Optimization Problems in Partial Differential Equations

Thesis submitted in accordance with the requirements of
the University of Liverpool for the degree of Doctor of Philosophy
by

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July 1, 2015

Abstract

The primary objective of this research is to investigate various optimization problems connected with partial differential equations (PDE). In chapter 2, we utilize the tool of tangent cones from convex analysis to prove the existence and uniqueness of a minimization problem. Since the admissible set considered in chapter 2 is a suitable convex set in $L^\infty(D)$, we can make use of tangent cones to derive the optimality condition for the problem. However, if we let the admissible set to be a rearrangement class generated by a general function (not a characteristic function), the method of tangent cones may not be applied. The central part of this research is Chapter 3, and it is conducted based on the foundation work mainly clarified by Geoffrey R. Burton with his collaborators near 90s, see [7, 8, 9, 10]. Usually, we consider a rearrangement class (a set comprising all rearrangements of a prescribed function) and then optimize some energy functional related to partial differential equations on this class or part of it. So, we call it rearrangement optimization problem (ROP). In recent years this area of research has become increasingly popular amongst mathematicians for several reasons. One reason is that many physical phenomena can be naturally formulated as ROPs. Another reason is that ROPs have natural links with other branches of mathematics such as geometry, free boundary problems, convex analysis, differential equations, and more. Lastly, such optimization problems also offer very challenging questions that are fascinating for researchers, see for example [2]. More specifically, Chapter 2 and Chapter 3 are prepared based on four papers [24, 40, 41, 42], mainly in collaboration with Behrouz Emamizadeh. Chapter 4 is inspired by [5]. In [5], the existence and uniqueness of solutions of various PDEs involving Radon measures are presented. In order to establish a connection between rearrangements and PDEs involving Radon measures, the author try to investigate a way to extend the notion of rearrangement of functions to rearrangement of Radon measures in Chapter 4.

Acknowledgements and dedications

First of all, I express my profound gratitude to Prof. Behrouz Emamizadeh who was my principal supervisor during the first two years of PhD, without his guidance, I could not enjoy the mathematics as it is now. Then, I would like to thank my current supervisors Dr. Hayk Mikayelyan, Dr. Zili Wu, and Prof. Alexander B. Movchan, for valuable discussions with them on this work. I also want to thank Xi'an Jiaotong-Liverpool University (XJTLU) and University of Liverpool (UoL) for their support on this research. Finally, I am grateful to my families, especially to my parents and my wife Ning Kang, for their patience, understanding and encouragement.

This thesis is dedicated to my grandparents Longying Ge and Honggui Sun. I hope they will be satisfied if they ever have a chance to read it.

Contents

Chapter 1

Introduction

The central part of the thesis is about rearrangements of functions, see Chapter 3, and this part of work is based on the basic theories developed by G. R. Burton near 90s. More specifically, in [7, 8], the author summarized the basic properties of rearrangement of functions, and gave a definition of a class of functions which are rearrangements of each other, called rearrangement class which is usually the admissible set in our problems. Then, a variety of properties of this class and its topological closure are investigated. At the same time, the author also applied these results to several real world problems connected with partial differential equations, both the unconstrained and constrained versions[1]. Amongst other things, the following is a main result of so called unconstrained version taken from [7]:

Theorem 1.0.1. [2] *Let $(\Omega, \mathcal{M}, \mu)$ be a finite, separable, non-atomic measure space. Let $f \in L^p(\mu)$ and $\mathcal{R}$ be a rearrangement class generated by f. Suppose Ψ is a real strictly convex functional on $L^p(\mu)$, sequentially continuous in the L^q-topology on $L^p(\mu)$. Then Ψ attains a maximum value relative to $\mathcal{R}$. If f^* is a maximizer and $g \in L^q(\mu)$ is a subgradient of Ψ at f^* (such a g must exist) then $f^* = \phi \circ g$ almost everywhere, for some increasing function ϕ.*

G. R. Burton's rearrangement theory including Theorem 1.0.1 was applied to real world problems and used as a tool in projects that were purely academic. For example, in [21], the authors investigated a physical problem which is to design a composite membrane with patches. The authors

[1] For unconstrained version, we mean doing the optimization on the pure rearrangement class, while for the constrained version, we focus on part of the rearrangement class.

[2] For definitions, see Preliminaries of Chapter 3.

model the problem into a rearrangement optimization problem where the goal function is the principal eigenvalue associated to a differential equation with Dirichlet Boundary condition. One can find similar papers such as [18, 25, 34] etc.

It is easily verified that a rearrangement class whose generator is a characteristic function χ_E, i.e. $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$, consists of χ_F with $|F| = |E|$. This simple fact makes it possible to investigate numerous shape optimization problems using the theory of rearrangements. As a result, one can establish an intimate connection between free boundary problems with rearrangement optimization problems, eg. Remark 3.2.1 in Chapter 3.

Since the space of L^∞ is excluded in a number of results in G. R. Burton's rearrangement theory, we apply the method of tangent cones in order to derive optimality conditions in particular problems. We learned this approach from the paper [32]. In [32], the authors firstly consider a different admissible set in L^∞ and then formulate the optimality condition with respect to this set. Finally, they invoke the optimality condition to prove the existence of the optimal shape. Chapter 2 describes a problem similar to [32].

In this research, we will utilize the tools of rearrangement theory and tangent cones to investigate various optimization problems involving partial differential equations. More precisely, the main result of Chapter 2 is Theorem 2.3.2, for which a similar result is discussed in [9], but the authors in [9] considered a different admissible set. For Chapter 3, Theorem 3.2.1 and Theorem 3.3.5 are extensions of Theorem 2.2 in [43] and Theorem 3.1 in [32] respectively. One point should be remarked in Chapter 3 is that Theorem 3.2.5 seems to be new in the existing literature, and we hope it will serve as a motivation for further research. Two stability results regarding rearrangement theory discussed in Section 3.4 are also new. Finally, a natural generalization of the idea of rearrangements to Radon measure will be presented in the last chapter. This development is novel (based on the author's knowledge) and the author hopes it will stimulate a new direction of research.

Chapter 2

Tangent Cones

In this chapter, we will utilize the tool of tangent cones from convex analysis to study a minimization problem, which models the minimum energy of displacement for an isotropic elastic membrane subjected to a vertical force, such as a load distribution. Before discussing the concrete problem, we needs some preliminaries about tangent cones.

2.1 Preliminaries

This section gathers the mathematical background for the sections to follow. In this chapter, we will denote by $|\cdot|$ the Lebesgue measure in $\mathbb{R}^N$. Let us begin with the definition of tangent cones.

Definition 2.1.1. *Let X be a normed linear space and C a nonempty set of X. The inner (intermediate, or derivable) tangent cone*[1] *of C at a, denoted $T'_C(a)$, is defined as follows: $v \in T'_C(a)$ if and only if for each $t_n \downarrow 0$ there exists a sequence $\{v_n\}_{n=1}^{\infty}$ in X satisfying*

(i) $\lim_{n\to\infty} v_n = v$,

(ii) $a + t_n v_n \in C$, $\forall\, n \in \mathbb{N}$.

As the notion of Gâteaux differentiability is used quit often in our analysis, we give its definition in the following for the convenience of the readers, see also [3] or [14]:

[1] We will call it the tangent cone for short in the following.

Definition 2.1.2. *Let X be a normed linear space, and U an open subset of X. Then, the functional $F : U \to \mathbb{R}$ is Gâteaux differentiable at $x \in U$ if there exists $F'(x) \in X^*$, the dual of X, such that, for all $v \in X$,*

$$\lim_{t \downarrow 0} \frac{F(x+tv) - F(x)}{t} = \langle F'(x), v \rangle.$$

$F'(x)$ is referred as Gâteaux derivative of F at x. Moreover, if F is Gâteaux differentiable at every point x of U, we say F is Gâteaux differentiable (on U).

The following two lemmata are useful for deriving the minimality conditions of the problem.

Lemma 2.1.1. *Let C and X be as in Definition 2.1.1, $\Phi : X \to \mathbb{R}$ a functional which is Gâteaux differentiable and Lipschitz continuous in an open set E containing C. If f is a minimizer of Φ in C, then*

$$\langle \Phi'(f), h \rangle \geq 0, \quad \forall\, h \in T'_C(f), \tag{2.1}$$

where $\langle \cdot\,, \cdot \rangle$ denotes the dual pairing between X^* and X. Here, $\Phi'(f)$ stands for the Gâteaux derivative of Φ at f.

Proof. Throughout the proof, k denotes the Lipschitz constant of Φ in E. We assume the assertion of the lemma is false, to derive a contradiction. So we can find $\hat{h} \in T'_C(f)$ with $\langle \Phi'(f), \hat{h} \rangle < 0$. Fix a sequence $t_n \downarrow 0$. From the definition, there exists a sequence $\{h_n\}$ in X such that $h_n \to \hat{h}$ and $f + t_n h_n \in C$, $\forall\, n \in \mathbb{N}$. Set $\delta = \langle \Phi'(f), \hat{h} \rangle$, and keeping in mind that δ is negative, there exists $N_1 \in \mathbb{N}$ such that

$$\frac{\Phi(f + t_n \hat{h}) - \Phi(f)}{t_n} < \frac{\delta}{2}, \quad \forall n \geq N_1.$$

Since $h_n \to \hat{h}$, we can find $N_2 \in \mathbb{N}$ such that $N_2 \geq N_1$, $f + t_{N_2}\hat{h} \in E$ and $\left\| h_{N_2} - \hat{h} \right\| < -\frac{\delta}{2k}$. In particular, we have

$$\Phi(f + t_{N_2}\hat{h}) - \Phi(f) < \frac{\delta}{2} t_{N_2}. \tag{2.2}$$

On the other hand, Φ is Lipschitz continuous in E, hence

$$\Phi(f + t_{N_2} h_{N_2}) - \Phi(f + t_{N_2}\hat{h}) \leq k t_{N_2} \left\| h_{N_2} - \hat{h} \right\| < -\frac{\delta}{2} t_{N_2}. \tag{2.3}$$

From (2.2) and (2.3), we deduce $\Phi(f + t_{N_2} h_{N_2}) < \Phi(f)$. This, recalling $f + t_{N_2} h_{N_2} \in C$, contradicts the minimality of f. $\square$

Lemma 2.1.2. *Let C be a nonempty convex set in X, and $\Phi : X \to \mathbb{R}$ a convex functional which is Gâteaux differentiable. If $\langle \Phi'(f), h\rangle \geq 0$ for all h in $T'_C(f)$, then f is a minimizer of Φ in C.*

Proof. To derive a contradiction, let us assume the assertion is false. So we can find $\hat{f} \in C$ such that $\Phi(\hat{f}) < \Phi(f)$. Observe that $\hat{f} - f \in T'_C(f)$ because C convex. Indeed, pick an arbitrary sequence $t_n \downarrow 0$, and set $v_n = \hat{f} - f$, $\forall n \in \mathbb{N}$. Clearly, $v_n \to \hat{f} - f$ and $f + t_n(\hat{f} - f) = t_n \hat{f} + (1 - t_n)f \in C$ as desired. Now by the assumption, $\langle \Phi'(f), \hat{f} - f\rangle \geq 0$. However,

$$\begin{aligned}\langle \Phi'(f), \hat{f} - f\rangle &= \lim_{t\downarrow 0} \frac{\Phi(f + t(\hat{f} - f)) - \Phi(f)}{t} \\ &\leq \lim_{t\downarrow 0} \frac{(1-t)\Phi(f) + t\Phi(\hat{f}) - \Phi(f)}{t} = \Phi(\hat{f}) - \Phi(f) < 0,\end{aligned}$$

which is a contradiction. Therefore, f is a minimizer of Φ in C. □

Remark 2.1.1. *The readers are encouraged to compare Lemmata 2.1.1 and 2.1.2 here with Propositions 1.39 and 2.25 in [14]. Note that there are two main differences:*

(i) The Bouligand tangent cone is used in [14] instead of the one in Definition 2.1.1.

(ii) The Fréchet differentiability of Φ is crucial when applying Propositions 1.39 and 2.25 in [14] to real problems.

For convenience, we fix $D \subseteq \mathbb{R}^N$ and denote

$$A_\alpha \equiv \left\{ f \in L^\infty(D) : 0 \leq f \leq 1, \int_D f(x)dx = \alpha \right\}, \tag{2.4}$$

for a given $0 < \alpha < |D|$. Note that the condition $\alpha \in (0, |D|)$ is to ensure the set A_α is not trivial. In order to determine the characteristics of tangent cones in A_α, let us introduce the following definition.

Definition 2.1.3. *For any function $f \in A_\alpha$, we define the sets D^0_f, D^*_f and D^1_f as follows*

(i) $D^0_f = \{x \in D : f(x) = 0\}$,

(ii) $D^*_f = \{x \in D : 0 < f(x) < 1\}$,

(iii) $D^1_f = \{x \in D : f(x) = 1\}$.

Lemma 2.1.3. *If $f \in A_\alpha$, then the tangent cone of A_α at f consists of functions $h \in L^\infty(D)$ such that*

(i) $\int_D h(x)dx = 0$,

(ii) $\lim_{n\to\infty} \left\|\chi_{Q_n^0} h^-\right\|_\infty = 0$,

(iii) $\lim_{n\to\infty} \left\|\chi_{Q_n^1} h^+\right\|_\infty = 0$,

where $Q_n^0 = \{x \in D : f(x) \leq 1/n\}$, $Q_n^1 = \{x \in D : f(x) \geq 1 - 1/n\}$, *and* h^+ *(resp.* h^-*) is the positive (resp. negative) part of* h.

Proof. See Proposition 2.1 in [4] and Proposition 4.5 in [16]. □

Lemma 2.1.4. *Let $f \in A_\alpha$. If $h \in T'_{A_\alpha}(f)$, then*

$$h(x) \geq 0 \text{ a.e. in } D_f^0, \quad h(x) \leq 0 \text{ a.e. in } D_f^1.$$

Proof. Observe that $D_f^0 \subseteq Q_n^0$ and $D_f^1 \subseteq Q_n^1$. Whence, the assertion readily follows from Lemma 2.1.3. □

Henceforth, we will denote positive real numbers by $\mathbb{R}^+$.

Definition 2.1.4. *We say the graph of f has no significant flat sections provided*

$$\left|\left\{x \in \mathbb{R}^N : f(x) = c\right\}\right| = 0, \ \forall\, c \in \mathbb{R}^+.$$

2.2 Mathematical model

In reality, the following Poisson boundary value problem models the stationary state of vibration of an isotropic elastic membrane, fixed around the boundary, and subjected to a vertical force $f(x)$:

$$\begin{cases} -\Delta u = f(x) & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{2.5}$$

where D is a smooth bounded domain in $\mathbb{R}^2$. Then, the function u stands for the displacement of the membrane from the rest position. Clearly, by changing f, u will change, so it makes sense to use u_f instead of u to stress the dependence of displacement on the force.

2.2.1 The linear case

First, we are interested in minimizing the *total displacement* of the membrane:

$$\inf_{f \in A_\alpha} \Phi_L(f) \equiv \int_D u_f dx. \tag{2.6}$$

Observe that we confine ourselves to forces described by A_α, referring to (2.4). In other words, the admissible forces are those that take values in $[0,1]$, and have fixed *total strength*, designated by α.

Lemma 2.2.1. *$\Phi_L(\cdot)$ is linear.*

Proof. Let us recall the Saint Venant's boundary value problem:

$$\begin{cases} -\Delta v = 1 & \text{in } D \\ v = 0 & \text{on } \partial D. \end{cases} \tag{2.7}$$

Multiplying the differential equation in (2.5) by v, integrating the result over D and after an application of divergence theorem, it yields

$$\int_D \nabla u_f \cdot \nabla v dx = \int_D f v dx. \tag{2.8}$$

Similarly, multiplying the differential equation in (2.7) by u_f, and integrating the result over D, yields

$$\int_D \nabla u_f \cdot \nabla v dx = \int_D u_f dx. \tag{2.9}$$

Comparing (2.8) and (2.9), we have

$$\Phi_L(f) = \int_D u_f dx = \int_D f v dx. \tag{2.10}$$

Thus, $\Phi_L(\cdot)$ is linear. □

Due to the linearity of $\Phi_L(\cdot)$, it is not necessary to utilize the method of tangent cones to derive the following result.

Theorem 2.2.2. *The minimization problem (2.6) has a unique solution $\hat{f} \in A_c$. Moreover, $\hat{f}$ minimizes $\Phi_L(f)$ in A_c if and only if*

(i) $\left|D^*_{\hat{f}}\right| = 0$,

(ii) $v(x_0) \geq v(x_1)$, $\forall\,(x_0, x_1) \in D^0_{\hat{f}} \times D^1_{\hat{f}}$.

Also, the unique minimizer $\hat{f}$ *is a characteristic function which can be identified with* $\chi_{\{v<c\}}$ *for some* $c > 0$.

Proof. We postpone addressing the uniqueness to the end of the proof. Let $\hat{f}$ be a solution of (2.6) in A_α and we first prove the assertion (i). In order to derive a contradiction, we suppose $|D^*_{\hat{f}}| > 0$. Then, we have $\beta \equiv \int_{D^*_{\hat{f}}} \hat{f}dx > 0$. By using strong maximum principle and Lemma 7.7 in [30], we deduce v is positive and has no significant flat sections in D. Observing that $|D^*_{\hat{f}}| > \int_{D^*_{\hat{f}}} \hat{f}dx = \beta$, there must exist positive c_1 such that $\left|\left\{x \in D^*_{\hat{f}} : v(x) < c_1\right\}\right| = \beta$. To this end, we set

$$\bar{f}(x) = \begin{cases} \hat{f}(x) & x \in D \setminus D^*_{\hat{f}} \\ 0 & x \in \left\{x \in D^*_{\hat{f}} : v(x) \geq c_1\right\} \\ 1 & x \in \left\{x \in D^*_{\hat{f}} : v(x) < c_1\right\}. \end{cases} \tag{2.11}$$

It is obvious that $\bar{f} \in A_\alpha$ and we evaluate

$$\begin{aligned} \Phi_L(\bar{f}) - \Phi_L(\hat{f}) &= \int_D \bar{f}vdx - \int_D \hat{f}vdx = \int_{D^*_{\hat{f}}} (\bar{f} - \hat{f})vdx \\ &= \int_{\left\{x\in D^*_{\hat{f}} : v<c_1\right\}} vdx - \int_{D^*_{\hat{f}}} \hat{f}vdx \\ &= \int_{\left\{x\in D^*_{\hat{f}} : v<c_1\right\}} (1-\hat{f})vdx - \int_{\left\{x\in D^*_{\hat{f}} : v\geq c_1\right\}} \hat{f}vdx \\ &< c_1 \left\{ \int_{\left\{x\in D^*_{\hat{f}} : v<c_1\right\}} (1-\hat{f})dx - \int_{\left\{x\in D^*_{\hat{f}} : v\geq c_1\right\}} \hat{f}dx \right\} = 0, \end{aligned} \tag{2.12}$$

which is a contradiction. So, we have $|D^*_{\hat{f}}| = 0$, i.e. $\hat{f}$ must be a characteristic function. Now, we proceed to test $\hat{f}$ for assertion (ii). In order to derive a contradiction, let us assume there exist $\omega_0 \subseteq D^0_{\hat{f}}$ and $\omega_1 \subseteq D^1_{\hat{f}}$ such that

$$|\omega_0| = |\omega_1| \quad \text{and} \quad \int_{\omega_0} vdx < \int_{\omega_1} vdx. \tag{2.13}$$

Then, we set

$$\tilde{f}(x) = \begin{cases} \hat{f}(x) & x \in D \setminus (\omega_0 \cup \omega_1) \\ 0 & x \in \omega_1 \\ 1 & x \in \omega_0. \end{cases}$$

Clearly, $\tilde{f} \in A_\alpha$ and we evaluate

$$\Phi_L(\tilde{f}) - \Phi_L(\hat{f}) = \int_D \tilde{f} v dx - \int_D \hat{f} v dx = \int_{\omega_0 \cup \omega_1} (\tilde{f} - \hat{f}) v dx$$
$$= \int_{\omega_0} v dx - \int_{\omega_1} v dx < 0,$$

which contradicts the minimality of $\hat{f}$. So, $\hat{f}$ satisfies condition (ii).

Conversely, let $\hat{f}$ satisfies (i) and (ii). By using elliptic regularity theory and Sobolev embedding theorem, see for example [26], we have $v \in C(\bar{D})$. Since v has no significant flat section in D, it follows from (i) and (ii) that $\hat{f} = \chi_{\hat{D}}$ with $\hat{D} = \{x \in D : v(x) < c_3\}$ for some positive c_3. We fix an arbitrary $g \in A_\alpha$ and construct $\bar{g}$ as in (2.11) if $|D_g^*| > 0$, otherwise, set $\bar{g} = g$. On the other hand, for an arbitrary $\chi_{\tilde{D}}$ with $\tilde{D} \subseteq D$ and $|\tilde{D}| = \alpha$, we have

$$\Phi_L(\chi_{\hat{D}}) - \Phi_L(\chi_{\tilde{D}}) = \int_{\hat{D}} v dx - \int_{\tilde{D}} v dx = \int_{\hat{D} \setminus \tilde{D}} v dx - \int_{\tilde{D} \setminus \hat{D}} v dx \leq 0. \tag{2.14}$$

Therefore, similarly to the calculations performed in (2.12), in conjunction with (2.14), we have $\Phi_L(g) \geq \Phi_L(\bar{g}) \geq \Phi_L(\hat{f})$. Since g is arbitrary, $\hat{f}$ is the minimizer of (2.6).

Finally, the uniqueness follows from assertion (i) and Lemma 2.2.1. □

Remark 2.2.1. *In fact, by using (2.10), we can apply the Bathtub principle, see Theorem 1.14 in [39], to prove Theorem 2.2.2.*

2.2.2 The nonlinear case

In contrast with the linear case, we study the following minimization problem

$$\inf_{f \in A_\alpha} \Phi_N(f) \equiv \int_D f u_f dx. \tag{2.15}$$

The nonlinear case above will be the focus of our analysis in the following sections. Let us briefly introduce the physical interpretation and give a general overview of the following sections in this chapter.

Physically, the quantity $\Phi_N(f)$ as defined in (2.15) measures the *energy of displacement* of the membrane. The dependence of Φ_N on f is obviously nonlinear. More specifically, the minimization problem (2.15) implies that we are interested in the minimum value of the energy of displacement given that the forces applied to the membrane are selected from A_α. We are particularly interested in force functions that achieve the minimum. Such forces are called *optimal solutions* of (2.15). Indeed, we shall prove that optimal solutions exist. In fact, we will show that there exists a unique optimal solution. Our next result is that the optimal solution has a distinguished characterization like linear case; namely, it is of bang-bang type. Such name is referred to two-valued functions. In our case, the optimal solution will turn out to be a $\{0, 1\}$-valued, hence a characteristic function. Another feature of the optimal solution is that its support contains a layer around the boundary of D, which is expected from the physical point of view. Indeed, a force acting at a location *near* the boundary, noting that the membrane is held fixed at the boundary, will result in small displacement in contrast to when the same amount of force is applied to points which are located *far* from the boundary.

After addressing the existence and uniqueness of optimal solutions (in Section 2.3), we present two monotonicity results (in Section 2.4), the first of which is physically quite interesting and in compliance with expectation. More precisely, we shall prove that by increasing the value of α, the support of the corresponding optimal solution increases in the sense of *nested* sets. The second monotonicity result is intriguing due to its physical interpretation that, the maximal distance from the rest position of the optimal solution is increasing with respect to α. In that section, we shall utilize some techniques from [22]. The last section of this chapter will be allocated to showing that the optimal solutions of (2.15) are stable, see Theorem 2.5.1.

Remark 2.2.2. *Since A_α is well structured, the tangent cone to A_α has a very convenient characterization. This characterization in conjunction with the convexity of A_α and Φ_N pave the way toward derivation of* necessary and sufficient *conditions for a function to be an optimal solution of (2.15). The method of tangent cones was recently used in [32], where the authors investigated a shape optimization problem. Surely, this method can also be applied to many other optimization problems, however, we should warn the readers that the method has its limitations. For example, in [11, 12], the authors explore the possibility of designing a membrane, fixed at the boundary, and made out of* two *materials, so that the corresponding frequency is maximal. This is shape optimization problem to which the method of tangent cones* *can certainly be applied. However, if we look at the same problem allowing* three or more *materials used in the design then the method of tangent cones can no longer be accessible.*

Let us mention that G. R. Burton and J. B. McLeod [9] amongst other things studied a similar optimization problem to (2.15). They used the same functional as $\Phi_N(f)$, but considered a different admissible set. The admissible set, used in [9], was a rearrangement class, see Definitions 3.1.1 and 3.1.3 . In that paper, the authors address existence and uniqueness of optimal solutions in general domains, and in radial domains in particular.

2.3 Existence and uniqueness of optimal solutions

This section is devoted to the minimization problem (2.15). But first, we need the following basic result regarding the energy functional Φ_N.

Lemma 2.3.1. *The functional Φ_N enjoys the following properties:*

(i) Φ_N *is weak*-continuous in* $L^\infty(D)$.

(ii) Φ_N *is strictly convex in* A_α.

(iii) Φ_N *is Gâteaux differentiable; moreover,* $\Phi'_N(f)$ *can be identified with* $2u_f$.

(iv) Φ_N *is Lipschitz continuous in* A_α.

Proof. For (i), (ii) and (iii), see Lemma 2.1 in [43]. We proceed to prove part (iv). To this end, we multiply the differential equation in (2.5) by u_f, integrate the result over D, and finally apply the divergence theorem to deduce

$$\int_D |\nabla u_f|^2 dx = \int_D f u_f dx. \tag{2.16}$$

An application of the Hölder's inequality to the right hand side of (2.16), coupled with the Poincaré inequality, leads to

$$\int_D |\nabla u_f|^2 dx \leq C \|f\|_2 \|u_f\|_{H_0^1(D)},$$

where C is a positive constant. From the last inequality, we infer

$$\|u_f\|_{H_0^1(D)} \leq C \|f\|_2 . \tag{2.17}$$

For all f and g in A_α, we have

$$\begin{aligned}
|\Phi_N(f) - \Phi_N(g)| &= \left| \int_D (f-g)u_f dx + \int_D g(u_f - u_g)dx \right| \\
&\leq \left| \int_D (f-g)u_f dx \right| + \left| \int_D g(u_f - u_g)dx \right| \\
&\leq C \|f-g\|_2 \|u_f\|_{H_0^1(D)} + C \|g\|_2 \|u_f - u_g\|_{H_0^1(D)} \\
&\leq C(\|f\|_2 + \|g\|_2) \|f-g\|_2 \\
&\leq C \|f-g\|_\infty ,
\end{aligned}$$

where we have used (2.17) in the third inequality, and the definition of A_α in the last inequality. □

Remark 2.3.1. *After revisiting the proof of (iv) in the Lemma above, it is not hard to see that the Lipschitz continuity can be extended to an open set* $\mathcal{O} \in L^\infty(D)$ *containing* A_α*, for example,* $\mathcal{O} \equiv \{f \in L^\infty : -1 < f < 2\}$.

The main result of this section is the following:

Theorem 2.3.2. *The minimization problem (2.15) has a unique solution* $\hat{f}$. *Moreover,* $\hat{f}$ *is characterized as follows:* $\hat{f}$ *minimizes* $\Phi_N(f)$ *relative to* A_α *if and only if:*

(i) $|D^*_{\hat{f}}| = 0$,

(ii) $u_{\hat{f}}(x_0) \geq u_{\hat{f}}(x_1)$, $\forall\, (x_0, x_1) \in D^0_{\hat{f}} \times D^1_{\hat{f}}$.

Indeed, $\hat{f}$ *is a characteristic function which is equal to* $\chi_{\left\{u_{\hat{f}} < c\right\}}$*, where* $c = \max_{\bar{D}} u_{\hat{f}} > 0$.

Proof. The proof is based on the notion of tangent cones, a tool that was also considered in [32]. We begin by observing that, A_α is closed in $L^\infty(D)$, and convex. Hence, A_α is weak*-closed. By Theorem 2.10.2 in [33], we infer A_α is in fact weak*-compact. On the other hand, by Lemma 2.3.1 (i), Φ_N is weak*-continuous, hence (2.15) is solvable. We postpone addressing the uniqueness to the end of the proof.

Let $\hat{f}$ be a solution of (2.15) in A_α, and set

$$D^*_n = \left\{ x \in D : 1/n \leq \hat{f} \leq 1 - 1/n \right\}.$$

We first prove $u_{\hat{f}}$ is constant on $D^*_{\hat{f}}$. To this end, observe that $D^*_{\hat{f}} = \bigcup_{n=1}^\infty D^*_n$, hence it suffices to prove $u_{\hat{f}}$ is constant on D^*_n. To derive a contradiction, suppose $u_{\hat{f}}$ is not constant on D_n^*, for some n. Thus, there exist two measurable sets ω_1 and ω_2 in D_n^* such that

$$|\omega_1| = |\omega_2| \quad \text{and} \quad \int_{\omega_1} u_{\hat{f}} dx < \int_{\omega_2} u_{\hat{f}} dx. \tag{2.18}$$

Now taking

$$h(x) = \begin{cases} 1 & x \in \omega_1 \\ -1 & x \in \omega_2 \\ 0 & x \in (\omega_1 \cup \omega_2)^c, \end{cases}$$

which belongs to $T'_{A_\alpha}(\hat{f})$ (see Lemma 2.1.3), yields

$$\langle \Phi'_N(\hat{f}), h \rangle = 2\int_D u_{\hat{f}} h dx = 2\int_{\omega_1} u_{\hat{f}} dx - 2\int_{\omega_2} u_{\hat{f}} dx < 0, \tag{2.19}$$

by (2.18). Recalling Lemma 2.3.1 and Remark 2.3.1, clearly, (2.19) contradicts the optimality condition (2.1). Thus, $u_{\hat{f}}$ is constant on $D_{\hat{f}}^*$. Next, from the differential equation (2.5), coupled with Lemma 7.7 in [30], we infer that the graph of $u_{\hat{f}}$ has no significant flat sections in $D_{\hat{f}}^*$. Therefore, $|D_{\hat{f}}^*| = 0$, as desired.

For part (ii), let us assume there exist two measurable sets $\omega_0 \subseteq D_{\hat{f}}^0$ and $\omega_1 \subseteq D_{\hat{f}}^1$ such that

$$|\omega_0| = |\omega_1| \quad \text{and} \quad \int_{\omega_0} u_{\hat{f}} dx < \int_{\omega_1} u_{\hat{f}} dx. \tag{2.20}$$

Next, we set:

$$h(x) = \begin{cases} 1 & x \in \omega_0 \\ -1 & x \in \omega_1 \\ 0 & x \in (\omega_0 \cup \omega_1)^c, \end{cases}$$

which belongs to $T'_{A_\alpha}(\hat{f})$. Similarly to the proof of part (i), the inequality in (2.20) leads to a contradiction of the optimality condition (2.1). Since $\left|D_{\hat{f}}^*\right| = 0$, we deduce that $\hat{f}$ must be a characteristic function $\chi_{\hat{D}}$, where $|\hat{D}| = \alpha$. By the elliptic regularity theory, see for example [26], we infer $u_{\hat{f}} \in H^2(D)$, thus, by the Sobolev embedding theorem, it follows that $u_{\hat{f}} \in C(\bar{D})$. Using part (ii), we deduce $\hat{D} = \left\{x \in D : u_{\hat{f}}(x) \leq c\right\}$, where

$$c = \sup_{x \in D_{\hat{f}}^1} u_{\hat{f}}(x) = \inf_{x \in D_{\hat{f}}^0} u_{\hat{f}}(x) > 0.$$

Clearly, we have $u_{\hat{f}} = c$ on $\partial D^0_{\hat{f}}$, which implies $u_{\hat{f}} = c$ in $D^0_{\hat{f}}$. Whence, $\hat{D} = \Big\{x \in D : u_{\hat{f}}(x) < c\Big\}$, where $c = \max_{\bar{D}} u_{\hat{f}} > 0$.

Conversely, let us assume the pair $(\hat{f}, u_{\hat{f}})$ satisfies (i) and (ii). Due to the continuity of $u_{\hat{f}}$, we deduce

$$c^* = \sup_{x \in D^1_{\hat{f}}} u_{\hat{f}}(x) = \inf_{x \in D^0_{\hat{f}}} u_{\hat{f}}(x) > 0.$$

Let us fix h in $T'_{A_c}(\hat{f})$. From Lemmata 2.1.3 and 2.1.4, we obtain

$$\begin{aligned}\langle \Phi'_N(\hat{f}), h\rangle = 2\int_D u_{\hat{f}} h dx &= 2\int_{D^0_{\hat{f}}} u_{\hat{f}} h dx + 2\int_{D^1_{\hat{f}}} u_{\hat{f}} h dx \\ &\geq 2\int_{D^0_{\hat{f}}} c^* h dx + 2\int_{D^1_{\hat{f}}} c^* h dx = 2c^* \int_D h dx = 0.\end{aligned}$$

Therefore, we infer from Lemma 2.1.2 that $\hat{f}$ is a minimizer.

Finally, we settle the issue of uniqueness. To this end, we assume $\hat{f}$ is a solution of (2.15). To derive a contradiction, let us assume $\tilde{f}$ is another solution of (2.15). We set $g = \frac{1}{2}(\hat{f} + \tilde{f})$ which belongs to A_α, because A_α is convex. Recalling that Φ_N is strict convex, it now follows that $\Phi_N(g) < \Phi_N(\hat{f})$, which contradicts the minimality of $\hat{f}$. □

Proposition 2.3.3. *If D is simply connected and $\chi_{\hat{D}}$ is the unique minimizer of the problem (2.15), then $\hat{D}$ is connected and contains a layer around ∂D.*

Proof. From Theorem 2.3.2, we infer $\hat{D} = \Big\{x \in D : u_{\hat{f}}(x) < c\Big\}$, where $c = \max_{\bar{D}} u_{\hat{f}} > 0$. This implies $\hat{D}$ contains a layer around ∂D, since $u_{\hat{f}} \in C(\bar{D})$ and $u_{\hat{f}}$ vanishes on ∂D. To prove $\hat{D}$ is connected, we assume otherwise and derive a contradiction. So let us assume there is a component of $\hat{D}$, denoted D', such that the intersection of $\partial D'$ and ∂D is empty. Observe that, $u_{\hat{f}} = c$ on $\partial D'$. Recalling the Poisson problem (2.5), it follows that

$$\left\{\begin{array}{ll} -\Delta u_{\hat{f}} = 1 & \text{in } D' \\ u_{\hat{f}} = c & \text{on } \partial D'. \end{array}\right. \tag{2.21}$$

Hence by the strong maximum principle, we deduce, from (2.21), $u_{\hat{f}} > c$ in D'. This clearly contradicts the fact that $u_{\hat{f}} < c$ in D'. □

2.4 Monotonicity results

In this section, we address two monotonicity results related to problem (2.15). Let us fix some notation. Similarly to A_α, we define A_β as follows:

$$A_\beta = \left\{ f \in L^\infty(D) : 0 \le f \le 1, \int_D f(x)dx = \beta \right\},$$

where $0 < \beta < |D|$. By Theorem 2.3.2, we know the two minimization problems

$$\inf_{f\in A_\alpha} \Phi_N(f) \quad \text{and} \quad \inf_{f\in A_\beta} \Phi_N(f),$$

have unique solutions, which we denote them by $\chi_{\hat{D}_\alpha}$ and $\chi_{\hat{D}_\beta}$, respectively. Furthermore, we have

$$\hat{D}_\alpha = \{x \in D : u_\alpha(x) < c_\alpha\} \quad \text{and} \quad \hat{D}_\beta = \{x \in D : u_\beta(x) < c_\beta\}, \tag{2.22}$$

for positive $c_\alpha = \max_{\bar{D}} u_\alpha$ and $c_\beta = \max_{\bar{D}} u_\beta$, where u_α and u_β satisfy:

$$\begin{cases} -\Delta u_\alpha = \chi_{\hat{D}_\alpha} & \text{in } D \\ u_\alpha = 0 & \text{on } \partial D, \end{cases} \tag{2.23}$$

and

$$\begin{cases} -\Delta u_\beta = \chi_{\hat{D}_\beta} & \text{in } D \\ u_\beta = 0 & \text{on } \partial D. \end{cases} \tag{2.24}$$

Our first monotonicity result is the following

Theorem 2.4.1. *If $0 < \beta \le \alpha < |D|$, then $c_\beta \le c_\alpha$.*

Proof. From (2.23) and (2.24), it follows that

$$\begin{cases} -\Delta(u_\alpha - u_\beta) = \chi_{\hat{D}_\alpha} - \chi_{\hat{D}_\beta} & \text{in } D \\ u_\alpha - u_\beta = 0 & \text{on } \partial D. \end{cases} \tag{2.25}$$

Multiplying the differential equation in (2.25) by $u_\alpha - u_\beta$, integrating the result over D, followed by an application of divergence theorem, yields

$$\begin{aligned} \int_D |\nabla(u_\alpha - u_\beta)|^2 \, dx &= \int_D (\chi_{\hat{D}_\alpha} - \chi_{\hat{D}_\beta})(u_\alpha - u_\beta)dx \\ &= \int_{\hat{D}_\alpha \setminus \hat{D}_\beta} (u_\alpha - u_\beta)dx + \int_{\hat{D}_\beta \setminus \hat{D}_\alpha} (u_\beta - u_\alpha)dx. \end{aligned} \tag{2.26}$$

From (2.22), we infer $u_\alpha - u_\beta < c_\alpha - c_\beta$ in $\hat{D}_\alpha \setminus \hat{D}_\beta$, and $u_\alpha - u_\beta > c_\alpha - c_\beta$ in $\hat{D}_\beta \setminus \hat{D}_\alpha$. Thus, equation (2.26) leads to

$$\begin{aligned}\int_D |\nabla(u_\alpha - u_\beta)|^2\,dx &\le |\hat{D}_\alpha \setminus \hat{D}_\beta|(c_\alpha - c_\beta) + |\hat{D}_\beta \setminus \hat{D}_\alpha|(c_\beta - c_\alpha)\\ &= (c_\alpha - c_\beta)\left(|\hat{D}_\alpha \setminus \hat{D}_\beta| - |\hat{D}_\beta \setminus \hat{D}_\alpha|\right).\end{aligned} \tag{2.27}$$

Because $|\hat{D}_\alpha \setminus \hat{D}_\beta| = |\hat{D}_\alpha| - |\hat{D}_\alpha \cap \hat{D}_\beta|$ and $|\hat{D}_\beta \setminus \hat{D}_\alpha| = |\hat{D}_\beta| - |\hat{D}_\alpha \cap \hat{D}_\beta|$, we infer

$$|\hat{D}_\alpha \setminus \hat{D}_\beta| - |\hat{D}_\beta \setminus \hat{D}_\alpha| = |\hat{D}_\alpha| - |\hat{D}_\beta|. \tag{2.28}$$

Substituting (2.28) into (2.27), it follows that

$$\int_D |\nabla(u_\alpha - u_\beta)|^2\,dx \le (c_\alpha - c_\beta)\left(|\hat{D}_\alpha| - |\hat{D}_\beta|\right) = (c_\alpha - c_\beta)(\alpha - \beta).$$

Since the left hand side of the last equation is nonnegative, and the fact that $\beta \le \alpha$, we infer $c_\beta \le c_\alpha$. □

Our second monotonicity result is as follows

Theorem 2.4.2. *If $0 < \beta \le \alpha < |D|$, then $\hat{D}_\beta \subseteq \hat{D}_\alpha$.*

Proof. Let us introduce the following subsets of D:

$$E = \{u_\alpha - u_\beta > c_\alpha - c_\beta\} \quad \text{and} \quad F = \{u_\alpha - u_\beta \le c_\alpha - c_\beta\}.$$

Similarly to the proof of Theorem 2.4.1, recalling (2.22), we infer $u_\alpha - u_\beta < c_\alpha - c_\beta$ in $\hat{D}_\alpha \setminus \hat{D}_\beta$, and $u_\alpha - u_\beta > c_\alpha - c_\beta$ in $\hat{D}_\beta \setminus \hat{D}_\alpha$. By the definition of E and F, it follows that

$$\hat{D}_\alpha \setminus \hat{D}_\beta \subseteq F, \tag{2.29}$$

and

$$\hat{D}_\beta \setminus \hat{D}_\alpha \subseteq E. \tag{2.30}$$

From (2.29), in conjunction with the fact that $F = D \setminus E$, we deduce $E \subseteq (D \setminus \hat{D}_\alpha) \cup \hat{D}_\beta$. From the differential equations in (2.23) and (2.24), we obtain

$$-\Delta(u_\alpha - u_\beta) = \chi_{\hat{D}_\alpha} - \chi_{\hat{D}_\beta} \le 0 \quad \text{in} \quad E \subseteq (D \setminus \hat{D}_\alpha) \cup \hat{D}_\beta.$$

On the other hand, by Theorem 2.4.1, we have $u_\alpha - u_\beta = c_\alpha - c_\beta$ on ∂E. Whence, by the weak maximum principle, $u_\alpha - u_\beta \le c_\alpha - c_\beta$ in E. Recalling the definition of E, we can conclude E must be empty. Furthermore, it follows from (2.30) that $\hat{D}_\beta \setminus \hat{D}_\alpha$ is empty as well. Hence, $\hat{D}_\beta \subseteq \hat{D}_\alpha$, as desired. □

2.5 A stability result

Before stating the result of this section, we fix some notation. Let

$$A_{\alpha_n} = \left\{ f \in L^\infty(D) : 0 \le f \le 1, \int_D f(x)dx = \alpha_n \right\},$$

where $0 < \alpha_n < |D|$. Let $\chi_{\hat{D}_n}$ denote the unique solution of the following minimization problem:

$$\inf_{f \in A_{\alpha_n}} \Phi_N(f).$$

The symmetric difference of two sets E and F is denoted by $E \triangle F$.

Our stability result is the following:

Theorem 2.5.1. *Let $\chi_{\hat{D}}$ denote the minimizer of problem (2.15), satisfying $|\hat{D}| = \alpha$. If $\alpha_n \to \alpha$, then $\chi_{\hat{D}_n} \to \chi_{\hat{D}}$ in $L^1(D)$. Moreover, $|\hat{D}_n \triangle \hat{D}| \to 0$.*

Proof. Since $\alpha_n \to \alpha$, we infer

$$\begin{aligned} |\alpha_n - \alpha| = \left| \int_D \chi_{\hat{D}_n} dx - \int_D \chi_{\hat{D}} dx \right| &= \left| \int_D (\chi_{\hat{D}_n} - \chi_{\hat{D}}) dx \right| \\ &= \int_D |\chi_{\hat{D}_n} - \chi_{\hat{D}}| dx \to 0, \end{aligned} \tag{2.31}$$

where the third equality in (2.31) holds as a result of Theorem 2.4.2. Hence, $\chi_{\hat{D}_n} \to \chi_{\hat{D}}$ in $L^1(D)$.

Since

$$\int_D |\chi_{\hat{D}_n} - \chi_{\hat{D}}| dx = |\hat{D}_n \triangle \hat{D}|,$$

we also deduce $|\hat{D}_n \triangle \hat{D}| \to 0$. This completes the proof of the theorem. $\square$

Chapter 3

Rearrangement of Functions

This chapter will be devoted to the theory of rearrangement of functions and its applications to partial differential equations. More specifically, we will present the mathematical background first. Then, two applications to real problems will follow. Finally, we will show two intriguing approximation results.

3.1 Preliminaries

This section gathers the background for the sections to follow. We begin by reviewing the relevant parts of the rearrangement theory attributed to G. R. Burton. The appropriate references for this section are [7, 8, 36]. We stress that the materials to follow are specialized to suit the purpose of the present chapter, hence they may not appeal in the most generality.

Definition 3.1.1. *Let X and X' be two measurable subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively. Suppose $\mathcal{L}_N(X) = \mathcal{L}_M(X') < \infty$, where $\mathcal{L}_N$ and $\mathcal{L}_M$ denote the Lebesgue measures in $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively. Suppose $f : X \to [0,\infty)$ and $g : X' \to [0,\infty)$ are measurable functions. We say f and g are rearrangements of each other if:*

$$\begin{aligned}\lambda_{f,\mathcal{L}_N}(\alpha) &\equiv \mathcal{L}_N\left(\{x \in X :\ f(x) \geq \alpha\}\right) \\ &= \mathcal{L}_M\left(\{x \in X' :\ g(x) \geq \alpha\}\right) \equiv \lambda_{g,\mathcal{L}_M}(\alpha), \quad \forall\, \alpha \geq 0.\end{aligned} \tag{3.1}$$

Definition 3.1.2. *Let f be a function as in Definition 3.1.1. The function $f^{\Delta} : (0, \mathcal{L}_N(X)) \to \mathbb{R}$ defined by*

$$f^{\Delta}(s) = \max\{\alpha : \lambda_{f,\mathcal{L}_N}(\alpha) \geq s\}$$

is called the decreasing rearrangement of f. Also, the function $f_\Delta(s) \equiv f^\Delta(\mathcal{L}_N(X) - s)$ is called the increasing rearrangement of f.

The following remark is useful.

Remark 3.1.1. *It is well known that when f is continuous and its graph has no significant flat sections in the sense that*

$$\mathcal{L}_N(\{x \in X : f(x) = c\}) = 0, \quad \forall c \in \mathbb{R}^+,$$

then f^Δ and f_Δ will be both continuous, moreover, f^Δ will be strictly decreasing, and f_Δ will be strictly increasing.

Definition 3.1.3. *Let f be as in Definition 3.1.1. The set $\mathcal{R}(f)$, called the rearrangement class generated by f, is defined as follows:*

$$\mathcal{R}(f) = \{g : X \to [0, \infty) : \ g \text{ and } f \text{ are rearrangements of each other}\}.$$

Remark 3.1.2. *As the weak closure of rearrangement classes is vital in rearrangement theory, we will utilize $\bar{\cdot}$ and $\bar{\cdot}^s$ to denote the corresponding weak and strong closure respectively in this chapter.*

One of the cornerstones in G. R. Burton's rearrangement theory is the following result.

Lemma 3.1.1. *Let $1 < p \le \infty$, and p' be the conjugate exponent of p, i.e. $1/p + 1/p' = 1$. Suppose $f \in L^{p'}(X)$, and $\mathcal{R} \equiv \mathcal{R}(f)$ is the rearrangement class generated by f. Then*

(i) $\mathcal{R} \subseteq L^{p'}(X)$, *and* $\|f\|_{p'} = \|g\|_{p'}$, *for every* $g \in \mathcal{R}$. *Here* $\|\cdot\|_{p'}$ *denotes the usual* $L^{p'}$*-norm*[1].

(ii) $\overline{\mathcal{R}}$, *the weak closure of* $\mathcal{R}$ *in* $L^{p'}(X)$, *is convex and weakly compact in* $L^{p'}(X)$. *Moreover,* $\overline{\mathcal{R}} = \overline{co}^s(\mathcal{R})$, *the closed convex hull of* $\mathcal{R}$.

(iii) For $\mathcal{A}$ *an affine subspace of finite codimension in* $L^{p'}(X)$, *ext*$(\overline{\mathcal{R}} \cap \mathcal{A})$, *the set of extreme points of* $\overline{\mathcal{R}} \cap \mathcal{A}$, *is equal to* $\mathcal{R} \cap \mathcal{A}$.

(iv) Let $\mathcal{A}$ *be as in (iii). Then* $\overline{\mathcal{R}} \cap \mathcal{A} = \overline{co}^s(\mathcal{R} \cap \mathcal{A})$.

(v) The relative weak and strong topologies on $\mathcal{R}$ *coincide.*

Proof. For (i), (ii) and (v), see [7, 8]. For (iii) and (iv), see [10]. □

[1] The integration is taken over the corresponding domain.

Lemma 3.1.2. *Let p' and f be as in Lemma 3.1.1. Then*

(i) There is a measure preserving map $\rho : D \to (0, |D|)$ such that $f = f^{\Delta} \circ \rho$.

(ii) $\left\|g^{\Delta} - h^{\Delta}\right\|_{p'} \leq \|g - h\|_{p'}$ for all g and h in $L^{p'}(D)$.

Proof. For (i), see Lemma 2.4 in [8] or Proposition 3 in [49]. For (ii), see Lemma 2.7 in [8] or Corollary 1 in [19]. □

In what follows we often write increasing instead of non-decreasing and decreasing instead of non-increasing.

Lemma 3.1.3. *Let $f : X \to (0, \infty)$ and $g : X \to (0, \infty)$ be measurable functions. Suppose the graph of g has no significant flat sections. Then there is a decreasing function ϕ such that $\phi \circ g$ and f are rearrangements of each other. In particular,*

$$\phi(s) = f_{\Delta} \circ \lambda_{g,\mathcal{L}_N}(s).$$

Moreover, there is a increasing function $\tilde{\phi}$ such that $\tilde{\phi} \circ g$ and f are rearrangements of each other.

Proof. The proof is similar to Lemma 2.9 in [8]. Recalling Remark 3.1.1, we infer g^{Δ} is strictly decreasing, hence it has a left inverse which coincides with $\lambda_{g,\mathcal{L}_N}$. We set $\phi(s) = f_{\Delta} \circ \lambda_{g,\mathcal{L}_N}(s)$. To see that $\phi \circ g$ and f are rearrangements of each other we first observe that $\phi \circ g$ and $\phi \circ g^{\Delta}$ are rearrangements of each other. This, in turn, implies $(\phi \circ g)_{\Delta} = (\phi \circ g^{\Delta})_{\Delta}$. However, $(\phi \circ g^{\Delta})_{\Delta} = \phi \circ g^{\Delta}$, hence $(\phi \circ g)_{\Delta} = \phi \circ g^{\Delta} = f_{\Delta} \circ \lambda_{g,\mathcal{L}_N} \circ g^{\Delta} = f_{\Delta}$. Therefore, $\phi \circ g$ and f are rearrangements of each other, as desired. The last assertion can be proved similarly by considering $\tilde{\phi}(s) = f^{\Delta} \circ \lambda_{g,\mathcal{L}_N}(s)$. □

Lemma 3.1.4. *Let $f \in L^{p'}(X)$ and $g \in L^{p}(X)$ be non-negative functions, where $1/p + 1/p' = 1$. Let $\mathcal{R}$ be the rearrangement class generated by f. Suppose there is a decreasing (or increasing) function ϕ such that $\phi \circ g \in \mathcal{R}$. Then $\phi \circ g$ is the unique minimizer (or maximizer) of the linear functional*

$$L(h) = \int_X hg d\mathcal{L}_N,$$

relative to $h \in \overline{\mathcal{R}}$.

Proof. The proof is a minor variant of the proof of Lemma 2.4 in [9]. □

Remark 3.1.3. *In Lemma 3.1.4, it is important to notice that $\phi \circ g$ is the unique minimizer (or maximizer) of the linear functional $L(h)$ relative to $\overline{\mathcal{R}}$, not only $\mathcal{R}$.*

Lemma 3.1.5. *Let $f \in L^{p'}(X)$ and $\mathcal{R}$ be the rearrangement class generated by f. Let $\overline{\mathcal{R}}$ be the weak closure of $\mathcal{R}$ in $L^{p'}(X)$. Then*

$$\overline{\mathcal{R}} = \left\{ g \in L^1(X) : \int_X g d\mathcal{L}_N = \int_X f d\mathcal{L}_N \text{ and } \int_0^s g^{\Delta} dt \leq \int_0^s f^{\Delta} dt, \ \forall\, s \in (0, \mathcal{L}_N(X)) \right\}.$$

Proof. See Lemma 2.2 in [9], or [51]. □

Corollary 3.1.6. *Suppose the hypotheses of Lemma 3.1.5 hold. Let $h \in \overline{\mathcal{R}}$, and $\mathcal{R}(h)$ denote the rearrangement class generated by h. Then, $\mathcal{R}(h)$ is contained in $\overline{\mathcal{R}}$.*

Proof. The proof follows immediately from Lemma 3.1.5. □

The next lemma is easy to prove:

Lemma 3.1.7. *Let $f : X \to [0, \infty)$ be a measurable function, then,*

$$\int_E f d\mathcal{L}_N \geq \int_0^{|E|} f_{\Delta} dt,$$

for every measurable subset $E \subseteq X$.

Henceforth, the support of f will be denoted by

$$S(f) \equiv \{x \in X : f(x) > 0\},$$

and the reader should distinguish this definition of support from the usual topological definition.

Lemma 3.1.8. *Let $f \in L^{p'}(X)$ and $\mathcal{R}$ be the rearrangement class generated by f. Let $\overline{\mathcal{R}}$ be the weak closure of $\mathcal{R}$ in $L^{p'}(X)$. For every g in $\overline{\mathcal{R}}$, we have $|S(f)| \leq |S(g)|$.*

Proof. In order to derive a contradiction, let us assume $|S(g)| < |S(f)|$. Hence, $\alpha \equiv \int_0^{|S(g)^c|} f_\Delta dt$ is positive. Since $g \in \overline{\mathcal{R}}$, there exists $\{g_n\} \subseteq \mathcal{R}$ such that $g_n \rightharpoonup g$ in $L^{p'}(X)$. Then, we have

$$
\begin{aligned}
(3.2) \quad \alpha = \int_0^{|S(g)^c|} f_\Delta dt &= \int_0^{|S(g)^c|} g_{n_\Delta} dt \\
&\leq \int_{S(g)^c} g_n d\mathcal{L}_N = \int_X g_n \chi_{S(g)^c} d\mathcal{L}_N \\
&\to \int_X g \chi_{S(g)^c} d\mathcal{L}_N = \int_{S(g)^c} g d\mathcal{L}_N = 0,
\end{aligned}
$$

which contradicts the positivity of α. The inequality in (3.2) is a consequence of Lemma 3.1.7. □

We will also need two results from functional analysis.

Lemma 3.1.9. *Suppose $1 < p \leq \infty$ and p' is the conjugate exponent of p. Suppose $f_n \rightharpoonup f$ in $L^{p'}(X)$, and $g_n \to g$ in $L^p(X)$. Then $\int_X f_n g_n d\mathcal{L}_N \to \int_X f g d\mathcal{L}_N$.*

Proof. By applying the Hölder's inequality we obtain

$$
\begin{aligned}
\left| \int_X f_n g_n d\mathcal{L}_N - \int_X f g d\mathcal{L}_N \right| &= \left| \int_X f_n (g_n - g) d\mathcal{L}_N + \int_X (f_n - f) g d\mathcal{L}_N \right| \\
&\leq \|f_n\|_{p'} \|g_n - g\|_p + \left| \int_X (f_n - f) g d\mathcal{L}_N \right|.
\end{aligned}
$$

Since $\{f_n\}$ is bounded in $L^{p'}(X)$, and $g_n \to g$ in $L^p(X)$, the first term on the right hand side of the inequality above tends to zero. Its second term tends to zero because f_n converges weakly to f. □

Lemma 3.1.10. *Let $\mathcal{C}$ be a convex set in a real vector space Y. Let l_1 and l_2 be linear functionals on Y, and I be a real number for which there exist y_1 and y_2 in $\mathcal{C}$ such that $l_1(y_1) < I < l_1(y_2)$. Moreover, suppose there exists $y_0 \in Y$ such that $l_2(y) \geq l_2(y_0)$, for all $y \in \mathcal{C}$ satisfying $l_1(y) = I$. Then there is a real number γ such that y_0 minimizes $l_2 + \gamma l_1$, relative to $\mathcal{C}$.*

Proof. The proof is a minor variant of the proof of Lemma 2.13 in [8]. □

We end this section by recalling the notion of subdifferentiability of convex functionals, see for example [15].

Definition 3.1.4. *For $1 \leq r < \infty$, let $\Psi : L^r(X) \to \overline{\mathbb{R}}$ be a convex functional. We assume Ψ is proper, i.e. $\Psi(u_0) < +\infty$, for some $u_0 \in L^r(X)$ and nowhere takes the value $-\infty$. For $u \in L^r(X)$, the subdifferential of Ψ at u is denoted $\partial\Psi(u)$, and defined as follows:*

$$\partial\Psi(u) = \left\{ w \in L^{r'}(X) : \ \Psi(v) \geq \Psi(u) + \int_X (v-u) w \, d\mathcal{L}_N, \ \forall v \in L^r(X) \right\}.$$

If $\partial\Psi(u) \neq \emptyset$, then we say Ψ is subdifferentiable at u.

Since $r < \infty$, it is well known that if Ψ is norm continuous, then an application of the Hahn-Banach theorem implies $\partial\Psi(u) \neq \emptyset$.

3.2 Rearrangement optimization problem 1

Consider the boundary value problem

$$\begin{cases} -\Delta u + h(x)u = f(x) & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{3.3}$$

where D is a smooth (C^2 is enough) bounded domain in $\mathbb{R}^2$. The functions $h(x)$ and $f(x)$ are non-negative and bounded. Physically, (3.3) models an elastic membrane which is fixed around the boundary, subject to a vertical force $f(x)$. The function $h(x)$ represents the density of the membrane, and u the displacement from the rest position. In case the membrane is isotropic, i.e. it is made of a single material, $h = 0$, hence (3.3) reduces to the classical Poisson's problem:

$$\begin{cases} -\Delta u = f(x) & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{3.4}$$

Henceforth, we will use $|\cdot|$ and dx instead of $\mathcal{L}_N(\cdot)$ and $d\mathcal{L}_N$ respectively in this chapter. The energy functional associated with (3.4) is defined by

$$\Phi(f) = \int_D f u_f \, dx, \tag{3.5}$$

where $u_f \in H_0^1(D)$ is the unique positive solution of (3.4). Two interesting optimization problems related to Φ are as follows:

$$\sup_{f \in \mathcal{R}} \Phi(f) \qquad \text{and} \qquad \inf_{f \in \mathcal{R}} \Phi(f),$$

where $\mathcal{R}$ denotes a rearrangement class generated by a known function. Both of these problems have been extensively investigated by G. R. Burton in [7, 8, 9]. In recent years a number of mathematicians have attempted to apply the tools introduced by G. R. Burton to various optimization problems similar to the ones mentioned above.

Let us now proceed to describe precisely the problems that will be discussed in this section. First, we consider the following boundary value problem:

$$\begin{cases} -\Delta_p u = f & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{3.6}$$

where Δ_p is the classical p-Laplace operator, i.e. $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$, with $1 < p < \infty$. Next, denoting the unique solution of (3.6) by $u_f \in W_0^{1,p}(D)$, and recalling that u_f is the unique minimizer of the functional

$$F(u) = \frac{1}{p}\int_D |\nabla u|^p dx - \int_D fu dx,$$

relative to $u \in W_0^{1,p}(D)$, we define the p-energy functional associated to (3.6), as follows:

$$\Phi_p(f) = \int_D f u_f dx. \tag{3.7}$$

We are interested in the following optimization problems:

$$\inf_{f \in \mathcal{R}} \Phi_p(f), \tag{3.8}$$

and

$$\inf_{f \in \mathcal{R} \cap \Lambda} \Phi_p(f), \tag{3.9}$$

where $\mathcal{R}$ denotes a class of rearrangements generated by a known function, and Λ an affine subspace of codimension one in an appropriate function space.

Let us describe the physical interpretation of (3.8) which is most realistic when $p = 2$. The goal is to identify a force function selected from $\mathcal{R}$, in such a way that the total energy of displacement of the membrane is as small as possible. A similar problem has been considered in [32]. In that paper the authors considered an elastic membrane made out of two materials with prescribed quantities, subject to a fixed vertical force. They proved the existence of the best possible design so that the corresponding total energy of displacement is minimal. The analysis conducted in [32] was based on tangent cones, but in the present work we follow the approach of [7, 8, 9, 43].

The physical relevance of (3.9) can be described similarly to the unconstrained problem. In this case, we are interested in minimizing the total energy of displacement of the membrane under the constraint that the vertical force is admissible provided it is applied to a location intersecting a prescribed set.

Problem (3.8) has been considered in [43], under very restrictive conditions on the generator of the rearrangement class. More precisely, the author imposed the generator to be *strictly* positive and *bounded*. In this section, we remove both of these conditions. In addition, we address the case where D is a ball, an interesting situation that is neglected in [43]. In [20], the authors discussed the maximization version of (3.8); that is,

$$\sup_{f\in\mathcal{R}} \Phi_p(f).$$

Motivated by [20], the paper [23] mainly discusses a maximization problem related to the following boundary value problem:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } D \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = f(x) & \text{on } \partial D, \end{cases} \tag{3.10}$$

where $f \in \mathcal{R}$, and $\dfrac{\partial}{\partial \nu}$ denotes the outward normal derivative to the boundary. More precisely, the authors investigate the following maximization problem:

$$\sup_{f\in\mathcal{R}} \mathcal{I}(f),$$

where

$$\mathcal{I}(f) = \int_{\partial D} f u_f \, d\mathcal{H}^{n-1}. \tag{3.11}$$

Here $u_f \in W^{1,p}(D)$ denotes the unique solution of (3.10), and $d\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure on ∂D.

Problem (3.9), to the best of our knowledge is new. We hope it will serve as a motivation for further research. Henceforth, we refer to (3.8) as the *unconstrained* problem, and (3.9) as the *constrained* problem.

The rest of the section is organized as follows. First, we will focus on the unconstrained problem (3.8). In that subsection we will also consider the case of D being a planar disk, and prove that the minimizer is radial and increasing. Secondly, we will consider the constrained problem (3.9), and show there exists a unique solution. Because of the presence of the constraint in this problem, the expectation of having a radially increasing minimizer, in case the domain is a disk, is no longer guaranteed.

3.2.1 The unconstrained minimization problem

This subsection is devoted to the unconstrained problem (3.8). Let us fix some notation. Consider $f_0 \in L^{p'}(D)$, assumed to be a non-negative and non-trivial function. Here the set D is assumed to be a smooth bounded domain in $\mathbb{R}^2$. We let $\mathcal{R}$ denote the rearrangement class generated by f_0. For $f \in L^{p'}(D)$, $u_f \in W_0^{1,p}(D)$, as before, denotes the unique positive solution of (3.6).

The first main result of this section is the following

Theorem 3.2.1. *The unconstrained problem (3.8) has a unique solution $\hat{f} \in \mathcal{R}$. Moreover, there exists a decreasing function ϕ such that*

$$\hat{f} = \phi(\hat{u}), \quad \text{a.e. in } D, \tag{3.12}$$

where $\hat{u} = u_{\hat{f}}$. The equation (3.12) is called the Euler-Lagrange equation for $\hat{f}$.

The second main result is

Theorem 3.2.2. *Let D be a disk centered at the origin with radius a. Then $\hat{f}$, the unique solution of (3.8), is radial, i.e. $\hat{f}$ is a function of $r = |x|$. Moreover, $\hat{f}$ is increasing in r.*

To prove the above theorems we need the following basic result.

Lemma 3.2.3. *The following statements are true.*

(i) Φ_p *is weakly sequentially continuous in* $L^{p'}(D)$.

(ii) Φ_p *is strictly convex.*

(iii) Φ_p *is Gâteaux differentiable. Moreover, the Gâteaux derivative of* Φ_p *at* f*, denoted* $\Phi_p'(f)$*, can be identified with* $\dfrac{p}{p-1}u_f$.

Proof. (i) Let us consider $f_n \rightharpoonup f$ in $L^{p'}(D)$. For simplicity, let us set $u_n = u_{f_n}$ and $u = u_f$. We claim

$$(p-1)\Phi_p(f) + p\int_D (f_n - f)u\,dx \le (p-1)\Phi_p(f_n) \le (p-1)\Phi_p(f) + p\int_D (f_n - f)u_n\,dx. \tag{3.13}$$

We only prove the first inequality in (3.13), since the second one can be proved similarly. To this end, we begin by observing that

$$(p-1)\Phi_p(g) = \sup_{v\in W_0^{1,p}(D)} \left\{ p\int_D gv\,dx - \int_D |\nabla v|^p dx \right\}, \tag{3.14}$$

for every $g \in L^{p'}(D)$, and that,

$$(p-1)\Phi_p(f) = p\int_D fu\,dx - \int_D |\nabla u|^p dx, \tag{3.15}$$

recalling that $u = u_f$. From (3.14), with $g = f_n$, we infer

$$(p-1)\Phi_p(f_n) \ge p\int_D f_n u\,dx - \int_D |\nabla u|^p dx.$$

This last inequality lends itself to

$$(p-1)\Phi_p(f_n) \ge p\int_D (f_n - f)u\,dx + p\int_D fu\,dx - \int_D |\nabla u|^p dx. \tag{3.16}$$

Finally, (3.16) in conjunction with (3.15) yields the first inequality in (3.13).

From (3.13), it is clear that in order to complete the proof of part (i), it suffices to show

$$\lim_{n\to\infty}\int_D (f_n - f)u\,dx = 0 \quad \text{and} \quad \lim_{n\to\infty}\int_D (f_n - f)u_n\,dx = 0. \tag{3.17}$$

The first limit in (3.17) follows from the weak convergence of $\{f_n\}$ in $L^{p'}(D)$, since $u \in L^p(D)$. However, the verification of the second limit in (3.17) requires more work. To this end, let us recall

$$\begin{cases} -\Delta_p u_n = f_n & \text{in } D \\ u_n = 0 & \text{on } \partial D. \end{cases} \tag{3.18}$$

Multiplying the differential equation in (3.18) by u_n, and integrating the result over D, yields

$$\int_D |\nabla u_n|^p dx = \int_D f_n u_n dx. \tag{3.19}$$

An application of Hölder's inequality to the right hand side of (3.19), followed by the Poincaré inequality, leads to

$$\int_D |\nabla u_n|^p dx \leq C\|f_n\|_{p'}\|u_n\|_{W_0^{1,p}(D)},$$

where C is a universal positive constant. Whence, $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(D)$. This, in turn, implies existence of a subsequence of $\{u_n\}$, still denoted $\{u_n\}$, and $w \in W_0^{1,p}(D)$, such that

$$u_n \rightharpoonup w \quad \text{in } W_0^{1,p}(D) \qquad \text{and} \qquad u_n \to w \quad \text{in } L^p(D).$$

Next we write

$$\int_D (f_n - f)u_n dx = \int_D (f_n - f)(u_n - w)dx + \int_D (f_n - f)w dx. \tag{3.20}$$

The first term on the right hand side of (3.20) tends to zero because of Lemma 3.1.9. The second term in (3.20) also tends to zero because of weak convergence of $\{f_n\}$ in conjunction with the fact that w belongs to $L^p(D)$. This completes the proof of part (i). Parts (ii) and (iii) have been proved in [43]. □

Proof of Theorem 3.2.1. We first relax the minimization problem (3.8), by extending the admissible set $\mathcal{R}$ to $\overline{\mathcal{R}}$, the weak closure of $\mathcal{R}$ in $L^{p'}(D)$. Whence, we obtain

$$\inf_{f \in \overline{\mathcal{R}}} \Phi_p(f). \tag{3.21}$$

Clearly, (3.21) is solvable, since Φ_p is weakly continuous, and $\overline{\mathcal{R}}$ is weakly compact. In addition, thanks to the strict convexity of Φ_p and convexity of $\overline{\mathcal{R}}$, the solution to (3.21) is unique. Let us denote this unique solution by $\hat{f}$. We now proceed to prove that in fact $\hat{f} \in \mathcal{R}$. To this end, we recall the necessary condition satisfied by $\hat{f}$; namely,

$$0 \in \partial\Phi_p(\hat{f}) + \partial\xi_{\overline{\mathcal{R}}}(\hat{f}), \tag{3.22}$$

where $\xi_{\overline{\mathcal{R}}}$ stands for the indicator function supported on $\overline{\mathcal{R}}$; i.e.,

$$\xi_{\overline{\mathcal{R}}}(g) = \begin{cases} 0 & g \in \overline{\mathcal{R}} \\ \infty & g \notin \overline{\mathcal{R}}, \end{cases}$$

see [15] or [14] for details. Since Φ_p is Gâteaux differentiable, $\partial\Phi_p(\hat{f}) = \{\Phi_p'(\hat{f})\}$. On the other hand, from Definition 3.1.4, we infer

$$\begin{aligned} &\partial\xi_{\overline{\mathcal{R}}}(\hat{f}) \\ &= \left\{ w \in L^p(D): \ \xi_{\overline{\mathcal{R}}}(f) \geq \xi_{\overline{\mathcal{R}}}(\hat{f}) + \int_D (f-\hat{f}) w dx, \ \forall f \in L^{p'}(D) \right\}. \end{aligned} \tag{3.23}$$

Note that from (3.23), we infer that for $(w, f) \in \partial\xi_{\overline{\mathcal{R}}}(\hat{f}) \times \overline{\mathcal{R}}$:

$$\int_D (f-\hat{f}) w dx \leq 0. \tag{3.24}$$

Also, from (3.22) we deduce

$$\Phi_p'(\hat{f}) + w = 0, \tag{3.25}$$

for some $w \in \partial\xi_{\overline{\mathcal{R}}}(\hat{f})$. The equation (3.25), in turn, implies

$$\int_D \Phi_p'(\hat{f})(f-\hat{f}) dx + \int_D w(f-\hat{f}) dx = 0, \quad \forall f \in L^{p'}(D). \tag{3.26}$$

Recalling $\Phi_p'(\hat{f}) = \dfrac{p}{p-1}\hat{u}$, in conjunction with (3.26) and (3.24), we obtain

$$\int_D (f-\hat{f})\hat{u} dx \geq 0, \quad \forall f \in \overline{\mathcal{R}}. \tag{3.27}$$

Whence, $\hat{f}$ minimizes the linear functional $L(h) = \int_D h\hat{u} dx$, relative to $h \in \overline{\mathcal{R}}$.

From the differential equation

$$-\Delta_p \hat{u} = \hat{f}, \quad \text{in } D,$$

coupled with Lemma 7.7 in [30], it follows that the graph of $\hat{u}_S$, the restriction of $\hat{u}$ to the set $S(\hat{f}) = \{x \in D : \ \hat{f}(x) > 0\}$, has no significant flat sections on $S(\hat{f})$. From Lemma 3.1.8, we know there exists a $f_1 \in \mathcal{R}$ such that $S(f_1) \subseteq S(\hat{f})$. Therefore, if we denote by $\mathcal{R}_S$, the functions which are rearrangements of f_1 on $S(\hat{f})$, then by Lemma 3.1.3 we infer existence of a decreasing function ϕ_S such that $\phi_S(\hat{u}_S) \in \mathcal{R}_S$. We now proceed to extend ϕ_S to a decreasing function ϕ in such a way that $\phi(\hat{u}) \in \mathcal{R}(f_1) = \mathcal{R}$. Let us assume for the moment that this task has been accomplished. Then, from Lemma 3.1.4, it follows that $\phi(\hat{u})$ is the unique minimizer of the functional L, whence we must have $\hat{f} = \phi(\hat{u})$, which is the desired result.

We now come to the issue of extending ϕ_S. This is done in two steps. The first step is to show that $\hat{u}$ achieves its smallest values on $S(\hat{f})$. To this end, it suffices to prove the following inequality

$$\alpha \equiv ess \sup_{S(\hat{f})} \hat{u} \leq ess \inf_{S(\hat{f})^c} \hat{u} \equiv \beta, \tag{3.28}$$

where $S(\hat{f})^c$ denotes the complement of $S(\hat{f})$. In order to prove (3.28), we assume it is false and will derive a contradiction. So let us suppose $\alpha > \beta$, for the moment. Whence, there exist constants γ, δ, and sets $A \subseteq S(\hat{f})$, $B \subseteq S(\hat{f})^c$, both of positive measure, such that $\beta < \gamma < \delta < \alpha$, and

$$\hat{u} \geq \delta \quad \text{on } A \qquad \text{and} \qquad \hat{u} \leq \gamma \quad \text{on } B.$$

We may assume $|A| = |B|$, otherwise we consider subsets of A or B, see [46]. Let $\eta : A \to B$ be a measure preserving bijection; such a map exists, see for example [46]. Next, we define a new function $\overline{f}$ as follows:

$$\overline{f}(x) = \begin{cases} \hat{f}(x) & x \in (A \cup B)^c \\ \hat{f}(\eta(x)) & x \in A \\ \hat{f}(\eta^{-1}(x)) & x \in B. \end{cases}$$

Clearly $\overline{f}$ is a rearrangement of $\hat{f}$. Since $\hat{f} \in \overline{\mathcal{R}}$, it follows from Corollary 3.1.6 that $\overline{f} \in \overline{\mathcal{R}}$. Thus,

$$\begin{aligned} \int_D \overline{f}\hat{u}dx - \int_D \hat{f}\hat{u}dx &= \int_{A\cup B} \overline{f}\hat{u}dx - \int_{A\cup B} \hat{f}\hat{u}dx = \int_B \overline{f}\hat{u}dx - \int_A \hat{f}\hat{u}dx \\ &= \int_B \hat{f}(\eta^{-1}(x))\hat{u}dx - \int_A \hat{f}\hat{u}dx \\ &= \int_A \hat{f}(x)\hat{u}(\eta(x))dx - \int_A \hat{f}\hat{u}dx \\ &\leq (\gamma - \delta)\int_A \hat{f}dx < 0, \end{aligned}$$

which contradicts the minimality of $\hat{f}$, relative to $\overline{\mathcal{R}}$.

In the second step, we give an explicit definition of the extended function. We denote the extended function by ϕ, and define it as follows:

$$\phi(t) = \begin{cases} \phi_S(t) & t < \alpha \\ 0 & t \geq \alpha, \end{cases}$$

where α is defined as in (3.28). Clearly, ϕ is decreasing, and $\phi(\hat{u}) \in \mathcal{R}$. Hence, the proof of the theorem is completed. $\square$

Remark 3.2.1. *If $f_0 = \chi_{D_0}$, the characteristic function of some measurable set $D_0 \subseteq D$, then, from Theorem 3.2.1, we can deduce $\hat{f} = \chi_{\hat{D}}$, for some $\hat{D} \subseteq D$, satisfying $|\hat{D}| = |D_0|$. In addition, from (3.12), it follows that $\hat{D} = \{x \in D : \hat{u}(x) < \beta\}$, for some $\beta > 0$. This, in turn, implies that $\hat{D}$ contains a layer around the boundary ∂D, since $\hat{u} \in C(\overline{D})$. If D is simply connected, we can additionally show that $\hat{D}$ is connected. To see this, assume the contrary. So, we assume there is a component of $\hat{D}$, say $\mathcal{U}$, such that the intersection of $\partial\mathcal{U}$ and ∂D is empty. Observe that, $\hat{u} = \beta$ on $\partial\mathcal{U}$. Thus $\hat{u}$ satisfies*

$$\begin{cases} -\Delta_p \hat{u} = \hat{f} & \text{in } \mathcal{U} \\ \hat{u} = \beta & \text{on } \partial\mathcal{U}. \end{cases} \tag{3.29}$$

Applying the strong maximum principle to (3.29), we find $\hat{u} > \beta$ in $\mathcal{U}$. This clearly contradicts the fact that $\hat{u} < \beta$ throughout $\hat{D}$, hence, $\hat{D}$ is connected. Since $\hat{D} = \{x \in D : \hat{u}(x) < \beta\}$, $\hat{u}$ satisfies

$$\begin{cases} -\Delta_p \hat{u} = \chi_{\{\hat{u} < \beta\}} & \text{in } D \\ \hat{u} = 0 & \text{on } \partial D. \end{cases}$$

By setting $v = \beta - \hat{u}$, we derive

$$\Delta_p v = \chi_{\{v > 0\}}, \tag{3.30}$$

which is the one phase obstacle problem for the p-Laplacian operator. Through a private communication with H. Shahgholian, we found that many questions related to the free boundary of (3.30) are yet to be settled, see [35, 38]. However, when $p = 2$, the free boundary of the problem (3.30) is extensively studied, see for example [44].

In order to prove Theorem 3.2.2, we need the following result.

Lemma 3.2.4. *Let $f \in L^{p'}(B)$, where B is a ball centered at the origin. Let R be a rotation map about the origin, i.e. $R(\theta) = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$, and let $f_R(x) = f(Rx)$. Let $u \in W_0^{1,p}(B)$ and $v \in W_0^{1,p}(B)$ satisfy*

$$\begin{cases} -\Delta_p u = f & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{3.31}$$

and

$$\begin{cases} -\Delta_p v = f_R & \text{in } B \\ v = 0 & \text{on } \partial B, \end{cases} \tag{3.32}$$

respectively. Then $v(x) = u(Rx)$, in B.

Proof. We set $w(x) = u(Rx)$. Consider a test function $\zeta \in C_0^\infty(B)$. It is easy to see that $\nabla w(x) = R^{-1}\nabla u(Rx)$, and that $|\nabla w(x)| = |\nabla u(Rx)|$. Whence

$$\begin{aligned} &\int_B |\nabla w(x)|^{p-2}\nabla w(x)\cdot\nabla\zeta(x)dx \\ &\qquad = \int_B |\nabla u(Rx)|^{p-2}\nabla u(Rx)\cdot R\nabla\zeta(x)dx, \end{aligned} \tag{3.33}$$

where we have used $R^{-1} = R^t$, the transpose of R. We next use the change of variables $y = Rx$ in (3.33), to obtain

$$\begin{aligned} &\int_B |\nabla w(x)|^{p-2}\nabla w(x)\cdot\nabla\zeta(x)dx \\ &\qquad = \int_B |\nabla u(y)|^{p-2}\nabla u(y)\cdot\nabla\tilde{\zeta}(y)dy, \end{aligned} \tag{3.34}$$

where $\tilde{\zeta}(y) = \zeta(x)$. Since, $\tilde{\zeta} \in C_0^\infty(B)$, from (3.34) and (3.31), we infer

$$\begin{aligned} &\int_B |\nabla w(x)|^{p-2}\nabla w(x)\cdot\nabla\zeta(x)dx \\ &\qquad = \int_B f(x)\tilde{\zeta}(x)dx = \int_B f_R(x)\zeta(x)dx. \end{aligned} \tag{3.35}$$

Whence, w is a solution of (3.32). By uniqueness we deduce $w(x) = v(x)$, as desired. □

Proof of Theorem 3.2.2. Let us first show that $\hat{f}$ is radial. To this end, we let R be a rotation map about the origin. Also, we set $\hat{f}_R(x) = \hat{f}(Rx)$. Let $\hat{u} \in W_0^{1,p}(D)$, and $v \in W_0^{1,p}(D)$, denote the solutions of (3.6), with $f = \hat{f}$ and $f = \hat{f}_R$, respectively. From Lemma 3.2.4, we infer $v(x) = \hat{u}(Rx)$. Whence

$$\Phi_p(\hat{f}_R) = \int_D \hat{f}_R v dx = \int_D \hat{f}(Rx)\hat{u}(Rx)dx = \int_D \hat{f}\hat{u}dx = \Phi_p(\hat{f}).$$

Thus, $\hat{f}_R$ is also a solution of (3.8). By uniqueness, we deduce $\hat{f} = \hat{f}_R$. Since R is arbitrary, we infer $\hat{f}$ is radial.

To prove $\hat{f}$ is increasing, we first need to show that $\hat{u}$ is radial and decreasing. To this end, it suffices to show the boundary value problem

$$\begin{cases} -\Delta_p u = \hat{f} & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{3.36}$$

has a radial solution. Thus, we need to prove the following initial value problem is solvable.

$$-\frac{1}{r}\left(r|u'|^{p-2}u'\right)' = \hat{f}(r), \qquad u'(0) = 0, \quad u(a) = 0.$$

By integrating the above ordinary differential equation from 0 to r, we derive

$$r|u'|^{p-2}u' = -\int_0^r s\hat{f}(s)ds. \tag{3.37}$$

Thus, $u' \leq 0$, since $\hat{f} \geq 0$. Hence u is decreasing, as expected. Now, integrating (3.37), from r to a, yields

$$u(r) = \int_r^a \left(\frac{1}{t}\int_0^t s\hat{f}(s)ds\right)^{\frac{1}{p-1}} dt.$$

Therefore, the unique solution of (3.36) is radial.

Now we apply Theorem 3.2.1, which ensures $\hat{f}$ satisfies (3.12), for some decreasing function ϕ. Therefore, $\hat{f}$ must be increasing. This completes the proof of the theorem. □

3.2.2 The constrained minimization problem

In this subsection, we prove the constrained problem (3.9) is solvable. But first we need some preliminaries. We fix a measurable set $K \subseteq D$ such that

$|K| = \alpha$ with $0 < \alpha < |D|$. Let $f_0 \in L^{p'}(D)$ be a *positive* function, which is different from the unconstrained case. Then, there exist positive β and γ such that

$$|E_1 \equiv \{x \in D : f_0(x) > \beta\}| \leq \alpha \leq |\{x \in D : f_0(x) \geq \beta\} \equiv E_2|, \tag{3.38}$$

and

$$|F_1 \equiv \{x \in D : f_0(x) < \gamma\}| \leq \alpha \leq |\{x \in D : f_0(x) \leq \gamma\} \equiv F_2|. \tag{3.39}$$

Furthermore, we suppose f_0 does not concentrate its largest or smallest values on K in the sense by satisfying:

(i) $|E_1 \setminus K| > 0$, or $|E_1 \setminus K| = 0$ and $|K \setminus E_2| > 0$;

(ii) $|F_1 \setminus K| > 0$, or $|F_1 \setminus K| = 0$ and $|K \setminus F_2| > 0$.

Let us set $\epsilon = \int_K f_0 dx$. The class of rearrangements of f_0 in D is denoted $\mathcal{R}(f_0)$, which for simplicity we use $\mathcal{R}$ instead of $\mathcal{R}(f_0)$. Finally, we set

$$\Lambda = \{f \in L^{p'}(D) : \int_K f dx = \epsilon\}.$$

Observe that Λ is an affine subspace of codimension one in $L^{p'}(D)$.

We are now ready to state the main result of this section.

Theorem 3.2.5. *Let f_0, $\mathcal{R}$, D, K and Λ be as described in the beginning of this section. Then, the constrained problem:*

$$\inf_{f \in \mathcal{R} \cap \Lambda} \Phi_p(f) \tag{3.40}$$

has a unique solution $\tilde{f}$. Moreover, $\tilde{f}$ satisfies the following Euler-Lagrange equation:

$$\tilde{f} = \phi(\tilde{u} + \lambda \chi_K), \quad \text{a.e. in } D, \tag{3.41}$$

for some $\lambda \in \mathbb{R}$, and a decreasing function ϕ, unknown a priori. Here $\tilde{u}$ stands for the solution of (3.6), with $f = \tilde{f}$.

The following result will be used in the proof of Theorem 3.2.5.

Lemma 3.2.6. *Let $\mathcal{R}$ and Λ be as in Theorem 3.2.5. Then, $\overline{\mathcal{R} \cap \Lambda} = \overline{\mathcal{R}} \cap \Lambda$, where the bar indicates the weak closure in $L^{p'}(D)$.*

Proof. Since $\overline{\mathcal{R}} \cap \Lambda$ is weakly closed, it follows that $\overline{\mathcal{R} \cap \Lambda} \subseteq \overline{\mathcal{R}} \cap \Lambda$. To prove the reverse inclusion, we fix $g \in \overline{\mathcal{R}} \cap \Lambda$, and consider a weakly open subbasis $\mathcal{N}_{\varrho,l}(g)$, containing g. Whence,

$$\mathcal{N}_{\varrho,l}(g) = \left\{ f \in L^{p'}(D) : \left| \int_D lf dx - \int_D lg dx \right| < \varrho \right\},$$

where $l \in L^p(D)$. We set $\mathcal{V} = \left\{ f \in L^{p'}(D) : \int_D lf dx = \int_D lg dx \right\}$, which is an affine subspace of codimension one in $L^{p'}(D)$, and observe that $\mathcal{V} \subseteq \mathcal{N}_{\varrho,l}(g)$. Let $\mathcal{K} = \mathcal{V} \cap \overline{\mathcal{R}} \cap \Lambda$, so $\mathcal{K}$ is convex, weakly compact and non-empty. Moreover, by the Krein-Milman theorem, see [17] or [48], we infer $\mathcal{K} = \overline{\text{co}}(\text{ext}(\mathcal{K}))$. Therefore, $\text{ext}(\mathcal{K})$ is not empty. However, $\text{ext}(\mathcal{K}) = \mathcal{V} \cap \mathcal{R} \cap \Lambda$, by Lemma 3.1.1 (iii). Whence, $\mathcal{N}_{\varrho,l}(g) \cap \mathcal{R} \cap \Lambda$ is not empty, which implies g must be a weak limit point of $\mathcal{R} \cap \Lambda$. Thus, $g \in \overline{\mathcal{R} \cap \Lambda}$, as desired. $\square$

Proof of Theorem 3.2.5. We begin by relaxing the problem (3.40). To this end, we extend $\mathcal{R} \cap \Lambda$ to $\overline{\mathcal{R} \cap \Lambda}$, and consider:

$$\inf_{f \in \overline{\mathcal{R} \cap \Lambda}} \Phi_p(f). \tag{3.42}$$

Since $\overline{\mathcal{R} \cap \Lambda}$ is weakly compact, and Φ_p is weakly continuous, the minimization problem (3.40) is solvable. Moreover, $\overline{\mathcal{R} \cap \Lambda} = \overline{\mathcal{R}} \cap \Lambda$, by Lemma 3.2.6, hence $\overline{\mathcal{R} \cap \Lambda}$ is convex. This, along with the fact that Φ_p is strictly convex imply that (3.40) has a unique solution. Let us denote the solution by $\tilde{f}$. We claim that, in fact, $\tilde{f} \in \mathcal{R} \cap \Lambda$. To prove the claim, we first write the necessary condition satisfied by $\tilde{f}$:

$$0 \in \partial \Phi_p(\tilde{f}) + \partial \xi_{\overline{\mathcal{R}} \cap \Lambda}(\tilde{f}), \tag{3.43}$$

where $\xi_{\overline{\mathcal{R}} \cap \Lambda}$ denotes the indicator function supported on $\overline{\mathcal{R}} \cap \Lambda$. From (3.43), we infer existence of $g \in \partial \xi_{\overline{\mathcal{R}} \cap \Lambda}(\tilde{f})$ such that

$$\frac{p}{p-1} \int_D \tilde{u}(f - \tilde{f}) dx + \int_D g(f - \tilde{f}) dx = 0, \quad \forall f \in L^{p'}(D), \tag{3.44}$$

where we have used $\partial \Phi_p(\tilde{f}) = \{\Phi_p'(\tilde{f})\}$; here $\tilde{u}$ denotes the solution of (3.6), with $f = \tilde{f}$. From (3.44), we deduce

$$\int_D \tilde{u}(f - \tilde{f}) dx \geq 0, \quad \forall f \in \overline{\mathcal{R}} \cap \Lambda. \tag{3.45}$$

The inequality (3.45) implies that $\tilde{f}$ minimizes the linear functional $\tilde{L}(f) = \int_D \tilde{u} f dx$, relative to $f \in \overline{\mathcal{R}} \cap \Lambda$. At this stage we utilize Lemma 3.1.10. For this purpose, we set $l_1(f) = \int_D \chi_K f dx$, $l_2(f) = \tilde{L}(f)$, $\mathcal{C} = \overline{\mathcal{R}}$ and $I = \epsilon$. In order to apply Lemma 3.1.10, we only need to verify existence of f_1 and f_2 in $\overline{\mathcal{R}}$ such that

$$\int_D \chi_K f_1 dx < \epsilon < \int_D \chi_K f_2 dx.$$

For simplicity, we suppose $|E_1 \setminus K| > 0$, $|F_1 \setminus K| = 0$ and $|K \setminus F_2| > 0$, and other cases can follow similarly. Furthermore, it follows from (3.38) and (3.39) that $|K \setminus E_1| > 0$ and $|F_2 \setminus K| > 0$. We construct f_1 as follows. Let $A \subseteq F_2 \setminus K$ and $B \subseteq K \setminus F_2$ be measurable sets with $|A| = |B|$. Let $\eta_1 : A \to B$ be a measure preserving bijection. Define

$$f_1(x) = \begin{cases} f_0(x) & x \in (A \cup B)^c \\ f_0(\eta_1(x)) & x \in A \\ f_0(\eta_1^{-1}(x)) & x \in B. \end{cases}$$

Clearly, $f_1 \in \mathcal{R}$, and $\int_D \chi_K f_1 dx < \int_D \chi_K f_0 dx = \epsilon$. Next, we construct f_2. Let $C \subseteq E_1 \setminus K$ and $D \subseteq K \setminus E_1$ be measurable sets with $|D| = |C|$. Let $\eta_2 : C \to D$ be a measure preserving bijection. Let

$$f_2(x) = \begin{cases} f_0(x) & x \in (C \cup D)^c \\ f_0(\eta_2(x)) & x \in C \\ f_0(\eta_2^{-1}(x)) & x \in D. \end{cases}$$

Then, $f_2 \in \mathcal{R}$, and $\int_D \chi_K f_2 dx > \int_D \chi_K f_0 dx = \epsilon$. Now we can apply Lemma 3.1.10 to infer existence of $\lambda \in \mathbb{R}$ such that $\tilde{f}$ minimizes the linear functional $M(f) = \int_D f(\tilde{u} + \lambda\chi_K) dx$, relative to $\overline{\mathcal{R}}$. Observe that $\tilde{u}$ has no significant flat sections on D, hence the same holds for $\tilde{u} + \lambda\chi_K$. Then, it follows from Lemma 3.1.3 there exists a decreasing function ϕ such that $\phi(\tilde{u} + \lambda\chi_K) \in \mathcal{R}$. Thus, by Lemma 3.1.4, we obtain

$$\tilde{f} = \phi(\tilde{u} + \lambda\chi_K) \quad \text{a.e. in } D.$$

This completes the proof of the theorem. □

Remark 3.2.2. *Interested readers are encouraged to use the ideas and tools presented in this chapter to investigate the following maximization problem:*

$$\sup_{f \in \mathcal{R} \cap \Lambda} \mathcal{I}(f),$$

where $\mathcal{I}(f)$ is defined as in (3.11). This problem will certainly be of interest to the authors of [23].

3.3 Rearrangement optimization problem 2

In [32], the authors consider the following boundary value problem:

$$\begin{cases} -\Delta u + \chi_E u = f & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{3.46}$$

in which D is a smooth bounded domain in $\mathbb{R}^N (N = 2, 3)$, f is a given function, E is a measurable subset of D, and χ_E is the characteristic function of E, i.e. $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. Denoting the unique solution of (3.46) by u_E, they investigate the following minimization problem:

$$\inf_{|E|=\alpha} \int_D f u_E dx, \tag{3.47}$$

where $|E|$ denotes the Lebesgue measure of E, and α is a given positive number. After proving (3.47) to be solvable, they set up the minimality condition in terms of the tangent cones. Since the underlying function space is $L^\infty(D)$, they are able to derive a convenient formulation of the tangent cone of an appropriate convex set. They prove, amongst other results, that solvability of (3.47) is ensured once certain conditions are satisfied by the force function f.

The main motivation of this research comes from the physical meaning of the minimization problem (3.47) which we briefly describe here. The boundary value problem (3.46) models an elastic membrane, constructed out of two different materials, fixed around the boundary, and subject to a vertical force $f(x)$ at each x. The function u_E denotes the displacement of the membrane from the rest position, and the quantity $\int_D f u_E dx$ measures the total energy of displacement. To ensure the membrane is as robust as possible, one naturally is led to the minimization problem (3.47). Clearly, any solution of (3.47) is a favorable design. A natural question that may arise is: *what if we want to use more than two different materials in our design?*[2]

We present an answer to this question under similar restrictions imposed on f as in [32]. The mathematical set up of the problem is as follows. Consider the boundary value problem

$$\begin{cases} -\Delta u + g(x)u = f(x) & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{3.48}$$

[2]say, due to cost restrictions.

in which $D \subseteq \mathbb{R}^N$, g is a non-negative function in $L^\infty(D)$, and f is a non-trivial non-negative function in $L^2(D)$. Associated with (3.48) a quantity called *energy* is defined by:

$$\Phi(g) = \int_D f u_g dx = \int_D |\nabla u_g|^2 dx + \int_D g u_g^2 dx, \tag{3.49}$$

where u_g is the unique solution of (3.48). Note that the following identity follows from the variational formulation of u_g:

$$\Phi(g) = \sup_{v \in H_0^1(D)} \left\{ 2 \int_D f v dx - \int_D (|\nabla v|^2 + g v^2) dx \right\} \tag{3.50}$$

Let us fix a non-trivial function g_0 such that $0 \leq g_0 \leq 1$, and let $\mathcal{R} \equiv \mathcal{R}(g_0)$ denote the *rearrangement* class generated by g_0. We are interested in the minimization problem:

$$\inf_{g \in \mathcal{R}} \Phi(g). \tag{3.51}$$

Remark 3.3.1. *If $g_0 = \chi_{E_0}$, for some E_0 with $|E_0| = \alpha$, then $\mathcal{R} = \{\chi_E : |E| = \alpha\}$. In this case, by identifying $\mathcal{R}$ with the set $\{E : |E| = \alpha\}$, we see that (3.51) reduces to (3.47).*

Remark 3.3.2. *Another interesting way to describe Problem (3.51) is that we can treat $g(x)$ as the spring factor. Then, the mathematical model for (3.48) is the elastic membrane supported by springs with fixed boundary. So, our target is to minimize the energy of displacement by rearranging the supports of springs*[3].

Our approach to proving the solvability of (3.51) is based on the well developed theory of rearrangements of functions by G. R. Burton [7, 8]. To this end, we first relax the minimization problem (3.51) by extending the admissible set $\mathcal{R}$ to its weak closure $\overline{\mathcal{R}}$ with respect to L^2-topology. Once the relaxed problem is shown to be solvable, we will demonstrate how the restrictions on the force function f imply that solutions of the relaxed problem are indeed solutions of the original problem (3.51).

An important feature of Φ which has been overlooked in [32] is its strict convexity. This crucial fact guarantees that the solution of (3.51), if it exists, is unique. Using the strict convexity of Φ we are able to recapture the results

[3] This physical interpretation comes from a discussion with A. B. Movchan, and he also pointed out that the differential equation in (3.48) can also be found in [13].

of [32], and reduce technicalities in the case of radial domains. Indeed, we will show that when D is a ball and f is radial then the solution of (3.51) is radial and non-increasing. We will also discuss the maximization problem.

The remaining part of this section concerns a free boundary result, for which we have used the method of domain derivatives to verify that the value of the solution of the state equation corresponding to an optimal shape is constant on the free boundary (the boundary of the optimal shape), and various monotonicity assertions pertaining to the density and the amount of materials used in the construction of the membrane.

3.3.1 More preliminaries

Before attacking the problem, we need to develop more backgrounds about rearrangement of functions. Let us use $\overline{\mathcal{R}}$ to denote the weak closure of $\mathcal{R}$ in $L^2(D)$. It is well-known that $\overline{\mathcal{R}}$ is convex, and weakly compact in $L^2(D)$, see Lemma 3.1.1.

Lemma 3.3.1. *Let $\overline{\mathcal{R}}$ be defined as above. Then, $\overline{\mathcal{R}} \subseteq L^\infty(D)$ and $\forall\, g \in \overline{\mathcal{R}} : \|g\|_\infty \leq \|g_0\|_\infty$.*

Proof. In order to derive a contradiction, we suppose $g \notin L^\infty(D)$. Hence, for every positive M, $|\{x \in D : g(x) > M\}| > 0$. Let us choose $M = \|g_0\|_\infty$, and set $E = \{x \in D : g(x) > \|g_0\|_\infty\}$. Since $g \in \overline{\mathcal{R}}$, there exists $\{g_n\} \subseteq \mathcal{R}$ such that $g_n \rightharpoonup g$ in $L^2(D)$. Then, we have

$$\int_E g_n dx = \int_D g_n \chi_E dx \to \int_D g \chi_E dx = \int_E g dx. \tag{3.52}$$

From the definition of E and the fact that $\int_E g_n dx \leq \|g_0\|_\infty |E|$, in conjunction with (3.52), we deduce

$$\|g_0\|_\infty |E| < \int_E g dx = \lim_{n\to\infty} \int_E g_n dx \leq \|g_0\|_\infty |E|. \tag{3.53}$$

Obviously, (3.53) is a contradiction. The above argument in particular implies the measure of E is zero. Hence, $\|g\|_\infty \leq \|g_0\|_\infty$. This completes the proof of the lemma. □

Lemma 3.3.2. *Suppose non-negative functions $\{g_n\} \subseteq L^\infty(D)$, and $g \in L^2(D)$. Suppose $g_n \rightharpoonup g$ in $L^2(D)$. Then, g is non-negative a.e. in D.*

Proof. This is an immediate consequence of Mazur Lemma. Indeed, by Mazur Lemma, there exists a sequence $\{v_n\}$ in the convex hull of the set

$\{g_n : n \in \mathbb{N}\}$ such that $v_n \to g$ in $L^2(D)$. Therefore, $v_n \to g$ in measure. Whence, there exists a subsequence of $\{v_n\}$ which converges to g a.e. in D. This completes the proof. □

We will also need the following rearrangement result for the Dirichlet integral (see e. g. [6]). Note that here v^* denotes the Schwarz symmetrization of v (see e. g. [36]):

Lemma 3.3.3.

(i) *If $v \in H_0^1(\mathbb{R}^N)$ is non-negative then $v^* \in H_0^1(\mathbb{R}^N)$, and the following inequality holds*

$$\int_{\mathbb{R}^N} |\nabla v^*|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla v|^2 \, dx. \tag{3.54}$$

(ii) *If $v \in H_0^1(\mathbb{R}^N)$ is non-negative, equality holds in (3.54), and $\{x \in \mathbb{R}^N : \nabla v = 0,\ 0 < v(x) < M\}$ has zero measure, then v is a translate of v^*.*

3.3.2 Existence and uniqueness of optimal solutions

This subsection is devoted to the minimization problem (3.51). But first, we need the following basic result regarding the energy functional Φ.

Lemma 3.3.4. *The energy functional Φ satisfies the following:*

(i) *Φ is weakly continuous on $\overline{\mathcal{R}}$ with respect to L^2-topology.*

(ii) *Φ is strictly convex on $\overline{\mathcal{R}}$.*

(iii) *Given g and h in $\overline{\mathcal{R}}$, the following formula holds*

$$\lim_{t \to 0^+} \frac{\Phi(\xi_t) - \Phi(g)}{t} = -\int_D (h-g)u^2 dx, \quad 0 < t < 1, \tag{3.55}$$

in which $\xi_t = g + t(h-g)$, and $u = u_g$.

Proof.

(i) Let $\{g_n\} \subseteq \overline{\mathcal{R}}$ and $g \in \overline{\mathcal{R}}$ such that $g_n \rightharpoonup g$ in $L^2(D)$. For simplicity, let us set $u_n = u_{g_n}$ and $u = u_g$. We have

$$\begin{cases} -\Delta u_n + g_n u_n = f & \text{in } D \\ u_n = 0 & \text{on } \partial D. \end{cases} \tag{3.56}$$

Multiplying the differential equation in (3.56) by u_n, and integrating the result over D, yields

$$\int_D |\nabla u_n|^2 dx + \int_D g_n u_n^2 dx = \int_D f u_n dx. \tag{3.57}$$

From Lemma 3.3.2, we know g_n are non-negative. Therefore (3.57) implies

$$\int_D |\nabla u_n|^2 dx \leq \int_D f u_n dx. \tag{3.58}$$

By applying Hölder's inequality and the Poincaré inequality to the right hand side of (3.58) we obtain

$$\int_D |\nabla u_n|^2 dx \leq C \left\|f\right\|_2 \left\|u_n\right\|_{H_0^1(D)}, \tag{3.59}$$

in which C is a positive constant. Whence, $\{u_n\}$ is a bounded sequence in $H_0^1(D)$. This in turn implies existence of a subsequence of $\{u_n\}$, still denoted $\{u_n\}$, and $w \in H_0^1(D)$, such that

$$u_n \rightharpoonup w \quad \text{in } H_0^1(D) \qquad \text{and} \qquad u_n \to w \quad \text{in } L^2(D).$$

Let us prove that $w = u$, where u is the solution of

$$\begin{cases} -\Delta u + gu = f & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{3.60}$$

Indeed, by (3.56) we have

$$\int_D \nabla u_n \cdot \nabla \phi \, dx + \int_D g_n u_n \phi dx = \int_D f \phi dx, \qquad \forall \phi \in C_0^\infty(D).$$

Since $u_n \rightharpoonup w$ in $H_0^1(D)$, $g_n \rightharpoonup g$ in $L^2(D)$ and $u_n \to w$ strongly in $L^2(D)$, from the latter equation we find

$$\int_D \nabla w \cdot \nabla \phi \, dx + \int_D gw \, \phi dx = \int_D f \phi dx, \qquad \forall \phi \in C_0^\infty(D).$$

This means that w is a solution of (3.60), and by uniqueness, we must have $w = u$. To prove (i), we observe that

$$\Big|\Phi(g_n) - \Phi(g)\Big| = \Big|\int_D f(u_n - u) dx\Big| \leq ||f||_2 \, ||u_n - u||_2$$

which together with the fact that $\lim_{n\to\infty} ||u_n - u||_2 = 0$ imply (i).

(ii) Let $h, g \in \overline{\mathcal{R}}$, $0 < t < 1$, and $\xi_t = th + (1-t)g$. For $v \in H_0^1(D)$, we have

$$
\begin{aligned}
(3.61)\quad 2\int_D fv\,dx - \int_D |\nabla v|^2 dx - \int_D \xi_t v^2 dx = \\
t\left(2\int_D fv\,dx - \int_D |\nabla v|^2 - \int_D hv^2 dx\right) \\
+ (1-t)\left(2\int_D fv\,dx - \int_D |\nabla v|^2 - \int_D gv^2 dx\right)
\end{aligned}
$$

By taking supremum of (3.61) with respect to $v \in H_0^1(D)$, we obtain

$$
(3.62)\qquad \Phi(th + (1-t)g) \le t\Phi(h) + (1-t)\Phi(g).
$$

This proves the convexity of Φ. We now show that Φ is in fact strictly convex, by contradiction. To this end, we assume that there exists $t \in (0,1)$ such that $\Phi(th + (1-t)g) = t\,\Phi(h) + (1-t)\Phi(g)$. For simplicity, we use u_t in place of $u_{th+(1-t)g}$. So, we have

$$
\begin{aligned}
(3.63)\quad 2\int_D fu_t\,dx - \int_D |\nabla u_t|^2 dx - \int_D \xi_t u_t^2 dx = \\
t\left(2\int_D fu_h\,dx - \int_D |\nabla u_h|^2 - \int_D hu_h^2 dx\right) \\
+ (1-t)\left(2\int_D fu_g\,dx - \int_D |\nabla u_g|^2 - \int_D gu_g^2 dx\right)
\end{aligned}
$$

From (3.63), we deduce the following equations

$$
\begin{aligned}
(3.64)\quad 2\int_D fu_h\,dx - \int_D |\nabla u_h|^2 - \int_D hu_h^2 dx = \\
2\int_D fu_t\,dx - \int_D |\nabla u_t|^2 dx - \int_D hu_t^2 dx
\end{aligned}
$$

and

$$
\begin{aligned}
(3.65)\quad 2\int_D fu_g\,dx - \int_D |\nabla u_g|^2 - \int_D gu_g^2 dx = \\
2\int_D fu_t\,dx - \int_D |\nabla u_t|^2 dx - \int_D gu_t^2 dx
\end{aligned}
$$

From the maximality of u_h coupled with (3.64), we infer $u_h = u_t$. Similarly, from the maximality of u_g and (3.65), we find $u_g = u_t$. Hence, $u_t = u_h = u_g$. On the other hand, from the differential equations

$$-\Delta u_h + hu_h = f, \quad \text{a.e. in } D,$$

and

$$-\Delta u_g + gu_g = f, \quad \text{a.e. in } D,$$

we infer $(h-g)u_h = 0$ almost everywhere in D. Since u_h is positive by the strong maximum principle, we must have $h = g$ almost everywhere in D. Therefore, the strict convexity is proved.

(iii) For simplicity, we set $u_t = u_{\xi_t}$. We know

$$\begin{cases} -\Delta u_t + \xi_t u_t = f & \text{in } D \\ u_t = 0 & \text{on } \partial D, \end{cases} \tag{3.66}$$

and

$$\begin{cases} -\Delta u + gu = f & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{3.67}$$

From (3.66) and (3.67), we obtain

$$-\Delta(u_t - u) + g(u_t - u) = gu_t - \xi_t u_t = (g - \xi_t)u_t \tag{3.68}$$

Multiplying (3.68) by $u_t + u$, and integrating the result over D, we get

$$\begin{aligned} &\int_D |\nabla u_t|^2 dx - \int_D |\nabla u|^2 + \int_D g u_t^2 dx - \int_D g u^2 dx \\ &\qquad = \int_D (g - \xi_t) u_t (u_t + u) dx = -t \int_D (h - g) u_t (u_t + u) dx \end{aligned} \tag{3.69}$$

From (3.69), we derive $\Phi(\xi_t) - \Phi(g) = -t\int_D (h-g)u_t u dx$, which in turn implies:

$$\begin{aligned} &\Phi(\xi_t) - \Phi(g) + t\int_D (h-g)u^2 dx \\ &\qquad = -t\int_D (h-g)u_t u dx + t\int_D (h-g)u^2 dx \\ &\qquad\qquad = -t\int_D (h-g)(u_t - u)u dx. \end{aligned} \tag{3.70}$$

By applying Hölder's inequality to the right hand side of (3.70), we find

$$\left|\Phi(\xi_t) - \Phi(g) + t\int_D (h-g)u^2 dx\right| \leq t\,\|h-g\|_\infty \,\|u_t - u\|_2 \,\|u\|_2 \,. \tag{3.71}$$

Since $\xi_t \rightharpoonup g$ weakly in $L^2(D)$ (and even strongly), by the proof of part (i) we have $||u_t - u||_2 \to 0$ as $t \to 0$. Hence, dividing by t in (3.71) and letting $t \to 0$ we get the desired result.

□

Remark 3.3.3. *By revisiting the proof of Lemma 3.3.4 (i), we actually have that Φ is strictly convex and weakly continuous on the set $\{f \in L^2(D) : f \geq 0\}$ with respect to L^2-topology.*

Before stating the main result of this section, we make some assumptions. To begin with, henceforth we will use $v_f \in H_0^1(D)$ to denote the unique solution of the Poisson boundary value problem:

$$\begin{cases} -\Delta v_f = f & \text{in } D \\ v_f = 0 & \text{on } \partial D \end{cases} \tag{3.72}$$

Here are the assumptions that we need:

A1: $v_f \leq f$ in D.

A2: $f \leq -\Delta f$ in D.

Remark 3.3.4. *Note that we can find a non-negative f satisfying* **A2***. Indeed, consider the boundary value problem*

$$\begin{cases} -\Delta u - u = N & \textit{in } D \\ u = 0 & \textit{on } \partial D \end{cases} \tag{3.73}$$

in which $N \in [0,\infty)$. The energy functional associated with (3.73) is

$$I(u) = \frac{1}{2}\int_D |\nabla u|^2 dx - \frac{1}{2}\int_D u^2 dx - \int_D N u dx$$

It is clear from the Poincaré inequality, see for example [1], that if D is thin, then $I(u)$ will be coercive. So, by an application of the direct method *of calculus of variations to the functional $I(u)$, we infer the existence of a critical point which is a solution of (3.73). In order to show that (3.73) has a non-negative solution, it suffices to point out that $I(|u|) \leq I(u)$.*

The main result of the section is the following:

Theorem 3.3.5. *Suppose that f satisfies one of the assumptions* **A1** *or* **A2**. *Then the minimization problem (3.51) has a unique solution $\hat{g} \in \mathcal{R}$. Moreover, there exists an increasing function ψ such that*

$$\hat{g} = \psi(\hat{u}) \quad a.e.\ in\ D, \tag{3.74}$$

where $\hat{u} = u_{\hat{g}}$.

To prove Theorem 3.3.5, we need the following lemma:

Lemma 3.3.6. *Suppose f satisfies one of the assumptions* **A1** *or* **A2**. *Suppose g is a measurable function such that $0 \leq g \leq 1$. Then, u_g has no significant flat sections on D.*

Proof. First, let us suppose the assumption **A1** apply. From the boundary value problems (3.48) and (3.72), we deduce

$$\begin{cases} -\Delta(u_g - v_f) + g(u_g - v_f) = -gv_f & \text{in } D \\ u_g - v_f = 0 & \text{on } \partial D, \end{cases}$$

Since g and v_f are non-negative, $u_g < v_f$ in D by the strong maximum principle. On the other hand, by the strong maximum principle, we have $f \geq v_f > 0$ in D.

In order to derive a contradiction, we assume that there exists $L \subseteq D$ such that the measure of L is positive, and u_g is constant on L. By applying Lemma 7.7 in [30], we infer $f = gu_g$ in L. To this end, by observing that $D = S(g) \cup S(g)^c$, let us divide the discussion into cases. If $|L \cap S(g)| > 0$, then we have

$$f = gu_g < gv_f \leq v_f \leq f \quad \text{in} \quad L \cap S(g)$$

which is a contradiction. Otherwise, we have $f = 0$ in $L \cap S(g)^c$ with $|L \cap S(g)^c| > 0$ which is absurd.

To show that the assertion of the lemma holds under **A2**, it suffices to prove that **A2** implies **A1**. To this end, notice that we have:

$$\begin{cases} -\Delta(v_f - f) = f + \Delta f & \text{in } D \\ v_f - f \leq 0 & \text{on } \partial D. \end{cases} \tag{3.75}$$

Since $f + \Delta f$ is non-positive, we can apply the maximum principle to (3.75) to deduce $v_f \leq f$. Hence, the proof is complete. $\square$

Proof of Theorem 3.3.5. We first relax the minimization problem (3.51) by extending the admissible set $\mathcal{R}$ to $\overline{\mathcal{R}}$. Thus, we get

$$\inf_{g\in\overline{\mathcal{R}}} \Phi(g). \tag{3.76}$$

By Lemma 3.3.4 (i), Φ is weakly continuous on $\overline{\mathcal{R}}$ with respect to L^2-topology. Hence, the minimization problem (3.76) is solvable. Furthermore, thanks to the strict convexity of Φ (Lemma 3.3.4 (ii)) the solution to (3.76) is unique. Let us denote this solution by $\hat{g}$.

Fix $g \in \overline{\mathcal{R}}$ and set $g_t = \hat{g} + t(g - \hat{g})$, for $t \in (0, 1)$. Due to the convexity of $\overline{\mathcal{R}}$, $g_t \in \overline{\mathcal{R}}$. From Lemma 3.3.4 (iii) we can derive $\int_D (g - \hat{g})\hat{u}^2 dx \le 0$. Whence, $\hat{g}$ maximizes the linear functional $L(h) = \int_D h\hat{u}^2 dx$, relative to $h \in \overline{\mathcal{R}}$. From Lemmata 3.3.1, 3.3.2, and 3.3.6, it follows that the graph of $\hat{u}$ has no significant flat sections on D. Then by Lemma 3.1.3 we infer existence of an increasing function $\hat{\psi}$ such that $\hat{\psi}((\hat{u})^2) \in \mathcal{R}$. Moreover, by setting $\psi(t) = \hat{\psi}(t^2)$, we have $\psi(\hat{u}) \in \mathcal{R}$ with ψ to be increasing. Therefore, from Lemma 3.1.4, it follows that $\psi(\hat{u})$ is the unique maximizer of the functional L, whence we must have $\hat{g} = \psi(\hat{u})$, which is the desired result. The proof of the theorem is completed. $\square$

An intriguing question arises at this point; namely, even though $\hat{g}$ in Theorem 3.3.5 is a global minimizer, is it possible for Φ to have local minimizers relative to $\mathcal{R}$? The answer to this question is negative. To prove this, we need a less restrictive version of Theorem 3.3 (iii) in [8] stated as follows:

Lemma 3.3.7. *Let $1 \le r < \infty$, $\mathcal{N} : L^r(D) \to \mathbb{R}$ be weakly sequentially continuous and $\mathcal{R} = \mathcal{R}(h_0)$ denote the rearrangement class generated by some $h_0 \in L^r(D)$. Assume that for every pair $(h_1, h_2) \in \overline{\mathcal{R}}\times\overline{\mathcal{R}}$ the following relation holds:*

$$\lim_{t\to 0^+} \frac{\mathcal{N}(th_2 + (1-t)h_1) - \mathcal{N}(h_1)}{t} = \int_D (h_2 - h_1)\,\mathcal{G}\,dx$$

for some $\mathcal{G} \in L^{r'}(D)$. Suppose $\mathcal{U}$ is a strong neighborhood (relative to $\mathcal{R}$) of $\hat{h} \in \mathcal{R}$, for which we have:

$$\forall\, h \in \mathcal{U} : \mathcal{N}(\hat{h}) \le \mathcal{N}(h).$$

Then, $\hat{h}$ minimizes the linear functional $\mathcal{L}(h) = \int_D h\,\mathcal{G}\,dx$, relative to $h \in \overline{\mathcal{R}}$.

Now we state our result concerning local minimizers.

Theorem 3.3.8. *Let the hypotheses of Theorem 3.3.5 hold. If g_1 and g_2 are two local minimizers of $\Phi(g)$ relative to $g \in \mathcal{R}$, then $g_1 = g_2$.*

Proof. For simplicity we set $u_1 = u_{g_1}$ and $u_2 = u_{g_2}$. Lemma 3.3.7 in conjunction with Lemma 3.3.4 implies that g_1 and g_2 are maximizers of the linear functionals:

$$\mathcal{L}_1(g) = \int_D g u_1^2 dx$$

and

$$\mathcal{L}_2(g) = \int_D g u_2^2 dx$$

relative to $g \in \overline{\mathcal{R}}$, respectively. In particular, we infer:

$$\int_D g_2 u_1^2 dx \leq \int_D g_1 u_1^2 dx \quad \text{and} \quad \int_D g_1 u_2^2 dx \leq \int_D g_2 u_2^2 dx. \tag{3.77}$$

Thus, we obtain:

$$\begin{aligned}
2\int_D f u_1 dx - \int_D (|\nabla u_1|^2 + g_1 u_1^2) dx &\leq 2\int_D f u_1 dx - \int_D (|\nabla u_1|^2 + g_2 u_1^2) dx \\
&\leq 2\int_D f u_2 dx - \int_D (|\nabla u_2|^2 + g_2 u_2^2) dx \\
&\leq 2\int_D f u_2 dx - \int_D (|\nabla u_2|^2 + g_1 u_2^2) dx \\
&\leq 2\int_D f u_1 dx - \int_D (|\nabla u_1|^2 + g_1 u_1^2) dx
\end{aligned}$$

where the first and third inequalities are consequences of (3.77), whereas the second and the fourth inequalities follow from (3.50). From the equation above, we see that all inequalities must in fact be equalities. This in turn implies $u_1 = u_2$, due to the uniqueness. Whence, we deduce $g_1 = g_2$ as desired. □

3.3.3 Radial domain

In this subsection we assume D is a ball, say $B(0, R)$. The following is our result regarding radial symmetry of solutions to the minimization problem (3.51).

Theorem 3.3.9. *Suppose f is radial and satisfies one of the assumptions* **A1** *or* **A2**. *Then the solution of (3.51) is radial and non-increasing.*

Proof. Let g denote the solution of (3.51) and let R be a rotational map about the origin. Since f is radial, we infer $u_g \circ R = u_{g\circ R}$. Thus, $\Phi(g \circ R) = \Phi(g)$ and $g \circ R$ is also a solution of (3.51). By uniqueness, we deduce $g \circ R = g$, for every rotational map R. Whence, g is radial, as desired. To prove that g is non-increasing we observe that, since $u = u_g$ is radial, we can write the equation in (3.48) as

$$-(r^{N-1}u')' = r^{N-1}(f - gu).$$

Since $f \geq v_f$ by **A1** and $g \leq 1$ by assumption, we have $f - gu \geq v_f - u$. Furthermore, $v_f - u > 0$ by the proof of Lemma 3.3.6. Hence,

$$-(r^{N-1}u')' > 0, \quad -r^{N-1}u' > 0, \quad u' < 0.$$

Since (by Theorem 3.3.5) $g = \psi(u)$ for some non-decreasing ψ, g is non-increasing as desired. □

3.3.4 Some remarks

Remark 3.3.5. *In addition to the minimization problem (3.51), one can also consider the maximization problem:*

$$\sup_{g\in\mathcal{R}} \Phi(g). \tag{3.78}$$

Since Φ is weakly continuous and convex, Φ reaches its maximum value at the extremal points of the convex set $\overline{\mathcal{R}}$ (i.e. the elements of $\mathcal{R}$). Hence, problem (3.78) is solvable (see Theorem 7 of [7] or Remark 3.1 of [32]). Moreover, if one of the assumptions **A1** *or* **A2** *holds, along the same lines as in the proof of Theorem 3.3.5, it can be shown that, if $\tilde{g}$ is a maximizer, then:*

$$\tilde{g} = \tilde{\psi}(\tilde{u}), \tag{3.79}$$

almost everywhere in D, for some decreasing function $\tilde{\psi}$. Here $\tilde{u} = u_{\tilde{g}}$, the solution of (3.48) with $g = \tilde{g}$.

Note that, for maximizers we do not have uniqueness in general. However, we are going to prove that in case D is a ball and f is radially symmetric and non-increasing, any maximizer is radially symmetric and non-decreasing, hence unique. Indeed, let $v = u_{\tilde{g}_*}$, where $\tilde{g}_*$ is the increasing Schwarz symmetrization of $\tilde{g}$ (see [36]). For simplicity we write u instead *of $u_{\tilde{g}}$. Then, by Lemma 3.3.3 (i), we have*

$$-\frac{1}{2}\Phi(\tilde{g}) = \frac{1}{2}\int_D |\nabla u|^2 dx + \frac{1}{2}\int_D \tilde{g}u^2 dx - \int_D fu dx \tag{3.80}$$
$$\geq \frac{1}{2}\int_D |\nabla u^*|^2 dx + \frac{1}{2}\int_D \tilde{g}u^2 dx - \int_D fu dx.$$

Now, applying the Hardy-Littlewood inequality, see for example [37], to the last two integrals in (3.80), keeping in mind that $f = f^*$, we obtain

$$-\frac{1}{2}\Phi(\tilde{g}) \geq \frac{1}{2}\int_D |\nabla u^*|^2 dx + \frac{1}{2}\int_D \tilde{g}_* \, {u^*}^2 dx - \int_D fu^* dx. \tag{3.81}$$

Recalling that v minimizes the functional:

$$\mathcal{I}(w) = \frac{1}{2}\int_D |\nabla w|^2 dx + \frac{1}{2}\int_D \tilde{g}_* w^2 dx - \int_D fw dx,$$

relative to $w \in H_0^1(D)$, we infer from (3.81):

$$-\frac{1}{2}\Phi(\tilde{g}) \geq \frac{1}{2}\int_D |\nabla v|^2 dx + \frac{1}{2}\int_D \tilde{g}_* v^2 dx - \int_D fv dx = -\frac{1}{2}\Phi(\tilde{g}_*). \tag{3.82}$$

As $\tilde{g}$ is maximal for Φ, then $\Phi(\tilde{g}_*) \leq \Phi(\tilde{g})$, which together with (3.80), (3.81) and (3.82) yield:

$$\int_D |\nabla u|^2 dx = \int_D |\nabla u^*|^2 dx$$

We now proceed to show that $u = u^*$. From Lemma 3.3.3 (ii), it suffices to verify that the set $\{x \in D : \ \nabla u = 0, \ 0 < u(x) < M\}$ has measure zero. Observe that, by the proof of Theorem 3.3.9, we have $-\Delta u = f - \tilde{g}u > 0$ almost everywhere. Thus, we deduce the measure of $\{x \in D : \ \nabla u = 0, \ 0 < u(x) < M\}$ is zero. This implies $u = u^*$, and by (3.79), $\tilde{g} = \tilde{\psi}(u^*)$ almost everywhere in D. Since $\tilde{\psi}$ is decreasing, $\tilde{g}$ is radial and non-decreasing, as claimed.

Remark 3.3.6. *A consequence of (3.74) is that the larger values of $\hat{u}$ is achieved where $\hat{g}$ is large. Whence, in case the set $\{\hat{g} = 0\}$ has positive measure, it will contain a layer around the boundary ∂D, since $\hat{u}$ is continuous (if $N = 2, 3$), and vanishes on ∂D. Physically, this means that in the construction of a robust membrane one should use the material with least density near the boundary. A similar conclusion can be drawn regarding the maximization problem (3.78).*

Remark 3.3.7. *Note that Theorem 3.3.9 can be improved. Indeed, if D is Steiner symmetric with respect to a hyperplane l (see e. g. [36]) and f is Steiner symmetric with respect to l, then $\hat{g}$ (the solution of (3.51)) will also be Steiner symmetric. Of course, in this case, one needs to use the inequality:*

$$\int_D |\nabla u|^2 dx \geq \int_D |\nabla u^\sharp|^2 dx, \tag{3.83}$$

instead of (3.54) in which $u^\sharp$ stands for the Steiner symmetrization of u. We apply well known techniques of symmetrization. From Theorem 3.3.5, we know $\hat{g} = \psi(u)$, almost everywhere in D, for some increasing function ψ. Here, we are using u in place of $u_{\hat{g}}$. Let us consider the auxiliary problem:

$$\begin{cases} -\Delta W + W\psi(W) = f(x) & \text{in } D \\ W = 0 & \text{on } \partial D. \end{cases} \tag{3.84}$$

Since $\Psi(t) = \int_0^t s\psi(s)ds$ is convex, (3.84) has a unique solution $\hat{W} \in H_0^1(D)$ which is the unique minimizer of the functional:

$$\mathcal{K}(W) = \frac{1}{2}\int_D |\nabla W|^2 dx + \int_D \Psi(W)dx - \int_D fW dx, \tag{3.85}$$

relative to $W \in H_0^1(D)$. Indeed, $\hat{W} = u$, since u is a solution of (3.84). From the inequality (3.83), the Hardy-Littlewood inequality, e.g. [31], and the fact that:

$$\int_D \Psi(u)dx = \int_D \Psi(u^\sharp)dx,$$

we deduce

$$\frac{1}{2}\int_D |\nabla u|^2 dx + \int_D \Psi(u)dx - \int_D fudx \geq \frac{1}{2}\int_D |\nabla u^\sharp|^2 dx + \int_D \Psi(u^\sharp)dx - \int_D fu^\sharp dx,$$

where we have used $f = f^\sharp$. The last inequality clearly implies $u^\sharp$ also minimizes $\mathcal{K}$ relative to $H_0^1(D)$. Hence, by uniqueness, we infer $u = u^\sharp$. Recalling the relation $\hat{g} = \psi(u)$, we obtain $\hat{g} = \psi(u^\sharp)$, almost everywhere in D. Since ψ is increasing, it follows that $\hat{g} = \hat{g}^\sharp$, as desired.

3.3.5 Free boundary, monotonicity and stability results

Consider the following boundary value problem:

$$\begin{cases} -\Delta u + (\alpha\chi_E + \beta\chi_{E^c})\, u = f & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{3.86}$$

where D is a smooth bounded domain in $\mathbb{R}^N$ $(N = 2,3)$, $f \in L^2(D)$ is a given non-negative function, $1 \geq \alpha > \beta \geq 0$, E is a measurable subset of D, and E^c is the complement of E in D. Denoting the unique solution of (3.86) by u_E, we are interested in the following minimization problem:

$$\inf_{|E|=\gamma} \int_D f u_E dx, \tag{3.87}$$

where $0 < \gamma < |D|$. By Theorem 3.3.5, we know (3.87) has a unique solution $\tilde{D} \subset D$ with $|\tilde{D}| = \gamma$ if f satisfies **A1** or **A2**. Also, we have $\tilde{D} = \{x \in D : u_{\tilde{D}}(x) > c\}$ for some positive c.

3.3.5.1 Free boundary

As mentioned above, $\tilde{D} = \{x \in D : u_{\tilde{D}}(x) > c\}$ for some positive c. This in turn implies:

$$u_{\tilde{D}}(x) = c, \quad \text{on } \partial\tilde{D}. \tag{3.88}$$

In this subsection we show that the free boundary result (3.88) could have been drawn *a priori* provided we were ensured that $\tilde{D}$ would be a smooth open set, positioned away from the fixed boundary ∂D. To this end, we use the technique of domain derivatives similar to those employed in [20]. Our presentation is merely aimed at highlighting a particular method of dealing with issues such as these, but will not be very rigorous since similar techniques already exist in the literature.

For simplicity we set $u = u_{\tilde{D}}$. So:

$$\begin{cases} -\Delta u + ((\alpha - \beta)\chi_{\tilde{D}} + \beta)u = f & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{3.89}$$

Let $V \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ be a vector field with compact support in D. Define $D^t = (Id + tV)(\tilde{D})$, the image of $\tilde{D}$ under the mapping $Id + tV$, which is a diffeomorphism for small t. Thus, D^t inherits the same properties retained by $\tilde{D}$; namely, smoothness, openness and being away from ∂D. Next we let $u^t \in H_0^1(D)$ be the function satisfying:

$$\begin{cases} -\Delta u^t + ((\alpha-\beta)\chi_{D^t} + \beta)u^t = f & \text{in } D \\ u^t = 0 & \text{on } \partial D. \end{cases} \tag{3.90}$$

Next, we define the domain derivative of u in the direction of V, denoted u':

$$u'(x) = \lim_{t\to 0^+} \frac{u^t(x) - u(x)}{t}, \quad x \in D. \tag{3.91}$$

The limit in (3.91) exists [50]. Moreover, $u' \in H^1(D)$ but falls short of being a member of $H_0^1(D)$. Using similar arguments as in [20], one can show $u^t \to u$ in $H_0^1(D)$ as $t \to 0$. Recalling that $\tilde{D}$ is the solution of the minimization problem:

$$\inf_{|E|=\gamma} \Phi(E) := \int_D f u_E dx, \tag{3.92}$$

we define the domain derivative of Φ, in the direction of V, as follows:

$$\Phi'(E) = \lim_{t\to 0^+} \frac{\Phi(E^t) - \Phi(E)}{t}. \tag{3.93}$$

The notation used in (3.93) is easy to understand. It is equally easy to verify that $\Phi'(\tilde{D}) = \int_D f u' dx$. From the Lagrange multiplier theorem applied to the minimization problem (3.92), we infer the existence of a constant $\tilde{c}$ such that:

$$\Phi'(\tilde{D}) = \tilde{c}\,\mathrm{Vol}'(\tilde{D}). \tag{3.94}$$

The right hand side of (3.94) is the domain derivative of the volume operator, which is easily computed, or using Theorem 6 on p.713 of [26]:

$$\mathrm{Vol}'(\tilde{D}) = \int_{\partial\tilde{D}} V \cdot \nu d\sigma,$$

where ν stands for the unit normal vector on $\partial\tilde{D}$. So, we derive

$$\int_D f u' dx = \tilde{c} \int_{\partial\tilde{D}} V \cdot \nu d\sigma, \quad \forall\, V \in C^2(\mathbb{R}^N, \mathbb{R}^N). \tag{3.95}$$

Now we proceed to find an equivalent expression for the left hand side of (3.95) which will lead us to our desired result. To this end, multiply the differential equation in (3.89) by u^t, the differential equation in (3.90) by u, subtract the resulting equations, and integrate over D to obtain:

$$(\alpha-\beta)\left(\int_{\tilde{D}} uu^t dx - \int_{D^t} uu^t dx\right) = \int_D f(u^t-u)dx. \tag{3.96}$$

On the other hand, we have:

$$\begin{aligned}\int_{D^t} uu^t dx &= \int_{\tilde{D}} u(x+tV)u^t(x+tV)\left|\det\left(\delta_{ij}+t\frac{\partial V_i}{\partial x_j}\right)\right| dx\\ &= \int_{\tilde{D}} u(x+tV)u^t(x+tV)(1+t\nabla\cdot V+O(t^2))dx,\end{aligned} \tag{3.97}$$

as $t\to 0^+$. Note that:

$$\begin{aligned}&u(x+tV)u^t(x+tV)\\ &=(u(x)+t\nabla u\cdot V+O(t^2))(u^t(x)+t\nabla u^t\cdot V+O(t^2))\\ &=uu^t+t\nabla(uu^t)\cdot V+O(t^2),\end{aligned} \tag{3.98}$$

as $t\to 0^+$. Thus, from (3.98), we derive

$$u(x+tV)u^t(x+tV)(1+t\nabla\cdot V+O(t^2)) = uu^t+t\nabla\cdot(uu^tV)+O(t^2), \tag{3.99}$$

as $t\to 0^+$. Finally, from (3.96), (3.97), (3.99) and the fact that $u^t\to u$ in $H_0^1(D)$, we obtain

$$-(\alpha-\beta)\int_{\tilde{D}}\nabla\cdot(u^2V)dx = \int_D fu'dx.$$

Thus, as $\tilde{D}$ is smooth, and recalling (3.95), we derive

$$-(\alpha-\beta)\int_{\partial\tilde{D}} u^2V\cdot\nu d\sigma = \tilde{c}\int_{\partial\tilde{D}} V\cdot\nu d\sigma, \quad \forall V\in C^2(\mathbb{R}^N,\mathbb{R}^N). \tag{3.100}$$

Clearly, from (3.100), we find

$$u=c=\left(\frac{-\tilde{c}}{\alpha-\beta}\right)^{1/2}, \quad \text{on } \partial\tilde{D},$$

as desired.

3.3.5.2 Monotonicity and stability results with respect to γ

We know that for each $0 < \gamma < |D|$ the minimization problem (3.87) has a unique solution. Now consider $0 < \gamma_1, \gamma_2 < |D|$ and their corresponding unique solutions:

$$\tilde{D}_{\gamma_1} = \{x \in D : u_{\gamma_1}(x) > c_{\gamma_1}\}, \quad \tilde{D}_{\gamma_2} = \{x \in D : u_{\gamma_2}(x) > c_{\gamma_2}\}, \tag{3.101}$$

for positive c_{γ_1} and c_{γ_2}, where u_{γ_1} and u_{γ_2} satisfy:

$$\begin{cases} -\Delta u_{\gamma_1} + (\alpha\chi_{\tilde{D}_{\gamma_1}} + \beta\chi_{\tilde{D}^c_{\gamma_1}})u_{\gamma_1} = f & \text{in } D \\ u_{\gamma_1} = 0 & \text{on } \partial D, \end{cases} \tag{3.102}$$

and

$$\begin{cases} -\Delta u_{\gamma_2} + (\alpha\chi_{\tilde{D}_{\gamma_2}} + \beta\chi_{\tilde{D}^c_{\gamma_2}})u_{\gamma_2} = f & \text{in } D \\ u_{\gamma_2} = 0 & \text{on } \partial D. \end{cases} \tag{3.103}$$

Theorem 3.3.10. *If $0 < \gamma_1 < \gamma_2 < |D|$ then $c_{\gamma_1} \geq c_{\gamma_2}$.*

Proof. From (3.102) and (3.103), we deduce

$$\begin{aligned} &- \Delta(u_{\gamma_1} - u_{\gamma_2}) + (\alpha\chi_{\tilde{D}_{\gamma_1}} + \beta\chi_{\tilde{D}^c_{\gamma_1}})(u_{\gamma_1} - u_{\gamma_2}) \\ &\qquad\qquad = -(\alpha - \beta)u_{\gamma_2}(\chi_{\tilde{D}_{\gamma_1}} - \chi_{\tilde{D}_{\gamma_2}}) \quad \text{in } D, \end{aligned} \tag{3.104}$$

with $u_{\gamma_1} - u_{\gamma_2} = 0$ on ∂D. Multiplying the differential equation in (3.104) by $u_{\gamma_1} - u_{\gamma_2}$, integrating the result over D, followed by an application of the divergence theorem yields:

$$\begin{aligned} &\int_D |\nabla(u_{\gamma_1} - u_{\gamma_2})|^2 dx + \int_D (\alpha\chi_{\tilde{D}_{\gamma_1}} + \beta\chi_{\tilde{D}^c_{\gamma_1}})(u_{\gamma_1} - u_{\gamma_2})^2 dx \\ &\quad = -(\alpha - \beta)\int_D u_{\gamma_2}(u_{\gamma_1} - u_{\gamma_2})(\chi_{\tilde{D}_{\gamma_1}} - \chi_{\tilde{D}_{\gamma_2}})\, dx \\ &\quad = -(\alpha - \beta)\int_D u_{\gamma_2}(u_{\gamma_1} - u_{\gamma_2})(\chi_{\tilde{D}_{\gamma_1}\setminus\tilde{D}_{\gamma_2}} - \chi_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}})\, dx \geq 0, \end{aligned} \tag{3.105}$$

where the last inequality is due to the fact that the left hand side is non-negative. Hence, since $\alpha > \beta$, (3.105) leads to

$$0 \geq \int_D u_{\gamma_2}(u_{\gamma_1} - u_{\gamma_2})(\chi_{\tilde{D}_{\gamma_1}\setminus\tilde{D}_{\gamma_2}} - \chi_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}})\, dx. \tag{3.106}$$

Changing γ_1 with γ_2 in (3.106) and adding this new inequality to (3.106) we find

$$0 \geq \int_D (u_{\gamma_1}^2 - u_{\gamma_2}^2)(\chi_{\tilde{D}_{\gamma_1}\setminus\tilde{D}_{\gamma_2}} - \chi_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}})\,dx. \tag{3.107}$$

From (3.101), we infer $u_{\gamma_1}^2 - u_{\gamma_2}^2 > c_{\gamma_1}^2 - c_{\gamma_2}^2$ in $\tilde{D}_{\gamma_1} \setminus \tilde{D}_{\gamma_2}$ and $u_{\gamma_1}^2 - u_{\gamma_2}^2 < c_{\gamma_1}^2 - c_{\gamma_2}^2$ in $\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1}$. Hence, from (3.107) we find

$$0 \geq (c_{\gamma_1}^2 - c_{\gamma_2}^2) \int_D (\chi_{\tilde{D}_{\gamma_1}\setminus\tilde{D}_{\gamma_2}} - \chi_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}})\,dx.$$

Finally, since $|\tilde{D}_{\gamma_1} \setminus \tilde{D}_{\gamma_2}| = |\tilde{D}_{\gamma_1}| - |\tilde{D}_{\gamma_1} \cap \tilde{D}_{\gamma_2}|$ and $|\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1}| = |\tilde{D}_{\gamma_2}| - |\tilde{D}_{\gamma_1} \cap \tilde{D}_{\gamma_2}|$, we get

$$\int_D (\chi_{\tilde{D}_{\gamma_1}\setminus\tilde{D}_{\gamma_2}} - \chi_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}})\,dx = \gamma_1 - \gamma_2.$$

Therefore,

$$0 \geq (c_{\gamma_1}^2 - c_{\gamma_2}^2)(\gamma_1 - \gamma_2).$$

Since $\gamma_1 - \gamma_2 < 0$, we obtain $c_{\gamma_1} - c_{\gamma_2} \geq 0$ as desired. $\square$

Theorem 3.3.11. *If* $0 < \gamma_1 < \gamma_2 < |D|$ *then* $\tilde{D}_{\gamma_1} \subseteq \tilde{D}_{\gamma_2}$.

Proof. Let us introduce the following subsets of D:

$$\begin{cases} E \equiv \{x \in D : u_{\gamma_1}(x) - u_{\gamma_2}(x) \leq c_{\gamma_1} - c_{\gamma_2}\} \\ F \equiv \{x \in D : u_{\gamma_1}(x) - u_{\gamma_2}(x) > c_{\gamma_1} - c_{\gamma_2}\}. \end{cases}$$

From (3.101), we deduce $u_{\gamma_1} - u_{\gamma_2} > c_{\gamma_1} - c_{\gamma_2}$ in $\tilde{D}_{\gamma_1} \setminus \tilde{D}_{\gamma_2}$, and $u_{\gamma_1} - u_{\gamma_2} < c_{\gamma_1} - c_{\gamma_2}$ in $\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1}$. By using the definitions of E and F, we have

$$\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1} \subseteq E, \tag{3.108}$$

and

$$\tilde{D}_{\gamma_1} \setminus \tilde{D}_{\gamma_2} \subseteq F. \tag{3.109}$$

Since $F = E^c$, by utilizing (3.108), we infer $F \subseteq (\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1})^c = \tilde{D}_{\gamma_2}^c \cup \tilde{D}_{\gamma_1}$. From (3.102) and (3.103) and observing that $\tilde{D}_{\gamma_2}^c \cup \tilde{D}_{\gamma_1} = (\tilde{D}_{\gamma_2} \cap \tilde{D}_{\gamma_1}^c)^c$ we infer

$$\begin{aligned} -\Delta(u_{\gamma_1} - u_{\gamma_2}) &+ (\alpha\chi_{\tilde{D}_{\gamma_1}} + \beta\chi_{\tilde{D}_{\gamma_1}^c \cap \tilde{D}_{\gamma_2}^c})(u_{\gamma_1} - u_{\gamma_2}) \\ &= \chi_{\tilde{D}_{\gamma_1}\setminus\tilde{D}_{\gamma_2}}(\beta - \alpha)u_{\gamma_2}, \quad \text{in} \quad F \subseteq \tilde{D}_{\gamma_2}^c \cup \tilde{D}_{\gamma_1}. \end{aligned} \tag{3.110}$$

On the other hand, we have $u_{\gamma_1} - u_{\gamma_2} = c_{\gamma_1} - c_{\gamma_2}$ on ∂F. Since $\beta < \alpha$ and $u_{\gamma_2} > 0$ in D, in conjunction with $c_{\gamma_1} - c_{\gamma_2} \geq 0$ from Theorem 3.3.10 and by using the maximum principle we deduce $u_{\gamma_1} - u_{\gamma_2} \leq c_{\gamma_1} - c_{\gamma_2}$ in F. By using the definition of F, it follows that $F = \emptyset$. From (3.109), we infer $\tilde{D}_{\gamma_1} \setminus \tilde{D}_{\gamma_2} = \emptyset$, i.e., $\tilde{D}_{\gamma_1} \subseteq \tilde{D}_{\gamma_2}$ as desired. $\square$

Corollary 3.3.12. *If $0 < \gamma_1 < \gamma_2 < |D|$, then $u_{\gamma_1} > u_{\gamma_2}$ in D.*

Proof. By Theorem 3.3.11, we have $-(\alpha - \beta)u_{\gamma_2}(\chi_{\tilde{D}_{\gamma_1}} - \chi_{\tilde{D}_{\gamma_2}}) \geq 0$ in D. Recalling the boundary value problem (3.104), followed by an application of the strong maximum principle, we infer $u_{\gamma_1} > u_{\gamma_2}$ in D as desired. $\square$

For the rest of this subsection, we will denote the minimization problem (3.87) as follows

$$\Psi(\gamma) \equiv \inf_{|E|=\gamma} \int_D f u_E dx. \tag{3.111}$$

Since f is non-negative and non-trivial, the following is an easy consequence of Corollary 3.3.12.

Corollary 3.3.13. *$\Psi(\gamma)$ is a decreasing function on $(0, |D|)$.*

Theorem 3.3.14. *If γ_1 converges to γ_2 in $(0, |D|)$ then u_{γ_1} converges to u_{γ_2} in $C(\bar{D})$. Moreover, c_{γ_1} converges to c_{γ_2}, where $c_{\gamma_1} = u_{\gamma_1}(\partial\tilde{D}_{\gamma_1})$ and $c_{\gamma_2} = u_{\gamma_2}(\partial\tilde{D}_{\gamma_2})$.*

Proof. Fix $0 < \gamma_2 < |D|$. Let γ_1 increase to γ_2, we claim that u_{γ_1} converges to u_{γ_2} in $C(\bar{D})$. From Theorem 3.3.11 we know that $\tilde{D}_{\gamma_1} \subseteq \tilde{D}_{\gamma_2}$. By applying general Hölder's inequality and Sobolev embedding theorem, (3.105) leads to

$$\begin{aligned}
&\int_D |\nabla(u_{\gamma_1} - u_{\gamma_2})|^2 dx + \int_D (\alpha\chi_{\tilde{D}_{\gamma_1}} + \beta\chi_{\tilde{D}^c_{\gamma_1}})(u_{\gamma_1} - u_{\gamma_2})^2 dx \\
&\quad = (\alpha - \beta) \int_D u_{\gamma_2}(u_{\gamma_1} - u_{\gamma_2})\chi_{\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1}} dx \\
&\quad \leq (\alpha - \beta) \|u_{\gamma_2}\|_4 \|u_{\gamma_1} - u_{\gamma_2}\|_4 |\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1}|^{\frac{1}{2}} \\
&\quad \leq C(\alpha - \beta) \|u_{\gamma_2}\|_{H^1_0(D)} \|u_{\gamma_1} - u_{\gamma_2}\|_{H^1_0(D)} |\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1}|^{\frac{1}{2}}
\end{aligned} \tag{3.112}$$

in which C is a positive constant. Since the second term of the first line of (3.112) is non-negative, we obtain

$$\|u_{\gamma_1} - u_{\gamma_2}\|_{H^1_0(D)} \leq C(\alpha - \beta) \|u_{\gamma_2}\|_{H^1_0(D)} |\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1}|^{\frac{1}{2}}. \tag{3.113}$$

Observing that $|\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1}| = |\tilde{D}_{\gamma_2}| - |\tilde{D}_{\gamma_1}| = \gamma_2 - \gamma_1$ by Theorem 3.3.11, from (3.113) we deduce that u_{γ_1} converges to u_{γ_2} in $H_0^1(D)$. By utilizing elliptic regularity theory and Sobolev embedding theorem (see e. g. [26]) we infer that u_{γ_1} converges to u_{γ_2} in $C(\bar{D})$. One can prove in a similar way that if γ_1 decreases to γ_2 then u_{γ_1} converges to u_{γ_2} in $C(\bar{D})$.

Let us proceed to proving the second assertion regarding the convergence of c_{γ_1} to c_{γ_2}. In order to derive a contradiction, we assume that there exists an $\epsilon > 0$ such that $|c_{\gamma_1} - c_{\gamma_2}| > \epsilon$ when γ_1 converges to γ_2. Furthermore, by Theorem 3.3.11, we have $|\tilde{D}_{\gamma_2} \vartriangle \tilde{D}_{\gamma_1}| = |\gamma_2 - \gamma_1|$, where $\vartriangle$ denotes the symmetric difference of sets. Since $|\tilde{D}_{\gamma_2} \vartriangle \tilde{D}_{\gamma_1}|$ converges to zero and u_{γ_2} is in $C(\bar{D})$, we deduce that there exist $x_1 \in \partial\tilde{D}_{\gamma_1}$ and $x_2 \in \partial\tilde{D}_{\gamma_2}$ such that $|u_{\gamma_2}(x_1) - u_{\gamma_2}(x_2)| < \frac{1}{2}\epsilon$ if $|\gamma_2 - \gamma_1| < \delta_1$, for some positive δ_1. On the other hand, since u_{γ_1} converges pointwise to u_{γ_2}, then $|u_{\gamma_1}(x_1) - u_{\gamma_2}(x_1)| < \frac{1}{2}\epsilon$ if $|\gamma_2 - \gamma_1| < \delta_2$, for some positive $\delta_2 < \delta_1$. Then, by using triangle inequality, we have

$$|c_{\gamma_1} - c_{\gamma_2}| = |u_{\gamma_1}(x_1) - u_{\gamma_2}(x_2)| \leq |u_{\gamma_2}(x_1) - u_{\gamma_2}(x_2)| + |u_{\gamma_1}(x_1) - u_{\gamma_2}(x_1)| < \epsilon$$

which is a contradiction. This completes the proof of the theorem. $\square$

Corollary 3.3.15. $\Psi(\gamma)$ *is continuous on* $(0, |D|)$.

Proof. Let γ_1 converge to γ_2 in $(0, |D|)$. By applying Hölder's inequality, we have

$$\begin{aligned} |\Psi(\gamma_1) - \Psi(\gamma_2)| &= \left| \int_D (u_{\gamma_1} - u_{\gamma_2}) f \, dx \right| \\ &\leq \|u_{\gamma_1} - u_{\gamma_2}\|_2 \, \|f\|_2 \\ &\leq \|u_{\gamma_1} - u_{\gamma_2}\|_\infty \, \|f\|_2 \, |D|^{\frac{1}{2}}. \end{aligned} \tag{3.114}$$

By Theorem 3.3.14 u_{γ_1} converges to u_{γ_2} in $C(\bar{D})$. Hence (3.114) implies that $\Psi(\gamma_1)$ converges to $\Psi(\gamma_2)$. $\square$

From Corollary 3.3.13 we infer that $\Psi(\gamma)$ is differentiable almost everywhere. However, the following theorem shows that it is actually continuously differentiable on $(0, |D|)$.

Theorem 3.3.16. $\Psi(\gamma)$ *is continuously differentiable on* $(0, |D|)$. *Moreover,*

$$\Psi'(\gamma) = -(\alpha - \beta) c_\gamma^2$$

in which $c_\gamma = u_\gamma(\partial\tilde{D}_\gamma)$.

Proof. Fix $0 < \gamma_2 < |D|$ and let γ_1 increase to γ_2. We claim that $\frac{\Psi(\gamma_1)-\Psi(\gamma_2)}{\gamma_1-\gamma_2}$ converges to $-(\alpha-\beta)c_{\gamma_2}^2$. Multiplying the differential equation in (3.104) by $u_{\gamma_1}+u_{\gamma_2}$, integrating the result over D, followed by an application of divergence theorem, in conjunction with $\tilde{D}_{\gamma_1} \subseteq \tilde{D}_{\gamma_2}$ (Theorem 3.3.11) yields

$$\begin{aligned}
&\int_D (|\nabla u_{\gamma_1}|^2 - |\nabla u_{\gamma_2}|^2)\,dx + \int_D (\alpha\chi_{\tilde{D}_{\gamma_1}} + \beta\chi_{\tilde{D}^c_{\gamma_1}})(u_{\gamma_1}^2 - u_{\gamma_2}^2)\,dx \\
&\qquad = -(\alpha-\beta)\int_D u_{\gamma_2}(u_{\gamma_1}+u_{\gamma_2})(\chi_{\tilde{D}_{\gamma_1}} - \chi_{\tilde{D}_{\gamma_2}})\,dx \\
&\qquad = (\alpha-\beta)\int_D u_{\gamma_2}(u_{\gamma_1}+u_{\gamma_2})\chi_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}}\,dx \\
&\qquad = (\alpha-\beta)\int_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}} u_{\gamma_2}(u_{\gamma_1}+u_{\gamma_2})\,dx.
\end{aligned} \tag{3.115}$$

Furthermore, from (3.102), (3.103) and (3.115), we deduce

$$\begin{aligned}
\Psi(\gamma_1)-\Psi(\gamma_2) &= \int_D u_{\gamma_1} f\,dx - \int_D u_{\gamma_2} f\,dx \\
&= \left[\int_D |\nabla u_{\gamma_1}|^2\,dx + \int_D (\alpha\chi_{\tilde{D}_{\gamma_1}} + \beta\chi_{\tilde{D}^c_{\gamma_1}})u_{\gamma_1}^2\,dx\right] \\
&\qquad - \left[\int_D |\nabla u_{\gamma_2}|^2\,dx + \int_D (\alpha\chi_{\tilde{D}_{\gamma_2}} + \beta\chi_{\tilde{D}^c_{\gamma_2}})u_{\gamma_2}^2\,dx\right] \\
&= \int_D (|\nabla u_{\gamma_1}|^2 - |\nabla u_{\gamma_2}|^2)\,dx + \int_D (\alpha\chi_{\tilde{D}_{\gamma_1}} + \beta\chi_{\tilde{D}^c_{\gamma_1}})(u_{\gamma_1}^2 - u_{\gamma_2}^2)\,dx \\
&\qquad - (\alpha-\beta)\int_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}} u_{\gamma_2}^2\,dx \\
&= (\alpha-\beta)\int_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}} u_{\gamma_2}(u_{\gamma_1}+u_{\gamma_2})\,dx - (\alpha-\beta)\int_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}} u_{\gamma_2}^2\,dx \\
&= (\alpha-\beta)\int_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}} u_{\gamma_2}u_{\gamma_1}\,dx,
\end{aligned} \tag{3.116}$$

where we have used the fact that $\tilde{D}_{\gamma_1} \subseteq \tilde{D}_{\gamma_2}$ in the third equality, and also applied (3.115) in the fourth equality. To this end, by using (3.116) and the fact that $|\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1}| = \gamma_2 - \gamma_1$, we calculate:

$$\begin{aligned}
\left|\frac{\Psi(\gamma_1)-\Psi(\gamma_2)}{\gamma_1-\gamma_2} - \left[-(\alpha-\beta)c_{\gamma_2}^2\right]\right| &= \frac{\alpha-\beta}{\gamma_2-\gamma_1}\left|\int_{\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}} (u_{\gamma_2}u_{\gamma_1} - c_{\gamma_2}^2)dx\right| \\
&\le (\alpha-\beta)\left\|u_{\gamma_2}u_{\gamma_1} - c_{\gamma_2}^2\right\|_{\infty,\tilde{D}_{\gamma_2}\setminus\tilde{D}_{\gamma_1}}.
\end{aligned} \tag{3.117}$$

By (3.101) and Corollary 3.3.12, in $\tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1}$ we have $c_{\gamma_2} < u_{\gamma_2} < u_{\gamma_1} \leq c_{\gamma_1}$. So, by applying Theorem 3.3.14 we infer

$$\left\| u_{\gamma_2} u_{\gamma_1} - c_{\gamma_2}^2 \right\|_{\infty, \tilde{D}_{\gamma_2} \setminus \tilde{D}_{\gamma_1}} \leq |c_{\gamma_1}^2 - c_{\gamma_2}^2|,$$

which converges to zero. From (3.117), we obtain the desired result.

Similarly, when γ_1 decreases to γ_2, the ratio $\frac{\Psi(\gamma_1) - \Psi(\gamma_2)}{\gamma_1 - \gamma_2}$ converges to $-(\alpha - \beta) c_{\gamma_2}^2$. By Theorem 3.3.14 we know that c_γ is continuous with respect to γ. Hence, we infer that $\Psi(\gamma)$ is continuously differentiable with $\Psi'(\gamma) = -(\alpha - \beta) c_\gamma^2$ on $(0, |D|)$. $\square$

3.3.5.3 Monotonicity and stability results with respect to α

Assume that $0 \leq \beta < \alpha_1, \alpha_2 \leq 1$. For each of α_1 and α_2, the minimization problem (3.87) has a unique solution, which we denote by $\tilde{D}_{\alpha_1}$ and $\tilde{D}_{\alpha_2}$, respectively. We know that $|\tilde{D}_{\alpha_1}| = |\tilde{D}_{\alpha_2}| = \gamma$ and

$$\tilde{D}_{\alpha_1} = \{x \in D : u_{\alpha_1}(x) > c_{\alpha_1}\}, \quad \tilde{D}_{\alpha_2} = \{x \in D : u_{\alpha_2}(x) > c_{\alpha_2}\}, \tag{3.118}$$

for c_{α_1} and c_{α_2} positive, where u_{α_1} and u_{α_2} satisfy:

$$\begin{cases} -\Delta u_{\alpha_1} + (\alpha_1 \chi_{\tilde{D}_{\alpha_1}} + \beta \chi_{\tilde{D}^c_{\alpha_1}}) u_{\alpha_1} = f & \text{in } D \\ u_{\alpha_1} = 0 & \text{on } \partial D, \end{cases} \tag{3.119}$$

and

$$\begin{cases} -\Delta u_{\alpha_2} + (\alpha_2 \chi_{\tilde{D}_{\alpha_2}} + \beta \chi_{\tilde{D}^c_{\alpha_2}}) u_{\alpha_2} = f & \text{in } D \\ u_{\alpha_2} = 0 & \text{on } \partial D. \end{cases} \tag{3.120}$$

From (3.119) and (3.120), we infer that:

$$\begin{aligned} & -\Delta(u_{\alpha_1} - u_{\alpha_2}) + (\alpha_1 \chi_{\tilde{D}_{\alpha_1}} + \beta \chi_{\tilde{D}^c_{\alpha_1}})(u_{\alpha_1} - u_{\alpha_2}) \\ & \qquad = u_{\alpha_2} \left[(\alpha_1 - \beta)(\chi_{\tilde{D}_{\alpha_2}} - \chi_{\tilde{D}_{\alpha_1}}) + (\alpha_2 - \alpha_1) \chi_{\tilde{D}_{\alpha_2}} \right] \quad \text{in } D, \end{aligned} \tag{3.121}$$

with $u_{\alpha_1} - u_{\alpha_2} = 0$ on ∂D.

In this subsection, we define the following subsets of D for convenience:

$$\begin{cases} E \equiv \{x \in D : u_{\alpha_1}(x) - u_{\alpha_2}(x) < c_{\alpha_1} - c_{\alpha_2}\} \\ F \equiv \{x \in D : u_{\alpha_1}(x) - u_{\alpha_2}(x) > c_{\alpha_1} - c_{\alpha_2}\} \\ G \equiv \{x \in D : u_{\alpha_1}(x) - u_{\alpha_2}(x) = c_{\alpha_1} - c_{\alpha_2}\}. \end{cases}$$

By (3.118), we infer that

$$\tilde{D}_{\alpha_2} \setminus \tilde{D}_{\alpha_1} \subseteq E \quad \text{and} \quad \tilde{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2} \subseteq F. \tag{3.122}$$

Theorem 3.3.17. *If* $0 < \alpha_1 < \alpha_2 \leq 1$ *then* $c_{\alpha_1} > c_{\alpha_2}$.

Proof. By (3.122), we get $E = (G \cup F)^c \subseteq F^c \subseteq (\tilde{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2})^c$. Then, the differential equation (3.121) leads to

$$\begin{aligned} & -\Delta(u_{\alpha_1} - u_{\alpha_2}) + (\alpha_1 \chi_{\tilde{D}_{\alpha_1} \cap \tilde{D}_{\alpha_2}} + \beta \chi_{\tilde{D}^c_{\alpha_1}})(u_{\alpha_1} - u_{\alpha_2}) \\ & \quad = u_{\alpha_2} \left[(\alpha_1 - \beta)\chi_{\tilde{D}_{\alpha_2} \setminus \tilde{D}_{\alpha_1}} + (\alpha_2 - \alpha_1)\chi_{\tilde{D}_{\alpha_2}} \right] \quad \text{in } E \subseteq (\tilde{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2})^c. \end{aligned} \tag{3.123}$$

We will use the method of contradiction. So let us assume $c_{\alpha_1} \leq c_{\alpha_2}$. Thus, $u_{\alpha_1} - u_{\alpha_2} = c_{\alpha_1} - c_{\alpha_2} \leq 0$ on ∂E. Since $\beta < \alpha_1 < \alpha_2$ and $u_{\alpha_2} > 0$ in D, the right hand side of (3.123) is non-negative. By using the maximum principle we deduce $u_{\alpha_1} - u_{\alpha_2} \geq c_{\alpha_1} - c_{\alpha_2}$ in E. This implies $E = \emptyset$ by using the definition of E. Hence, by (3.122), $\tilde{D}_{\alpha_2} \setminus \tilde{D}_{\alpha_1} = \emptyset$, i.e., $\tilde{D}_{\alpha_2} \subseteq \tilde{D}_{\alpha_1}$. By using the fact that $|\tilde{D}_{\alpha_1}| = |\tilde{D}_{\alpha_2}| = \gamma$, we infer $\tilde{D}_{\alpha_2} = \tilde{D}_{\alpha_1}$. With this condition, (3.121) leads to

$$\begin{aligned} & -\Delta(u_{\alpha_1} - u_{\alpha_2}) + (\alpha_1 \chi_{\tilde{D}_{\alpha_1}} + \beta \chi_{\tilde{D}^c_{\alpha_1}})(u_{\alpha_1} - u_{\alpha_2}) \\ & \qquad\qquad = u_{\alpha_2}(\alpha_2 - \alpha_1)\chi_{\tilde{D}_{\alpha_2}} \text{ in } D, \end{aligned} \tag{3.124}$$

with $u_{\alpha_1} - u_{\alpha_2} = 0$ on ∂D. After applying the strong maximum principle to (3.124), we deduce $u_{\alpha_1} > u_{\alpha_2}$ in D. Let us denote the distribution functions of u_{α_1} and u_{α_2} as $\lambda_{u_{\alpha_1}}(t) \equiv |\{x \in D : u_{\alpha_1}(x) > t\}|$ and $\lambda_{u_{\alpha_2}}(t) \equiv |\{x \in D : u_{\alpha_2}(x) > t\}|$, respectively. Since $u_{\alpha_1} > u_{\alpha_2}$ in D, we infer

$$\lambda_{u_{\alpha_1}}(t) > \lambda_{u_{\alpha_2}}(t), \quad \forall\, 0 < t < \|u_{\alpha_1}\|_\infty . \tag{3.125}$$

Recalling (3.118), we have $\lambda_{u_{\alpha_1}}(c_{\alpha_1}) = |\tilde{D}_{\alpha_1}| = \gamma = |\tilde{D}_{\alpha_2}| = \lambda_{u_{\alpha_2}}(c_{\alpha_2})$. As $u_{\alpha_1} \in C(\bar{D})$, we know that $\lambda_{u_{\alpha_1}}(\cdot)$ is decreasing on $(0, \|u_{\alpha_1}\|_\infty)$. By using (3.125), we deduce that $c_{\alpha_1} > c_{\alpha_2}$ which is a contradiction. This completes the proof. $\square$

Corollary 3.3.18. *If* $0 < \alpha_1 < \alpha_2 \leq 1$, *then* $u_{\alpha_1} > u_{\alpha_2}$ *in* D.

Proof. By Theorem 3.3.17, we have $u_{\alpha_1} - u_{\alpha_2} \geq c_{\alpha_1} - c_{\alpha_2} > 0$ in $F \cup G$. Then, let us focus on the subset E. Since $c_{\alpha_1} > c_{\alpha_2}$, we deduce $\partial E \subseteq \partial D \cup G$. Also, we have $u_{\alpha_1} - u_{\alpha_2} \geq 0$ on ∂E. By applying the strong maximum principle to (3.123), we infer $u_{\alpha_1} - u_{\alpha_2} > 0$ in E. Therefore, $u_{\alpha_1} - u_{\alpha_2} > 0$ in $E \cup F \cup G = D$ as desired. $\square$

Proposition 3.3.19. *If* $0 < \alpha_1 < \alpha_2 \le 1$*, then* $\tilde{D}_{\alpha_1} \cap \tilde{D}_{\alpha_2} \neq \emptyset$.

Proof. By (3.122) we have $F = (G \cup E)^c \subseteq E^c \subseteq (\tilde{D}_{\alpha_2} \setminus \tilde{D}_{\alpha_1})^c$. From the differential equation in (3.121), it follows that

$$\begin{aligned}(3.126)\quad & -\Delta(u_{\alpha_1} - u_{\alpha_2}) + (\alpha_1 \chi_{\tilde{D}_{\alpha_1}} + \beta \chi_{\tilde{D}^c_{\alpha_1} \cap \tilde{D}^c_{\alpha_2}})(u_{\alpha_1} - u_{\alpha_2}) \\ &= u_{\alpha_2}\left[-(\alpha_1 - \beta)(\chi_{\tilde{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2}}) + (\alpha_2 - \alpha_1)\chi_{\tilde{D}_{\alpha_1} \cap \tilde{D}_{\alpha_2}}\right] \quad \text{in } F \subseteq (\tilde{D}_{\alpha_2} \setminus \tilde{D}_{\alpha_1})^c.\end{aligned}$$

In order to derive a contradiction, we assume $\tilde{D}_{\alpha_1} \cap \tilde{D}_{\alpha_2} = \emptyset$. Then, (3.126) leads to

$$\begin{aligned}(3.127)\quad & -\Delta(u_{\alpha_1} - u_{\alpha_2}) + (\alpha_1 \chi_{\tilde{D}_{\alpha_1}} + \beta \chi_{\tilde{D}^c_{\alpha_1} \cap \tilde{D}^c_{\alpha_2}})(u_{\alpha_1} - u_{\alpha_2}) \\ &= -u_{\alpha_2}(\alpha_1 - \beta)\chi_{\tilde{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2}} \quad \text{in } F \subseteq (\tilde{D}_{\alpha_2} \setminus \tilde{D}_{\alpha_1})^c.\end{aligned}$$

Since $c_{\alpha_1} > c_{\alpha_2}$ (Theorem 3.3.17), from the definition of F we infer $u_{\alpha_1} - u_{\alpha_2} = c_{\alpha_1} - c_{\alpha_2} > 0$ on ∂F. Clearly, the right hand side of (3.127) is non-positive. Thus, by applying the maximum principle, we infer $u_{\alpha_1} - u_{\alpha_2} \le c_{\alpha_1} - c_{\alpha_2}$ in F. Recalling the definition of F, this implies $F = \emptyset$. By (3.122), we deduce $\tilde{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2} = \emptyset$, i. e. $\tilde{D}_{\alpha_1} \subseteq \tilde{D}_{\alpha_2}$. Since $|\tilde{D}_{\alpha_1}| = |\tilde{D}_{\alpha_2}| = \gamma$, we infer $\tilde{D}_{\alpha_2} = \tilde{D}_{\alpha_1} \neq \emptyset$ which is a contradiction. □

Theorem 3.3.20. *Let* $0 \le \beta < \alpha_1, \alpha_2 \le 1$*. If* α_1 *converges to* α_2 *in* $(\beta, 1]$*, then* $|\tilde{D}_{\alpha_1} \triangle \tilde{D}_{\alpha_2}|$ *converges to zero.*

Proof. Fix $\beta < \alpha_2 \le 1$ and let α_1 increase to α_2. We claim that $|\tilde{D}_{\alpha_1} \triangle \tilde{D}_{\alpha_2}|$ converges to zero. First, let us introduce the following auxiliary boundary value problem

$$(3.128)\qquad \begin{cases} -\Delta \hat{u}_{\alpha_1} + (\alpha_1 \chi_{\tilde{D}_{\alpha_2}} + \beta \chi_{\tilde{D}^c_{\alpha_2}})\hat{u}_{\alpha_1} = f & \text{in } D \\ \hat{u}_{\alpha_1} = 0 & \text{on } \partial D. \end{cases}$$

From (3.120) and (3.128), we deduce

$$\begin{aligned}(3.129)\quad & -\Delta(\hat{u}_{\alpha_1} - u_{\alpha_2}) + (\alpha_1 \chi_{\tilde{D}_{\alpha_2}} + \beta \chi_{\tilde{D}^c_{\alpha_2}})(\hat{u}_{\alpha_1} - u_{\alpha_2}) \\ &= (\alpha_2 - \alpha_1)u_{\alpha_2}\chi_{\tilde{D}_{\alpha_2}} \text{ in } D,\end{aligned}$$

with $\hat{u}_{\alpha_1} - u_{\alpha_2} = 0$ on ∂D. Since $\alpha_2 > \alpha_1$, we infer that $(\alpha_2 - \alpha_1)u_{\alpha_2}\chi_{\tilde{D}_{\alpha_2}}$ is non-negative. So, by applying the strong maximum principle to (3.129), we obtain $\hat{u}_{\alpha_1} > u_{\alpha_2}$ in D. Furthermore, by (3.118), we have

$$(3.130)\quad \hat{D}_{\alpha_1} \equiv \{x \in D : \hat{u}_{\alpha_1}(x) > c_{\alpha_2}\} \supseteq \{x \in D : u_{\alpha_2}(x) > c_{\alpha_2}\} = \tilde{D}_{\alpha_2}.$$

Multiplying the differential equation in (3.129) by $\hat{u}_{\alpha_1} - u_{\alpha_2}$, integrating the result over D, followed by an application of divergence theorem yields

$$
\begin{aligned}
&\int_D |\nabla(\hat{u}_{\alpha_1} - u_{\alpha_2})|^2 dx + \int_D (\alpha_1 \chi_{\tilde{D}_{\alpha_2}} + \beta \chi_{\tilde{D}^c_{\alpha_2}})(\hat{u}_{\alpha_1} - u_{\alpha_2})^2 dx \\
&\qquad = (\alpha_2 - \alpha_1) \int_D u_{\alpha_2}(\hat{u}_{\alpha_1} - u_{\alpha_2})\chi_{\tilde{D}_{\alpha_2}} dx \\
&\qquad \le (\alpha_2 - \alpha_1) \|u_{\alpha_2}\|_4 \|\hat{u}_{\alpha_1} - u_{\alpha_2}\|_4 |\tilde{D}_{\alpha_2}|^{\frac{1}{2}} \\
&\qquad \le C(\alpha_2 - \alpha_1) \|u_{\alpha_2}\|_{H^1_0(D)} \|\hat{u}_{\alpha_1} - u_{\alpha_2}\|_{H^1_0(D)} |\tilde{D}_{\alpha_2}|^{\frac{1}{2}},
\end{aligned} \tag{3.131}
$$

where we have used general Hölder's inequality in the first inequality and Sobolev embedding theorem in the second inequality. Since the second term of the first line of (3.131) is non-negative, we obtain

$$
\|\hat{u}_{\alpha_1} - u_{\alpha_2}\|_{H^1_0(D)} \le C(\alpha_2 - \alpha_1) \|u_{\alpha_2}\|_{H^1_0(D)} |\tilde{D}_{\alpha_2}|^{\frac{1}{2}}. \tag{3.132}
$$

Noting that α_1 increases to α_2, we infer $\hat{u}_{\alpha_1}$ converges to u_{α_2} in $H^1_0(D)$. By utilizing elliptic regularity theory and Sobolev embedding theorem, we infer $\hat{u}_{\alpha_1}$ converges to u_{α_2} in $C(\bar{D})$. So, from (3.130) and the fact that $|\tilde{D}_{\alpha_2}| = \gamma$, in conjunction with Lemma 3.3.6, we deduce that $|\hat{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2}|$ decreases to zero, and

$$
|\hat{D}_{\alpha_1}| \to \gamma^+. \tag{3.133}
$$

On the other hand, from (3.119) and (3.128), we have

$$
\begin{aligned}
&-\Delta(u_{\alpha_1} - \hat{u}_{\alpha_1}) + (\alpha_1 \chi_{\tilde{D}_{\alpha_1}} + \beta \chi_{\tilde{D}^c_{\alpha_1}})(u_{\alpha_1} - \hat{u}_{\alpha_1}) \\
&\qquad = (\alpha_1 - \beta)\hat{u}_{\alpha_1}(\chi_{\tilde{D}_{\alpha_2} \setminus \tilde{D}_{\alpha_1}} - \chi_{\tilde{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2}}) \quad \text{in } D,
\end{aligned} \tag{3.134}
$$

with $u_{\alpha_1} - \hat{u}_{\alpha_1} = 0$ on ∂D. Now, let us introduce the following subsets of D:

$$
\begin{cases}
\hat{E} \equiv \{x \in D : u_{\alpha_1}(x) - \hat{u}_{\alpha_1}(x) \le c_{\alpha_1} - c_{\alpha_2}\} \\
\hat{F} \equiv \{x \in D : u_{\alpha_1}(x) - \hat{u}_{\alpha_1}(x) > c_{\alpha_1} - c_{\alpha_2}\}.
\end{cases}
$$

Using (3.118) and (3.130), we infer $\tilde{D}_{\alpha_1} \setminus \hat{D}_{\alpha_1} \subseteq \hat{F}$ and $\hat{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_1} \subseteq \hat{E}$. Moreover, by (3.130), we have $\hat{F} = (\hat{E})^c \subseteq (\hat{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_1})^c \subseteq (\tilde{D}_{\alpha_2} \setminus \tilde{D}_{\alpha_1})^c$. So, (3.134) leads to

$$
\begin{aligned}
&-\Delta(u_{\alpha_1} - \hat{u}_{\alpha_1}) + (\alpha_1 \chi_{\tilde{D}_{\alpha_1}} + \beta \chi_{\tilde{D}^c_{\alpha_1} \cap \tilde{D}^c_{\alpha_2}})(u_{\alpha_1} - \hat{u}_{\alpha_1}) \\
&\qquad = -(\alpha_1 - \beta)\hat{u}_{\alpha_1}\chi_{\tilde{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2}} \quad \text{in } \hat{F} \subseteq (\tilde{D}_{\alpha_2} \setminus \tilde{D}_{\alpha_1})^c.
\end{aligned} \tag{3.135}
$$

Since $c_{\alpha_1} > c_{\alpha_2}$ (by Theorem 3.3.17), we have $u_{\alpha_1} - \hat{u}_{\alpha_1} = c_{\alpha_1} - c_{\alpha_2} > 0$ on $\partial\hat{F}$. By applying the maximum principle to (3.135), we deduce $u_{\alpha_1} - \hat{u}_{\alpha_1} \le c_{\alpha_1} - c_{\alpha_2}$ in $\hat{F}$. Recalling the definition of $\hat{F}$, we have $\hat{F} = \emptyset$. Since $\tilde{D}_{\alpha_1} \setminus \hat{D}_{\alpha_1} \subseteq \hat{F}$, we infer $\tilde{D}_{\alpha_1} \setminus \hat{D}_{\alpha_1} = \emptyset$, i. e. $\tilde{D}_{\alpha_1} \subseteq \hat{D}_{\alpha_1}$. So, from (3.133) and the fact that $|\tilde{D}_{\alpha_1}| = \gamma$, we deduce $|\hat{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_1}|$ decreases to zero. Furthermore, recalling that $|\hat{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2}|$ decreases to zero, from (3.130) we have

$$|\tilde{D}_{\alpha_1} \vartriangle \tilde{D}_{\alpha_2}| = |(\tilde{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2}) \cup (\tilde{D}_{\alpha_2} \setminus \tilde{D}_{\alpha_1})| \le |\hat{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_2}| + |\hat{D}_{\alpha_1} \setminus \tilde{D}_{\alpha_1}| \to 0^+$$

when α_1 increases to α_2 as desired. Similarly, when α_1 decreases to α_2 with $\beta < \alpha_2 < 1$, we will have $|\tilde{D}_{\alpha_1} \vartriangle \tilde{D}_{\alpha_2}|$ converging to zero. This completes the proof. $\square$

For the rest of this subsection, we will denote the minimization problem (3.87) as follows

$$\Psi(\alpha) \equiv \inf_{|E|=\gamma} \int_D f u_{E,\alpha}\, dx = \int_D f u_\alpha\, dx \tag{3.136}$$

Theorem 3.3.21. *Let $0 \le \beta < \alpha_1, \alpha_2 \le 1$. If α_1 converges to α_2 in $(\beta, 1]$, then u_{α_1} converges to u_{α_2} in $C(\bar{D})$.*

Proof. Fix $\beta < \alpha_2 \le 1$. Multiplying the differential equation in (3.121) by $u_{\alpha_1} - u_{\alpha_2}$, integrating the result over D, followed by an application of divergence theorem yields

$$\begin{aligned}
&\int_D (|\nabla(u_{\alpha_1} - u_{\alpha_2})|^2) dx + \int_D (\alpha_1 \chi_{\tilde{D}_{\alpha_1}} + \beta \chi_{\tilde{D}^c_{\alpha_1}})(u_{\alpha_1} - u_{\alpha_2})^2 dx \\
&= (\alpha_1 - \beta) \int_D u_{\alpha_2}(u_{\alpha_1} - u_{\alpha_2})(\chi_{\tilde{D}_{\alpha_2}} - \chi_{\tilde{D}_{\alpha_1}}) dx \\
&\qquad + (\alpha_1 - \alpha_2) \int_D u_{\alpha_2}(u_{\alpha_1} - u_{\alpha_2}) \chi_{\tilde{D}_{\alpha_2}} dx \\
&\le (\alpha_1 - \beta)\, \|u_{\alpha_2}\|_4 \, \|u_{\alpha_1} - u_{\alpha_2}\|_4 \, |\tilde{D}_{\alpha_1} \vartriangle \tilde{D}_{\alpha_2}|^{\frac{1}{2}} \\
&\qquad + |\alpha_1 - \alpha_2| \, \|u_{\alpha_2}\|_4 \, \|u_{\alpha_1} - u_{\alpha_2}\|_4 \, |\tilde{D}_{\alpha_2}|^{\frac{1}{2}} \\
&\le C \, \|u_{\alpha_2}\|_{H^1_0(D)} \, \|u_{\alpha_1} - u_{\alpha_2}\|_{H^1_0(D)} \\
&\qquad \times \left[(\alpha_1 - \beta) |\tilde{D}_{\alpha_1} \vartriangle \tilde{D}_{\alpha_2}|^{\frac{1}{2}} + |\alpha_1 - \alpha_2| |\tilde{D}_{\alpha_2}|^{\frac{1}{2}} \right],
\end{aligned} \tag{3.137}$$

where we have used general Hölder's inequality in the first inequality and Sobolev embedding theorem in the second inequality. Since the second term of the first line of (3.137) is non-negative, we deduce

$$\begin{aligned}(3.138)\quad &\|u_{\alpha_1} - u_{\alpha_2}\|_{H_0^1(D)} \\ &\leq C\,\|u_{\alpha_2}\|_{H_0^1(D)} \left[(\alpha_1 - \beta)\,|\tilde{D}_{\alpha_1} \vartriangle \tilde{D}_{\alpha_2}|^{\frac{1}{2}} + |\alpha_1 - \alpha_2|\,|\tilde{D}_{\alpha_2}|^{\frac{1}{2}}\right].\end{aligned}$$

By using Theorem 3.3.20 the right hand side of (3.138) converges to zero when α_1 converges to α_2. Hence, from (3.138), we infer that u_{α_1} converges to u_{α_2} in $H_0^1(D)$. Furthermore, by applying elliptic regularity theory and Sobolev embedding theorem, we infer that u_{α_1} converges to u_{α_2} in $C(\bar{D})$. □

Corollary 3.3.22. *Let $0 \leq \beta < \alpha_1, \alpha_2 \leq 1$. If α_1 converges to α_2 in $(\beta, 1]$, then $\Psi(\alpha_1)$ converges to $\Psi(\alpha_2)$.*

Proof. From (3.136), by using Hölder's inequality, we calculate:

$$\begin{aligned}|\Psi(\alpha_1) - \Psi(\alpha_2)| &= \left|\int_D f(u_{\alpha_1} - u_{\alpha_2})dx\right| \\ &\leq \|f\|_2\,\|u_{\alpha_1} - u_{\alpha_2}\|_2 \\ &\leq \|f\|_2\,\|u_{\alpha_1} - u_{\alpha_2}\|_\infty\,|D|^{\frac{1}{2}}.\end{aligned} \tag{3.139}$$

Since u_{α_1} converges to u_{α_2} in $C(\bar{D})$ (Theorem 3.3.21), from (3.139) we infer that $\Psi(\alpha_1)$ converges to $\Psi(\alpha_2)$, as desired. □

3.4 Approximation and stability results

In reality, it is not easy to find the exact solution of the minimization problems (3.8) and (3.51). Usually, we need numerical simulations, in this case, if the generator of the rearrangement class is a simple function, it will simplify the computations by the computer. As every measurable function can be approximated by simple functions in an appropriate sense, it is interesting to address the following question:

Question 3.4.1. *If f_n converges to f in an appropriate L^p space, does $\hat{f}_n$ converge to $\hat{f}$ in the same space, where $\hat{\cdot}$ denotes the corresponding unique minimizer of (3.8) or (3.51) in $\mathcal{R}(\cdot)$?*

The weak closure of rearrangement class is of great importance in rearrangement theory, for example, see Lemma 3.1.1. Motivated by Question 3.4.1, we are interested in the following stability question.

Question 3.4.2. *If f_n converges to f in an appropriate L^p space, does the Hausdorff distance*[4] *between $\overline{\mathcal{R}(f_n)}$ and $\overline{\mathcal{R}(f)}$ diminish, where $\overline{(\cdot)}$ denotes the corresponding weak closure of $(\cdot)$ in $L^p(D)$?*

The structure of this section will be organized as follows. In the following subsection, we collect some well-known results. The last subsection contains the main results where answers to the two aforementioned questions will be given.

3.4.1 More preliminaries

First, we present the definition of Hausdorff distance for convenience.

Definition 3.4.1. *Let (X, d) be a metric space. Suppose L and K are two non-empty subsets of X. Then, the Hausdorff distance between L and K is defined by*

$$d_H(L, K) = \max\left\{\sup_{x\in K}\left(\inf_{y\in L} d(x,y)\right), \sup_{y\in L}\left(\inf_{x\in K} d(x,y)\right)\right\}.$$

Then, let us recall the Radon-Riesz Theorem.

Theorem 3.4.1. *Let $1 < p < \infty$, $f \in L^p(D)$ and $\{f_n\} \subseteq L^p(D)$. If $f_n \rightharpoonup f$ in $L^p(D)$ and $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$, then $f_n \to f$ in $L^p(D)$.*

Proof. See section 37 in [45]. □

3.4.2 Main results

We give an affirmative answer to Question 3.4.1 by introducing the following result.

Theorem 3.4.2. *Let $1 < p < \infty$, $f_0 \in L^p(D)$, $\{f_n\} \subseteq L^p(D)$ and Φ be a functional on $L^p(D)$. Suppose f_0, $\{f_n\}$ and Φ satisfy*

(i) $f_n \to f_0$ in $L^p(D)$,

(ii) Φ is strictly convex and weakly continuous on $\overline{co}^s\left(\cup_{n=0}^{\infty}\mathcal{R}(f_n)\right)$,

(iii) There exists unique $\hat{f}_n \in \mathcal{R}(f_n)$ such that $\Phi(\hat{f}_n) = \inf_{f\in\mathcal{R}(f_n)} \Phi(f) = \inf_{f\in\overline{\mathcal{R}(f_n)}} \Phi(f)$ for all $n \in \mathbb{N}\cup\{0\}$.

[4]See Definition 3.4.1 below.

Then, $\hat{f}_n \to \hat{f}_0$ in $L^p(D)$.

Remark 3.4.1. *From Lemma 3.1.1 (ii), we know*

$$\bigcup_{n=0}^{\infty} \overline{\mathcal{R}(f_n)} \subseteq \overline{co}^s \left(\bigcup_{n=0}^{\infty} \overline{\mathcal{R}(f_n)} \right) = \overline{co}^s \left(\bigcup_{n=0}^{\infty} \mathcal{R}(f_n) \right).$$

Usually, we can prove that Φ is strictly convex and weakly continuous on a larger set $F \supseteq \overline{co}^s\left(\cup_{n=0}^{\infty}\mathcal{R}(f_n)\right)$, see Lemma 3.2.3, Lemma 3.3.4 and Remark 3.3.3. For condition (iii) in Theorem 3.4.2, by observing that Φ is strictly convex and $\overline{\mathcal{R}(f_{(\cdot)})}$ is convex, the uniqueness of minimizer is ensured. Since the approach of proving the existence and uniqueness of (3.8) or (3.51) is to first relax the problem by extending the rearrangement class to its weak closure, the second equality in (iii) is usually the situation, see the proof of Theorem 3.2.1 or Theorem 3.3.5 for details. Finally, if the existence and uniqueness of solution are ensured for the corresponding maximization problem[5], then we could change 'inf' into 'sup' in condition (iii).

We break the proof of Theorem 3.4.2 into several lemmas.

Lemma 3.4.3. *Let E be a bounded subset of $L^p(D)$, and Φ be a weakly continuous functional on $\overline{E}$. Then, Φ is uniformly continuous on $\overline{E}$.*

Proof. We argue by contradiction. Suppose Φ is not uniformly continuous on $\overline{E}$, i.e. there exists $\epsilon > 0$ such that

$$\forall\, n \in \mathbb{N},\ \exists\, x_n, y_n \in \overline{E} \text{ for which } \|x_n - y_n\|_p < \frac{1}{n} \text{ and } |\Phi(x_n) - \Phi(y_n)| \geq \epsilon. \tag{3.140}$$

So, we have $x_n - y_n \to 0$ in $L^p(D)$. Since E is bounded, $\overline{E}$ is also bounded in $L^p(D)$. Hence, there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that $x_{n_k} \rightharpoonup \hat{x}$ and $y_{n_k} \rightharpoonup \hat{y}$. Obviously, $\hat{x}, \hat{y} \in \overline{E}$. By using the weak continuity of Φ and the last inequality in (3.140), we have

$$|\Phi(\hat{x}) - \Phi(\hat{y})| \geq \epsilon > 0. \tag{3.141}$$

Recalling the facts that $x_n - y_n \to 0$, $x_{n_k} \rightharpoonup \hat{x}$ and $y_{n_k} \rightharpoonup \hat{y}$, we must have $\hat{x} = \hat{y}$ which contradicts (3.141). This completes the proof of the lemma. □

Lemma 3.4.4. *Let $f, g \in L^p(D)$ and $\tilde{f} \in \mathcal{R}(f)$. Then, there exists $\tilde{g} \in \mathcal{R}(g)$ such that*

$$\left\| \tilde{g} - \tilde{f} \right\|_p = \left\| g^{\Delta} - f^{\Delta} \right\|_p \leq \|g - f\|_p\,. \tag{3.142}$$

[5] Usually, this happens in radial domain, see for example Theorem 3.5 in [20].

Proof. By applying Lemma 3.1.2 (i), we infer the existence of a measure preserving map $\rho : D \to (0, |D|)$ such that $\tilde{f} = f^{\Delta} \circ \rho$. After defining $\tilde{g} = g^{\Delta} \circ \rho$, we have $\tilde{g} \in \mathcal{R}(g)$ and $\left\| \tilde{g} - \tilde{f} \right\|_p = \left\| g^{\Delta} - f^{\Delta} \right\|_p$. By applying Lemma 3.1.2 (ii), the assertion follows. □

Lemma 3.4.5. *Let f_0 and Φ be as in Theorem 3.4.2. For $\alpha > 0$ and $h \in L^p(D)$, we define:*

$$A(\alpha, h) = \{g \in \mathcal{R}(f_0) : \ \|g - h\|_p \geq \alpha\},$$

and

$$\gamma(\alpha, h) = \inf_{g \in A(\alpha, h)} \Phi(g) - \Phi(h). \tag{3.143}$$

If $\alpha > 0$ and $A(\alpha, \hat{f}_0)$ is not empty[6]*, then $\gamma(\alpha, \hat{f}_0)$ is positive.*

Proof. Let $\{g_n\} \subseteq A(\alpha, \hat{f}_0)$ be a minimizing sequence such that

$$\Phi(g_n) \leq \inf_{g \in A(\alpha, \hat{f}_0)} \Phi(g) + \frac{1}{n}.$$

By Lemma 3.1.1 (i), we have $\|g_n\|_p = \|f_0\|_p$ for every $n \in \mathbb{N}$. Hence, there exists a subsequence, still denoted $\{g_n\}$, such that $g_n \rightharpoonup \bar{g}$ in $L^p(D)$. Observe that $\bar{g} \in \overline{\mathcal{R}(f_0)}$. By weak continuity of Φ, we deduce

$$\Phi(\bar{g}) \leq \inf_{g \in A(\alpha, \hat{f}_0)} \Phi(g).$$

Then, we claim $\bar{g} \neq \hat{f}_0$. Suppose not, let us assume $\bar{g} = \hat{f}_0$. Since $\hat{f}_0 \in \mathcal{R}(f_0)$ by condition (iii) in Theorem 3.4.2, we are in a position to apply Theorem 3.4.1. So, we have $g_n \to \hat{f}_0$ in $L^p(D)$ which contradicts the definition of $A(\alpha, \hat{f}_0)$. Whence, by condition (iii) in Theorem 3.4.2, it follows that

$$\Phi(\hat{f}_0) < \Phi(\bar{g}) \leq \inf_{g \in A(\alpha, \hat{f}_0)} \Phi(g).$$

This completes the proof of the lemma. □

Remark 3.4.2. *Actually, we can utilize the tools from rearrangement theory instead of Theorem 3.4.1 to prove Lemma 3.4.5. We present this alternative proof here to show the powerfulness of rearrangement theory.*

[6] In case $\alpha > 0$ and $A(\alpha, \hat{f}_0)$ is empty, $\gamma(\alpha, \hat{f}_0) = \infty$.

Alternative proof of Lemma 3.4.5. We begin the proof with two observations. Firstly, from condition (iii) in Theorem 3.4.2, $\hat{f}_0$ is the unique minimizer of Φ relative to $\overline{\mathcal{R}(f_0)}$. Secondly, $\gamma(\alpha, \hat{f}_0)$ is already non-negative, so to finish the proof of the lemma we only need to rule out the possibility of $\gamma(\alpha, \hat{f}_0)$ being zero.

For simplicity, we set $A \equiv A(\alpha, \hat{f}_0)$. Then, note that A^c, the complement of A relative to $\mathcal{R}(f_0)$, is equal to the set $\{g \in \mathcal{R}(f_0) : \|g - \hat{f}_0\|_p < \alpha\}$; which is a strongly open subset of $\mathcal{R}(f_0)$. By Lemma 3.1.1 (v), there exists an open weak set W such that $\hat{f} \in W \subseteq A^c$. Without loss of generality, we can choose $W = \{g \in \mathcal{R}(f_0) : |l(g) - l(\hat{f}_0)| < \epsilon\}$, for some $\epsilon > 0$ and $l \in (L^p)^* = L^{p'}$. Since $A \subseteq W^c$, clearly $\inf_A \Phi(g) \geq \inf_{W^c} \Phi(g)$. Hence, it suffices to show that $\inf_{W^c} \Phi(g) > \Phi(\hat{f}_0)$. To seek a contradiction we assume $\inf_{W^c} \Phi(g) = \Phi(\hat{f}_0)$, and let $\{g_n\} \subseteq W^c$ be a minimizing sequence. After passing to a subsequence, if necessary, and still denoted $\{g_n\}$, we infer $g_n \rightharpoonup \bar{g}$, for some $\bar{g} \in \overline{\mathcal{R}(f_0)}$. Since $l(g_n) \to l(\bar{g})$, we have $\bar{g} \in E \equiv \{g \in \overline{\mathcal{R}(f_0)} : |l(g) - l(\hat{f}_0)| \geq \epsilon\}$. On the other hand, by the weak continuity of Φ, we get $\Phi(g_n) \to \Phi(\bar{g})$. So we must have $\Phi(\bar{g}) = \Phi(\hat{f}_0)$. Since Φ is strictly convex, $\bar{g} = \hat{f}_0$. Whence, $\hat{f}_0 \in E$, which is a contradiction. $\square$

Proof of Theorem 3.4.2. In order to derive a contradiction, we assume there exist $\epsilon > 0$ and a subsequence of $\{f_n\}$, still denoted $\{f_n\}$, such that $\left\|\hat{f}_n - \hat{f}_0\right\|_p \geq \epsilon$ for all $n \in \mathbb{N}$. Then, by Lemma 3.4.4, there exist $\{g_n\} \subseteq \mathcal{R}(f_0)$ and $h_n \in \mathcal{R}(f_n)$ such that

$$\begin{cases} \left\|\hat{f}_n - g_n\right\|_p = \left\|f_n^\Delta - f_0^\Delta\right\|_p \leq \|f_n - f_0\|_p\,, \\ \left\|h_n - \hat{f}_0\right\|_p = \left\|f_n^\Delta - f_0^\Delta\right\|_p \leq \|f_n - f_0\|_p\,. \end{cases} \tag{3.144}$$

Since $f_n \to f_0$ in $L^p(D)$, there exists $N_1 \in \mathbb{N}$ such that $\left\|\hat{f}_n - g_n\right\|_p \leq \frac{\epsilon}{2}$ for all $n \geq N_1$. Recalling $\left\|\hat{f}_n - \hat{f}_0\right\|_p \geq \epsilon$, we have

$$\left\|g_n - \hat{f}_0\right\|_p \geq \left\|\hat{f}_n - \hat{f}_0\right\|_p - \left\|\hat{f}_n - g_n\right\|_p \geq \frac{\epsilon}{2}, \quad \forall\, n \geq N_1. \tag{3.145}$$

Since $\{g_n\} \subseteq \mathcal{R}(f_0)$, by Lemma 3.4.5, we have $0 < \gamma(\frac{\epsilon}{2}, \hat{f}_0) < \infty$, otherwise, replace ϵ by a smaller positive value. Then, by using condition (i), (3.144) and Lemma 3.4.3, there exists $N_2 \in \mathbb{N}$ with $N_2 \geq N_1$ such that

$$\begin{cases} \left|\Phi(\hat{f}_n) - \Phi(g_n)\right| < \frac{1}{2}\gamma\left(\frac{\epsilon}{2}, \hat{f}_0\right) \\ \left|\Phi(h_n) - \Phi(\hat{f}_0)\right| < \frac{1}{2}\gamma\left(\frac{\epsilon}{2}, \hat{f}_0\right), \end{cases} \tag{3.146}$$

for all $n \geq N_2$. Therefore, by using (3.145), (3.146) and Lemma 3.4.5, we infer

$$\Phi(\hat{f}_n) > \Phi(g_n) - \frac{1}{2}\gamma\left(\frac{\epsilon}{2}, \hat{f}_0\right) \geq \Phi(\hat{f}_0) + \frac{1}{2}\gamma\left(\frac{\epsilon}{2}, \hat{f}_0\right) > \Phi(h_n), \quad \forall\, n \geq N_2.$$

By recalling $h_n \in \mathcal{R}(f_n)$, the equation above is obviously a contradiction. This completes the proof of the theorem. □

Remark 3.4.3. *By examining the proof above, the condition (i) of Theorem 3.4.2 can be relaxed to $f_n^\Delta \to f_0^\Delta$ in $L^p(0, |D|)$.*

Regarding to Question 3.4.2, we have the following:

Theorem 3.4.6. *Let $1 < p < \infty$. Suppose $f_n \to f$ in $L^p(D)$. Then $d_H(\overline{\mathcal{R}_n}, \overline{\mathcal{R}}) \to 0$, as $n \to \infty$. Here, $\overline{\mathcal{R}_n}$ and $\overline{\mathcal{R}}$ are the weak closure of $\mathcal{R}_n \equiv \mathcal{R}(f_n)$ and $\mathcal{R} \equiv \mathcal{R}(f)$ in $L^p(D)$ respectively.*

Proof. Consider $\xi_n \in \mathcal{R}_n$. So, by Lemma 3.1.2 (i), $\xi_n = \xi_n^\Delta \circ \rho_n$, for some measure preserving map ρ_n. Thus,

$$\|\xi_n^\Delta \circ \rho_n - f^\Delta \circ \rho_n\|_p = \|\xi_n^\Delta - f^\Delta\|_p = \|f_n^\Delta - f^\Delta\|_p \leq \|f_n - f\|_p, \tag{3.147}$$

where the inequality in (3.147) follows from Lemma 3.1.2 (ii). Let us fix $\epsilon > 0$. Since $f_n \to f$, in $L^p(D)$, we infer existence of $N \in \mathbb{N}$ such that:

$$\|\xi_n - f^\Delta \circ \rho_n\|_p \leq \|f_n - f\|_p < \epsilon, \quad \forall n \geq N, \tag{3.148}$$

where we have used (3.147) and $\xi_n = \xi_n^\Delta \circ \rho_n$. Note that $f^\Delta \circ \rho_n \in \mathcal{R}$, hence from (3.148), we deduce $\xi_n \in \mathcal{R} + B_\epsilon(0)$, where: $B_\epsilon(0) = \{h \in L^p(D) : \|h\|_p < \epsilon\}$. Hence, trivially, we obtain $\xi_n \in \overline{\mathcal{R}} + B_\epsilon(0)$ for all $n \geq N$. Thus, $\mathcal{R}_n \subseteq \overline{\mathcal{R}} + B_\epsilon(0)$ for all $n \geq N$.

Let us fix $n \geq N$, and consider $\eta \in \overline{\mathcal{R}_n}$. Then, there exists a sequence $\{\eta_i\} \subseteq \mathcal{R}_n$ such that $\eta_i \rightharpoonup \eta$, in $L^p(D)$. Note that $\eta_i \in \overline{\mathcal{R}} + B_\epsilon(0)$ for all $i \in \mathbb{N}$. Therefore there exists $g_i \in \overline{\mathcal{R}}$ such that $\|\eta_i - g_i\|_p < \epsilon$. Since $\{g_i\}$ is bounded in $L^p(D)$, we can pass to a subsequence, if necessary, still denoted $\{g_i\}$, such that $g_i \rightharpoonup g$ in $L^p(D)$. This, in turn, implies that $g \in \overline{\mathcal{R}}$. Further, we have $\eta_i - g_i \rightharpoonup \eta - g$ in $L^p(D)$. Thus, from the weakly lower semicontinuity of the L^p-norm we obtain: $\|\eta - g\|_p \leq \liminf_{i\to\infty} \|\eta_i - g_i\|_p < 2\epsilon$. Whence, $\eta \in g + B_{2\epsilon}(0) \subseteq \overline{\mathcal{R}} + B_{2\epsilon}(0)$. This shows that:

$$\overline{\mathcal{R}_n} \subseteq \overline{\mathcal{R}} + B_{2\epsilon}(0), \quad \forall n \geq N. \tag{3.149}$$

Similarly, one can prove:

$$\overline{\mathcal{R}} \subseteq \overline{\mathcal{R}_n} + B_{2\epsilon}(0), \quad \forall n \geq N. \tag{3.150}$$

From (3.149), (3.150) and Definition 3.4.1, we find $\mathrm{d}_H(\overline{\mathcal{R}_n}, \overline{\mathcal{R}}) < 2\epsilon$ for all $n \geq N$. This completes the proof of the theorem. □

Chapter 4

Rearrangement of Measures

The aim of this chapter is to explore a way to generalize the notion of rearrangement of functions to rearrangement of Radon measures. Since this line of research seems to be blank in the existing literature, we will develop the theory from basics. For more information about Radon measures, we refer to [5, 28, 29].

4.1 Extension to Radon measures

Let us start with the following definition of Radon measures. Henceforth, D will be a smooth bounded domain in $\mathbb{R}^N$ and $\mathbb{R}_{\geq 0}$ will denote non-negative real numbers.

Definition 4.1.1. *Let $\mathcal{B}(\bar{D})$ be the Borel σ-algebra on $\bar{D}$. By a Radon measure, we mean a finite signed Borel measure $\mu : \mathcal{B}(\bar{D}) \to (-\infty, \infty)$ with its total variation $|\mu|$, i.e.*

$$|\mu|(E) \equiv \sup \left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : \{E_n\} \subseteq \mathcal{B}(\bar{D}) \text{ partition of } E \right\} \tag{4.1}$$

for all $E \in \mathcal{B}(\bar{D})$, satisfying

(i) $|\mu|(\partial D) = 0$, and $|\mu|(K) < \infty$ for every compact set $K \subseteq \bar{D}$.

(ii) every open set $O \subseteq \bar{D}$ is inner regular, i.e.

$$|\mu|(O) = \sup \{|\mu|(K) : K \subseteq O, K \text{ compact}\}.$$

(iii) every set $E \in \mathcal{B}(\bar{D})$ is outer regular, i.e.

$$|\mu|(E) = \inf \{|\mu|(O) : O \supseteq E, O \text{ open}\}.$$

Moreover, by Reisz representation theorem, every Radon measure is identified with a unique continuous linear functional on $C_0(\bar{D})$, i.e. dual of $C_0(\bar{D})$, where $C_0(\bar{D}) \equiv \{\zeta \in C(\bar{D}) : \zeta = 0 \text{ on } \partial D\}$.

Remark 4.1.1. *Since every open set $O \subseteq \bar{D}$ is σ-compact, by Proposition 1.60 in [29], every Borel set E is inner regular. In addition, as $|\mu|(\partial D) = 0$, we are allowed to replace $\bar{D}$ by D when μ is involved.*

Lemma 4.1.1. *For every $E \in \mathcal{B}(D)$, the Borel σ-algebra on D, we have*

$$|\mu|(E) = \sup \left\{ \int_E \zeta d\mu : \zeta \in C_0(\bar{D}), \|\zeta\|_\infty \leq 1 \right\}. \tag{4.2}$$

Proof. Let us fix $E \in \mathcal{B}(D)$. To simplify the notation, we will use A and B to denote the values of (4.1) and (4.2) respectively. By Hahn decomposition theorem (see [47]), there exist Borel sets D^+ and D^- such that $D = D^+ \cup D^-$, $D^+ \cap D^- = \emptyset$, and such that the positive and negative variations[1] of μ satisfy

$$\mu^+(E) = \mu(D^+ \cap E), \quad \mu^-(E) = -\mu(D^- \cap E), \quad \forall E \in \mathcal{B}(D).$$

Then, for every $\zeta \in C_0(\bar{D})$ with $\|\zeta\|_\infty \leq 1$, we have

$$\begin{aligned}\int_E \zeta d\mu &= \int_{D^+ \cap E} \zeta d\mu + \int_{D^- \cap E} \zeta d\mu \\ &\leq \mu^+(D^+ \cap E) + \mu^-(D^- \cap E) = |\mu(D^+ \cap E)| + |\mu(D^- \cap E)|.\end{aligned}$$

Therefore, we deduce $B \leq A$. Since D^+ and D^- are Borel, in conjunction with the inner regularity of Borel sets by Remark 4.1.1, it follows that for every $\epsilon > 0$ there exist compact sets $K^+ \subseteq D^+$ and $K^- \subseteq D^-$ satisfying

$$\mu^+(D^+ \setminus K^+) < \frac{\epsilon}{2}, \quad \mu^-(D^- \setminus K^-) < \frac{\epsilon}{2}.$$

[1] The positive and negative variations of μ are defined as follows:

$$\mu^+ \equiv \frac{1}{2}(|\mu| + \mu) \text{ and } \mu^- \equiv \frac{1}{2}(|\mu| - \mu).$$

Evidently, they are positive measures. Note that, in [29], they are called upper and lower variations instead.

So, by Urysohn's lemma (see [47]), there is $\zeta \in C_0(\bar{D})$ such that $\zeta = 1$ on K^+, $\zeta = -1$ on K^-, and $|\zeta| \leq 1$ on D. Whence, we deduce

$$\begin{aligned} A &= \mu^+(D^+ \cap E) + \mu^-(D^- \cap E) \\ &< \mu^+(K^+ \cap E) + \mu^-(K^- \cap E) + \epsilon \leq \int_E \zeta d\mu + 2\epsilon \leq B + 2\epsilon. \end{aligned}$$

By arbitrariness of ϵ, we have $A \leq B$. This completes the proof of the lemma. □

Lemma 4.1.2. *Let $f \in L^1(D)$, where $L^1(D) \equiv L^1(D, \mathcal{L}^N)$*[2] *with $\mathcal{L}^N$ denoting N-dimensional Lebesgue measure. For every $E \in \mathcal{B}(D)$, we have*

$$\int_E |f| d\mathcal{L}_N = \sup \left\{ \int_E \zeta f d\mathcal{L}_N : \zeta \in C_0(\bar{D}), \|\zeta\|_\infty \leq 1 \right\}.$$

Proof. Since f is measurable, it is possible to decompose D by Borel sets D^+ and D^- with $D = D^+ \cup D^-$ and $D^+ \cap D^- = \emptyset$ such that $f \geq 0$ a.e. on D^+ and $f \leq 0$ a.e. on D^-. Then, recalling $f \in L^1(D)$, by using similar technicalities applied in the proof of Lemma 4.1.1, the conclusion follows. □

Remark 4.1.2. *The space generated by Radon measures is denoted by $\mathcal{M}(\mathcal{D})$ and is equipped with the total variation norm*

$$\|\mu\|_{\mathcal{M}(\mathcal{D})} = \sup \left\{ \int_D \zeta d\mu : \zeta \in C_0(\bar{D}), \|\zeta\|_\infty \leq 1 \right\}.$$

By Lemma 4.1.1, we infer $\|\mu\|_{\mathcal{M}(\mathcal{D})} = |\mu|(D)$.

In the spirit of Definition 3.1.1, we define the distribution function of the Radon measure μ as follows:

$$\begin{aligned} (4.3) \quad \mathcal{T}_\mu(\beta) \equiv \sup \Bigg\{ \int_E \zeta d\mu :\ & \zeta \in C_0(\bar{D}),\ \|\zeta\|_\infty \leq 1, \\ & E \in \mathcal{B}(D),\ \mathcal{L}_N(E) = \beta \Bigg\}. \end{aligned}$$

Furthermore, by Lemma 4.1.1, we can write (4.3) in the following equivalent form

$$\mathcal{T}_\mu(\beta) = \sup \{ |\mu|(E) : E \in \mathcal{B}(D), \mathcal{L}_N(E) = \beta \}. \tag{4.4}$$

[2] We will adopt this notation in this chapter.

Since the Lebesgue measure is non-atomic, by Proposition 1.20 in [29], we infer $\mathcal{T}_\mu(\beta) > -\infty$ if $\beta \in [0, \mathcal{L}_N(D)]$. In addition, from (4.4), we also have $T_\mu(\mathcal{L}_N(D)) = |\mu|(D)$.

Proposition 4.1.3. *Let $f \in L^1(D)$. We define*

$$\mu_f(E) = \int_E f d\mathcal{L}_N, \quad \forall\, E \in \mathcal{B}(D), \quad \textit{with} \quad |\mu_f|(\partial D) = 0. \tag{4.5}$$

Then, μ_f is a Radon measure. Moreover, we can identify f with μ_f, i.e. $f \in \mathcal{M}(\mathcal{D})$, and $\|\mu_f\|_{\mathcal{M}(\mathcal{D})} = \|f\|_{L^1(D)}$. In addition, we also have

$$\mathcal{T}_{\mu_f}(\beta) = \mathcal{T}_f(\beta) \equiv \sup\left\{ \int_E \zeta f d\mathcal{L}_N : \zeta \in C_0(\bar{D}),\, \|\zeta\|_\infty \le 1, \right. \\ \left. E \in \mathcal{B}(D),\, \mathcal{L}_N(E) = \beta \right\}. \tag{4.6}$$

Proof. Since $f \in L^1(D)$, it is easy to check μ_f is indeed a Radon measure. Then, we define the following two functionals on $C_0(\bar{D})$:

$$\begin{cases} L_1(\zeta) = \int_{\bar{D}} \zeta d\mu_f = \int_D \zeta d\mu_f \\ L_2(\zeta) = \int_{\bar{D}} \zeta f d\mathcal{L}_N = \int_D \zeta f d\mathcal{L}_N, \end{cases} \tag{4.7}$$

for all $\zeta \in C_0(\bar{D})$. Clearly, L_1 and L_2 are both linear. On the one hand, by Remark 4.1.2, we infer L_1 is a continuous linear functional on $C_0(\bar{D})$ with $\|L_1\|_{(C_0(\bar{D}))'} = \|\mu_f\|_{\mathcal{M}(\mathcal{D})}$. On the other hand, recalling $f \in L^1(D)$, by Lemma 4.1.2, we have L_2 is also a continuous linear functional on $C_0(\bar{D})$ with $\|L_2\|_{(C_0(\bar{D}))'} = \|f\|_{L^1(D)}$.

At this stage, we claim $L_1(\zeta) = L_2(\zeta)$ for all $\zeta \in C_0(\bar{D})$. First, let us assume f is non-negative. Observe that L_1 and L_2 are well-defined for characteristic functions χ_E, where $E \in \mathcal{B}(D)$, $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$. From (4.5) and (4.7), it follows that

$$L_1(\chi_E) = \mu_f(E) = \int_E f d\mathcal{L}_N = L_2(\chi_E), \quad \forall\, E \in \mathcal{B}(D).$$

Without loss of generality, we assume ζ is non-negative. Otherwise, we have $\zeta = \zeta^+ - \zeta^-$, where $\zeta^+ \equiv \max\{f, 0\}$ and $\zeta^- \equiv -\min\{f, 0\}$. By using linearity, the claim will follow from the non-negative case. By Theorem 1.74 in [29], ζ can be approximated by an increasing sequence $\{s_n\}$ of non-negative simple functions. Since $0 \le s_n \le \zeta$ and $|s_n f| \le |\zeta f|$, by Lebesgue dominated convergence theorem, the claim follows. Now we start to remove the extra assumption $f \geq 0$. Indeed, we can write $f = f^+ - f^-$. From the first step, we infer the existence of continuous linear functionals L^+, L^- and Radon measures μ_{f^+}, μ_{f^-} such that

$$\begin{cases} L^+(\zeta) = \int_{\bar{D}} \zeta d\mu_{f^+} = \int_{\bar{D}} \zeta f^+ d\mathcal{L}_N \\ L^-(\zeta) = \int_{\bar{D}} \zeta d\mu_{f^-} = \int_{\bar{D}} \zeta f^- d\mathcal{L}_N, \end{cases}$$

for all $\zeta \in C_0(\bar{D})$. Therefore, we have

$$L(\zeta) \equiv L^+(\zeta) - L^-(\zeta) = \int_{\bar{D}} \zeta d\mu_f = \int_{\bar{D}} \zeta f d\mathcal{L}_N, \quad \forall \zeta \in C_0(\bar{D}),$$

where $\mu_f = \mu_{f^+} - \mu_{f^-}$ by (4.5). After an application of Reisz representation theorem, we can identify f with μ_f and $\|\mu_f\|_{\mathcal{M}(\mathcal{D})} = \|f\|_{L^1(D)}$. From (4.5), by a decomposition argument, we have

$$|\mu_f|(E) = \int_E |f| d\mathcal{L}_N, \quad \forall\, E \in \mathcal{B}(D).$$

So, by (4.4) and Lemma 4.1.2, we infer (4.6). □

Remark 4.1.3. *To understand (4.7), we may define a linear isometry* $I : L^1(D) \to L^1(\bar{D})$ *by* $I(f) = f$ *on* D *and* $I(f) = 0$ *on* ∂D*. By Proposition 4.1.3, we can embed* $L^1(D)$ *into* $\mathcal{M}(\mathcal{D})$*. If* $f \in L^1(D)$*, we will always abuse* f *with* μ_f *by denoting* $\mu_f \sim f$*.*

The following two definitions are consistent with Definitions 3.1.1 and 3.1.3 in certain sense.

Definition 4.1.2. *Let* D' *be a smooth bounded domain in* $\mathbb{R}^M$ *with* $\mathcal{L}_M(D') = \mathcal{L}_N(D)$*. Suppose* $\mu \in \mathcal{M}(\mathcal{D})$ *and* $\nu \in \mathcal{M}(\mathcal{D}')$*. We say* ν *is a rearrangement of* μ *if and only if*

$$\mathcal{T}_\nu(\beta) = \mathcal{T}_\mu(\beta), \quad \forall\, 0 \leq \beta \leq \mathcal{L}_N(D).$$

Definition 4.1.3. *Let* $\mu \in \mathcal{M}(\mathcal{D})$*. The rearrangement class generated by* μ*, denoted by* $\mathcal{R}(\mu)$*, is defined as follows:*

$$\mathcal{R}(\mu) = \{\nu \in \mathcal{M}(\mathcal{D}) : \nu \textit{ is a rearrangement of } \mu\}.$$

The following is a useful property of the distribution function $\mathcal{T}_\mu(\cdot)$.

Proposition 4.1.4. *Let* $\mu \in \mathcal{M}(D)$*. Then,* *(i)* $\mathcal{T}_\mu(\cdot)$ *is increasing on* $[0, \mathcal{L}_N(D)]$.

(ii) $\mathcal{T}_\mu(\cdot)$ *is continuous on* $[0, \mathcal{L}_N(D)]$.

Proof.

(i) Let us fix $0 \leq \beta_1 < \beta_2 \leq \mathcal{L}_N(D)$. From (4.4), it follows that for every $\epsilon > 0$ there exists $E_1 \in \mathcal{B}(D)$ with $\mathcal{L}_N(E_1) = \beta_1$ such that $\mathcal{T}_\mu(\beta_1) \leq |\mu|(E_1) + \epsilon$. By using Proposition 1.20 in [29], we infer the existence of $E_2 \in \mathcal{B}(D)$ with $E_1 \subseteq E_2$ and $\mathcal{L}_N(E_2) = \beta_2$. Then, we have

$$\mathcal{T}_\mu(\beta_1) \leq |\mu|(E_1) + \epsilon \leq |\mu|(E_2) + \epsilon \leq \mathcal{T}_\mu(\beta_2) + \epsilon.$$

By arbitrariness of ϵ, we deduce $\mathcal{T}_\mu(\beta_1) \leq \mathcal{T}_\mu(\beta_2)$ as desired.

(ii) In order to derive a contradiction, we suppose there exist $0 \leq \beta \leq \mathcal{L}_N(D)$ and a convergent sequence $\{\beta_n\} \subseteq [0, \mathcal{L}_N(D)]$ such that $\beta_n \to \beta$ and $|\mathcal{T}_\mu(\beta_n) - \mathcal{T}_\mu(\beta)| > \epsilon > 0$. Without loss of generality, we assume β_n decreases to β. So, by Part (i), we have $\mathcal{T}_\mu(\beta_n) > \mathcal{T}_\mu(\beta) + \epsilon$ and $\mathcal{T}_\mu(\beta_n)$ is monotonically decreasing. Furthermore, we deduce $T_\mu(\beta_n)$ converges to l with

$$T_\mu(\beta) + \epsilon \leq l \leq \mathcal{T}_\mu(\mathcal{L}_N(D)) = |\mu|(D). \tag{4.8}$$

On the other hand, from (4.4), we infer for every $n \in \mathbb{N}$ there exists $E_n \in \mathcal{B}(D)$ with $\mathcal{L}_N(E_n) = \beta_n$ such that

$$|\mu|(E_n) \geq \mathcal{T}_\mu(\beta_n) - \frac{\epsilon}{2} \geq l - \frac{\epsilon}{2}. \tag{4.9}$$

Combining (4.8) with (4.9), we have $|\mu|(E_n) - \mathcal{T}_\mu(\beta) \geq \frac{\epsilon}{2}$. By using (4.4), it follows that

$$\begin{aligned} \frac{\epsilon}{2} &\leq |\mu|(E_n) - \mathcal{T}_\mu(\beta) \\ &\leq |\mu|(E_n) - \sup\{|\mu|(F) : F \in \mathcal{B}(D), F \subseteq E_n, \mathcal{L}_N(F) = \beta\} \\ &\leq \inf\{|\mu|(F) : F \in \mathcal{B}(D), F \subseteq E_n, \mathcal{L}_N(F) = \beta_n - \beta\}. \end{aligned} \tag{4.10}$$

Since β_n decreases to β, we infer there exists $N \in \mathbb{N}$ such that

$$\frac{\beta_n}{\beta_n - \beta} \geq \frac{\beta}{\beta_n - \beta} > \frac{2|\mu|(D)}{\epsilon} + 1, \quad \forall n \geq N.$$

Whence, recalling (4.10), we have

$$|\mu|(E_n) \geq \frac{\epsilon}{2} \left\lceil \frac{\beta_n}{\beta_n - \beta} \right\rceil > |\mu|(D), \quad \forall n \geq N, \tag{4.11}$$

where $\lceil \cdot \rceil$ denotes the integer part of the corresponding real number. Clearly, (4.11) is a contradiction as desired.

□

Remark 4.1.4. *Recalling Definition 3.1.1, it is not hard to see that the distribution function $\lambda_f(\cdot)$ defined there is only left continuous in general. So the continuity of this new definition of distribution function seems to be a significant improvement.*

The following proposition shows a way to characterize $L^1(D)$ as a subspace of $\mathcal{M}(D)$.

Proposition 4.1.5. *Let $\mu \in \mathcal{M}(D)$. Then, μ can be identified with a function $g \in L^1(D)$ if and only if $\mathcal{T}_\mu(0) = 0$.*

Proof. Assume that μ can be identified with a function $g \in L^1(D)$, by Proposition 4.1.3, we have $\mathcal{T}_\mu(0) = \mathcal{T}_g(0) = 0$ as desired. To prove the converse, let us suppose $\mathcal{T}_\mu(0) = 0$. Then, from (4.4), we have

$$\sup\{|\mu|(E) : E \in \mathcal{B}(D), \mathcal{L}_N(E) = 0\} = 0. \tag{4.12}$$

First, let us assume μ is non-negative. From (4.12), we infer μ is absolutely continuous with respect to $\mathcal{L}_N$. By Radon-Nikodym theorem (Theorem 1.101 in [29]), there exists a unique measurable function $g : D \to \mathbb{R}_{\geq 0}$ such that

$$\mu(E) = \int_E g d\mathcal{L}_N, \quad \forall\, E \in \mathcal{B}(D).$$

Since $\int_D g d\mathcal{L}_N = \mu(D) < \infty$, we have $g \in L^1(D)$. Secondly, let us consider the general case, i.e. μ is a signed Radon measure. By Hahn decomposition theorem, there exist Borel sets D^+ and D^- such that $D = D^+ \cup D^-$, $D^+ \cap D^- = \emptyset$, and such that the positive and negative variations μ^+ and μ^- of μ satisfy

$$\mu^+(E) = \mu(D^+ \cap E), \quad \mu^-(E) = -\mu(D^- \cap E), \quad \forall\, E \in \mathcal{B}(D). \tag{4.13}$$

Then, by the first step, there exist measurable functions $g^+ : D \to \mathbb{R}_{\geq 0}$ and $g^- : D \to \mathbb{R}_{\geq 0}$ such that

$$\mu^+(E) = \int_E g^+ d\mathcal{L}_N, \quad \mu^-(E) = \int_E g^- d\mathcal{L}_N, \quad \forall\, E \in \mathcal{B}(D), \tag{4.14}$$

and $\int_D g^+ d\mathcal{L}_N = \mu^+(D) \le |\mu|(D) < \infty$, $\int_D g^- d\mathcal{L}_N = \mu^-(D) \le |\mu|(D) < \infty$. Therefore, we infer

$$\begin{aligned} \mu(E) &= \mu^+(E) - \mu^+(E) \\ &= \int_E g^+ d\mathcal{L}_N - \int_E g^- d\mathcal{L}_N = \int_E g d\mathcal{L}_N, \quad \forall\, E \in \mathcal{B}(D), \end{aligned} \tag{4.15}$$

where $g \equiv g^+ - g^-$. On the other hand, from (4.13) and (4.14), we deduce $g^+ = 0$ on D^- and $g^- = 0$ on D^+ pointwise $\mathcal{L}_N$ almost everywhere. So, we have $\|g\|_{L^1(D)} = \int_{D^+} g^+ d\mathcal{L}_N + \int_{D^-} g^- d\mathcal{L}_N = |\mu|(D) < \infty$. Whence, by Proposition 4.1.3, it follows from (4.15) that μ can be identified with g as desired. $\square$

In this chapter, we define the rearrangement of functions in a slight different way compared to Definition 3.1.1.

Definition 4.1.4. *Let D' be a smooth bounded domain in $\mathbb{R}^M$ with $\mathcal{L}_M(D') = \mathcal{L}_N(D)$. Suppose $f : D \to \mathbb{R}$ and $g : D' \to \mathbb{R}$ are two measurable functions, we say f is a rearrangement of g if and only if*

$$\begin{aligned} \lambda_{f,\mathcal{L}_N}(\alpha) &\equiv \mathcal{L}_N(\{x \in D : |f(x)| \ge \alpha\}) \\ &= \mathcal{L}_M(\{x \in D' : |g(x)| \ge \alpha\}) \equiv \lambda_{g,\mathcal{L}_M}(\alpha), \quad \forall\, \alpha \in \mathbb{R}_{\ge 0}. \end{aligned}$$

Remark 4.1.5. *If f, g are non-negative functions, then Definition 4.1.4 is reduced to Definition 3.1.1.*

Henceforth, we will use the notation λ_f instead of $\lambda_{f,\mathcal{L}_N}$ if the reference measure for distribution function is Lebesgue measure. The following is a useful consequence of Definition 4.1.4.

Lemma 4.1.6. *Let D', f and g be as in Definition 4.1.4. Then, we have*

$$\begin{aligned} &\lambda_f(\alpha) = \lambda_g(\alpha), \quad \forall\, \alpha \in \mathbb{R}_{\ge 0}, \quad \textit{if and only if} \\ &\qquad \mathcal{L}_N(\{|f(x)| > \alpha\}) = \mathcal{L}_M(\{|g(x)| > \alpha\}), \quad \forall\, \alpha \in \mathbb{R}_{\ge 0}. \end{aligned}$$

Proof. Observing that

$$\begin{cases} \{|f(x)| > \alpha\} = \bigcup_{n=1}^{\infty} \left\{|f(x)| \ge \alpha + \frac{1}{n}\right\} \\ \{|f(x)| \ge \alpha\} = \bigcap_{n=1}^{\infty} \left\{|f(x)| > \alpha - \frac{1}{n}\right\}, \end{cases}$$

and $\mathcal{L}_N(D) < \infty$, by using monotone convergence of Lebesgue measure, the assertion in the lemma follows. $\square$

At this stage, we intend to show that two definitions of rearrangement, i.e. Definition 4.1.2 and Definition 4.1.4, are consistent when $\mathcal{M}(\cdot)$ is restricted to $L^1(\cdot)$. Let us start with the following two lemmata. In addition, the sign function will be utilized in the proof of the following lemma, and it is defined as follows:

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases}$$

Lemma 4.1.7. *Let $f \in L^1(D)$. If $0 < \beta \le \mathcal{L}_N(D)$, then there exists $\alpha \in \mathbb{R}_{\ge 0}$ such that $\mathcal{L}_N(\{|f(x)| > \alpha\}) \le \beta \le \lambda_f(\alpha)$. Moreover, we have*

$$\mathcal{T}_f(\beta) = \int_{\{|f|>\alpha\}} |f| d\mathcal{L}_N + \alpha \mathcal{L}_N(F), \tag{4.16}$$

where $F \in \mathcal{B}(D)$, $F \subseteq \{|f(x)| = \alpha\}$ and $\mathcal{L}_N(F) + \mathcal{L}_N(\{|f(x)| > \alpha\}) = \beta$. Or, equivalently, we have

$$\mathcal{T}_f(\beta) = \int_\alpha^\infty \mathcal{L}_N(\{|f(x)| > t\}) dt + \alpha\beta. \tag{4.17}$$

Proof. Since $\lambda_f(\cdot)$ is decreasing, the first assertion easily follows. Recalling (4.6), we set $E = F \cup \{|f(x)| > \alpha\}$, where F is the Borel subset of $\{|f(x)| = \alpha\}$ as described in the lemma. Note that F is admissible here because Lebesgue measure is Borel regular. We claim that

$$\mathcal{T}_f(\beta) \ge \int_E |f| d\mathcal{L}_N = \int_{\{|f|>\alpha\}} |f| d\mathcal{L}_N + \alpha \mathcal{L}_N(F). \tag{4.18}$$

Indeed, since $f \in L^1(D)$, there exists $\delta > 0$ such that

$$\int_G |f| d\mathcal{L}_N < \frac{\epsilon}{2}, \quad \forall G \subseteq D \text{ with } \mathcal{L}_N(G) < \delta. \tag{4.19}$$

On the other hand, since $\operatorname{sign}(f)$ is a measurable function, by Lusin's theorem, see for example [27], there exists compact set $K \subsetneq D$ such that $\mathcal{L}_N(D \setminus K) < \delta$ and the restriction of $\operatorname{sign}(f)$ on K is continuous. So, we can find $\zeta \in C_0(\bar{D})$ such that $\zeta = \operatorname{sign}(f)$ on K. Whence, recalling (4.19), we have

$$\begin{aligned}\mathcal{T}_f(\beta) \geq \int_E \zeta f d\mathcal{L}_N &= \int_{E\setminus K} \zeta f d\mathcal{L}_N + \int_{E\cap K} |f| d\mathcal{L}_N \\ &\geq \int_{E\setminus K} |f| d\mathcal{L}_N - \epsilon + \int_{E\cap K} |f| d\mathcal{L}_N \\ &= \int_{\{|f|>\alpha\}} |f| d\mathcal{L}_N + \alpha \mathcal{L}_N(F) - \epsilon,\end{aligned}$$

where we have utilized the fact $\mathcal{L}_N(D \setminus K) < \delta$ and (4.19) in the second inequality. By the arbitrariness of ϵ, (4.18) follows. On the other hand, by (4.6), for any $\epsilon > 0$, there exist $E \in \mathcal{B}(D)$ with $\mathcal{L}_N(E) = \beta$ and $\zeta \in C_0(\bar{D})$ with $\|\zeta\|_\infty \leq 1$ such that

$$\mathcal{T}_f(\beta) - \epsilon \leq \int_E \zeta f d\mathcal{L}_N \leq \int_E |f| d\mathcal{L}_N \leq \int_{\{|f|>\alpha\}} |f| d\mathcal{L}_N + \alpha \mathcal{L}_N(F).$$

By the arbitrariness of ϵ, we have

$$\mathcal{T}_f(\beta) \leq \int_{\{|f|>\alpha\}} |f| d\mathcal{L}_N + \alpha \mathcal{L}_N(F). \tag{4.20}$$

By using (4.18) and (4.20), (4.16) follows.

Let us start to prove the third assertion. First, by using Fubini's theorem, it follows that

$$\begin{aligned}&\int_{\{|f|>\alpha\}} |f| d\mathcal{L}_N = \int_{\{|f|>\alpha\}} \int_0^{|f(x)|} dt d\mathcal{L}_N(x) \\ &= \int_{\{|f|>\alpha\}} \int_\alpha^{|f(x)|} dt d\mathcal{L}_N(x) + \int_{\{|f|>\alpha\}} \int_0^\alpha dt d\mathcal{L}_N \\ &= \int_{\{|f|>\alpha\}} \int_\alpha^\infty \chi_{\{y\in D: |f(y)|>t\}}(x) dt d\mathcal{L}_N(x) + \alpha \mathcal{L}_N(\{|f(x)| > \alpha\}) \\ &= \int_\alpha^\infty \int_{\{|f|>\alpha\}} \chi_{\{y\in D: |f(y)|>t\}}(x) d\mathcal{L}_N(x) dt + \alpha \mathcal{L}_N(\{|f(x)| > \alpha\}) \\ &= \int_\alpha^\infty \mathcal{L}_N(\{|f(x)| > t\}) dt + \alpha \mathcal{L}_N(\{|f(x)| > \alpha\}).\end{aligned} \tag{4.21}$$

Therefore, by (4.16) and (4.21), we have

$$\begin{aligned}\mathcal{T}_f(\beta) &= \int_{\{|f|>\alpha\}} |f| d\mathcal{L}_N + \alpha\mathcal{L}_N(F) \\ &= \int_{\alpha}^{\infty} \mathcal{L}_N(\{|f(x)| > t\})dt + \alpha\mathcal{L}_N(\{|f(x)| > \alpha\}) + \alpha\mathcal{L}_N(F) \\ &= \int_{\alpha}^{\infty} \mathcal{L}_N(\{|f(x)| > t\})dt + \alpha\beta,\end{aligned}$$

as desired. □

Lemma 4.1.8. *Let $f \in L^1(D)$. Suppose $0 < \beta < \mathcal{L}_N(D)$ and $\lambda_f(\alpha) = \beta$, then*

(i) For every $0 \leq \tilde{\beta} < \beta$, we have

$$\mathcal{T}_f(\tilde{\beta}) = \int_E |f| d\mathcal{L}_N,$$

where $E \in \mathcal{B}(D)$, $E \subseteq \{|f(x)| \geq \alpha\}$, and $\mathcal{L}_N(E) = \tilde{\beta}$.

(ii) For every $\beta < \tilde{\beta} \leq \mathcal{L}_N(D)$, we have

$$\mathcal{T}_f(\tilde{\beta}) = \int_{\{|f|\geq\alpha\}} |f| d\mathcal{L}_N + \int_F |f| d\mathcal{L}_N,$$

where $F \in \mathcal{B}(D)$, $F \subseteq \{|f(x)| < \alpha\}$, and $\mathcal{L}_N(F) = \tilde{\beta} - \beta$.

Proof. (i) If $\tilde{\beta} = 0$, we set $E = \emptyset$ and the assertion follows. So let us fix $0 < \tilde{\beta} < \beta$. Then, by Lemma 4.1.7, there exists $\tilde{\alpha} \geq \alpha$ such that $\mathcal{L}_N(\{|f(x)| > \tilde{\alpha}\}) \leq \tilde{\beta} \leq \lambda_f(\tilde{\alpha})$. Moreover, we have

$$\mathcal{T}_f(\tilde{\beta}) = \int_{\{|f|>\tilde{\alpha}\}} |f| dx + \tilde{\alpha}\mathcal{L}_N(F) = \int_{\{|f|>\tilde{\alpha}\}\cup F} |f| dx,$$

where $F \in \mathcal{B}(D)$, $F \subseteq \{|f(x)| = \tilde{\alpha}\}$ and $\mathcal{L}_N(F) + \mathcal{L}_N(\{|f(x)| > \tilde{\alpha}\}) = \tilde{\beta}$. By setting $E = \{|f(x)| > \tilde{\alpha}\} \cup F$, the assertion follows. Part (ii) can be proved in a similar way. □

The following two Propositions clarify that Definition 4.1.2 is a reasonable generalization of Definition 4.1.4.

Proposition 4.1.9. *Let D' be as in Definition 4.1.4. Suppose $f \in L^1(D)$ and $g \in L^1(D')$. Then, we have*

$$\mathcal{T}_f(\beta) = \mathcal{T}_g(\beta), \quad \forall\, 0 \leq \beta \leq \mathcal{L}_N(D), \quad \textit{if and only if} \quad \lambda_f(\alpha) = \lambda_g(\alpha), \quad \forall\, \alpha \in \mathbb{R}_{\geq 0}.$$

Proof. First, let us suppose $\lambda_f(\alpha) = \lambda_g(\alpha)$, $\forall\, \alpha \in \mathbb{R}_{\geq 0}$. We claim that $\mathcal{T}_f(\beta) = \mathcal{T}_g(\beta)$, $\forall\, 0 \leq \beta \leq \mathcal{L}_N(D)$. If $\beta = 0$ or $\beta = \mathcal{L}_N(D)$, the assertion follows trivially. So, we fix $0 < \beta < \mathcal{L}_N(D)$. By Lemma 4.1.7, there exists $\alpha \in \mathbb{R}_{\geq 0}$ such that $\mathcal{L}_N(\{|f(x)| > \alpha\}) \leq \beta \leq \lambda_f(\alpha)$. Moreover, we have

$$\mathcal{T}_f(\beta) = \int_\alpha^\infty \mathcal{L}_N(\{|f(x)| > t\})dt + \alpha\beta. \tag{4.22}$$

By Lemma 4.1.6, we also have $\mathcal{L}_M(\{|g(x)| > \alpha\}) \leq \beta \leq \lambda_g(\alpha)$. Similarly, it follows from Lemma 4.1.7 that

$$\mathcal{T}_g(\beta) = \int_\alpha^\infty \mathcal{L}_M(\{|g(x)| > t\})dt + \alpha\beta. \tag{4.23}$$

Therefore, by using (4.22) and (4.23), in conjunction with the fact that $\lambda_f(\alpha) = \lambda_g(\alpha)$, $\forall\, \alpha \in \mathbb{R}_{\geq 0}$ and Lemma 4.1.6, we infer $\mathcal{T}_f(\beta) = \mathcal{T}_g(\beta)$ as desired.

Secondly, we suppose $\mathcal{T}_f(\beta) = \mathcal{T}_g(\beta)$, $\forall\, 0 \leq \beta \leq \mathcal{L}_N(D)$. Then, we claim that $\lambda_f(\alpha) = \lambda_g(\alpha)$, $\forall\, \alpha \in \mathbb{R}_{\geq 0}$. If $\alpha = 0$, the assertion follows trivially. So, we consider the case $\alpha > 0$. In order to derive a contradiction, let us assume there exists $\hat{\alpha} > 0$ such that $\lambda_f(\hat{\alpha}) \neq \lambda_g(\hat{\alpha})$. Without loss of generality, we assume $\lambda_f(\hat{\alpha}) > \lambda_g(\hat{\alpha})$. On the one hand, we have $\mathcal{T}_f(\lambda_g(\hat{\alpha})) = \mathcal{T}_g(\lambda_g(\hat{\alpha}))$. More specifically, by Lemma 4.1.8 (i) and Lemma 4.1.7, we infer

$$\int_E |f| d\mathcal{L}_N = \int_{\{|g| \geq \hat{\alpha}\}} |g| d\mathcal{L}_M, \tag{4.24}$$

where $E \in \mathcal{B}(D)$, $E \subseteq \{|f(x)| \geq \hat{\alpha}\}$ and $\mathcal{L}_N(E) = \lambda_g(\hat{\alpha})$. On the other hand, we also have $\mathcal{T}_f(\lambda_f(\hat{\alpha})) = \mathcal{T}_g(\lambda_f(\hat{\alpha}))$. By applying Lemma 4.1.7 and Lemma 4.1.8 (ii), we deduce

$$\int_{\{|f| \geq \hat{\alpha}\}} |f| d\mathcal{L}_N = \int_{\{|g| \geq \hat{\alpha}\}} |g| d\mathcal{L}_M + \int_F |g| d\mathcal{L}_M, \tag{4.25}$$

where $F \in \mathcal{B}(D)$, $F \subseteq \{|g(x)| < \hat{\alpha}\}$ and $\mathcal{L}_M(F) = \lambda_f(\hat{\alpha}) - \lambda_g(\hat{\alpha})$. From (4.24) and (4.25), we infer

$$\int_{\{|f| \geq \hat{\alpha}\} \setminus E} |f| d\mathcal{L}_N = \int_F |g| d\mathcal{L}_M. \tag{4.26}$$

Since $|f| \geq \hat{\alpha}$ on $\{|f(x)| \geq \hat{\alpha}\} \setminus E$ and $|g| < \hat{\alpha}$ on F, from (4.26), we must have $\mathcal{L}_N(\{|f(x)| \geq \hat{\alpha}\} \setminus E) = \mathcal{L}_M(F) = \lambda_f(\hat{\alpha}) - \lambda_g(\hat{\alpha}) = 0$ which is a contradiction. This completes the proof of the proposition. □

Proposition 4.1.10. *Let D' be as in Definition 4.1.4. Suppose $f \in L^1(D)$ and $\mu \in \mathcal{M}(\mathcal{D}')$. If μ is a rearrangement of f, then μ can be identified with a function $g \in L^1(D')$ and $\lambda_\alpha(f) = \lambda_\alpha(g)$ for all non-negative α.*

Proof. Let μ be a rearrangement of f, then we have $\mathcal{T}_\mu(\beta) = \mathcal{T}_f(\beta)$ for all $0 \leq \beta \leq \mathcal{L}_N(D)$. Since $f \in L^1(D)$, it follows that $\mathcal{T}_\mu(0) = \mathcal{T}_f(0) = 0$. Whence, by Proposition 4.1.5, Proposition 4.1.3 and Proposition 4.1.9, the conclusion follows. □

4.2 Further investigations

In this section, we intend to explore more properties of Rearrangements of Radon measures. Let us start with the following proposition. To simplify the notation, we set

$$\begin{cases} \bar{D}^* = [-\frac{\mathcal{L}_N(D)}{2}, \frac{\mathcal{L}_N(D)}{2}] \\ D^* = (-\frac{\mathcal{L}_N(D)}{2}, \frac{\mathcal{L}_N(D)}{2}). \end{cases}$$

Proposition 4.2.1. *Let $\mu \in \mathcal{M}(D)$. We define $\mu^* : \mathcal{B}(\bar{D}^*) \to \mathbb{R}_{\geq 0}$ by*

(i) $\mu^*(\emptyset) = \mu^*(\partial \bar{D}^*) = 0$.

(ii) $\mu^*((-\frac{\beta}{2}, 0]) = \mu^*([0, \frac{\beta}{2})) = \frac{1}{2}\mathcal{T}_\mu(\beta)$, *for every* $0 < \beta \leq \mathcal{L}_N(D)$.

(iii) $\mu^*(F \setminus E) = \mu^*(F) - \mu^*(E)$, *for every* $E, F \in \mathcal{B}(\bar{D}^*)$ *and* $E \subseteq F$.

(iv) $\mu^*(\bigcup_{n=1}^\infty E_n) = \Sigma_{n=1}^\infty \mu^*(E_n)$, *for all pairwise disjoint collection* $\{E_n\} \subseteq \mathcal{B}(\bar{D}^*)$.

(v) $\mu^*(\bigcap_{n=1}^\infty E_n) = \lim_{n\to\infty} \mu^*(E_n)$, *for every decreasing sequence* $\{E_n\} \subseteq \mathcal{B}(\bar{D}^*)$.

Then, μ^* is a non-negative Radon measure on $\bar{D}^*$. Moreover, μ^* is a rearrangement of μ.

Proof. First, we must check if μ^* is well-defined, i.e. every $E \in \mathcal{B}(\bar{D}^*)$ is associated to a unique non-negative real number by μ^*. Let us start with the existence of this association. Recalling that Borel σ-algebra is generated by open sets and every open set on D^* is a countable union of disjoint open intervals, it suffices to show every open interval $(a, b) \subsetneq \bar{D}^*$ is admissible by this construction. If $ab < 0$, it is trivial. So we consider $ab \geq 0$, without loss of generality, let us assume $b > a \geq 0$. Then, we have $(a, b) = [0, b) \setminus (\bigcap_{n=1}^{\infty} [0, a + \frac{1}{n}))$ as desired. By the structure of this construction and the continuity of $\mathcal{T}_\mu(\cdot)$ by Proposition 4.1.4 (ii), uniqueness follows. Furthermore, from (4.4), we have $\mathcal{T}_\mu(\cdot)$ is non-negative. Whence, by using Proposition 4.1.4, we readily deduce that μ^* is a non-negative Radon measure on $\bar{D}^*$ and μ^* is a rearrangement of μ. □

Remark 4.2.1. *We will call μ^* the symmetrically decreasing rearrangement of μ.*

Lemma 4.2.2. *Let $f \in L^1(D)$ with μ_f defined in Proposition 4.1.3. Then, μ_f^* can be identified with a function $f^* \in L^1(D^*)$ and $\lambda_f(\alpha) = \lambda_{f^*}(\alpha)$ for all non-negative α. Moreover, f^* is symmetrically decreasing.*

Proof. From Proposition 4.2.1, we know μ_f^* is a rearrangement of μ_f (also f). By Proposition 4.1.10, the first assertion follows. The second assertion can be deduced from Propositions 4.2.1 and 4.1.4. □

Lemma 4.2.3. *Let $f, g \in L^1(D)$. If $|f| \leq |g|$ on D, then $f^* \leq g^*$ on D^*.*

Proof. Since $|f| \leq |g|$ on D, we have $\lambda_f(\alpha) \leq \lambda_g(\alpha)$ for all $\alpha \in \mathbb{R}_{\geq 0}$. By Lemma 4.2.2, we deduce $\lambda_{f^*}(\alpha) \leq \lambda_{g^*}(\alpha)$ for all $\alpha \in \mathbb{R}_{\geq 0}$. By symmetry, we readily deduce $f^* \leq g^*$ on D^*. □

Remark 4.2.2. *In $L^1(D)$, by $f \leq g$, we mean $f \leq g$ pointwise $\mathcal{L}_N$ almost everywhere.*

Note that since $\|\zeta\|_{L^1(\bar{D})} = \|\zeta\|_{L^1(D)} < \infty$ for every $\zeta \in C_0(\bar{D})$, by restricting ζ to D, we can embed $C_0(\bar{D})$ into $\mathcal{M}(D)$. Then, it is reasonable to have the following result.

Proposition 4.2.4. *Let $\zeta \in C_0(\bar{D})$. Then, we have $\zeta^* \in C_0(\bar{D}^*)$.*

Proof. By Proposition 4.2.1 and Lemma 4.2.2, we have

$$\lim_{x \to (\frac{\mathcal{L}_N(D)}{2})^-} \zeta^*(x) = 0, \qquad \lim_{x \to (-\frac{\mathcal{L}_N(D)}{2})^+} \zeta^*(x) = 0.$$

By defining $\zeta^*(x) = 0$ on $\partial \bar{D}^*$, it follows that ζ^* is continuous at $\partial \bar{D}^*$. Then, it suffices to show ζ^* is continuous on D^*. We argue by contradiction and assume there exist $\beta_1, \beta_2 \in \mathbb{R}_{\geq 0}$ such that $\beta_2 - \beta_1 > 0$ and

$\lambda_{\zeta^*}(\beta_1) = \lambda_{\zeta^*}(\beta_2) > 0$. By Lemma 4.2.2, we have $\lambda_\zeta(\beta_1) = \lambda_\zeta(\beta_2)$. Since $\lambda_\zeta(\cdot)$ is decreasing on $\mathbb{R}_{\geq 0}$, we deduce $\mathcal{L}_N(\{\beta_1 < |\zeta(x)| < \beta_2\}) = 0$. On the other hand, by using the continuity of ζ, we have $\{\beta_1 < |\zeta(x)| < \beta_2\}$ is non-empty and open. So, we must have $\mathcal{L}_N(\{\beta_1 < \zeta(x) < \beta_2\}) > 0$ which is a contradiction. $\square$

Remark 4.2.3. *Since ζ^* is constructed as a element in $L^1(D^*)$, the meaning of continuity of it should be dealt with more care. Here, by $\zeta^* \in C_0(\bar{D}^*)$, we mean there exists $\eta \in C_0(\bar{D}^*)$ such that $\zeta^* = \eta$ pointwise $\mathcal{L}_1$ almost everywhere.*

The following is a variant of Hardy-Littlewood inequality.

Proposition 4.2.5. *Let $\mu \in \mathcal{M}(D)$ and $\zeta \in C_0(\bar{D})$. Then, we have*

$$\int_E \zeta d\mu \leq \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} \zeta^* d\mu^*, \quad \forall\, E \in \mathcal{B}(D). \tag{4.27}$$

Proof. Since $\int_E \zeta d\mu \leq \int_E |\zeta| d|\mu|$, it suffices to prove the inequality when ζ and μ are non-negative. Let us fix $E \in \mathcal{B}(D)$. By using (4.4), Proposition 4.1.4 (i), and Proposition 4.2.1, we infer

$$\begin{aligned}
\int_E \chi_F d\mu = \mu(E \cap F) &\leq \mathcal{T}_\mu(\mathcal{L}_N(E \cap F)) \\
&\leq \min\{\mathcal{T}_\mu(\mathcal{L}_N(E)), \mathcal{T}_\mu(\mathcal{L}_N(F))\} \\
&\leq \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} \chi_{(-\frac{\mathcal{L}_N(F)}{2}, \frac{\mathcal{L}_N(F)}{2})} d\mu^* \\
&= \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} \chi_F^* d\mu^*, \quad \forall\, F \in \mathcal{B}(\bar{D}).
\end{aligned} \tag{4.28}$$

To this end, let us define

$$A_n = \left\{ \sum_{k=1}^n c_k \chi_{E_k} : c_k \in \mathbb{R}_+ \text{ is strictly increasing with respect to } k, \right.$$
$$\left. \{E_k\}_{1\leq k\leq n} \subseteq \mathcal{B}(\bar{D}) \text{ are pairwise disjoint} \right\}.$$

Then, from (4.28), we have

$$\int_E s_1 d\mu \leq \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} s_1^* d\mu^*, \quad \forall\, s_1 \in A_1.$$

Let us assume

$$\int_E s_M d\mu \leq \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} s_M^* d\mu^*, \quad \forall\, s_M \in A_M, \tag{4.29}$$

and fix $s_{M+1} \in A_{M+1}$. If $\mathcal{L}_N(E) \leq \sum_{k=2}^{M+1} \mathcal{L}_N(E_k)$, then we infer

$$\begin{aligned}
\int_E s_{M+1} d\mu = \int_E \sum_{k=1}^{M+1} c_k \chi_{E_k} d\mu &\leq \int_E \left(\sum_{k=3}^{M+1} c_k \chi_{E_k} + c_2 \chi_{E_1 \cup E_2} \right) d\mu \\
&\leq \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} \left(\sum_{k=3}^{M+1} c_k \chi_{E_k} + c_2 \chi_{E_1 \cup E_2} \right)^* d\mu^* \\
&= \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} s_{M+1}^* d\mu^*,
\end{aligned}$$

where we have utilized (4.29) in the second inequality.

If $\mathcal{L}_N(E) > \sum_{k=2}^{M+1} \mathcal{L}_N(E_k)$, then we deduce

$$\begin{aligned}
\int_E s_{M+1} d\mu &= \sum_{k=1}^{M+1} c_k \mu(E_k \cap E) \\
&= c_1 \mu \left(E \cap \left(\bigcup_{i=1}^{M+1} E_i \right) \right) + \sum_{k=2}^{M+1} (c_k - c_{k-1}) \mu \left(E \cap \left(\bigcup_{i=k}^{M+1} E_i \right) \right) \\
&\leq c_1 \mu(E) + \sum_{k=2}^{M+1} (c_k - c_{k-1}) \mu \left(E \cap \left(\bigcup_{i=k}^{M+1} E_i \right) \right) \\
&\leq c_1 \mathcal{T}_\mu(\mathcal{L}_N(E)) + \sum_{k=2}^{M+1} (c_k - c_{k-1}) \mathcal{T}_\mu \left(\mathcal{L}_N \left(\bigcup_{i=k}^{M+1} E_i \right) \right) \\
&= \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} s_{M+1}^* d\mu^*,
\end{aligned}$$

where we have utilized (4.4) and Proposition 4.1.4 (i) in the second inequality. So, by the principle of mathematical induction, we have

$$\int_E s_n d\mu \leq \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} s_n^* d\mu^*, \quad \forall\, s_n \in A_n,\, n \in \mathbb{N}. \tag{4.30}$$

By utilizing Theorem 1.74 in [29], we can approximate ζ uniformly by a increasing sequence of non-negative simple functions $\{s_n\}$ such that $s_n \in A_n$

and $s_n \leq \zeta$. Furthermore, we deduce $\chi_E s_n$ converges to $\chi_E \zeta$ uniformly and $\chi_E s_n \leq \chi_E \zeta$. By Lebesgue dominated convergence theorem, we infer

$$\lim_{n\to\infty} \int_E s_n d\mu = \lim_{n\to\infty} \int_{\bar{D}} \chi_E s_n d\mu = \int_{\bar{D}} \chi_E \zeta d\mu = \int_E \zeta d\mu. \tag{4.31}$$

On the other hand, by Lemma 4.2.3, it follows from (4.30) that

$$\int_E s_n d\mu \leq \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} s_n^* d\mu^* \leq \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} \zeta^* d\mu^*. \tag{4.32}$$

Whence, from (4.31) and (4.32), the inequality (4.27) follows. □

By examining the proof of Proposition 4.2.5, we have the following result.

Proposition 4.2.6. *Let $\mu \in \mathcal{M}(D)$ and $g \in L^1(D) \cap L^1(D, \mu)$. Then, we have*

$$\int_E g d\mu \leq \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} g^* d\mu^*, \quad \forall\, E \in \mathcal{B}(D). \tag{4.33}$$

Remark 4.2.4. *If $f \in L^1(D)$ and $\mu \sim f$, we can write (4.33) as*

$$\int_E g f d\mathcal{L}_N \leq \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} g^* f^* d\mathcal{L}^1, \quad \forall\, E \in \mathcal{B}(D).$$

Moreover, if $g \in L^\infty(D, \mathcal{L}_N)$, then $g \in L^1(D) \cap L^1(D, \mu)$. So, we have

$$\int_E g f d\mathcal{L}_N \leq \int_{-\frac{\mathcal{L}_N(E)}{2}}^{\frac{\mathcal{L}_N(E)}{2}} g^* f^* d\mathcal{L}^1, \quad \forall\, E \in \mathcal{B}(D),\ f \in L^1(D),\ g \in L^\infty(D, \mathcal{L}_N).$$

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