Backward Stochastic Differential Equations with Unbounded Coefficients and Their Applications

Thesis

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Contents

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Two years just passed, a new life is about to start.

Declarations

I hereby declare that this thesis is based on my own research in accordance with the regulations of the University of Liverpool. The work is original expect when indicated otherwise. This thesis has not been submitted for examination at any other university.

Chapter 3 is based on a submitted paper (Gashi and Li [24]). Chapter 4, Chapter 5 and Chapter 6 are based on preprint papers (Gashi and Li [25], [26], [27]) that are ready to but not yet be submitted for publication by the time of submission of this thesis.

Abstract

In this thesis, we focus on problems on the theory of Backward Stochastic Differential Equations (BSDEs). In particular, BSDEs with an unbounded generator are considered, under various conditions (on the generator). Using more general (or weaker) conditions, the classical results on BSDEs are improved and some associated problems on mathematical finance are resolved. Chapter 1 introduces some of the literature, general setting and ideas in this field and emphasises the motivations which has led to the study of these equations. In addition, some mathematical preliminaries we used throughout this thesis are included in Chapter 2.

In Chapter 3, we consider nonlinear BSDEs with an unbounded generator. Under a Lipschitz-type condition, we show sufficient conditions for the existence and uniqueness of solutions to nonlinear BSDEs, which are weaker than the existing ones. We also give a comparison theorem as a generalisation of Peng's result.

Chapter 4 studies a class of backward stochastic differential equations whose generator satisfies linear growth and continuity conditions, which can also be unbounded. We prove the existence of the solution pair for this class of equations which is more general than the existing ones.

In Chapter 5, we consider the problem of solvability for linear backward stochastic differential equations with unbounded coefficients. New and weaker sufficient conditions for the existence of a unique solution pair are given. It is shown that certain exponential processes have stronger integrability in this case. As applications, we solve the problems of completeness in a market with a possibly unbounded coefficients and optimal investment with power utility in a market with unbounded coefficients.

Chapter 6 studies the classical Stochastic Differential Equations where the drift and diffusion coefficients satisfy Lipschitz-type and linear growth conditions, which can also be unbounded. We give sufficient conditions for the existence of a unique solution to unbounded SDEs. The method of proof is that of Picard iterations and the resulting conditions are new. We also prove a comparison theorem.

Chapter 7 summaries the results in this thesis and outlines possible directions for future works based on current results.

Chapter 1

Introduction

1.1 Overview

Backward stochastic differential equation (BSDE) is known as a special type of stochastic differential equations (SDEs) because its value of solution is prescribed at the final point (rather than initial point) of the time interval. A linear backward stochastic differential equation was firstly studied by Bismut [8] who attempted to solve a stochastic optimal control problem by stochastic version of Pontryagin maximum principle. There are some other works on maximum principle which are also studied using linear BSDEs, e.g. Arkin and Saksonov [2], Kabanov [34] and more recently, Cadenillas and Karatzas [11]. The general theory of nonlinear BSDEs with Lipschitz condition was first introduced by Pardoux and Peng [61], and, later independently, by Duffie and Epstein [20] in a more financial context (theory of recursive utilities).

The theory of backward stochastic differential equations has been extensively studied in so many different research areas, such as stochastic control, mathematical finance and partial differential equations (PDEs) etc., in last 25 years. To be more specific, BSDEs can be applied to solve stochastic optimal problem, establish probabilistic representations of solutions to PDEs and define nonlinear expectations and so on. In mathematical finance, the theory of hedging and pricing strategy of a contingent claim can be formulated in terms of a linear BSDE or nonlinear BSDE when portfolio constraints are taken into account. As many financial problems are relevant to optimal control problems, BSDEs become a useful mathematical tool to study mathematical finance and hence many new financial and actuarial applications are found and developed. See [41] for fundamental study on applications to finance and [19] for more practical ingredients to finance and insurance. In the book edited by El Karoui and Mazliak [40], the relationship between BSDEs and stochastic control are discussed, especially presenting a link between BSDEs and martingale methods in stochastic control. Peng [64] introduces a notion of g-expectation (nonlinear expectation) by BSDEs. This notion provides us a way to establish a nonlinear g-martingale theory which later is shown to be as importance as the classical martingale theory in probability. A systematic study on recent development of theory of nonlinear expectation can be found in [67].

Another direction connected with applications in this area is how to improve the conditions of existence and uniqueness of a solution for BSDEs, which is also the main subject of this thesis. Many articles focus on weakening the existence conditions of a solution for BSDEs. In general, they assume that the drift coefficient is continuous and satisfies a linear or a quadratic growth condition. See [50] and [46] for instance. In addition, those works are all based on the comparison theorem of solutions of BSDEs, which is also discussed in this thesis. Note that, however, in general we do not have uniqueness of the solution.

1.2 Backward Stochastic Differential Equations

In this section, we briefly introduce the theory of general BSDEs (see Yong and Zhou [77] in detail). Under the usual Lipschitz condition, both initial and terminal value problem for ordinary differential equations (ODEs) are well-posed. Accordingly, the terminal value problem of ODEs on time interval $[0, T]$ is equivalent to its initial value problem on the same time interval but reversing time backwardly $t \mapsto T - t$. Nevertheless, as we need its solution is adapted to a given filtration, BSDE encounters a completely different situation. For the case of stochastic differential equations (SDEs), it is not allowed to simply reverse the time to obtain a solution for its terminal value problem because it would ruin the adaptiveness of a solution. Therefore, it is necessary to reformulate a terminal value problem for SDEs.

The solution of a BSDE is regarded as a pair of adapted (stochastic) processes, which the second part of the solution rectifies the non-adaptiveness generated by the "backward" essence of the equations. Also the second part of the solution reveals uncertainty of the dynamics happened between now and terminal time, while the first part of the solution indicates the mean evolution of the dynamics. The following simple example (also see Yong and Zhou [77] in detail) describes how the uncertainty be involved as a part of the adapted solution.

Let us consider an investor is going to invest two assets, a bond and a stock, in a financial market. Assume that the bond's annual return rate is 8% and the stock has the following expected annual return rate: 16% (resp. −16%) if it will be a bull (resp. bear) market in next year. The corresponding financial goal of the investment is to achieve amount m £ (resp. n £) if it will be bull (reps. bear) market in next year ($m \geq n$). Now we are going to figure out how to achieve these financial goals. Assume that $(y - z)$ £ is invested in the bond and z £ is invested in the stock. Hence one has the following system of equations:

$$\begin{cases} 1.08(y-z) + 1.16z = m, \\ 1.08(y-z) + 0.84z = n, \end{cases}$$

which has a unique solution

$$\begin{cases} y = \frac{75m+25n}{108}, \\ z = \frac{25(m-n)}{8}. \end{cases}$$

From above setting, it is worth noting that the financial goal (future wealth) is given by a random variable which takes two values m and n, depending on following year's market performance. In addition, the investment is divided into two parts: $y - z$ and z, where z controls the uncertainty of the investment from now to the end of next year (z is also be seen as the amount invested into the risky asset, i.e. a stock). In particular, when m is equivalent to n, s/he is intent to have a certain amount of wealth in the next year, and hence the second part of the investment (solution) z becomes zero. In other words, all the investment will put into the riskless asset, a bond.

Now we introduce the setting of general BSDEs. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t, t \geq 0), \mathbb{P})$ be a given complete filtered probability space on which a k-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $\mathscr{F}_t$ is the augmentation of $\sigma\{W(s) : 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathscr{F}$. Consider the backward stochastic differential equation:

$$y(t) = \xi + \int_t^T f(s, y(s), z(s))ds - \int_t^T z(s)dW(s), \quad t \in [0, T], \tag{1.2.1}$$

where

- terminal condition $\xi \in M^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}^d)$ (the space of all $\mathscr{F}_T$-measurable $\mathbb{R}^d$-valued random variables ζ such that $\mathbb{E}[|\zeta|^2] < \infty$) is a given $\mathscr{F}_T$-measurable $\mathbb{R}^d$-valued random variable;
- generator (coefficient) $f : \Omega \times (0, T) \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \to \mathbb{R}^d$ is a progressively measurable

function and there exists a real constant $c > 0$ such that

$$f(\cdot, 0, 0) \in M^2(0, T; \mathbb{R}^d),$$

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq c(|y_1 - y_2| + |z_1 - z_2|),$$

for all $y_1, y_2 \in \mathbb{R}^d$ and $z_1, z_2 \in \mathbb{R}^{d\times k}$, (t, ω) a.e. and $M^2(0, T; \mathbb{R}^d)$ is the space of $\mathscr{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\varphi(\cdot)$ such that $\mathbb{E}\int_0^T |\varphi(t)|^2 dt < \infty$. Then a pair of adapted processes $(y(\cdot), z(\cdot)) \in M^2(0, T; \mathbb{R}^d) \times M^2(0, T; \mathbb{R}^{d\times k})$ is called s solution of BSDE (1.2.1).

Let us make some comments on the solution of a backward stochastic differential equation (see [53] and [60] in detail). As we mentioned before, unlike deterministic equations (ODEs) where forward and backward problems are similar, in general, a stochastic differential equation with given terminal value does not admit a non-anticipating solution. For instance, a constant solution $V(t) = \xi$ is anticipating. A possible adapted approximation is the martingale y with terminal value ξ. Hence we could formulate it by introducing a process z as a martingale representation of y:

$$-dy(t) = -z(t)dW(t), \quad y(T) = \xi.$$

In a more general setting, a coefficient f (linear or nonlinear) will be considered.

In addition, it is not natural to have a constraint that requires the solution of a backward equation to be adapted to the past of $W(\cdot)$ for every time t. That is why it is essential to have some freedom to select z independently (of y).

Moreover, the randomness of solution comes from the randomness of ξ and f. In order to ensure y is progressively measurable, the stochastic integral is introduced, which, in other words, reduces the randomness of y. In particular, for some stopping time τ, if ξ and f are $\mathscr{F}_\tau$-measurable, then $z = 0$ on time interval $[\tau, T]$ and hence y is the solution of a ODE:

$$dy(t) = -f(t, y(t), 0)dt, \quad y(T) = \xi.$$

1.3 Linear Backward Stochastic Differential Equations

In this section, we introduce a special and important form of BSDEs, linear backward stochastic differential equations (LBSDEs), which have many interesting applications on mathematical finance, stochastic control and so on. Let us start with the following simple example (see e.g. [53] and [77]):

Consider the following terminal value problem

$$\begin{cases} dy(t) &= 0, \quad t \in [0,T], \\ y(T) &= \xi, \end{cases} \tag{1.3.1}$$

where $\xi \in M^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}^d)$.

Our goal is to find an $\{\mathscr{F}_t\}_{t\geq 0}$-adapted solution $y(\cdot)$. However, as there is only one solution

$$y(t) = \xi, \quad \forall\, t \in [0,T], \tag{1.3.2}$$

which is not necessarily $\{\mathscr{F}_t\}_{t\geq 0}$-adapted (unless ξ is a constant, i.e. $\mathscr{F}_0$-measurable). Therefore, we need to reformulate equation (1.3.1) in order to obtain a $\{\mathscr{F}_t\}_{t\geq 0}$-adapted solution.

Now let us reformulate equation (1.3.1) by redefining $y(\cdot)$ in the following way:

$$y(t) = \mathbb{E}[\,\xi\,|\mathscr{F}_t] \quad t \in [0,T]. \tag{1.3.3}$$

Then $y(\cdot)$ is $\{\mathscr{F}_t\}_{t\geq 0}$-adapted and (as ξ is $\mathscr{F}_T$-measurable) it satisfies the terminal condition $y(T) = \xi$. Nevertheless, $y(\cdot)$ (given by (1.3.3)) does not satisfy (1.3.1). Hence our next goal is to reformulate the equation such that $y(\cdot)$ is indeed one of $\{\mathscr{F}_t\}_{t\geq 0}$-adapted solutions to (1.3.1).

Note that $y(\cdot)$ (given by (1.3.3)) is a square integrable $\{\mathscr{F}_t\}_{t\geq 0}$ martingale. By (Brownian) martingale representation theorem (see Theorem 2.4.2), one has

$$y(t) = y(0) + \int_0^t z(s)dW(s), \quad t \in [0,T], \tag{1.3.4}$$

and also

$$\xi = y(T) = y(0) + \int_0^T z(s)dW(s), \quad t \in [0,T].$$

Hence

$$y(t) = \xi - \int_0^T z(s)dW(s), \quad t \in [0,T],$$

which its differential form is

$$\begin{cases} dy(t) &= z(t)dW(t), \quad t \in [0,T], \\ y(T) &= \xi. \end{cases} \tag{1.3.5}$$

From above reformulation, an extra term $z(t)dW(t)$ is introduced and it is the difference between equations (1.3.5) and (1.3.1). Its appearance repairs the non-adaptiveness of the $y(\cdot)$ given by (1.3.2). Accordingly, an adapted solution of (1.3.5) is a pair of $\{\mathscr{F}_t\}_{t\geq 0}$-adapted processes $(y(\cdot), z(\cdot))$ satisfying (1.3.5), where $y(\cdot)$ is given by (1.3.3) and $z(\cdot)$ is introduced in (1.3.4) by (Brownian) martingale representation theorem.

Since a BSDE is an equation with two unknowns, it is nature to ask if the adapted solution is unique. In fact, the adaptiveness provides another condition that guarantees the uniqueness of the solution of a BSDE. Applying the Itô formula to $|y(t)|^2$, we have

$$\mathbb{E}\,|\xi|^2 = \mathbb{E}\,|y(t)|^2 + \int_t^T \mathbb{E}\,|z(s)|^2 ds, \quad t \in [0, T].$$

Due to the linearity of equation (1.3.5), together with the above addition relation, the uniqueness of the solution of equation (1.3.5) is obtained.

Now let us introduce the general linear backward stochastic differential equations:

$$\begin{cases} dy(t) &= \left[A(t)y(t) + \sum_{i=1}^n B_i(t)z_i(t) + f(t)\right] dt + z(t)dW(t), \quad t \in [0, T], \\ y(T) &= \xi, \end{cases} \tag{1.3.6}$$

where $A(\cdot), B_i(\cdot)\,(i = 1, \cdots, n)$ are bounded and $\mathbb{R}^{d\times d}$-valued $\{\mathscr{F}_t\}_{t\geq 0}$-adapted processes, $f(\cdot) \in M^2(0, T; \mathbb{R}^d)$ and $\xi \in M^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}^d)$. Similar to above simplest case (equation (1.3.5)), the pair of processes $(y(\cdot), z(\cdot))$ is required to be forward (i.e. $\{\mathscr{F}_t\}_{t\geq 0}$) adapted, while the process $y(\cdot)$ is given at the terminal time T. Therefore we say the equation is solved backwardly.

See El Karoui [38], Yong and Zhou [77] etc. for further results, e.g. well-posedness of solutions, comparison theorem, on LBSDEs in detail.

1.4 Application to Finance

In finance, one of the most fundamental and important problems is to consider how to price an option. The simplest example is that of the well known Black-Scholes model and an European call option, which is also an example of linear backward stochastic differential equations. Let us first see the following example (see [65] for a more general version in detail):

1.4.1 Pricing and Hedging

Consider a stock price in a financial market is given by the following SDE:

$$dS(t) = S(t)[\mu dt + \sigma dW(t)],$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$, volatility of the stock. Solving the above SDE, for any $t \geq 0$, we have

$$S(t) = S(0)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}.$$

In addition, let us consider the price of a risk-free asset:

$$dR(t) = rR(t)dt,$$

which gives

$$R(t) = R(0)\, e^{rt}.$$

The strategy is given by a pair of adapted processes $(m(t), n(t))$, where $m(t)$ is the number of stock, and $n(t)$ is the number of risk-free asset. Assume that the strategy is *self-financing*, then the value of the portfolio is given by

$$dY(t) = m(t)dS(t) + n(t)dR(t) = m(t)S(t)[\mu dt + \sigma dW(t)] + rn(t)R(t)dt.$$

Let us denote the amount of money invested in risky assets by $\pi(t) = m(t)S(t)$. Substituting it into above equation and using equality $Y(t) = m(t)S(t) + n(t)R(t)$, we obtain

$$dY(t) = rY(t)dt + \frac{\mu - r}{\sigma}\sigma\pi(t)dt + \sigma\pi(t)dW(t).$$

Setting $Z(t) = \sigma\pi(t)$ and $\theta = \frac{\mu - r}{\sigma}$ (the risk premium),

$$dY(t) = [rY(t) + \theta Z(t)]dt + Z(t)dW(t).$$

Now let us see how to price a European call option. Recall that a European call option with maturity T and exercise price K is a contract which gives the right but no obligation to its holder to buy one share of the stock at the exercise price K. On the other hand, the seller of the option has to pay $(S(T) - K)^+$ to its holder, which equals the profit that permits the exercise of the option. Let us consider a claim which profit is given by a non-negative random variable depending on $S(\cdot)$. Is it natural to ask at which price y should one sell the option?

In order to find price y, the *replication* is one fundamental idea. The seller sells the option at the price y and invests this amount in the financial market following the strategy Z (to be found). The value of his portfolio is given by the following SDE:

$$dY(t) = [rY(t) + \theta Z(t)]dt + Z(t)dW(t), \quad Y(0) = y.$$

The problem is then to find y and Z such that the solution of above SDE verifies $Y(T) = \xi$, (We say that in this case y is the *fair price*) i.e. can we find adapted (Y, Z) such that

$$dY(t) = [rY(t) + \theta Z(t)]dt + Z(t)dW(t), \quad Y(T) = \xi.$$

In this case it suffices to sell the option at the price $y = Y(0)$. Hence the pricing problem is to solve the linear BSDE.

Let see an example for nonlinear BSDE. Assume that the regulator of the market imposes to avoid short-selling of the stock. Hence this kind of transactions is discouraged by penalizing investors by a proportional cost $\beta\pi(t)^- = \gamma Z(t)^-(\gamma > 0)$. In this case, replicating a claim is to solve the following BSDE

$$dY(t) = \left[rY(t) + \theta Z(t) - \gamma Z(t)^-\right] dt + Z(t)dW(t), \quad Y(T) = \xi,$$

which is nonlinear BSDE but it still verifies the conditions f and ξ in (1.2.1). Another example of nonlinear BSDEs appeared in finance is as follows:

$$dY(t) = [rY(t) + \theta Z(t)]dt + Z(t)dW(t) - (R - r)\left[Y(t) - \frac{Z(t)}{\sigma}\right]^- dt, \quad Y(T) = \xi,$$

where $[\,\cdot\,]^- = \min\{[\,\cdot\,], 0\}$. In this case, we are going to solve this BSDE, replicating a claim when the borrowing rate R is bigger than the lending rate r.

Note that the strategies here are *admissible* ($Y(t) \geq 0$), which follows from the comparison theorem (see Theorem 3.5) as $\xi \geq 0$ and $f(t, 0, 0) \geq 0$.

1.4.2 Stochastic Differential Utility

Duffie and Epstein [20] introduce a stochastic differential formulation of recursive utility where the information is generated by Brownian motion. Recursive utility is an extension of the standard additive utility with the instantaneous utility not only depending on the instantaneous consumption rate $c(t)$ but also depending on the future utility. In fact it is related to the solution of a special type of BSDE which its drift coefficient does not depend on z:

$$-dY(t) = \left[f(c(t), Y(t) - A(Y(t))\frac{1}{2}Z^*Z(t)\right] dt - Z^*(t)dW(t), \quad Y(T) = Y,$$

where $c(\cdot)$ is a positive consumption rate and $A(Y(\cdot))$ is the "variance multiplier". Hence the utility at time t of the future consumption $c(s)$, $t \leq s \leq T$:

$$Y(t) = \mathbb{E}\left[Y + \int_t^T \left[f(c(s), Y(s)) - A(Y(s))\frac{1}{2}Z^*Z(s)\right] ds \middle| \mathscr{F}_t\right].$$

Also Duffie and Epstein [20] show that if coefficient f is Lipschitz with respect to y, then there exists a unique solution of equation

$$-dY(t) = f(t, c(t), Y(t))dt - Z^*dW(t), \quad Y(T) = Y.$$

Moreover, El Karoui, Peng and Quenez [41] study a more general class of recursive utilities where f also depends on z:

$$-dY(t) = f(t, c(t), Y(t), Z(t))dt - Z^*dW(t), \quad Y(T) = Y.$$

Here we state some examples of recursive utilities provided in [20]:

(i) *Standard Additive Utility.* The coefficient f of the standard additive utility is given by: $f(c, y) = u(c) - \beta y$, where $\beta \geq 0$ is a discounting rate. The recursive utility is

$$Y(t) = \mathbb{E}\left[Ye^{-\beta(T-t)} + \int_t^T u(c(s))e^{-\beta(s-t)}ds \middle| \mathscr{F}_t\right].$$

(ii) *Uzawa Utility.* In this case, the discounting rate β depends on the consumption rate $c(t)$. So the coefficient f becomes $f(c, y) = u(c) - \beta(c)y$. The recursive utility remains the same.

(iii) *Kreps-Porteus Utility.* The coefficient f is give by

$$f(c, y) = \frac{\beta}{\rho}\frac{c^\rho - y^\rho}{y^{\rho-1}}.$$

Although the closed-form expression for resulting utility function is not available, this example still provide more flexibility and nice features. See [20] for details.

1.5 Application to Stochastic Control

Many stochastic control problems are considered in the following criteria (see [17] etc). Consider the following stochastic control problem that the law of the controlled process belong to a family of equivalent measures whose densities are given by a (forward) SDE

$$\Gamma_t^u = 1 + \int_0^t \Gamma_s^u [\beta(s, u(s))ds + \gamma(s, u(s))^* dW(s)], \quad t \in [0, T],$$

where a feasible control $u(t) \in \mathcal{U}$ is a predictable process and $\beta(t, u(t))$ and $\gamma(t, u(t))$ are uniformly bounded predictable processes. The problem is to minimise the following cost function over $\mathcal{U}$:

$$J(u) = \mathbb{E}\left[\int_0^T \Gamma_t^u F(t, u(t))dt + \Gamma_T^u F(u(T))\right],$$

where $F(\cdot, u(\cdot))$ is the running cost function and $F(u(T))$ is the terminal condition. By the Proposition 2.2 in [41], $J(u) = Y_0^u$, where (Y^u, Z^u) is the solution of the linear BSDE with standard data (f^u, ξ^u), where

$$f^u(t, Y, Z) = F(t, u(t)) + \beta(t, u(t)) \cdot Y + \gamma(t, u(t))^* \cdot Z, \quad \xi^u = F(u(T)).$$

Let $(Y^u(t), Z^u(t))$ be the unique solution of a BSDE with with standard data (f^u, ξ^u). Then for each $t \in [0, T]$, $Y^u(t)$ is the cost function of the stochastic control problem in the sense that

$$Y^u(t) = \inf_{u(\cdot) \in \mathcal{U}} \mathbb{E}\left[\int_t^T \Gamma_s^u F(s, u(s))ds + \Gamma_T^u \xi^u \middle| \mathscr{F}_t\right].$$

1.6 Nonlinear Expectation

Apart from applications to finance and stochastic control theory, backward stochastic differential equations are also used to define nonlinear expectations, so called g-expectations. A nonlinear expectation is regarded as an operator which preserves most of essential properties of the standard mathematical expectations except the linearity property. The original motivation to introduce and study nonlinear expectations goes back to the theory of decision making. In fact, it was proved that there exist contradictions between decisions made in our real life and theory of optimal decisions based on additive probabilities and the expected utility theory. Accordingly, economists and mathematicians started to seek a new notion of expectation. The g-expectation, which is defined by a BSDE with a nonlinear coefficient g, is the fundamental example of a nonlinear expectation. The g-expectation gradually becomes an important concept in probability theory and stochastic analysis because it generates g-martingales, g-supermartingales, g-submartingales etc. and nonlinear versions of some classical results, such as nonlinear Feynman-Kac formula, Doob-

Meyer decomposition, etc. Now we briefly introduce the g-expectation which is induced by BSDE (see [64] for more details).

For any given $X \in L_0^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}) = \bigcup_{0\le n<\infty} L^2(\Omega, \mathscr{F}_n, \mathbb{P}; \mathbb{R})$ and $t \geq 0$. Let $T \in [0, \infty)$ be such that X is $\mathscr{F}_T$-adapted. Then the following BSDE

$$\begin{cases} -dy_t &= g(t, y(t), z(t))dt - z(t)dW(t), \quad t \in [0, T], \\ y(T) &= X, \end{cases}$$

is well-defined:

$$(y(t), z(t))_{0\le t\le T} \in L^2_{\mathscr{F}}(0, T; \mathbb{R}) \times L^2_{\mathscr{F}}(0, T; \mathbb{R}^d).$$

Note that $y(0)$ is a deterministic number that depends on X. Then we define the g-expectation of X.

Definition 1.6.1. *For each $X \in L_0^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R})$, we say*

$$\mathcal{E}_g[X] := y(0),$$

the g-expectation of X related to g.

Since X is $\mathscr{F}_T$-adapted, then for any $T_1 > T$, X is also $\mathscr{F}_{T_1}$-measurable, i.e $X \in L^2(\Omega, \mathscr{F}_{T_1}, \mathbb{P}; \mathbb{R})$, which indicates that $\mathcal{E}_g[X]$ could be also defined by

$$\mathcal{E}_g[X] := y_1(0),$$

where $(y_1(t))_{0\le t\le T_1}$ solves the following BSDE

$$\begin{cases} -dy_1(t) &= g(t, y_1(t), z_1(t))dt - z_1(t)dW(t), \quad t \in [0, T_1], \\ y_1(T_1) &= X, \end{cases}$$

Due to assumption that $g(y, 0, \cdot) \equiv 0$, $\forall y \in \mathbb{R}$, it is easy to check that $y(0) = y_1(0)$. Indeed, since X is $\mathscr{F}_T$-measurable, we obtain

$$(y_1(t), z_1(t)) = \begin{cases} (y(t), z(t)), & 0 \le t \le T, \\ (X, 0), & T \le t \le T_1, \end{cases}$$

which means $(y(t), z(t))$ and $(y_1(t), z_1(t))$ are identical to each other in $[0, T]$. Moreover, it is natural to ask if every nonlinear expectation can be derived as the solution of a BSDE. Coquet *et al.* [14] provides a positive answer for all nonlinear expectations which satisfy a certain boundedness condition.

In financial and insurance applications, g-expectations are used to define dynamic risk measures. Static risk measures, such as Value-at-Risk (VaR), are well-understood. Nevertheless, it is still difficult to model dynamic risk measures which continuously quantify the riskiness of financial positions during a specified period of time. It is clear that financial positions should be consistently valued over time until they are liquidated. Properties of BSDEs implies that g-expectations can be a useful tool for modelling dynamic risk measures. Delong [19] illustrates some good examples of modelling dynamic risk measures.

1.7 Motivations and Contributions

In general, this thesis focuses on theory of Backward Stochastic Differential Equations with unbounded generators (coefficients), which is motivated by the following example:

The classical pricing problem is equivalent to solve a one dimension linear BSDE:

$$-dY(t) = (r(t)Y(t) + Z(t)\theta(t))dt - Z(t)dW(t), \qquad Y(T) = \xi, \tag{1.7.1}$$

where ξ is the contingent claim to price and to hedge, $r(\cdot)$ is the short rate of interest and $\theta(\cdot)$ is the risk premium vector. The classical results on the existence and uniqueness of BSDE assume that the coefficients are uniformly Lipchitz with bounded constants. However, in the above model, the assumption that $r(\cdot)$ and $\theta(\cdot)$ are uniformly bounded is rarely satisfied in a financial market. Hence we are interested in studying the wellposed-ness of solution of BSDEs with an unbounded generator.

In addition, consider some short term structure SDEs as follows:

$$\begin{cases} dr(t) &= [\alpha(t) - \beta(t)r(t)]dt + r(t)^{\lambda}\langle \nu(t), dW(t)\rangle, \quad t \geq 0, \\ r(0) &= r_0, \end{cases} \tag{1.7.2}$$

where α, $\beta : [0, \infty) \to (0, \infty)$, $\nu : [0, \infty) \to \mathbb{R}^d$ are given deterministic functions, and $\lambda \in [0, 1]$, $r_0 > 0$. It is known that for case that $d = 1$ and $\lambda = 0$, above SDE is called the (generalised) Vasicek's model; for the case that $d = 1$ and $\lambda = \frac{1}{2}$, it is called the Cox-Ingersoll-Ross (CIR) model. If the interest rate follows some short term structure SDEs as above, it might be unbounded (any strong solution to (1.7.2) could be unbounded in general). Hence it is nature to ask under what conditions, linear BSDE (1.7.1) are solvable, i.e. in what sense, the financial market is complete.

Finally, we gives a summary of contributions of our results in this thesis.

In Chapter 3, we show the unique solvability of nonlinear BSDEs that satisfies Lipschitz condition, but under weaker assumptions on the processes $c_1(\cdot)$ and $c_2(\cdot)$ as compared to El Karoui and Huang [39]. Moreover, the unique solvability is shown under novel conditions on $c_1(\cdot)$ and $c_2(\cdot)$, which in general are not comparable to those in [39], by different approaches. A comparison theorem more general than that of Peng [62], [66], is also given.

In Chapter 4, we consider a generator which is continuous in y and z, but with a weaker linear growth condition than existing ones. Note that in this case , we also relax the Lipschitz condition proposed in Chapter 3. By using the results of Chapter 3 (existence theorem and comparison theorem), and appropriately modifying the approach of [4], [50], we prove the existence of a solution pair for nonlinear BSDEs.

Chapter 5 studies the problem of solvability for linear backward stochastic differential equations with unbounded coefficients under various assumptions. New and weaker sufficient conditions for the existence of a unique solution pair are given. Comparing to the existing results (i.e. Yong [78]), we obtain the solvability of a bigger class of linear backward stochastic differential equations. We also obtain stronger integrability of certain exponential processes under various assumptions. As applications, we solve the problems of completeness in a market with a possibly unbounded coefficients and optimal investment with power utility in a market with unbounded coefficients.

Chapter 6 discusses the problem of existence and uniqueness of a solution to nonlinear SDE under certain new conditions on the coefficients $c_1(\cdot)$, $c_2(\cdot)$, which have a similarity of those of Delbaen and Tang [18]. Moreover, our method of proof is different, since it is a modification of the Picard iteration procedure rather than being based on a fixed point theorem in [18]. In addition, we consider a comparison theorem for this class of equations with same diffusion coefficients. This generalises the classical comparison result of SDEs to the case of possibly unbounded coefficients. We expect the results on this class of SDE will play an essential role on further study of fully coupled Forward-Backward Stochastic Differential Equations (FBSDEs) with unbounded coefficients.

1.8 Notations

The following is the list of the main notations used throughout this thesis.

- $|\cdot|$ is the Euclidian norm.

- $\mathbb{R}^d$ is d-dimensional real Euclidean space.

- $L^p(\Omega;\mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued random variables ζ with $\mathbb{E}[|\zeta|^p] < \infty$, for $p \in [0,\infty)$.

- $M^p(0,T;\mathbb{R}^d)$ is the space of $\mathscr{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\varphi(\cdot)$ such that $\mathbb{E}\int_0^T |\varphi(t)|^p dt < \infty$, for $p \in [0,\infty)$.

- $M^2(\Omega,\mathscr{F}_T,\mathbb{P};\mathbb{R}^d)$ is the space of all $\mathscr{F}_T$-measurable $\mathbb{R}^d$-valued random variables ζ such that $\mathbb{E}[|\zeta|^2] < \infty$.

- $M^2(0,T;\mathbb{R}^d)$ is the space of $\mathscr{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\varphi(\cdot)$ such that $\|\varphi\| \equiv \mathbb{E}\int_0^T |\varphi(t)|^2 dt < \infty$.

- $\widehat{M}_i^2(\Omega,\mathscr{F}_T,\mathbb{P};\mathbb{R}^d)$ is the space of all $\mathscr{F}_T$-measurable $\mathbb{R}^d$-valued random variables ζ such that $\mathbb{E}[p(T)|\zeta|^2] < \infty$, $i = 1,2$.

- $\widehat{M}_i^2(0,T;\mathbb{R}^d)$ is the space of $\mathscr{F}_t$-progressively measurable $\mathbb{R}^d$-valued processes $\varphi(\cdot)$ such that $\mathbb{E}\int_0^T p(t)|\varphi(t)|^2 dt < \infty$, $i = 1,2$.

- $\widehat{H}_i^2(0,T;\mathbb{R}^d)$ is the space of càdlàg $\mathscr{F}_t$-adapted $\mathbb{R}^d$-valued processes $\varphi(\cdot)$ such that $\mathbb{E}\left[\sup_{t\in[0,T]} p(t)|\varphi(t)|^2\right] < \infty$, $i = 1,2$.

Chapter 2

Some Mathematical Preliminaries

In this chapter, we describe fundamental concepts of probability theory and stochastic calculus that appears throughout this thesis. This is only a brief description, and more systematic and detailed introductions are given in some classic textbooks on stochastic calculus and stochastic control, e.g. Karatzas and Shreve [36], Mao [54] and Yong and Zhou [77] etc.

2.1 Stochastic Processes

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A *filtration* is a collection of $\{\mathscr{F}_t\}_{t\geq 0}$ of increasing sub σ-algebras of $\mathscr{F}$, i.e. $\mathscr{F}_t \subset \mathscr{F}_s \subset \cdots \subset \mathscr{F}, \forall\, 0 \leq t \leq s \leq \infty$. A probability space is said to be *complete* if for any $\mathbb{P}$-null set A (i.e. $\mathbb{P}(A) = 0$ for a set $A \in \mathscr{F}$), one has another $\mathbb{P}$-null set $B \subseteq \mathscr{F}$ whenever $B \subseteq A$. A *filtered* probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ is a probability space equipped with the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ of its σ-algebra $\mathscr{F}$, which consists of: a sample space of elementary events, a field of events, a probability defined on that field, and a filtration of increasing subfields.

A stochastic process with state space $\mathbb{R}^d$ is a collection $(X(t), t \geq 0)$ of $\mathbb{R}^d$-valued random variables. The stochastic process can be regarded as a function of two variables (t, ω) from $I \times \Omega$ to $\mathbb{R}^d$, where I is a index set. From now on, unless otherwise specified, we write a stochastic process $(X(t), t \geq 0)$ as $X(t)$.

A stochastic process is said to be *continuous* if for almost all $\omega \in \Omega$, function $X(t, \omega)$ is continuous on $t \geq 0$. A stochastic process is said to be *measurable* if it is regarded as a function of two variables (t, ω) from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}^d$ is $\mathcal{B}(\mathbb{R}_+) \times \mathscr{F}$-measurable, where $\mathcal{B}(\mathbb{R}_+)$ is a collection of all Borel subsets of $\mathbb{R}_+$ ($\mathbb{R}_+ = [0, \infty)$). It is said to be *progressively measurable* if for each $T \geq 0$, $(X(t), 0 \leq t \leq T)$ is regarded as a function of two variables (t, ω) from $[0,T] \times \Omega$ to $\mathbb{R}^d$ is $\mathcal{B}([0,T]) \times \mathscr{F}_T$-measurable, where $\mathcal{B}([0,T])$ is a collection of all Borel subsets of $[0,T]$. It is said to *adapted* to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ if for all t, $X(t)$ is $\mathscr{F}_t$-measurable. Note that if $X(t)$ is said to be $\{\mathscr{F}_t\}_{t\geq 0}$-progressively measurable , it has to be measurable and $\{\mathscr{F}_t\}_{t\geq 0}$-adapted. A process is called *predictable* if it is measurable with respect to predictable σ-field, i.e. a σ-field generated by adapted left-continuous processes.

Remark 2.1.1. *If we need the (Itô) integrals have desired properties, e.g. interchange of the expectation and integral, X is only adapted is not enough, and hence a stronger assumption, that of progressively measurable process is required.*

2.2 Convergence

We describe the definitions of some main convergence of a sequence of functions and random variables and some results on their relations. Convergence of Expectations are also discussed.

2.2.1 Convergence of Functions

Definition 2.2.1 (Pointwise Convergence). *$f_n(x)$ converge pointwisely to $f(x)$ if for any $x > 0$ and $\epsilon > 0$, there exists a natural number $N = N(\epsilon, x)$ such that for all $n > N$, $|f_n(x) - f(x)| < \epsilon$.*

It is worth noting that in above definition, N depends on x. In other words, for a given positive ϵ, a value of N that makes above statement hold for some x but might not work for some other x.

Definition 2.2.2 (Uniform Convergence). *$f_n(x)$ converge uniformly to $f(x)$ if for any $\epsilon > 0$, there exists a natural number $N = N(\epsilon)$ such that for all $n > N$ and for all x, $|f_n(x) - f(x)| < \epsilon$.*

An equivalent definition of uniform convergence is given as follows:

Theorem 2.2.1. *$f_n(x)$ converge uniformly to $f(x)$ if and only if*

$$\sup_x |f_n(x) - f(x)| \longrightarrow 0.$$

Note that the idea of uniform convergence is that one can choose N without regard to the value of x. An important result about uniform convergence is as follows:

Theorem 2.2.2 (Uniform Convergence Theorem). *If f_n is a sequence of continuous functions which converges uniformly to the function f on an interval S, then f is continuous on S as well.*

Note that pointwise convergence of continuous functions is not enough to guarantee continuity of the limit function. This theorem shows that the uniform limit of uniformly continuous functions is uniformly continuous.

Theorem 2.2.3 (Monotone Convergence Theorem). *If a sequence f_n is monotone and bounded, then it converges.*

Theorem 2.2.4 (Dominated Convergence Theorem). *Assume that $f_n : \mathbb{R} \to [-\infty, \infty]$ are (Lebesgue) measurable functions such that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists and there is an integrable $g : \mathbb{R} \to [0, \infty]$ with $|f_n(x)| \leq g(x)$ for each $x \in \mathbb{R}$. Then f is integrable as is f_n for each n, and*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n d\mu = \int_{\mathbb{R}} f d\mu.$$

Theorem 2.2.5 (Dini's Theorem). *Let K be a compact metric space. Let $f : K \to \mathbb{R}$ be a continuous function and $f_n : K \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of continuous functions. If $\{f_n\}_{n\in\mathbb{N}}$ converges pointwisely to f and if $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$ and all $n \in \mathbb{N}$, then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f*

This is one of the few situations in mathematics where pointwise convergence implies uniform convergence.

2.2.2 Convergence of Random Variables

Definition 2.2.3 (Convergence in probability). *$\{X_n\}$ converge in probability to X if for any $\epsilon > 0$, $\mathbb{P}|X_n - X| > \epsilon \to 0$ as $n \to \infty$.*

Definition 2.2.4 (Convergence almost surely). *$\{X_n\}$ converge almost surely (a.s.) to X if for any ω outside a set of zero probability, $X_n(\omega) \to X(\omega)$ as $n \to \infty$.*

Almost sure convergence implies convergence in probability. Convergence in probability implies almost sure convergence on a subsequence, i.e. if $\{X_n\}$ converge to X in probability, then there is a subsequence n_k that converges almost surely to the same limit.

2.3 Itô Calculus

In this section, we introduce the stochastic integrals, also called Itô integrals, with respect to Brownian motion and their properties. The corresponding calculus is called Itô calculus.

2.3.1 Itô Integral

Assume that $X(t)$ is a regular adapted process such that $\int_0^t X^2(s)ds < \infty$ holds with probability one and $\int_0^t \mathbb{E}[X^2(s)]ds < \infty$. Then the Itô integral $\int_0^t X(s)dW(s)$ is defined and has the following properties:

(i) *Zero mean property.*

$$\mathbb{E}\left[\int_0^t X(s)dW(s)\right] = 0.$$

(ii) *Isômetry property.*

$$\mathbb{E}\left[\int_0^t X(s)dW(s)\right]^2 = \int_0^t \mathbb{E}[X^2(s)]ds. \tag{2.3.1}$$

Remark 2.3.1. *If $\int_0^t \mathbb{E}[X^2(s)]ds = \infty$, then the Itô integral may fail to be a martingale but it is always a local martingale.*

The Itô integral $M(t) = \int_0^t X(s)dW(s), 0 \leq t \leq T$ is a random function of t. The quadratic variation of M is defined by

$$[M, M](t) = \lim_{P} \sum_{i=1}^{n} [M(t_i^n) - M(t_{i-1}^n)]^2,$$

where the limit is taken over all partitions $P : 0 = t_0 < t_1 < \cdots < t_n = t$ with mesh size $\max_{0\leq i\leq n}(t_i^n - t_{i-1}^n) \to 0$ as $n \to \infty$.

2.3.2 Itô Formula for Itô Process

Theorem 2.3.1. *Assume that $X(t)$ have a stochastic differential for $0 \leq t \leq T$:*

$$dX(t) = \mu(t)dt + \sigma(t)dW(t).$$

Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a smooth function for which the partial derivatives $f_t(t, X(t))$, $f_x(t, X(t))$ and $f_{xx}(t, X(t))$ are defined and continuous, then the *stochastic differential of the process $f(t, X(t))$ exists and is given by*

$$df(t,X(t)) = \left[f_t(t,X(t))\mu(t) + \frac{1}{2}f_{xx}(t,X(t))\sigma^2(t)\right]dt + f_x(t,X(t))\sigma(t)dW(t).$$

The formula for integration by parts (Itô product rule) in differential notation is given by

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + d[X,Y](t).$$

If

$$dX(t) = \mu_1(t)dt + \sigma_1(t)dW(t),$$

$$dY(t) = \mu_2(t)dt + \sigma_2(t)dW(t),$$

then their quadratic covariation becomes

$$d[X,Y](t) = dX(t)dY(t) = \sigma_1(t)\sigma_2(t)(dW(t))^2 = \sigma_1(t)\sigma_2(t)dt.$$

which makes the formula

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + \sigma_1(t)\sigma_2(t)dt.$$

2.3.3 Change of Measure

We illustrate that what will happen to processes if the original probability measure is replaced by an equivalent probability measure. In general, changer of measure for a process is undertook by using the Girsanov's theorem.

Let $\mathbb{P}$ and $\mathbb{Q}$ are two probability measures on the same space.

Definition 2.3.1. *$\mathbb{Q}$ is called absolutely continuous with respect to $\mathbb{P}$, denoted by $\mathbb{Q} \ll \mathbb{P}$, if $\mathbb{Q}(A) = 0$ whenever $\mathbb{P}(A) = 0$. $\mathbb{P}$ and $\mathbb{Q}$ are called equivalent if $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$.*

Theorem 2.3.2 (Radon-Nikodym Theorem)**.** *If $\mathbb{Q} \ll \mathbb{P}$, then there exists a random variable Λ such that $\Lambda \geq 0$, $\mathbb{E}_{\mathbb{P}}\Lambda = 1$, and*

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\Lambda \mathbb{1}(A)] = \int_A \Lambda d\mathbb{P},$$

for any measurable set A. $\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}}$ is $\mathbb{P}$-almost surely unique. Conversely, if there exists a random variable Λ with the above properties and $\mathbb{Q}$ is defined as above, then it is a probability measure and $\mathbb{Q} \ll \mathbb{P}$.

The Λ is called the Radon-Nikodym derivative. It follows from above theorem that if $\mathbb{Q} \ll \mathbb{P}$, then the relation between expectations under $\mathbb{P}$ and $\mathbb{Q}$ is given by

$$\mathbb{E}_Q[X] = \mathbb{E}_P[\Lambda X],$$

for any random variable X integrable with respect to $\mathbb{Q}$.

So if $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, i.e. they have the same null sets, then there exists a random variable Λ such that the probabilities under $\mathbb{Q}$ are given by $\mathbb{Q}(A) = \int_A \Lambda d\mathbb{P}$. The Girsanov's theorem gives the form of Λ:

Theorem 2.3.3 (Girsanov's Theorem)**.** *Let $W(t), t \in [0, T]$ be a Brownian motion under probability $\mathbb{P}$ and $H(t), t \in [0, T]$ be an adapted process such that an equivalent measure $\mathbb{Q}$ is given by*

$$\Lambda(t) = \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{1}{2}\int_0^t H^2(s)ds - \int_0^t H(s)dW(s)},$$

and

$$\mathbb{E}\left[\int_0^T H^2(s)\Lambda^2(s)ds\right] < \infty.$$

Then the process

$$\widetilde{W}(t) = W(t) + \int_0^t H(s)ds,$$

is a Brownian motion, and $\mathbb{E}[\Lambda(T)] = 1$ under the probability measure $\mathbb{Q}$.

2.4 Martingales

Martingales play a significant role in modern theory of stochastic processes and stochastic calculus. In this thesis, we consider the martingales are constructed from a Brownian motion.

Definition 2.4.1 (Martingales, Supermartingales, Submartingales)**.** *A stochastic process $X(t)$ adapted to a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is a supermartingale (resp. submartingale) if for any t, it is integrable, $\mathbb{E}[|X(t)|] < \infty$, and for any $s < t$,*

$$\mathbb{E}[X(t)|\mathscr{F}_s] \leq X(s), \quad (resp.\, \mathbb{E}[X(t)|\mathscr{F}_s] \geq X(s)),\ a.s..$$

If $\mathbb{E}[X(t)|\mathscr{F}_s] = X(s)$, a.s., the process $X(t)$ is said to be a martingale.

Square integrable martingales play a important role in theory of integration.

Definition 2.4.2. *A random variable X is square integrable if $\mathbb{E}[X^2] < \infty$. A process $X(t)$ on a time interval $[0, T]$ (where T can be infinite) is square integrable if $\sup_{t\in[0,T]} \mathbb{E}[X^2(t)] < \infty$.*

In order to introduce the concepts of localization and local martingale, we recall the definition of *stopping times*:

Definition 2.4.3 (Stopping Times). *A non-negative random variable τ (which is allowed to take values at ∞) is said to be a stopping times (w.r.t a filtration $\{\mathscr{F}_t\}_{t\geq 0}$) if for any t, the event $\{\tau \leq t\} \in \mathscr{F}_t$ holds.*

In other words, a random time τ is called a stopping time if for any t it is possible to determine if τ has occurred or not by observing the process $X(s)$, $0 \leq s \leq t$.

2.4.1 Local Martingales

In general, stochastic integrals with respect to martingales are local martingales, not true martingales, which makes it necessary to introduce the concept of local martingales.

Definition 2.4.4. *An adapted process $M(t)$ is said to be a local martingale if there exists an increasing sequence of stopping times τ_n and for each n the stopped process $M(t \wedge \tau_n)$ is a martingale.*

Any martingale is a local martingale. However, since its expectation can be distorted by large values of small probability, in general, a local martingale is not a martingale. The following corollary shows that under certain condition, a local martingale becomes a true martingale.

Corollary 2.4.1. *Let $M(t)$, $0 \leq t < \infty$ be a local martingale such that for all t, $\mathbb{E}[\sup_{s\leq t} |M(s)|] < \infty$. Then it is a martingale.*

Positive local martingales are studied especially in application to finance.

Theorem 2.4.1. *A non-negative local martingale $M(t)$, $0 \leq t \leq T$, is a supermartingale, i.e. $\mathbb{E}[M(t)] < \infty$ and for any $s < t$, $\mathbb{E}[M(t)|\mathscr{F}_s] \leq M(s)$.*

Now we state the representation of martingales by stochastic integrals of predictable processes. Let $M(t)$, $t \in [0, T]$ be a martingale adapted to filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$, and $H(t)$ be a predictable process satisfying $\int_0^T H^2(s)d\langle M, M\rangle(s) < \infty$ with probability one. Then $\int_0^t H(s)dM(s)$ is a local martingale.

Theorem 2.4.2 (Brownian Martingale Representation)**.** *Let $X(t)$, $t \in [0,T]$ be a local martingale adapted to filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$. Then there exists a predicable process $H(t)$ such that $\int_0^T H^2(s)ds < \infty$ with probability one and*

$$X(t) = X(0) + \int_0^t H(s)dB(s).$$

Moreover, if Y is an integrable $\mathscr{F}_T$-measurable random variable and $\mathbb{E}|Y| < \infty$, then

$$Y = \mathbb{E}[Y] + \int_0^T H(t)dB(t).$$

The martingale representation theorem shows that if the filtration is generated by a Brownian motion, then every martingale with respect to this filtration is an initial condition (of the martingale) plus an Itô integral with respect to the Brownian motion. Hence Brownian motion is the only source of uncertainty to be removed by hedging in the model.

2.4.2 Semimartingales

Semimartingales are considered as the most general processes for which stochastic calculus is developed. It is a process including a sum of a local martingale and finite variation process, i.e.

$$\lim_{P_n} \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)| < \infty,$$

where $P_n = \max_{0\leq i\leq n}(t_i^n - t_{i-1}^n)$. More precisely,

Definition 2.4.5 (Semimartingales)**.** *A* càdlàg *(i.e. right continuous with left limits) adapted process $S(t)$ is a semimartingale if it can expressed as a sum of two processes: a local martingale $M(t)$ and a process of finite variation $A(t)$ with $M(0) = A(0) = 0$, and*

$$S(t) = S(0) + M(t) + A(t).$$

Definition 2.4.6 (Local Time)**.** *Assume that $(W(t), t \geq 0)$ is a diffusion process (e.g. a Brownian motion). A local time of $W(t)$ at point a is a stochastic process*

$$L^a(t) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{|W(s)-a|<\varepsilon\}} ds,$$

Note that local time can be regarded as a measure of time $W(t)$ spent at point a up to time t.

With concepts of semimartingale and local time, we introduce an application of Itô's formula to the function sign(x), which is called the *Tanaka's formula.*

Theorem 2.4.3 (Tanaka's Formula)**.** *Assume that $X(t)$ is a continuous semimartingale. Then for any $a \in \mathbb{R}$, there exists a continuous non-decreasing adapted process $L^a(t)$, called the local time at a of X, such that*

$$|X(t) - a| = |X(0) - a| + \int_0^t sign((X(s) - a)dX(s) + L^a(t).$$

As a function in a, $L^a(t)$ is right continuous with left limits. For any fixed a as a function in t, $L^a(t)$ increases only when $X(t) = a$, i.e.

$$L^a(t) = \int_0^t \mathbb{1}(X(s) = a)dL^a(s).$$

It is common to associate a random measure $dL^a(t)$ on $\mathbb{R}_+$ with the increasing $L^a(t)$. Roughly speaking, it measures the time spent at point a by the semimartingale X:

Proposition 2.4.1. *The measure $dL^a(t)$ is a.s. carried by the set $\{t : X(t) = a\}$, i.e. $\int_0^t |X(s) - a|dL^a(s) = 0$, a.s..*

2.4.3 Stochastic Exponential

The stochastic exponential, which also known as the semimartingale, is a stochastic version of exponential function. For a semimartingale M, its stochastic exponential $\mathcal{E}(M)(t) = U(t)$ is the unique solution to

$$U(t) = 1 + \int_0^t U(s-)dM(s),$$

where $U(s-) \equiv \lim_{s\uparrow t} U(s)$ denotes the left continuous process.

Theorem 2.4.4. *Let M be a continuous semimartingale. Then its stochastic exponential is given by*

$$\mathcal{E}(M)(t) = e^{M(t)-M(0)-\frac{1}{2}[M,M](t)}.$$

$M(t)$ is a stochastic integral with respect to $M(t)$. $\mathcal{E}(M)$ is a local martingale because stochastic integrals with respect to martingales or local martingales are local martingales. In applications, it is important to have conditions for stochastic exponential $\mathcal{E}(M)$ to be a true martingale.

Theorem 2.4.5 (Kazamaki's Condition)**.** *Let M be a continuous local martingale with $M(0) = 0$. If $\mathbb{E}\left[e^{\frac{1}{2}M(t)}\right] < \infty$, then $\mathcal{E}(M)$ is a martingale on $[0, T]$.*

Moreover, we have the following stronger condition for $\mathcal{E}(M)$ to be a true martingale.

Theorem 2.4.6 (Novikov's Condition)**.** *Let M be a continuous local martingale with $M(0) = 0$. If $\mathbb{E}\left[e^{\frac{1}{2}[M,M](t)}\right] < \infty$, then $\mathcal{E}(M)$ is a martingale with mean one on $[0, T]$.*

2.4.4 Martingales Inequalities

Theorem 2.4.7 (Doob's martingale Inequalities)**.** *Let $X(t)$ be an $\mathbb{R}^d$-valued martingale, and $[s, T]$ be a bounded interval in $\mathbb{R}_+$. If $p > 1$ and $X(t) \in L^p(\Omega; \mathbb{R}^d)$, then*

$$\mathbb{E}\left[\sup_{s \le t \le T} |X(t)|^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X(T)|^p].$$

Theorem 2.4.8 (Burkholder-Gundy Inequality)**.** *There are positive constants c_p and C_p depending only on $p > 0$ such that for any local martingale $M(t)$, null at zero,*

$$c_p \mathbb{E}\left([M, M](T)^{p/2}\right) \le \mathbb{E}\left[\sup_{t \le T} |M(t)|^p\right] \le C_p \mathbb{E}\left([M, M](T)^{p/2}\right),$$

for $1 \le p < \infty$. Moreover, if $M(t)$ is continuous, the it holds for $0 < p < 1$ as well.

In addition, if $M(t) = \int_0^t \sigma(s)dW(s)$, where $W(t)$ be a standard Brownian motion, then it is a Itô integral and its quadratic variation is given by $\int_0^t \sigma^2(s)ds$. In this case for any $p > 0$ and any stopping time τ, the Burkholder-Gundy inequality gives:

$$\frac{1}{K_p}\mathbb{E}\left[\int_0^\tau |\sigma(s)|^2 ds\right]^p \le \mathbb{E}\left[\sup_{0 \le t \le \tau}\left|\int_0^t \sigma(s)dW(s)\right|^{2p}\right] \le K_p\, \mathbb{E}\left[\int_0^\tau |\sigma(s)|^2 ds\right]^p,$$

where K_p is a positive constant.

2.5 Other Results

Theorem 2.5.1 (Gronwall's Inequality)**.** *Assume that $g(t)$ and $h(t)$ be regular non-negative functions on $[0, T]$, then for any regular $f(t) \ge 0$ satisfying the inequality* *for all* $t \in [0,T]$,

$$f(t) \leq g(t) + \int_0^t h(s)f(s)ds,$$

we have

$$f(t) \leq g(t) + \int_0^t h(s)g(s)e^{\int_s^t h(u)du}ds.$$

In particular, if g(t) is a non-decreasing function, the integral above simplifies to

$$f(t) \leq g(t)e^{\int_0^t h(s)ds}.$$

Theorem 2.5.2 (Fubini's Theorem)**.** *Assume that* $X(t)$ *is a stochastic process with regular sample paths (i.e. for all* ω *at any point* t*,* $X(t)$ *has left and right limits). Then*

$$\mathbb{E}\left[\int_0^t |X(s)|ds\right] = \int_0^t \mathbb{E}[|X(s)|]ds.$$

Moreover, if the quantity is finite, then

$$\mathbb{E}\left[\int_0^t X(s)ds\right] = \int_0^t \mathbb{E}[X(s)]ds.$$

Fubini's Theorem permits to interchange integrals (summations) and expectations.

The following inequalities are frequently used throughout later analysis.

(i) *Hölder's inequality.* If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $X \in L^p(\Omega; \mathbb{R}^d)$, $Y \in L^q(\Omega; \mathbb{R}^d)$, then

$$|\mathbb{E}(XY)| \leq (\mathbb{E}|X|^p)^{\frac{1}{p}} \cdot (\mathbb{E}|Y|^q)^{\frac{1}{q}}.$$

An integral version of this is

$$\left|\int_0^t f(s)g(s)ds\right| \leq \left[\int_0^t f^p(s)ds\right]^{1/p} \cdot \left[\int_0^t g^q(s)ds\right]^{1/q}.$$

In addition, a simple generalization of Hölder's inequality is

$$(\mathbb{E}|X|^p)^{\frac{1}{p}} \leq (\mathbb{E}|X|^q)^{\frac{1}{q}},$$

if $0 < p < q < \infty$ and $X \in L^p(\Omega; \mathbb{R}^d)$.

(ii) *Markov's inequality.* Let X be a non-negative random variable and assume that $E(X)$ exists. For any $t > 0$,

$$\mathbb{P}(\{\omega : X(\omega) \geq t\}) \leq \frac{\mathbb{E}(X)}{t}.$$

(iii) *Cauchy-Schwartz inequality.* If $X \in L^2(\Omega; \mathbb{R}^d)$ and $Y \in L^2(\Omega; \mathbb{R}^d)$ have finite variances, then

$$\mathbb{E}|XY| \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}.$$

An integral version of this is

$$\left| \int_0^t f(s)g(s)ds \right|^2 \leq \left[\int_0^t f^2(s)ds \right] \cdot \left[\int_0^t g^2(s)ds \right].$$

(iv) *Minkowskis inequality.* If $f, g : \mathbb{R}^n \to \mathbb{R}$ are Lebesgue measurable. Then for $p < \infty$,

$$\left[\int_0^t |f(s) + g(s)|^p ds \right]^{\frac{1}{p}} \leq \left[\int_0^t |f(s)|^p ds \right]^{\frac{1}{p}} + \left[\int_0^t |g(s)|^p ds \right]^{\frac{1}{p}}.$$

Chapter 3

Backward Stochastic Differential Equations with an Unbounded Generator

3.1 Abstract

We consider a class of backward stochastic differential equations with a possibly unbounded generator. Under a Lipschitz-type condition, we give sufficient conditions for the existence of a unique solution pair, which are weaker than the existing ones. We also give a comparison theorem as a generalisation of Peng's result. This work is based on a submitted paper [24].

3.2 Introduction

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t, t \geq 0), \mathbb{P})$ be a given complete filtered probability space on which a k-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $(\mathscr{F}_t, t \geq 0)$ is the augmentation of $\sigma\{W(s) : 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathscr{F}$. Consider the backward stochastic differential equation (BSDE):

$$y(t) = \xi + \int_t^T f(s, y(s), z(s))ds - \int_t^T z(s)dW(s), \quad t \in [0, T], \tag{3.2.1}$$

where ξ is a given $\mathscr{F}_T$-measurable $\mathbb{R}^d$-valued random variable, and the generator $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^d$ is a progressively measurable function.

Linear equations of the type (3.2.1) were introduced by Bismut [9] in the context of stochastic linear quadratic control. The nonlinear equations (3.2.1) were introduced by Pardoux and Peng [61]. Under the global Lipschitz condition on f, i.e. under the assumption that there exists a real constant $c > 0$ such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq c(|y_1 - y_2| + |z_1 - z_2|), \tag{3.2.2}$$

for all $y_1, y_2 \in \mathbb{R}^d$, $z_1, z_2 \in \mathbb{R}^{d\times k}$, (t, ω) *a.e.*, they prove the existence of a unique solution pair $(y(\cdot), z(\cdot))$. BSDEs have been studied extensively since then, and have found wide applicability in areas such as mathematical finance, stochastic control, and stochastic controllability; see, for example, [10], [23], [41], [53], [55], [77], [63], [73], and the references therein. The global Lipschitz condition (3.2.2) has been weakened to local Lipschitz condition in [3], and to non-Lipschitz condition of a particular type in [54], [71].

The BSDEs with a possibly *unbounded* generator f are particulary important in mathematical finance. Several important interest rate models are solutions to stochastic differential equations. Such solutions are unbounded in general (see, for example, [7], [16], [79]). The problem of *market completeness* in that case gives rise to BSDEs with unbounded coefficients (see [78] for details). This has motivated [39] (see also [18]) to weaken the Lipschitz condition (3.2.2) to

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq c_1(t)|y_1 - y_2| + c_2(t)|z_1 - z_2|, \tag{3.2.3}$$

for all $y_1, y_2 \in \mathbb{R}^d$, $z_1, z_2 \in \mathbb{R}^{d\times k}$, (t, ω) *a.e.*, for some non-negative processes $c_1(\cdot)$ and $c_2(\cdot)$. Here the processes $c_1(\cdot)$ and $c_2(\cdot)$ can be *unbounded*. In [39], under certain conditions on the processes $c_1(\cdot)$ and $c_2(\cdot)$, the solvability of (3.2.1) is shown. The linear BSDEs with possibly unbounded coefficients are considered in [78], where only scalar equations are considered by exploiting their explicit solvability.

In this chapter we also show the unique solvability of (3.2.1) that satisfies Lipschitz condition (3.2.3), but under weaker assumptions on the processes $c_1(\cdot)$ and $c_2(\cdot)$ as compared to [39]. Moreover, the unique solvability of (3.2.1) is shown under novel conditions on $c_1(\cdot)$ and $c_2(\cdot)$, which in general are not comparable to those in [39]. A comparison theorem more general than that of Peng [62], [66], is also given.

3.3 Notations and Assumptions

The following is the list of the additional notations used in this chapter.

• $c_1(\cdot)$, $c_2(\cdot)$, $\gamma(\cdot)$, $\overline{\gamma}(\cdot)$, are given positive $\mathbb{R}$-valued progressively measurable processes.

• $1 < \beta_1 \in \mathbb{R}$, $1 < \beta_2 \in \mathbb{R}$, are given constants.

• $4 < \overline{\beta_1} \in \mathbb{R}$, $1 < 90\overline{\beta_1}^2/(\overline{\beta_1}^2 - 16) < \overline{\beta_2} \in \mathbb{R}$, are given constants.

• $\alpha_1(t) \equiv \gamma(t) + \beta_1 c_1^2(t) + \beta_2 c_2^2(t)$, $\alpha_2(t) \equiv \overline{\gamma(t)} + \overline{\beta_1} c_1(t) + \overline{\beta_2} c_2^2(t)$, are assumed positive.

• $p_1(t) \equiv \exp\left[\int_0^t \alpha_1(s)ds\right]$, $p_2(t) \equiv \exp\left[\int_0^t \alpha_2(s)ds\right]$.

We say that the progressively measurable function f and the random variable ξ, or the pair (f, ξ), satisfies *conditions A1* (resp. *conditions A2*) if:

(i) $\xi \in \widehat{M_1^2}(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}^d)$ (resp. $\xi \in \widehat{M_2^2}(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}^d)$);

(ii) $|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq c_1(t)|y_1 - y_2| + c_2(t)|z_1 - z_2|$, for all $y_1, y_2 \in \mathbb{R}^d$, $z_1, z_2 \in \mathbb{R}^{d\times k}$, (t, ω) *a.e.*;

(iii) $\left[f(\cdot, 0, 0)\alpha_1(\cdot)^{-\frac{1}{2}}\right] \in \widehat{M_1^2}(0, T; \mathbb{R}^d)$ (resp. $\left[f(\cdot, 0, 0)\alpha_2(\cdot)^{-\frac{1}{2}}\right] \in \widehat{M_2^2}(0, T; \mathbb{R}^d)$).

The sufficient conditions for the solvability of (3.2.1), as given in [39], are similar to our conditions A2. Indeed, if we choose $\overline{\gamma}(t) = 0$, $\overline{\beta}_1 = \overline{\beta}_2 \equiv \beta$, where β is *large enough*, then conditions A2 are those of [39]. Clearly, due to the process $\overline{\gamma}(t)$ our conditions A2 are more general then those of [39]. The importance of this process is that assumption (iii) above can be suitable weakened by choosing large values for this process, which is not an option in [39]. Moreover, even if we take $\overline{\gamma}(t) = 0$, our assumption (i) is weaker than that of [39]. Indeed, the parameter β of [39] should be bigger than 446.05 (in [39] it is only claimed that this coefficient should be *large enough*, but a straightforward calculation included in our Appendix, gives this numerical lower bound). This is clearly not the case in conditions A2 where the coefficient $\overline{\beta}_1$ is only required to be greater than 4.

The conditions A1 are new. In general, these are not comparable with conditions A2. However, in certain special cases we can compare them. For example, if $c_1(t) = 0$, $1 < \beta_2 < \overline{\beta}_2$, $\gamma(t) = \overline{\gamma}(t)$, then $\widehat{M}_1^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}^d) \subset \widehat{M}_2^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}^d)$, and thus the above assumption (i) on the random variable ξ is weaker in the case of conditions A1. Similarly, if $c_2(t) = 0$, $\gamma(t) = \overline{\gamma}(t)$, $\overline{\beta}_1 = 2\beta_1$, then the above assumption (i) on the random variable ξ is weaker in the case of conditions A2.

3.4 Solvability

In this section we give sufficient conditions for the existence and uniqueness of a solution pair for (3.2.1). Our method of proof is different from [39] being based on Picard iterations, and similarly to [61], we begin with a simpler form of (3.2.1) and progress towards the general case. The proofs of the results under conditions A1 and A2 are different and are thus given separately in most cases, but there are also similarities between them.

Lemma 3.4.1. *Let $\phi(\cdot) \in \widehat{H}_1^2(0,T;\mathbb{R}^d)$, $\psi(\cdot) \in \widehat{M}_1^2(0,T;\mathbb{R}^{d\times k})$ be given, and assume that $\sqrt{\alpha_1(\cdot)}\phi(\cdot) \in \widehat{M}_1^2(0,T;\mathbb{R}^d)$. If the pair (f,ξ) satisfies the conditions A1, then:*
(i) there exists a unique solution pair $(y(\cdot), z(\cdot)) \in \widehat{H}_1^2(0,T;\mathbb{R}^d) \times \widehat{M}_1^2(0,T;\mathbb{R}^{d\times k})$ of equation

$$y(t) = \xi + \int_t^T f(s,\phi(s),\psi(s))ds - \int_t^T z(s)dW(s), \quad t \in [0,T], \tag{3.4.1}$$

and $\sqrt{\alpha_1(\cdot)}y(\cdot) \in \widehat{M}_1^2(0,T;\mathbb{R}^d)$.

(ii) if $y^+(t) \equiv \mathbb{1}_{[y(t)>0]}y(t)$, the processes

$$\int_t^T p_1(s)y(s)z(s)dW(s) \quad \text{and} \quad \int_t^T p_1(s)y^+(s)z(s)dW(s),$$

are martingales.

Proof. (i) By making use of the Cauchy-Schwartz inequality, we first show that

$\int_0^T f(s,\phi(s),\psi(s))ds$ belongs to $M^2(\Omega,\mathscr{F}_T,\mathbb{P};\mathbb{R}^d)$:

$$\mathbb{E}\bigg|\int_0^T f(s,\phi(s),\psi(s))ds\bigg|^2 = \mathbb{E}\bigg|\int_0^T \sqrt{p_1^{-1}(s)\alpha_1(s)}\frac{\sqrt{p_1(s)}f(s,\phi(s),\psi(s))}{\sqrt{\alpha_1(s)}}ds\bigg|^2$$

$$= \mathbb{E}\left\{\left[\int_0^T p_1^{-1}(s)\alpha_1(s)ds\right]\left[\int_0^T \frac{p_1(s)|f(s,\phi(s),\psi(s))|^2}{\alpha_1(s)}ds\right]\right\}$$

$$\leq \mathbb{E}\int_0^T \frac{p_1(s)|f(s,\phi(s),\psi(s))|^2}{\alpha_1(s)}ds$$

$$= \mathbb{E}\int_0^T \frac{p_1(s)}{\alpha_1(s)}|f(s,\phi(s),\psi(s) - f(s,0,0) + f(s,0,0)|^2 ds$$

$$\leq \mathbb{E}\int_0^T \frac{p_1(s)}{\alpha_1(s)}[|f(s,\phi(s),\psi(s) - f(s,0,0)| + |f(s,0,0)|]^2 ds$$

$$\leq \mathbb{E}\int_0^T \frac{p_1(s)}{\alpha_1(s)}[c_1(s)|\phi(s)| + c_2(s)|\psi(s)| + |f(s,0,0)|]^2 ds$$

$$\leq \mathbb{E}\int_0^T \frac{p_1(s)}{\alpha_1(s)}[3c_1^2(s)|\phi(s)|^2 + 3c_2^2(s)|\psi(s)|^2 + 3|f(s,0,0)|^2]ds$$

$$= \mathbb{E}\int_0^T \frac{3p_1(s)}{\beta_1}\frac{\beta_1 c_1^2(s)}{\gamma(s)+\beta_1 c_1^2(s)+\beta_2 c_2^2(s)}|\phi(s)|^2 ds$$

$$+ \frac{3p_1(s)}{\beta_2}\frac{\beta_2 c_2^2(s)}{\gamma(s)+\beta_1 c_1^2(s)+\beta_2 c_2^2(s)}|\psi(s)|^2 ds + 3\,\mathbb{E}\int_0^T \frac{p_1(s)|f(s,0,0)|^2}{\alpha_1(s)}ds$$

$$\leq \frac{3}{\beta_1}\mathbb{E}\int_0^T p_1(s)|\phi(s)|^2 ds + \frac{3}{\beta_2}\mathbb{E}\int_0^T p_1(s)|\psi(s)|^2 ds + 3\mathbb{E}\int_0^T \frac{p_1(s)|f(s,0,0)|^2}{\alpha_1(s)}ds$$

$$<\infty.$$

Since $\xi \in \widehat{M}_1^2(\Omega,\mathscr{F}_T,\mathbb{P};\mathbb{R}^d)$ implies that $\xi \in M^2(\Omega,\mathscr{F}_T,\mathbb{P};\mathbb{R}^d)$, it follows from

Lemma 2.1 of [61] that (3.4.2) has a unique solution pair $(y(\cdot), z(\cdot)) \in M^2(0,T;\mathbb{R}^d) \times M^2(0,T;\mathbb{R}^{d\times k})$. Moreover, since we proved that

$$\mathbb{E}\int_0^T \frac{p_1(s)|f(s,\phi(s),\psi(s))|^2}{\alpha_1(s)}ds,$$

is finite, it follows from Lemma 6.2 [1] of [39] that in fact $(y(\cdot), z(\cdot)) \in \widehat{H}_1^2(0,T;\mathbb{R}^d) \times \widehat{M}_1^2(0,T;\mathbb{R}^{d\times k})$ and $[\sqrt{\alpha_1(\cdot)}y(\cdot)] \in \widehat{M}_1^2(0,T;\mathbb{R}^d)$.

(ii) The proof follows closely that in [12] (pp. 307), and since it is short, we include it here for completeness. From the Burkholder-Davis-Gundy inequality (see, for example, Theorem 2.4.8), there exists a constant K such that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t p_1(s)y(s)z(s)dW(s)\right|\right] \\
\leq &K\,\mathbb{E}\left[\int_0^T |\sqrt{p_1(s)}y(s)|^2|\sqrt{p_1(s)}z(s)|^2 ds\right]^{\frac{1}{2}} \\
\leq &K\,\mathbb{E}\left[\sup_{t\in[0,T]}|\sqrt{p_1(t)}y(t)|^2\int_0^T \sqrt{p_1(s)}z(s)|^2 ds\right]^{\frac{1}{2}} \\
\leq &\frac{K}{2}\,\mathbb{E}\left[\sup_{t\in[0,T]}|\sqrt{p_1(t)}y(t)|^2 + \int_0^T |\sqrt{p_1(s)}z(s)|^2 ds\right] < \infty,
\end{aligned}$$

where the last step follows from the fact that $y(\cdot) \in \widehat{H}_1^2(0,T;\mathbb{R}^d)$ and $z(\cdot) \in \widehat{M}_1^2(0,T;\mathbb{R}^{d\times k})$, proved in part (i). The conclusion then follows from Corollary 2.4.1. Since $\sup_{t\in[0,T]}|\sqrt{p_1(t)}y^+(t)|^2 \leq \sup_{t\in[0,T]}|\sqrt{p_1(t)}y(t)|^2$, the conclusion follows even for $\int_0^t p_1(s)y^+(s)z(s)dW(s)$.

□

Lemma 3.4.2. *Let $\phi(\cdot) \in \widehat{H}_2^2(0,T;\mathbb{R}^d)$, $\psi(\cdot) \in \widehat{M}_2^2(0,T;\mathbb{R}^{d\times k})$ be given, and assume that $\sqrt{\alpha_2(\cdot)}\phi(\cdot) \in \widehat{M}_2^2(0,T;\mathbb{R}^d)$. If the pair (f,ξ) satisfies the conditions A2, then: (i) there exists a unique solution pair $(y(\cdot), z(\cdot)) \in \widehat{H}_2^2(0,T;\mathbb{R}^d) \times \widehat{M}_2^2(0,T;\mathbb{R}^{d\times k})$ of equation*

$$y(t) = \xi + \int_t^T f(s,\phi(s),\psi(s))ds - \int_t^T z(s)dW(s), \quad t\in[0,T], \tag{3.4.2}$$

and $\sqrt{\alpha_2(\cdot)}y(\cdot) \in \widehat{M}_2^2(0,T;\mathbb{R}^d)$.

[1] Note that the results in Lemma 6.2 of [39] is valid for any $\alpha(t)$ (in the sense of [39]).

(ii) if $y^+(t) \equiv \mathbb{1}_{[y(t)>0]} y(t)$, the processes

$$\int_t^T p_2(s) y(s) z(s) dW(s) \quad \text{and} \quad \int_t^T p_2(s) y^+(s) z(s) dW(s),$$

are martingales.

Proof. The proof of part (ii) is the same as the proof of part (ii) of the previous lemma. We thus focus on part (i). We have

$$\begin{aligned}
&\mathbb{E}\left|\int_0^T f(s,\phi(s),\psi(s))ds\right|^2 \\
\leq &\mathbb{E}\int_0^T \frac{p_2(s)}{\alpha_2(s)}[3c_1^2(s)|\phi(s)|^2 + 3c_2^2(s)|\psi(s)|^2 + 3|f(s,0,0)|^2]ds \\
\leq &\mathbb{E}\int_0^T \frac{3p_2(s)}{\overline{\beta_1}^2} \frac{\overline{\beta_1}c_1(s)}{\overline{\gamma(s)} + \overline{\beta_1}c_1(s) + \overline{\beta_2}c_2^2(s)}(\overline{\gamma(s)} + \overline{\beta_1}c_1(s) + \overline{\beta_2}c_2^2(s))|\phi(s)|^2 ds \\
&+ \mathbb{E}\int_0^T \frac{3p_2(s)}{\overline{\beta_2}} \frac{\overline{\beta_2}c_2^2(s)}{\overline{\gamma(s)} + \overline{\beta_1}c_1(s) + \overline{\beta_2}c_2^2(s)}|\psi(s)|^2 ds + 3\mathbb{E}\int_0^T \frac{p_2(s)|f(s,0,0)|^2}{\alpha_2(s)}ds \\
\leq &\frac{3}{\overline{\beta_1}^2}\mathbb{E}\int_0^T p_2(s)\alpha_2(s)|\phi(s)|^2 ds + \frac{3}{\overline{\beta_2}}\mathbb{E}\int_0^T p_2(s)|\psi(s)|^2 ds \\
&+ 3\mathbb{E}\int_0^T \frac{p_2(s)|f(s,0,0)|^2}{\alpha_2(s)}ds < \infty.
\end{aligned}$$

The rest of the proof is the same as in the proof of part (i) of the previous lemma. □

Lemma 3.4.3. *(i) Let $\phi(\cdot) \in \widehat{H}_1^2(0,T;\mathbb{R}^d)$ be given. If the pair (f,ξ) satisfies conditions A1, then there exists a unique solution pair $(y(\cdot),z(\cdot)) \in \widehat{H}_1^2(0,T;\mathbb{R}^d) \times \widehat{M}_1^2(0,T;\mathbb{R}^{d\times k})$ of equation*

$$y(t) = \xi + \int_t^T f(s,\phi(s),z(s))ds - \int_t^T z(s)dW(s), \quad t \in [0,T], \tag{3.4.3}$$

and $\sqrt{\alpha_1(\cdot)}y(\cdot) \in \widehat{M}_1^2(0,T;\mathbb{R}^d)$.

(ii) Let $\phi(\cdot) \in \widehat{H}_2^2(0,T;\mathbb{R}^d)$ *be given. If the pair* (f,ξ) *satisfies conditions A2, then there exists a unique solution pair* $(y(\cdot),z(\cdot)) \in \widehat{H}_2^2(0,T;\mathbb{R}^d) \times \widehat{M}_2^2(0,T;\mathbb{R}^{d\times k})$ *of equation*

$$y(t) = \xi + \int_t^T f(s,\phi(s),z(s))ds - \int_t^T z(s)dW(s), \quad t \in [0,T],$$

and $\sqrt{\alpha_2(\cdot)}y(\cdot) \in \widehat{M}_2^2(0,T;\mathbb{R}^d)$.

Proof. (i) (*Uniqueness*) Let $(y_1(\cdot),z_1(\cdot))$ and $(y_2(\cdot),z_2(\cdot))$ be two solution pairs of (3.4.3) with the claimed properties. Then

$$\begin{aligned}
&- d\,p_1(t)|y_1(t)-y_2(t)|^2 \\
=&\{-\alpha_1(t)p_1(t)\,|y_1(t)-y_2(t)|^2 \\
&+ 2p_1(t)(y_1(t)-y_2(t))'\,[f(t,\phi(t),z_1(t)) - f(t,\phi(t),z_2(t))] \\
&- p_1(t)|z_1(t)-z_2(t)|^2\}dt - 2p_1(t)(y_1(t)-y_2(t))'(z_1(t)-z_2(t))dW(t) \\
\le&[-\alpha_1(t)p_1(t)\,|y_1(t)-y_2(t)|^2 - p_1(t)|z_1(t)-z_2(t)|^2]dt \\
&- 2p_1(t)(y_1(t)-y_2(t))'(z_1(t)-z_2(t))dW(t) \\
&+ 2p_1(t)|y_1(t)-y_2(t)||f(t,\phi(t),z_1(t)) - f(t,\phi(t),z_2(t))|dt \\
\le&[-\alpha_1(t)p_1(t)\,|y_1(t)-y_2(t)|^2 - p_1(t)|z_1(t)-z_2(t)|^2]dt \\
&- 2p_1(t)(y_1(t)-y_2(t))'(z_1(t)-z_2(t))dW(t)
\end{aligned}$$

$$
\begin{aligned}
&+ 2p_1(t)c_2(t)|y_1(t) - y_2(t)||z_1(t) - z_2(t)|dt \\
\leq &[-\alpha_1(t)p_1(t)\,|y_1(t) - y_2(t)|^2 - p_1(t)|z_1(t) - z_2(t)|^2]dt \\
&- 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
&+ \beta_2 c_2^2(t)p_1(t)|y_1(t) - y_2(t)|^2 dt + \beta_2^{-1} p_1(t)|z_1(t) - z_2(t)|^2 dt \\
\leq &- 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t),
\end{aligned}
$$

which in integral form becomes

$$
p_1(t)|y_1(t) - y_2(t)|^2 \leq \int_t^T -2p_1(s)(y_1(s) - y_2(s))'(z_1(s) - z_2(s))dW(s). \quad (3.4.4)
$$

The stochastic integral in (3.4.4) is a local martingale that is clearly lower bounded by zero, and is thus a supermartingale (see, for example, Theorem 2.4.1). Taking the expectation of both sides of (3.4.4) results in

$$
\mathbb{E}\left[p_1(t)|y_1(t) - y_2(t)|^2\right] \leq -\,\mathbb{E}\left[\int_t^T 2p_1(s)(y_1(s) - y_2(s))'(z_1(s) - z_2(s))dW(s)\right] \leq 0.
$$

Since $p_1(t) > 0$, it follows that $y_1(t) = y_2(t)$, $\forall\ t \in [0,T]$, a.s., which proves the uniqueness of $y(\cdot)$. Due to this fact, the integral form of (3.4.4) becomes

$$
0 = \int_t^T p_1(s)|z_1(s) - z_2(s)|^2 ds,
$$

which implies that $z_1(t) = z_2(t)$ for *a.e.* $t \in [0,T]$, and thus proves the uniqueness of $z(\cdot)$.

(*Existence*) Let $z_0(t) \equiv 0$, $\forall t \in [0,T]$, and for $n \geq 1$ consider the following sequence of equations:

$$
y_n(t) = \xi + \int_t^T f(s, \phi(s), z_{n-1}(s))ds - \int_t^T z_n(s)dW(s), \quad t \in [0,T]. \qquad (3.4.5)
$$

From Lemma 3.4.1 we know that these equations have unique solution pairs $\{(y_n(\cdot), z_n(\cdot)) \in \widehat{H}_1^2(0,T;\mathbb{R}^d) \times \widehat{M}_1^2(0,T;\mathbb{R}^{d\times k})\}_{n\geq 1}$, for which it also holds that

$\{\sqrt{\alpha_1(\cdot)}y_n(\cdot) \in \widehat{M}_1^2(0,T;\mathbb{R}^d)\}_{n\geq 1}$. Similarly to the proof of uniqueness, we have

$$
\begin{aligned}
& - d\, p_1(t)|y_{n+1}(t) - y_n(t)|^2 \\
=& \{-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 \\
& + 2p_1(t)(y_{n+1}(t) - y_n(t))' \left[f(t,\phi(t),z_n(t)) - f(t,\phi(t),z_{n-1}(t))\right] \\
& - p_1(t)|z_{n+1}(t) - z_n(t)|^2\}dt - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\
\leq & [-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 - p_1(t)|z_{n+1}(t) - z_n(t)|^2]dt \\
& - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\
& + 2p_1(t)|y_{n+1}(t) - y_n(t)|\,|f(t,\phi(t),z_n(t)) - f(t,\phi(t),z_{n-1}(t))|\,dt \\
\leq & [-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 - p_1(t)|z_{n+1}(t) - z_n(t)|^2]dt \\
& - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\
& + 2p_1(t)c_2(t)|y_{n+1}(t) - y_n(t)||z_n(t) - z_{n-1}(t)|dt \\
\leq & [-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 - p_1(t)|z_{n+1}(t) - z_n(t)|^2]dt \\
& - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\
& + \beta_2 c_2^2(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2dt + \beta_2^{-1}p_1(t)|z_n(t) - z_{n-1}(t)|^2dt \\
\leq & [-p_1(t)|z_{n+1}(t) - z_n(t)|^2 + \beta_2^{-1}p_1(t)|z_n(t) - z_{n-1}(t)|^2]dt \\
& -2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t),
\end{aligned}
$$

which in integral form becomes

$$
\begin{aligned}
&p_1(t)|y_{n+1}(t)-y_n(t)|^2+\int_t^T p_1(s)|z_{n+1}(s)-z_n(s)|^2 ds\\
\le&-\int_t^T 2p_1(s)(y_{n+1}(s)-y_n(s))'(z_{n+1}(s)-z_n(s))dW(s)\\
&+\beta_2^{-1}\int_t^T p_1(s)|z_n(s)-z_{n-1}(s)|^2]ds.
\end{aligned}
$$

From Lemma 3.4.1 (ii), it is clear that the stochastic integral on the right hand side is a martingale. Taking the expected values of both sides gives

$$
\begin{aligned}
&\mathbb{E}\left[p_1(t)|y_{n+1}(t)-y_n(t)|^2\right]+\mathbb{E}\int_t^T p_1(s)|z_{n+1}(s)-z_n(s)|^2 ds\\
\le&\beta_2^{-1}\mathbb{E}\int_t^T p_1(s)|z_n(s)-z_{n-1}(s)|^2]ds.
\end{aligned}
$$

Let us define

$$
\eta_n(t)\equiv\mathbb{E}\int_t^T p_1(s)|y_n(s)-y_{n-1}(s)|^2 ds,
$$

and

$$
\mu_n(t)\equiv\mathbb{E}\int_t^T p_1(s)|z_n(s)-z_{n-1}(s)|^2 ds.
$$

Using the same argument as in the last part of the proof of Proposition 2.2 in [61], we obtain $\eta_{n+1}(0)\le\beta_2^{-n}\mathbb{E}\int_0^T p_1(s)|z_1(s)|^2ds$ and $\mu_n(0)\le\beta_2^{-n}\mu_1(0)$. Since the right-hand sides of these two inequalities decrease with n, it follows that $\{y_n\}_{n\ge1}$ is a Cauchy sequence in $\widehat{M}_1^2(0,T;\mathbb{R}^d)$, and $\{z_n\}_{n\ge1}$ is a Cauchy sequence in $\widehat{M}_1^2(0,T;\mathbb{R}^{d\times k})$. Moreover, this also implies that $\{\sqrt{\alpha_1}y_n\}_{n\ge1}$ is a Cauchy sequence in $\widehat{M}_1^2(0,T;\mathbb{R}^d)$. Hence, the limiting processes $y^*=\lim_{n\to\infty}y_n$ and $z^*=\lim_{n\to\infty}z_n$ are the solution pair of (3.4.3). In addition, when such a pair of processes is substituted in (3.4.3), then (3.4.3) becomes an example of (3.4.2) with $\psi(\cdot)=z^*(\cdot)$. Therefore, Lemma 3.4.1 applies, and we have that $y^*(\cdot)\in\widehat{H}_1^2(0,T;\mathbb{R}^{d\times k})$.

(ii) Due to Lemma 3.4.2, the proof in this case is identical to the proof of part (i) (with an obvious change of notation), and is thus omitted. □

Theorem 3.4.1. *(i) If the pair (f,ξ) satisfies conditions A1, then the BSDE* (3.2.1) *has a unique solution pair $(y(\cdot),z(\cdot))\in\widehat{H}_1^2(0,T;\mathbb{R}^d)\times\widehat{M}_1^2(0,T;\mathbb{R}^{d\times k})$, and $\sqrt{\alpha_1(\cdot)}y(\cdot)\in\widehat{M}_1^2(0,T;\mathbb{R}^d)$.*

(ii) If the pair (f, ξ) satisfies conditions A2, then the BSDE (3.2.1) *has a unique solution pair* $(y(\cdot), z(\cdot)) \in \widehat{H}_2^2(0,T;\mathbb{R}^d) \times \widehat{M}_2^2(0,T;\mathbb{R}^{d\times k})$, *and* $\sqrt{\alpha_2(\cdot)}y(\cdot) \in \widehat{M}_2^2(0,T;\mathbb{R}^d)$.

Proof. (i) (*Uniqueness*) Let $(y_1(\cdot), z_1(\cdot))$ and $(y_2(\cdot), z_2(\cdot))$ be two solution pairs of (3.2.1) with the claimed properties. Then we have

$$
\begin{aligned}
&- d\,p_1(t)|y_1(t) - y_2(t)|^2 \\
\leq &[-\alpha_1(t)p_1(t)\,|y_1(t) - y_2(t)|^2 - p_1(t)|z_1(t) - z_2(t)|^2]dt \\
&- 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
&+ 2p_1(t)|y_1(t) - y_2(t)|\,|f(t, y_1(t), z_1(t)) - f(t, y_2(t), z_2(t))|\,dt \\
\leq &[-\alpha_1(t)p_1(t)\,|y_1(t) - y_2(t)|^2 - p_1(t)|z_1(t) - z_2(t)|^2]dt \\
&- 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
&+ 2p_1(t)|y_1(t) - y_2(t)|\,[c_1(t)|y_1(t) - y_2(t)| + c_2(t)|z_1(t) - z_2(t)|]\,dt \\
\leq &[-\alpha_1(t)p_1(t)\,|y_1(t) - y_2(t)|^2 - p_1(t)|z_1(t) - z_2(t)|^2]dt \\
&- 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
&+ \beta_1 c_1^2 p_1(t)|y_1(t) - y_2(t)|^2 dt + \beta_1^{-1}p_1(t)|y_1(t) - y_2(t)|^2 dt \\
&+ \beta_2 c_2^2 p_1(t)|y_1(t) - y_2(t)|^2 dt + \beta_2^{-1}p_1(t)|z_1(t) - z_2(t)|^2 dt \\
\leq &\left[\beta_1^{-1}p_1(t)|y_1(t) - y_2(t)|^2 + (\beta_2^{-1} - 1)p_1(t)|z_1(t) - z_2(t)|^2\right] dt \\
&- 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
\leq &\beta_1^{-1}p_1(t)|y_1(t) - y_2(t)|^2 dt - 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t),
\end{aligned}
$$

and, with the help of Lemma 3.4.1 (ii), the conclusion follows similarly to the proof of uniqueness in Lemma 3.4.3 by the use of Gronwall's lemma.

(*Existence*) Let $y_0(t) \equiv 0$, $\forall t \in [0,T]$, and for $n \geq 1$ consider the sequence of equations:

$$y_n(t) = \xi + \int_t^T f(s, y_{n-1}(s), z_n(s))ds - \int_t^T z_n(s)dW(s), \quad t \in [0,T]. \tag{3.4.6}$$

From Lemma 3.4.3 we know that these equations have unique solution pairs $\{(y_n(\cdot), z_n(\cdot)) \in \widehat{H}_1^2(0,T;\mathbb{R}^d) \times \widehat{M}_1^2(0,T;\mathbb{R}^{d\times k})\}_{n\geq 1}$. Then

$$\begin{aligned}
&- d\, p_1(t)|y_{n+1}(t) - y_n(t)|^2 \\
=&\{-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 \\
&+ 2p_1(t)(y_{n+1}(t) - y_n(t))' \left[f(t, y_n(t), z_{n+1}(t)) - f(t, y_{n-1}(t), z_n(t))\right] \\
&- p_1(t)|z_{n+1}(t) - z_n(t)|^2\}dt - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t). \\
\leq&[-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 - p_1(t)|z_{n+1}(t) - z_n(t)|^2]dt \\
&- 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\
&+ 2p_1(t)|y_{n+1}(t) - y_n(t)|\, |f(t, y_n(t), z_{n+1}(t)) - f(t, y_{n-1}(t), z_n(t))|\, dt \\
\leq&[-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 - p_1(t)|z_{n+1}(t) - z_n(t)|^2]dt \\
&- 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\
&+ 2p_1(t)|y_{n+1}(t) - y_n(t)|\left[c_1(t)|y_n(t) - y_{n-1}(t)| + c_2(t)|z_{n+1}(t) - z_n(t)|\right] dt
\end{aligned}$$

$$\leq [-\alpha_1(t)p_1(t)|y_{n+1}(t)-y_n(t)|^2 - p_1(t)|z_{n+1}(t)-z_n(t)|^2]dt$$

$$- 2p_1(t)(y_{n+1}(t)-y_n(t))'(z_{n+1}(t)-z_n(t))dW(t)$$

$$+ \beta_1 c_1^2(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 dt + \beta_1^{-1} p_1(t)|y_n(t) - y_{n-1}(t)|^2 dt$$

$$+ \beta_2 c_2^2(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 dt + \beta_2^{-1} p_1(t)|z_{n+1}(t) - z_n(t)|^2 dt$$

$$\leq \beta_1^{-1} p_1(t)|y_n(t) - y_{n-1}(t)|^2 dt + (\beta_2^{-1} - 1)p_1(t)|z_{n+1}(t) - z_n(t)|^2 dt$$

$$- 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t).$$

Then the expectation of the integral-form of this inequity becomes

$$\mathbb{E}\left[p_1(t)|y_{n+1}(t) - y_n(t)|^2\right]$$

$$\leq \mathbb{E}\int_t^T \beta_1^{-1} p_1(s)|y_n(s) - y_{n-1}(s)|^2 ds + \mathbb{E}\int_t^T (\beta_2^{-1} - 1)p_1(s)|z_{n+1}(s) - z_n(s)|^2 ds, \tag{3.4.7}$$

due to Lemma 3.4.1 (ii). Using the notation $\nu_{n+1}(t) = \mathbb{E}\int_t^T p_1(t)|y_{n+1}(s) - y_n(s)|^2 ds$, and similarly to the last part of the proof of Theorem 3.1 of [61], we obtain $\nu_{n+1}(0) \leq \beta_1^{-n}\frac{1}{n!}\nu_1(0)$. Since the sum of the right-hand sides of these inequalities converges, we conclude, together with (3.4.7), that $\{y_n\}$ is a Cauchy sequence in $\widehat{M}_1^2(0,T;\mathbb{R}^d)$, and $\{z_n\}$ is a Cauchy sequence in $\widehat{M}_1^2(0,T;\mathbb{R}^{d\times k})$. Moreover, this also implies that $\{\sqrt{\alpha}y_n\}_{n\geq 1}$ is a Cauchy sequence in $\widehat{M}_1^2(0,T;\mathbb{R}^d)$. Thus the limiting processes $y^* = \lim_{n\to\infty} y_n$ and $z^* = \lim_{n\to\infty} z_n$ are the solution pair to (3.2.1). In addition, when such a pair of processes is substituted in (3.2.1), then (3.2.1) becomes an example of (3.4.2) with $\phi(\cdot) = y^*(\cdot)$ and $\psi(\cdot) = z^*(\cdot)$. Therefore, Lemma 3.4.1 applies, and we have that $y^*(\cdot) \in \widehat{H}_1^2(0,T;\mathbb{R}^{d\times k})$.

(ii) *(Uniqueness)* Let $(y_1(\cdot), z_1(\cdot))$ and $(y_2(\cdot), z_2(\cdot))$ be two solution pairs of (3.2.1) with the claimed properties. Similarly to the proof of uniqueness for part (i), we have

$$
\begin{aligned}
&-d\,p_2(t)|y_1(t)-y_2(t)|^2\\
\leq&[-\alpha_2(t)p_2(t)\,|y_1(t)-y_2(t)|^2-p_2(t)|z_1(t)-z_2(t)|^2]dt\\
&-2p_2(t)(y_1(t)-y_2(t))'(z_1(t)-z_2(t))dW(t)\\
&+2p_2(t)|y_1(t)-y_2(t)|\left[c_1(t)|y_1(t)-y_2(t)|+c_2(t)|z_1(t)-z_2(t)|\right]dt\\
\leq&[-\alpha_2(t)p_2(t)\,|y_1(t)-y_2(t)|^2-p_2(t)|z_1(t)-z_2(t)|^2]dt\\
&-2p_2(t)(y_1(t)-y_2(t))'(z_1(t)-z_2(t))dW(t)+2c_1p_2(t)|y_1(t)-y_2(t)|^2dt\\
&+\overline{\beta_2}c_2^2p_2(t)|y_1(t)-y_2(t)|^2dt+\overline{\beta_2}^{-1}p_2(t)|z_1(t)-z_2(t)|^2dt\\
\leq&(\beta_2^{-1}-1)p_2(t)|z_1(t)-z_2(t)|^2dt-2p_2(t)(y_1(t)-y_2(t))'(z_1(t)-z_2(t))dW(t)\\
\leq&-2p_2(t)(y_1(t)-y_2(t))'(z_1(t)-z_2(t))dW(t).
\end{aligned}
$$

Then with the help of Lemma 3.4.1 (ii), the conclusion follows similarly to the proof of uniqueness in Lemma 3.4.3.

(Existence) Let $y_0(t)\equiv 0$, $\forall t\in[0,T]$, and for $n\geq 1$ consider the sequence of equations:

$$
y_n(t)=\xi+\int_t^T f(s,y_{n-1}(s),z_n(s))ds-\int_t^T z_n(s)dW(s),\quad t\in[0,T]. \tag{3.4.8}
$$

From Lemma 3.4.3 we know that these equations have unique solution pairs $\{(y_n(\cdot),z_n(\cdot))\in\widehat{H}_2^2(0,T;\mathbb{R}^d)\times\widehat{M_2^2}(0,T;\mathbb{R}^{d\times k})\}_{n\geq 1}$. By Lemma 6.2 of [39], we have following estimates:

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T p_2(t)|y_{n+1}(t)-y_n(t)|^2\alpha_2(t)dt\right]\\
\leq&\,8\,\mathbb{E}\left[\int_0^T p_2(t)\frac{|f(t,y_n(t),z_{n+1}(t))-f(t,y_{n-1}(t),z_n(t))|^2}{\alpha_2(t)}dt\right],
\end{aligned}
\tag{3.4.9}
$$

and

$$\begin{aligned}&\mathbb{E}\left[\int_0^T p_2(t)|z_{n+1}(t)-z_n(t)|^2dt\right]\\&\leq 45\,\mathbb{E}\left[\int_0^T p_2(t)\frac{|f(t,y_n(t),z_{n+1}(t))-f(t,y_{n-1}(t),z_n(t))|^2}{\alpha_2(t)}dt\right].\end{aligned}\tag{3.4.10}$$

By the Lipschitz condition, we have

$$\mathbb{E}\left[\int_0^T p_2(t)\frac{|f(t,y_n(t),z_{n+1}(t))-f(t,y_{n-1}(t),z_n(t))|^2}{\alpha_2(t)}dt\right]$$

$$\leq \mathbb{E}\int_0^T \frac{p_2(t)}{\alpha_2(t)}[c_1(t)|y_n(t)-y_{n-1}(t)|+c_2(t)|z_{n+1}(t)-z_n(t)|]^2dt$$

$$\leq 2\,\mathbb{E}\int_0^T \frac{p_2(t)}{\alpha_2(t)}\left[c_1^2(t)|y_n(t)-y_{n-1}(t)|^2+c_2^2(t)|z_{n+1}(t)-z_n(t)|^2\right]dt$$

$$\leq 2\,\mathbb{E}\int_0^T \frac{p_2(t)}{\overline{\beta_1}^2}\frac{\overline{\beta_1}c_1(t)}{\overline{\gamma(t)}+\overline{\beta_1}c_1(t)+\overline{\beta_2}c_2^2(t)}(\overline{\gamma(t)}+\overline{\beta_1}c_1(t)+\overline{\beta_2}c_2^2(t))|y_n(t)-y_{n-1}(t)|^2dt$$

$$+\ 2\,\mathbb{E}\int_0^T \frac{p_2(t)}{\overline{\beta_2}}\frac{\overline{\beta_2}c_2^2(t)}{\overline{\gamma(t)}+\overline{\beta_1}c_1(t)+\overline{\beta_2}c_2^2(t)}|z_{n+1}(t)-z_n(t)|]^2ds$$

$$\leq \frac{2}{\overline{\beta_1}^2}\mathbb{E}\int_0^T p_2(t)\alpha_2(t)|y_n(t)-y_{n-1}(t)|^2dt+\frac{2}{\overline{\beta_2}}\mathbb{E}\int_0^T p_2(t)|z_{n+1}(t)-z_n(t)|]^2dt.$$

Substituting it into (3.4.10), we have

$$\mathbb{E}\left[\int_0^T p_2(t)|z_{n+1}(t)-z_n(t)|^2dt\right]$$

$$\leq \frac{90}{\overline{\beta_1}^2}\mathbb{E}\int_0^T p_2(t)\alpha_2(t)|y_n(t)-y_{n-1}(t)|^2dt+\frac{90}{\overline{\beta_2}}\mathbb{E}\int_0^T p_2(t)|z_{n+1}(t)-z_n(t)|]^2dt.$$

Let $\overline{\beta_2} > 90$. Then we have

$$\mathbb{E}\left[\int_0^T p_2(t)|z_{n+1}(t) - z_n(t)|^2 dt\right] \leq \frac{\frac{90}{\overline{\beta_1}^2}}{\left(1 - \frac{90}{\overline{\beta_2}}\right)} \mathbb{E}\int_0^T p_2(t)\alpha_2(t)|y_n(t) - y_{n-1}(t)|^2 dt.$$

Substituting it into (3.4.10), we obtain

$$\mathbb{E}\left[\int_0^T p_2(t)|y_{n+1}(t) - y_n(t)|^2 \alpha_2(t) dt\right]$$

$$\leq \frac{16}{\overline{\beta_1}^2}\mathbb{E}\int_0^T p_2(t)\alpha_2(t)|y_n(t) - y_{n-1}(t)|^2 dt + \frac{16}{\overline{\beta_2}}\mathbb{E}\int_0^T p_2(t)|z_{n+1}(t) - z_n(t)|]^2 dt$$

$$\leq \frac{16}{\overline{\beta_1}^2}\mathbb{E}\int_0^T p_2(t)\alpha_2(t)|y_n(t) - y_{n-1}(t)|^2 dt$$

$$+ \frac{16}{\overline{\beta_2}} \frac{\frac{90}{\overline{\beta_1}^2}}{\left(1 - \frac{90}{\overline{\beta_2}}\right)} \mathbb{E}\int_0^T p_2(t)\alpha_2(t)|y_n(t) - y_{n-1}(t)|^2 dt$$

$$= \left[\frac{16}{\overline{\beta_1}^2} + \frac{16}{\overline{\beta_2}} \frac{\frac{90}{\overline{\beta_1}^2}}{\left(1 - \frac{90}{\overline{\beta_2}}\right)}\right] \mathbb{E}\int_0^T p_2(t)\alpha_2(t)|y_n(t) - y_{n-1}(t)|^2 dt.$$

Let $\kappa = \left[\frac{16}{\overline{\beta_1}^2} + \frac{16}{\overline{\beta_2}} \frac{\frac{90}{\overline{\beta_1}^2}}{\left(1 - \frac{90}{\overline{\beta_2}}\right)}\right] < 1$, i.e. $\overline{\beta}_1 > 4$ and $\overline{\beta}_2 > \frac{90\overline{\beta}_1^2}{\overline{\beta}_1^2 - 16}$. Then the conclusion follows similarly to the proof of existence in Lemma 3.4.3. □

3.5 Comparison theorem

The following results generalise Peng's comparison theorem ([62],[66]) to equations with a possibly unbounded generator. Similarly to [62], [66], we assume that $d = 1$.

In addition to equation (3.2.1), let us consider two further equations

$$\widehat{y}_1(t) = \widehat{\xi}_1 + \int_t^T [\widehat{f}_1(s, \widehat{y}_1(s), \widehat{z}_1(s))]ds - \int_t^T \widehat{z}_1(s)dW(s), \quad t \in [0, T],$$

$$\widehat{y}_2(t) = \widehat{\xi}_2 + \int_t^T [\widehat{f}_2(s, \widehat{y}_2(s), \widehat{z}_2(s))]ds - \int_t^T \widehat{z}_2(s)dW(s), \quad t \in [0, T].$$

We assume that the pair $(\widehat{f}_1, \widehat{\xi}_1)$ satisfies conditions A1, whereas the pair $(\widehat{f}_2, \widehat{\xi}_2)$ satisfies conditions A2. Based on Theorem 3.4.1, this means that there exist unique solution pairs $(\widehat{y}_1(\cdot), \widehat{z}_1(\cdot)) \in \widehat{H}_1^2(0, T; \mathbb{R}) \times \widehat{M}_1^2(0, T; \mathbb{R}^{1\times k})$ and $(\widehat{y}_2(\cdot), \widehat{z}_2(\cdot)) \in \widehat{H}_2^2(0, T; \mathbb{R}) \times \widehat{M}_2^2(0, T; \mathbb{R}^{1\times k})$. The following differences will appear in the proof:

$$\begin{aligned} Y_1(t) &\equiv y(t) - \widehat{y}_1(t), \quad Z_1(t) \equiv z(t) - \widehat{z}_1(t), \\ Y_2(t) &\equiv y(t) - \widehat{y}_2(t), \quad Z_2(t) \equiv z(t) - \widehat{z}_2(t). \end{aligned}$$

Theorem 3.5.1. *(Comparison theorem) (i) If $\widehat{\xi}_1 \geq \xi$ and $\widehat{f}_1(t, y, z) \geq f(t, y, z)$, a.s. $\forall\ (t, y, z) \in ([0, T] \times \mathbb{R} \times \mathbb{R}^{1\times k})$, then $\widehat{y}_1(t) \geq y(t)$, $\forall\ t \in [0, T]$, a.s..*

(ii) If $\widehat{\xi}_2 \geq \xi$ and $\widehat{f}_2(t, y, z) \geq f(t, y, z)$, a.s. $\forall\ (t, y, z) \in ([0, T] \times \mathbb{R} \times \mathbb{R}^{1\times k})$, then $\widehat{y}_2(t) \geq y(t)$, $\forall\ t \in [0, T]$, a.s..

Proof. (i) The equation of the difference $Y_1(t)$ is

$$-dY_1(t) = [f(t, y(t), z(t)) - \widehat{f}_1(t, \widehat{y}_1(t), \widehat{z}_1(t))]dt - Z_1(t)dW(t).$$

Denoting by $Y_1^+(t) \equiv \mathbb{1}_{[Y_1(t)>0]} Y_1(t)$, and using Tanaka-Meyer formula (see Theorem 2.4.3), we obtain

$$-dY_1^+(t) = -\mathbb{1}_{[Y_1(t)>0]} dY_1(t) - \frac{1}{2} dL(t),$$

where $L(t)$ is the local time of $Y_1(\cdot)$ at 0. Since $\int_0^T |Y_1(t)| dL(t) = 0$, a.s. (see Proposition 2.4.1), we have

$$\begin{aligned} -d[Y_1^+(t)]^2 = &\, 2\, Y_1^+(t) \mathbb{1}_{[Y_1(t)>0]} [f(t, y(t), z(t)) - \widehat{f}_1(t, \widehat{y}_1(t), \widehat{z}_1(t))]dt \\ &- \mathbb{1}_{[Y_1(t)>0]} Z_1^2(t)dt - \mathbb{1}_{[Y_1(t)>0]} 2Y_1^+(t) Z_1(t) dW(t). \end{aligned}$$

Using Itô formula, we obtain

$$
\begin{aligned}
&- d\, p_1(t)[Y_1^+(t)]^2 \\
=& - \alpha_1(t)p_1(t)[Y_1^+(t)]^2 dt - \mathbb{1}_{[Y_1(t)>0]} p_1(t) Z_1^2(t) dt - 2p_1(t)Y_1^+(t)Z_1(t)dW(t) \\
&+ 2p_1(t)Y_1^+(t)\mathbb{1}_{[Y_1(t)>0]}[f(t,y(t),z(t)) - \widehat{f}_1(t,\widehat{y}_1(t),\widehat{z}_1(t))]dt \\
\le& -\alpha_1(t)p_1(t)[Y_1^+(t)]^2 dt + 2p_1(t)Y_1^+(t)\mathbb{1}_{[Y_1(t)>0]}[f(t,y(t),z(t)) - \widehat{f}_1(t,y(t),z(t)) \\
&+ \widehat{f}_1(t,y(t),z(t)) - \widehat{f}_1(t,\widehat{y}_1(t),\widehat{z}_1(t))]dt \\
&- \mathbb{1}_{[Y_1(t)>0]} p_1(t) Z_1^2(t) dt - 2p_1(t)Y_1^+(t)Z_1(t)dW(t) \\
\le& [-\alpha_1(t)p_1(t)[Y_1^+(t)]^2 + 2p_1(t)Y_1^+(t)\mathbb{1}_{[Y_1(t)>0]}[f(t,y(t),z(t)) - \widehat{f}_1(t,y(t),z(t))] \\
&- \mathbb{1}_{[Y_1(t)>0]} p_1(t) Z_1^2(t)]dt - \mathbb{1}_{[Y_1(t)>0]} 2p_1(t)Y_1^+(t)Z_1(t)dW(t) \\
&+ \beta_1 p_1(t) c_1^2(t)[Y_1^+(t)]^2 dt + \beta_1^{-1} p_1(t)[Y_1^+(t)]^2 dt + \beta_2 c_2^2(t) p_1(t)[Y_1^+(t)]^2 dt \\
&+ \beta_2^{-1} \mathbb{1}_{[Y_1(t)>0]} p_1(t) Z_1^2(t) dt \\
\le& \beta_1^{-1} p_1(t)[Y_1^+(t)]^2 dt - 2p_1(t)Y_1^+(t)Z_1(t)dW(t),
\end{aligned}
$$

which in integral form becomes

$$
p_1(t)[Y_1^+(t)]^2 \le \int_t^T \beta_1^{-1} p_1(s)[Y_1^+(s)]^2 ds - \int_t^T 2p_1(s)Y_1^+(s)Z_1(s)dW(s).
$$

The stochastic integral on the right-hand side is a martingale due to Lemma 3.4.1 (ii). Therefore,

$$
\mathbb{E}[p_1(t)[Y_1^+(t)]^2] \le \mathbb{E}\int_t^T \beta_1^{-1} p_1(s)[Y_1^+(s)]^2 ds,
$$

and the conclusion follows from Gronwall's lemma.

(ii) In a similar way to the proof of part (i), we have

$$
\begin{aligned}
&- d\, p_2(t)[Y_2^+(t)]^2 \\
=& - \alpha_2(t)p_2(t)[Y_2^+(t)]^2dt + 2p_2(t)Y_2^+(t)\mathbb{1}_{[Y_2(t)>0]}[f(t,y(t),z(t)) - \widehat{f}_2(t,\widehat{y}_2(t),\widehat{z}_2(t))]dt \\
&- \mathbb{1}_{[Y_2(t)>0]}p_2(t)Z_2^2(t)dt - 2p_2(t)Y_2^+(t)Z_2(t)dW(t) \\
\le& -\alpha_2(t)p_2(t)[Y_2^+(t)]^2dt + 2p_2(t)Y_2^+(t)\mathbb{1}_{[Y_2(t)>0]}[f(t,y(t),z(t)) - \widehat{f}_2(t,y(t),z(t)) \\
&+ \widehat{f}_2(t,y(t),z(t)) - \widehat{f}_2(t,\widehat{y}_2(t),\widehat{z}_2(t))]dt \\
&- \mathbb{1}_{[Y_2(t)>0]}p_2(t)Z_2^2(t)dt - 2p_2(t)Y_2^+(t)Z_2(t)dW(t) \\
\le& [-\alpha_2(t)p_2(t)[Y_2^+(t)]^2 + 2p_2(t)Y_2^+(t)\mathbb{1}_{[Y_2(t)>0]}[f(t,y(t),z(t)) - \widehat{f}_2(t,y(t),z(t))] \\
&- \mathbb{1}_{[Y_2(t)>0]}p_2(t)Z_2^2(t)]dt - \mathbb{1}_{[Y_2(t)>0]}2p_2(t)Y_2^+(t)Z_2(t)dW(t) + 2p_2(t)c_1(t)[Y_2^+(t)]^2dt \\
&+ \overline{\beta}_2c_2^2(t)p_2(t)[Y_2^+(t)]^2dt + \overline{\beta}_2^{-1}\mathbb{1}_{[Y_2(t)>0]}p_2(t)Z_2^2(t)dt \\
\le& - 2\mathbb{1}_{[Y_2(t)>0]}p_1(t)Y_2^+(t)Z_2(t)dW(t),
\end{aligned}
$$

which in integral form becomes

$$
p_2(t)[Y_2^+(t)]^2 \le -\int_t^T 2\mathbb{1}_{[Y_2(s)>0]}p_2(s)Y_2^+(s)Z_2(s)dW(s).
$$

Since the stochastic integral on the right-hand side is a martingale due to Lemma 3.4.1 (ii), we have

$$
\mathbb{E}[p_2(t)[Y_2^+(t)]^2] \le 0,
$$

which concludes the proof. □

3.6 Conclusion

We have considered BSDEs with a possibly unbounded generator. Under two cases of Lipschitz-type generator, we give sufficient conditions for the existence of unique solution pairs. These are novel conditions as compared to existing ones, and are either weaker or not comparable (in general) with the existing ones. A comparison theorem is also given. It is to be expected that these results will be useful in tackling more difficult problems with unbounded generator, such as the BSDEs with a quadratic growth and the Riccati BSDE, which play a fundamental role in stochastic control.

Appendix

Here we include the derivation of the lower bound for the parameter β that appears in [39]. We do so for the completeness of the chapter. Since in Theorem 6.1 of [39] no explicit lower bound is given, it is only assumed that parameter β should be *large enough*. The notation of [39] will be used.

Equation (6.5) of [39] states that for some constants k and k' the following holds

$$\|(y,\eta)\|_\beta^2 = k\|\xi\|_\beta^2 + \frac{k'}{\beta}\left\|\frac{f}{\alpha}\right\|_\beta^2, \tag{3.6.1}$$

where the definitions of these norms are given in [39], and are just weighted Euclidian norms. From equation (5.5) of [39], which gives the definition of the norm $\|(y,\eta)\|_\beta^2$, and the conclusions of Lemma 6.2 of [39], we obtain

$$\begin{aligned}\|(y,\eta)\|_\beta^2 &= \|\alpha y\|_\beta^2 + \|\eta\|_\beta^2 \\ &\leq \frac{2}{\beta}\|\xi\|_\beta^2 + \frac{8}{\beta^2}\left\|\frac{f}{\alpha}\right\|_\beta^2 + 18\|\xi\|_\beta^2 + \frac{45}{\beta}\left\|\frac{f}{\alpha}\right\|_\beta^2 \\ &= \left(18 + \frac{2}{\beta}\right)\|\xi\|_\beta^2 + \left(\frac{45}{\beta} + \frac{8}{\beta^2}\right)\left\|\frac{f}{\alpha}\right\|_\beta^2. \end{aligned} \tag{3.6.2}$$

Comparing (3.6.1) and (3.6.2) gives $k' = 45 + \frac{8}{\beta}$.

Equation (6.16) of [39] states that for some constants $\tilde{k}$ and $\tilde{k}'$ the following holds

$$\|(\delta Y, \delta Z, \delta N)\|_\beta^2 \leq \tilde{k}\|\delta\xi\|_\beta^2 + \tfrac{\tilde{k}'}{\beta}\left\|\tfrac{\delta_2 f}{\alpha}\right\|_\beta^2. \tag{3.6.3}$$

Here, different from [39], we have used the *tilde* notation for the constants k and k' in order to avoid the clash of notation with these constants introduced in the previous paragraph. By inequality (6.5) of [39], we obtain that

$$\begin{aligned}
\|(\delta Y, \delta Z, \delta N)\|_\beta^2 &\leq k\|\delta\xi\|_\beta^2 + \frac{k'}{\beta}\left\|\frac{\varphi_t}{\alpha_t^2}\right\|_\beta^2 \\
&\leq k\|\delta\xi\|_\beta^2 + \frac{3k'}{\beta}\left(\|\alpha\delta Y\|_\beta^2 + \|m^*\delta Z\|_\beta^2 + \left\|\frac{\delta_2 f}{\alpha}\right\|_\beta^2\right) \\
&\leq k\|\delta\xi\|_\beta^2 + \frac{3k'}{\beta}\left(k\,\|\delta\xi\|_\beta^2 + \frac{k'}{\beta}\left\|\frac{\delta_2 f}{\alpha}\right\|_\beta^2 + \left\|\frac{\delta_2 f}{\alpha}\right\|_\beta^2\right) \\
&= \left(k + \frac{3kk'}{\beta}\right)\|\delta\xi\|_\beta^2 + \frac{3k'}{\beta}\left(\frac{k'}{\beta} + 1\right)\left\|\frac{\delta_2 f}{\alpha}\right\|_\beta^2. \qquad (3.6.4)
\end{aligned}$$

Comparing (3.6.3) and (3.6.4) gives $\widetilde{k'} = 3k'\left(\frac{k'}{\beta} + 1\right)$.

The inequality at the end of page 35 of [39] is

$$\|\alpha\delta Y\|_\beta^2 + \|m^*\delta Z\|_\beta^2 \leq \frac{\widehat{k'}}{\beta}\|\alpha\delta y\|_\beta^2 + \|m^*\delta z\|_\beta^2, \qquad (3.6.5)$$

where we have used the *hat* notation for the constant k' in order to avoid the clash of notation with this constant introduced earlier. Similarly to the previous paragraph we obtain

$$\begin{aligned}
\|(\delta Y, \delta Z)\|_\beta^2 &= \|\alpha\delta Y\|_\beta^2 + \|m^*\delta Z\|_\beta^2 \leq \frac{\widetilde{k'}}{\beta}\left\|\frac{\varphi}{\alpha}\right\|_\beta^2 \\
&\leq \frac{3\widetilde{k'}}{\beta}\|\alpha\delta y\|_\beta^2 + \|m^*\delta z\|_\beta^2. \qquad (3.6.6)
\end{aligned}$$

Comparing (3.6.5) and (3.6.6) gives $\widehat{k'} = 3\widetilde{k'} = 9k'\left(\frac{k'}{\beta} + 1\right)$. In order to apply the contraction mapping principle, it is necessary to have $\frac{\widehat{k'}}{\beta} < 1$, i.e.

$$9\left(\frac{45}{\beta} + \frac{8}{\beta^2}\right)\left(\frac{45}{\beta} + \frac{8}{\beta^2} + 1\right) < 1.$$

By solving above inequality for $\beta > 0$, we obtain that by *large enough* in [39] it is meant that $\beta > 446.05$.

Chapter 4

Backward Stochastic Differential Equations with a Continuous and Unbounded Generator

4.1 Abstract

In this chapter, we consider a class of backward stochastic differential equations whose generator are continuous and satisfies the linear growth condition, which can also be unbounded. We prove the existence of the solution pair (also a maximal solution) for this class of equations, which is more general than the existing ones. This work is based on a preprint paper [25].

4.2 Introduction

Similar to Chapter 3, we consider the backward stochastic differential equation (i.e. (3.2.1)):

$$y(t) = \xi + \int_t^T f(s, y(s), z(s))ds - \int_t^T z(s)dW(s), \quad t \in [0, T], \tag{4.2.1}$$

under the same mathematical setting with a weaker assumption.

An important weakening of the assumptions on the generator, as compared to [61], was given in [50] (see also [29]). There it is assumed that the generator is continuous with respect to y and z, and it satisfies the linear growth condition

$$|f(t,y,z)| \leq c\,(1+|y|+|z|), \tag{4.2.2}$$

for all $y \in \mathbb{R}$, $z \in \mathbb{R}^k$, (t,ω) *a.e.*. Under such conditions, it was shown that equation (4.2.1) admits a solution pair but a non-unique one in general. In a more recent papers [71], [72], the linear growth condition (4.2.2) has been generalised to

$$|f(t,y,z)| \leq c\,[q(t)+|y|+|z|], \tag{4.2.3}$$

for all $y \in \mathbb{R}$, $z \in \mathbb{R}^k$, (t,ω) *a.e.*. Here, different from (4.2.2), the process $q(\cdot)$ is not assumed to be bounded. BSDEs with continuous quadratic generator have been considered in [4], where the unbounded generators are also considered.

In this chapter, we consider a generator which is continuous in y and z, but with a weaker linear growth condition than (4.2.3). We assume that

$$|f(t,y,z)| \leq c_0(t)+c_1(t)|y|+c_2(t)|z|, \tag{4.2.4}$$

for all $y \in \mathbb{R}$, $z \in \mathbb{R}^k$, (t,ω) *a.e.*. Here the processes $c_0(\cdot), c_1(\cdot), c_2(\cdot)$ are not assumed to be bounded. By using the results in Chapter 3, and appropriately modifying the approach of [4], [50], we prove the existence of a solution pair for (4.2.1).

4.3 Notations and Assumptions

The following is the list of the main notations used in this chapter.

- $c_1(\cdot)$, $c_2(\cdot)$, $\gamma(\cdot)$ are given $\mathbb{R}$-valued progressively measurable processes.

- $1 < \beta_1 \in \mathbb{R}$, $1 < \beta_2 \in \mathbb{R}$, are given constants.

- $\alpha(t) \equiv \gamma(t) + 2\beta_1 c_1^2(t) + 2\beta_2 c_2^2(t)$ is assumed to be positive.

- $p(t) \equiv \exp\left[\int_0^t \alpha(s)ds\right]$.

We say that the progressively measurable function f and the random variable ξ, or the pair (f,ξ), satisfies *conditions A* if:

(i) $d = 1$ and $f(t, y, z)$ is a continuous function of y and z;

(ii) $|f(t, y, z)| \le c_0(t) + c_1(t)|y| + c_2(t)|z|$, for all $y \in \mathbb{R}$, $z \in \mathbb{R}^k$, (t, ω) *a.e.*;

(iii) $\xi \in M^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R})$;

(iv) $c_0(\cdot) \in M^2(0, T; \mathbb{R})$ and $\left[c_0(\cdot)\widetilde{\alpha}(\cdot)^{-\frac{1}{2}}\right] \in M^2(0, T; \mathbb{R})$.

4.4 Unbounded Continuous Generator

In this section, we consider the one-dimensional version of equation (4.2.1) with a continuous f with respect to y and z, which satisfies a linear growth condition rather than the Lipschitz-type condition. The equation that we consider is more general than the existing ones, and the derivations rely on the main results from the Chapter 3. The main idea here, as in [50], is to approximate the generator f by an infinite sequence of Lipschitz-type approximating functions. Each such a function generates a BSDE, and we show that the solutions to such a sequence of BSDEs converge to the solution of (4.2.1). As we already mentioned in introduction, our assumption (ii) permits for random and possibly unbounded coefficients $c_0(\cdot)$, $c_1(\cdot)$ and $c_2(\cdot)$, which is not the case in [71] and [72].

We introduce the sequence of functions

$$f_n(t, y, z) \equiv \sup_{(u,v) \in \mathbb{R}^{1+k}} \{f(t, u, v) - [c_1(t) + n]|u - y| - [c_2(t) + n]|v - z|\}, \quad n \ge 1,$$

which are clearly well-defined. Their main properties are summarized in the following Lemma:

Lemma 4.4.1. *(i) Linear growth: for any $y \in \mathbb{R}$, $z \in \mathbb{R}^k$, $|f_n(t, y, z)| \le c_0(t) + c_1(t)|y| + c_2(t)|z|$;*

(ii) Monotonicity: f_n is a decreasing function of n;

(iii) Lipschitz condition: for any $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^k$,

$$|f_n(t, y_1, z_1) - f_n(t, y_2, z_2)| \le [c_1(t) + n]|y_1 - y_2| + [c_2(t) + n]|z_1 - z_2|;$$

(iv) Convergence: for any $y \in \mathbb{R}$, $z \in \mathbb{R}^k$, $\lim_{n\to\infty} f_n(t,y,z) = f(t,y,z)$.

Proof. (i) By the linear growth of f, for all $y \in \mathbb{R}$, $z \in \mathbb{R}^k$, we have

$$f_n(t,y,z) \le \sup_{(u,v)\in\mathbb{R}^{1+k}} \{|f(t,u,v)| - [c_1(t)+n]|u-y| - [c_2(t)+n]|v-z|\}$$

$$\le c_0(t) + \sup_{(u,v)\in\mathbb{R}^{1+k}} \{c_1(t)|u| + c_2(t)|v| - [c_1(t)+n]|u-y| - [c_2(t)+n]|v-z|\}$$

$$\le c_0(t) + \sup_{(u,v)\in\mathbb{R}^{1+k}} \{c_1(t)|u| + c_2(t)|v| - c_1(t)(|u|-|y|) - c_2(t)(|v|-|z|)\}$$

$$\le c_0(t) + c_1(t)|y| + c_2(t)|z|.$$

The inequality $f_n(t,y,z) \ge -c_0(t) - c_1(t)|y| - c_2(t)|z|$ can be proved similarly.

(ii) This follows from the definition of f_n itself.

(iii) By inequality $|\sup_{i\in I} a_i - \sup_{i\in I} b_i| \le \sup_{i\in I} |a_i - b_i|$, with I being an arbitrary index set, we have

$$|f_n(t,y_1,z_1) - f_n(t,y_2,z_2)|$$

$$= \Bigg| \sup_{(u,v)\in\mathbb{R}^{1+k}} \{f(t,u,v) - [c_1(t)+n]|u-y_1| - [c_2(t)+n]|v-z_1|\}$$

$$- \sup_{(u,v)\in\mathbb{R}^{1+k}} \{f(t,u,v) - [c_1(t)+n]|u-y_2| - [c_2(t)+n]|v-z_2|\} \Bigg|$$

$$\le \sup_{(u,v)\in\mathbb{R}^{1+k}} \big|[c_1(t)+n](|u-y_2| - |u-y_1|) + [c_2(t)+n](|v-z_2| - |v-z_1|)\big|$$

$$\le \sup_{(u,v)\in\mathbb{R}^{1+k}} \big|[c_1(t)+n]|u-y_2-u+y_1| + [c_2(t)+n]|v-z_2-v+z_1|\big|$$

$$= [c_1(t)+n]|y_1-y_2| + [c_2(t)+n]|z_1-z_2|.$$

(iv) For any $n \geq 1$, there exists $(u_n, v_n) \in \mathbb{R}^{1+k}$ such that

$$f_n(t, y, z) \leq f(t, u_n, v_n) - [c_1(t) + n]|u_n - y| - [c_2(t) + n]|v_n - z| + n^{-1}.$$

In other words,

$$f_n(t, y, z) + [c_1(t) + n]|u_n - y| + [c_2(t) + n]|v_n - z| \leq f(t, u_n, v_n) + n^{-1}.$$

Note that in order to make the left-hand side of above inequality finite as $n \to \infty$, it is necessary to have $\lim_{n\to\infty}(u_n, v_n) = (y, z)$. And then

$$\lim_{n\to\infty} f_n(t, y, z) \leq f(t, y, z).$$

On the other hand, by the definition of f_n, we have $f_n(t, y, z) \geq f(t, y, z)$. Hence

$$\lim_{n\to\infty} f_n(t, y, z) = f(t, y, z).$$

Hence the results follow. □

Using functions $\{f_n\}_{n\geq 1}$ as generators, we introduce the following sequence of equations

$$y_n(t) = \xi + \int_t^T f_n(s, y_n(s), z_n(s))ds - \int_t^T z_n(s)dW(s), \quad t \in [0, T]. \tag{4.4.1}$$

Lemma 4.4.2. *For any $n \geq 1$, assume that conditions A hold, the BSDEs* (4.4.1) *have unique solution pairs $(y_n(\cdot), z_n(\cdot)) \in H^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^k)$.*

Proof. We only need to show that the assumptions of the previous section, which ensures the applicability of Theorem 3.4.1, hold. Thus,

$$\mathbb{E}\left[e^{\int_0^T \{\gamma(t)+\beta_1[c_1(t)+n]^2+\beta_2[c_2(t)+n]^2\}dt}|\xi|^2\right] \leq e^{2Tn^2(\beta_1+\beta_2)}\, \mathbb{E}[p(T)|\xi|^2] < \infty,$$

and

$$\mathbb{E}\int_0^T e^{\int_0^t \{\gamma(s)+\beta_1[c_1(s)+n]^2+\beta_2[c_2(s)+n]^2\}ds} \frac{|f_n(t, 0, 0)|^2}{\gamma(t) + \beta_1[c_1(t) + n]^2 + \beta_2[c_2(t) + n]^2}dt$$

$$\leq \mathbb{E}\int_0^T e^{2tn^2(\beta_1+\beta_2)}\, p(t)\, \frac{|c_0(t)|^2}{\gamma(t) + \beta_1 c_1^2(t) + \beta_2 c_2^2(t)}dt$$

$$\leq 2\,\mathbb{E}\int_0^T e^{2tn^2(\beta_1+\beta_2)}\,p(t)\,\frac{|c_0(t)|^2}{\gamma(t)+2\beta_1 c_1^2(t)+2\beta_2 c_2^2(t)}dt$$

$$\leq 2\,e^{2Tn^2(\beta_1+\beta_2)}\mathbb{E}\int_0^T p(t)\,\frac{|c_0(t)|^2}{\alpha(t)}dt < \infty.$$

□

Our main task now is to prove that the sequence of solutions $\{y_n(\cdot), z_n(\cdot)\}_{n\geq 1}$, converges to the solution $(y(\cdot), z(\cdot))$ of (4.2.1). We first present two useful lemmas.

Lemma 4.4.3. *Assume that conditions A hold. There exists a constant κ, independent of n, such that $\|y_n\| \leq \kappa$ and $\|z_n\| \leq \kappa$, for all $n \geq 1$.*

Proof. From the fact that the sequence f_n is decreasing, and Theorem 3.5.1, we know that $y_1(t) \geq y_2(t) \geq ...$, $\forall t \in [0,T]$ *a.s.*. Hence, there exists a constant κ_1 such that

$$\kappa_1 \geq \|y_1\| \geq \|y_2\| \geq ....$$

By making use the linear growth property of f_n, we obtain

$$\begin{aligned}
&-dp(t)|y_n(t)|^2\\
=&-\alpha(t)p(t)|y_n(t)|^2dt - p(t)|z_n(t)|^2dt - 2p(t)y_n(t)z_n(t)dW(t)\\
&+2p(t)y_n(t)f_n(t,y_n(t),z_n(t))dt\\
\leq&-\alpha(t)p(t)|y_n(t)|^2dt - p(t)|z_n(t)|^2dt - 2p(t)y_n(t)z_n(t)dW(t)\\
&+2p(t)|y_n(t)||f_n(t,y_n(t),z_n(t))|dt\\
\leq&-\alpha(t)p(t)|y_n(t)|^2dt - p(t)|z_n(t)|^2dt - 2p(t)y_n(t)z_n(t)dW(t)\\
&+2p(t)|y_n(t)|[c_0(t)+c_1(t)|y_n(t)|+c_2(t)|z_n(t)|]dt\\
\leq&-\alpha(t)p(t)|y_n(t)|^2dt - p(t)|z_n(t)|^2dt - 2p(t)y_n(t)z_n(t)dW(t)
\end{aligned}$$

$$+ p(t)|y_n(t)|^2dt + p(t)c_0^2(t)dt + 2\beta_1 c_1^2(t)p(t)|y_n(t)|^2dt + (2\beta_1)^{-1}p(t)|y_n(t)|^2dt$$

$$+ 2\beta_2 c_2^2(t)p(t)|y_n(t)|^2 + (2\beta_2)^{-1}p(t)|z_n(t)|^2dt$$

$$\le -\left[1-(2\beta_2)^{-1}\right]p(t)|z_n(t)|^2dt + p(t)c_0^2(t)dt$$

$$+ \left[1+(2\beta_1)^{-1}\right]p(t)|y_n(t)|^2dt - 2p(t)y_n(t)z_n(t)dW(t),$$

which in integral form becomes

$$\begin{aligned}\int_t^T p(s)|z_n(s)|^2ds \le & \frac{p(T)\xi^2}{[1-(2\beta_2)^{-1}]} + \frac{\int_t^T p(s)c_0^2(s)ds}{[1-(2\beta_2)^{-1}]} \\ & + \frac{\left[1+(2\beta_1)^{-1}\right]}{[1-(2\beta_2)^{-1}]}\int_t^T p(s)|y_n(s)|^2ds \\ & - \frac{2\int_t^T p(s)y_n(s)z_n(s)dW(s)}{[1-(2\beta_2)^{-1}]}.\end{aligned}$$

From Lemma 3.4.1, we know that the stochastic integral on the right-hand side is a martingale. Taking the expectation of both sides gives

$$\begin{aligned}\|z_n\| &\le \frac{\mathbb{E}[p(T)\xi^2]}{[1-(2\beta_2)^{-1}]} + \frac{\mathbb{E}\int_0^T p(s)c_0^2(s)ds}{[1-(2\beta_2)^{-1}]} + \frac{\left[1+(2\beta_1)^{-1}\right]}{[1-(2\beta_2)^{-1}]}\|y_n\| \qquad (4.4.2)\\ &\le \frac{\mathbb{E}[p(T)\xi^2]}{[1-(2\beta_2)^{-1}]} + \frac{\mathbb{E}\int_0^T p(s)c_0^2(s)ds}{[1-(2\beta_2)^{-1}]} + \frac{\left[1+(2\beta_1)^{-1}\right]}{[1-(2\beta_2)^{-1}]}\kappa_1 = \kappa_2.\end{aligned}$$

Finally, $\kappa = \max(\kappa_1, \kappa_2)$. □

Lemma 4.4.4. *Assume that conditions A hold. Then the pair of processes* $(y_n(\cdot), z_n(\cdot))_{n\ge1}$ *converges to* $(y(\cdot), z(\cdot))$ *in* $M^2(0,T;\mathbb{R}) \times M^2(0,T;\mathbb{R}^k)$.

Proof. Let us consider a progressively measurable and Lipchitz function

$$g(t,y,z) = -[c_0(t) + c_1(t)|y| + c_2(t)|z|].$$

From Lemma 4.4.1 (iii) and Theorem 3.4.1, we know the following BSDEs have a unique adapted solution on $M^2(0,T;\mathbb{R}) \times M^2(0,T;\mathbb{R}^k)$:

$$y_n(t) = \xi + \int_t^T f_n(s, y_n(s), z_n(s))ds - \int_t^T z_n(s)dW(s), \quad t\in[0,T],$$

and

$$K(t) = \xi + \int_t^T g(s, K(s), L(s))ds - \int_t^T L(s)dW(s), \quad t \in [0, T].$$

By Theorem 3.5.1, we have

$$K(t) \le y_n(t) \le y_{n-1}(t) \le y_1(t), \ \forall n \ge 1.$$

Hence $\{y_n(t)\}_{n\ge1}$ is decreasing and bounded in $M^2(0, T; \mathbb{R})$. Then by dominated convergence theorem (see Theorem 2.2.4), we know that $\{y_n(t)\}_{n\ge1}$ converges pointwisely to $y(t)^*$ in $M^2(0, T; \mathbb{R})$.

Applying the Itô's formula to $p(t)|y_n(t) - y_m(t)|^2$, we obtain

$$\begin{aligned} &p(t)|y_n(t) - y_m(t)|^2 + \int_t^T p(s)|z_n(s) - z_m(s)|^2 ds \\ &= \int_t^T 2p(s)(y_n(s) - y_m(s))[f_n(s, y_n(s), z_n(s)) - f_m(s, y_m(s), z_m(s))]ds \qquad (4.4.3) \\ &- \int_t^T 2p(s)(y_n(s) - y_m(s))(z_n(s) - z_m(s))dW(s). \end{aligned}$$

Taking expectations on both sides, by Lemma 3.4.1 (ii) and using the linear growth property of f_n and f_m, we obtain

$$\begin{aligned} &\mathbb{E}\left[\int_t^T p(s)|z_n(s) - z_m(s)|^2 ds\right] \\ &\le 2\mathbb{E}\left[\int_t^T p(s)(y_n(s) - y_m(s))[f_n(s, y_n(s), z_n(s)) - f_m(s, y_m(s), z_m(s))]ds\right] \\ &\le 2\mathbb{E}\Bigg[\int_t^T \sqrt{\alpha(s)p(s)}(y_n(s) - y_m(s))\frac{\sqrt{p(s)}}{\sqrt{\alpha(s)}}\times \\ &\qquad [f_n(s, y_n(s), z_n(s)) - f_m(s, y_m(s), z_m(s))]ds\Bigg] \end{aligned}$$

$$\leq 2\left(\mathbb{E}\left[\int_t^T \alpha(s)p(s)|y_n(s)-y_m(s)|^2 ds\right]\right)^{\frac{1}{2}}$$

$$\times\left(\mathbb{E}\left[\int_t^T \frac{p(s)}{\alpha(s)}|f_n(s,y_n(s),z_n(s))-f_m(s,y_m(s),z_m(s))|^2 ds\right]\right)^{\frac{1}{2}}$$

$$\leq 2\sqrt{2}\left(\mathbb{E}\left[\int_t^T \alpha(s)p(s)|y_n(s)-y_m(s)|^2 ds\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_t^T \frac{p(s)}{\alpha(s)}\left[|c_0(s)+c_1(s)|y_n(s)|\right.\right.\right.$$

$$\left.\left.\left.+c_2(s)|z_n(s)||^2+|c_0(s)+c_1(s)|y_m(s)|+c_2(s)|z_m(s)||^2\right]ds\right]\right)^{\frac{1}{2}}$$

$$\leq 2\sqrt{6}\left(\mathbb{E}\left[\int_t^T \alpha(s)p(s)|y_n(s)-y_m(s)|^2 ds\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_t^T \frac{p(s)}{\alpha(s)}\left[2c_0^2(s)+c_1^2(s)|y_n(s)|^2\right.\right.\right.$$

$$\left.\left.\left.+c_2^2(s)|z_n(s)|^2+c_1^2(s)|y_m(s)|^2+c_2^2(s)|z_m(s)|^2\right]ds\right]\right)^{\frac{1}{2}}$$

$$\leq 2\sqrt{6}\left(\mathbb{E}\left[\int_t^T \alpha(s)p(s)|y_n(s)-y_m(s)|^2 ds\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_t^T p(s)\left[\frac{2c_0^2(s)}{\alpha(s)}+|y_n(s)|^2\right.\right.\right.$$

$$\left.\left.\left.+|z_n(s)|^2+|y_m(s)|^2+|z_m(s)|^2\right]ds\right]\right)^{\frac{1}{2}}$$

$$\leq 2\sqrt{6}\,\widetilde{\kappa}\left(\mathbb{E}\left[\int_t^T \alpha(s)p(s)|y_n(s)-y_m(s)|^2 ds\right]\right)^{\frac{1}{2}},$$

where $\widetilde{\kappa} \equiv \left(4\kappa+\left\|c_0(\cdot)\alpha(\cdot)^{-\frac{1}{2}}\right\|\right)^{\frac{1}{2}}$. Therefore, this, together with the fact that $\{y_n(t)\}_{n\geq 1}$ pointwisely converges in $M^2(0,T;\mathbb{R})$, implies that $\{z_n(t)\}_{n\geq 1}$ is a Cauchy sequence in $M^2(0,T;\mathbb{R}^k)$ and then converges to $z(t)^*$ in the same space. □

Now we present the main result in this chapter.

Theorem 4.4.1. *(Existence) The BSDE* (4.2.1) *has an adapted solution* $(y(\cdot), z(\cdot)) \in H^2(0,T;\mathbb{R}) \times M^2(0,T;\mathbb{R}^k)$*, which is also a maximal solution, i.e. for any other solution* $(\bar{y}(\cdot), \bar{z}(\cdot))$ *of equation* (4.2.1)*, we have* $y(\cdot) \geq \bar{y}(\cdot)$*.*

Proof. Similar to previous deviations in Lemma 4.4.4, taking supremum over t for equation (4.4.3) and using the Burkholder-Davis-Gundy's inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} p(t)|y_n(t) - y_m(t)|^2\right] \\
\leq\,&\mathbb{E}\left[\sup_{t\in[0,T]} \int_t^T 2p(s)(y_n(s) - y_m(s))[f_n(s, y_n(s), z_n(s)) - f_m(s, y_m(s), z_m(s))]ds\right] \\
&+ \mathbb{E}\left[\sup_{t\in[0,T]} \int_t^T -2p(s)(y_n(s) - y_m(s))(z_n(s) - z_m(s))dW(s)\right] \\
\leq\,&2\,\mathbb{E}\left[\int_t^T p(s)|y_n(s) - y_m(s)||f_n(s, y_n(s), z_n(s)) - f_m(s, y_m(s), z_m(s))|ds\right] \\
&+ K\,\mathbb{E}\left[\int_0^T |\sqrt{p(s)}(y_n(s) - y_m(s))|^2|\sqrt{p(s)}(z_n(s) - z_m(s))|^2 ds\right]^{\frac{1}{2}} \\
\leq\,&2\left(\mathbb{E}\left[\int_t^T \alpha(s)p(s)|y_n(s) - y_m(s)|^2 ds\right]\right)^{\frac{1}{2}} \\
&\times\left(\mathbb{E}\left[\int_t^T \frac{p(s)}{\alpha(s)}|f_n(s, y_n(s), z_n(s)) - f_m(s, y_m(s), z_m(s))|^2 ds\right]\right)^{\frac{1}{2}} \\
&+ \frac{K}{2}\mathbb{E}\left[\sup_{t\in[0,T]} |\sqrt{p(s)}(y_n(s) - y_m(s))|^2 + \int_t^T |\sqrt{p(s)}(z_n(s) - z_m(s))|^2 ds\right] \\
\leq\,&2\sqrt{6}\,\widetilde{\kappa}\left(\mathbb{E}\left[\int_t^T \alpha(s)p(s)|y_n(s) - y_m(s)|^2 ds\right]\right)^{\frac{1}{2}}
\end{aligned}
$$

$$+\ \frac{K}{2}\mathbb{E}\left[\sup_{t\in[0,T]} p(s)|y_n(s)-y_m(s)|^2+\int_t^T p(s)|z_n(s)-z_m(s)|^2ds\right].$$

Hence, together with Lemma 4.4.4, for any $t\in[0,T]$, $\{y_n(t)\}_{n\geq 1}$ converges uniformly in $H^2(0,T;\mathbb{R})$ to $y(t)$ in the same space. Since $\{y_n(t)\}_{n\geq 1}$ is continuous, by the uniform convergence theorem (see Theorem 2.2.2), $y(t)$ is a continuous process.

Now we show that the sequence of processes $\{f_n(t,y_n(t),z_n(t))\}_{n\geq 1}$ converges to $\{f(t,y(t),z(t))\}$ in $M^1(0,T;\mathbb{R})$. Note that for any $\delta\geq 1$,

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T |f_n(s,y_n(s),z_n(s))-f(s,y(s),z(s))|ds\right]\\
=\ &\mathbb{E}\left[\int_t^T |f_n(s,y_n(s),z_n(s))-f(s,y(s),z(s))|\mathbb{1}_{\left\{\frac{c_1(s)|y_n(s)|+c_2(s)|z_n(s)|}{c_0^2(s)+c_1^2(s)+c_2^2(s)}\leq\delta\right\}}ds\right]\\
&+\mathbb{E}\left[\int_t^T |f_n(s,y_n(s),z_n(s))-f(s,y(s),z(s))|\mathbb{1}_{\left\{\frac{c_1(s)|y_n(s)|+c_2(s)|z_n(s)|}{c_0^2(s)+c_1^2(s)+c_2^2(s)}>\delta\right\}}ds\right]\\
\leq\ &\mathbb{E}\left[\int_t^T |f_n(s,y_n(s),z_n(s))-f(s,y_n(s),z_n(s))|\mathbb{1}_{\left\{\frac{c_1(s)|y_n(s)|+c_2(s)|z_n(s)|}{c_0^2(s)+c_1^2(s)+c_2^2(s)}\leq\delta\right\}}ds\right]\\
&+\mathbb{E}\left[\int_t^T |f(s,y_n(s),z_n(s))-f(s,y(s),z(s))|\mathbb{1}_{\left\{\frac{c_1(s)|y_n(s)|+c_2(s)|z_n(s)|}{c_0^2(s)+c_1^2(s)+c_2^2(s)}\leq\delta\right\}}ds\right]\\
&+\mathbb{E}\left[\int_t^T |f_n(s,y_n(s),z_n(s))-f(s,y(s),z(s))|\mathbb{1}_{\left\{\frac{c_1(s)|y_n(s)|+c_2(s)|z_n(s)|}{c_0^2(s)+c_1^2(s)+c_2^2(s)}>\delta\right\}}ds\right].
\end{aligned}\tag{4.4.4}$$

By (ii), (iii) and (iv) in Lemma 4.4.1, assumption (i), and then by the Dini's Theorem (see Theorem 2.2.5), as $n\to\infty$, we have

$$\sup_{\left\{\frac{c_1(s)|y_n(s)|+c_2(s)|z_n(s)|}{c_0^2(s)+c_1^2(s)+c_2^2(s)}\leq\delta\right\}} |f_n(s,y(s),z(s))-f(s,y(s),z(s))|\longrightarrow 0.$$

Therefore by the dominated convergence theorem, the first term in right hand side uniformly converges to 0. Due to assumption (i), at least along a subsequence, the second term in the right hand side converges to 0. For the final term in the right hand side, by Lemma 4.4.1 (iii), assumption (ii) and Lemma 4.4.3, together with the fact that $a\mathbb{1}_{\{X>a\}} < X$ for any nonnegative random variable X and $a > 0$, we obtain

$$\mathbb{E}\left[\int_t^T |f_n(s,y_n(s),z_n(s)) - f(s,y(s),z(s))|\mathbb{1}_{\left\{\frac{c_1(s)|y_n(s)|+c_2(s)|z_n(s)|}{c_0^2(s)+c_1^2(s)+c_2^2(s)}>\delta\right\}}ds\right]$$

$$\le \mathbb{E}\Bigg[\int_t^T [2c_0(s) + c_1(s)|y_n(s)| + c_2(s)|z_n(s)| + c_1(s)|y(s)| + c_2(s)|z(s)|]$$
$$\times \mathbb{1}_{\left\{\frac{c_1(s)|y_n(s)|+c_2(s)|z_n(s)|}{c_0^2(s)+c_1^2(s)+c_2^2(s)}>\delta\right\}}ds\Bigg]$$

$$\le \mathbb{E}\Bigg[\int_t^T [2c_0(s) + c_1(s)|y_n(s)| + c_2(s)|z_n(s)| + c_1(s)|y(s)| + c_2(s)|z(s)|]$$
$$\times \frac{c_1(s)|y_n(s)| + c_2(s)|z_n(s)|}{\delta[c_0^2(s) + c_1^2(s) + c_2^2(s)]}ds\Bigg]$$

$$= \frac{1}{\delta}\,\mathbb{E}\Bigg[\int_t^T \frac{1}{c_0^2(s) + c_1^2(s) + c_2^2(s)}\big[2c_0(s)c_1(s)|y_n(s)| + c_1^2(s)|y_n(s)|^2$$

$$+ c_1(s)c_2(s)|y_n(s)||z_n(s)| + c_1^2(s)|y_n(s)||y(s)| + c_1(s)c_2(s)|y_n(s)||z(s)|$$

$$+ 2c_0(s)c_2(s)|z_n(s)| + c_1(s)c_2(s)|y_n(s)||z_n(s)| + c_2^2(s)|z_n(s)|^2$$

$$+ c_1(s)c_2(s)|y(s)||z_n(s)| + c_2^2(s)|z_n(s)||z(s)|\big]ds\Bigg]$$

$$\le \frac{1}{\delta}\,\mathbb{E}\Bigg[\int_t^T \frac{1}{c_0^2(s) + c_1^2(s) + c_2^2(s)}\big[2c_0^2(s) + 4c_1^2(s)|y_n(s)|^2 + 4c_2^2(s)|z_n(s)|^2$$

$$+ c_1^2(s)|y(s)|^2 + c_2^2(s)|z(s)|^2\big]ds\Bigg]$$

$$\le \frac{1}{\delta}\,\mathbb{E}\Bigg[\int_t^T [2 + 4|y_n(s)|^2 + 4|z_n(s)|^2 + |y(s)|^2 + |z(s)|^2]ds\Bigg] \le \frac{\bar{\kappa}}{\delta},$$

where $\bar{\kappa}$ is a constant independent of n. Hence taking limits in the following equation

$$y_n(t) = \xi + \int_t^T f_n(s, y_n(s), z_n(s))ds - \int_t^T z_n(s)dW(s), \quad t \in [0,T], .$$

we deduce that $(y(t), z(t)) \in H^2(0,T;\mathbb{R}) \times M^2(0,T;\mathbb{R}^k)$ is an adapted solution of equation (4.2.1).

Furthermore, suppose that $(\bar{y}(t), \bar{z}(t)) \in H^2(0,T;\mathbb{R}) \times M^2(0,T;\mathbb{R}^k)$ is any solution of equation (4.2.1). By Theorem 3.5.1, we have $y_n(t) \geq \bar{y}(t)$ for any $n \geq 1$, and then $y(t) \geq \bar{y}(t)$. Hence $y(t)$ is a maximal solution of equation (4.2.1). □

4.5 Conclusion

Based on the results on nonlinear BSDEs in Chapter 3, we have considered BSDEs with a possibly unbounded and continuous generator. Under linear growth condition, we give sufficient conditions for the existence of a solution pair (which is also a maximal solution) to this class of BSDEs. These are novel conditions as compared to existing ones. In addition, as Chapter 3 shown, we could also consider the case with p_2 (see page 29) for this class of BSDEs, which is expected to be solved in the final version of preprint paper [25].

Chapter 5

Linear Backward Stochastic Differential Equations with Unbounded Coefficients

5.1 Abstract

We consider the problem of solvability for linear backward stochastic differential equations with unbounded coefficients. New and weaker sufficient conditions for the existence of a unique solution pair are given. It is shown that certain exponential processes have stronger integrability in this case. As applications, we solve the problems of completeness in a market with a possibly unbounded coefficients and optimal investment with power utility in a market with unbounded coefficients. This work is based on a printed paper [26].

5.2 Introduction

We consider the following linear backward stochastic differential equation (LBSDE):

$$\begin{cases} dY(t) = [r(t)Y(t) + \theta'(t)Z(t)]dt + Z'(t)dW(t), \quad t \in [0,T], \\ \\ Y(T) = \xi, \quad a.s., \end{cases} \tag{5.2.1}$$

under the same mathematical setting as those in Chapter 3 and 4, whereas $r(\cdot)$ and $\theta(\cdot) \equiv [\theta_1(\cdot), ..., \theta_d(\cdot)]'$ are given $\mathcal{F}_t$-adapted processes. The problem *solvability* for (5.2.1) is the problem of existence of the pair of adapted processes $Y(\cdot)$ and $Z(\cdot) \equiv [Z_1(\cdot), ..., Z_d(\cdot)]'$ such that (5.2.1) holds. The pair of processes $(Y(\cdot), Z(\cdot))$ in then called the *solution pair*.

The BSDEs with a possibly *unbounded* generator are important in mathematical finance. When the interest rate is modeled as a solution to a stochastic differential equation (e.g. [7], [16], [79]), which in general are unbounded processes, it gives rise to various problems in a market with unbounded coefficients. One such a problem is the *market completeness* (see, e.g. [7]). This has motivated [18], [24], [39], [78], to consider the problem of solvability of BSDEs with unbounded coefficients. In [24], [39], general BSDEs are considered, and solution pairs are shown to exist in certain *weighted* spaces. Different from these papers, Yong [78] considers linear BSDEs with unbounded coefficients, under different assumptions on the terminal value ξ. His approach is based on establishing the integrability of exponential processes and the reduction of the linear BSDE to a more basic form. The solution pairs in this case belong to *non-weighted* spaces. As an application, he resolves the basic problem of market completeness under various assumptions on ξ.

In this chapter, we show the existence of a unique solution pair of (5.2.1) under weaker assumptions on ξ as compared to Yong [78]. Moreover, we obtain stronger integrability of the exponential process under weaker assumptions than that of [78]. As applications, we extend the results of [78] on market completeness and solve the optimal investment with power utility in a market with unbounded coefficients.

In order to easily compare our results with those in [78], we keep the structure of this chapter similar to that of [78].

5.3 Preliminaries

5.3.1 Notations for Some Spaces

- H is any finite dimensional Euclidian space with norm $|\cdot|$.

- $\mathcal{L}^0_{\mathcal{F}_T}(\Omega; H)$ is the set of all $\mathcal{F}_T$-measurable H-valued random variables.

- $\mathcal{L}^p_{\mathcal{F}_T}(\Omega; H)$ is the set of all random variables $\xi \in \mathcal{L}^0_{\mathcal{F}_T}(\Omega; H)$ which for some $p \in (0, \infty)$ satisfy the condition $\mathbb{E}|\xi|^p < \infty$.

- $\mathcal{L}^0_{\mathcal{F}}(0, T; H)$ is the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $\psi : [0, T] \to H$.

- $\mathcal{L}^p_{\mathcal{F}}(\Omega; \mathcal{L}^q(0, T; H))$ is the set of all processes $\psi(\cdot) \in \mathcal{L}^0_{\mathcal{F}}(0, T; H)$ which for some

$p, q \in (0, \infty)$ satisfy the condition

$$\mathbb{E}\left[\int_0^T |\psi(t)|^q dt\right]^{p/q} < \infty.$$

• $\mathcal{L}^p(0, T; H) \equiv \mathcal{L}^p_{\mathcal{F}}(\Omega; \mathcal{L}^p(0, T; H))$, for some $p \in (0, \infty)$.

• $\mathcal{L}^0_{\mathcal{F}}(\Omega; C([0, T]; H))$ is the set of all processes $\psi(\cdot) \in \mathcal{L}^0_{\mathcal{F}}(0, T; H)$ with almost all paths continuous.

• $\mathcal{L}^p_{\mathcal{F}}(\Omega; C([0, T]; H))$ is the set of all processes $\psi(\cdot) \in \mathcal{L}^0_{\mathcal{F}}(\Omega; C([0, T]; H))$ which for some $p > 0$ satisfy the condition

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\psi(t)|^p\right] < \infty.$$

• $\mathcal{L}^q_{\mathcal{F}}(0, T; \mathcal{L}^p(\Omega; H))$ is the set of all processes $\psi(\cdot) \in \mathcal{L}^0_{\mathcal{F}}(0, T; H)$ which for some $p, q \in (0, \infty)$ satisfy the condition

$$\int_0^T [\mathbb{E}|\psi(t)|^p]^{q/p}\, dt < \infty.$$

• $\mathcal{L}^{q+}_{\mathcal{F}_T}(\Omega; H) = \bigcup_{p\in(q,\infty]} \mathcal{L}^p_{\mathcal{F}_T}(\Omega; H)$ for some $q > 0$.

• $\mathcal{L}^{q-}_{\mathcal{F}_T}(\Omega; H) = \bigcap_{p\in(0,q)} \mathcal{L}^p_{\mathcal{F}_T}(\Omega; H)$ for some $q > 0$.

• $\mathcal{L}^{p\pm}_{\mathcal{F}}(\Omega; \mathcal{L}^{q\pm}(0, T; H))$ and $\mathcal{L}^{p\pm}_{\mathcal{F}}(\Omega; C([0, T]; H))$ are defined in a similar way to the above.

5.3.2 BSDE Reduction

Here we describe the approach of Yong [78] to solving the BSDE (5.2.1), which is based on reducing such an equation to a more basic BSDE.

Suppose that $(Y(\cdot), Z(\cdot))$ is an adapted solution of (5.2.1). Let $M(\cdot)$ be the solution of following:

$$\begin{cases} dM(t) = & -r(t)M(t)dt - \theta(t)M(t)dW(t), \quad t \in [0,T], \\ \\ M(0) = & 1, \end{cases} \tag{5.3.1}$$

which is given by

$$M(t) \equiv M(t\,; r(\cdot), \theta(\cdot)) = e^{-\int_0^t \left[r(s)+\frac{1}{2}\theta^2(s)\right]ds + \int_0^t \theta(s)dW(s)}, \quad t \in [0,T]. \tag{5.3.2}$$

The process $M(\cdot)$ is called the *exponential process* in [78], and it is a generalisation of the so-called *exponential supermartingale* (see [36]):

$$M(t\,; 0, \theta(\cdot)) = e^{-\frac{1}{2}\int_0^t \theta^2(s)ds + \int_0^t \theta(s)dW(s)},$$

for which it holds that

$$\sup_{t\in[0,T]} \mathbb{E}[M(t\,; 0, \theta(\cdot))] \leq 1. \tag{5.3.3}$$

Applying Itô's formula to $M(\cdot)Y(\cdot)$, we have

$$d[M(t)Y(t)] = M(t)[Z(t) - Y(t)\theta(t)]dW(t), \quad t \in [0,T].$$

If we denote

$$\begin{cases} \widetilde{Y}(t) = & M(t)Y(t), \quad t \in [0,T], \\ \widetilde{Z}(t) = & M(t)[Z(t) - Y(t)\theta(t)], \end{cases} \tag{5.3.4}$$

then

$$\begin{cases} d\widetilde{Y}(t) = & \widetilde{Z}(t)dW(t), \quad t \in [0,T], \\ \widetilde{Y}(T) = & M(T)\xi \equiv \tilde{\xi}. \end{cases} \tag{5.3.5}$$

Note that (5.3.5) admits a unique adapted solution $(\widetilde{Y}(\cdot), \widetilde{Z}(\cdot))$ if $\tilde{\xi}$ has some integrability. Then we define

$$\begin{cases} Y(t) = & M(t)^{-1}\widetilde{Y}(t), \quad t \in [0,T], \\ Z(t) = & M(t)^{-1}[\widetilde{Z}(t) + \widetilde{Y}(t)\theta(t)]. \end{cases} \tag{5.3.6}$$

Accordingly, we expect that $(Y(\cdot), Z(\cdot))$ given by (5.3.6) is an adapted solution to (5.2.1). In the following sections, we will investigate some estimate of exponential processes $M(T)$ and $M(\cdot)^{-1}$ and the solvability of (5.3.5), i.e. (5.2.1) under various conditions on processes $r(\cdot)$, $\theta(\cdot)$ and ξ.

5.3.3 Assumptions

In Yong [78], the integrability of the exponential process is considered. Based on that and the above reduction method, the solvability of the linear BSDE (5.2.1) is also addressed in [78]. In this chapter we develop analogs of all results in Yong [78] under weaker assumptions. We compare our results with those of [78] throughout the paper. Here we make a comparison of the main assumptions.

One of the main results in [78] is Theorem 4.1 which gives sufficient conditions for the solvability of (5.2.1). It makes the following assumptions on the given data, i.e. ξ, $r(\cdot)$ and $\theta(\cdot)$.

(Y1) ξ is any random variable of $\mathcal{L}^{p(Y)}_{\mathcal{F}_T}(\Omega;\mathbb{R})$, for some constant $p(Y) > 1$.

(Y2) The process $r(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega;\mathcal{L}^1(0,T;\mathbb{R}))$ is such that

$$\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\alpha\int_0^t r(s)ds}\right] < \infty, \tag{5.3.7}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right] < \infty, \tag{5.3.8}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha_0\int_0^t r(s)ds}\right] < \infty, \tag{5.3.9}$$

for some constants $\alpha > 0$ and $\alpha_0 > 0$.

(Y3) The process $\theta(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega;\mathcal{L}^2(0,T;\mathbb{R}^d))$ is such that

$$\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2 ds}\right] < \infty, \tag{5.3.10}$$

for some constant $\beta > 1$.

The value of $p(Y)$ is determined by α, α_0, and β. There is a slip in the statement of Theorem 4.1 of [78] where assumption (5.3.8) does not appear. However, this theorem makes use of the conclusions of Theorem 3.5 of [78], which in turn is based on Theorem 3.4 (i) of [78], which does make assumption (5.3.8).

Our analog to Theorem 4.1 of [78] is Theorem 5.5.1, which makes the following assumptions.

(A1) ξ is any random variable of $\mathcal{L}^{p}_{\mathcal{F}_T}(\Omega;\mathbb{R})$, for some constant $p > 1$.

(A2) The process $r(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^1(0,T;\mathbb{R}))$ is such that

$$\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\alpha_0 \int_0^t r(s)ds}\right] < \infty, \tag{5.3.11}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha \int_0^t r(s)ds}\right] < \infty, \tag{5.3.12}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha \int_0^t [r(s)+|\theta(s)|^2]ds}\right] < \infty, \tag{5.3.13}$$

for some constants $\alpha > 0$ and $\alpha_0 > 0$.

(A3) The process $\theta(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^2(0,T;\mathbb{R}^d))$ is such that

$$\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\frac{\beta}{2} \int_0^t \theta'(s)dW(s)}\right] < \infty, \tag{5.3.14}$$

$$\sup_{t\in[0,T]} \mathbb{E}\left[e^{\frac{\beta_0}{2} \int_0^t \theta'(s)dW(s)}\right] < \infty, \tag{5.3.15}$$

$$\theta(\cdot) \in \mathcal{L}^{2+}_{\mathcal{F}}(\Omega; \mathcal{L}^2([0,T];\mathbb{R}^d), \tag{5.3.16}$$

for some constants $\beta > 1$ and $\beta_0 > 1$.

Let us compare these two sets of assumptions. The value of the coefficient p is determined by the values of $\alpha, \alpha_0, \beta, \beta_0$, and it is shown in the proof of Theorem 5.5.1 that it is smaller than $p(Y)$, i.e. our results apply to a wider class of BSDEs than those of Yong [78]. Next, the assumption (5.3.10), which is a 'Novikov' type condition, is always stronger than our assumptions (5.3.14), (5.3.15), which are 'Kazamaki' type conditions. In fact, this particular weakening of the assumptions of Yong was the main motivation for the current chapter. It remains to compare assumptions on the process $r(\cdot)$. Assumptions (5.3.7) and (5.3.8) are the same as assumptions (5.3.11) and (5.3.12). The assumption (5.3.9) is more difficult to compare with (5.3.13), but this can be done in special cases. For example, if $r(t) < 0$, then (5.3.13) is implied (5.3.10).

5.4 Integrability of Exponential Processes

Now we discuss some estimates of exponential process $M(t\,; r(\cdot), \theta(\cdot))$, under various integrability conditions on $r(\cdot)$ and $\theta(\cdot)$. This will lead to the solvability of linear BSDEs with possibly unbounded coefficients. Before studying the integrability of exponential process, we first present the following lemmas which are essential to obtain the integrability of exponential process. Their proofs are given in the Appendix.

Lemma 5.4.1. *Suppose that α_0, β, p, q and γ are positive constants satisfying the following system of equations and inequalities:*

$$\begin{cases} \alpha_0 = \frac{pq\gamma}{(\gamma-1)(q-1)}, \\ \\ \beta = \frac{2q\gamma}{\gamma-1}(\frac{\sqrt{p\gamma}}{\gamma} + p), \\ \\ p > 0, q > 1, \gamma > 1. \end{cases} \tag{5.4.1}$$

Then $p^*(\gamma) = \frac{\alpha_0\beta}{\beta+2\alpha_0}$ is the global maximum of $p(\gamma)$.

Lemma 5.4.2. *Suppose that β , p and γ are positive constants satisfying the following system of equations and inequalities:*

$$\begin{cases} \frac{\beta}{2} = \frac{\gamma}{\gamma-1}(\frac{\sqrt{p\gamma}}{\gamma} + p), \\ \\ p > 0, \gamma > 1. \end{cases} \tag{5.4.2}$$

Then when $\gamma = 2\beta - 1$, $p^*(\gamma) = \frac{\beta^2}{2\beta-1}$ is the global maximum of $p(\gamma)$.

Now we present the first result on the integrability of exponential process $M(t\,; r(\cdot), \theta(\cdot))$.

Theorem 5.4.1 (Upper bound of $\sup_{t\in[0,T]} \mathbb{E}\,(M(t\,; r(\cdot), \theta(\cdot))^p)$). *Suppose that $r(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^1(0, T; \mathbb{R}))$ and $\theta(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^2(0, T; \mathbb{R}^d))$.*

(i) Suppose that conditions (5.3.11) *and* (5.3.14) *hold. Then*

$$\sup_{t\in[0,T]} \mathbb{E}\left[M(t\,;r(\cdot),\theta(\cdot))^{\frac{\alpha_0\beta}{\beta+2\alpha_0}}\right]$$
$$\leq \left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\alpha_0\int_0^t r(s)ds}\right]\right)^{\frac{\beta}{\beta+2\alpha_0}} \left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t \theta'(s)dW(s)}\right]\right)^{\frac{2\alpha_0}{\beta+2\alpha_0}}.$$

(ii) Let

$$\inf_{t\in[0,T]}\int_0^t r(s)ds \geq -\kappa, \quad a.s., \tag{5.4.3}$$

for some $\kappa\in\mathbb{R}$ *and* (5.3.14) *holds for some* $\beta>1$*, then*

$$\sup_{t\in[0,T]} \mathbb{E}\left[M(t\,;r(\cdot),\theta(\cdot))^{\frac{\beta^2}{2\beta-1}}\right] \leq e^{\frac{\kappa\beta^2}{2\beta-1}}\left\{\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{2\beta-2}{2\beta-1}}. \tag{5.4.4}$$

Proof. (i) Let $p>0$, $q>1$, $\gamma>1$. By the Hölder's inequality (see Section 2.5) and (5.3.3), we obtain

$$\mathbb{E}[M(t\,;r(\cdot),\theta(\cdot))^p]$$

$$=\mathbb{E}\left[e^{-p\int_0^t\left[r(s)+\frac{1}{2}\theta^2(s)\right]ds-p\int_0^t\theta'(s)dW(s)}\right]$$

$$=\mathbb{E}\left[e^{-p\int_0^t r(s)ds-(\frac{\sqrt{p\gamma}}{\gamma}+p)\int_0^t\theta'(s)dW(s)}\cdot e^{\frac{1}{\gamma}\int_0^t\frac{1}{2}(\sqrt{p\gamma}\theta(s))^2ds-\frac{1}{\gamma}\int_0^t\sqrt{p\gamma}(-\theta'(s))dW(s)}\right]$$

$$=\mathbb{E}\left[e^{-p\int_0^t r(s)ds-(\frac{\sqrt{p\gamma}}{\gamma}+p)\int_0^t\theta'(s)dW(s)}\cdot M(t\,;0,-\sqrt{p\gamma}\theta(\cdot))^{\frac{1}{\gamma}}\right]$$

$$\leq\left\{\mathbb{E}\left[e^{-\frac{p\gamma}{\gamma-1}\int_0^t r(s)ds-\frac{\gamma}{\gamma-1}(\frac{\sqrt{p\gamma}}{\gamma}+p)\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{\gamma-1}{\gamma}}\left\{\mathbb{E}\left[M(t\,;0,-\sqrt{p\gamma}\theta(\cdot))^{\frac{1}{\gamma}\cdot\gamma}\right]\right\}^{\frac{1}{\gamma}}$$

$$\leq\left\{\mathbb{E}\left[e^{-\frac{p\gamma}{\gamma-1}\frac{q}{q-1}\int_0^t r(s)ds}\right]\right\}^{\frac{(q-1)(\gamma-1)}{q\gamma}}\left\{\mathbb{E}\left[e^{-\frac{q\gamma}{\gamma-1}(\frac{\sqrt{p\gamma}}{\gamma}+p)\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{\gamma-1}{q\gamma}}$$

$$=\left\{\mathbb{E}\left[e^{-\alpha_0\int_0^t r(s)ds}\right]\right\}^{\frac{(q-1)(\gamma-1)}{q\gamma}}\cdot\left\{\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{\gamma-1}{q\gamma}},$$

where $\alpha_0 = \frac{pq\gamma}{(\gamma-1)(q-1)}$ and $\beta = \frac{2q\gamma}{\gamma-1}(\frac{\sqrt{p\gamma}}{\gamma} + p)$.

Therefore by Lemma 5.4.1, when $\gamma \to \infty$, $p = \frac{\alpha_0\beta}{\beta+2\alpha_0}$ is the global maximum of $p(\gamma)$. Furthermore we have $\frac{(q-1)(\gamma-1)}{q\gamma} = \frac{\beta}{\beta+2\alpha_0}$ and $\frac{\gamma-1}{q\gamma} = \frac{2\alpha_0}{\beta+2\alpha_0}$. Thus the result follows.

(ii) From (i) and (5.4.3), we obtain

$$\begin{aligned}
&\mathbb{E}[M(t\,; r(\cdot), \theta(\cdot))^p] \\
\leq &\left\{\mathbb{E}\left[e^{-\frac{p\gamma}{\gamma-1}\int_0^t r(s)ds - \frac{\gamma}{\gamma-1}(\frac{\sqrt{p\gamma}}{\gamma}+p)\int_0^t \theta'(s)dW(s)}\right]\right\}^{\frac{\gamma-1}{\gamma}} \\
\leq &e^{\kappa p}\left\{\mathbb{E}\left[e^{-\frac{\gamma}{\gamma-1}(\frac{\sqrt{p\gamma}}{\gamma}+p)\int_0^t \theta'(s)dW(s)}\right]\right\}^{\frac{\gamma-1}{\gamma}}.
\end{aligned}$$

By Lemma 5.4.2, we know when $\gamma = 2\beta - 1$, $p = \frac{\beta^2}{2\beta-1}$ is the global maximum of $p(\gamma)$ and thus (5.4.4) follows. □

Now let us compare our result with Theorem 3.2 of [78]. From now on we denote p in this chapter and in [78] respectively by p and $p(Y)$. Recall that $p(Y) = \frac{\alpha_0\beta}{\beta+\alpha_0(2\sqrt{\beta}-1)}$ in Theorem 3.2 (ii) of [78] (here we assume $\alpha = \alpha_0$). Then it is easy to see when $\beta \in (1, \frac{9}{4})$, $p < p(Y)$; when $\beta = \frac{9}{4}$, $p = p(Y)$; when $\beta > \frac{9}{4}$, $p > p(Y)$. Hence we conclude that under weaker condition, stronger integrability is obtained for any $\beta > \frac{9}{4}$. Similarly, note that $p(Y) = \frac{\beta}{2\sqrt{\beta}-1}$ in Theorem 3.2 (iii) of [78]. When $\beta > 1$, we always have $p > p(Y)$.

Now we combine the result in [78] with ours as follows.

Theorem 5.4.2. *Let $r(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^1(0, T; \mathbb{R}))$ and $\theta(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^2(0, T; \mathbb{R}^d))$.*

(i) Suppose that conditions (5.3.10) *and* (5.3.11) *hold for some $\alpha_0 > 0$ and $\beta > 1$. Then for $\beta \in \left(1, \frac{9}{4}\right)$, we have*

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left[M(t\,; r(\cdot), \theta(\cdot))^{\frac{\alpha_0\beta}{\beta+\alpha_0(2\sqrt{\beta}-1)}}\right] \\
\leq &\left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\alpha_0\int_0^t r(s)ds}\right]\right)^{\frac{\beta}{\beta+\alpha_0(2\sqrt{\beta}-1)}} \left(\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^t |\theta(s)|^2 ds}\right]\right)^{\frac{\alpha_0(\sqrt{\beta}-1)}{\beta+\alpha_0(2\sqrt{\beta}-1)}},
\end{aligned}$$

and for $\beta \in \left[\frac{9}{4}, \infty\right)$, *we have*

$$\sup_{t\in[0,T]} \mathbb{E}\left[M(t\,;r(\cdot),\theta(\cdot))^{\frac{\alpha_0\beta}{\beta+2\alpha_0}}\right]$$
$$\leq \left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\alpha_0\int_0^t r(s)ds}\right]\right)^{\frac{\beta}{\beta+2\alpha_0}} \left(\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^t |\theta(s)|^2 ds}\right]\right)^{\frac{2\alpha_0}{\beta+2\alpha_0}}.$$

(ii) Let

$$\inf_{t\in[0,T]}\int_0^t r(s)ds \geq -\kappa, \quad a.s.,$$

for some $\kappa \in \mathbb{R}$ *and* (5.3.10) *holds for some* $\beta > 1$, *then*

$$\sup_{t\in[0,T]} \mathbb{E}\left[M(t\,;r(\cdot),\theta(\cdot))^{\frac{\beta^2}{2\beta-1}}\right] \leq e^{\frac{\kappa\beta^2}{2\beta-1}} \left\{\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^t |\theta(s)|^2 ds}\right]\right\}^{\frac{2\beta-2}{2\beta-1}}.$$

Proof. By [42], we know (5.3.10) implies (5.3.14). Thus the result follows.

□

Theorem 5.4.3 (Upper bound of $\mathbb{E}(\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^p)$). *Suppose that* $r(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega;\mathcal{L}^1(0,T;\mathbb{R}))$ *and* $\theta(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega;\mathcal{L}^2(0,T;\mathbb{R}^d))$.

(i) Suppose that condition (5.3.12) *holds for some* $\alpha > 0$ *and* (5.3.14) *holds for some* $\beta > 1$. *Then*

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{\frac{\alpha\beta^2}{\beta^2+2\alpha\beta-\alpha}}\right]$$
$$\leq C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right]\right)^{\frac{\beta^2}{\beta^2+2\alpha\beta-\alpha}} \left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t \theta'(s)dW(s)}\right]\right)^{\frac{2\alpha(\beta-1)}{\beta^2+2\alpha\beta-\alpha}},$$

where $C = \left(\frac{\alpha\beta^2}{\alpha\beta^2-2\beta^2-2\alpha\beta+\alpha}\right)^{\frac{\alpha\beta^2}{\beta^2+2\alpha\beta-\alpha}}$.

(ii) Suppose that (5.3.12) *holds for some* $\alpha > p > 0$ *and* (5.3.14) *holds for some* $\beta \in (0,1]$. *Then for any* $p \in \left(\frac{\alpha\beta}{2\alpha+\beta}, \frac{\alpha\beta}{\alpha+\beta}\right)$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^p\right]$$
$$\leq C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right]\right)^{\frac{p}{\alpha}} \left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t \theta'(s)dW(s)}\right]\right)^{\frac{2\alpha\beta-2p(\alpha+\beta)}{\alpha\beta}},$$

where $C = \left(\frac{\alpha^2 p^2}{\alpha^2 p^2 - \beta(\alpha-p)[(\beta+2\alpha)p-\alpha\beta]}\right)^{\frac{\alpha p^2}{\beta[(2\alpha+\beta)p-\alpha\beta]}}$.

(iii) (a) Suppose that (5.3.14) *holds for some* $\beta > 1$ *and* (5.4.3) *holds for some* $\kappa \in \mathbb{R}$ *, then*

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{\frac{\beta^2}{2\beta-1}}\right] \le C\left\{\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t \theta'(s)dW(s)}\right]\right\}^{\frac{2\beta-2}{2\beta-1}},$$

where $C = e^{\frac{\kappa\beta^2}{2\beta-1}}\left[\frac{\beta^2}{(\beta-1)^2}\right]^{\frac{\beta^2}{2\beta-1}}$.

(b) In addition, if (5.3.14) *holds for some* $\beta \in (0,1]$*, then for any* $p \in (\frac{\beta}{2},\beta)$

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^p\right] \le C\left\{\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t \theta'(s)dW(s)}\right]\right\}^{\frac{2(\beta-p)}{\beta}},$$

where $C = e^{\kappa p}\left[\frac{p^2}{p^2-\beta(2p-\beta)}\right]^{\frac{p^2}{\beta(2p-\beta)}}$.

Proof. (i) Note that $M(t\,;0,\theta(\cdot))$ is a martingale when (5.3.14) holds for $\beta > 1$. Let $\alpha > p$. By the Hölder's inequality and the Doob's martingale inequality (see, for example, Theorem 2.4.7), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^p\right]\\
=&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,\theta(\cdot))^p\cdot e^{-p\int_0^t r(s)ds}\right]\\
\le&\left\{\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,\theta(\cdot))^{\frac{p\alpha}{\alpha-p}}\right]\right\}^{\frac{\alpha-p}{\alpha}}\left\{\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right]\right\}^{\frac{p}{\alpha}}\\
\le&\left(\frac{\frac{p\alpha}{\alpha-p}}{\frac{p\alpha}{\alpha-p}-1}\right)^{\frac{p\alpha}{\alpha-p}\cdot\frac{\alpha-p}{\alpha}}\left\{\mathbb{E}\left[M(T\,;0,\theta(\cdot))^{\frac{p\alpha}{\alpha-p}}\right]\right\}^{\frac{\alpha-p}{\alpha}}\\
&\cdot\left\{\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right]\right\}^{\frac{p}{\alpha}}
\end{aligned}$$

From (5.4.4), when $r(\cdot)=0$, we obtain

$$\sup_{t\in[0,T]}\mathbb{E}\left[M(t\,;0,\theta(\cdot))^{\frac{\beta^2}{2\beta-1}}\right] \le \sup_{t\in[0,T]}\left\{\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t \theta'(s)dW(s)}\right]\right\}^{\frac{2\beta-2}{2\beta-1}}. \tag{5.4.5}$$

Then

$$\frac{p\alpha}{\alpha-p}=\frac{\beta^2}{2\beta-1}\implies p=\frac{\alpha\beta^2}{\beta^2+2\alpha\beta-\alpha}\quad and\quad \frac{p}{\alpha}=\frac{\beta^2}{\beta^2+2\alpha\beta-\alpha}.$$

and

$$\frac{2\beta-2}{2\beta-1}\cdot\frac{\alpha-p}{\alpha}=\frac{2\beta-2}{2\beta-1}\cdot\frac{(2\beta-1)p}{\beta^2}=\frac{2\alpha(\beta-1)}{\beta^2+2\alpha\beta-\alpha}.$$

Thus the result follows.

(ii) Let us denote $\beta'=\frac{\beta}{p\gamma}>1$. By (5.4.5), we have

$$\frac{\alpha}{\gamma(\alpha-p)}=\frac{\beta'^2}{2\beta'-1}\implies\gamma=\frac{\beta[(2\alpha+\beta)p-\alpha\beta]}{\alpha p^2}>0.$$

So it is necessary to have $(2\alpha+\beta)p-\alpha\beta>0$, i.e. $p>\frac{\alpha\beta}{2\alpha+\beta}$. On the other hand, again by (5.4.5), we have

$$\frac{2\beta'-2}{2\beta'-1}\cdot\frac{\alpha-p}{\alpha}=\frac{2\alpha\beta-2p(\alpha+\beta)}{\alpha+\beta}>0,\quad\implies\quad p<\frac{\alpha\beta}{\alpha+\beta}.$$

Hence we have $p\in(\frac{\alpha\beta}{2\alpha+\beta},\frac{\alpha\beta}{\alpha+\beta})$ and

$$0<p\gamma=\frac{\beta[(2\alpha+\beta)p-\alpha\beta]}{\alpha p}<\beta\leq 1. \tag{5.4.6}$$

Note that

$$\sup_{t\in[0,T]}\mathbb{E}\left[e^{-\frac{\beta'}{2}\int_0^t p\gamma\theta'(s)dW(s)}\right]=\sup_{t\in[0,T]}\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t\theta'(s)dW(s)}\right]<\infty.$$

By [42], we know $M(t;0,p\gamma\theta(\cdot))$ is a martingale if $\beta'>1$. By (5.4.5) and (5.4.6), the Hölder's inequality and the Doob's martingale inequality, we obtain

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}M(t;r(\cdot),\theta(\cdot))^p\right]\\ =&\mathbb{E}\left[\sup_{t\in[0,T]}\left(e^{-\int_0^t\frac{1}{2}p^2\gamma^2|\theta(s)|^2ds}\right)^{\frac{1}{p\gamma}\frac{1}{\gamma}}\cdot e^{-p\int_0^t r(s)ds}\cdot\left(e^{-\int_0^t p\gamma\theta'(s)dW(s)}\right)^{\frac{1}{\gamma}}\right]\\ \leq&\mathbb{E}\left[\sup_{t\in[0,T]}\left(e^{-\int_0^t\frac{1}{2}p^2\gamma^2|\theta(s)|^2ds}\right)^{\frac{1}{\gamma}}\cdot e^{-p\int_0^t r(s)ds}\cdot\left(e^{-\int_0^t p\gamma\theta'(s)dW(s)}\right)^{\frac{1}{\gamma}}\right]\end{aligned}$$

$$= \mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,p\gamma\theta(\cdot))^{\frac{1}{\gamma}} \cdot e^{-p\int_0^t r(s)ds}\right]$$

$$\leq \left\{\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,p\gamma\theta(\cdot))^{\frac{\alpha}{\gamma(\alpha-p)}}\right]\right\}^{\frac{\alpha-p}{\alpha}} \left\{\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right]\right\}^{\frac{p}{\alpha}}$$

$$\leq \left[\frac{\frac{\alpha}{\gamma(\alpha-p)}}{\frac{\alpha}{\gamma(\alpha-p)}-1}\right]^{\frac{\alpha}{\gamma(\alpha-p)}\cdot\frac{\alpha-p}{\alpha}} \left\{\mathbb{E}\left[M(T\,;0,p\gamma\theta(\cdot))^{\frac{\alpha}{\gamma(\alpha-p)}}\right]\right\}^{\frac{\alpha-p}{\alpha}} \left\{\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right]\right\}^{\frac{p}{\alpha}}$$

$$\leq C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right]\right)^{\frac{p}{\alpha}} \left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\frac{\beta'}{2}\int_0^t p\gamma\theta'(s)dW(s)}\right]\right)^{\frac{2\alpha\beta-2p(\alpha+\beta)}{\alpha\beta}}$$

$$= C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right]\right)^{\frac{p}{\alpha}} \left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t \theta'(s)dW(s)}\right]\right)^{\frac{2\alpha\beta-2p(\alpha+\beta)}{\alpha\beta}}.$$

Thus the result follows.

(iii) (a) Note that $M(t\,;0,\theta(\cdot))$ is a martingale if $\beta > 1$. By (5.4.4) and (5.4.5) and the Doob's martingale inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^p\right] \\
&= \mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,\theta(\cdot))^p \cdot e^{-p\int_0^t r(s)ds}\right] \\
&\leq e^{\kappa p}\,\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,\theta(\cdot))^p\right] \\
&\leq e^{\kappa p}\left(\frac{p}{p-1}\right)^p \mathbb{E}[M(T\,;0,\theta(\cdot))^p] \\
&\leq e^{\frac{\kappa\beta^2}{2\beta-1}}\left[\frac{\beta^2}{(\beta-1)^2}\right]^{\frac{\beta^2}{2\beta-1}} \left\{\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t \theta'(s)dW(s)}\right]\right\}^{\frac{2\beta-2}{2\beta-1}}.
\end{aligned}$$

Thus the result follows.

(b) Let us denote $\beta' = \frac{\beta}{p\gamma} > 1$. By (5.4.5), we have

$$\frac{1}{\gamma} = \frac{\beta'^2}{2\beta'-1} \implies \gamma = \frac{\beta(2p-\beta)}{p^2} > 0.$$

So it is necessary to have $2p-\beta>0$, i.e. $p>\frac{\beta}{2}$. On the other hand, again by (5.4.5), we have

$$\frac{2\beta'-2}{2\beta'-1}=\frac{2(\beta-p)}{\beta}>0, \quad \Longrightarrow \quad p<\beta.$$

Hence we have $p\in\left(\frac{\beta}{2},\beta\right)$ and

$$0<\gamma=\frac{\beta(2p-\beta)}{p^2}<1.$$

So

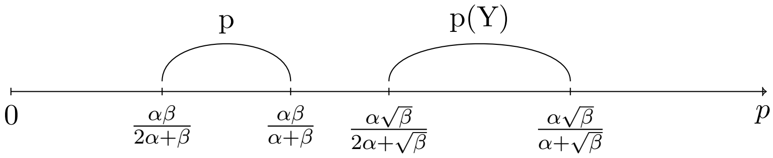
$$0<p\gamma=\frac{\beta(2p-\beta)}{p}<\beta\leq 1.$$

Again $M(t\,;0,p\gamma\theta(\cdot))$ is a martingale if $\beta'>1$. By the Doob's martingale inequality, we have

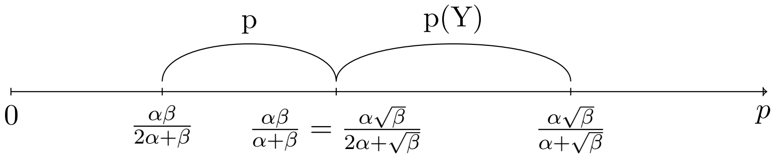
$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^p\right]\\
=&\,\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,p\gamma\theta(\cdot))^p\cdot e^{-p\int_0^t r(s)ds}\right]\\
\leq&\, e^{\kappa p}\,\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,p\gamma\theta(\cdot))^p\right]\\
\leq&\, e^{\kappa p}\left(\frac{\frac{1}{\gamma}}{\frac{1}{\gamma}-1}\right)^{\frac{1}{\gamma}}\mathbb{E}\left[M(T\,;0,p\gamma\theta(\cdot))^{\frac{1}{\gamma}}\right]\\
\leq&\, e^{\kappa p}\left[\frac{p^2}{p^2-\beta(2p-\beta)}\right]^{\frac{p^2}{\beta(2p-\beta)}}\left\{\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{2(\beta-p)}{\beta}}.
\end{aligned}$$

Thus the result follows. □

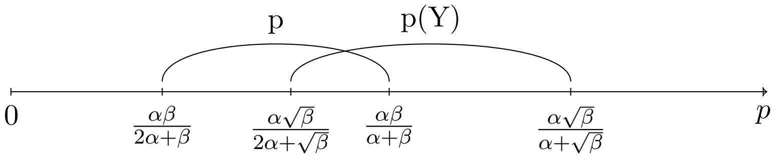
Now let us compare our result with Theorem 3.4 of [78]. Recall that $p(Y)=\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}$ in Theorem 3.4 (i) of [78]. It is easy to see that, in (i), for any $\beta>1$, $p=\frac{\alpha\beta^2}{\beta^2+2\alpha\beta-\alpha}>p(Y)=\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}$; Similarly, in (iii)(a), for any $\beta>1$, $p=\frac{\beta^2}{2\beta-1}>p(Y)=\frac{\beta}{2\sqrt{\beta}-1}$. In addition, in the case of (ii), we can compare the range of value p and $p(Y)$ as follows:
When $\beta\in(0,\frac{1}{4})$,



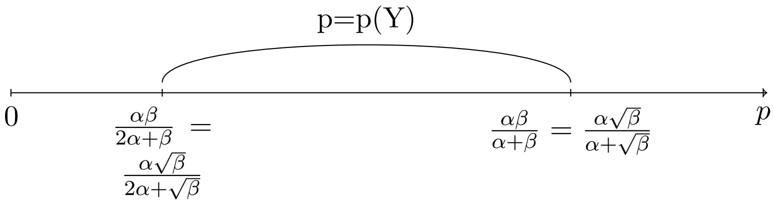
When $\beta = \frac{1}{4}$,



When $\beta \in (\frac{1}{4}, 1)$,



When $\beta = 1$,



For (iii) (b), we have similar relations as above.

Now we combine the result in [78] with ours as follows:

Theorem 5.4.4. *Let* $r(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^1(0,T;\mathbb{R}))$ *and* $\theta(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^2(0,T;\mathbb{R}^d))$.

(i) Suppose that (5.3.10) *holds for some* $\beta > 1$ *and* (5.3.12) *holds for some* $\alpha > 0$. *Then*

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,; r(\cdot),\theta(\cdot))^{\frac{\alpha\beta^2}{\beta^2+2\alpha\beta-\alpha}}\right]$$
$$\leq C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right]\right)^{\frac{\beta^2}{\beta^2+2\alpha\beta-\alpha}}\left(\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^t |\theta(s)|^2 ds}\right]\right)^{\frac{2\alpha(\beta-1)}{\beta^2+2\alpha\beta-\alpha}},$$

where $C = \left(\frac{\alpha\beta^2}{\alpha\beta^2-2\beta^2-2\alpha\beta+\alpha}\right)^{\frac{\alpha\beta^2}{\beta^2+2\alpha\beta-\alpha}}$.

(ii) Suppose that (5.3.10) *holds for some* $\beta \in (0,1]$ *and* (5.3.12) *holds for some* $\alpha > 0$. *Then for any* $p \in \left(\frac{\alpha\sqrt{\beta}}{2\alpha+\sqrt{\beta}}, \frac{\alpha\sqrt{\beta}}{\alpha+\sqrt{\beta}}\right)$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^p\right]$$
$$\leq C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right]\right)^{\frac{p}{\alpha}}\left(\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2ds}\right]\right)^{1-\frac{\alpha p}{(\alpha-p)\sqrt{\beta}}},$$

where $C=\left(\frac{\alpha^2p^2}{[\alpha\sqrt{\beta}-p(\alpha+\sqrt{\beta})]^2}\right)^{\frac{\alpha p^2}{\sqrt{\beta}[(2\alpha+\sqrt{\beta})p-\alpha\sqrt{\beta}]}}$ *and then for any* $p\in\left(\frac{\alpha\beta}{2\alpha+\beta},\frac{\alpha\beta}{\alpha+\beta}\right)$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^p\right]$$
$$\leq C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)ds}\right]\right)^{\frac{p}{\alpha}}\left(\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2ds}\right]\right)^{\frac{2\alpha\beta-2p(\alpha+\beta)}{\alpha\beta}},$$

where $C=\left(\frac{\alpha^2p^2}{\alpha^2p^2-\beta(\alpha-p)[(\beta+2\alpha)p-\alpha\beta]}\right)^{\frac{\alpha p^2}{\beta[(2\alpha+\beta)p-\alpha\beta]}}$.

(iii) (a) Suppose that (5.3.10) *holds for some* $\beta>1$ *and* (5.4.3) *holds for some* $\kappa\in\mathbb{R}$ *, then*

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{\frac{\beta^2}{2\beta-1}}\right]\leq C\left\{\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^t|\theta(s)|^2ds}\right]\right\}^{\frac{2\beta-2}{2\beta-1}},$$

where $C=e^{\frac{\kappa\beta^2}{2\beta-1}}\left[\frac{\beta^2}{(\beta-1)^2}\right]^{\frac{\beta^2}{2\beta-1}}$.

(b) Suppose that (5.3.10) *holds for* $\beta\in(0,1]$ *and* (5.4.3) *holds for some* $\kappa\in\mathbb{R}$, *then for any* $p\in(\frac{\sqrt{\beta}}{2},\sqrt{\beta})$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^p\right]\leq C\left\{\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2ds}\right]\right\}^{\frac{(\sqrt{\beta}-p)}{\sqrt{\beta}}},$$

where $C=e^{\kappa p}\left[\frac{p^2}{(\sqrt{\beta}-1)^2}\right]^{\frac{p^2}{\sqrt{\beta}(2p-\sqrt{\beta})}}$ *and then for any* $p\in(\frac{\beta}{2},\beta)$

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^p\right]\leq C\left\{\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2ds}\right]\right\}^{\frac{2(\beta-p)}{\beta}},$$

where $C=e^{\kappa p}\left[\frac{p^2}{p^2-\beta(2p-\beta)}\right]^{\frac{p^2}{\beta(2p-\beta)}}$.

Proof. By [42], we know (5.3.10) implies (5.3.14). Thus the result follows. □

Theorem 5.4.5 (Upper bound of $\mathbb{E}(\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-p})$)**.** *Suppose that $r(\cdot)\in\mathcal{L}^1_{\mathcal{F}}(\Omega;\mathcal{L}^1(0,T;\mathbb{R}))$ and $\theta(\cdot)\in\mathcal{L}^1_{\mathcal{F}}(\Omega;\mathcal{L}^2(0,T;\mathbb{R}^d))$.*

(i) Suppose that condition (5.3.13) *holds for some $\alpha>0$ and* (5.3.15) *holds for some $\beta_0>1$. Then*

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-\frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}\right]\\ \le& C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha\int_0^t[r(s)+|\theta(s)|^2]ds}\right]\right)^{\frac{\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}\left(\sup_{t\in[0,T]}\mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t\theta'(s)dW(s)}\right]\right)^{\frac{2\alpha(\beta_0-1)}{\beta_0^2+2\alpha\beta_0-\alpha}},\end{aligned}$$

where $C=\left(\frac{\alpha\beta_0^2}{\alpha\beta_0^2-2\beta_0^2-2\alpha\beta_0+\alpha}\right)^{\frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}$.

(ii) Suppose that (5.3.15) *holds for some $\beta_0\in(0,1]$ and* (5.3.13) *holds for some $\alpha>0$. Then for any $p\in\left(\frac{\alpha\beta_0}{2\alpha+\beta_0},\frac{\alpha\beta_0}{\alpha+\beta_0}\right)$,*

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-p}\right]\\ \le& C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha\int_0^t[r(s)+|\theta(s)|^2]ds}\right]\right)^{\frac{p}{\alpha}}\left(\sup_{t\in[0,T]}\mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t\theta'(s)dW(s)}\right]\right)^{\frac{2\alpha\beta_0-2p(\alpha+\beta_0)}{\alpha\beta_0}},\end{aligned}$$

where $C=\left(\frac{\alpha^2p^2}{\alpha^2p^2-\beta_0(\alpha-p)[(\beta_0+2\alpha)p-\alpha\beta_0]}\right)^{\frac{\alpha p^2}{\beta_0[(2\alpha+\beta_0)p-\alpha\beta_0]}}$.

(iii) (a) Suppose that

$$\sup_{t\in[0,T]}\int_0^t[r(s)+|\theta(s)|^2]ds\le\kappa_1,\ a.s.,\tag{5.4.7}$$

holds for some $\kappa_1\in\mathbb{R}$ and (5.3.15) *holds for some $\beta_0>1$, then*

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-\frac{\beta_0^2}{2\beta_0-1}}\right]\le C\left(\sup_{t\in[0,T]}\mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t\theta'(s)dW(s)}\right]\right)^{\frac{2\alpha(\beta_0-1)}{\beta_0^2+2\alpha\beta_0-\alpha}},$$

where $C = e^{\frac{\kappa_1 \alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}} \left(\frac{\alpha\beta_0^2}{\alpha\beta_0^2-2\beta_0^2-2\alpha\beta_0+\alpha}\right)^{\frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}$.

(b) In addition, if (5.3.15) *holds for some* $\beta_0 \in (0,1]$*, then for any* $p \in \left(\frac{\alpha\beta_0}{2\alpha+\beta_0}, \frac{\alpha\beta_0}{\alpha+\beta_0}\right)$

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-p}\right] \le C\left(\mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t \theta'(s)dW(s)}\right]\right)^{\frac{2\alpha\beta_0-2p(\alpha+\beta_0)}{\alpha\beta_0}},$$

where $C = e^{\kappa_1 p}\left(\frac{\alpha^2p^2}{\alpha^2p^2-\beta_0(\alpha-p)[(\beta_0+2\alpha)p-\alpha\beta_0]}\right)^{\frac{\alpha p^2}{\beta_0[(2\alpha+\beta_0)p-\alpha\beta_0]}}$.

Proof. (i) By Theorem 5.4.3 (i), when $\beta_0 > 1$ we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-p}\right]\\
=&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;-r(\cdot)-|\theta(\cdot)|^2,-\theta(\cdot))^{p}\right]\\
\le& C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha\int_0^t[r(s)+|\theta(s)|^2]ds}\right]\right)^{\frac{\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}\left(\sup_{t\in[0,T]}\mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t\theta'(s)dW(s)}\right]\right)^{\frac{2\alpha(\beta_0-1)}{\beta_0^2+2\alpha\beta_0-\alpha}},
\end{aligned}$$

where $C = \left(\frac{\alpha\beta_0^2}{\alpha\beta_0^2-2\beta_0^2-2\alpha\beta_0+\alpha}\right)^{\frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}$.

(ii) By Theorem 5.4.3 (ii), for any $p \in \left(\frac{\alpha\beta_0}{2\alpha+\beta_0}, \frac{\alpha\beta_0}{\alpha+\beta_0}\right)$ when $\beta_0 \in (0,1]$ we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-p}\right]\\
=&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;-r(\cdot)-|\theta(\cdot)|^2,-\theta(\cdot))^{p}\right]\\
\le& C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha\int_0^t[r(s)+|\theta(s)|^2]ds}\right]\right)^{\frac{p}{\alpha}}\left(\sup_{t\in[0,T]}\mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t\theta'(s)dW(s)}\right]\right)^{\frac{2\alpha\beta_0-2p(\alpha+\beta_0)}{\alpha\beta_0}},
\end{aligned}$$

where $C = \left(\frac{\alpha^2p^2}{\alpha^2p^2-\beta_0(\alpha-p)[(\beta_0+2\alpha)p-\alpha\beta_0]}\right)^{\frac{\alpha p^2}{\beta_0[(2\alpha+\beta_0)p-\alpha\beta_0]}}$.

(iii) (a) By (5.4.7), we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha\int_0^t [r(s)+|\theta(s)|^2]ds}\right] \le e^{\alpha\kappa_1}. \tag{5.4.8}$$

By Theorem 5.4.3 (i), when $\beta_0 > 1$ we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-p}\right] \\
=&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;-r(\cdot)-|\theta(\cdot)|^2,-\theta(\cdot))^{p}\right] \\
\le&\overline{C}\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha\int_0^t [r(s)+|\theta(s)|^2]ds}\right]\right)^{\frac{\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}\left(\sup_{t\in[0,T]}\mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t\theta'(s)dW(s)}\right]\right)^{\frac{2\alpha(\beta_0-1)}{\beta_0^2+2\alpha\beta_0-\alpha}} \\
\le&\overline{C}\; e^{\frac{\kappa_1\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}\left(\sup_{t\in[0,T]}\mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t\theta'(s)dW(s)}\right]\right)^{\frac{2\alpha(\beta_0-1)}{\beta_0^2+2\alpha\beta_0-\alpha}},
\end{aligned}$$

where $\overline{C} = \left(\frac{\alpha\beta_0^2}{\alpha\beta_0^2-2\beta_0^2-2\alpha\beta_0+\alpha}\right)^{\frac{\alpha\beta_0^2}{\beta^2+2\alpha\beta_0-\alpha}}$.

(b) In the case that $\beta_0\in(0,1]$, by Theorem 5.4.3 (ii), for any $p\in\left(\frac{\alpha\beta_0}{2\alpha+\beta_0},\frac{\alpha\beta_0}{\alpha+\beta_0}\right)$ we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-p}\right] \\
=&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;-r(\cdot)-|\theta(\cdot)|^2,-\theta(\cdot))^{p}\right] \\
\le&\overline{C}\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha\int_0^t [r(s)+|\theta(s)|^2]ds}\right]\right)^{\frac{p}{\alpha}}\left(\sup_{t\in[0,T]}\mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t\theta'(s)dW(s)}\right]\right)^{\frac{2\alpha\beta_0-2p(\alpha+\beta_0)}{\alpha\beta_0}}
\end{aligned}$$

$$\leq \overline{C}\, e^{\kappa_1 p} \left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t \theta'(s)dW(s)} \right] \right)^{\frac{2\alpha\beta_0 - 2p(\alpha+\beta_0)}{\alpha\beta_0}},$$

where $\overline{C} = \left(\frac{\alpha^2 p^2}{\alpha^2p^2 - \beta_0(\alpha-p)[(\beta_0+2\alpha)p-\alpha\beta_0]}\right)^{\frac{\alpha p^2}{\beta_0[(2\alpha+\beta_0)p-\alpha\beta_0]}}$. Thus the result follows. $\square$

Remark 5.4.1. *Suppose that $\beta_0 = \beta$ and $\alpha_0 = \alpha$. Then same comparison results can be obtained by similar analysis after Theorem 5.4.3.*

Now we combine the result in [78] with ours as follows.

Theorem 5.4.6. *Let $r(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^1(0,T;\mathbb{R}))$ and $\theta(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^2(0,T;\mathbb{R}^d))$.*

(i) Suppose that (5.3.9) holds for some $\alpha_0 > 0$ and (5.3.10) holds for some $\beta > 1$. Then for some $\alpha > 0$ given by (5.3.13),

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-\frac{\alpha\beta^2}{\beta^2+2\alpha\beta-\alpha}} \right]$$
$$\leq C \left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha_0\int_0^t r(s)ds} \right] \right)^{\frac{\alpha\beta^2}{\alpha_0[\beta^2+2\alpha\beta-\alpha]}} \left(\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^t |\theta(s)|^2 ds} \right] \right)^{\frac{2\alpha\beta}{\beta^2+2\alpha\beta-\alpha} + \frac{2\alpha(\beta-1)}{\beta^2+2\alpha\beta-\alpha}},$$

where $C = \left(\frac{\alpha\beta^2}{\alpha\beta^2 - 2\beta^2 - 2\alpha\beta + \alpha}\right)^{\frac{\alpha\beta^2}{\beta^2+2\alpha\beta-\alpha}}$.

(ii) Suppose that (5.3.9) holds for some $\alpha_0 > 0$ and (5.3.10) holds for some $\beta \in (0,1]$. Then for any $p \in \left(\frac{\alpha_0\beta}{\beta+2\alpha_0(\sqrt{\beta}+1)}, \frac{\alpha_0\beta}{\beta+\alpha_0(\sqrt{\beta}+2)}\right)$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-p} \right]$$
$$\leq C \left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha_0\int_0^t r(s)ds} \right] \right)^{\frac{p}{\alpha_0}} \left(\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2 ds} \right] \right)^{1+\frac{2p}{\beta} - \frac{\alpha_0\sqrt{\beta}p^2}{\alpha_0\beta - p(2\alpha_0+\beta)}},$$

where $C = \left(\frac{\alpha_0^2\beta^2p^2}{\alpha_0\beta(\sqrt{\beta}-p-2\sqrt{\beta}p)(2\alpha_0+\beta) - p\beta(2\alpha_0+\beta)^2}\right)^{\frac{\alpha_0\beta p}{\sqrt{\beta}\{[2\alpha_0\beta+\sqrt{\beta}(2\alpha_0+\beta)]p - \alpha_0\sqrt{\beta}\beta\}}}$ and *then for some* $\alpha > 0$ *given by* (5.3.13) *and for any* $p \in \left(\frac{\alpha\beta}{2\alpha+\beta}, \frac{\alpha\beta}{\alpha+\beta}\right)$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,; r(\cdot), \theta(\cdot))^{-p}\right] \\
&\leq C\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha_0\int_0^t r(s)ds}\right]\right)^{\frac{p}{\alpha_0}} \left(\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2 ds}\right]\right)^{2-\frac{2p(2\alpha+\beta)}{\alpha\beta}+\frac{2\alpha p}{\alpha_0\beta}},
\end{aligned}$$

where $C = \left(\frac{\alpha_0^2 p^2}{\alpha_0^2 p^2 - (2\alpha_0+\beta)^2[\alpha_0\beta - p(\alpha_0+\beta)][(4\alpha_0+\beta)p - 2\alpha_0\beta]}\right)^{\frac{\alpha_0 p^2}{\beta p(4\alpha_0+\beta)-\alpha_0\beta}}$.

Proof. Let $\alpha_0 > \alpha$ and $\beta = \beta_0 > 2\alpha$. By the Hölder's inequality, we know that (5.3.10) and (5.3.9) imply (5.3.13). Thus the results follow. □

5.5 Solvability of Linear BSDEs

We first recall the following result for a generalization of the Hölder's inequality.

Lemma 5.5.1. *Assume that* $r \in (0,\infty)$ *and* $p_1, p_2, \cdots, p_n \in (0,\infty)$ *such that*

$$\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{r}.$$

Then for all measurable real valued functions $f_1, f_2, \cdots, f_n$ *define on measurable space* S*, if* $f_i \in \mathcal{L}^{p_i}(\mu)$*, for all* $k = 1, 2, \cdots, n$*, we have*

$$\prod_{i=1}^n f_i \in \mathcal{L}^r(\mu),$$

where $\mathcal{L}^{p_i}(\mu)$ *is the space of* p_i*-th power integrable functions together with norm* $\|\cdot\|_{p_i}$*, i.e.* $\|f\|_{p_i} = \left(\int_S |f|^{p_i} d\mu\right)^{\frac{1}{p_i}} < \infty$.

Based on the estimate of exponential processes in Section 5.4, we now present the results on solvability of linear backward stochastic differential equations under various assumptions on the parameters.

Theorem 5.5.1. *Suppose that* $\theta(\cdot) \in \mathcal{L}_{\mathcal{F}}^{2+}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)$ *and* (5.3.11), (5.3.12), (5.3.13), (5.3.14) *and* (5.3.15) *hold with positive constants* α_0, α, β *and* β_0 *satisfying:*

$$\begin{cases} \alpha_0 > \frac{\alpha\beta\beta_0^2}{\alpha\beta\beta_0^2+\alpha\beta-2\alpha\beta\beta_0-2\alpha\beta_0^2-\beta\beta_0^2}, \\ \alpha > \frac{\beta_0^2}{(\beta_0-1)^2}, \\ \beta_0 \geq 1, \beta > 2. \end{cases} \tag{5.5.1}$$

Then for any $\xi \in \mathcal{L}^{p+}_{\mathcal{F}_T}(\Omega;\mathbb{R})$ *with*

$$p = \frac{\alpha_0\alpha\beta\beta_0^2}{\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0 - 2\alpha_0\alpha\beta_0^2 - \alpha\beta\beta_0^2} > 1, \tag{5.5.2}$$

BSDE (5.2.1) *admits a unique adapted solution* $(Y(\cdot), Z(\cdot))$ *with*

$$Y(\cdot) \in \mathcal{L}^{1+}_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R})), \quad Z(\cdot) \in \mathcal{L}^{1+}_{\mathcal{F}}(\Omega; \mathcal{L}^2([0,T];\mathbb{R}^d)).$$

Proof. For simplicity, from now on we denote $M(t) = M(t\,; r(\cdot), \theta(\cdot))$. By Theorem 5.4.1 (i), we know that if

$$\alpha_0 > \frac{\beta}{\beta-2}, \tag{5.5.3}$$

and $\beta > 2$, then

$$M(\cdot) \in \mathcal{L}^{p_1}_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R})), \tag{5.5.4}$$

where

$$p_1 = \frac{\alpha_0\beta}{2\alpha_0+\beta} > 1. \tag{5.5.5}$$

By Lemma 5.5.1, for any $\xi \in \mathcal{L}^{p_2}_{\mathcal{F}_T}(\Omega)$, we have

$$\tilde{\xi} = M(T)\xi \in \mathcal{L}^{p_3}_{\mathcal{F}_T}(\Omega;\mathbb{R}),$$

where

$$p_3 = \frac{p_1p_2}{p_1+p_2} = \frac{\alpha_0\beta p_2}{\alpha_0\beta + (2\alpha_0+\beta)p_2}. \tag{5.5.6}$$

Suppose $p_3 > 1$, $\beta > 2$ and (5.5.3), then we have

$$p_2 > \frac{\alpha_0\beta}{\alpha_0\beta - 2\alpha_0 - \beta} > 1. \tag{5.5.7}$$

By Theorem 5.4.5, if

$$\alpha > \frac{\beta_0^2}{(\beta_0-1)^2}, \tag{5.5.8}$$

and $\beta_0 > 1$, we have

$$M(\cdot)^{-1} \in \mathcal{L}_{\mathcal{F}}^{p_4}(\Omega; C([0,T];\mathbb{R})), \tag{5.5.9}$$

where

$$p_4 = \frac{\alpha\beta_0^2}{\beta_0^2 + 2\alpha\beta_0 - \alpha} > 1.$$

By BSDE (5.3.5), Theorem 5.4.1, Theorem 5.4.5, the Hölder's inequality and the Doob's martingale inequality with $p_1, p_2, p_3, p_4, q_1, q_2 > 1$, we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y(t)|^{p_5}\right] = \mathbb{E}\left[\sup_{t\in[0,T]} \left|M(t)^{-1}\widetilde{Y}(t)\right|^{p_5}\right]$$

$$\leq \left\{\mathbb{E}\left[\sup_{t\in[0,T]} |M(t)|^{-q_1\cdot p_5}\right]\right\}^{\frac{1}{q_1}} \cdot \left\{\mathbb{E}\left[\sup_{t\in[0,T]} \left|\widetilde{Y}(t)\right|^{\frac{q_1}{q_1-1}p_5}\right]\right\}^{\frac{q_1-1}{q_1}}$$

$$\leq \left\{\mathbb{E}\left[\sup_{t\in[0,T]} |M(t)|^{-p_4}\right]\right\}^{\frac{p_5}{p_4}} \cdot \left\{\mathbb{E}\left[\sup_{t\in[0,T]} \left|\widetilde{Y}(t)\right|^{p_3}\right]\right\}^{\frac{p_5}{p_3}}$$

$$\leq C_{p_3,p_5}\left\{\mathbb{E}\left[\sup_{t\in[0,T]} |M(t)|^{-p_4}\right]\right\}^{\frac{p_5}{p_4}} \cdot \{\mathbb{E}\,|M(T)|^{q_2\cdot p_3}\}^{\frac{1}{q_2}\frac{p_5}{p_3}} \cdot \left\{\mathbb{E}\,|\xi|^{\frac{q_2}{q_2-1}p_3}\right\}^{\frac{q_2-1}{q_2}\frac{p_5}{p_3}}$$

$$\leq C_{p_3,p_5}\left\{\mathbb{E}\left[\sup_{t\in[0,T]} |M(t)|^{-p_4}\right]\right\}^{\frac{p_5}{p_4}} \cdot \{\mathbb{E}\,|M(T)|^{p_1}\}^{\frac{p_5}{p_1}} \cdot \{\mathbb{E}\,|\xi|^{p_2}\}^{\frac{p_5}{p_2}} < \infty,$$

where

$$p_5 = \frac{p_3p_4}{p_3+p_4} = \frac{\frac{\alpha_0\beta p_2}{\alpha_0\beta+(2\alpha_0+\beta)p_2} \cdot \frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}{\frac{\alpha_0\beta p_2}{\alpha_0\beta+(2\alpha_0+\beta)p_2} + \frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}$$

$$= \frac{\alpha_0\alpha\beta\beta_0^2p_2}{\alpha_0\beta p_2(\beta_0^2+2\alpha\beta_0-\alpha)+\alpha\beta_0^2[\alpha_0\beta+(2\alpha_0+\beta)p_2]}.$$

So in order to ensure $p_5 > 1$, it is necessary to have

$$(\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0 - 2\alpha_0\alpha\beta_0^2 - \alpha\beta\beta_0^2)p_2 > \alpha_0\alpha\beta\beta_0^2.$$

Since $p_2 > 1$, in other words it is necessary to have

$$\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0 - 2\alpha_0\alpha\beta_0^2 - \alpha\beta\beta_0^2 > 0,$$

i.e. if $\alpha > \frac{\beta_0^2}{(\beta_0-1)^2}$ holds, then

$$\alpha_0 > \frac{\alpha\beta\beta_0^2}{(\alpha\beta\beta_0^2 + \alpha\beta - 2\alpha\beta\beta_0 - 2\alpha\beta_0^2 - \beta\beta_0^2)}, \tag{5.5.10}$$

and thus

$$p_2 > \frac{\alpha_0\alpha\beta\beta_0^2}{\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0 - 2\alpha_0\alpha\beta_0^2 - \alpha\beta\beta_0^2}. \tag{5.5.11}$$

It is easy to see that if $\beta_0 > 1$, then (5.5.10) is bigger than (5.5.3) and (5.5.11) is also bigger than (5.5.7). Hence when (5.5.8) and (5.5.10) are satisfied, we always have (5.5.11). Substituting (5.5.11) into (5.5.6), if $\beta_0 > 1$ and (5.5.8) hold, then

$$\begin{aligned} p_3 =& \frac{\alpha_0\beta p_2}{\alpha_0\beta + (2\alpha_0 + \beta)p_2} \\ >& \frac{\alpha_0\alpha\beta\beta_0^2}{\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0} > 1. \end{aligned}$$

Accordingly, by Theorem 5.1 in [41], we know that BSDE (5.3.5) admits a unique solution

$$\widetilde{Y}(\cdot) \in \mathcal{L}_{\mathcal{F}}^{p_3}(\Omega; C([0,T];\mathbb{R})), \quad \widetilde{Z}(\cdot) \in \mathcal{L}_{\mathcal{F}}^{p_3}(\Omega; \mathcal{L}^2([0,T];\mathbb{R}^d)),$$

and the following estimate holds

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]} |\widetilde{Y}(t)|^{p_3} + \left[\int_0^T |\widetilde{Z}(t)|^2 dt\right]^{\frac{p_3}{2}}\right] \\ =& C_{p_3}\,\mathbb{E}\left|\widetilde{\xi}\right|^{p_3} = C_{p_3}\,\mathbb{E}\,|M(T)\xi|^{p_3} \\ \leq& C_{p_3}\left\{\mathbb{E}\,|M(T)|^{q_3\cdot p_3}\right\}^{\frac{1}{q_3}} \cdot \left\{\mathbb{E}\,|\xi|^{\frac{q_3}{q_3-1}p_3}\right\}^{\frac{q_3-1}{q_3}} \\ \leq& C_{p_3}\left\{\mathbb{E}\,|M(T)|^{p_1}\right\}^{\frac{p_3}{p_1}} \cdot \left\{\mathbb{E}\,|\xi|^{p_2}\right\}^{\frac{p_3}{p_2}} < \infty, \end{aligned}$$

where $p_1, p_2, p_3, q_3 > 1$ and C_{p_3} is some constant. Note that

$$p_5 = \frac{p_3 p_4}{p_3 + p_4} > \frac{\alpha_0^2\alpha^2\beta_2\beta_0^4}{\alpha_0^2\alpha\beta^2\beta_0^2(\beta_0^2 + 2\alpha\beta_0 - \alpha) + \alpha_0^2\alpha\beta^2\beta_0^2(\alpha\beta_0^2 + \alpha - \beta_0^2 - 2\alpha\beta_0)} = 1.$$

Finally, taking a constant $\varepsilon \in (0, p_5)$, using the Hölder's inequality and Minkowski's inequality, we obtain

$$\mathbb{E}\left[\int_0^T |Z(t)|^2\right]^{\frac{p_5-\varepsilon}{2}}$$

$$=\mathbb{E}\left[\int_0^T \left|M(t)^{-1}\left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]\right|^2 dt\right]^{\frac{p_5-\varepsilon}{2}}$$

$$\leq \mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-1}\right|^2 \int_0^T \left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{p_5-\varepsilon}{2}}$$

$$=\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-1}\right|^{p_5-\varepsilon}\left(\int_0^T \left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2 dt\right)^{\frac{p_5-\varepsilon}{2}}\right]$$

$$\leq \left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-1}\right|^{(p_5-\varepsilon)\frac{p_4}{p_5-\varepsilon}}\right]\right)^{\frac{p_5-\varepsilon}{p_4}} \cdot \left(\mathbb{E}\left[\int_0^T \left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{p_5-\varepsilon}{2}\frac{p_4}{p_4-p_5+\varepsilon}}\right)^{\frac{p_4-p_5+\varepsilon}{p_4}}$$

$$\leq \left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}} \cdot \left(\mathbb{E}\left[\int_0^T \left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{p_4(p_5-\varepsilon)}{2(p_4-p_5+\varepsilon)}\frac{p_4-p_5+\varepsilon}{p_4(p_5-\varepsilon)}}\right)^{\frac{p_4-p_5+\varepsilon}{p_4}\frac{p_4(p_5-\varepsilon)}{p_4-p_5+\varepsilon}}$$

$$=\left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}}\left(\mathbb{E}\left[\int_0^T \left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{1}{2}}\right)^{p_5-\varepsilon}$$

$$\leq \left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}}$$
$$\cdot\left\{\left(\mathbb{E}\left[\int_0^T\left[\widetilde{Z}(t)\right]^2 dt\right]^{\frac{1}{2}}\right)+\left(\mathbb{E}\left[\int_0^T\left[\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{1}{2}}\right)\right\}^{p_5-\varepsilon}$$

$$\leq \left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}}\left\{\left(\mathbb{E}\left[\int_0^T\left[\widetilde{Z}(t)\right]^2 dt\right]^{\frac{p_3}{2}}\right)^{\frac{1}{p_3}}\right.$$
$$\left.+\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left|\widetilde{Y}(t)\right|\left(\int_0^T[\theta(t)]^2\,dt\right)^{\frac{1}{2}}\right]\right)\right\}^{p_5-\varepsilon}$$

$$\leq \left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}}\left\{\left(\mathbb{E}\left[\int_0^T\left[\widetilde{Z}(t)\right]^2 dt\right]^{\frac{p_3}{2}}\right)^{\frac{1}{p_3}}\right.$$
$$\left.+\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left|\widetilde{Y}(t)\right|^{p_3}\right]\right)^{\frac{1}{p_3}}\left(\mathbb{E}\left[\int_0^T[\theta(t)]^2\,dt\right]^{\frac{p_3}{p_3-1}}\right)^{\frac{p_3-1}{p_3}}\right\}^{p_5-\varepsilon}$$

$$<\infty$$

where the last step follows from the assumption that $\theta(\cdot)\in\mathcal{L}_{\mathcal{F}}^{2+}(\Omega;\mathcal{L}^2([0,T];\mathbb{R}^d)$.

□

In order to compare our result to Theorem 4.1 of [78], let us denote parameters α and β in [78] by $\alpha(Y)$ and $\beta(Y)$ respectively. We need to make both parameters identical, i.e. $\alpha_0=\alpha=\alpha(Y)$ and $\beta_0=\beta=\beta(Y)$. Hence our $p=\frac{\alpha\beta^2}{\alpha\beta^2+\alpha-2\beta^2-4\alpha\beta}$ and $p(Y)=\frac{\alpha\beta^2}{\alpha\beta^2-2\beta^2-4\alpha\beta\sqrt{\beta}}$. Now it is easy to see that when condition (5.5.1) and condition (4.1) in [78] are satisfied, our p is always less than $p(Y)$, which implies our space of terminal value is always bigger than that of [78].

Now we present a similar result to the above theorem but only assuming $\theta(\cdot)\in\mathcal{L}_{\mathcal{F}}^{2}(\Omega;\mathcal{L}^2([0,T];\mathbb{R}^d)$.

Theorem 5.5.2. *Suppose that* $\theta(\cdot)\in\mathcal{L}_{\mathcal{F}}^{2}(\Omega;\mathcal{L}^2([0,T];\mathbb{R}^d)$ *and* (5.3.11), (5.3.12), (5.3.13), (5.3.14) *and* (5.3.15) *hold with positive constants* α_0, α, β *and* β_0 *satisfying:*

$$\begin{cases} \alpha_0 > \max\left\{\frac{2\beta}{\beta-4}, \frac{\alpha\beta\beta_0^2}{\alpha\beta\beta_0^2+\alpha\beta-2\alpha\beta\beta_0-2\alpha\beta_0^2-\beta\beta_0^2}\right\}, \\ \alpha > \frac{\beta_0^2}{(\beta_0-1)^2}, \\ \beta_0 \geq 1, \beta > 4. \end{cases} \tag{5.5.12}$$

Then for any $\xi \in \mathcal{L}^{p+}_{\mathcal{F}_T}(\Omega;\mathbb{R})$ *with*

$$p = \max\left\{\frac{2\alpha_0\beta}{\alpha_0\beta-4\alpha_0-2\beta}, \frac{\alpha_0\alpha\beta\beta_0^2}{\alpha_0\alpha\beta\beta_0^2+\alpha_0\alpha\beta-\alpha_0\beta\beta_0^2-2\alpha_0\alpha\beta\beta_0-2\alpha_0\alpha\beta_0^2-\alpha\beta\beta_0^2}\right\} >1, \tag{5.5.13}$$

BSDE (5.2.1) *admits a unique adapted solution* $(Y(\cdot), Z(\cdot))$ *with*

$$Y(\cdot) \in \mathcal{L}^{0+}_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R})), \quad Z(\cdot) \in \mathcal{L}^{0+}_{\mathcal{F}}(\Omega; \mathcal{L}^2([0,T];\mathbb{R}^d)).$$

Proof. For simplicity, from now on we denote $M(\cdot) = M(\cdot\,; r(\cdot), \theta(\cdot))$. By Theorem 5.4.1 (i), we know that if $\beta > 4$ and

$$\alpha_0 > \frac{2\beta}{\beta-4}, \tag{5.5.14}$$

then

$$M(\cdot) \in \mathcal{L}^{p_1}_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R})), \tag{5.5.15}$$

where

$$p_1 = \frac{\alpha_0\beta}{2\alpha_0+\beta} > 1. \tag{5.5.16}$$

By Lemma 5.5.1, for any $\xi \in \mathcal{L}^{p_2}_{\mathcal{F}_T}(\Omega)$, we have

$$\tilde{\xi} = M(T)\xi \in \mathcal{L}^{p_3}_{\mathcal{F}_T}(\Omega;\mathbb{R}),$$

where

$$p_3 = \frac{p_1p_2}{p_1+p_2} = \frac{\alpha_0\beta p_2}{\alpha_0\beta + (2\alpha_0+\beta)p_2}. \tag{5.5.17}$$

Suppose $p_3 \geq 2$, $\beta > 4$ and (5.5.14), then we have

$$p_2 > \frac{2\alpha_0\beta}{\alpha_0\beta - 4\alpha_0 - 2\beta} > 1. \tag{5.5.18}$$

By Theorem 5.4.5, if

$$\alpha > \frac{\beta_0^2}{(\beta_0 - 1)^2}, \tag{5.5.19}$$

and $\beta_0 > 1$, we have

$$M(\cdot)^{-1} \in \mathcal{L}_{\mathcal{F}}^{p_4}(\Omega; C([0,T]; \mathbb{R})), \tag{5.5.20}$$

where

$$p_4 = \frac{\alpha\beta_0^2}{\beta_0^2 + 2\alpha\beta_0 - \alpha} > 1.$$

By BSDE (5.3.5), Theorem 5.4.1, Theorem 5.4.5 and the Hölder's inequality with $p_1, p_2, p_3, p_4, q_1, q_2 > 1$, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y(t)|^{p_5}\right] = \mathbb{E}\left[\sup_{t\in[0,T]} |M(t)^{-1}\widetilde{Y}(t)|^{p_5}\right]$$

$$\leq \left\{\mathbb{E}\left[\sup_{t\in[0,T]} |M(t)|^{-q_1\cdot p_5}\right]\right\}^{\frac{1}{q_1}} \cdot \left\{\mathbb{E}\left[\sup_{t\in[0,T]} \left|\widetilde{Y}(t)\right|^{\frac{q_1}{q_1-1}p_5}\right]\right\}^{\frac{q_1-1}{q_1}}$$

$$\leq \left\{\mathbb{E}\left[\sup_{t\in[0,T]} |M(t)|^{-p_4}\right]\right\}^{\frac{p_5}{p_4}} \cdot \left\{\mathbb{E}\left[\sup_{t\in[0,T]} \left|\widetilde{Y}(t)\right|^{p_3}\right]\right\}^{\frac{p_5}{p_3}}$$

$$\leq \left\{\mathbb{E}\left[\sup_{t\in[0,T]} |M(t)|^{-p_4}\right]\right\}^{\frac{p_5}{p_4}} \cdot \left\{\mathbb{E}\,|M(T)|^{q_2\cdot p_3}\right\}^{\frac{1}{q_2}\frac{p_5}{p_3}} \cdot \left\{\mathbb{E}\,|\xi|^{\frac{q_2}{q_2-1}p_3}\right\}^{\frac{q_2-1}{q_2}\frac{p_5}{p_3}}$$

$$\leq \left\{\mathbb{E}\left[\sup_{t\in[0,T]} |M(t)|^{-p_4}\right]\right\}^{\frac{p_5}{p_4}} \cdot \left\{\mathbb{E}\,|M(T)|^{p_1}\right\}^{\frac{p_5}{p_1}} \cdot \left\{\mathbb{E}\,|\xi|^{p_2}\right\}^{\frac{p_5}{p_2}} < \infty,$$

where

$$p_5 = \frac{p_3 p_4}{p_3 + p_4} = \frac{\frac{\alpha_0\beta p_2}{\alpha_0\beta + (2\alpha_0+\beta)p_2} \cdot \frac{\alpha\beta_0^2}{\beta_0^2 + 2\alpha\beta_0 - \alpha}}{\frac{\alpha_0\beta p_2}{\alpha_0\beta + (2\alpha_0+\beta)p_2} + \frac{\alpha\beta_0^2}{\beta_0^2 + 2\alpha\beta_0 - \alpha}}$$

$$= \frac{\alpha_0\alpha\beta\beta_0^2 p_2}{\alpha_0\beta p_2(\beta_0^2 + 2\alpha\beta_0 - \alpha) + \alpha\beta_0^2[\alpha_0\beta + (2\alpha_0+\beta)p_2]}.$$

So in order to ensure $p_5 > 1$, it is necessary to have

$$(\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0 - 2\alpha_0\alpha\beta_0^2 - \alpha\beta\beta_0^2)p_2 > \alpha_0\alpha\beta\beta_0^2.$$

Since $p_2 > 1$, in other words it is necessary to have

$$\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0 - 2\alpha_0\alpha\beta_0^2 - \alpha\beta\beta_0^2 > 0,$$

i.e. if (5.5.19) holds, then

$$\alpha_0 > \frac{\alpha\beta\beta_0^2}{(\alpha\beta\beta_0^2 + \alpha\beta - 2\alpha\beta\beta_0^2 - 2\alpha\beta_0^2 - \beta\beta_0^2)}, \tag{5.5.21}$$

and thus

$$p_2 > \frac{\alpha_0\alpha\beta\beta_0^2}{\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0 - 2\alpha_0\alpha\beta_0^2 - \alpha\beta\beta_0^2}. \tag{5.5.22}$$

Therefore we obtain that

$$\alpha_0 > \max\left\{\frac{2\beta}{\beta - 4}, \frac{\alpha\beta\beta_0^2}{\alpha\beta\beta_0^2 + \alpha\beta - 2\alpha\beta\beta_0 - 2\alpha\beta_0^2 - \beta\beta_0^2}\right\},$$

and

$$p = \max\left\{\frac{2\alpha_0\beta}{\alpha_0\beta - 4\alpha_0 - 2\beta}, \frac{\alpha_0\alpha\beta\beta_0^2}{\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0 - 2\alpha_0\alpha\beta_0^2 - \alpha\beta\beta_0^2}\right\}$$

Substituting (5.5.18) and (5.5.22) into (5.5.17), if $\beta_0 > 1$ and (5.5.19) hold, then

$$\begin{aligned} p_3 &= \frac{\alpha_0\beta p_2}{\alpha_0\beta + (2\alpha_0 + \beta)p_2} \\ &> \max\left\{2, \frac{\alpha_0\alpha\beta\beta_0^2}{\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0}\right\} \\ &> 1. \end{aligned}$$

Accordingly, by Theorem 5.1 in [41], we know that BSDE (5.3.5) admits a unique solution

$$\widetilde{Y}(\cdot) \in \mathcal{L}_{\mathcal{F}}^{p_3}(\Omega; C([0,T]; \mathbb{R})), \quad \widetilde{Z}(\cdot) \in \mathcal{L}_{\mathcal{F}}^{p_3}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)),$$

and the following estimate holds

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}|\widetilde{Y}(t)|^{p_3}+\left[\int_0^T|\widetilde{Z}(t)|^2dt\right]^{\frac{p_3}{2}}\right]\\
=&C_{p_3}\mathbb{E}\left|\widetilde{\xi}\right|^{p_3}=C_{p_3}\mathbb{E}\left|M(T)\xi\right|^{p_3}\\
\leq&C_{p_3}\left\{\mathbb{E}\left|M(T)\right|^{q_3\cdot p_3}\right\}^{\frac{1}{q_3}}\cdot\left\{\mathbb{E}\left|\xi\right|^{\frac{q_3}{q_3-1}p_3}\right\}^{\frac{q_3-1}{q_3}}\\
\leq&C_{p_3}\left\{\mathbb{E}\left|M(T)\right|^{p_1}\right\}^{\frac{p_3}{p_1}}\cdot\left\{\mathbb{E}\left|\xi\right|^{p_2}\right\}^{\frac{p_3}{p_2}}<\infty,
\end{aligned}$$

where $p_1, p_2, p_3, q_3 > 1$ and C_{p_3} is some constant. Note that

$$p_5=\frac{p_3p_4}{p_3+p_4}>\frac{2\alpha\beta_0^2}{2(\beta_0^2+2\alpha\beta_0-\alpha)+\alpha\beta_0^2},$$

which is not necessary bigger or equal to 1.

Finally, taking a constant $\varepsilon\in(0,p_5)$, using the Hölder's inequality and Minkowski's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T|Z(t)|^2\right]^{\frac{p_5-\varepsilon}{2}}\\
=&\mathbb{E}\left[\int_0^T\left|M(t)^{-1}\left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]\right|^2dt\right]^{\frac{p_5-\varepsilon}{2}}\\
\leq&\mathbb{E}\left[\sup_{t\in0,T}\left|M(t)^{-1}\right|^2\int_0^T\left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2dt\right]^{\frac{p_5-\varepsilon}{2}}\\
=&\mathbb{E}\left[\sup_{t\in0,T}\left|M(t)^{-1}\right|^{p_5-\varepsilon}\left(\int_0^T\left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2dt\right)^{\frac{p_5-\varepsilon}{2}}\right]\\
\leq&\left(\mathbb{E}\left[\sup_{t\in0,T}\left|M(t)^{-1}\right|^{(p_5-\varepsilon)\frac{p_4}{p_5-\varepsilon}}\right]\right)^{\frac{p_5-\varepsilon}{p_4}}\\
&\cdot\left(\mathbb{E}\left[\int_0^T\left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2dt\right]^{\frac{p_5-\varepsilon}{2}\frac{p_4}{p_4-p_5+\varepsilon}}\right)^{\frac{p_4-p_5+\varepsilon}{p_4}}
\end{aligned}$$

$$
\le \left(\mathbb{E}\left[\sup_{t\in 0,T} \left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}}
$$

$$
\cdot \left(\mathbb{E}\left[\int_0^T \left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{p_4(p_5-\varepsilon)}{2(p_4-p_5+\varepsilon)}\frac{p_4-p_5+\varepsilon}{p_4(p_5-\varepsilon)}}\right)^{\frac{p_4-p_5+\varepsilon}{p_4}\frac{p_4(p_5-\varepsilon)}{p_4-p_5+\varepsilon}}
$$

$$
= \left(\mathbb{E}\left[\sup_{t\in 0,T} \left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}} \left(\mathbb{E}\left[\int_0^T \left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{1}{2}}\right)^{p_5-\varepsilon}
$$

$$
\le \left(\mathbb{E}\left[\sup_{t\in 0,T} \left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}}
$$

$$
\cdot \left\{\left(\mathbb{E}\left[\int_0^T \left[\widetilde{Z}(t)\right]^2 dt\right]^{\frac{1}{2}}\right) + \left(\mathbb{E}\left[\int_0^T \left[\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{1}{2}}\right)\right\}^{p_5-\varepsilon}
$$

$$
\le \left(\mathbb{E}\left[\sup_{t\in 0,T} \left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}} \left\{\left(\mathbb{E}\left[\int_0^T \left[\widetilde{Z}(t)\right]^2 dt\right]^{\frac{p_3}{2}}\right)^{\frac{1}{p_3}}\right.
$$

$$
\left. + \left(\mathbb{E}\left[\sup_{t\in[0,T]} \left|\widetilde{Y}(t)\right| \left(\int_0^T [\theta(t)]^2 dt\right)^{\frac{1}{2}}\right]\right)\right\}^{p_5-\varepsilon}
$$

$$
\le \left(\mathbb{E}\left[\sup_{t\in 0,T} \left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}} \left\{\left(\mathbb{E}\left[\int_0^T \left[\widetilde{Z}(t)\right]^2 dt\right]^{\frac{p_3}{2}}\right)^{\frac{1}{p_3}}\right.
$$

$$
\left. + \frac{1}{2}\left(\mathbb{E}\left[\sup_{t\in[0,T]} \left|\widetilde{Y}(t)\right|^2\right] + \mathbb{E}\int_0^T [\theta(t)]^2 dt\right)\right\}^{p_5-\varepsilon} < \infty,
$$

where the last step follows from the assumption that $\theta(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)$ and $p_3 \ge 2$. □

Similar to last theorem, in this case we have $p = \frac{2\alpha\beta^2}{\alpha\beta^2 - 4\alpha\beta - 2\beta^2}$ and $p(Y) = \frac{2\alpha\beta^2}{2\alpha\beta^2 - 4\beta^2 - 8\alpha\beta\sqrt{\beta}}$. When condition (5.5.1) and condition (4.1) in [78] are satisfied, our p is always less than $p(Y)$, which also implies our space of terminal value is always bigger than that of [78].

Now we study the case that (5.5.1) does not hold, which might lead to no adapted solution $(Y(\cdot), Z(\cdot))$ in $\mathcal{L}_{\mathcal{F}}^{1+}(\Omega; C([0,T]; \mathbb{R})) \times \mathcal{L}_{\mathcal{F}}^{1+}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d))$. However, adapted solutions with less integrability can be obtained as follows.

Theorem 5.5.3. *Suppose that $\theta(\cdot) \in \mathcal{L}_{\mathcal{F}}^{2+}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)$ and* (5.3.11), (5.3.12), (5.3.13), (5.3.14) *and* (5.3.15) *hold with positive constants α_0, α, β and β_0 satisfying:*

$$\alpha_0 > \frac{\beta}{\beta - 2}, \quad \beta_0 \geq 1, \quad \beta > 2. \tag{5.5.23}$$

Then for any $\xi \in \mathcal{L}_{\mathcal{F}_T}^{p+}(\Omega; \mathbb{R})$ with

$$p = \frac{\alpha_0 \beta}{\alpha_0 \beta - 2\alpha_0 - \beta} > 1, \tag{5.5.24}$$

BSDE (5.2.1) *admits a unique adapted solution $(Y(\cdot), Z(\cdot))$ with*

$$Y(\cdot) \in \mathcal{L}_{\mathcal{F}}^{q+}(\Omega; C([0,T]; \mathbb{R})), \quad Z(\cdot) \in \mathcal{L}_{\mathcal{F}}^{q+}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)),$$

with

$$q = \frac{\alpha \beta_0^2}{(\alpha+1)\beta_0^2 + 2\alpha\beta_0 - \alpha} \in (0,1). \tag{5.5.25}$$

Proof. By Theorem 5.5.1, we have (5.5.4) and (5.5.5). Then for any $\xi \in \mathcal{L}_{\mathcal{F}_T}^{p_2}(\Omega; \mathbb{R})$ with $p_2 > p = \frac{\alpha_0\beta}{\alpha_0\beta - 2\alpha_0 - \beta} > 1$ given by (5.5.24), we have $\tilde{\xi} = M(T)\xi \in \mathcal{L}_{\mathcal{F}_T}^{p_3}(\Omega; \mathbb{R})$, where

$$\begin{aligned} p_3 = \frac{p_1 p_2}{p_1 + p_2} &= \frac{\alpha_0 \beta p_2}{\alpha_0 \beta + (2\alpha_0 + \beta) p_2} > \frac{\alpha_0 \beta \frac{\alpha_0\beta}{\alpha_0\beta - 2\alpha_0 - \beta}}{\alpha_0 \beta + (2\alpha_0 + \beta)\frac{\alpha_0\beta}{\alpha_0\beta - 2\alpha_0 - \beta}} \\ &= \frac{\alpha_0\beta}{\alpha_0\beta - 2\alpha_0 - \beta + 2\alpha_0 + \beta} = 1. \end{aligned}$$

Hence (5.5.12) holds. Furthermore,

$$p_5 = \frac{p_3 p_4}{p_3 + p_4} > \frac{p_4}{1 + p_4} = \frac{\alpha\beta_0^2}{(\alpha+1)\beta_0^2 + 2\alpha\beta_0 - \alpha} = q,$$

where $p_4 = \frac{\alpha\beta_0^2}{\beta_0^2 + 2\alpha\beta_0^2 - \alpha} > 0$ given by Theorem (5.4.5). Note that here p_4 is not necessary bigger than 1.

The rest of proof is similar to that of Theorem 5.5.1. □

Let us compare out result with Theorem 4.2 of [78]. We know that $p =$

$\frac{\alpha\beta}{\alpha\beta-2\alpha-\beta}$ and $p(Y)=\frac{\alpha\beta}{\alpha\beta+\alpha-\beta-2\alpha\sqrt{\beta}}$. It is easy to see that when $\beta>2$, we always have $p<p(Y)$ which implies our space of terminal condition is bigger than that of [78]. Similarly, regarding to the comparison of q, we conclude that when $\beta>2$, $q>q(Y)$ which implies our space of solution guarantees stronger integrability than that of [78].

Note that if (5.5.23) are not satisfied, then $M(T)$ is not necessary in $\mathcal{L}^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$. The following results deals with this case.

Theorem 5.5.4. *Suppose that* $\theta(\cdot)\in\mathcal{L}^{2+}_{\mathcal{F}}(\Omega;\mathcal{L}^2([0,T];\mathbb{R}^d)$ *and* (5.3.11), (5.3.12), (5.3.13), (5.3.14) *and* (5.3.15) *hold with positive constants* α_0, α, β *and* $\beta_0\geq\frac{1}{2}$. *Then for any* $\xi\in\mathcal{L}^p_{\mathcal{F}_T}(\Omega;\mathbb{R})$ *with* $p>0$, *BSDE* (5.2.1) *admits a unique adapted solution* $(Y(\cdot),Z(\cdot))$ *with*

$$Y(\cdot)\in\mathcal{L}^q_{\mathcal{F}}(\Omega;C([0,T];\mathbb{R})),\quad Z(\cdot)\in\mathcal{L}^{q-}_{\mathcal{F}}(\Omega;\mathcal{L}^2([0,T];\mathbb{R}^d)),$$

with

$$q=\frac{\alpha_0\alpha\beta_0^2p}{\alpha_0\beta_0^2p+2\alpha_0\alpha\beta_0p-\alpha_0\alpha p+\alpha_0\alpha\beta_0^2+\alpha_0\alpha\beta_0^2p+\alpha\beta_0^2p}\in(0,1).\qquad(5.5.26)$$

Proof. By Theorem 3.2 (i) in [78], we have $M(\cdot)\in\mathcal{L}^{p_1}_{\mathcal{F}}(\Omega;C([0,T];\mathbb{R}))$ with $p_1=\frac{\alpha_0}{\alpha_0+1}$. Then for any $\xi\in\mathcal{L}^p_{\mathcal{F}_T}(\Omega;\mathbb{R})$ with $p_2=p$, we have $\tilde{\xi}\equiv M(T)\xi\in\mathcal{L}^{p_3}_{\mathcal{F}_T}(\Omega;\mathbb{R})$, where

$$p_3=\frac{p_1p_2}{p_1+p_2}=\frac{\frac{\alpha_0}{\alpha_0+1}p}{\frac{\alpha_0}{\alpha_0+1}+p}=\frac{\alpha_0p}{\alpha_0+p(\alpha_0+1)}\in(0,1).$$

By Lemma 4.3 in [78], BSDE (5.3.5) admits a unique adapted solution

$$\widetilde{Y}(\cdot)\in\mathcal{L}^{p_3}_{\mathcal{F}}(\Omega;C([0,T];\mathbb{R})),\quad\widetilde{Z}(\cdot)\in\mathcal{L}^{p_3}_{\mathcal{F}}(\Omega;\mathcal{L}^2([0,T];\mathbb{R}^d)).$$

By Theorem 5.4.5 (i), we have (5.5.9) with $p_4=\frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}\in(0,1)$.

Moreover,

$$p_5=\frac{p_3p_4}{p_3+p_4}=\frac{\alpha_0\alpha\beta_0^2p}{\alpha_0\beta_0^2p+2\alpha_0\alpha\beta_0p-\alpha_0\alpha p+\alpha_0\alpha\beta_0^2+\alpha_0\alpha\beta_0^2p+\alpha\beta_0^2p}=q\in(0,1),$$

The rest of proof is similar to that of Theorem (5.5.1). $\square$

Let us compare out result with Theorem 4.4 of [78]. Note that our

$$q = \frac{\alpha\beta^2 p}{\alpha\beta^2 + \alpha\beta^2 p + 2\alpha\beta p + 2\beta^2 p - \alpha p},$$

and

$$q(Y) = \frac{\alpha\beta^2 p}{\alpha\beta^2 + \alpha\beta^2 p + 2\alpha\beta\sqrt{\beta}p + 2\beta^2 p + \alpha\beta p}.$$

Hence we always have $q > q(Y)$, which implies our space of solution guarantees stronger integrability than that of [78].

5.6 Applications to Mathematical Finance

Here we give two applications of our results to mathematical finance. Let us consider a market with one bond and n stocks, the prices of which are, respectively,

$$\begin{cases} dP_0(t) = & r(t)P_0(t)dt, \\ dP_i(t) = & P_i(t)[b_i(t)dt + \sigma_i'(t)dW(t)], \quad i = 1, ..., n, \\ P_i(0) > & 0, \quad i = 0, 1, ..., n. \end{cases} \tag{5.6.1}$$

The process $r(\cdot)$ is the interest rate, the processes $b_i(\cdot)$, $i = 1, ..., n$, are called the appreciation rates, and the processes $\sigma_i'(t) = [\sigma_{i1}(t), ..., \sigma_{1d}(t)]$, $i = 1, ..., n$, are the volatilities of the stocks. If we denote by $\pi_i(t)$ the value of the holdings in asset i, then it can be shown (see, e.g. [7], [77]) that the value of a *self-financing portfolio* is

$$\begin{cases} dY(t) = & [r(t)Y(t) + \pi'(t)(b(t) - r(t)\mathbb{1})]dt + \pi'(t)\sigma(t)dW(t), \quad t \in [0, T], \\ Y(0) = & Y_0 \end{cases} \tag{5.6.2}$$

Here $\pi(t) \equiv [\pi_1(t), ..., \pi_n]'$, $b(t) \equiv [b_1(t), ..., b_n(t)]'$, $\sigma(t) \equiv [\sigma_1(t), ..., \sigma_n(t)]'$, and Y_0 is the investors initial wealth. We assume that *rank* $\sigma(t) = d$, *a.e.* $t \in [0, T]$ *a.s.*, which ensures that $n \geq d$ and $[\sigma'(t)\sigma(t)]^{-1}$ exists. If we define the processes $\theta(\cdot)$ and $Z(\cdot)$ as

$$\begin{aligned} \theta(t) &\equiv [\sigma'(t)\sigma(t)]^{-1}\sigma'(t)[b(t) - r(t)\mathbb{1}], \\ Z(t) &\equiv \sigma'(t)\pi(t), \end{aligned} \tag{5.6.3}$$

we can rewrite (5.6.2) as

$$\begin{cases} dY(t) = & [r(t)Y(t) + \theta'(t)Z(t)]dt + Z'(t)dW(t), \quad t \in [0, T], \\ Y(0) = & Y_0. \end{cases} \tag{5.6.4}$$

5.6.1 Market Completeness

The problem of pricing and hedging contingent claims is the most fundamental in mathematical finance. For a given terminal payoff $\xi \in L^0_{\mathcal{F}_T}(\Omega, \mathbb{R})$, it is required to find the initial wealth Y_0 and the trading strategy $\pi(\cdot)$ such that $Y(T) = \xi$. In this case, the initial value Y_0 is the *price* of the contingent claim at time zero, whereas the treading strategy $\pi(\cdot)$ represents the *hedging* strategy. In this case we say that the portfolio (5.6.2) *replicates* the contingent claim. The replicating portfolio does not always exist. If the market is such that for all ξ from a certain set there exist a replicating portfolio, we say that the market is *complete*. The completeness of the markets with bounded coefficients is well studied. However, in many important cases, some of the market coefficients can be unbounded. A typical example is when the interest rate $r(\cdot)$ is modeled by a stochastic differential equation. In this case much less in known about market completeness, and thus of pricing and hedging. Here we apply the results of the previous section to give alterative conditions for market completeness as compared to those of Yong [78]. We first need the following two definitions adapted from [78]

$$\begin{aligned} \Pi^p[0,T] \quad \equiv \quad & \Big\{ \pi(\cdot) \in \mathcal{L}^0_{\mathcal{F}}(0,T;\mathbb{R}^n) : \pi'(\cdot)(b(\cdot) - r(\cdot)\mathbb{1}) \in \mathcal{L}^p_{\mathcal{F}}(\Omega; \mathcal{L}^1(0,T;\mathbb{R})) \\ & \textit{and} \quad \sigma'(\cdot)\pi(\cdot) \in \mathcal{L}^p_{\mathcal{F}}(\Omega; \mathcal{L}^1(0,T;\mathbb{R}^d) \Big\}. \end{aligned}$$

Definition 5.6.1. *Let $\mathcal{H} \subset L^0_{\mathcal{F}_T}(\Omega, \mathbb{R})$ and $\Pi \subset \Pi^0[0,T]$. The market (5.6.1) is $(\mathcal{H}, \Pi)$-complete if for any $\xi \in \mathcal{H}$ there exists a solution pair $(Y(\cdot), Z(\cdot))$ to the linear BSDE*

$$\begin{cases} dY(t) = & [r(t)Y(t) + \theta'(t)Z(t)]dt + Z'(t)dW(t), \quad t \in [0,T], \\ Y(T) = & \xi \quad a.s., \end{cases} \tag{5.6.5}$$

where the process $Z(\cdot)$ is such that $\pi(\cdot)$ as given by (5.6.3) belongs to the set Π.

Since the BSDE (5.6.5) is the same as the one considered in the previous section, the proof of the next theorem is immediate from the results of that section.

Theorem 5.6.1. *Let the market coefficients be such that the processes $r(\cdot)$ and $\theta(\cdot)$ satisfy the conditions* (5.3.11), (5.3.12), (5.3.13), (5.3.14) *and* (5.3.15) *for some positive constants α_0, α, β and β_0.*

(i) If $\theta(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)$ and (5.5.12) holds, then the market (5.6.1) is $(L^{p^+}_{\mathcal{F}_T}(\Omega, \mathbb{R}), \Pi^{0^+})$-complete, with p given by (5.5.5).

(ii) If $\theta(\cdot) \in \mathcal{L}^{2+}_{\mathcal{F}}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)$ *and (5.5.1) holds, then the market (5.6.1) is* $(L^{p^+}_{\mathcal{F}_T}(\Omega, \mathbb{R}), \Pi^{1^+})$*-complete, with* p *given by (5.5.2).*

(iii) If $\theta(\cdot) \in \mathcal{L}^{2+}_{\mathcal{F}}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)$ *and (5.5.23) holds, then the market (5.6.1) is* $(L^{p^+}_{\mathcal{F}_T}(\Omega, \mathbb{R}), \Pi^{q^+})$*-complete, with* p *given by (5.5.24) and* q *given by (5.5.25).*

(iv) If $\theta(\cdot) \in \mathcal{L}^{2+}_{\mathcal{F}}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)$ *and* $\beta_0 \geq 1/2$*, then for any* $p > 0$ *the market (5.6.1) is* $(L^{p}_{\mathcal{F}_T}(\Omega, \mathbb{R}), \Pi^{q^-})$*-complete, where* q *is given by (5.5.26).*

5.6.2 Optimal investment

The problem of *optimal investment* is also fundamental in mathematical finance. Here we have an investor endowed with an initial wealth $Y(0)$ that aims to invest in a certain optimal way. For a market with deterministic coefficients, this problem has been initiated and solved by Merton [56], [57], [58], [59], whereas in a market with random coefficients and under a rather general setting, the problem has been solved in [68], [15], [35] (see also [37] and [47] for a textbook account). Explicit solutions are known only in special cases, such as for the mean-variance (quadratic) [52], exponential and power utility [22]. In all of these cases the coefficients are assumed to be *bounded*. The problem of optimal investment with a possibly unbounded coefficients has been studied to a much lesser extent (see, for example, [5], [6], [48], [49]). The unboundedness of the coefficients is due to the modeling of the interest rate (or the volatility) as solutions to stochastic differential equations with deterministic coefficients. Due to this *Markovian* nature of the model, explicit solutions are found in some cases under a further assumption that the coefficient $\theta(\cdot)$ is bounded, which enforces a special structure for the appreciation rates $b_i(\cdot)$.

Here we give the solution to the optimal investment problem with power utility in a market with *unbounded* coefficients and without the assumption of a Markovian nature of the coefficients or of a bounded coefficient $\theta(\cdot)$. We find an explicit solution to this problem using the results of the previous sections, a Riccati BSDE with unbounded coefficients, and a combination of ideas from [52] and [22]. Thus, consider an investor with an initial wealth $Y(0)$ and the power utility

$$J(Z(\cdot)) \equiv -\mathbb{E}[Y^{\lambda}(T)], \quad \lambda \in (0,1).$$

The optimal investment problem is the following *optimal control* problem:

$$\begin{cases} \min_{Z(\cdot)\in\mathcal{A}} J(Z(\cdot)), \\ s.t. \quad (5.6.4), \end{cases} \tag{5.6.6}$$

where $\mathcal{A}$ is the admissible set of processes to be defined precisely after the following two results.

Lemma 5.6.1. *Let* $\eta \equiv 2\lambda(1-\lambda)^{-1}+1$ *and the processes* $\widehat{r}(\cdot)$, $\widehat{\theta}(\cdot)$, *be defined as*

$$\begin{aligned} \widehat{r}(t) &\equiv -\lambda\eta r(t) - \frac{\lambda\eta\theta'(t)\theta(t)}{1-\lambda}, \\ \widehat{\theta}(t) &\equiv -\frac{2\lambda}{1-\lambda}\theta(t). \end{aligned}$$

Let $\widehat{\theta}(\cdot) \in \mathcal{L}^{2+}_{\mathcal{F}}(\Omega; \mathcal{L}^2([0,T];\mathbb{R}^d)$ *and the processes* $\widehat{r}(\cdot)$, $\widehat{\theta}(\cdot)$, *satisfy the conditions* (5.3.11), (5.3.12), (5.3.13), (5.3.14) *and* (5.3.15), *for some positive constants* α_0, α, β *and* β_0 *satisfying (5.5.1). Then the equation*

$$\begin{cases} dR(t) = R_1(t)dt + R_2'(t)dW(t), \quad t\in[0,T], \\ R_1(t) \equiv \widehat{r}(t)R(t) + \widehat{\theta}'(t)R_2(t), \\ R(T) = 1. \end{cases} \tag{5.6.7}$$

has a unique solution pair $R(\cdot)\in\mathcal{L}^{1+}_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}))$, $R_2(\cdot)\in\mathcal{L}^{1+}_{\mathcal{F}}(\Omega;\mathcal{L}^2([0,T];\mathbb{R}^d))$, *and* $R(t) > 0$ $\forall t\in[0,T]$ *a.s.. If* $\widehat{r}(t) < 0$ *a.e.* $t\in[0,t]$ *a.s., then* $R(t)\geq 1$ $\forall t \in [0,T]$ *a.s..*

Proof. Theorem 5.5.1 ensures the existence of a unique solution pair $R(\cdot) \in \mathcal{L}^{1+}_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}))$, $R_2(\cdot) \in \mathcal{L}^{1+}_{\mathcal{F}}(\Omega;\mathcal{L}^2([0,T];\mathbb{R}^d))$. Let $\widehat{M}(\cdot)$ denote the solution to the equation

$$\begin{cases} d\widehat{M}(t) = & -\widehat{r}(t)\widehat{M}(t)dt - \widehat{\theta}'(t)\widehat{M}(t)dW(t), \quad t\in[0,T], \\ \widehat{M}(0) = & 1, \end{cases} \tag{5.6.8}$$

and $(\widehat{R}(\cdot), \widehat{R}_2(\cdot))$ be the unique solution pair of the equation

$$\begin{cases} d\widehat{R}(t) = & \widehat{R}_2'(t)dW(t), \quad t\in[0,T], \\ \widehat{R}(T) = & \widehat{M}(T). \end{cases}$$

Due to $\widehat{R}(t) = \mathbb{E}\left[\widehat{M}(T)|\mathscr{F}_t\right]$ and (5.3.6), we have

$$\begin{aligned} R(t) &= \widehat{M}(t)^{-1}\widehat{R}(t) = \widehat{M}(t)^{-1}\mathbb{E}\left[\widehat{M}(T)|\mathscr{F}_t\right] \\ &= E\left[e^{-\int_t^T[\widehat{r}(s)+\frac{1}{2}\widehat{\theta}'(s)\widehat{\theta}(s)]ds-\int_t^T\widehat{\theta}'(s)dW(s)}\middle|\mathscr{F}_t\right]. \end{aligned} \tag{5.6.9}$$

Since the process $\widehat{\theta}(\cdot)$ is assumed to satisfy the Kazamaki condition (5.3.14), the following is a probability measure

$$\widehat{\mathbb{P}}(A) = \int_A N(T)d\mathbb{P}(\omega), \quad \forall A \in \mathscr{F},$$

where

$$N(t) \equiv e^{\frac{-1}{2}\int_0^t\widehat{\theta}'(s)\widehat{\theta}(s)ds-\int_0^t\widehat{\theta(s)}'dW(s)}.$$

We can now write (5.6.9) as

$$R(t) = \widehat{\mathbb{E}}\left[e^{-\int_t^T[\widehat{r}(s)]ds}\middle|\mathscr{F}_t\right] > 0, \tag{5.6.10}$$

where $\widehat{\mathbb{E}}[\,\cdot\,]$ is the expectation under the new probability measure $\widehat{\mathbb{P}}$. It is clear that $R(t) > 0$, $\forall t \in [0,T]$ *a.s.*, and if $\widehat{r}(t) \le 0$, *a.e.* $t \in [0,t]$ *a.s.*, then $R(t) \ge 1$, $\forall t \in [0,T]$ *a.s.*. □

Remark 5.6.1. *The condition $\widehat{r}(t) \le 0$ a.e. $t \in [0,T]$ a.s., is very reasonable from the applications point of view (e.g. the interest rate $r(t) \ge 0$ a.e. $t \in [0,T]$ a.s.), and thus we assume it for the remainder of this section to ensure that $R(t) \ge 1$, $\forall t \in [0,T]$ a.s..*

Lemma 5.6.2. *Let the conditions of Lemma 5.6.1 hold. The processes $Q(t) \equiv R^{1/\eta}(t)$ and $Q_2(t) \equiv \eta^{-1}Q^{1-\eta}(t)R_2(t)$ are a solution pair to the Riccati equation*

$$\begin{cases} dQ(t) = Q_1(t)dt + Q_2'(t)dW(t), \quad t \in [0,T], \\ Q_1(t) \equiv -\lambda r(t)Q(t) - \dfrac{\lambda(Q_2(t)+\theta(t)Q(t))'(Q_2(t)+\theta(t)Q(t))}{2(1-\lambda)Q(t)}, \\ Q(T) = 1, \\ Q(t) > 0, \quad \forall t \in [0,T] \quad a.s.. \end{cases} \tag{5.6.11}$$

Proof. The differential of $R(\cdot)$ is

$$
\begin{aligned}
dR(t) =& dQ(t)^{\eta} \\
=& \Bigg\{ \eta Q(t)^{\eta-1} \left[-\lambda r(t) Q(t) - \frac{\lambda (Q_2(t) + \theta(t)Q(t))'(Q_2(t) + \theta(t)Q(t))}{2(1-\lambda)Q(t)} \right] \\
& + \frac{\eta(\eta-1)}{2} Q(t)^{\eta-2} Q_2'(t) Q_2(t) \Bigg\} dt + \eta Q(t)^{\eta-1} Q_2'(t) dW(t) \\
=& \Bigg\{ - \lambda \eta r(t) Q(t)^{\eta} - \frac{\lambda\eta}{1-\lambda} Q(t)^{\eta-2} \big[Q_2'(t) Q_2(t) + 2 Q_2'(t) Q(t) \theta(t) \\
& + \theta'(t)\theta(t)Q(t)^2 \big] + \frac{\eta(\eta-1)}{2} Q(t)^{\eta-2} Q_2'(t) Q_2(t) \Bigg\} dt + \eta Q(t)^{\eta-1} Q_2'(t) dW(t) \\
=& \left[- \left(\lambda \eta r(t) + \frac{\lambda \eta \theta'(t)\theta(t)}{1-\lambda} \right) R(t) - \frac{2\lambda}{1-\lambda} \theta'(t) R_2(t) \right] dt + R_2'(t) dW(t), \\
=& R_1(t) dt + R_2'(t) dW(t),
\end{aligned}
$$

which has a solution for $R(T) = 1$, as shown in the previous lemma. □

The admissible set $\mathcal{A}$ is defined as:

$$
\begin{aligned}
\mathcal{A} \ = \ & \{ Z(\cdot) \in \mathcal{L}_{\mathcal{F}}^{0+}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)) | \quad Y(t) > 0 \quad \forall t \in [0,T] \quad a.s., \quad and \\
& [\lambda Q(\cdot) Z(\cdot) Y^{-1}(\cdot) + Q_2(\cdot)] Y(\cdot)^{\lambda} \in \mathcal{L}_{\mathcal{F}}^2(0,T;\mathbb{R}^d) \}.
\end{aligned}
$$

The requirement of $Y(\cdot)$ being positive prevents bankruptcy, whereas the second requirement on the admissible set is a technical one implied by the method we use to solve the optimal investment problem.

Lemma 5.6.3. *Let the processes $\overline{r}(\cdot)$ and $\overline{\theta}(\cdot)$, defined as*

$$
\begin{aligned}
\overline{r}(t) &\equiv \frac{1}{1-\lambda} r(t) + \frac{2+3\lambda}{2(1-\lambda)^2} \theta'(t)\theta(t), \\
\overline{\theta}(t) &\equiv \frac{1}{1-\lambda} \theta(t),
\end{aligned}
$$

be such that $\overline{r}(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^1([0,T];\mathbb{R}))$, $\overline{\theta}(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(\Omega; \mathcal{L}^2([0,T];\mathbb{R}^d))$. *Let* $\overline{r}(\cdot)$ *satisfy (5.3.12) and* $\overline{\theta}(\cdot)$ *satisfy (5.3.14) for some constants* $\overline{\alpha} > 0$ *and* $\overline{\beta} > 1$, *respectively, such that* $\overline{\alpha}\overline{\beta}^2 = 4(\overline{\beta}^2 + 2\overline{\alpha}\overline{\beta} - \overline{\alpha})$. *If the assumptions of Lemma 5.6.1 hold, then* $[(Q_2(\cdot) + \theta(\cdot)Q(\cdot))](1-\lambda)^{-1}Q^{-1}(\cdot)Y(\cdot) \in \mathcal{A}$.

Proof. If we choose $Z(t) = [(Q_2(t) + \theta(t)Q(t))](1-\lambda)^{-1}Q^{-1}(t)Y(t)$, then (5.6.4) becomes

$$\begin{cases} dY(t) = \left[r(t) + \theta'(t)\frac{(Q_2(t)+\theta(t)Q(t))}{(1-\lambda)Q(t)}\right]Y(t)dt + \frac{(Q_2'(t)+\theta'(t)Q(t))}{(1-\lambda)Q(t)}Y(t)dW(t), \ t \in [0,T], \\ Y(0) = Y_0. \end{cases} \tag{5.6.12}$$

We first show that this linear equation, the coefficients of which depend on the solution pair $(Q(\cdot), Q_2(\cdot))$, has a solution such that $Y(t) > 0$, $\forall t \in [0,T]$ *a.s.*. Let $\mu \in (0,1)$, and consider the equation

$$\begin{cases} dX(t) = \overline{r}(t)X(t)dt + \overline{\theta}'(t)X(t)dW(t), \quad t \in [0,T] \\ X(0) = Y(0)Q(0)^{-\mu}. \end{cases} \tag{5.6.13}$$

The process

$$X(t) \equiv e^{\int_0^t [\overline{r}(s) - \frac{1}{2}\overline{\theta}'(s)\overline{\theta}(s)]ds + \int_0^t \overline{\theta}'(s)dW(s)}, \quad t \in [0,T]. \tag{5.6.14}$$

is a solution to (6.2.1) if it has enough integrability. It is sufficient for this purpose to show that $\overline{\theta}'(\cdot)X(\cdot)$ is a square integrable process. Thus, from Theorem 5.4.3 (i), it follows that

$$\begin{aligned} \mathbb{E}\left[\int_0^T X(t)^2\overline{\theta}'(t)\overline{\theta}(t)dt\right] &\leq \mathbb{E}\left[\sup_{t\in[0,T]} X(t)^2 \int_0^T \overline{\theta}'(t)\overline{\theta}(t)dt\right] \\ &\leq \frac{1}{2}\left(\mathbb{E}\left[\sup_{t\in[0,T]} X(t)^4\right] + \mathbb{E}\left[\int_0^T \overline{\theta}'(t)\overline{\theta}(t)dt\right]^2\right) < \infty. \end{aligned}$$

In order to show that (5.6.14) is the unique solution to (6.2.1), let us assume that $\overline{X}(\cdot)$ is another solution to (6.2.1). By Itô's formula we obtain

$$\begin{aligned} dX(t)^{-1}\overline{X}(t) &= [-r(t) + \theta'(t)\theta(t)]M(t)^{-1}\overline{M}(t)dt - \theta'(t)M(t)^{-1}\overline{M}(t)dW(t) \\ &\quad + M(t)^{-1}[r(t)\overline{M}(t)dt + \theta'(t)\overline{M}(t)dW(t)] \\ &\quad - \theta'(t)\theta(t)M(t)^{-1}\overline{M}(t)dt = 0. \end{aligned} \tag{5.6.15}$$

Hence we have

$$\begin{cases} dM(t)^{-1}\overline{M}(t) = 0, \quad t \in [0,T], \\ M(0)^{-1}\overline{M}(0) = 1, \end{cases}$$

which gives $M(t)^{-1}\overline{M}(t) = 1$, i.e. $M(t) = \overline{M}(t)$, for all $t \in [0,T]$, a.s..

By an application of Itô's formula to the process $Y(t) \equiv X(t)Q(t)^{\mu}$, it can be shown that it satisfies (5.6.17). Moreover, since both $X(\cdot)$ and $Q(\cdot)$ are positive, so is $Y(\cdot)$. The proof of the uniqueness of $Y(\cdot)$ can be shown in a similar way to the uniqueness of $X(\cdot)$.

To conclude the proof, we now show that the process

$$[(Q_2(\cdot) + \theta(\cdot)Q(\cdot))](1-\lambda)^{-1}Q^{-1}(\cdot)Y(\cdot)$$

is square integrable:

$$\mathbb{E}\left[\int_0^T [\lambda Q(t)U(t) + Q_2(t)]'[\lambda Q(t)U(t) + Q_2(t)]Y(t)^{2\lambda}dt\right]$$

$$\leq \mathbb{E}\left[\int_0^T \sup_{t\in[0,T]} Y(t)^{2\lambda}[\lambda Q(t)U(t) + Q_2(t)]'[\lambda Q(t)U(t) + Q_2(t)]dt\right]$$

$$= \mathbb{E}\left[\sup_{t\in[0,T]} Y(t)^{2\lambda}\int_0^T [\lambda Q(t)U(t) + Q_2(t)]'[\lambda Q(t)U(t) + Q_2(t)]dt\right]$$

$$\leq \frac{1}{2}\,\mathbb{E}\left[\sup_{t\in[0,T]} Y(t)^{4\lambda}\right] + \frac{1}{2}\,\mathbb{E}\left\{\int_0^T [\lambda Q(t)U(t) + Q_2(t)]'[\lambda Q(t)U(t) + Q_2(t)]dt\right\}^2.$$

For the first term on the right hand side, let $Y(t) = X(t)Q(t)^{\mu}$ for any $\mu = \frac{1}{1-\lambda} \in (0,1)$, we obtain

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[0,T]} Y(t)^{4\lambda}\right] &= \mathbb{E}\left[\sup_{t\in[0,T]} X(t)^{4\lambda}Q(t)^{4\mu\lambda}\right] \\ &= \mathbb{E}\left[\sup_{t\in[0,T]} X(t)^{4\lambda}R(t)^{\frac{4\mu\lambda}{\eta}}\right] \end{aligned}$$

$$\leq \mathbb{E}\left[\sup_{t\in[0,T]} X(t)^{4\lambda}\cdot \sup_{t\in[0,T]} R(t)^{\frac{4\lambda}{1+\lambda}}\right]$$

$$\leq \frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T]} X(t)^{8} + \sup_{t\in[0,T]} R(t)^{8}\right] < \infty,$$

where the last inequality holds due to the fact that $X(t)$ is the solution of

$$\begin{cases} dX(t) = -\overline{r}(t)X(t)dt - \overline{\theta'}(t)X(t)dW(t), \\ X(0) = Y(0)Q(0)^{-\mu}, \end{cases}$$

where $\overline{r}(t) = -\left(1+\frac{\lambda}{1-\lambda}\right)r(t) - \left(\frac{\lambda}{2(1-\lambda)^2}+\frac{1}{1-\lambda}\right)\theta(t)^2$ and $\overline{\theta'}(t) = -\frac{1}{1-\lambda}\theta'(t)$, and hence Theorem 5.4.3 (i) is applied with $\frac{\alpha\beta^2}{\beta^2+2\alpha\beta-\alpha} = 4$. For the second term on the right hand side,

$$\mathbb{E}\left\{\int_0^T [\lambda Q(t)U(t)+Q_2(t)]'[\lambda Q(t)U(t)+Q_2(t)]dt\right\}^2$$

$$= \mathbb{E}\left\{\int_0^T \left[1+\frac{2\lambda}{1-\lambda}+\frac{\lambda^2}{(1-\lambda)^2}\right]Q_2'(t)Q_2(t) + \left[\frac{2\lambda}{1-\lambda}+\frac{2\lambda^2}{(1-\lambda)^2}\right]Q_2'(t)\theta(t)Q(t) + \frac{\lambda^2}{(1-\lambda)^2}\theta'(t)\theta(t)Q(t)^2dt\right\}^2$$

$$\leq 3k_1^2\,\mathbb{E}\left[\int_0^T Q_2'(t)Q_2(t)dt\right]^2 + 3k_2^2\,\mathbb{E}\left[\int_0^T Q_2'(t)Q(t)\theta(t)dt\right]^2 + 3k_3^2\,\mathbb{E}\left[\int_0^T \theta'(t)\theta(t)Q(t)^2dt\right]^2$$

$$\leq \frac{3k_1^2}{\eta^4}\,\mathbb{E}\left[\int_0^T R_2'(t)R_2(t)dt\right]^2 + \frac{3k_3^2}{4}\,\mathbb{E}\left[\int_0^T \left[(\theta'(t)\theta(t))^2 + R(t)^{\frac{4}{\eta}}\right]dt\right]^2 + \frac{3k_2^2}{4\eta^2}\,\mathbb{E}\left[\int_0^T \left[R_2'(t)R(t)+\frac{1}{2}(\theta'(t)\theta(t))^2+\frac{1}{2}R(t)^{\frac{4}{\eta}}\right]dt\right]^2$$

$$\leq c_1\,\mathbb{E}\left[\int_0^T R_2'(t)R_2(t)dt\right]^2 + c_2\,\mathbb{E}\left[\int_0^T \frac{1}{2}(\theta'(t)\theta(t))^2dt\right]^2 + c_3\,\mathbb{E}\left[\int_0^T R(t)^{\frac{4}{\eta}}dt\right]^2$$

$$\leq c_1\, \mathbb{E}\left[\int_0^T R_2'(t)R_2(t)dt\right]^2 + c_2\, \mathbb{E}\left[\int_0^T \frac{1}{2}(\theta'(t)\theta(t))^2 dt\right]^2 + c_3 T^2\, \mathbb{E}\left[\sup_{t\in[0,T]} R(t)^8 dt\right]$$

$$< \infty,$$

where $k_1 = 1 + \frac{2\lambda}{1-\lambda} + \frac{\lambda^2}{(1-\lambda)^2}$, $k_2 = \frac{2\lambda}{1-\lambda} + \frac{2\lambda^2}{(1-\lambda)^2}$ $k_3 = \frac{\lambda^2}{(1-\lambda)^2}$ and $c_1 = \frac{3k_1^2}{\eta^4} + \frac{9k_2^2}{4\eta^2}$, $c_2 = \frac{9k_2^2}{4\eta^2} + 6k_3^2$, $c_3 = \frac{3k_3^2}{2} + \frac{9k_2^2}{16\eta^2}$. Therefore the process $[\lambda Q(t)U(t) + Q_2(t)]Y(t)^\lambda$ is square integrable. □

Theorem 5.6.2. *Let the conditions of Lemma 5.6.3 hold. The optimal investment problem (5.6.6) has a unique solution given by*

$$Z^*(t) = \frac{(Q_2(t) + \theta(t)Q(t))}{(1-\lambda)Q(t)} Y(t). \tag{5.6.16}$$

The corresponding optimal cost is $J(Z^*(\cdot)) = -Q(0)Y^\lambda(0)$.

Proof. We first show that $Z^*(\cdot) \in \mathcal{A}$. Substituting (5.6.16) into (5.6.4) gives

$$\begin{cases} dY(t) = \left[r(t) + \theta'(t)\dfrac{(Q_2(t) + \theta(t)Q(t))}{(1-\lambda)Q(t)}\right] Y(t)dt \\ \qquad\qquad + \dfrac{(Q_2'(t) + \theta'(t)Q(t))}{(1-\lambda)Q(t)} Y(t)dW(t), \quad t \in [0,T], \\ Y(0) = Y_0. \end{cases} \tag{5.6.17}$$

This is a linear stochastic differential equation with a possibly unbounded coefficients. Moreover, the coefficients depend on the solution pair $(Q(\cdot), Q_2(\cdot))$ of the Riccati equation (5.6.11), and thus the known results on the existence of solutions to the linear stochastic differential equations, do not apply (see, for example, [77] for the equations with bounded coefficients). However, due to the estimates, one can apply Itô's formula to the process $Y^*(\cdot) \equiv X(\cdot)Q(\cdot)^{1/1-\lambda}$ to show that it is a solution to (5.6.17) (we omit the details for brevity). In order to show that $Y^*(\cdot)$ is the unique solution of (5.6.17), let $\overline{Y}(\cdot)$ be another solution of (5.6.17). The differential of the process $\overline{Y}(\cdot)(Y^*(\cdot))^{-1}$ is $d\overline{Y}(t)(Y^*(t))^{-1} = 0$, and since $\overline{Y}(0)(Y^*(0))^{-1} = 1$, we conclude that $\overline{Y}(t) = Y^*(t)$, $\forall t \in [0,T]$, *a.s.*. We have thus shown that (5.6.17) has a unique positive solution. Due to the estimate, we conclude that $Z^*(\cdot) \in \mathcal{A}$.

We now show that $Z^*(\cdot)$ is the unique minimizer of $J(Z(\cdot))$. Thus for any

$Z(\cdot) \in \mathcal{A}$, the differential of $Q(\cdot)Y^{\lambda}(\cdot)$ is

$$\begin{aligned}
dQ(t)Y(t)^{\lambda} =& Q_1(t)Y(t)^{\lambda}dt + Q_2'(t)Y(t)^{\lambda}dW(t) + \lambda Y(t)^{\lambda-1}Q_2'(t)Z(t)dt \\
&+ Q(t)[\lambda r(t)Y(t)^{\lambda} + \lambda Y(t)^{\lambda-1}\theta'(t)Z(t) \\
&+ \frac{1}{2}\lambda(\lambda-1)Y(t)^{\lambda-2}Z'(t)Z(t)]dt + \lambda Q(t)Y(t)^{\lambda-1}Z'(t)dW(t) \\
=& \Bigg\{ Q_1(t)Y(t)^{\lambda} + \lambda r(t)Q(t)Y(t)^{\lambda} + Y(t)^{\lambda}\Bigg[\lambda Q_2'(t)U(t) \\
&+ \lambda Q(t)\theta'(t)U(t) + \frac{\lambda(\lambda-1)}{2}Q(t)U'(t)U(t)\Bigg]\Bigg\}dt \\
&+ \left[Q_2'(t)Y(t)^{\lambda} + \lambda Q(t)Y(t)^{\lambda}U'(t)\right]dW(t),
\end{aligned}$$

where $U(t) \equiv Z(t)/Y(t)$. After integration and taking the expectation, this becomes

$$\begin{aligned}
&-\mathbb{E}\left[Y(T)^{\lambda}\right] \\
=& -Q(0)Y(0) - \mathbb{E}\int_0^T [Q_1(t)Y(t)^{\lambda} + \lambda r(t)Q(t)Y(t)^{\lambda}]dt \\
&- \mathbb{E}\int_0^T Y(t)^{\lambda}\left[(\lambda Q_2'(t) + \lambda\theta'(t)Q(t))U(t) + \frac{\lambda(\lambda-1)}{2}Q(t)U'(t)U(t)\right]dt.
\end{aligned}$$

By the completion of squares method, we obtain

$$\begin{aligned}
-\mathbb{E}\left[Y(T)^{\lambda}\right] &= -Q(0)Y(0) \\
&- \mathbb{E}\int_0^T Y(t)^{\lambda}\left[Q_1(t) + \lambda r(t)Q(t) + \frac{\lambda(Q_2(t)+\theta(t)Q(t))'(Q_2(t)+\theta(t)Q(t))}{2(1-\lambda)Q(t)}\right]dt \\
&+ \frac{\lambda(1-\lambda)}{2}\mathbb{E}\int_0^T Y(t)^{\lambda}Q(t)\left[U(t) - \frac{Q(t)^{-1}(Q_2(t)+\theta(t)Q(t))}{1-\lambda}\right]' \\
&\times \left[U(t) - \frac{Q(t)^{-1}(Q_2(t)+\theta(t)Q(t))}{1-\lambda}\right]dt
\end{aligned}$$

$$
\begin{aligned}
&= - Q(0)Y(0)\\
&\quad + \frac{\lambda(1-\lambda)}{2}\mathbb{E}\int_0^T Y(t)^\lambda Q(t)\left[U(t) - \frac{(Q_2(t)+\theta(t)Q(t))}{(1-\lambda)Q(t)}\right]'\\
&\quad \times \left[U(t) - \frac{(Q_2(t)+\theta(t)Q(t))}{(1-\lambda)Q(t)}\right] dt\\
&\geq - Q(0)Y(0),
\end{aligned}
$$

with equality if and only if $U(t) = \frac{Q_2(t)+\theta(t)Q(t)}{(1-\lambda)Q(t)}$, *a.e.* $t \in [0,T]$ *a.s.*. □

5.7 Conclusion

In this chapter, we have given new sufficient conditions for the integrability of an exponential process, and have applied such results to the solvability of linear backward stochastic differential equations with a possibly unbounded coefficients. The solvability is proved under various conditions on the terminal value and on the coefficients of the equation. Applications to market completeness, pricing, and optimal investment are also given. In these applications we have used only some of our results, and similar applications can be obtained by applying our other solvability results.

Appendix

Proof of Lemma 5.4.1

Firstly let check if $q > 1$ is always satisfied. From the second equation in (5.4.1), we have

$$q = \frac{\beta(\gamma-1)}{2(p\gamma+\sqrt{p\gamma})} > 1 \Rightarrow \frac{\beta(\gamma-1)}{2} > p\gamma + \sqrt{p\gamma}.$$

Let $\sqrt{p\gamma} = z > 0$, we have

$$z^2 + z - \frac{\beta(\gamma-1)}{2} < 0$$

and

$$z_{1,2} = \frac{-1 \pm \sqrt{1+2\beta(\gamma-1)}}{2}.$$

In order to have $(z-z_1)(z-z_2) < 0$, it is necessary to have

$$z_2 < z < z_1 = \frac{-1 + \sqrt{1+2\beta(\gamma-1)}}{2},$$

i.e.

$$z = \sqrt{p\gamma} < \frac{-1 + \sqrt{1+2\beta(\gamma-1)}}{2}. \tag{5.7.1}$$

Thus, in order to have $q > 1$, (5.7.1) should be satisfied.

Substituting z into (5.4.1) together with $z > 0$, we have,

$$z = \frac{-1 + \sqrt{1+2\beta(\gamma-1) + \frac{\beta^2}{\alpha_0}(\gamma-1)}}{2 + \frac{\beta}{\alpha_0}}. \tag{5.7.2}$$

Note that clearly $\Delta = 1 + 2\beta(\gamma-1) + \frac{\beta^2}{\alpha_0}(\gamma-1) > 0$ holds. So (5.7.1) can be written as

$$\frac{-1 + \sqrt{1+2\beta(\gamma-1) + \frac{\beta^2}{\alpha_0}(\gamma-1)}}{2 + \frac{\beta}{\alpha_0}} < \frac{-1 + \sqrt{1+2\beta(\gamma-1)}}{2}. \tag{5.7.3}$$

Denote that $A = 2\beta(\gamma-1) > 0$, $B = \frac{\beta}{2\alpha_0} + 1 > 1$. Then (5.7.3) becomes

$$\frac{-1+\sqrt{1+AB}}{2B} < \frac{-1+\sqrt{1+A}}{2}$$

$$\Rightarrow \sqrt{1+AB} < -\frac{\beta}{2\alpha_0} + \sqrt{\frac{\beta^2}{4\alpha_0^2} + \frac{\beta}{\alpha_0} + 1 + AB^2}.$$

Clearly, the above inequality always holds for any $\gamma > 1$. Hence $q > 1$ is always satisfied.

Now let us analyse the range of value of p. By (5.7.2), we have

$$p = \frac{\left[-1 + \sqrt{1+2\beta(\gamma-1) + \frac{\beta^2}{\alpha_0}(\gamma-1)}\right]^2}{\left(2 + \frac{\beta}{\alpha_0}\right)^2 \gamma}. \tag{5.7.4}$$

Therefore for $\gamma \in (1, \infty)$, we have

$$p(1) = \lim_{\gamma \to 1^+} p(\gamma) = \frac{-1+1}{(2+\frac{\beta}{\alpha_0})^2} = 0.$$

$$p(\infty) = \lim_{\gamma\to\infty} p(\gamma) = \frac{\alpha_0\beta}{\beta + 2\alpha_0}.$$

Setting $\frac{dp}{d\gamma} = 0$, we have

$$\gamma^* = \frac{-\alpha_0 + 2\beta\alpha_0 + \beta^2}{\alpha_0}. \tag{5.7.5}$$

Note that when $\beta > 1$, then $\gamma^* > 1$ always holds. Therefore,

$$\begin{aligned} p(\gamma^*) &= \frac{(-2\alpha_0 + 2\beta\alpha_0 + \beta^2)^2\alpha_0}{(-\alpha_0 + 2\beta\alpha_0 + \beta^2)(2\alpha_0 + \beta)^2} < \frac{(-2\alpha_0 + 2\beta\alpha_0 + \beta^2)^2\alpha_0}{(-2\alpha_0 + 2\beta\alpha_0 + \beta^2)(2\alpha_0 + \beta)^2} \\ &= \frac{(-2\alpha_0 + 2\beta\alpha_0 + \beta^2)\alpha_0}{(2\alpha_0 + \beta)^2} = \frac{\frac{\beta^2}{\alpha_0} + 2\beta - 2}{\left(\frac{\beta}{\alpha_0} + 2\right)^2} \\ &< \frac{\frac{\beta^2}{\alpha_0} + 2\beta}{\left(\frac{\beta}{\alpha_0} + 2\right)^2} = p(\infty). \end{aligned}$$

Therefore when $\gamma \to \infty$, $p = \frac{\alpha_0\beta}{\beta + 2\alpha_0}$ is the global maximum of $p(\gamma)$. Thus the result follows. □

Proof of Lemma 5.4.2

From system (5.4.2) we have

$$p_{1,2} = \frac{\beta(\gamma - 1) + 1 \pm \sqrt{2\beta(\gamma - 1) + 1}}{2\gamma} \tag{5.7.6}$$

Therefore for $\gamma \in (1, \infty)$, we have

$$p_1(1) = \lim_{\gamma\to 1^+} p_1(\gamma) = \frac{1+1}{2} = 1.$$

$$p_1(\infty) = \lim_{\gamma\to\infty} p_1(\gamma) = \frac{\beta}{2}.$$

$$p_2(1) = \lim_{\gamma\to 1^+} p_2(\gamma) = \frac{1-1}{2} = 0.$$

$$p_2(\infty) = \lim_{\gamma\to\infty} p_2(\gamma) = \frac{\beta}{2}.$$

Setting $\frac{dp_1}{d\gamma} = 0$, we have $\gamma^* = 2\beta - 1$. So if $\beta > 1$, then

$$p_1(\gamma^*) = \frac{\beta^2}{2\beta - 1}.$$

Hence we have $p_1(\gamma^*) > p_1(\infty) = \frac{\beta^2}{2\beta}$. Similarly, we obtain $p_2(\gamma^*) = \frac{(\beta-1)^2}{2\beta-1} < p_1(\gamma^*)$. Hence the result follows. □

Chapter 6

Stochastic Differential Equations with Unbounded Coefficients

6.1 Abstract

In this chapter, we consider a type of nonlinear stochastic differential equations (SDEs) where the drift and diffusion coefficients are under Lipschitz-type and linear growth condition, which can also be unbounded. We give sufficient conditions for the existence of a unique solution to this type of SDE. The method of proof is that of Picard iterations and the resulting conditions are new. A new comparison theorem is also given. Note that this is a preliminary result so far. Also this work is based on a preprint paper [27].

6.2 Introduction

We consider the following stochastic differential equation (SDE):

$$x(t) = x(0) + \int_0^t f(s, x(s))ds + \int_0^t g(s, x(s))dW(s), \quad t \in [0, T], \tag{6.2.1}$$

under similar mathematical setting assumed in previous chapters.

In mathematics, the theory of stochastic differential equations (SDEs) has systematically formulated by Itô in 1940s. By using Picard's method of successive approximations, Itô [33] establishes the existence and uniqueness of solutions to SDEs under a sufficient condition, Lipschitz condition, i.e. under the assumption that there exists positive real constants c_1 and c_2 such that

$$|f(t,x_1) - f(t,x_2)| \leq c_1|x_1 - x_2|, \quad |g(t,x_1) - g(t,x_2)| \leq c_2|x_1 - x_2|, \qquad (6.2.2)$$

for all $x_1, x_2 \in \mathbb{R}^d$, (t,ω) *a.e.*. Since then, the SDEs have been studied extensively, and have been widely accepted as an important mathematical tool in areas such as mathematical finance, stochastic control, engineering etc. See textbooks, for example, [45] , [55] and [69].

One direction of research has been to weaken the assumption of global Lipschitz condition (6.2.2) by assuming only local Lipschitz condition, or non-Lipschitz condition of a particular form (see [13], [74]), which generalise Itô criterion. In particular, in [74], they assume that f and g satisfy moduli of continuity conditions in the second variable, x, i.e.

$$|f(t,x_1) - f(t,x_2)| \leq \kappa(|x_1 - x_2|), \ \forall x_1, x_2 \in \mathbb{R}^d,$$

$$|g(t,x_1) - g(t,x_2)| \leq \rho(|x_1 - x_2|), \ \forall x_1, x_2 \in \mathbb{R}^d,$$

where $\kappa(u)$, $u \in (0,\infty)$ is a positive increasing concave function such that $\int_{0+} \kappa^{-1}(u)du = +\infty$ and $\rho(u)$, $u \in (0,\infty)$ is a positive increasing function such that $\int_{0+} \rho^{-2}(u)du = +\infty$. The above conditions are known as Yamada-Watanabe's condition, which is weaker than the Lipschitz condition.

However very few effort has been put in studying the SDEs with possible unbounded coefficients. The interest in these equations is not only theoretical, but is also motivated by applications in mathematical finance. Indeed, some very important interest rate models are given by stochastic differential equations (see, for example, [7], [16], [79]). Moreover, solutions to these SDEs can be unbounded. Gyöngy and Martńez [28] present the existence and uniqueness theorem for SDE with locally unbounded drift coefficient but bounded diffusion coefficient. In [18], by the BMO martingale theory, Delbaen and Tang study nonlinear SDEs with the unbounded coefficients f and g. To be more specific, they assume the following global Lipschitz-type condition: there exist non-negative adapted processes $c_1(\cdot)$ and $c_2(\cdot)$ such that

$$|f(t,x_1) - f(t,x_2)| \leq c_1(t)|x_1 - x_2|, \quad |g(t,x_1) - g(t,x_2)| \leq c_2(t)|x_1 - x_2|, \quad (6.2.3)$$

for all $x_1, x_2 \in \mathbb{R}^d$, (t,ω) *a.e.*. Clearly, these conditions have great similarity with (6.2.2). However, different from (6.2.2), here the processes $c_1(\cdot)$ and $c_2(\cdot)$ are not assumed to be bounded. The main idea in [18] is using the contraction mapping principle to look for a fix point.

The first contribution of the present chapter, which is contained in Section 6.3, is to consider the problem of existence and uniqueness of a solution to (6.2.1) under the condition (6.2.3). We do so under certain new conditions on the coefficients $c_1(\cdot)$, $c_2(\cdot)$, which have a similarity of those of [18]. Moreover, our method of proof is different, since it is a modification of the Picard iteration procedure rather than being based on a fixed point theorem in [18].

In addition, the second contribution of the present chapter, which is contained in Section 6.4, is to consider a comparison theorem for this class of equations with same diffusion coefficients. This generalises the classical comparison result of SDEs (see, for example, [1], [32], [75]) to the case of possibly unbounded coefficients.

We conclude this introductory section with some notations:

- $0 < \kappa, C \in \mathbb{R}$ are given constant.

- $c_i(\cdot) \in M^2(0,T;\mathbb{R}^d)$, $i = 1,3,4$ are given positive $\mathbb{R}$-valued progressively measurable processes and $c_2(\cdot) \le C$ is a given positive $\mathbb{R}$-valued measurable process;

- $\alpha(t) = c_1^2(t) + 2c_1(t) + c_2^2(t)$;

- $p(t) \equiv \exp\Big[-\int_0^t \alpha(s)ds\Big]$.

6.3 Unbounded Lipschitz-type Drift and Diffusion Coefficients

In this section, we give sufficient conditions for the existence and uniqueness of a solution to

$$x(t) = x(0) + \int_0^t f(s,x(s))ds + \int_0^t g(s,x(s))dW(s), \quad t \in [0,T], \tag{6.3.1}$$

under the following assumptions A on f and g:

(i) $x(0) \in M^2(\Omega, \mathscr{F}_0, \mathbb{P}; \mathbb{R}^d)$;

(ii) $g(t,x) \equiv g(t) + h(t,x)$, where $g(t) \in M^2(0,T;\mathbb{R}^d)$ is a possibly unbounded process and $h(t,x) \in M^2(0,T;\mathbb{R}^d)$ is an adapted process satisfies the following the Lipschitz condition;

(iii) Lipschitz condition:

$$|f(t,x_1) - f(t,x_2)| \leq c_1(t)|x_1 - x_2|,$$

$$|h(t,x_1) - h(t,x_2)| \leq c_2(t)|x_1 - x_2|,$$

for all $x_1, x_2 \in \mathbb{R}^d$, (t,ω) *a.e.*;

(iv) Linear growth bound:

$$|f(t,x)|^2 \leq c_3^2(t) + \kappa|x|^2,$$

$$|g(t,x)|^2 \leq c_4^2(t) + \kappa|x|^2,$$

for all $x \in \mathbb{R}^d$, (t,ω) *a.e.*;

(v) $f(\cdot,0) \in M^2(0,T;\mathbb{R}^d)$;

An interesting example of an SDE that satisfies the above conditions is the *self-financing portfolio.* In this case, for some constants r, b, σ, and some suitable process $u(t)$, we have

$$dx(t) = [rx(t) + bu(t)]dt + \sigma u(t)dW(t), \tag{6.3.2}$$

which clearly satisfies the above conditions. Thus, the equations considered in this chapter can be seen as generalisations of (6.3.2).

Now we begin with a simpler form of (6.2.1) and then progress towards a more general case.

Lemma 6.3.1. *Let $\phi(\cdot) \in M^2(0,T;\mathbb{R}^d)$ be given and assume that f and g satisfy the assumptions A. Then*
(i) There exists a unique solution $x(\cdot) \in M^2(0,T;\mathbb{R}^d)$ of equation

$$x(t) = x(0) + \int_0^t f(s,\phi(s))ds + \int_0^t g(s,\phi(s))dW(s), \quad t \in [0,T]. \tag{6.3.3}$$

(ii) The process

$$\int_0^t p(s)x(s)g(s,x(s))dW(s)$$

is a martingale.

Proof. (i) We first show that $\int_0^t f(s,\phi(s))ds$ is well-defined. Note that

$$\mathbb{E}\int_0^t |f(s,\phi(s))|ds \leq \mathbb{E}\int_0^t [|f(s,\phi(s))|]\, ds$$

$$\leq \mathbb{E}\int_0^t [|f(s,\phi(s)) - f(s,0)| + |f(s,0)|]\, ds + \frac{1}{2}\,\mathbb{E}\int_0^t |g(s,\phi(s))|^2 ds$$

$$\leq \frac{1}{2}\,\mathbb{E}\int_0^t c_1^2(s)ds + \frac{1}{2}\,\mathbb{E}\int_0^t c_4^2(s)ds + \left[\frac{1+\kappa}{2}\right]\mathbb{E}\int_0^t |\phi(s)|^2 ds$$

$$+\,\mathbb{E}\int_0^t |f(s,0)|ds < \infty.$$

Therefore $\int_0^t f(s,\phi(s))ds$ is well-defined.

Secondly, we show that if $\phi(\cdot) \in M^2(0,T;\mathbb{R}^d)$, then $x(\cdot) \in M^2(0,T;\mathbb{R}^d)$ as well. Using the elementary inequality $|a+b+c|^2 \leq 3|a|^2 + 3|b|^2 + 3|c|^2$, the Cauchy-Schwartz inequality, Itô isometry (see, for example, (2.3.1)) and condition (iv), we obtain

$$\mathbb{E}|x(t)|^2 \leq 3\,\mathbb{E}|x(0)|^2 + 3\,\mathbb{E}\left|\int_0^t f(s,\phi(s))ds\right|^2 + 3\,\mathbb{E}\left|\int_0^t g(s,\phi(s))dW(s)\right|^2$$

$$\leq 3\,\mathbb{E}|x(0)|^2 + 6\,\mathbb{E}\left[\left|\int_0^t f(s,\phi(s))ds\right|^2\right] + 3\,\mathbb{E}\int_0^t |g(s,\phi(s))|^2 ds$$

$$\leq 3\,\mathbb{E}|x(0)|^2 + 6\,\mathbb{E}\left[\left|\int_0^t |f(s,\phi(s))|^2 ds\right|\left|\int_0^t 1^2 ds\right|\right]$$

$$+\,3\,\mathbb{E}\int_0^t |g(s,\phi(s))|^2 ds$$

$$\leq 3\,\mathbb{E}|x(0)|^2 + 6T\,\mathbb{E}\left[\int_0^t [c_3^2(s) + \kappa|\phi(s)|^2]ds\right]$$

$$+\,3\,\mathbb{E}\left[\int_0^t [c_4^2(s) + \kappa|\phi(s)|^2]ds\right] < \infty \tag{6.3.4}$$

which implies that $x(\cdot) \in M^2(0,T;\mathbb{R}^d)$. Therefore equation (6.3.3) admits a unique solution $x(\cdot) \in M^2(0,T;\mathbb{R}^d)$.

(ii) From the Burkholder-Davis-Gundy inequality (see, for example, Theorem 2.4.8) and the elementary inequality $|ab| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$, there exists a constant K such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t p(s)x(s)g(s,x(s))dW(s)\right|\right]$$

$$\leq K\,\mathbb{E}\left[\int_0^T |\sqrt{p(s)}x(s)|^2|\sqrt{p(s)}g(s,x(s))|^2 ds\right]^{\frac{1}{2}}$$

$$= K\,\mathbb{E}\left[\sup_{t\in[0,T]} |\sqrt{p(t)}x(t)|^2 \int_0^T \sqrt{p(s)}g(s,x(s))|^2 ds\right]^{\frac{1}{2}}$$

$$\leq \frac{K}{2}\,\mathbb{E}\left[\sup_{t\in[0,T]} |\sqrt{p(t)}x(t)|^2 + \int_0^T |\sqrt{p(s)}g(s,x(s))|^2 ds\right]$$

$$\leq \frac{K}{2}\,\mathbb{E}\left[\sup_{t\in[0,T]} |x(t)|^2 + \int_0^T |g(s,x(s))|^2 ds\right] < \infty,$$

where the last step follows from the fact (similar to part (i) above) that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |x(t)|^2\right]$$

$$\leq 3\,\mathbb{E}\left[\sup_{t\in[0,T]} |x(0)|^2\right] + 3\,\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t f(s,\phi(s))ds\right|^2\right]$$

$$+ 3\,\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t g(s,\phi(s))dW(s)\right|^2\right]$$

$$\le 3\,\mathbb{E}|x(0)|^2 + 6\,\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t f(s,\phi(s))ds\right|^2\right] + 3\,\frac{8}{2(2-1)}\mathbb{E}\int_0^T |g(s,\phi(s))|^2 ds$$

$$\le 3\,\mathbb{E}|x(0)|^2 + 6\,\mathbb{E}\left[\left|\int_0^T |f(s,\phi(s))|^2 ds\right|\left|\int_0^T 1^2 ds\right|\right]$$

$$+ 12\,\mathbb{E}\int_0^T |g(s,\phi(s))|^2 ds$$

$$\le 3\,\mathbb{E}|x(0)|^2 + 6T\,\mathbb{E}\left[\int_0^T [c_3^2(s)+\kappa|\phi(s)|^2]ds\right] + 12\,\mathbb{E}\left[\int_0^T [c_4^2(s)+\kappa|\phi(s)|^2]ds\right] < \infty,$$

and

$$\mathbb{E}\int_0^t |g(s,x(s))|^2 ds \le \mathbb{E}\left[\int_0^t [c_4^2(s)+\kappa|x(s)|^2]ds\right] < \infty.$$

The conclusion then follows from Corollary 2.4.1. □

Now we present our main result in this chapter.

Theorem 6.3.1. *The SDE* (6.2.1) *has a unique solution* $x(\cdot) \in \widehat{M}^2(0,T;\mathbb{R}^d)$.

Proof. (*Uniqueness*) Let $x_1(\cdot)$ and $x_2(\cdot)$ be two solutions of (6.2.1) with the claimed properties. Then by Itô product rule and the Lipschitz property of f and g, we obtain

$$d\,p(t)|x_1(t)-x_2(t)|^2$$

$$= -\,\alpha(t)p(t)\,|x_1(t)-x_2(t)|^2 dt + 2p(t)(x_1(t)-x_2(t))'\,[f(t,x_1(t))-f(t,x_2(t))]\,dt$$

$$+ p(t)\,|g(t,x_1(t))-g(t,x_2(t))|^2\,dt$$

$$+ 2p(t)(x_1(t)-x_2(t))'\,[g(t,x_1(t))-g(t,x_2(t))]\,dW(t)$$

$$\le -\,\alpha(t)p(t)\,|x_1(t)-x_2(t)|^2 dt + 2c_1(t)p(t)\,|x_1(t)-x_2(t)|^2 dt$$

$$+ c_2^2(t)p(t)\,|x_1(t) - x_2(t)|^2 dt$$

$$+ 2p(t)(x_1(t) - x_2(t))'\,[g(t, x_1(t)) - g(t, x_2(t))]\,dW(t)$$

$$\le 2p(t)(x_1(t) - x_2(t))'\,[g(t, x_1(t)) - g(t, x_2(t))]\,dW(t)$$

which in integral form becomes

$$p(t)|x_1(t) - x_2(t)|^2 \le \int_0^t 2p(s)(x_1(s) - x_2(s))'\,[g(t, x_1(s)) - g(t, x_2(s))]\,dW(s). \tag{6.3.5}$$

From Lemma 3.4.1 (ii), it is clear that the stochastic integral on the right hand side is a martingale. Taking the expectation of both sides of (6.3.5) results in

$$\mathbb{E}\left[p(t)|x_1(t) - x_2(t)|^2\right] \le \mathbb{E}\left[\int_0^t 2p(s)(x_1(s) - x_2(s))'\,[g(t, x_1(s)) - g(t, x_2(s))]\,dW(s)\right]$$
$$= 0.$$

Since $p(t) > 0$, it follows that $x_1(t) = x_2(t)$, $\forall\ t \in [0, T]$, a.s., which proves the uniqueness of $x(\cdot)$.

(*Existence*) Let $x_0(t) = 0$, for $n \ge 1$, consider the following sequence of equations:

$$x_n(t) = x(0) + \int_0^t f(s, x_{n-1}(s))ds + \int_0^t g(s, x_{n-1}(s))d(s), \quad t \in [0, T]. \tag{6.3.6}$$

From Lemma 6.3.1 we know that these equations have unique solutions $\{x_n(\cdot) \in \widehat{M}^2(0, T; \mathbb{R}^d)\}_{n \ge 1}$. Similarly to the proof of uniqueness, we obtain

$$d\,p(t)|x_{n+1}(t) - x_n(t)|^2$$

$$= -\alpha(t)p(t)|x_{n+1}(t) - x_n(t)|^2 dt$$

$$+ 2p(t)(x_{n+1}(t) - x_n(t))'\,[f(t, x_n(t)) - f(t, x_{n-1}(t))]\,dt$$

$$+ p(t)\,|g(t, x_n(t)) - g(t, x_{n-1}(t))|^2\,dt$$

$$+ 2p(t)(x_{n+1}(t) - x_n(t))'\,[g(t, x_n(t)) - g(t, x_{n-1}(t))]\,dW(t)$$

$$
\begin{aligned}
\leq & -\alpha(t)p(t)|x_{n+1}(t)-x_n(t)|^2dt + 2c_1(t)p(t)|x_{n+1}(t)-x_n(t)||x_n(t)-x_{n-1}(t)|dt \\
& + c_2^2(t)p(t)|x_n(t)-x_{n-1}(t)|^2dt \\
& + 2p(t)(x_{n+1}(t)-x_n(t))'\left[g(t,x_n(t))-g(t,x_{n-1}(t))\right]dW(t) \\
\leq & -\alpha(t)p(t)|x_{n+1}(t)-x_n(t)|^2dt + c_1^2(t)p(t)|x_{n+1}(t)-x_n(t)|^2dt \\
& + p(t)|x_n(t)-x_{n-1}(t)|^2dt + c_2^2(t)\alpha_2^2(t)p(t)|x_{n+1}(t)-x_n(t)|^2dt \\
& + p(t)|x_n(t)-x_{n-1}(t)|^2dt + c_2^2(t)p(t)|x_{n+1}(t)-x_n(t)|^2dt \\
& + 2p(t)(x_{n+1}(t)-x_n(t))'\left[g(t,x_{n+1}(t))-g(t,x_n(t))\right]dW(t) \\
\leq & \left[2+C^2\right]p(t)|x_n(t)-x_{n-1}(t)|^2dt \\
& + 2p(t)(x_{n+1}(t)-x_n(t))'\left[g(t,x_n(t))-g(t,x_{n-1}(t))\right]dW(t)
\end{aligned}
$$

which in integral form becomes

$$
\begin{aligned}
& p(t)|x_{n+1}(t)-x_n(t)|^2 \\
\leq & \left[2+C^2\right]\int_0^t p(s)|x_n(s)-x_{n-1}(s)|^2]ds \\
& + \int_0^t 2\,p(s)(x_{n+1}(s)-x_n(s))'\left[g(s,x_n(s))-g(s,x_{n-1}(s))\right]dW(s).
\end{aligned} \tag{6.3.7}
$$

From Lemma 6.3.1 (ii), it is clear that the stochastic integral on the right hand side is a martingale. Taking the expected values of both sides gives

$$
\begin{aligned}
\mathbb{E}\left[p(t)|x_{n+1}(t)-x_n(t)|^2\right] \leq & \overline{C}\,\mathbb{E}\int_0^t p(s)|x_n(s)-x_{n-1}(s)|^2ds \\
= & \overline{C}\int_0^t \mathbb{E}\left[p(s)|x_n(s)-x_{n-1}(s)|^2\right]ds,
\end{aligned} \tag{6.3.8}
$$

where $\overline{C} \equiv [2+C^2]$ and the equality follows from the Fubini's Theorem (see, for example, Theorem 2.5.2). Let us define $\eta_n(t) \equiv \mathbb{E}\int_0^t p(s)|x_n(s)-x_{n-1}(s)|^2ds$. Using the Cauchy formula

$$\int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_1} h(s)dsdt_1 \cdots dt_{n-1} = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} h(s)ds$$

in repeated iterations of (6.3.8), we obtain

$$\eta_{n+1}(0) \leq \frac{(\overline{C}t)^n}{(n)!} \int_0^t \eta_1(0)ds.$$

Since the right-hand sides of above inequality decreases with $n \to \infty$, it follows that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\widehat{M}^2(0,T;\mathbb{R}^d)$. Hence, the limiting processes $x^* = \lim_{n\to\infty} x_n$ is the solution of (6.2.1). □

6.4 A Comparison Theorem

The following result generalises a classical comparison theorem for solution of stochastic differential equations with possibly unbounded coefficients.

Theorem 6.4.1. *(Comparison theorem) Let $d = 1$ and $x(0) \leq \widehat{x}(0) \in M^2(\Omega, \mathscr{F}_0, \mathbb{P}; \mathbb{R})$ be given. Also let $\widehat{f}(t,x) : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ and $\widehat{g}(t,x) : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}^{1\times k}$ be progressively measurable functions such that:*

$$|\widehat{f}(t,x_1) - \widehat{f}(t,x_2))| \leq c_1(t)|x_1 - x_2|,$$

$$|\widehat{g}(t,x_1)) - \widehat{g}(t,x_2)| \leq c_2(t)|x_1 - x_2|,$$

for all x_1, $x_2 \in \mathbb{R}$ and

$$|\widehat{f}(t,x)|^2 \leq c_3^2(t) + \kappa|x|^2,$$

$$|\widehat{g}(t,x)|^2 \leq c_4^2(t) + \kappa|x|^2,$$

for all $x \in \mathbb{R}$ and

$$f(t,x) \leq \widehat{f}(t,x),\ g(t,x) = \widehat{g}(t,x),\ a.s.\ \forall\, t \in [0,T],$$

for any $x \in \mathbb{R}$. Then $x(t) \leq \widehat{x}(t)$, $\forall t \in [0,T]$, a.s., where $\widehat{x}(\cdot)$ is the solution of equation

$$\widehat{x}(t) = \widehat{x}(0) + \int_0^t \widehat{f}(s,\widehat{x}(s))ds + \int_0^t \widehat{g}(s,\widehat{x}(s))dW(s), \quad t \in [0,T]. \tag{6.4.1}$$

Proof. From Theorem 6.3.1, it is clear that there exists a unique solution

$\widehat{x}(\cdot) \in \widehat{M}^2(0,T;\mathbb{R})$ to (6.4.1). Denoting the difference by $X(t) \equiv x(t) - \widehat{x}(t)$, we obtain

$$dX(t) = \left[f(t,x(t)) - \widehat{f}(t,\widehat{x}(t))\right] dt + [g(t,x(t)) - \widehat{g}(t,\widehat{x}(t))]dW(t).$$

Denoting by $X^+(t) \equiv \mathbb{1}_{[X(t)>0]}X(t)$, and using Tanaka-Meyer formula (see Theorem 2.4.3), we obtain

$$dX^+(t) = \mathbb{1}_{[X(t)>0]}dX(t) + \frac{1}{2}dL^0(t),$$

where $L^0(t)$ is the local time of $X(t)$ at point 0. Since $\int_0^T |X(t)|dL^0(t) = 0$, a.s. (see Proposition 2.4.1), we have

$$\begin{aligned} d[X^+(t)]^2 = & 2X^+(t)\mathbb{1}_{[X(t)>0]}[f(t,x(t)) - \widehat{f}(t,\widehat{x}(t))]dt \\ & + \mathbb{1}_{[X(t)>0]}|g(t,x(t)) - \widehat{g}(t,\widehat{x}(t))|^2 dt \\ & + 2X^+(t)\mathbb{1}_{[X(t)>0]}[g(t,x(t)) - \widehat{g}(t,\widehat{x}(t))]dW(t). \end{aligned}$$

Using Itô formula, we obtain

$$\begin{aligned} & d\,p(t)[X^+(t)]^2 = -\alpha(t)p(t)[X^+(t)]^2 dt \\ & + 2p(t)X^+(t)\mathbb{1}_{[X(t)>0]}[f(t,x(t)) - \widehat{f}(t,\widehat{x}(t))]dt \\ & + \mathbb{1}_{[X(t)>0]}p(t)|g(t,x(t)) - \widehat{g}(t,\widehat{x}(t))|^2 dt \\ & + 2\mathbb{1}_{[X(t)>0]}p(t)X^+(t)[g(t,x(t)) - \widehat{g}(t,\widehat{x}(t))]dW(t) \\ \le & -\alpha(t)p(t)[X^+(t)]^2 dt + 2p(t)X^+(t)\mathbb{1}_{[X(t)>0]}[f(t,x(t)) - \widehat{f}(t,x(t)) \\ & + \widehat{f}(t,x(t)) - \widehat{f}(t,\widehat{x}(t))]dt \\ & + \mathbb{1}_{[X(t)>0]}p(t)|g(t,x(t)) - \widehat{g}(t,x(t)) + \widehat{g}(t,x(t)) - \widehat{g}(t,\widehat{x}(t))|^2 dt \end{aligned}$$

$$+ 2\mathbb{1}_{[X(t)>0]}p(t)X^+(t)[g(t,x(t)) - \widehat{g}(t,\widehat{x}(t))]dW(t)$$

$$\leq -\alpha(t)p(t)[X^+(t)]^2dt + 2p(t)X^+(t)\mathbb{1}_{[X(t)>0]}[c_1(t)|X(t)|]dt$$

$$+ \mathbb{1}_{[X(t)>0]}p(t)c_2^2(t)|X(t)|^2dt$$

$$+ 2\mathbb{1}_{[X(t)>0]}p(t)X^+(t)[g(t,x(t)) - \widehat{g}(t,\widehat{x}(t))]dW(t)$$

$$\leq 2\,p(t)X^+(t)[g(t,x(t)) - \widehat{g}(t,\widehat{x}(t))]dW(t),$$

which in integral form becomes

$$p(t)[X^+(t)]^2 \leq \int_0^t 2p(s)X^+(s)[g(s,x(s)) - \widehat{g}(s,\widehat{x}(s))]dW(s). \tag{6.4.2}$$

Similar to Lemma 6.3.1 (ii), we now show that the stochastic integral on the right-hand side is a martingale. In fact by the Burkholder-Davis-Gundy inequality (see Theorem 2.4.8), the elementary inequality $|ab| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$ and $|a-b|^2 \leq 2|a|^2+2|b|^2$, and linear growth condition on g and $\widehat{g}$, there exists a constant K such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t 2p(s)X^+(s)[g(s,x(s)) - \widehat{g}(s,\widehat{x}(s))]dW(s)\right|\right]$$

$$\leq K\,\mathbb{E}\left[\int_0^T \left|\sqrt{p(s)}X^+(s)\right|^2 \left|\sqrt{p(s)}[g(s,x(s)) - \widehat{g}(s,\widehat{x}(s))]\right|^2 ds\right]^{\frac{1}{2}}$$

$$\leq K\,\mathbb{E}\left[\sup_{t\in[0,T]}\left|\sqrt{p(t)}X^+(t)\right|^2 \int_0^T \left|\sqrt{p(s)}[g(s,x(s)) - \widehat{g}(s,\widehat{x}(s))]\right|^2 ds\right]^{\frac{1}{2}}$$

$$\leq \frac{K}{2}\,\mathbb{E}\left[\sup_{t\in[0,T]}\sqrt{p(t)}\left|X^+(t)\right|^2 + \int_0^T \sqrt{p(s)}[g(s,x(s)) - \widehat{g}(s,\widehat{x}(s))]^2ds\right]$$

$$\leq \frac{K}{2}\,\mathbb{E}\left[\sup_{t\in[0,T]}\left|X^+(t)\right|^2\right] + K\,\mathbb{E}\left[\int_0^T \left[|g(s,x(s))|^2 + |\widehat{g}(s,\widehat{x}(s))|^2\right] ds\right]$$

$$\leq K\,\mathbb{E}\left[\sup_{t\in[0,T]}|x(t)|^2\right]+K\,\mathbb{E}\left[\sup_{t\in[0,T]}|\widehat{x}(t)|^2\right]+K\,\mathbb{E}\left[\int_0^T\left[c_4^2(s)+\kappa|x(s)|^2\right]ds\right]$$

$$+K\,\mathbb{E}\left[\int_0^T\left[c_4^2(s)+\kappa|\widehat{x}(s)|^2\right]ds\right]<\infty, \tag{6.4.3}$$

where the last step follows from the fact that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|x(t)|^2\right]<\infty \quad \textit{and} \quad \mathbb{E}\left[\sup_{t\in[0,T]}|\widehat{x}(t)|^2\right]<\infty,$$

which are shown in Lemma 6.3.1 (ii) (simply replace $\phi(\cdot)$ by $x(\cdot)$ and $\widehat{x}(\cdot)$ respectively). Hence the stochastic integral on the right-hand side is a martingale due to Corollary 2.4.1. Taking the expected values of both sides gives,

$$\mathbb{E}[p(t)[X^+(t)]^2]\leq 0.$$

Hence the conclusion follows from the definition of $X^+(\cdot)$. □

6.5 Conclusion

In this chapter, we consider a type of nonlinear stochastic differential equations (SDEs) where the drift and diffusion coefficients are under Lipschitz-type and linear growth condition, which can also be unbounded. We give sufficient conditions for the existence of a unique solution to this type of SDEs. We propose different approach from the existing one and the resulting conditions are also new. Finally, we prove a comparison theorem for SDEs with unbounded coefficients. These results, together with results on BSDEs in Chapter 3, are expected to play an essential role in solving similar problems on forward-backward stochastic differential equations with possibly unbounded coefficients.

Chapter 7

Conclusion and Future Work

Through the various sections of this thesis, we mainly study the theory of Backward Stochastic Differential Equations with unbounded generators (coefficients) in a variety of contexts. We discuss these equations, where the generators are possibly unbounded, under different type of conditions which are shown to be more general or weaker. Moreover, classical (forward) Stochastic Differential Equations with unbounded drift and diffusion generators are also considered. Finally, based on above results, we describe some possible future works.

In each of these situations, we develop fundamental results for the existence (and/or uniqueness) of solutions, and prove an appropriate version of the comparison theorem. We show that under appropriate assumptions, versions of these results exist, even in more general situations. We obtain the sufficient conditions for the existence and uniqueness of solutions to unbounded BSDEs under Lipschitz condition on the generator; we derive the existence of solutions to unbounded BSDEs under linear growth and continuity conditions on the generator; we also give new sufficient conditions for existence of a unique solution pair to linear BSDEs which are weaker than previously known. Stronger integrability of solution is also ensured under such conditions. In addition, we consider the (forward) Stochastic Differential Equations in similar fashion and generalise a revelent comparison theorem.

Furthermore, using these results, we consider some fundamental problems in mathematical finance, which are also our motivations to study the theory of BSDEs with possibly unbounded generators. As applications, two problems of mathematical finance are considered. Namely, we resolve the question of completeness in a market with possibly unbounded coefficients in such a market and the optimal investment problem with power utility using Riccati BSDE with unbounded coefficients.

Various possible extensions to this scenario remain open. In particular, we have tried but not completely solved the well-posedness of solutions to BSDEs under non-Lipshcitz conditions (see, for instance, [54]) with unbounded generators. Such equations have shown of importance interest in various problems. Other extensions apart from the classical theory of BSDEs are also possible. Forward-Backward Stochastic Differential Equations (FBSDEs) are of significant interest in many practical problems. We expect that the our results on existence and uniqueness of solutions to nonlinear BSDEs and SDEs, i.e. in Chapter 3 and Chapter 6 will contribute to tackle the similar problems on fully-coupled FBSDEs. Another specially important model of symmetric matrix valued BSDEs with a quadratic growth in (y, z) is called the Backward Stochastic Riccati Equation. In fact, the comparison theorem of BSDE plays essential role in studying existence and uniqueness theorem in which the drift coefficient satisfies quadratic growth in z and some local Lipschitz conditions, which was studied by [46]. Hence our comparison theorem obtained in Chapter 3 will also expected to contribute to study the Backward Stochastic Riccati Equation with possibly unbounded generators. We did not consider the computational problems which arise from the general theory of BSDEs and requires further study on the convergence of solutions that is not addressed in this thesis.

The theory of BSDEs is still developing. This thesis presents results on the theory of BSDEs from the classical situation but under more weaker conditions. This provides groundwork for the extension of many existing results, both theoretical and practical, to more general situations.

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