_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. Some examples of the sequences $(a_n)_{n=1}^{\infty}$ can be found in

Section 2.4. These include the sequences $(P(n))_{n=1}^{\infty}$ and $(P(p_n))_{n=1}^{\infty}$, where $P(x)$ is a polynomial mapping $\mathbb{N}$ into itself and p_n denotes the nth prime number.

Recall the elementary identities $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ and $\sum_{n=1}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}$ for $|x| < 1$. Also, as is easily verified, a simple computation shows that

$$\int_{B(0;1)} F(|A_1(x)|)\, d\mu(x) = (q-1)\sum_{n=1}^{\infty} \frac{F(q^n)}{q^n};$$
$$\int_{B(0;1)} F(|A_1(x)|)^2\, d\mu(x) = (q-1)\sum_{n=1}^{\infty} \frac{F(q^n)^2}{q^n}$$

for every function F defined as in Theorem 4.6.1. These two identities, in the light of the results in this section, indicate the relation between the expectation of the variable $|\alpha|$ and the frequency with which it takes a specific value for μ-almost everywhere α in $B(0;1)$. Analogous observations hold for other variables in this section. Particularly, the valuation $|\cdot|$ is in $L^2(\mu)$, and so we can now employ the subsequence ergodic theory.

Theorem 4.6.1. [I] *Let $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function such that*

$$\int_{B(0;1)} |F(|A_1(x)|)|^2\, d\mu(x) < \infty.$$

Then we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(|A_{a_n}(\alpha)|) = (q-1)\sum_{n=1}^{\infty} \frac{F(q^n)}{q^n}$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Theorem 4.6.1. Note that $A_{a_n+1}(x) = A_1(T^{a_n}x)$. Apply Theorem 2.3.1 with $f(x) = F(|A_1(x)|)$. ■

Theorem 4.6.2. [I] *Let $H : \mathbb{N}^m \to \mathbb{R}$ be a function such that*

$$\int_{B(0;1)} |H(|A_1(x)|, \ldots, |A_m(x)|)|^2\, d\mu(x) < \infty.$$

Then we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} H(|A_{a_n}(\alpha)|, \ldots, |A_{a_n+m-1}(\alpha)|) = \sum_{i_1,\ldots,i_m\in\mathbb{N}} H(q^{i_1}, \ldots, q^{i_m})\left(\frac{(q-1)^m}{q^{i_1+\cdots+i_m}}\right)$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Theorem 4.6.2. Note that $A_{a_n+m}(x) = A_m(T^{a_n}x)$. Apply Theorem 2.3.1 with $f(x) = H(|A_1(x)|, \ldots, |A_m(x)|)$. ■

Theorems 4.6.1 and 4.6.2 are general results for calculating means. They both readily extend from L^2 to L^p $(p > 1)$, though this is primarily of technical interest.

Specializing for instance to the case $F(x) = \log_q x$, we establish the subsequential Khinchin's constant in the positive characteristic setting:

$$\lim_{N\to\infty} |A_{a_1}(\alpha)\cdots A_{a_N}(\alpha)|^{\frac{1}{N}} = q^{\frac{q}{q-1}}$$

μ-almost everywhere $\alpha \in B(0;1)$. Results for means other than the geometric mean can be obtained by making different choices of F and H, see [24, p 230–232] for more details. In addition, the following three results can be viewed as corollaries of Theorem 4.6.1.

Corollary 4.6.3. [I] *We have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \deg(A_{a_n}(\alpha)) = \frac{q}{q-1}$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Corollary 4.6.3. Apply Theorem 4.6.1 with $F(x) = \log_q x$. ■

Corollary 4.6.4. [I] *Given any $A \in \mathbb{F}_q[Z]^*$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\cdot \#\{1 \le n \le N\colon A_{a_n}(\alpha) = A\} = |A|^{-2}$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Corollary 4.6.4. Apply Theorem 4.6.1 with $F(x) = \frac{1}{(q-1)|A|}\mathbb{1}_{\{|A|\}}(x)$. See also the proof of Corollary 4.5.4. ■

Corollary 4.6.5. [I] *Given any two natural numbers $k < l$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\cdot \#\{1 \le n \le N\colon \deg(A_{a_n}(\alpha)) = l\} = \frac{q-1}{q^l},$$
$$\lim_{N\to\infty} \frac{1}{N}\cdot \#\{1 \le n \le N\colon \deg(A_{a_n}(\alpha)) \ge l\} = \frac{1}{q^{l-1}},$$
$$\lim_{N\to\infty} \frac{1}{N}\cdot \#\{1 \le n \le N\colon k \le \deg(A_{a_n}(\alpha)) < l\} = \frac{1}{q^{k-1}}\left(1 - \frac{1}{q^{l-k}}\right)$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Corollary 4.6.5. In view of Theorem 4.6.1, we consider $F_1(x) = \mathbb{1}_{\{q^l\}}(x)$, $F_2(x) = \mathbb{1}_{[q^l,\infty)}(x)$ and $F_3(x) = \mathbb{1}_{[q^k,q^l)}(x)$, respectively. ■

4.7 On the metric theory of continued fractions in positive characteristic II

In this section, we state the moving average variants of those results in Section 4.6. The proofs, which are very similar to those in the previous section, are foregone. Note that we use Theorem 2.6.3 for the calculations in this section, and we assume that $(a_n, b_n)_{n=1}^{\infty}$ is a Stoltz sequence.

Theorem 4.7.1. [I] *Let $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function such that*

$$\int_{B(0;1)} |F(|A_1(x)|)|\, d\mu(x) < \infty.$$

Then we have

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{b_n} F(|A_{a_n+j}(\alpha)|) = (q-1) \sum_{n=1}^{\infty} \frac{F(q^n)}{q^n}$$

μ-almost everywhere $\alpha \in B(0;1)$.

Theorem 4.7.2. [I] *Let $H : \mathbb{N}^m \to \mathbb{R}$ be a function such that*

$$\int_{B(0;1)} |H(|A_1(x)|, \ldots, |A_m(x)|)|\, d\mu(x) < \infty.$$

Then we have

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{b_n} H(|A_{a_n+j}(\alpha)|, \ldots, |A_{a_n+j+m-1}(\alpha)|) = \sum_{i_1,\ldots,i_m \in \mathbb{N}} H(q^{i_1}, \ldots, q^{i_m}) \left(\frac{(q-1)^m}{q^{i_1+\cdots+i_m}} \right)$$

μ-almost everywhere $\alpha \in B(0;1)$.

Corollary 4.7.3. [I] *We have*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{b_n} \deg(A_{a_n+j}(\alpha)) = \frac{q}{q-1}$$

μ-almost everywhere $\alpha \in B(0;1)$.

Corollary 4.7.4. [I] *Given any $A \in \mathbb{F}_q[Z]^*$, we have*

$$\lim_{n\to\infty} \frac{1}{b_n} \cdot \#\{1 \leq j \leq b_n \colon A_{a_n+j}(\alpha) = A\} = |A|^{-2}$$

μ-almost everywhere $\alpha \in B(0;1)$.

Corollary 4.7.5. [I] *Given any two natural numbers $k < l$, we have*

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{b_n} \cdot \#\{1 \leq j \leq b_n \colon \deg(A_{a_n+j}(\alpha)) = l\} = \frac{q-1}{q^l}, \\
&\lim_{n\to\infty} \frac{1}{b_n} \cdot \#\{1 \leq j \leq b_n \colon \deg(A_{a_n+j}(\alpha)) \geq l\} = \frac{1}{q^{l-1}}, \\
&\lim_{n\to\infty} \frac{1}{b_n} \cdot \#\{1 \leq j \leq b_n \colon k \leq \deg(A_{a_n+j}(\alpha)) < l\} = \frac{1}{q^{k-1}} \left(1 - \frac{1}{q^{l-k}} \right)
\end{aligned}$$

μ-almost everywhere $\alpha \in B(0;1)$.

4.8 Gál and Koksma's method

In this section, we introduce Gál and Koksma's method on determining the error term of ergodic averages. The method is slightly technical but it is very useful for establishing our quantitative metric theory of continued fractions in positive characteristic. The following lemma appears in [17, Théorèm 3] in slightly different language.

Lemma 4.8.1 (Gál and Koksma's Method)**.** [17] *Let S be a measurable subset of a Euclidean space. For any non-negative integers M and N, let $\varphi(M,N;x) \geq 0$ be a function defined on S such that*

(1) $\varphi(M,0;x) = 0$ for every non-negative integer $M \geq 0$;

(2) $\varphi(M,N;x) \leq \varphi(M,N';x) + \varphi(M+N',N-N';x)$ for every non-negative integers $M,N \geq 0$ and $0 \leq N' \leq N$.

Suppose that, for all $M \geq 0$,

$$\int_S \varphi(M,N;x)^p \, dx = O(\phi(N)),$$

where $\phi(N)/N$ is a non-decreasing function. Then, given any $\epsilon > 0$, we have

$$\varphi(0,N;x) = o\Big(\phi(N)^{\frac{1}{p}} (\log N)^{1+\frac{1}{p}+\epsilon}\Big)$$

almost everywhere $x \in S$ with respect to Lebesgue measure.

Before proceeding, we give the following two remarks on Lemma 4.8.1. First, Gál and Koksma stated their results in the setting where the set S is a measurable subset of a Euclidean space. However, none of the proofs in [17] depend on the Euclidean setting. In fact, their results are true more generally. We are interested in the case where $S = B(0;1)$, for which the result is also true. Second, the function φ can be viewed as a generalization of the difference of two functions in a sequence:

$$\varphi(M,N;x) = |\varphi_{M+N}(x) - \varphi_M(x)|,$$

where the condition (2) is just a generalization of the triangle inequality

$$|\varphi_{M+N}(x) - \varphi_M(x)| \leq |\varphi_{M+N'}(x) - \varphi_M(x)| + |\varphi_{M+N}(x) - \varphi_{M+N'}(x)|.$$

Particularly, we focus on the case where $\varphi_N(x) = \sum_{n=1}^{N} F_n(x)$, that is,

$$\varphi(M,N;x) = \sum_{n=M+1}^{M+N} F_n(x).$$

The next lemma is useful when we would like to change variables in an integration. This result is an immediate consequence of Lemma 4.4.2 that the dynamical system $(B(0;1), \mathcal{B}, \mu, T)$ is measure-preserving.

Lemma 4.8.2. [J] *For every $n \in \mathbb{N}$, we have $d\mu(T^{-n}\alpha) = d\mu(\alpha)$.*

4.9 Quantitative metric theory of continued fractions in positive characteristic

In this final section, we give a quantitative version of the metrical results appeared in Section 4.5 by employing Gál and Koksma's method. We start with two general theorems for calculating the quantitative ergodic averages.

Theorem 4.9.1. [J] *Let $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function such that*

$$\int_{B(0;1)} |F(|A_1(x)|)|^2 \, d\mu(x) < \infty.$$

Then, given any $\epsilon > 0$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(|A_n(\alpha)|) = (q-1) \sum_{n=1}^{\infty} \frac{F(q^n)}{q^n} + o\Big(N^{-\frac{1}{2}} (\log N)^{\frac{3}{2}+\epsilon}\Big)$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Theorem 4.9.1. Apply Gál and Koksma's method with $S = B(0;1)$,

$$\varphi(M,N;\alpha) = \Big| \sum_{n=M+1}^{M+N} \Big(F(|A_1(T^n\alpha)|) - \int_{B(0;1)} F(|A_1(x)|) \, d\mu(x) \Big) \Big|,$$

$\phi(N) = N$ and $p = 2$. Then the proof is reduced to showing that, for any pair of integers $M \geq 0$ and $N \geq 1$, we have

$$I = \int_{B(0;1)} \Big| \sum_{n=M+1}^{M+N} \Big(F(|A_1(T^n\alpha)|) - \int_{B(0;1)} F(|A_1(x)|) \, d\mu(x) \Big) \Big|^2 \, d\mu(\alpha) \leq KN,$$

where K is a constant depending only on $F(x)$.

Put

$$P_1 = \int_{B(0;1)} F(|A_1(x)|) \, d\mu(x) \qquad \text{and} P_2 = \int_{B(0;1)} F(|A_1(x)|)^2 \, d\mu(x).$$

Then it is not hard to calculate P_1 and P_2 explicitly:

$$\begin{aligned} P_1 &= (q-1)qq^{-2}F(q) + (q-1)q^2q^{-4}F(q^2) + (q-1)q^3q^{-6}F(q^3) + \cdots \\ &= (q-1) \sum_{n=1}^{\infty} \frac{F(q^n)}{q^n}; \end{aligned} \tag{4.9.1}$$

and

$$\begin{aligned} P_2 &= (q-1)q^{-1}F(q)^2 + (q-1)q^{-2}F(q^2)^2 + (q-1)q^{-3}F(q^3)^2 + \cdots \\ &= (q-1) \sum_{n=1}^{\infty} \frac{F(q^n)^2}{q^n}. \end{aligned} \tag{4.9.2}$$

Working out $I = \int_{B(0;1)} \left(\sum_{n=M+1}^{M+N} (F(|A_1(T^n\alpha)|) - P_1) \right)^2 d\mu(\alpha)$, we get

$$\begin{aligned} I = &\sum_{n=M+1}^{M+N} \int_{B(0;1)} (F(|A_1(T^n\alpha)|) - P_1)^2 \, d\mu(\alpha) \\ &+ 2 \sum_{n=M+1}^{M+N-1} \sum_{\substack{m=M+2 \\ m>n}}^{M+N} \int_{B(0;1)} (F(|A_1(T^n\alpha)|) - P_1)(F(|A_1(T^m\alpha)|) - P_1) \, d\mu(\alpha). \end{aligned} \tag{4.9.3}$$

By Lemma 4.8.2, we can use the change of variables formula to obtain

$$\begin{aligned} &\sum_{n=M+1}^{M+N} \int_{B(0;1)} (F(|A_1(T^n\alpha)|) - P_1)^2 \, d\mu(\alpha) \\ &= \sum_{n=M+1}^{M+N} \left(\int_{B(0;1)} F(|A_1(T^n\alpha)|)^2 \, d\mu(\alpha) - 2P_1 \int_{B(0;1)} F(|A_1(T^n\alpha)|) \, d\mu(\alpha) + P_1^2 \right) \\ &= \sum_{n=M+1}^{M+N} \left(\int_{B(0;1)} F(|A_1(\alpha)|)^2 \, d\mu(T^{-n}\alpha) - 2P_1 \int_{B(0;1)} F(|A_1(\alpha)|) \, d\mu(T^{-n}\alpha) + P_1^2 \right) \\ &= \sum_{n=M+1}^{M+N} \left(\int_{B(0;1)} F(|A_1(\alpha)|)^2 \, d\mu(\alpha) - 2P_1 \int_{B(0;1)} F(|A_1(\alpha)|) \, d\mu(\alpha) + P_1^2 \right) \\ &= \sum_{n=M+1}^{M+N} (P_2 - 2P_1^2 + P_1^2) = N(P_2 - P_1^2), \end{aligned} \tag{4.9.4}$$

and

$$\begin{aligned} &\sum_{n=M+1}^{M+N-1} \sum_{\substack{m=M+2 \\ m>n}}^{M+N} \int_{B(0;1)} (F(|A_1(T^n\alpha)|) - P_1)(F(|A_1(T^m\alpha)|) - P_1) \, d\mu(\alpha) \\ &= \sum_{n=M+1}^{M+N-1} \sum_{\substack{m=M+2 \\ m>n}}^{M+N} \left(\int_{B(0;1)} F(|A_1(T^n\alpha)|) F(|A_1(T^m\alpha)|) \, d\mu(\alpha) \right. \\ &\quad \left. - P_1 \int_{B(0;1)} F(|A_1(T^n\alpha)|) \, d\mu(\alpha) - P_1 \int_{B(0;1)} F(|A_1(T^m\alpha)|) \, d\mu(\alpha) + P_1^2 \right) \\ &= \sum_{n=M+1}^{M+N-1} \sum_{\substack{m=M+2 \\ m>n}}^{M+N} \left(\int_{B(0;1)} F(|A_1(\alpha)|) F(|A_1(T^{m-n}\alpha)|) \, d\mu(\alpha) - P_1^2 \right) \\ &= \sum_{n=1}^{N-1} (N-n) \left(\int_{B(0;1)} F(|A_1(\alpha)|) F(|A_1(T^n\alpha)|) \, d\mu(\alpha) - P_1^2 \right). \end{aligned} \tag{4.9.5}$$

Combining (4.9.3), (4.9.4) and (4.9.5), we now have

$$I = N(P_2 - P_1^2) + 2 \sum_{n=1}^{N-1} (N-n) \left(\int_{B(0;1)} F(|A_1(\alpha)|) F(|A_1(T^n\alpha)|) \, d\mu(\alpha) - P_1^2 \right). \tag{4.9.6}$$

We can calculate $\int_{B(0;1)} F(|A_1(\alpha)|)F(|A_1(T^n\alpha)|)\,d\mu(\alpha)$ explicitly as follows:

$$\begin{aligned}
&\int_{B(0;1)} F(|A_1(\alpha)|)F(|A_1(T^n\alpha)|)\,d\mu(\alpha)\\
&= \sum_{A_2,\dots,A_n\in\mathbb{F}_q[Z]^*} |A_2\cdots A_n|^{-2}\Big(\sum_{i=1}^{\infty}\Big(\frac{q-1}{q^i}\Big)F(q^i)\Big)\Big(\sum_{j=1}^{\infty}\Big(\frac{q-1}{q^j}\Big)F(q^j)\Big) \qquad (4.9.7)\\
&= (q-1)^2\Big(\sum_{i=1}^{\infty}\frac{F(q^i)}{q^i}\Big)\Big(\sum_{j=1}^{\infty}\frac{F(q^j)}{q^j}\Big).
\end{aligned}$$

By (4.9.1) and (4.9.7), we see that $\int_{B(0;1)} F(|A_1(\alpha)|)F(|A_1(T^n\alpha)|)\,d\mu(\alpha) = P_1^2$. Thus, by (4.9.6), we arrive at the hypothesis of Lemma 4.8.1 that $I = O(N)$, and hence this completes the proof of Theorem 4.9.1. ■

Theorem 4.9.2. [J] *Let $H : \mathbb{N}^m \to \mathbb{R}$ be a function such that*

$$\int_{B(0;1)} |H(|A_1(x)|,\dots,|A_m(x)|)|^2\,d\mu(x) < \infty.$$

Then, given any $\epsilon > 0$, we have

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} H(|A_n(\alpha)|,\dots,|A_{n+m-1}(\alpha)|)\\
&\qquad = \sum_{i_1,\dots,i_m\in\mathbb{N}} H(q^{i_1},\dots,q^{i_m})\Big(\frac{(q-1)^m}{q^{i_1+\cdots+i_m}}\Big) + o\Big(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon}\Big)
\end{aligned}$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Theorem 4.9.2. The proof is similar to that of Theorem 4.9.1, so we shall give only an outline. First of all, we apply Lemma 4.8.1 with $S = B(0;1)$,

$$\begin{aligned}
&\varphi(M,N;\alpha)\\
&= \Big|\sum_{n=M+1}^{M+N}\Big(H(|A_1(T^n\alpha)|,\dots,|A_m(T^n\alpha)|) - \sum_{i_1,\dots,i_m\in\mathbb{N}} H(q^{i_1},\dots,q^{i_m})\Big(\frac{(q-1)^m}{q^{i_1+\cdots+i_m}}\Big)\Big)\Big|,
\end{aligned}$$

$\phi(N) = N$ and $p = 2$. Next, it is not hard to show that

$$\int_{B(0;1)} H(|A_1(x)|,\dots,|A_m(x)|)\,d\mu(x) = \sum_{i_1,\dots,i_m\in\mathbb{N}} H(q^{i_1},\dots,q^{i_m})\Big(\frac{(q-1)^m}{q^{i_1+\cdots+i_m}}\Big).$$

Finally, if we put $P = \int_{B(0;1)} H(|A_1(x)|,\dots,|A_m(x)|)\,d\mu(x)$, then

$$\int_{B(0;1)} H(|A_1(\alpha)|,\dots,|A_m(\alpha)|)H(|A_1(T^n\alpha)|,\dots,|A_m(T^n\alpha)|)\,d\mu(\alpha) = P^2.$$

These observations lead to Theorem 4.9.2. ■

Again, Theorems 4.9.1 and 4.9.2 are general results for calculating means. When $F(x) = \log_q x$, we establish the positive characteristic analogue of the quantitative version of Khinchin's famous result namely that

$$|A_1(\alpha)\cdots A_N(\alpha)|^{\frac{1}{N}} = q^{\frac{q}{q-1}} + o\Big(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon}\Big)$$

μ-almost everywhere $\alpha \in B(0;1)$, [11]. In addition, the following three results can be viewed as corollaries of Theorem 4.9.1.

Corollary 4.9.3. [J] *Given any $\epsilon > 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \deg(A_n(\alpha)) = \frac{q}{q-1} + o\Big(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon}\Big)$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Theorem 4.9.3. Apply Theorem 4.9.1 with $F(x) = \log_q x$. ■

Corollary 4.9.4. [J] *Given any $A \in \mathbb{F}_q[Z]^*$ and $\epsilon > 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\cdot \#\{1 \le n \le N\colon A_n(\alpha) = A\} = |A|^{-2} + o\Big(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon}\Big)$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Theorem 4.9.4. Apply Theorem 4.9.1 with $F(x) = \frac{1}{(q-1)|A|}\mathbb{1}_{\{|A|\}}(x)$. See also the proof of Corollary 4.5.4. ■

Corollary 4.9.5. [J] *Given any two natural numbers $k < l$ and $\epsilon > 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\cdot \#\{1 \le n \le N\colon \deg(A_n(\alpha)) = l\} = \frac{q-1}{q^l} + o\Big(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon}\Big),$$
$$\lim_{N\to\infty} \frac{1}{N}\cdot \#\{1 \le n \le N\colon \deg(A_n(\alpha)) \ge l\} = \frac{q-1}{q^{l-1}} + o\Big(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon}\Big),$$
$$\lim_{N\to\infty} \frac{1}{N}\cdot \#\{1 \le n \le N\colon k \le \deg(A_n(\alpha)) < l\} = \frac{1}{q^{k-1}}\left(1 - \frac{1}{q^{l-k}}\right) + o\Big(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon}\Big)$$

μ-almost everywhere $\alpha \in B(0;1)$.

Proof of Theorem 4.9.5. In view of Theorem 4.9.1, we consider $F_1(x) = \mathbb{1}_{\{q^l\}}(x)$, $F_2(x) = \mathbb{1}_{[q^l,\infty)}(x)$ and $F_3(x) = \mathbb{1}_{[q^k,q^l)}(x)$, respectively. ■

Chapter 5

Metric theory of p-adic continued fractions

Let p be a prime number. An analogue of the regular continued fraction expansion for the p-adic numbers was given by T. Schneider such that, for each $\alpha \in p\mathbb{Z}_p$, outside a countable set of the p-adic numbers with finite continued fraction expansion, we have uniquely determined two sequences $(a_n(\alpha) \in \mathbb{N})_{n=0}^{\infty}$ and $(b_n(\alpha) \in \{1, 2, \ldots, p-1\})_{n=1}^{\infty}$ such that

$$\alpha = \cfrac{p^{a_1(\alpha)}}{b_1(\alpha) + \cfrac{p^{a_2(\alpha)}}{b_2(\alpha) + \cfrac{p^{a_3(\alpha)}}{b_3(\alpha) + \ddots}}}.$$

In this chapter, we shall first prove the exactness of the p-adic continued fraction map. This fact implies a number of strictly weaker properties. Particularly, we then use the weak-mixing property and ergodicity, together with some subsequence and moving average ergodic theorems, to establish various metrical results regarding the averages of partial quotients of p-adic continued fraction expansions. A sample result that we prove is that if $(p_n)_{n=1}^{\infty}$ denotes the sequence of prime numbers, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_{p_n}(\alpha) = \frac{p}{p-1} \qquad \text{and} \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} b_{p_n}(\alpha) = \frac{p}{2}$$

for almost everywhere α with respect to Haar measure.

5.1 Introduction

The theory of continued fractions is one of the most important tools in number theory, analysis, probability theory and other areas of mathematics. It provides a powerful apparatus not only in the classical real numbers but also in the p-adic numbers. In this

chapter, we shall investigate the metrical theory of p-adic continued fractions. Some motivation of the metrical study of continued fractions can be found in Chapter 4.

In this case of p-adic numbers, a couple of candidate algorithms that might play the role of the role of the regular continued fraction expansion on $\mathbb{Q}_p$ are known to the author. The first one is Ruban's algorithm whose metric theory was studied in [48]. The algorithm studied in [48] is formally similar to the Gauss continued fraction map and analogues are given of the results of Khinchin for this map. A deficiency of this algorithm however, arising in part from the fact that $\mathbb{Q}_p$ is not a Euclidean domain, is that the sequence of rational approximations to a p-adic number arising from this algorithm is not in general convergent. This limits the arithmetic usefulness of Ruban's algorithm. In this paper, we instead study Schneider's algorithm [51] which does not suffer from this deficiency and, in our view, is more arithmetically useful. The metric theory of this algorithm was initiated in [22]. The form of the algorithm is however quite different from that of the Gauss transformation.

We now summarize the contents of this chapter. In Section 5.2, we introduce the Schneider's continued fraction algorithm. Then we define a cylinder set in Section 5.3 and use it to prove in Section 5.4 the exactness of the Schneider's continued fraction map. In Section 5.5, we use the ergodicity of the p-adic continued fraction map to establish the metric theory of p-adic continued fractions. In Section 5.6 and Section 5.7, we further investigate the metric theory of p-adic continued fractions by employing subsequence and moving average ergodic theorems.

5.2 Continued fraction algorithm

To study the regular continued fraction expansion on $\mathbb{Q}_p$, we use Schneider's algorithm [51] which can be described as follows. Let $\alpha \in p\mathbb{Z}_p$ with $|\alpha|_p = p^{-v(\alpha)}$. By ignoring a countable set, we may assume that the p-adic continued fraction expansion of α is non-terminating. The Schneider's *continued fraction map* $T_p : p\mathbb{Z}_p \to p\mathbb{Z}_p$ is defined by

$$T_p\alpha = \frac{p^a}{\alpha} - b = \frac{p^{v(\alpha)}}{\alpha} - \left(\frac{p^{v(\alpha)}}{\alpha} \bmod p\right),$$

where $a = v(\alpha) \in \mathbb{N}$ and $b \in \{1, \ldots, p-1\}$ is uniquely chosen such that $|\frac{p^a}{\alpha} - b|_p < 1$. Applying the Schneider's map T_p repeatedly, we see that

$$\alpha = \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{b_3 + \ddots}}}.$$

Therefore, the algorithm outputs a sequence of pairs $(a_n, b_n)_{n=1}^{\infty}$ in $\mathbb{N} \times \{1, \ldots, p-1\}$. The convergents of α arise in a manner similar to that in the case of the real numbers.

In particular, we define two integer sequences $(P_n)_{n=0}^{\infty}$ and $(Q_n)_{n=0}^{\infty}$ by

$$P_n = b_n P_{n-1} + p^{a_n} P_{n-2} \qquad \text{and} \qquad Q_n = b_n Q_{n-1} + p^{a_n} Q_{n-2},$$

with the initial conditions $P_0 = 0$, $Q_0 = 1$, $P_1 = p^{a_1}$ and $Q_1 = b_1$. A simple inductive argument gives that $Q_n P_{n-1} - Q_{n-1} P_n = (-1)^n p^{a_1 + \cdots + a_n}$. Since p does not divide Q_n, it follows that P_n and Q_n are coprime. Also, we have

$$\frac{P_n}{Q_n} = \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{b_3 + \ddots + \cfrac{p^{a_n}}{b_n}}}}.$$

A more thorough account of p-adic continued fraction algorithms can be found in [32].

We note that if we represent the regular continued fraction expansion of $\alpha \in p\mathbb{Z}_p$ by the sequence $(a_n(\alpha), b_n(\alpha))_{n=1}^{\infty}$ in $\mathbb{N} \times \{1, \ldots, p-1\}$, then the iteration of T_p produces

$$T_p^n \alpha = (a_{n+1}(\alpha), b_{n+1}(\alpha))_{n=1}^{\infty}.$$

This implies that $a_n(\alpha) = a_1(T_p^{n-1}\alpha)$ and $b_n(\alpha) = b_1(T_p^{n-1}\alpha)$ for all $n \geq 1$.

5.3 Cylinder sets

To prove some properties which hold for every set in a σ-algebra, it is enough to show that the properties hold on an easily managed sub-collection of subsets which can be extended to the required σ-algebra by using the Kolmogorov extension theorem. Cylinder sets are those subsets mentioned earlier which are employed to prove some metrical properties regarding continued fractions.

Let $n \in \mathbb{N}$, and let $x_j = (a_j, b_j) \in \mathbb{N} \times \{1, \ldots, p-1\}$ for each $1 \leq j \leq n$. Define

$$\Delta_{x_1} = \left\{ \alpha \in p\mathbb{Z}_p \colon v(\alpha) = a_1,\ \left(\frac{p^{v(\alpha)}}{\alpha} \bmod p \right) = b_1 \right\}.$$

Also, we define

$$\Delta_{x_1, x_2} = \{\alpha \in p\mathbb{Z}_p \colon \alpha \in \Delta_{x_1},\ T_p\alpha \in \Delta_{x_2}\}.$$

Proceeding inductively, we define

$$\Delta_{x_1, \ldots, x_n} = \{\alpha \in p\mathbb{Z}_p \colon \alpha \in \Delta_{x_1},\ T_p\alpha \in \Delta_{x_2},\ \ldots,\ T_p^{n-1}\alpha \in \Delta_{x_n}\}.$$

In other words, the *cylinder set* $\Delta_{x_1, \ldots, x_n}$ of length n is defined to be the set

$$\Delta_{x_1, \ldots, x_n} = \left\{ \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \ddots + \cfrac{p^{a_n}}{b_n + \beta}}} \colon \beta \in p\mathbb{Z}_p \right\}.$$

We note an easily verified fact that two cylinder sets $\Delta_{x_1,\dots,x_n}$ and $\Delta_{y_1,\dots,y_n}$ are disjoint if and only if $x_j \neq y_j$ for some $1 \leq j \leq n$. Let $\mathcal{A}$ denote the algebra of finite unions of cylinder sets. Then $\mathcal{A}$ generates the Borel σ-algebra of subsets of $p\mathbb{Z}_p$. This follows from the fact that the cylinder sets are clear Borel sets themselves and that they separate points, i.e., if $\alpha \neq \beta$, then there are two disjoint cylinder sets Δ_1 and Δ_2 such that $\alpha \in \Delta_1$ and $\beta \in \Delta_2$.

Now we define a measure μ^* on $\mathcal{A}$ by setting

$$\mu^*(\Delta_{x_1,\dots,x_n}) = p^{-(a_1+\cdots+a_n)}.$$

It is not hard to check that this measure coincides with the Haar measure μ on $\mathcal{A}$. In addition, it is plain that $p\mathbb{Z}_p = \bigcup_{x\in\mathbb{N}\times\{1,\dots,p-1\}} \Delta_x$. By the Kolmogorov extension theorem, the extension of measure μ^* is exactly the Haar measure μ on $p\mathbb{Z}_p$.

5.4 Exactness and weak mixing

In [22], Hirsh and Washington proved that the dynamical system $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ is measure-preserving and ergodic. Nevertheless, in order to calculate the more general averages of convergents of p-adic continued fraction expansions, we need subsequence ergodic theory which requires a stronger property of the dynamical system, called weak mixing. Indeed, we shall prove that the Schneider's continued fraction map is exact with respect to Haar measure. This fact of exactness implies all mixing properties and ergodicity as mentioned in the end of Section 4.4.

Recall that a measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$ is *exact* if we have

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \mathcal{N} = \big\{E \in \mathcal{B} \colon \mu(E) \in \{0,1\}\big\},$$

where $T^{-n}\mathcal{B} = \{T^{-n}E \colon E \in \mathcal{B}\}$.

Theorem 5.4.1. [A] *The dynamical system $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ is exact.*

In order to prove the exactness, we need the following two lemmas. Note that this proof is different from the one presented in [A].

Lemma 5.4.2. [22] *The dynamical system $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ is measure-preserving.*

Lemma 5.4.3. *For the dynamical system $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$, let $E \in \mathcal{B}$. Suppose that n is a natural number and $\Delta_{x_1,\dots,x_n}$ is a cylinder set with $x_j = (a_j, b_j) \in \mathbb{N} \times \{1, \dots, p-1\}$ for each $1 \leq j \leq n$. Then we have*

$$\mu(\Delta_{x_1,\dots,x_n} \cap T_p^{-n}E) = \mu(\Delta_{x_1,\dots,x_n})\mu(E).$$

Proof of Lemma 5.4.3. By the Kolmogorov extension theorem, we need only to prove the this lemma for the case that $E = \Delta_{y_1,\ldots,y_m}$, where $y_j = (a'_j, b'_j) \in \mathbb{N} \times \{1, \ldots, p-1\}$ for each $1 \leq j \leq m$, is an arbitrary cylinder set. We first observe that

$$T_p^{-n}\Delta_{y_1,\ldots,y_m} = \bigcup_{z_1,\ldots,z_n \in \mathbb{N}\times\{1,\ldots,p-1\}} \Delta_{z_1,\ldots,z_n,y_1,\ldots,y_m}.$$

By the disjointness of cylinder sets, it follows immediately that

$$\Delta_{x_1,\ldots,x_n} \cap T_p^{-n}\Delta_{y_1,\ldots,y_m} = \Delta_{x_1,\ldots,x_n,y_1,\ldots,y_m}.$$

Therefore, we conclude that

$$\mu(\Delta_{x_1,\ldots,x_n} \cap T_p^{-n}\Delta_{y_1,\ldots,y_m}) = p^{-(a_1+\cdots+a_n+a'_1+\cdots+a'_m)} = \mu(\Delta_{x_1,\ldots,x_n})\mu(\Delta_{y_1,\ldots,y_m}).$$

This completes the proof of Lemma 5.4.3. ■

We are now in a position to prove that the Schneider's continued fraction map T_p on $p\mathbb{Z}_p$ is exact with respect to Haar measure.

Proof of Theorem 5.4.1. Clearly, it suffices to prove only the inclusion $\bigcap_{n=1}^{\infty} T_p^{-n}\mathcal{B} \subseteq \mathcal{N}$. Let $E \in \cap_{n=1}^{\infty} T_p^{-n}\mathcal{B}$. It follows immediately that, for each $n \geq 1$, there exists an $E_n \in \mathcal{B}$ such that $E = T_p^{-n}E_n$ and $\mu(E_n) = \mu(E)$. Then, for each cylinder set $\Delta_{x_1,\ldots,x_n}$,

$$\mu(E \cap \Delta_{x_1,\ldots,x_n}) = \mu(T_p^{-n}E_n \cap \Delta_{x_1,\ldots,x_n}) = \mu(E)\mu(\Delta_{x_1,\ldots,x_n}).$$

It now follows from Lemma 4.4.4 that $\mu(E) \in \{0, 1\}$, and whence $E \in \mathcal{N}$. This shows the exactness of T_p. ■

5.5 Metric theory of p-adic continued fractions

The most basic implication of exactness is ergodicity. In this section, we shall use the fact that the continued fraction map is ergodic to give the answers to Gauss' metrical problems concerning the averages of partial quotients of continued fraction expansions. Indeed, for all $\alpha = (a_n(\alpha), b_n(\alpha))_{n=1}^{\infty} \in p\mathbb{Z}_p$ outside a set of measure zero, we would like to identify for instance the limits:

(1) $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n(\alpha)$;

(2) $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} b_n(\alpha)$;

(3) for each $a \in \mathbb{N}$, $\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \leq n \leq N \colon a_n(\alpha) = a\}$;

(4) for each $b \in \{1, \ldots, p-1\}$, $\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \leq n \leq N \colon b_n(\alpha) = b\}$.

Theorem 5.5.1. *Let $F_A : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $F_B : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be two functions such that*

$$\int_{p\mathbb{Z}_p} |F_A(a_1(x))|\, d\mu(x) < \infty \qquad \text{and} \qquad \int_{p\mathbb{Z}_p} |F_B(b_1(x))|\, d\mu(x) < \infty.$$

Then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F_A(a_n(\alpha)) = (p-1) \sum_{n=1}^{\infty} \frac{F_A(n)}{p^n},$$
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F_B(b_n(\alpha)) = \frac{1}{p-1} \sum_{n=1}^{p-1} F_B(n)$$

μ-almost everywhere $\alpha \in p\mathbb{Z}_p$.

Proof of Theorem 5.5.1. Note that $a_n(x) = a_1(T_p^{n-1}x)$ and $b_n(x) = b_1(T_p^{n-1}x)$. Apply Birkhoff's ergodic theorem with $f_A(x) = F_A(a_1(x))$ and $f_B(x) = F_B(b_1(x))$. Also, it is plain that, for each $n \in \mathbb{N}$ and $m \in \{1, \ldots, p-1\}$, we have

$$\mu(\{x \in p\mathbb{Z}_p \colon v(x) = n\}) = \left(\frac{1}{p^{n-1}}\right)\left(\frac{p-1}{p}\right) = \frac{p-1}{p^n},$$
$$\mu(\{x \in p\mathbb{Z}_p \colon b_1(x) = m\}) = \frac{1}{p-1}.$$

Then it follows that

$$\int_{p\mathbb{Z}_p} F_A(a_1(x))\, d\mu(x) = \sum_{n=1}^{\infty} F_A(n)\left(\frac{p-1}{p^n}\right) = (p-1)\sum_{n=1}^{\infty} \frac{F_A(n)}{p^n},$$
$$\int_{p\mathbb{Z}_p} F_B(b_1(x))\, d\mu(x) = \sum_{n=1}^{p-1} \frac{F_B(n)}{p-1} = \frac{1}{p-1}\sum_{n=1}^{p-1} F_B(n).$$

This completes the proof of Theorem 5.5.1. ■

Theorem 5.5.2. *Let $H_A : \mathbb{N}^m \to \mathbb{R}$ and $H_B : \mathbb{N}^m \to \mathbb{R}$ be two functions such that*

$$\int_{p\mathbb{Z}_p} |H_A(a_1(x), \ldots, a_m(x))|\, d\mu(x) < \infty \quad \text{and} \quad \int_{p\mathbb{Z}_p} |H_B(b_1(x), \ldots, b_m(x))|\, d\mu(x) < \infty.$$

Then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} H_A(a_n(\alpha), \ldots, a_{n+m-1}(\alpha)) = (p-1)^m \sum_{i_1,\ldots,i_m \in \mathbb{N}} \frac{H_A(i_1, \ldots, i_m)}{p^{i_1+\cdots+i_m}},$$
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} H_B(b_n(\alpha), \ldots, b_{n+m-1}(\alpha)) = \frac{1}{(p-1)^m} \sum_{i_1,\ldots,i_m \in \{1,\ldots,p-1\}} H_B(i_1, \ldots, i_m)$$

μ-almost everywhere $\alpha \in p\mathbb{Z}_p$.

Proof of Theorem 5.5.2. It is plain that $a_{n+m}(x) = a_m(T_p^n x)$ and $b_{n+m}(x) = b_m(T_p^n x)$. In view of the Birkhoff ergodic theorem, we consider $f_A(x) = H_A(a_1(x), \ldots, a_m(x))$ and $f_B(x) = H_B(b_1(x), \ldots, b_m(x))$. Then the results follow. ■

Theorems 5.5.1 and 5.5.2 are general results for calculating means. Note that, in the p-adic case, the set of values for b_n is restricted, but the numerators p^{a_n} determine the rate of convergence to $\alpha = (a_n(\alpha), b_n(\alpha))_{n=1}^{\infty} \in p\mathbb{Z}_p$. Specializing to the case $F_A(x) = x$, we establish the p-adic analogue of *Khinchin's constant*

$$\lim_{N\to\infty} \left(p^{a_1(\alpha)} \cdots p^{a_N(\alpha)}\right)^{\frac{1}{N}} = p^{\frac{p}{p-1}}$$

μ-almost everywhere $\alpha \in p\mathbb{Z}_p$. Results for means other than the geometric mean can be obtained by making different choices of F and H, see [24, p 230–232] for more details. In addition, the following three results can be viewed as corollaries of Theorem 5.5.1.

Corollary 5.5.3. [22] *We have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n(\alpha) = \frac{p}{p-1} \qquad \textit{and} \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} b_n(\alpha) = \frac{p}{2}$$

μ-almost everywhere $\alpha \in p\mathbb{Z}_p$.

Proof of Corollary 5.5.3. Apply Theorem 5.5.1 with $F_A(x) = x$ and $F_B(x) = x$. ■

Corollary 5.5.4. *Given any two natural numbers $a < a'$, we have*

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon a_n(\alpha) = a\} = \frac{p-1}{p^a}, \\
&\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon a_n(\alpha) \ge a\} = \frac{1}{p^{a-1}}, \\
&\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon a \le a_n(\alpha) < a'\} = \frac{1}{p^{a-1}}\left(1 - \frac{1}{p^{a'-a}}\right)
\end{aligned}$$

μ-almost everywhere $\alpha \in p\mathbb{Z}_p$.

Proof of Corollary 5.5.4. In view of Theorem 5.5.1, we can consider $F_A^1(x) = \mathbb{1}_{\{a\}}(x)$, $F_A^2(x) = \mathbb{1}_{[a,\infty)}(x)$ and $F_A^3(x) = \mathbb{1}_{[a,a')}(x)$, respectively. ■

Corollary 5.5.5. *Given any $b, b' \in \{1, \ldots, p-1\}$ with $b < b'$, we have*

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon b_n(\alpha) = b\} = \frac{1}{p-1}, \\
&\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon b_n(\alpha) \ge b\} = \frac{p-b}{p-1}, \\
&\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon b \le b_n(\alpha) < b'\} = \frac{b'-b}{p-1}
\end{aligned}$$

μ-almost everywhere $\alpha \in p\mathbb{Z}_p$.

Proof of Corollary 5.5.5. In view of Theorem 5.5.1, we can consider $F_B^1(x) = \mathbb{1}_{\{b\}}(x)$, $F_B^2(x) = \mathbb{1}_{[b,p-1]}(x)$ and $F_B^3(x) = \mathbb{1}_{[b,b')}(x)$, respectively. ■

Now we would like ask some further general questions. Given any sequence $(k_n)_{n=1}^{\infty}$ of natural numbers, we wish to identify, for a typical point $\alpha = (a_n(\alpha), b_n(\alpha))_{n=1}^{\infty} \in p\mathbb{Z}_p$, the subsequential limits:

(1) $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_{k_n}(\alpha)$;

(2) $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} b_{k_n}(\alpha)$;

(3) for each $a \in \mathbb{N}$, $\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon a_{k_n}(\alpha) = a\}$;

(4) for each $b \in \{1, \dots, p-1\}$, $\lim_{N\to\infty} \frac{1}{N} \cdot \#\{1 \le n \le N \colon b_{k_n}(\alpha) = b\}$;

(5) given another sequence $(l_n)_{n=1}^{\infty}$ of natural numbers, we would like to calculate the moving averages of the same quantities as in (1)–(4), for instance, the limit

$$\lim_{n\to\infty} \frac{1}{l_n} \sum_{j=1}^{l_n} a_{k_n+j}(\alpha) \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{l_n} \sum_{j=1}^{l_n} b_{k_n+j}(\alpha).$$

We shall answer these questions in Sections 5.6 and 5.7 for a large class of the sequences $(k_n)_{n=1}^{\infty}$ and $(l_n)_{n=1}^{\infty}$ by employing the subsequence and moving average ergodic theory.

5.6 On the metric theory of p-adic continued fractions I

In this section, we assume that the sequence $(k_n)_{n=1}^{\infty}$ of natural numbers is L^2-good universal. We also suppose, for any irrational number γ, that the sequence $(\gamma k_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. Some examples of the sequences $(k_n)_{n=1}^{\infty}$ can be found in Section 2.4. These include the sequences $(P(n))_{n=1}^{\infty}$ and $(P(p_n))_{n=1}^{\infty}$, where $P(x)$ is a polynomial mapping $\mathbb{N}$ into itself and p_n denotes the nth prime number.

Recall the elementary identities $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ and $\sum_{n=1}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}$ for $|x| < 1$. Also, as is easily verified, a simple computation shows that

$$\int_{p\mathbb{Z}_p} F_A(a_1(x))\, d\mu(x) = (p-1) \sum_{n=1}^{\infty} \frac{F_A(n)}{p^n};$$
$$\int_{p\mathbb{Z}_p} F_A(a_1(x))^2\, d\mu(x) = (p-1) \sum_{n=1}^{\infty} \frac{F_A(n)^2}{p^n},$$

for every function F_A defined as in Theorem 5.6.1. These two identities, in the light of the results in this section, indicate the relation between the expectation of the variable $v(\alpha)$ and the frequency with which it takes a specific value for μ-almost everywhere α in $p\mathbb{Z}_p$. Analogous observations hold for other variables in this section. Particularly, the valuation $v(\cdot)$ is in $L^2(\mu)$, and so we can now employ the subsequence ergodic theory.

Theorem 5.6.1. [A] *Let $F_A : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $F_B : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be two functions such that*

$$\int_{p\mathbb{Z}_p} |F_A(a_1(x))|^2 \, d\mu(x) < \infty \qquad \text{and} \qquad \int_{p\mathbb{Z}_p} |F_B(b_1(x))|^2 \, d\mu(x) < \infty.$$

Then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F_A(a_{k_n}(\alpha)) = (p-1) \sum_{n=1}^{\infty} \frac{F_A(n)}{p^n},$$
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F_B(b_{k_n}(\alpha)) = \frac{1}{p-1} \sum_{n=1}^{p-1} F_B(n)$$

μ-almost everywhere $\alpha \in p\mathbb{Z}_p$.

Proof of Theorem 5.6.1. Note that $a_{k_n}(x) = a_1(T_p^{k_n-1}x)$ and $b_{k_n}(x) = b_1(T_p^{k_n-1}x)$. Apply Theorem 2.3.1 with $f_A(x) = F_A(a_1(x))$ and $f_B(x) = F_B(b_1(x))$. ■

Theorem 5.6.2. [A] *Let $H_A : \mathbb{N}^m \to \mathbb{R}$ and $H_B : \mathbb{N}^m \to \mathbb{R}$ be two functions such that*

$$\int_{p\mathbb{Z}_p} H_A(a_1(x), \ldots, a_m(x))^2 \, d\mu(x) < \infty \quad \text{and} \quad \int_{p\mathbb{Z}_p} H_B(b_1(x), \ldots, b_m(x))^2 \, d\mu(x) < \infty.$$

Then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} H_A(a_{k_n}(\alpha), \ldots, a_{k_n+m-1}(\alpha)) = (p-1)^m \sum_{i_1,\ldots,i_m \in \mathbb{N}} \frac{H_A(i_1, \ldots, i_m)}{p^{i_1+\cdots+i_m}},$$
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} H_B(b_{k_n}(\alpha), \ldots, b_{k_n+m-1}(\alpha)) = \frac{1}{(p-1)^m} \sum_{i_1,\ldots,i_m \in \{1,\ldots,p-1\}} H_B(i_1, \ldots, i_m)$$

μ-almost everywhere $\alpha \in p\mathbb{Z}_p$.

Proof of Theorem 5.6.2. Note that $a_{k_n+m}(x) = a_m(T_p^{k_n}x)$ and $b_{k_n+m}(x) = b_m(T_p^{k_n}x)$. In view of Theorem 2.3.1, we can now consider $f_A(x) = H_A(a_1(x), \ldots, a_m(x))$ and $f_B(x) = H_B(b_1(x), \ldots, b_m(x))$. ■

Theorems 5.6.1 and 5.6.2 are general results for calculating means. They both readily extend to L^p $(p > 1)$ whenever $(k_n)_{n=1}^{\infty}$ is an L^p-good universal sequence, though this is primarily of technical interest. Specializing for instance to the case $F_A(x) = x$, we establish the positive characteristic subsequential Khinchin's constant

$$\lim_{N\to\infty} \left(p^{a_{k_1}(\alpha)} \cdots p^{a_{k_N}(\alpha)}\right)^{\frac{1}{N}} = p^{\frac{p}{p-1}}$$

μ-almost everywhere $\alpha \in p\mathbb{Z}_p$. Results for means other than the geometric mean can be obtained by making different choices of F and H, see [24, p 230–232] for more details. In addition, the following three results can be viewed as corollaries of Theorem 5.6.1.

Corollary 5.6.3. [A] *We have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a_{k_n}(\alpha)=\frac{p}{p-1}\qquad\text{and}\qquad\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}b_{k_n}(\alpha)=\frac{p}{2}$$

μ-almost everywhere $\alpha\in p\mathbb{Z}_p$.

Proof of Corollary 5.6.3. Apply Theorem 5.6.1 with $F_A(x)=x$ and $F_B(x)=x$. ■

Corollary 5.6.4. [A] *Given any two natural numbers $a<a'$, we have*

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\cdot\#\{1\le n\le N\colon a_{k_n}(\alpha)=a\}=\frac{p-1}{p^a},\\
&\lim_{N\to\infty}\frac{1}{N}\cdot\#\{1\le n\le N\colon a_{k_n}(\alpha)\ge a\}=\frac{1}{p^{a-1}},\\
&\lim_{N\to\infty}\frac{1}{N}\cdot\#\{1\le n\le N\colon a\le a_{k_n}(\alpha)<a'\}=\frac{1}{p^{a-1}}\left(1-\frac{1}{p^{a'-a}}\right)
\end{aligned}$$

μ-almost everywhere $\alpha\in p\mathbb{Z}_p$.

Proof of Corollary 5.6.4. In view of Theorem 5.6.1, we can consider $F_A^1(x)=\mathbb{1}_{\{a\}}(x)$, $F_A^2(x)=\mathbb{1}_{[a,\infty)}(x)$ and $F_A^3(x)=\mathbb{1}_{[a,a')}(x)$, respectively. ■

Corollary 5.6.5. *Given any $b,b'\in\{1,\dots,p-1\}$ with $b<b'$, we have*

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\cdot\#\{1\le n\le N\colon b_{k_n}(\alpha)=b\}=\frac{1}{p-1},\\
&\lim_{N\to\infty}\frac{1}{N}\cdot\#\{1\le n\le N\colon b_{k_n}(\alpha)\ge b\}=\frac{p-b}{p-1},\\
&\lim_{N\to\infty}\frac{1}{N}\cdot\#\{1\le n\le N\colon b\le b_{k_n}(\alpha)<b'\}=\frac{b'-b}{p-1}
\end{aligned}$$

μ-almost everywhere $\alpha\in p\mathbb{Z}_p$.

Proof of Corollary 5.6.5. In view of Theorem 5.6.1, we can consider $F_B^1(x)=\mathbb{1}_{\{b\}}(x)$, $F_B^2(x)=\mathbb{1}_{[b,p-1]}(x)$ and $F_B^3(x)=\mathbb{1}_{[b,b')}(x)$, respectively. ■

5.7 On the metric theory of p-adic continued fractions II

In this section, we state the moving average variants of those results in Section 5.6. The proofs, which are very similar to those in the previous section, are foregone. Note that we use Theorem 2.6.3 for the calculations in this section, and we assume that $(k_n,l_n)_{n=1}^{\infty}$ is a Stoltz sequence.

Theorem 5.7.1. [A] *Let $F_A:\mathbb{R}_{\ge0}\to\mathbb{R}$ and $F_B:\mathbb{R}_{\ge0}\to\mathbb{R}$ be two functions such that*

$$\int_{p\mathbb{Z}_p}|F_A(a_1(x))|\,d\mu(x)<\infty\qquad\text{and}\qquad\int_{p\mathbb{Z}_p}|F_B(b_1(x))|\,d\mu(x)<\infty.$$

Then we have

$$\lim_{n\to\infty}\frac{1}{l_n}\sum_{j=1}^{l_n}F_A(a_{k_n+j}(\alpha))=(p-1)\sum_{n=1}^{\infty}\frac{F_A(n)}{p^n},$$
$$\lim_{n\to\infty}\frac{1}{l_n}\sum_{j=1}^{l_n}F_B(b_{k_n+j}(\alpha))=\frac{1}{p-1}\sum_{n=1}^{p-1}F_B(n)$$

μ-almost everywhere $\alpha\in p\mathbb{Z}_p$.

Theorem 5.7.2. [A] *Let $H_A:\mathbb{N}^m\to\mathbb{R}$ and $H_B:\mathbb{N}^m\to\mathbb{R}$ be two functions such that*

$$\int_{p\mathbb{Z}_p}|H_A(a_1(x),\dots,a_m(x))|\,d\mu(x)<\infty\quad\text{and}\quad\int_{p\mathbb{Z}_p}|H_B(b_1(x),\dots,b_m(x))|\,d\mu(x)<\infty.$$

Then we have

$$\lim_{n\to\infty}\frac{1}{l_n}\sum_{j=1}^{l_n}H_A(a_{k_n+j}(\alpha),\dots,a_{k_n+j+m-1}(\alpha))=(p-1)^m\sum_{i_1,\dots,i_m\in\mathbb{N}}\frac{H_A(i_1,\dots,i_m)}{p^{i_1+\dots+i_m}},$$
$$\lim_{n\to\infty}\frac{1}{l_n}\sum_{j=1}^{l_n}H_B(b_{k_n+j}(\alpha),\dots,b_{k_n+j+m-1}(\alpha))=\frac{1}{(p-1)^m}\sum_{i_1,\dots,i_m\in\{1,\dots,p-1\}}H_B(i_1,\dots,i_m)$$

μ-almost everywhere $\alpha\in p\mathbb{Z}_p$.

Corollary 5.7.3. [A] *We have*

$$\lim_{n\to\infty}\frac{1}{l_n}\sum_{j=1}^{l_n}a_{k_n+j}(\alpha)=\frac{p}{p-1}\qquad\text{and}\qquad\lim_{n\to\infty}\frac{1}{l_n}\sum_{j=1}^{l_n}b_{k_n+j}(\alpha)=\frac{p}{2}$$

μ-almost everywhere $\alpha\in p\mathbb{Z}_p$.

Corollary 5.7.4. [A] *Given any two natural numbers $a<a'$, we have*

$$\lim_{n\to\infty}\frac{1}{l_n}\cdot\#\{1\le j\le l_n\colon a_{k_n+j}(\alpha)=a\}=\frac{p-1}{p^a},$$
$$\lim_{n\to\infty}\frac{1}{l_n}\cdot\#\{1\le j\le l_n\colon a_{k_n+j}(\alpha)\ge a\}=\frac{1}{p^{a-1}},$$
$$\lim_{n\to\infty}\frac{1}{l_n}\cdot\#\{1\le j\le l_n\colon a\le a_{k_n+j}(\alpha)<a'\}=\frac{1}{p^{a-1}}\left(1-\frac{1}{p^{a'-a}}\right)$$

μ-almost everywhere $\alpha\in p\mathbb{Z}_p$.

Corollary 5.7.5. *Given any $b,b'\in\{1,\dots,p-1\}$ with $b<b'$, we have*

$$\lim_{n\to\infty}\frac{1}{l_n}\cdot\#\{1\le j\le l_n\colon b_{k_n+j}(\alpha)=b\}=\frac{1}{p-1},$$
$$\lim_{n\to\infty}\frac{1}{l_n}\cdot\#\{1\le j\le l_n\colon b_{k_n+j}(\alpha)\ge b\}=\frac{p-b}{p-1},$$
$$\lim_{n\to\infty}\frac{1}{l_n}\cdot\#\{1\le j\le l_n\colon b\le b_{k_n+j}(\alpha)<b'\}=\frac{b'-b}{p-1}$$

μ-almost everywhere $\alpha\in p\mathbb{Z}_p$.

Chapter 6

Distribution functions for the a-adic van der Corput sequence

For an integer $b > 1$, let $(\phi_b(n))_{n=1}^{\infty}$ denote the base b van der Corput sequence in $[0, 1)$. Answering a question of O. Strauch, Aistleitner and Hofer calculated the asymptotic distribution function of $(\phi_b(n), \phi_b(n+1), \ldots, \phi_b(n+s-1))_{n=1}^{\infty}$ on $[0, 1)^s$ and showed that it is a copula. In this chapter, we introduce a generalised version of a van der Corput sequence, called the a-adic van der Corput sequence, which provides a wealth of low-discrepancy sequences. Then we shall see that the phenomenon extends not only to a broad class of subsequences of the van der Corput sequences but also to a more general setting in the a-adic integers. Indeed, we use the subsequence characterization of unique ergodicity, together with the fact that the van der Corput sequence can be seen as the orbit of the origin under the ergodic Kakutani-von Neumann transformation.

6.1 Introduction

In 1935, van der Corput introduced a procedure to generate low-discrepancy sequences on $[0, 1)$, see [57]. These sequences are considered the best distributed over $[0, 1)$, since no sequence has yet been found with discrepancy of smaller order of magnitude than the van der Corput sequences, and hence the van der Corput sequences are considered very important in the quasi-Monte Carlo method. The technique of van der Corput is based on a very simple idea. Let $b > 1$ be a natural number. The *van der Corput sequence* $(\phi_b(n))_{n=1}^{\infty}$ in base b is constructed by reversing the base b representation of the sequence of natural numbers.

In a collection of unsolved problems in uniform distribution theory[1], Strauch asked a question on the limit distribution of consecutive elements of the van der Corput sequence. Precisely, the question asked us to find all the distribution functions for the sequence $(\phi_b(n), \phi_b(n+1), \ldots, \phi_b(n+s-1))_{n=1}^{\infty}$ on $[0, 1)^s$.

[1] Problem 1.12 in [55] as of 11 December 2011.

The case $s = 2$ was solved by Fialová and Strauch in [13]. They showed that every point $(\phi_b(n), \phi_b(n+1))_{n=1}^{\infty}$ lies on the line segment

$$y = x - 1 + b^{-k} + b^{-k-1} \qquad \text{for } x \in [1 - b^{-k}, 1 - b^{-k-1}) \qquad (k = 0, 1, 2, \dots).$$

Then they gave an explicit formula for the asymptotic distribution function $\nu(x, y)$ of $(\phi_b(n), \phi_b(n+1))_{n=1}^{\infty}$ to calculate for instance the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\phi_b(n) - \phi_b(n+1)| = \int_0^1 \int_0^1 |x - y| \, d_x\nu(x, y) \, d_y\nu(x, y) = \frac{2(b-1)}{b^2},$$

which was previously demonstrated by Pillichshammer and Steinerberger in [46]. They also noted that the limit distribution $\nu(x, y)$ is a copula.

Recently, Aistleitner and Hofer [1] solved the general problem for $s > 2$ and showed that the asymptotic distribution function of $(\phi_b(n), \phi_b(n+1), \dots, \phi_b(n+s-1))_{n=1}^{\infty}$ is also a copula. They employed some ergodic properties of low-discrepancy sequences; in particular, they used the fact that the van der Corput sequence can be seen as the orbit of the origin under the ergodic Kakutani-von Neumann odometer.

In this chapter, we introduce a generalization of van der Corput sequences, called the a-adic van der Corput sequence. These generalised van der Corput sequences also belong to the class of low-discrepancy sequences. It is natural to ask whether the question of Strauch is still true in this general setting or not. Indeed, we demonstrate that this phenomenon holds in general and extends to a broad class of subsequences of the a-adic van der Corput sequence by utilizing the subsequential characterization of unique ergodicity, together with the fact that the a-adic van der Corput sequence can also be seen as the orbit of the origin under the generalised Kakutani-von Neumann odometer.

The outline of this chapter is as follows. In Section 6.2, we give an overview of the classical van der Corput sequences. We first provide a primitive definition of van der Corput sequences using the radical-inverse function, and then we introduce the Kakutani-von Neumann odometer and use it to give a modern definition of van der Corput sequences. In Section 6.3, we extend the classical van der Corput sequence with fixed base b to a more general setting with the base from the a-adic integers. Also, we prove that the newly defined a-adic van der Corput sequence satisfies the condition of low discrepancy and exhibit that it can be constructed by using a generalization of the Kakutani-von Neumann odometer, which possesses unique ergodicity. In Section 6.4, we answer the question of Strauch in this setting of the a-adic van der Corput sequence and show that this phenomenon also holds for a broad class of subsequences of the a-adic van der Corput sequence.

In this chapter, we reserve b for a natural number greater than 1 and $a = (a_n)_{n=1}^{\infty}$ for a sequence of natural numbers greater than 1.

6.2 van der Corput sequence and Kakutani-von Neumann odometer

Every nonnegative integer n has a unique b-adic representation of the form

$$n = \sum_{i=0}^{\infty} n_i b^i,$$

where $n_i \in \{0, 1, \dots, b-1\}$ and at most a finite number of n_i are non-zero. We note also that every real number $x \in [0, 1)$ has a b-adic representation of the form

$$x = \sum_{i=0}^{\infty} x_i b^{-i-1},$$

where $x_i \in \{0, 1, \dots, b-1\}$; however, this representation is not unique. More precisely, there are exactly two b-adic representations for each $x \in (0, 1)$, one with $x_i = 0$ $(i \geq i_0)$ and one with $x_i = b-1$ $(i \geq i_0)$ for some sufficiently large number i_0. If we restrict ourselves to the representations with $x_i \neq b-1$ for infinitely many i, then the coefficients x_i are uniquely determined for every $x \in [0, 1)$. Define the *radical-inverse function* $\phi_b : \mathbb{N}_0 \to [0, 1)$ by

$$\phi_b(n) = \phi_b\left(\sum_{i=0}^{\infty} n_i b^i\right) = \sum_{i=0}^{\infty} n_i b^{-i-1}.$$

The *van der Corput sequence* in base b is defined as $(\phi_b(n))_{n=1}^{\infty}$. It is a classical result that the van der Corput sequence $\omega = (\phi_b(n))_{n=1}^{\infty}$ is uniformly distributed on $[0, 1)$ and has extremely low discrepancy such that

$$ND_N(\omega) \leq \frac{\log N}{3 \log 2} + 1 \qquad \text{and} \qquad \limsup_{N \to \infty} \left(ND_N(\omega) - \frac{\log N}{3 \log 2} \right) \geq \frac{4}{9} + \frac{\log 3}{3 \log 2},$$

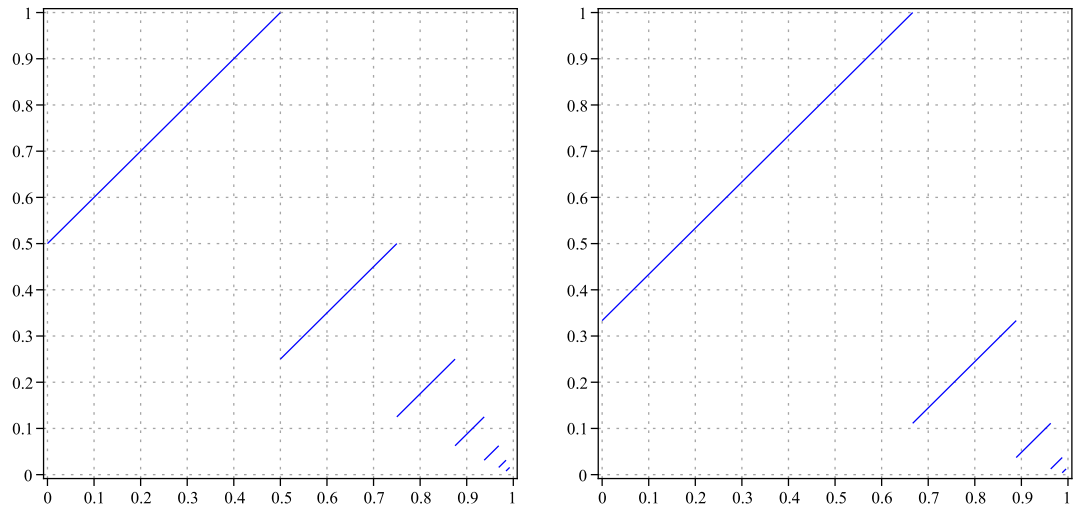
see [27, p 127–130] for more details. Its s-dimensional extension is the *Halton sequence* given by $(\phi_{b_1}(n), \dots, \phi_{b_s}(n))_{n=1}^{\infty}$ for pairwise coprime natural numbers $b_1, \dots, b_s \in \mathbb{N}_{>1}$. Properties of the van der Corput sequence and the Halton sequence are well-understood and they are examples of low-discrepancy sequences valuable in numerical integration.

A second approach to obtain the van der Corput sequence is to consider the orbit of the origin under the *Kakutani-von Neumann odometer* $T_b : [0, 1) \to [0, 1)$, which is a piecewise translation map given by

$$T_b x = x - 1 + b^{-k} + b^{-k-1} \qquad \text{for } x \in [1 - b^{-k}, 1 - b^{-k-1}) \qquad (k = 0, 1, 2, \dots).$$

That is, the base b van der Corput sequence is $(\phi_b(n))_{n=1}^{\infty} = (T_b^n 0)_{n=1}^{\infty}$. The name arises from the fact that T_b is a "Euclidean model" for the map $\tau(x) = x + 1$ on the ring of b-adic integers; in other words, it is a b-adic adding machine transformation.

The second construction provides a wealth of new constructions of low-discrepancy sequences, see e.g. [18]. In [15], Friedman proved the ergodicity and measure-preserving



(a) T_2 (b) T_3

Figure 6.1: The graphs of the Kakutani-von Neumann odometer for $b = 2$ and $b = 3$

property of the Kakutani-von Neumann odometer. It follows from the ergodicity of the Kakutani-von Neumann odometer and the Weyl criterion that $(T_b^n x)_{n=1}^{\infty}$ is uniformly distributed for almost everywhere $x \in [0, 1)$. Furthermore, it can be shown that the Kakutani-von Neumann odometer is uniquely ergodic, which implies that $(T_b^n x)_{n=1}^{\infty}$ is uniformly distributed for every $x \in [0, 1)$, see [18]. The sequence $(T_b^n x)_{n=1}^{\infty}$ for an arbitrary $x \in [0, 1)$ is called the *generalised van der Corput sequence*, and it can be seen as an example of randomized low-discrepancy sequences, see [21].

6.3 a-adic van der Corput sequence

In this section, we extend the definition of van der Corput sequences mentioned in the previous section to the most possible general setting, and we show that this new construction provides a broad class of low-discrepancy sequences.

Let $a = (a_n)_{n=1}^{\infty}$ be a sequence of natural numbers greater than 1, and let Δ_a denote the a-adic integers. Define the *Monna map* $\phi_a : \Delta_a \to [0, 1)$ by

$$\phi_a(x) = \sum_{n=1}^{\infty} \frac{x_n}{a_1 \cdots a_n}$$

for every $x = (x_n)_{n=1}^{\infty} \in \Delta_a$. We identify each natural number n with the element nu of Δ_a and have therefore defined $\phi_a(n)$. The Monna map is continuous and surjective, but not injective. We shall call the maximal subset of Δ_a on which ϕ_a is injective its *regular set*, and we shall call the sequence $a = (a_n)_{n=1}^{\infty}$ *useful* if the regular set contains all natural numbers. The *a-adic van der Corput sequence* is defined as $(\phi_a(n))_{n=1}^{\infty}$ for a useful sequence a. In the case where $a_n = b$ for all $n \in \mathbb{N}$, the sequence $(\phi_a(n))_{n=1}^{\infty}$

coincides with the base b van der Corput sequence; that is, the Monna map can be seen as a generalization of the radical-inverse function. One checks readily that, on the regular set, the map ϕ_a is a bijection and that the image of a uniformly distributed sequence on Δ_a is uniformly distributed on $[0,1)$. More generally, if a sequence is asymptotically distributed on Δ_a with respect to a measure ρ, then the image of the sequence is asymptotically distributed on $[0,1)$ with respect to the push forward of the measure ρ onto $[0,1)$. Now we show that this generalised version of van der Corput sequences gives us a wealth of low-discrepancy sequences. The aim of the following lemma is to show that the a-adic van der Corput sequence satisfies the low-discrepancy criterion, though it is not a sharp result and it can be further improved. We note also that the idea of the proof is developed from the classical dyadic case in [27, p 127–128].

Lemma 6.3.1. *Let $\omega = (\phi_a(n))_{n=1}^{\infty}$ be the a-adic van der Corput sequence. Then the discrepancy $D_N(\omega)$ of the a-adic van der Corput sequence satisfies*

$$ND_N(\omega) \leq \frac{\log N}{\log 2} + 1.$$

Proof of Lemma 6.3.1. We can always represent a given $N \in \mathbb{N}$ by its a-adic expansion

$$N = N_k a_1 \cdots a_{k-1} + N_{k-1} a_1 \cdots a_{k-2} + \cdots + N_2 a_1 + N_1, \tag{6.3.1}$$

where $k \in \mathbb{N}$ and $N_j \in \{0, 1, \ldots, a_j - 1\}$ $(1 \leq j \leq k)$. Partition the interval $[0, N]$ of integers into k subintervals $M_1, \ldots, M_k$ as follows. First, put $M_1 = [0, N_k a_1 \cdots a_{k-1}]$. Then, for each $1 < j \leq k$, we define M_j as the interval

$$[N_k a_1 \cdots a_{k-1} + \cdots + N_{k-j+2} a_1 \cdots a_{k-j+1} + 1, N_k a_1 \cdots a_{k-1} + \cdots + N_{k-j+1} a_1 \cdots a_{k-j}]$$

An integer $n \in M_j$ can be written in the form

$$n = N_k a_1 \cdots a_{k-1} + \cdots + N_{k-j+2} a_1 \cdots a_{k-j+1} + 1 + \sum_{i=1}^{k-j+1} n_i a_0 \cdots a_{i-1}, \tag{6.3.2}$$

where $a_0 = 1$ and $n_i \in \{0, 1, \ldots, a_i - 1\}$ $(1 \leq i \leq k-j+1)$ such that $n_{k-j+1} < N_{k-j+1}$. In fact, we get all $N_{k-j+1} a_1 \cdots a_{k-j}$ integers in M_j if we let the n_i run through all possible combinations. Also, we can combine the last part on the right side of (6.3.2):

$$n = N_k a_1 \cdots a_{k-1} + \cdots + N_{k-j+2} a_1 \cdots a_{k-j+1} + \sum_{i=1}^{k-j+1} n_i a_0 \cdots a_{i-1},$$

where $a_0 = 1$ and $n_i \in \{0, 1, \ldots, a_i - 1\}$ $(1 \leq i \leq k-j+1)$ such that $n_{k-j+1} \leq N_{k-j+1}$. It now follows that

$$\phi_a(n) = \frac{N_k}{a_1 \cdots a_k} + \cdots + \frac{N_{k-j+2}}{a_1 \cdots a_{k-j+2}} + \sum_{i=1}^{k-j+1} \frac{n_i}{a_1 \cdots a_i} = x_j + \sum_{i=1}^{k-j+1} \frac{n_i}{a_1 \cdots a_i},$$

where x_j only depends on j, and not on n. If n runs through M_j, then $\sum_{i=1}^{k-j+1} \frac{n_i}{a_1 \cdots a_i}$ runs through all fractions $0, \frac{1}{a_1}, \ldots, \frac{a_1-1}{a_1}, \frac{1}{a_1 a_2}, \ldots, \sum_{i=1}^{k-j} \frac{a_i-1}{a_1 \cdots a_i} + \frac{N_{k-j+1}}{a_1 \cdots a_{k-j+1}}$ in some order. Moreover, we note that

$$0 \le x_j = \frac{N_k}{a_1 \cdots a_k} + \cdots + \frac{N_{k-j+2}}{a_1 \cdots a_{k-j+2}} \le \frac{a_k \cdots a_{k-j+2} - 1}{a_1 \cdots a_k} \le \frac{1}{a_1 \cdots a_{k-j+1}}$$

We deduce that if the $\phi_a(n)$ $(n \in M_j)$ are ordered according to their magnitude, then we obtain a sequence ω_j of $N_{k-j+1} a_1 \cdots a_{k-j}$ elements that is an almost-arithmetic progression[2] with parameters $\delta_j = 0$ and $\eta_j = \frac{1}{a_1 \cdots a_{k-j+1}}$. It now follows immediately from [40, Theorem 2.1][3] that the discrepancy of each ω_j, multiplied by the number of elements in ω_j, is at most 1. Combining this with [27, Theorem 2.6, Ch 2][4] and the fact that $\phi_a(1), \ldots, \phi_a(N)$ is decomposed into k sequences ω_j, we obtain $N D_N(\omega) \le k$.

It remains to estimate k in terms of N. By (6.3.1), we have $N \ge a_1 \cdots a_{k-1} \ge 2^{s-1}$, and so we obtain $k \le (\log N / \log 2) + 1$. This completes the proof of Lemma 6.3.1. ■

It is an immediate consequence of Lemma 6.3.1 that $D_N(\omega) = O(N^{-1} \log N)$, where ω denotes any a-adic van der Corput sequence, and we have the following result.

Theorem 6.3.2. *The a-adic van der Corput sequence is a low-discrepancy sequence.*

We now give another construction of the a-adic van der Corput sequence using a generalization of the Kakutani-von Neumann odometer. Define $T_a : [0,1) \to [0,1)$ by

$$T_a x = x - 1 + \frac{1}{a_0 \cdots a_{k-1}} + \frac{1}{a_0 \cdots a_k} \qquad \text{for } x \in \left[1 - \frac{1}{a_0 \cdots a_{k-1}}, 1 - \frac{1}{a_0 \cdots a_k}\right),$$

[2]For $\delta \ge 0$ and $\epsilon > 0$, a sequence $c_1 < c_2 < \cdots < c_L$ of points from the interval $[0,1]$ is called an *almost-arithmetic progression*-(δ, ϵ) if there exists an η, with $0 < \eta \le \epsilon$, such that

(1) $0 \le c_1 \le \eta + \delta\eta$;

(2) $\eta - \delta\eta \le c_{i+1} - c_i \le \eta + \delta\eta$ for $1 \le i \le L-1$;

(3) $1 - \eta - \delta\eta \le c_L \le 1$.

Almost-arithmetic progressions were introduced by O'Neil in [44]. Their theoretical importance stems from the following criterion. The sequence $(c_n)_{n=1}^{\infty}$ of points in $[0,1]$ is uniformly distributed mod 1 if and only if the following condition holds: given $\delta \ge 0$, $\epsilon > 0$ and $\epsilon' > 0$, there exists $\bar{L} = \bar{L}(\delta, \epsilon, \epsilon')$ such that, for all $L > \bar{L}$, the initial segment $c_1, \ldots, c_L$ can be decomposed into almost-arithmetic progressions-(δ, ϵ) with at most L_0 elements left over, where $L_0 < \epsilon' L$.

[3]For an arbitrary almost-arithmetic progression-(δ, ϵ) sequence $c_1, \ldots, c_L$, we have

$$D_L(c_1, \ldots, c_L) \le \begin{cases} 1/L + \delta/(1 + \sqrt{1 - \delta^2}) & \text{for } \delta > 0, \\ \min(\eta, 1/L) & \text{for } \delta = 0. \end{cases}$$

[4]For $1 \le j \le m$, let ω_j be a sequence of N_j elements from $\mathbb{R}$ with discrepancy $D_{N_j}(\omega_j)$. Let ω be a superposition of $\omega_1, \ldots, \omega_m$, that is, a sequence obtained by listing in some order the terms of the ω_j. We set $N = N_1 + \cdots + N_m$, which will be the number of elements of ω. Then we have

$$D_N(\omega) \le \sum_{j=1}^{m} \frac{N_j}{N} D_{N_j}(\omega_j).$$

where $k \in \mathbb{N}$ and $a_0 = 1$. Observe that, under the transformation T_a,

$$0 \mapsto \frac{1}{a_1} \mapsto \cdots \mapsto \frac{a_1 - 1}{a_1} \mapsto \frac{1}{a_1 a_2} \mapsto \frac{1}{a_1} + \frac{1}{a_1 a_2} \mapsto \cdots \mapsto \frac{a_1 - 1}{a_1} + \frac{1}{a_1 a_2} \mapsto \frac{2}{a_1 a_2} \mapsto \cdots .$$

We see that the orbit of 0 under the generalised Kakutani-von Neumann odometer T_a is precisely the a-adic van der Corput sequence, so we have $(\phi_a(n))_{n=1}^{\infty} = (T_a^n 0)_{n=1}^{\infty}$.

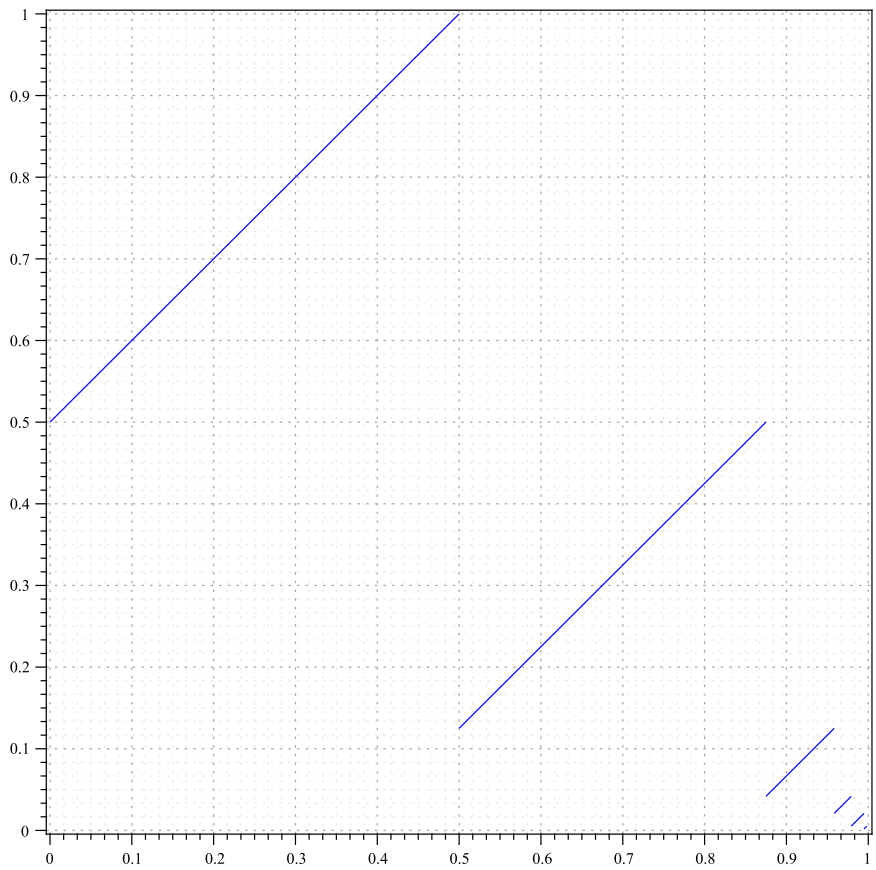


Figure 6.2: The graph of the Kakutani-von Neumann odometer for $a = (\overline{243})$

One readily checks that the generalised Kakutani-von Neumann odometer preserves the Lebesgue measure λ on $[0, 1)$. Moreover, the Monna map defines an isomorphism between the dynamical system $(\Delta_a, \mathcal{B}, \mu, \tau)$, where μ denotes the Haar measure and $\tau(x) = x + u$ $(x \in \Delta_a)$ is the a-adic adding machine, and the dynamical system $([0, 1), \mathcal{B}, \lambda, T_a)$. Because of the density of the natural numbers in Δ_a, it follows that the dynamical system $(\Delta_a, \mathcal{B}, \mu, \tau)$ is a uniquely ergodic group rotation, see [18, p 27] and [58, p 162–163]. Moreover, via the Monna map, the two dynamical systems are metrically isomorphic. This means that $([0, 1), \mathcal{B}, \lambda, T_a)$ is a uniquely ergodic dynamical system, and whence we have the following result.

Theorem 6.3.3. [G] *Suppose that $a = (a_n)_{n=1}^{\infty}$ is a useful sequence of natural numbers greater than 1. Then the dynamical system $([0, 1), \mathcal{B}, \lambda, T_a)$ is uniquely ergodic.*

6.4 Asymptotic distribution of the a-adic van der Corput sequence

In this final section, we return to the question of Strauch in a more general setting. Indeed, given $m_1, \ldots, m_s \in \mathbb{N}_0$ and a Hartman uniformly distributed sequence $(k_n)_{n=1}^{\infty}$, we find all the distribution functions for the sequence $(\phi_a(k_n+m_1), \ldots, \phi_a(k_n+m_s))_{n=1}^{\infty}$. This result includes the case $k_n = n$ and $m_1 = 0$, $m_2 = 1$, $\ldots$, $m_s = s-1$; i.e., we can calculate the asymptotic distribution function for $(\phi_a(n), \phi_a(n+1), \ldots, \phi_a(n+s-1))_{n=1}^{\infty}$.

Define a map $\gamma : [0,1) \to [0,1)^s : x \mapsto (T_a^{m_1}x, \ldots, T_a^{m_s}x)$, and let

$$\Gamma = \{\gamma(x) \colon x \in [0,1)\}.$$

The Lebesgue measure λ on $[0,1)$ induces a measure ℓ on Γ by setting

$$\ell(E) = \lambda(\{x \in [0,1) \colon \gamma(x) \in E\})$$

for each $E \subseteq \Gamma$. Moreover, ℓ induces a measure ν on $[0,1)^s$ by embedding Γ into $[0,1)^s$. More precisely, for every Jordan-measurable set $B \subseteq [0,1)^s$, we set

$$\nu(B) = \ell(B \cap \Gamma).$$

Theorem 6.4.1. [G] *Let $a = (a_n)_{n=1}^{\infty}$ be a useful sequence of natural numbers greater than 1, and let $m_1, \ldots, m_s \in \mathbb{N}_0$. Suppose $(k_n)_{n=1}^{\infty}$ is a Hartman uniformly distributed sequence. Then the following are true:*

(1) *The asymptotic distribution of $(\phi_a(k_n+m_1), \ldots, \phi_a(k_n+m_s))_{n=1}^{\infty}$ is ν.*

(2) *The measure ν is a copula on $[0,1)^s$.*

(3) *The sequence $(\phi_a(k_n+m_1), \ldots, \phi_a(k_n+m_s))_{n=1}^{\infty}$ is uniformly distributed on Γ.*

Proof of Theorem 6.4.1(1). We first note that

$$(\phi_a(k_n+m_1), \ldots, \phi_a(k_n+m_s))_{n=1}^{\infty} = (T_a^{k_n+m_1}0, \ldots, T_a^{k_n+m_s}0)_{n=1}^{\infty}.$$

Also, we know by the construction that, for each $n \in \mathbb{N}$,

$$(T_a^{k_n+m_1}0, \ldots, T_a^{k_n+m_s}0) = (T_a^{m_1}(T_a^{k_n}0), \ldots, T_a^{m_s}(T_a^{k_n}0)) \in \Gamma.$$

Now consider a Jordan measurable set $B \subseteq [0,1)^s$. We define the empirical measure of the first N points of $(T_a^{k_n+m_1}0, \ldots, T_a^{k_n+m_s}0)_{n=1}^{\infty}$ by

$$\nu_N(B) = \frac{1}{N} \cdot \#\big\{1 \le n \le N \colon (T_a^{k_n+m_1}0, \ldots, T_a^{k_n+m_s}0) \in B\big\}.$$

To prove that ν is the unique asymptotic distribution of the sequence, it suffices to show that ν_N converges to ν as N tends to infinity.

$$\begin{aligned}\lim_{N\to\infty} \nu_N(B) &= \lim_{N\to\infty} \frac{1}{N} \cdot \#\big\{1 \le n \le N \colon (T_a^{k_n+m_1}0, \dots, T_a^{k_n+m_s}0) \in B\big\} \\ &= \lim_{N\to\infty} \frac{1}{N} \cdot \#\big\{1 \le n \le N \colon (T_a^{k_n+m_1}0, \dots, T_a^{k_n+m_s}0) \in B \cap \Gamma\big\} \\ &= \lim_{N\to\infty} \frac{1}{N} \cdot \#\big\{1 \le n \le N \colon T_a^{k_n}(T_a^{m_1}0) \in \mathrm{proj}_1(B \cap \Gamma)\big\},\end{aligned}$$

where proj_1 denotes the projection onto the first coordinate of $[0,1)^s$. By Theorem 6.3.3, T_a is uniquely ergodic. It now follows from Theorem 2.5.1 and the Weyl criterion that $(T_a^{k_n}(T_a^{m_1}0))_{n=1}^{\infty}$ is uniformly distributed mod 1. Furthermore, the fact that a is a useful sequence indicates the bijection of the map $x \mapsto T_a(x)$. Therefore, we have

$$\lim_{N\to\infty} \nu_N(B) = \lambda(\mathrm{proj}_1(B \cap \Gamma)) = \ell(B \cap \Gamma) = \nu(B).$$

This shows ν is the asymptotic distribution of $(\phi_a(k_n+m_1), \dots, \phi_a(k_n+m_s))_{n=1}^{\infty}$. ■

Proof of Theorem 6.4.1(2). To show that the measure ν is a copula on $[0,1)^s$, we argue that the sequence $(\phi_a(k_n+m_j))_{n=1}^{\infty}$ is uniformly distributed mod 1 for every $1 \le j \le s$. We know that $\phi_a(k_n+m_j) = T_a^{k_n}(T_a^{m_j}0)$ for each $n \in \mathbb{N}$. Since T_a is uniquely ergodic, it follows from Theorem 2.5.1 and the Weyl criterion that $(T_a^{k_n}(T_a^{m_j}0))_{n=1}^{\infty}$ is uniformly distributed mod 1, as required. ■

Proof of Theorem 6.4.1(3). This is an immediate consequence of Theorem 6.4.1(1). ■

Theorem 6.4.1 is a more general answer to the question of Strauch. In particular, when $a_n = b$ for all $n \in \mathbb{N}$, this result refines the work of Aistleitner and Hofer [1] for a broad class of subsequences of the classical van der Corput sequence in base b.

Corollary 6.4.2. [G] *Let $b > 1$ be a natural number, and let $m_1, \dots, m_s \in \mathbb{N}_0$. Suppose $(k_n)_{n=1}^{\infty}$ is a Hartman uniformly distribution sequence. Then the following are true:*

(1) *The asymptotic distribution of $(\phi_b(k_n+m_1), \dots, \phi_b(k_n+m_s))_{n=1}^{\infty}$ is ν.*

(2) *The measure ν is a copula on $[0,1)^s$.*

(3) *The sequence $(\phi_b(k_n+m_1), \dots, \phi_b(k_n+m_s))_{n=1}^{\infty}$ is uniformly distributed on Γ.*

Proof of Corollary 6.4.2. Apply Theorem 6.4.1 with $a_n = b$ for all $n \in \mathbb{N}$. ■

Appendix A

Summary of excluded papers

In this appendix, we give a summary of each excluded paper.

[B] J. Hančl, A. Jaššová, P. Lertchoosakul and R. Nair. *Polynomial actions in positive characteristic*. Proc. Steklov Inst. Math., 280(suppl.2):37–42, 2013.
Summary: We prove the positive characteristic analogue of D.J. Rudolph's result regarding the famous H. Furstenberg's conjecture $2\times$ and $3\times$ invariant measure. Indeed, we show that either the entropy of the maps is zero or the non-atomic measure is Haar measure.

[C] J. Hančl, A. Jaššová, P. Lertchoosakul and R. Nair. *On the quantitative metric theory of continued fractions*. Preprint 2014.
Summary: Quantitative versions of the central results of the metric theory of continued fractions were given primarily by C. de Vroedt. We give improvements of the bounds involved by using a quantitative L^2 ergodic theorem.

[D] J. Hančl, A. Jaššová, P. Lertchoosakul and R. Nair. *Polynomial actions in positive characteristic II*. Preprint 2014.
Summary: We prove the positive characteristic analogue of E. Lindenstrauss' result regarding the famous H. Furstenberg's conjecture $2\times$ and $3\times$ invariant measure. Indeed, we improve our previous result in [B] by adjusting the condition that p and q are coprime to the condition that p does not divide any positive power of q.

[E] A. Jaššová, S. Kristensen, P. Lertchoosakul and R. Nair. *On recurrence in positive characteristic*. Preprint 2014.
Summary: Let $P-1$ denote the set of primes minus 1. A classical theorem of A. Sárközy says that any set of natural numbers of positive density contains a pair of elements whose difference belongs to the set $P-1$. We investigate the positive characteristic analogue of questions of this type, building on work of H. Furstenberg, by using an ergodic approach.

[F] A. Jaššová, P. Lertchoosakul and R. Nair. *On variants of the Halton sequence.* Preprint 2014.

Summary: We use some ergodic methods to prove the uniform distribution of a large class of subsequences of a generalization of the classical Halton sequences. This builds on earlier work of M. Hofer, M.R. Iacò and R. Tichy in the case of a generalization of the classical Halton sequences.

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