Geometric symmetric powers in the homotopy categories of schemes over a field

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by

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Abstract

In this PhD thesis, we investigate the lambda-structure of geometric symmetric powers in both the unstable and the stable $\mathbb{A}^1$-homotopy category of schemes over a field. We also establish a comparison between categoric, geometric, homotopy and projector symmetric powers in the rational stable $\mathbb{A}^1$-homotopy category of schemes over a field.

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Introduction

In motivic theory, symmetric powers are an important tool that encode (co)-homological information of motivic spaces. Generally speaking, motivic spaces depend on two coordinates: one simplicial coordinate and one geometric coordinate, i.e. the category of schemes. This suggests the possibility of defining symmetric powers of motivic spaces with a different approach than the categoric ones. In [40], Voevodsky proved a motivic version of the Dold-Thom's theorem. The symmetric powers considered in his work are what we call geometric symmetric powers, as they are induced from the geometric coordinate.

An admissible category [1] is a subcategory of schemes over a base field, containing the affine line and it is closed under finite products, coproducts and quotients of schemes by finite groups. A typical example of an admissible category is the category of quasi-projective schemes over a field. Geometric symmetric powers are left Kan extensions of the symmetric powers of schemes considered in an admissible category [40]. Categoric symmetric powers are the quotients of Cartesian powers of motivic spaces by the action of symmetric groups. A λ-structure on a model category, or on its homotopy category, is a categoric version of a λ-structure on commutative rings. As functors, categoric symmetric powers preserve $\mathbb{A}^1$-weak equivalences, and their left derived functors provide a λ-structure on the pointed and unpointed (unstable) motivic homotopy categories of an admissible category, [13]. The aim of the present work is to develop a systematic study of symmetric powers in the unstable and stable homotopy category of an admissible category over a field k.

Our first goal is to prove that geometric symmetric powers provide a λ-structure on the pointed unstable motivic homotopy category of an admissible category. For this purpose we first consider the projective cofibrant resolution on the category of simplicial Nisnevich sheaves on an admissible category, deduced from the small object argument applied to the class of morphisms resulting by multiplying representable sheaves with the generating cofibrations of the category of simplicial sets. This allows us to deduce that every motivic space is $\mathbb{A}^1$-weak equivalent to a simplicial sheaf, given termwise by coproducts of representable sheaves, as it was shown by Voevodsky in the context of radditive functors, see [40, 41]. The key point is that geometric symmetric powers

[1] f-admissible in [40].

of morphisms of simplicial sheaves that are directed colimits of termwise coprojections have canonical filtrations, called Künneth towers, and they provide a λ-structure on the motivic homotopy category. This gives the following result (Theorem 4.1.4 in the text):

> *The left derived geometric symmetric powers provide a λ-structure on the pointed unstable motivic homotopy category of an admissible category of schemes over a field.*

On the other hand, in both the unstable and the stable case, there is a natural transformation from the categoric symmetric power Sym^n to the geometrical symmetric power Sym^n_g. Let E be a functor from an admissible category to the unstable (or stable) $\mathbb{A}^1$-homotopy category on an admissible category. An interesting problem is to investigate whether the canonical morphisms $\vartheta^n_X : \mathrm{Sym}^n E(X) \to \mathrm{Sym}^n_g E(X)$ are isomorphisms for all schemes X in an admissible category. It turns out that, in the unstable case, ϑ^n_X is not always an isomorphism, for example this is the case when X is the 2-dimensional affine space $\mathbb{A}^2$ and $n = 2$, cf. Proposition 4.1.12. Our second goal is to show that these canonical morphisms become isomorphisms in the rational stable $\mathbb{A}^1$-homotopy category of schemes. However, the same result is not true on the stable $\mathbb{A}^1$-homotopy category of schemes with integral coefficients (see Remark 3.3.14).

Let us explain our approach towards the second goal. The rationalization of a stable homotopy category causes the loss of information of torsion objects. However, it allows us to think of a rational stable homotopy category as a derived category of chain complexes, and the latter is, philosophically, more accessible to understand. Morel predicted that rational stable $\mathbb{A}^1$-homotopy category of schemes is equivalent to the triangulated category of unbounded motives with rational coefficients, cf. [29].

An important ingredient to be used in this text is the notion of transfer of a morphism. This notion appears naturally in algebraic topology. For instance, let us consider a positive integer n and an n-sheeted covering $\pi\colon \tilde{X} \to X$. This covering induces a homomorphism of cohomology groups $\pi^*\colon H^r(X;\mathbb{Z}) \to H^r(\tilde{X};\mathbb{Z})$ for $r \in \mathbb{N}$. A transfer for π^* is a homomorphism $\mathrm{tr}\colon H^r(\tilde{X};\mathbb{Z}) \to H^r(X;\mathbb{Z})$ such that the composite $\mathrm{tr} \circ \pi^*$ is the multiplication by n. Voevodosky proved the existence of transfers for morphisms of qfh-sheaves induced by finite surjective morphisms of normal connected schemes. As a result, this implies the existence of transfers for morphisms of qfh-motives induced by such finite morphisms of schemes, see [39]. We use this notion in order to get transfers for the morphisms in the rational stable $\mathbb{A}^1$-homotopy category which are induced by the canonical morphism $X^n \to X^n/\Sigma_n$ for X a quasi-projective scheme.

Let T be the projective line $\mathbb{P}^1$ pointed at ∞, and let $E_{\mathbb{Q}}$ be the canonical functor from the category of quasi-projective schemes over a field k to the rational stable $\mathbb{A}^1$-homotopy category of T-spectra. We denote by Sym^n_T the nth fold categoric symmetric power on the category of symmetric T-spectra. Since the rational stable homotopy category of schemes is pseudo-abelian, one can use projectors in order to define projector symmetric powers, denoted by $\mathrm{Sym}^n_{\mathrm{pr}}$. As a result, we obtain that if -1 is a sum of squares then the categoric, geometric and projector symmetric powers of a quasi-projective scheme are isomorphic in rational stable $\mathbb{A}^1$-homotopy category. More precisely, our result is the following (Theorem 4.3.20 in the text):

Let k be a field such that -1 is a sum of squares in it. Then, for any quasi-projective k-scheme X, we have the following isomorphisms

$$L\mathrm{Sym}^n_T E_{\mathbb{Q}}(X) \simeq E_{\mathbb{Q}}(\mathrm{Sym}^n X) \simeq \mathrm{Sym}^n_{\mathrm{pr}} E_{\mathbb{Q}}(X)\,.$$

Another type of symmetric power is the nth fold homotopy symmetric power of a symmetric T-spectrum, defined as a homotopy quotient of the nth fold smash product of this spectrum by the symmetric group Σ_n; we denote by $\mathrm{Sym}^n_{h,T}$ the corresponding endofunctor on the category of motivic symmetric T-spectra. There are natural transformations $\mathrm{Sym}^n_{h,T} \to \mathrm{Sym}^n_T$ for $n \in \mathbb{N}$. It turns out that they induce a morphism of λ-structures on the category of symmetric T-spectra, and it becomes an isomorphism in the stable homotopy category, [12]. Consequently, for a quasi-projective k-scheme X, the nth fold homotopy symmetric power $\mathrm{Sym}^n_{h,T} E_{\mathbb{Q}}(X)$ is isomorphic to $L\mathrm{Sym}^n_T E_{\mathbb{Q}}(X)$. Thus we get a comparison of four types of symmetric powers in the rational stable $\mathbb{A}^1$-homotopy category.

In this thesis, we construct a stable geometric symmetric power $\mathrm{Sym}^n_{g,T}$ having the property that the composite $\mathrm{Sym}^n_{g,T} \circ \Sigma^\infty_T$ is isomorphic to $\Sigma^\infty_T \circ \mathrm{Sym}^n_g$, where Σ^∞_T is the T-suspension functor, see Section 3.3 for a detailed exposition. This property allows to deduce that $\mathrm{Sym}^n_{g,T}$ preserves stable $\mathbb{A}^1$-weak equivalences between T-spectra that are the T-suspension of nice motivic spaces, but this fact does not suffice to deduce the existence of the left derived functor of $\mathrm{Sym}^n_{g,T}$ for $n > 1$. This problem will remain open in the text. On the other hand, there is a natural transformation $\mathrm{Sym}^n_T \to \mathrm{Sym}^n_{g,T}$ for every $n \in \mathbb{N}$. Assuming the existence of left derived functors of the stable geometric symmetric powers, we show that the endofunctors $L\mathrm{Sym}^n_{g,T}$, for $n \in \mathbb{N}$, induce a λ-structure on the stable motivic homotopy category (Theorem 4.2.9) and the natural transformations $L\mathrm{Sym}^n_T \to L\mathrm{Sym}^n_{g,T}$ induce a morphism of λ-structures (Theorem 4.2.13).

Although in this thesis we are limited to work only over a base field, our constructions might be generalized to a broader class of nice base schemes. It would be interesting to investigate how to construct categoric (resp. geometric) symmetric powers in a more general framework, namely on the premotivic categories (resp. premotivic categories with geometric sections) defined in [6]; but we leave this question for a future project.

Organization of the thesis

In Chapter 1, we recall useful tools of homotopical algebra. We also outline important results on the category of symmetric spectra developed in [19]. In Chapter 2, we give a survey of both the unstable and the stable $\mathbb{A}^1$-homotopy theory of schemes over a field, [30]. Here, we also study simplicial radditive functors, [41]. Chapter 3 contains the essential part of the thesis. In it we construct Künneth towers of geometric symmetric powers of motivic spaces in both the unstable and stable set-up. In Chapter 4, we present our main results: Theorem 4.1.4 (for the unstable case) and Theorem 4.3.20 (for the stable case).

Chapter 1

Categoric and homotopic aspects

This chapter contains preliminary materials of abstract homotopical algebra which are the basis and foundation of the next chapters.

1.1 Rudiments of Model categories

According to D. Quillen, a "model category" means a category of "models" for a homotopy category. The original reference for model categories is the well known book titled "Homotopical algebra" published in 1967, see [32].

1.1.1 Preliminaries

In this section, we recall basics on model categories, their fundamental properties, such as, lifting properties, retract arguments, etc.

Definition 1.1.1. Let $\mathscr{C}$ be a category. We denote by $\operatorname{Map}\mathscr{C}$ the category whose objects are morphisms of $\mathscr{C}$ and whose morphisms are commutative squares. The *domain* and *codomain* functors

$$\operatorname{dom}, \operatorname{codom} : \operatorname{Map}\mathscr{C} \to \mathscr{C} ,$$

assign a morphism in $\mathscr{C}$, respectively, to its domain and codomain, that is,

$$\operatorname{dom}(X \xrightarrow{f} Y) = X , \qquad \operatorname{codom}(X \xrightarrow{f} Y) = Y .$$

If a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[\psi]{} & Y' \end{array}$$

is a morphism in $\operatorname{Map}\mathscr{C}$ from $f\colon X \to Y$ to $f'\colon X' \to Y'$, then we set

$$\operatorname{dom}\left(\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ Y & \xrightarrow[\psi]{} & Y' \end{array}\right) := (\, X \xrightarrow{\varphi} X' \,)\,, \quad \operatorname{codom}\left(\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ Y & \xrightarrow[\psi]{} & Y' \end{array}\right) :=(\, Y \xrightarrow{\psi} Y' \,)\,.$$

Definition 1.1.2. A *functorial factorization* on a category $\mathscr{C}$ is a pair (α, β) of functors α and β from $\operatorname{Map}\mathscr{C}$ to itself, such that

(1) $\operatorname{dom} \circ \alpha = \operatorname{dom}\,,$

(2) $\operatorname{codom} \circ \beta = \operatorname{codom}\,,$

(3) $\operatorname{codom} \circ \alpha = \operatorname{dom} \circ \beta$, and

(4) $\beta \circ \alpha = \mathrm{id}_{\operatorname{Map}\mathscr{C}}$.

In other terms, for every morphism f in $\mathscr{C}$ we have a commutative triangle,

$$\begin{array}{ccc} \operatorname{dom}\alpha(f) = \operatorname{dom} f & \xrightarrow{\quad f \quad} & \operatorname{codom}\beta(f) = \operatorname{codom} f \\ & {\scriptstyle \alpha(f)}\searrow \qquad \nearrow{\scriptstyle \beta(f)} & \\ & \operatorname{codom}\alpha(f) = \operatorname{dom}\beta(f) & \end{array}$$

or simply, $\alpha(f)$ and $\beta(f)$ are composable morphisms, and $f = \beta(f) \circ \alpha(f)$.

Definition 1.1.3. We say that a morphism f in $\mathscr{C}$ is a *retract* of a morphism g in $\mathscr{C}$ if f is a retract of g as objects in $\operatorname{Map}\mathscr{C}$, in other words, there exists a commutative diagram of the form

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

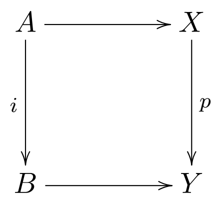
where the horizontal composites are identities.

Example 1.1.4. If p is the retraction of a morphism $s\colon A \to B$ in a category [26, p. 19], then the diagram

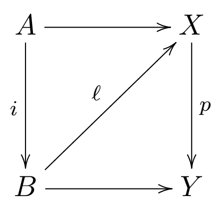
$$\begin{array}{ccccc} A & \xrightarrow{s} & B & \xrightarrow{p} & A \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle s\circ p} & & \big\downarrow{\scriptstyle s} \\ B & = & B & = & B \end{array}$$

displays s as a retract of $s \circ p$.

Definition 1.1.5. Suppose that $i\colon A \to B$ and $p\colon X \to Y$ are two morphisms in $\mathscr{C}$. We say that i has the *left lifting property* with respect to p, or p has the *right lifting property* with respect to i, if for any commutative square,



there exists a morphism $\ell\colon B \to X$, called *lifting*, such that the following diagram



is commutative.

Definition 1.1.6. Let I be a class of morphism in a category $\mathscr{C}$. A morphism is called:

(1) *I-injective*, if it has the right lifting property with respect to every morphism in I. We denote the class of I-injective morphisms by I-inj.

(2) *I-projective*, if it has the left lifting property with respect to every morphism in I. We denote the class of I-projective morphisms by I-proj.

(3) *I-cofibration*, if it is a morphism in (I-inj)-proj. We denote the class of I-cofibrations by I-cof.

Remark 1.1.7. From the definition it follows that for any two classes of morphisms I and J, one has the following:

(i) $I \subset I$-cof.

(ii) If $I \subset J$, then we have two inclusions I-inj $\supset J$-inj and I-proj $\supset J$-proj. Hence, one has I-cof $\subset J$-cof.

Definition 1.1.8. A *model category* is a category $\mathscr{C}$ provided of a *model structure*, that is, three classes of morphisms in $\mathscr{C}$:

(1) a class of *weak equivalences*,

(2) a class of *fibrations*,

(3) a class of *cofibrations*,

and two functorial factorizations (α, β), (γ, δ) satisfying the following axioms:

(MC1) *(limits)* $\mathscr{C}$ is complete and cocomplete.

(MC2) *(2-out-of-3)* If f and g are two composable morphisms in $\mathscr{C}$ such that two of f, g and $g \circ f$ are weak equivalences, then so is the third.

(MC3) *(retracts)* If f and g are morphisms in $\mathscr{C}$ such that f is a retract of g and g is a weak equivalence, cofibration or fibration, then so is f.

(MC4) *(lifting)* Trivial cofibrations have the left lifting property respect to fibrations, and trivial fibrations have the right lifting property respect to cofibrations.

(MC5) *(factorization)* Every morphism f in $\mathscr{C}$ has two factorizations:

$$f = \beta(f) \circ \alpha(f) \,,$$

$$f = \delta(f) \circ \gamma(f) \,,$$

where

(i) $\alpha(f)$ is a cofibration,

(ii) $\beta(f)$ is a trivial fibration,

and

(iii) $\gamma(f)$ is a trivial cofibration,

(iv) $\delta(f)$ is a fibration.

Remark 1.1.9. A category may have more than one model structure.

Remark 1.1.10. A model category has an initial and a terminal object, because it is complete and cocomplete. In fact, its initial object (resp. terminal object) is the colimit (resp. limit) of the empty diagram.

Example 1.1.11. The Quillen model structure on the category of simplicial sets $\Delta^{\mathrm{op}}\mathscr{S}ets$ has the following structure:

(1) a cofibration is a monomorphism,

(2) a weak equivalence is a weak homotopy equivalence, i.e. a morphism f such that its geometric realization $|f|$ induces bijections of homotopy groups, see [15, p. 352].

(3) a fibration is a Kan fibration, i.e. a morphism that has the right lifting property with respect to all horns $\Lambda^r[n] \hookrightarrow \Delta[n]$ for $n > 0$ and $0 \leq r \leq n$.

Example 1.1.12. The category of simplicial (pre-)sheaves has various model structures, see Section 2.1.2.

Example 1.1.13. The category of symmetric spectra has a projective model structure (see Theorem 1.4.30) and a stable model structure (see page 62).

Terminology. The initial object of a category will be denoted by $\emptyset$ (sometimes by 0) and the terminal object by $*$ (sometimes by pt or by 1) .

Definition 1.1.14. Let $\mathscr{C}$ be a model category and let X be an object of $\mathscr{C}$. We say that X is *cofibrant* if the morphism $\emptyset \to X$ is a cofibration, and X is *fibrant* if the morphism $X \to *$ is a fibration.

Lemma 1.1.15 (Ken Brown's lemma)**.** *Suppose that $\mathscr{C}$ is a model category and $\mathscr{D}$ is a category with a subcategory of weak equivalences satisfying the* 2*-out-of-*3 *axiom. If $F\colon \mathscr{C} \to \mathscr{D}$ is a functor which takes trivial cofibration between cofibrant objects to weak equivalences, then F takes all weak equivalences between cofibrant objects to weak equivalences.*

Proof. See [18, Lemma 1.1.12] □

Pointed model categories

Definition 1.1.16. A category with an initial $\emptyset$ and terminal object $*$ is called *pointed* if the canonical morphism $\emptyset \to *$ is an isomorphism.

Example 1.1.17. Additive categories are pointed, the zero object is both an initial and a terminal object.

Let $\mathscr{C}$ be a category. We denote

$$\mathscr{C}_* := * \downarrow \mathscr{C}$$

the category whose objects are morphisms $* \xrightarrow{v} X$ of $\mathscr{C}$. As in topology, it is sometimes denoted by (X, v) an element of $\mathscr{C}_*$, and call it object X with *base point* v. From the definition, it follows that the category $\mathscr{C}_*$ is pointed.

Suppose that $\mathscr{C}$ is a category with a terminal object $*$. For every object X of a category $\mathscr{C}$, we set

$$X_+ := X \amalg * .$$

We denote by $\mathscr{C}_+$ the full subcategory of $\mathscr{C}_*$ generated by objects of the form X_+ for all objects X in $\mathscr{C}$. Let us denote by

$$(-)_+ : \mathscr{C} \to \mathscr{C}_*$$

the composition of the functor $\mathscr{C} \to \mathscr{C}_+$, given by $X \mapsto X_+$, with the full embedding $\mathscr{C}_+ \hookrightarrow \mathscr{C}_*$. The functor $(-)_+$ is left adjoint to the forgetful functor $U \colon \mathscr{C}_* \to \mathscr{C}$,

$$(-)_+ : \mathscr{C} \rightleftarrows \mathscr{C}_* : U \tag{1.1}$$

If $\mathscr{C}$ is a pointed category, then the functors $(-)_+$ and U define an equivalence of categories between $\mathscr{C}$ and $\mathscr{C}_*$.

Lemma 1.1.18. *Let $\mathscr{C}$ be a model category. Then, the model structure on $\mathscr{C}$ induces a model structure on $\mathscr{C}_*$, where a morphism f in $\mathscr{C}_*$ is a cofibration (fibration, weak equivalence) if and only if $U(f)$ is a cofibration (fibration, weak equivalence) in $\mathscr{C}$.*

Proof. Notice that axioms (MC1), (MC2) and (MC3) for $\mathscr{C}_*$ follow immediately from the corresponding axioms of $\mathscr{C}$. To prove the lifting axiom (MC4), we give a commutative square in $\mathscr{C}_*$

$$\begin{array}{ccc} (A,a) & \longrightarrow & (X,x) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ (B,b) & \longrightarrow & (Y,y) \end{array} \tag{1.2}$$

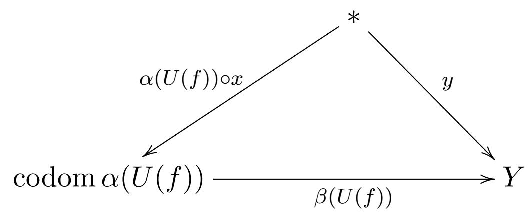
where i is a trivial cofibration and p is a fibration (the other case is similar). By the axiom (MC4) on $\mathscr{C}$, the square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle U(i)}\downarrow & & \downarrow{\scriptstyle U(p)} \\ B & \longrightarrow & Y \end{array}$$

has a lifting, say $\ell \colon B \to X$. We observe that, by diagram chasing, we have $\ell \circ b = x$. Hence, ℓ induces a morphism of pointed objects $(B,b) \to (X,x)$ which is a lifting of the square (1.2). Finally, let us prove the factorization axiom (MC5). Let (α,β) be a functorial factorization of $\mathscr{C}$. We define a functorial factorization (α_*,β_*) of $\mathscr{C}_*$ as follows. For a morphism $f \colon (X,x) \to (Y,y)$ in $\mathscr{C}_*$, we define $\alpha_*(f)$ to be the commutative triangle

$$\begin{array}{ccc} & * & \\ {\scriptstyle x}\swarrow & & \searrow{\scriptstyle \alpha(U(f))\circ x} \\ X & \xrightarrow[\alpha(U(f))]{} & \operatorname{codom}\alpha(U(f)) \end{array}$$

and we define $\beta_*(f)$ to be the commutative triangle



Since $\beta(U(f)) \circ \alpha(U(f)) = U(f)$ for every morphism f in $\mathscr{C}_*$, the pair (α_*, β_*) is a functorial factorization of $\mathscr{C}_*$. □

1.1.2 Cellular complexes

We start this section recalling some basics on ordered sets, ordinals and cardinals.

Definition 1.1.19.

(1) A *preorder* on a set is a binary relation that is reflexive and transitive. A *preordered set* is a set provided of a preorder.

(2) A *partial order* on a set is a binary relation that is reflexive, antisymmetric and transitive. An *ordered set* is a set provided of a partial order.

(3) A partially ordered set S, say with a order $\leq$, is called *totally ordered*, if every pair of elements $(a, b) \in S \times S$ is comparable, that is, $a \leq b$ or $b \leq a$.

(4) A totally ordered set S is called *well-ordered*, if S has a minimum element, that is, an element $b \in S$ such that $b \leq a$ for all $a \in S$.

Definition 1.1.20. A preordered set S is a *directed* set if every pair of elements has an upper bound, i.e. for every pair of elements $a, b \in S$ there exists an element c such that $a \leq c$ and $b \leq c$.

Example 1.1.21. Totally ordered sets are directed sets, but partially ordered sets are not necessarily directed sets.

Theorem 1.1.22 (Zermelo's Well-Ordering Theorem)**.** *Every nonempty set can be well-ordered.*

Proof. This theorem is equivalent to the Axiom of Choice. The reader may consult [24, Th. 5.1]. □

Definition 1.1.23. A set A is called *transitive*, if every element of A is a subset of A.

Example 1.1.24.

(1) By vacuity, $\emptyset$ is transitive.

(2) The sets $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$ are transitive.

(3) The set $\{\{\emptyset\}\}$ is not transitive, because $\{\emptyset\}$ is an element of $\{\{\emptyset\}\}$ but it is not a subset of $\{\{\emptyset\}\}$.

Definition 1.1.25. A set is called *ordinal*, if it is transitive and well-ordered by the set-membership order $\in$.

Example 1.1.26.

(1) $0{:=}\emptyset$ is an ordinal.

(2) The sets $1{:=}\{\emptyset\}$ and $2{:=}\{\emptyset, \{\emptyset\}\}$ are ordinals.

(3) If α is an ordinal, then $\alpha + 1{:=}\alpha \cup \{\alpha\}$ is an ordinal.

(4) The set $\big\{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\big\}$ is transitive but it is not an ordinal, because $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are not comparable by $\in$.

Definition 1.1.27. For any two ordinals α and β, we denote $\alpha < \beta$ to mean that $\alpha \in \beta$, and by $\alpha \leq \beta$ to mean that $\alpha \in \beta$ or $\alpha = \beta$.

Proposition 1.1.28. *We have the following statements:*

(*a*) *Every ordinal α is equal to the set of ordinals β such that $\beta < \alpha$.*

(*b*) *If α is an ordinal and β is a set such that $\beta \in \alpha$, then β is an ordinal.*

(*c*) *If $\alpha \neq \beta$ are two ordinals such that $\alpha \subset \beta$, then $\alpha \in \beta$.*

(*d*) *Let α and β be two ordinals. If $f\colon \alpha \to \beta$ is an isomorphism of ordered sets, then $\alpha = \beta$ and $f = \mathrm{id}$.*

(*e*) *Let α and β be two ordinals. Then exactly one of the following cases holds: $\alpha = \beta$, $\alpha < \beta$ or $\beta < \alpha$.*

(*f*) *If A is a set of ordinals, then the union of the elements of A, usually denoted by $\sup A$ or by $\bigcup A$, is an ordinal.*

Proof. See [24]. □

Definition 1.1.29. By the Zermelo's Well-Ordering Theorem 1.1.22, every set is in bijection with a certain ordinal. The *cardinal* of a set A is the smallest ordinal that is bijective to A. The cardinal of A is usually denoted by $|A|$.

Definition 1.1.30. An ordinal κ is called *cardinal*, if $|\kappa| = \kappa$; in other words, if κ is not bijective to any ordinal strictly less than κ.

Remark 1.1.31. Notice that there is no redundancy in Definition 1.1.29 and Definition 1.1.30.

Example 1.1.32.

(1) Finite ordinals are cardinals.

(2) The ordinal ω is a cardinal, it is usually denote by $\aleph_0$.

(3) The ordinal $\omega + 1$ is not a cardinal. Indeed, we have $\omega + 1 = \omega \cup \{\omega\}$, hence the function $f\colon \omega \cup \{\omega\} \to \omega$ given by

$$f(\beta) = \begin{cases} \beta + 1\,, & \text{if } \beta < \omega\,, \\ 0\,, & \text{if } \beta = \omega\,, \end{cases}$$

is a bijection, but ω is strictly less than $\omega + 1$.

Definition 1.1.33. Let κ be a cardinal. An ordinal λ is *κ-filtered*, if:

(1) it is a limit ordinal, and

(2) it satisfies the following property: if A is a set such that $A \subset \lambda$ and $|A| \leq \kappa$, then $\sup A < \lambda$.

Remark 1.1.34. The condition (2) in the previous definition implies that a κ-filtered ordinal is necessarily a limit ordinal.

Definition 1.1.35. An infinite cardinal κ is called *regular*, if it satisfies the following axiom: for every set A such that $|A| < \kappa$ and for every family $\{S_a\}_{a \in A}$ such that $|S_a| < \kappa$, one has $|\bigcup_{a \in A} S_a| < \kappa$.

Example 1.1.36. If κ is a finite cardinal, then the countable ordinal ω is κ-filtered.

Proposition 1.1.37. *If κ is infinite and successor cardinal, then κ is regular.*

Proof. See [17, Proposition 10.1.14]. □

Definition 1.1.38. Suppose $\mathscr{C}$ is a cocomplete category and λ is an ordinal. A *λ-sequence* in $\mathscr{C}$ is a colimit-preserving functor $X\colon \lambda \to \mathscr{C}$ in the following sense: for all limit ordinal $\gamma < \lambda$, the induced morphism

$$\operatorname{colim}_{\beta < \gamma} X_\beta \to X_\gamma$$

is an isomorphism. The morphism $X_0 \to \operatorname{colim}_{\beta < \lambda} X_\beta$ is called *transfinite composition* of the λ-sequence X.

Definition 1.1.39. Let I a class of morphisms of a cocomplete category $\mathscr{C}$ and let κ be a cardinal.

(1) An object A of $\mathscr{C}$ is called *κ-small relative* to I, if for all κ-filtered ordinals λ and λ-sequences

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots,$$

such that every morphism $X_\beta \to X_{\beta+1}$ is in I for $\beta+1 < \lambda$, the induced morphism of sets

$$\operatorname{colim}_{\beta<\lambda} \mathrm{Hom}_{\mathscr{C}}(A, X_\beta) \to \mathrm{Hom}_{\mathscr{C}}(A, \operatorname{colim}_{\beta<\lambda} X_\beta)$$

is bijective.

(2) An object $A \in \mathscr{C}$ is called *small relative* to I if it is *κ-small relative* to I for some cardinal κ.

(3) An object $A \in \mathscr{C}$ is called *small*, if it is small relative to the class to all morphisms of $\mathscr{C}$.

Definition 1.1.40. Let $\mathscr{C}$ be a cocomplete category and let I be a class of morphisms of $\mathscr{C}$.

(1) An object A of $\mathscr{C}$ is called *finite relative* to I, if there is a finite cardinal κ such that A is κ-small relative to I.

(2) An object A of $\mathscr{C}$ is called *finite*, if it is finite relative to the class of all morphisms of $\mathscr{C}$.

Example 1.1.41.

(i) Every set is small in the category of sets.

(ii) In the category of sets, a set is finite (in the sense of Definition 1.1.40) if and only if it is a finite set, i.e. a set with finitely many elements.

(iii) In the category of topological spaces $\mathscr{T}op$, a compact topological space may not be small: the space $X = \{0, 1\}$ with the trivial topology is compact but not small in $\mathscr{T}op$. This counterexample was given by Don Stanley (see Errata of [18]).

Definition 1.1.42. Let I be a set of morphisms in a cocomplete category $\mathscr{C}$. A morphism f in $\mathscr{C}$ is a *relative I-cell complex* if there exists an ordinal λ and a λ-sequence $X\colon \lambda \to \mathscr{C}$ such that f is the transfinite composition of X and such that, for each ordinal β with $\beta + 1 < \alpha$, there is a pushout square

$$\begin{array}{ccc} C_\beta & \longrightarrow & X_\beta \\ {\scriptstyle g_\beta}\downarrow & & \downarrow \\ D_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

such that $g_\beta \in I$. The class of relative I-cell complexes is denoted by I-cell. We say that an object $A \in \mathscr{C}$ is an *I-cell complex* if the morphism $0 \to A$ is a relative I-cell complex.

Lemma 1.1.43. *Suppose that I is a class of morphisms in a cocomplete category $\mathscr{C}$. We have the following assertions:*

(a) I-inj *and* I-proj *are closed under compositions.*

(b) I-inj *and* I-proj *are closed under retracts.*

(c) I-proj *is closed under pushouts and* I-inj *is closed under pullbacks.*

(d) I-proj *is closed under transfinite compositions.*

(e) $I\text{-cell} \subset I\text{-cof}$.

(f) *I-cell is closed under transfinite compositions.*

(g) *Any pushout of coproducts of morphisms of I is in* I-cell.

Proof. Each statement follows from the definitions, see [17] or in [18]. □

Proposition 1.1.44. *Let $\mathscr{C}$ be a category cocomplete and let I be a set of morphisms in $\mathscr{C}$. Let κ be a regular cardinal such that the domains of morphisms of I are κ-small-relative to* I-cell. *Then there exists a functorial factorization (γ, δ) on $\mathscr{C}$ such that for every morphism f in $\mathscr{C}$, we can write*

$$f = \delta(f) \circ \gamma(f)$$

where $\gamma(f)$ is a transfinite composition of a κ-sequence of pushouts of coproducts of elements in I, and $\delta(f)$ in I-inj.

Proof. The transfinite induction allows one to construct a suitable functorial factorization, and the regularity property on the cardinal κ permits to obtain the required properties of the factorization, see [17]. □

We following definition is due to D.M. Kan.

Definition 1.1.45. If $\mathscr{C}$ is a category and I is a set of morphisms in $\mathscr{C}$, we say that I *permits the small object argument*, if the domain of every element of I is small relative to I-cell.

Theorem 1.1.46 (The small object argument)**.** *Let $\mathscr{C}$ be a category cocomplete and let I be a set of morphisms in $\mathscr{C}$. Suppose that I permits the small object argument. Then there exists a functorial factorization (γ, δ) on $\mathscr{C}$ such that for every morphism f in $\mathscr{C}$, we can write*

$$f = \delta(f) \circ \gamma(f)$$

with $\gamma(f)$ in I-cell *and $\delta(f)$ in* I-inj.

Proof. By hypothesis, every object in dom (I) is small-relative to I-cell, then for every A in dom (I), there is a cardinal κ_A such that A is κ_A-small relative to I-cell. We consider the cardinal,

$$\kappa := \bigcup_{A \in \mathrm{dom}\,(I)} \kappa_A \,.$$

Since $\kappa_A < \kappa$ for every A in dom (I), every object A in dom (I) is κ-small relative to I-cell. Hence, by Proposition 1.1.44, there exists a functorial factorization (γ, δ) on $\mathscr{C}$ such that for every morphism f in $\mathscr{C}$, we can write

$$f = \delta(f) \circ \gamma(f)$$

with $\gamma(f)$ is a transfinite composition of a κ-sequence of pushouts of coproducts of elements in I, and $\delta(f)$ in I-inj. In particular, $\gamma(f)$ is in I-cell, this proves the theorem. □

Corollary 1.1.47. *Let I be a set of morphism in a cocomplete category $\mathscr{C}$. Suppose that I permits the small object argument. Then every morphism $f\colon A \to B$ in I-*cof*, there is a morphism $g\colon A \to C$ in I-*cell *such that f is a retract of g by a morphism which fixes A, that is, there is commutative diagram*

$$\begin{array}{ccccc} A & = & A & = & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & C & \longrightarrow & B \end{array}$$

where the horizontal composites are the identities.

Proof. See [18, Corollary 2.1.15]. □

1.1.3 Cofibrantly generated model categories

In practice, most of the interesting model categories have a class of cofibrations and a class of trivial cofibrations that are generated by sets of morphisms in the sense of the following definition.

Definition 1.1.48. A model category $\mathscr{C}$ is called *cofibrantly generated*, if there are two sets I and J of morphisms of $\mathscr{C}$ such that we have the following axioms:

(1) I permit the small object argument.

(2) J permit the small object argument.

(3) The class of fibrations in $\mathscr{C}$ is J-inj.

(4) The class of trivial fibrations in $\mathscr{C}$ is I-inj.

The set I is called *set of generating cofibrations* and the set J is called *set of generating trivial cofibrations* of $\mathscr{C}$.

Example 1.1.49. In the category of topological spaces $\mathscr{T}op$, the sets

$$I = \left\{S^{n-1} \hookrightarrow D^n \mid n \geq 0\right\} ,$$
$$J = \left\{[0,1]^{n-1} \times \{0\} \hookrightarrow [0,1]^n \mid n \geq 1\right\}$$

generate a model structure, see [32], [18] or [8]. Here, $S^{n-1} \hookrightarrow D^n$ is the inclusion of the $(n-1)$-dimensional sphere into the n-dimensional unit disc. In $\Delta^{\mathrm{op}}\mathscr{S}ets$, the sets

$$I = \{\partial\Delta[n] \hookrightarrow \Delta[n] \,|\, n \geq 0\} ,$$
$$J = \{\Lambda^r[n] \hookrightarrow \Delta[n] \,|\, n > 0,\, 0 \leq r \leq n\}$$

generate the Quillen model structure on the category of simplicial sets, see [32], [18], or [11]

Proposition 1.1.50. *Suppose $\mathscr{C}$ is a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J. Then, the following conditions are satisfied:*

(a) *The cofibrations form the class I-*cof*.*

(b) *Every cofibration is a retract of a relative I-*cell *complex.*

(c) *The domains of morphisms in I are small relative to the cofibrations.*

(d) *The trivial cofibrations form the class J-*cof*.*

(e) *Every trivial cofibration is a retract of a relative J-*cell *complex.*

(f) *The domains of morphisms in J are small relative to the trivial cofibrations.*

Proof. See [17] or [18]. □

The following theorem is known as the "recognition theorem", which gives us a necessary and sufficient condition on a complete and cocomplete category to be a cofibrantly generated model category.

Theorem 1.1.51 (Recognition theorem)**.** *Suppose $\mathscr{C}$ is a complete and cocomplete category and suppose that W is a class of morphisms in $\mathscr{C}$ and I, J are two sets of morphisms of $\mathscr{C}$. Then there exists a cofibrantly generated model structure on $\mathscr{C}$, with I as the set of generating cofibrations, J as the set of generating trivial cofibrations and W as the class of weak equivalences, if and only if the following conditions are satisfied:*

(1) *The class W has the 2-out-of-3 property (MC2) and it is closed under retracts (MC3).*

(2) *I permit the small object argument.*

(3) *J permit the small object argument.*

(4) $J\text{-cell} \subset W \cap I\text{-cof}$.

(5) $I\text{-inj} \subset W \cap J\text{-inj}$.

(6) *Either $W \cap I\text{-cof} \subset J\text{-cof}$ or $W \cap J\text{-inj} \subset I\text{-inj}$.*

Proof. See [18, Theorem 2.1.19]. □

The following lemma, due to A. Joyal, has sometimes a practical use, as it implies the lifting axiom (MC4) of Definition 1.1.8 in a category satisfying some axioms of lifting properties and functorial factorization .

Lemma 1.1.52 (Joyal's trick)**.** *Suppose that $\mathscr{C}$ is a category with a class of weak equivalences, a class of fibrations and a class of cofibrations satisfying axioms (MC1) and (MC2), and in addition, suppose that one has the following properties:*

(1) *The cofibrations are stable by compositions and pushouts.*

(2) *The fibrations have the right lifting property with respect to trivial cofibration.*

(3) *All morphism f can be functorially factored as $f = p \circ i$, with p a trivial fibration and i a cofibration.*

Then, the axiom (MC4) is also satisfied for $\mathscr{C}$.

Proof. See [21]. □

1.1.4 Homotopy categories

In topology, the classification of topological spaces up to homeomorphisms is considered as a difficult problem. However, the notion of homotopy provides a coarser but a clearer classification of such spaces. The homotopy category of a model category is the category resulting by inverting the weak equivalences. A generalization to model categories of the celebrated Whitehead's theorem asserts that a weak equivalence between fibrant-cofibrant objects is a homotopy equivalence.

Definition 1.1.53. Let $\mathscr{C}$ be a category and let $\mathcal{W}$ be a class of morphisms in $\mathscr{C}$. A *localization* of $\mathscr{C}$ with respect to $\mathcal{W}$ is a category $\mathscr{C}[\mathcal{W}^{-1}]$ together with a functor $\gamma\colon \mathscr{C} \to \mathscr{C}[\mathcal{W}^{-1}]$ such that

(1) for every $f \in \mathcal{W}$, the morphism $\gamma(f)$ is an isomorphism, and

(2) if $\mathscr{D}$ is another category and $\xi \colon \mathscr{C} \to \mathscr{D}$ is a functor such that $\xi(f)$ is an isomorphism for every $f \in \mathcal{W}$, then there is a unique functor $\delta \colon \mathscr{C}[\mathcal{W}^{-1}] \to \mathscr{D}$ such that we have a commutative diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\ \xi\ } & \mathscr{D} \\ & {\scriptstyle\gamma}\searrow \quad \nearrow{\scriptstyle\delta} & \\ & \mathscr{C}[\mathcal{W}^{-1}] & \end{array}$$

Theorem 1.1.54. *If $\mathscr{C}$ is a model category with a class of weak equivalences $\mathcal{W}$, then the localization of $\mathscr{C}$ with respect to $\mathcal{W}$ exists.*

Proof. See [17, Theorem 8.3.5]. □

Definition 1.1.55. If $\mathscr{C}$ is a model category with a class of weak equivalences $\mathcal{W}$. We denote $\mathscr{C}[\mathcal{W}^{-1}]$ by $\operatorname{Ho}\mathscr{C}$ and call it the *homotopy category* of $\mathscr{C}$.

Let $\mathscr{C}$ be a a model category. We denote by $\mathscr{C}_c$ (resp. $\mathscr{C}_f$, $\mathscr{C}_{cf}$) the full subcategory of cofibrant (resp. fibrant, cofibrant and fibrant) objects of $\mathscr{C}$. A morphism $f \colon X \to Y$ in $\mathscr{C}_c$ (resp. $\mathscr{C}_f$, $\mathscr{C}_{cf}$) is a *weak equivalence* if it is a weak equivalence in $\mathscr{C}$. We shall construct two natural functors

$$Q, R \colon \mathscr{C} \to \mathscr{C}\,,$$

called cofibrant (resp. fibrant) *replacement functor*. They are constructed as follows. For any X object of $\mathscr{C}$, we consider the morphism $\emptyset \to X \to *$. Since $\mathscr{C}$ is a model category, we have two functorial factorizations (α, β) and (γ, δ), see 1.1.2. Hence, defining

$$Q(X) := \operatorname{codom} \alpha(\emptyset \to X)$$

and

$$R(X) := \operatorname{codom} \gamma(X \to *)\,,$$

we obtain a sequence

$$\emptyset \xrightarrow{\alpha(0 \to X)} Q(X) \xrightarrow{\beta(\emptyset \to X)} X \xrightarrow{\gamma(X \to *)} R(X) \xrightarrow{\delta(X \to *)} *\,,$$

where

- $\alpha(\emptyset \to X)$ is a cofibration,

- $\beta(\emptyset \to X)$ is a trivial fibration,

- $\gamma(X \to *)$ is a trivial cofibration, and

- $\delta(X \to *)$ is a fibration.

In particular, $Q(X)$ is cofibrant and $R(X)$ is fibrant. If $f\colon X \to Y$ is a morphism in $\mathscr{C}$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q(X) & \longrightarrow & X & \longrightarrow & R(X) & \longrightarrow & 1 \\ \| & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \| \\ 0 & \longrightarrow & Q(Y) & \longrightarrow & Y & \longrightarrow & R(Y) & \longrightarrow & 1 \end{array}$$

because α, β, γ and δ are functors $\operatorname{Map}\mathscr{C} \to \operatorname{Map}\mathscr{C}$. Moreover, by functoriality, we get two functors

$$Q\colon \mathscr{C} \to \mathscr{C}_c\,,$$

$$R\colon \mathscr{C} \to \mathscr{C}_f\,,$$

called *cofibrant replacement* and *fibrant replacement* respectively. We shall denote by $i_c\colon \mathscr{C}_c \to \mathscr{C}$, and by $i_f\colon \mathscr{C}_f \to \mathscr{C}$, the corresponding inclusion functors. Notice that the morphisms $Q(X) \to X$, for X in $\mathscr{C}$, induce two natural transformations

$$Q \circ i_c \Rightarrow \mathrm{id}_{\mathscr{C}_c}\,, \qquad i_c \circ Q \Rightarrow \mathrm{id}_{\mathscr{C}}\,,$$

and the morphisms $X \to R(X)$, for X in $\mathscr{C}$, induce two natural transformations

$$id_{\mathscr{C}_f} \Rightarrow R \circ i_f\,, \qquad id_{\mathscr{C}} \Rightarrow i_f \circ R\,.$$

Lemma 1.1.56. *Suppose $\mathscr{C}$ is a model category. The replacement functors $Q\colon \mathscr{C} \to \mathscr{C}_c$ and $R\colon \mathscr{C} \to \mathscr{C}_f$ preserve weak equivalences.*

Proof. For any morphism $f\colon X \to Y$ in $\mathscr{C}$, we have a commutative diagram

$$\begin{array}{ccccc} Q(X) & \xrightarrow{\beta(0\to X)} & X & \xrightarrow{\gamma(X\to 1)} & R(X) \\ {\scriptstyle Q(f)}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle R(f)} \\ Q(Y) & \xrightarrow[\beta(0\to X)]{} & Y & \xrightarrow[\gamma(Y\to 1)]{} & R(Y) \end{array}$$

where $\beta(0 \to X), \beta(0 \to Y)$ are trivial fibrations and $\gamma(X \to 1), \gamma(Y \to 1)$ are trivial cofibrations. Now, if f is a weak equivalence, by 2-out-of-3 axiom, we deduce from the above diagram that $Q(f)$ and $R(f)$ are weak equivalences.

□

Proposition 1.1.57. *Suppose $\mathscr{C}$ is a model category. Then the inclusion functors i_c and i_f induce equivalences of categories*

$$\operatorname{Ho}\mathscr{C}_{cf} \to \operatorname{Ho}\mathscr{C}_c \to \operatorname{Ho}\mathscr{C}$$

and

$$\operatorname{Ho}\mathscr{C}_{fc} \to \operatorname{Ho}\mathscr{C}_f \to \operatorname{Ho}\mathscr{C}\,.$$

Proof. Note that is it enough to show that $\operatorname{Ho}\mathscr{C}_c \to \operatorname{Ho}\mathscr{C}$ and $\operatorname{Ho}\mathscr{C}_f \to \operatorname{Ho}\mathscr{C}$ are equivalences of categories. Let us prove that the first one is an equivalence of categories. By definition, the inclusion $i_c \colon \mathscr{C}_c \to \mathscr{C}$ preserves weak equivalences, so it induces a functor

$$\operatorname{Ho} i_c \colon \operatorname{Ho}\mathscr{C}_c \to \operatorname{Ho}\mathscr{C}\,.$$

On the other hand, Lemma 1.1.56 says that Q preserves weak equivalences, so it induces a functor

$$\operatorname{Ho} Q \colon \operatorname{Ho}\mathscr{C} \to \operatorname{Ho}\mathscr{C}_c\,,$$

moreover, the natural transformations

$$Q \circ i_c \Rightarrow \mathrm{id}_{\mathscr{C}_c} \quad , \qquad i_c \circ Q \Rightarrow \mathrm{id}_{\mathscr{C}}\,,$$

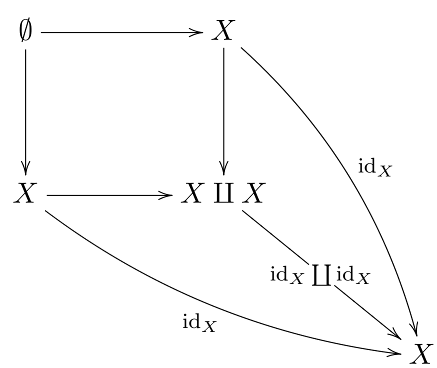
induce two natural isomorphisms

$$\operatorname{Ho} Q \circ \operatorname{Ho} i_c \Rightarrow \mathrm{id}_{\operatorname{Ho}\mathscr{C}_c} \quad , \qquad \operatorname{Ho} i_c \circ \operatorname{Ho} Q \Rightarrow \mathrm{id}_{\operatorname{Ho}\mathscr{C}}\,.$$

This proves that the functor $\operatorname{Ho} i_c \colon \mathscr{C}_c \to \operatorname{Ho}\mathscr{C}$ is an equivalence of categories. Similarly we prove that $\operatorname{Ho}\mathscr{C}_f \to \operatorname{Ho}\mathscr{C}$ is an equivalence of categories. □

Definition 1.1.58. Suppose $\mathscr{C}$ is a model category.

(1) For an object $X \in \mathscr{C}$, the *fold morphism* $\mathrm{id}_X \amalg \mathrm{id}_X \colon X \amalg X \to X$ is defined from the cocartesian diagram

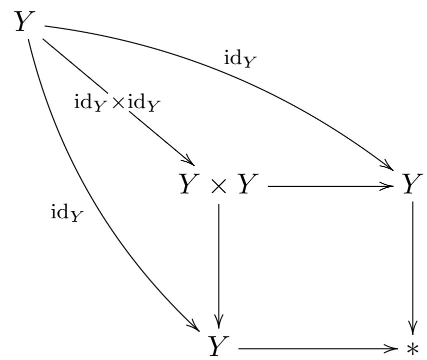


A *cylinder object* for X is a factorization of the fold morphism $\mathrm{id}_X \coprod \mathrm{id}_X$,

$$X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X\,,$$

where $i_0 \coprod i_1$ is a cofibration and p is a weak equivalence.

(2) For an object $Y \in \mathscr{C}$, the *diagonal morphism* $\mathrm{id}_Y \times \mathrm{id}_Y \colon Y \to Y \times Y$ is defined from the cartesian diagram



A *path object* for Y is a factorization axiom of the diagonal morphism $\mathrm{id}_Y \times \mathrm{id}_Y$ is factored as

$$Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y\,,$$

where s is a weak equivalence and $p_0 \times p_1$ is a fibration.

Remark 1.1.59. In a model category $\mathscr{C}$, by the factorization axiom, cylinder and path objects always exist.

In the next paragraphs we give the definition of left and right homotopy.

Definition 1.1.60. Suppose $\mathscr{C}$ is a model category and let $f, g \colon X \to Y$ be two morphisms in $\mathscr{C}$.

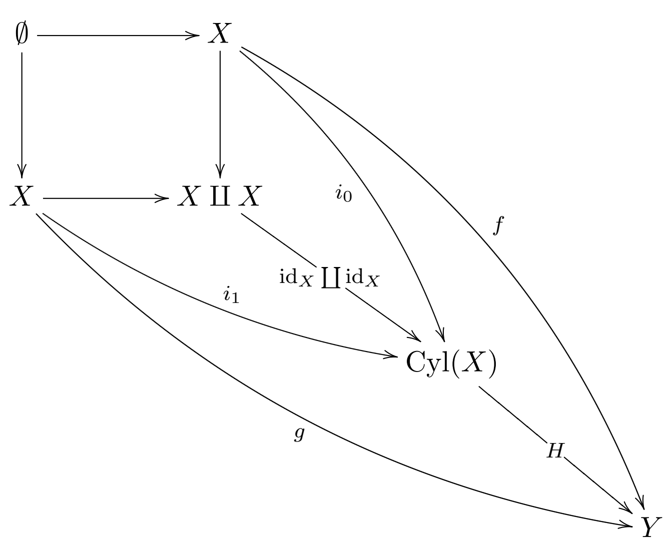
(1) A *left homotopy* from f to g is a pair (C, H), where C is a cylinder object

$$C: \qquad X \amalg X \xrightarrow{i_0 \coprod i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$$

for X, and H is a morphism

$$H \colon \mathrm{Cyl}(X) \to Y\,,$$

such that $H \circ i_0 = f$ and $H \circ i_1 = g$, as shown in the following diagram



We say that f is a *left homotopic* to g if there exists a left homotopy from f to g, it is denoted by $f \overset{l}{\simeq} g$.

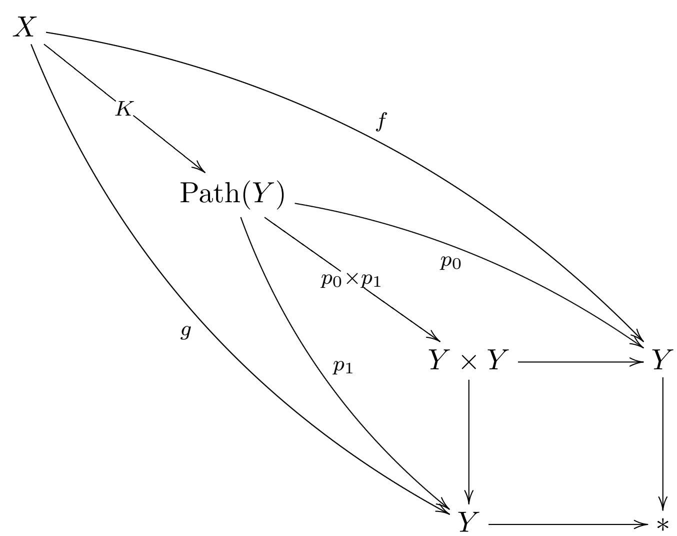
(2) A *right homotopy* from f to g is a pair (P, K) a path object

$$P: \qquad Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$$

for Y, and a K is a morphism

$$K\colon X \to \mathrm{Path}(Y)\,,$$

such that $p_0 \circ K = f$ and $p_1 \circ K = g$, as shown in the following diagram



We say that f is a *right homotopic* to g if there exists a right homotopy from f to g, it is denoted by $f \overset{r}{\simeq} g$.

(3) We say that f is *homotopic* to g, if f is both left homotopic and right homotopic to g, it is denoted by $f \simeq g$.

Theorem 1.1.61 (Whitehead's theorem)**.** *Let $\mathscr{C}$ be a model category and let X, Y be two fibrant cofibrant objects of $\mathscr{C}$. Then $f\colon X \to Y$ is a weak equivalence if and only if f is a homotopy equivalence.*

Proof. See [17, Theorem 7.5.10] or [18, Theorem 1.2.10]. □

1.2 Properties

In this section, we shall recall important properties of model categories and homotopy categories.

1.2.1 Quillen functors

Definition 1.2.1. Let $\mathscr{C}, \mathscr{D}$ be two model categories.

(1) A functor $F\colon \mathscr{C} \to \mathscr{D}$ is called *left Quillen functor*, if F is a left adjoint and preserves cofibrations and trivial cofibrations.

(2) A functor $U\colon \mathscr{C} \to \mathscr{D}$ is called *right Quillen functor*, if U is a right adjoint and preserves fibrations and trivial fibrations.

(3) Suppose that (F, U, φ) is an adjunction, where φ is an isomorphism of bi-functors

$$\mathrm{Hom}_{\mathscr{D}}(F(-), -) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(-, U(-)) .$$

The triplet (F, U, φ) is called is called *Quillen adjunction* if F is a left Quillen functor and U is a right Quillen functor.

Example 1.2.2. Let $\mathscr{C}$ be a model category. The adjunction (1.1) induced by the functor $(-)_+ : \mathscr{C} \to \mathscr{C}_*$ is a Quillen adjunction.

Lemma 1.2.3. *Suppose that $(F, U, \varphi)\colon \mathscr{C} \to \mathscr{D}$ between two model categories $\mathscr{C}$ and $\mathscr{D}$. If F is a left Quillen functor or U is a right Quillen functor, then (F, U, φ) is a Quillen adjunction.*

Proof. See [18, Lemma 1.3.4]. □

Derived functors

In the following definitions we use the notion of left and right Kan extensions. We refer the reader to [26] for a precise definition of these concepts.

Definition 1.2.4. Let $\mathscr{C}$ be a model category, let $\mathscr{D}$ be an arbitrary category and let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor.

(1) The *left derived functor* of F is the right Kan extension $LF\colon \mathrm{Ho}(\mathscr{C}) \to \mathscr{D}$ of F along the localization functor $\gamma_{\mathscr{C}}\colon \mathscr{C} \to \mathrm{Ho}(\mathscr{C})$.

(2) The *right derived functor* of F is the left Kan extension $LF\colon \mathrm{Ho}(\mathscr{C}) \to \mathscr{D}$ of F along the localization functor $\gamma_{\mathscr{C}}\colon \mathscr{C} \to \mathrm{Ho}(\mathscr{C})$.

Proposition 1.2.5. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor between a model category $\mathscr{C}$ and an arbitrary category $\mathscr{D}$. If F sends trivial cofibrations between cofibrant objects to isomorphisms, then the left derived functor of F exists.*

Proof. See [17, Proposition 8.4.4]. □

Definition 1.2.6. Let $\mathscr{C}$ and $\mathscr{D}$ be two model categories and let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor. The *(total) left derived functor* of F is the left derived functor of the composition $\mathscr{C} \xrightarrow{F} \mathscr{D} \xrightarrow{\gamma_{\mathscr{D}}} \mathrm{Ho}(\mathscr{D})$. In other words, the total left derived functor of F is the functor $LF\colon \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$ which is the right Kan extension of the composition

$$\mathscr{C} \xrightarrow{F} \mathscr{D} \xrightarrow{\gamma_{\mathscr{D}}} \mathrm{Ho}(\mathscr{D})$$

along $\gamma_{\mathscr{C}}\colon \mathscr{C} \to \mathrm{Ho}(\mathscr{C})$.

Proposition 1.2.7. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor between two model categories $\mathscr{C}$ and $\mathscr{D}$. If F sends trivial cofibrations between cofibrant objects to weak equivalences, then the left derived functor of $LF\colon \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$ exists.*

Proof. See [17, Proposition 8.4.8]. □

Definition 1.2.8. A Quillen adjunction $(F, U, \varphi)\colon \mathscr{C} \to \mathscr{D}$ is called *Quillen equivalence* if for all cofibrant object X in $\mathscr{C}$ and fibrant object Y in $\mathscr{D}$, a morphism $f \in \mathrm{Hom}_{\mathscr{D}}(F(X), Y)$ is a weak equivalence in $\mathscr{D}$ if and only if $\varphi(f) \in \mathrm{Hom}_{\mathscr{D}}(X, U(Y))$ is a weak equivalence in $\mathscr{C}$. In other words, if every cofibrant object X in $\mathscr{C}$ and fibrant object Y in $\mathscr{D}$, a morphism $f\colon F(X) \to Y$ is a weak equivalence in $\mathscr{D}$ if and only if $\varphi(f)\colon X \to U(Y)$ is a weak equivalence in $\mathscr{C}$.

Proposition 1.2.9. *Let $(F, U, \varphi)\colon \mathscr{C} \to \mathscr{D}$ be a Quillen adjunction. The following statements are equivalent:*

(a) *(F, U, φ) is a Quillen equivalence.*

(b) *For every cofibrant object X in $\mathscr{C}$, the composite*

$$X \xrightarrow{\eta} (U \circ F)(X) \xrightarrow{(U \circ r \circ F)(X)} (U \circ R \circ F)(X) \ ,$$

and for every fibrant object Y in $\mathscr{D}$, the composite

$$(F \circ Q \circ U)(Y) \xrightarrow{(F \circ q \circ U)(Y)} (F \circ U)(Y) \xrightarrow{\varepsilon} Y \, .$$

(c) *$L(F, U, \varphi)$ is an adjoint equivalence of categories.*

Proof. See [18, Proposition 1.3.13]. □

Proposition 1.2.10. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a left Quillen equivalence, and suppose that the terminal object $*$ of $\mathscr{C}$ is cofibrant and F preserves terminal object. Then $F_*\colon \mathscr{C}_* \to \mathscr{D}_*$ is a Quillen equivalence.*

Proof. See [18, Proposition 1.3.17]. □

1.2.2 Simplicial model categories

Some model categories can be seen as categories of modules over the category of simplicial sets, such model categories are known as simplicial model categories.

Definition 1.2.11. A category $\mathscr{C}$ is a *simplicial category*, if there is a bifunctor

$$\mathrm{Map}\,(-,-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \Delta^{\mathrm{op}}\mathscr{S}ets\,,$$

called *space functor*, satisfying the following properties:

(1) For two objects X and Y of $\mathscr{C}$, we have

$$\mathrm{Map}\,(X,Y)_0 = \mathrm{Hom}_{\mathscr{C}}(X,Y)\,.$$

(2) For each object X of $\mathscr{C}$, the functor $\mathrm{Map}\,(X,-)\colon \mathscr{C} \to \Delta^{\mathrm{op}}\mathscr{S}ets$ has a left adjoint functor

$$X \otimes -\colon \Delta^{\mathrm{op}}\mathscr{S}ets \to \mathscr{C}\,,$$

which is *associative*, that is, there is an isomorphism

$$X \otimes (K \times L) \xrightarrow{\sim} (X \otimes K) \otimes L\,,$$

functorial in X and in $K, L \in \Delta^{\mathrm{op}}\mathscr{S}ets$.

(3) For each object Y of $\mathscr{C}$, the functor $\mathrm{Map}\,(-,Y)\colon \mathscr{C}^{\mathrm{op}} \to \Delta^{\mathrm{op}}\mathscr{S}ets$ has a right adjoint functor

$$Y^{(-)}\colon \Delta^{\mathrm{op}}\mathscr{S}ets \to \mathscr{C}\,.$$

Definition 1.2.12. A *simplicial model category* $\mathscr{C}$ is a model category that is also simplicial such that

(M7) if $i\colon A \to B$ is a cofibration and $p\colon X \to Y$ is a fibration, then the morphism of simplicial sets

$$\mathrm{Map}\,(B,X) \xrightarrow{i^* \times p_*} \mathrm{Map}\,(A,X) \times_{\mathrm{Map}\,(A,X)} \mathrm{Map}\,(B,Y)$$

is a fibration, and it is a trivial fibrations if either i or p is a weak equivalence.

Remark 1.2.13. Let $\mathscr{C}$ be a simplicial model category. By adjointness in (2) and (3) of Definition 1.2.11, we have two isomorphisms

$$\mathrm{Hom}_{\mathscr{C}}(X \otimes K, Y) \simeq \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathscr{S}ets}(K, \mathrm{Map}\,(X,Y)) \simeq \mathrm{Hom}_{\mathscr{C}}(X, Y^K)\,,$$

functorial in $X, Y \in \mathscr{C}$ and $K \in \Delta^{\mathrm{op}}\mathscr{S}ets$. Notice that on the second isomorphism, to observe that, the functor $\mathrm{Map}\,(-,Y)\colon \mathscr{C}^{\mathrm{op}} \to \Delta^{\mathrm{op}}\mathscr{S}ets$ can be viewed as a functor $\mathscr{C} \to (\Delta^{\mathrm{op}}\mathscr{S}ets)^{\mathrm{op}}$. The above isomorphisms are known as axiom M6. The axiom (M7) is equivalent to say that the functor $\mathrm{Map}\,(X,-)$ of (2) is a left Quillen functor and the functor $\mathrm{Map}\,(-,Y)$ of (3) is a right Quillen functor.

Some properties

Definition 1.2.14. Let $\mathscr{C}$ be a cocomplete category. For every object X of $\Delta^{\mathrm{op}}\mathscr{C}$ and every simplicial set K, we define $X \otimes K$ to be the functor $X \otimes K \colon \Delta^{\mathrm{op}} \to \mathscr{C}$ given by

$$[n] \mapsto \coprod_{K_n} X_n$$

where $\coprod$ denotes the coproduct in $\mathscr{C}$. If $\theta \colon [m] \to [n]$ is a morphism in Δ, then θ induces a morphism $\theta^* \colon (X \otimes K)_n \to (X \otimes K)_m$ given by the following composite

$$\coprod_{K_n} X_n \xrightarrow{\coprod_X \theta^*} \coprod_{K_n} X_m \longrightarrow \coprod_{K_m} X_m$$

where the first arrow is the morphism induced by ${}_X\theta^* \colon X_n \to X_m$ and the second is induced by ${}_K\theta^* \colon K_n \to K_m$. We have a bi-functor

$$- \otimes - \colon \Delta^{\mathrm{op}}\mathscr{C} \times \Delta^{\mathrm{op}}\mathscr{S}ets \to \Delta^{op}\mathscr{C}$$

defined by $(X, K) \mapsto X \otimes K$.

Definition 1.2.15. For two objects X and Y of $\Delta^{op}\mathscr{C}$, we define a simplicial set $\mathrm{Map}_{\otimes}(X, Y)$ to be the contravariant functor

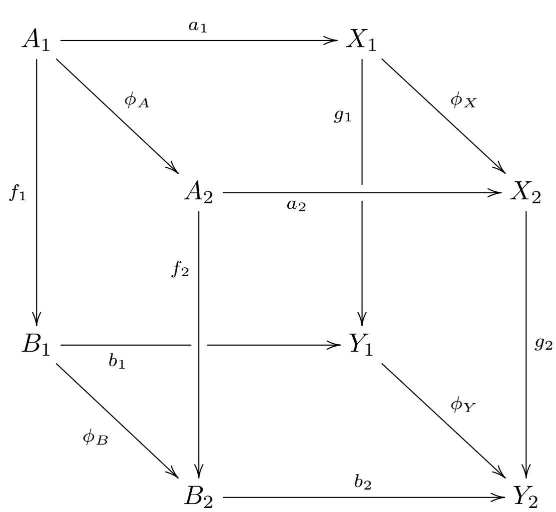
$$[n] \mapsto \mathrm{Hom}_{\Delta^{op}\mathscr{C}}(X \otimes \Delta[n], Y)\,,$$

where $\otimes$ is defined in Definition 1.2.14.

Theorem 1.2.16. *Let $\mathscr{C}$ be a complete and cocomplete category. Then $\Delta^{op}\mathscr{C}$ together with the bi-functor $- \otimes -$ and $\mathrm{Map}_{\otimes}(-,-)$ (see Definition 1.2.14 and 1.2.15) is a simplicial category.*

Proof. See [11]. □

Lemma 1.2.17 (Cube lemma)**.** *Let $\mathscr{C}$ be a model category. Suppose we have commutative cube of cofibrant objects*



(1.3)

where the faces on the back and front are cocartesian squares and suppose a_1, a_2 are monomorphisms. If ϕ_A, ϕ_B and ϕ_X are weak equivalences, then ϕ_Y is a weak equivalence too.

Proof. Let $\mathcal{B}$ be the category $\{a, b, c\}$ with three objects and non identity morphisms $a \to b$ and $a \to c$,

$$c \leftarrow a \to b\,.$$

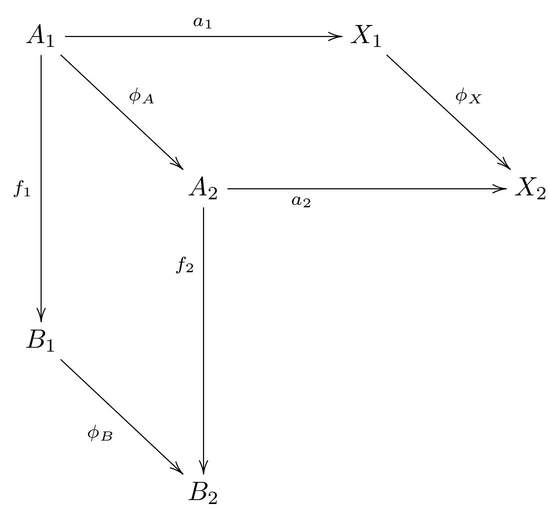
We choose a function $d\colon \mathrm{obj}\,(\mathcal{B}) \to \mathbb{N}$ such that $d(a) < d(b)$ and $d(a) > d(c)$, so that $\mathcal{B}$ becomes a Reedy category with $\mathcal{B}_+ = \{a, b\}$ and $\mathcal{B}_- = \{a, c\}$. The category $\mathscr{C}^{\mathcal{B}}$ is provided with the Reedy model structure (see [18]). We recall that the constant functor $R\colon \mathscr{C} \to \mathscr{C}^{\mathcal{B}}$ is the right adjoint functor of the colimit functor

$$\mathrm{colim} : \mathscr{C}^{\mathcal{B}} \to \mathscr{C}\,.$$

We claim that colim is a left Quillen functor. By Lemma 1.2.3, it is enough to prove that R is a right Quillen functor. Indeed, observe that R preserves weak equivalences. Notice that R also preserves fibrations, because a morphism from a diagram $C \leftarrow A \to B$ into a diagram $C' \leftarrow A' \to B'$ is a fibration in $\mathscr{C}^{\mathcal{B}}$ if and only if $B \to B'$, $C \to C'$ and $A \to A' \times_{C'} C$ are fibrations in $\mathscr{C}$. Hence, we deduce that R is a right Quillen functor. Now, a cofibrant object in $\mathscr{C}^{\mathcal{B}}$ has the form

$$C \leftarrow A \xrightarrow{f} B\,,$$

where A, B and C are cofibrant objects in $\mathscr{C}$, and f is a cofibration. By hypothesis we have a diagram



(1.4)

where a_1, a_2 are monomorphisms and ϕ_A, ϕ_B, ϕ_X are weak equivalences. Notice that this diagram is a morphism from $B_1 \leftarrow A_1 \xrightarrow{a_1} X_1$ to $B_2 \leftarrow A_2 \xrightarrow{a_2} X_2$, thus the triplet (ϕ_B, ϕ_A, ϕ_X) defines a weak equivalence between cofibrant objects in $\mathscr{C}^{\mathcal{B}}$. Observe that ϕ_Y is the colimit of (ϕ_B, ϕ_A, ϕ_X), see diagram (1.4). The Ken Brown's lemma allows us to deduce that ϕ_Y is a weak equivalence. $\square$

Theorem 1.2.18. *Let $\mathscr{C}$ be a model category. Then the total left derived functors of*

$$- \otimes -\colon \mathscr{C} \times \Delta^{\mathrm{op}}\mathscr{S}ets \to \mathscr{C}$$

and

$$\mathrm{Hom}(-,-)\colon \Delta^{\mathrm{op}}\mathscr{S}ets \times \mathscr{C}^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$$

exist.

Proof. See [18]. □

Definition 1.2.19. We denote by

$$- \otimes^L -\colon \mathrm{Ho}\,(\mathscr{C}) \times \mathrm{Ho}\,(\Delta^{\mathrm{op}}\mathscr{S}ets) \to \mathrm{Ho}\,(\mathscr{C})$$

the total left derived functor of $- \otimes -\colon \mathscr{C} \times \Delta^{\mathrm{op}}\mathscr{S}ets \to \mathscr{C}$ and by

$$R\mathrm{Hom}(-,-)\colon \mathrm{Ho}\,(\mathscr{C}) \times \mathrm{Ho}\,(\Delta^{\mathrm{op}}\mathscr{S}ets) \to \mathrm{Ho}\,(\mathscr{C})$$

the total left derived functor of $- \otimes -\colon \mathscr{C} \times \Delta^{\mathrm{op}}\mathscr{S}ets \to \mathscr{C}$.

1.2.3 Homotopy colimits and limits

If $\mathscr{C}$ is a cofibrantly generated model category and if $\mathcal{B}$ is a small category, then the category of functors $\mathscr{C}^{\mathcal{B}}$ has a projective model structure, i.e. a weak equivalence is an objectwise weak equivalence and a fibration is an objectwise fibration, see [17, Theorem 11.6.1]. In general, the functors

$$\operatorname*{colim}_{\mathcal{B}} : \mathscr{C}^{\mathcal{B}} \to \mathscr{C}\,, \qquad \lim_{\mathcal{B}} : \mathscr{C}^{\mathcal{B}} \to \mathscr{C}\,,$$

do not necessarily send objectwise weak equivalence in $\mathscr{C}^{\mathcal{B}}$ to weak equivalences in $\mathscr{C}$. However, their total derived functor

$$L\operatorname*{colim}_{\mathcal{B}} : \mathrm{Ho}\,(\mathscr{C}^{\mathcal{B}}) \to \mathrm{Ho}\,(\mathscr{C})\,, \qquad R\lim_{\mathcal{B}} : \mathrm{Ho}\,(\mathscr{C}^{\mathcal{B}}) \to \mathrm{Ho}\,(\mathscr{C})\,,$$

exist. More precisely, we have the following:

Proposition 1.2.20. *Let $\mathscr{C}$ be a cofibrantly generated model category and let $\mathcal{B}$ be a small category. Then, the adjoint functors*

$$\operatorname*{colim}_{\mathcal{B}} : \mathscr{C}^{\mathcal{B}} \rightleftarrows \mathscr{C} : \mathrm{Const}\,, \qquad \mathrm{Const} : \mathscr{C}^{\mathcal{B}} \rightleftarrows \mathscr{C} : \lim_{\mathcal{B}}\,,$$

induce adjoint pairs of total derived functors

$$L\operatorname*{colim}_{\mathcal{B}} : \mathrm{Ho}\,(\mathscr{C}^{\mathcal{B}}) \rightleftarrows \mathrm{Ho}\,(\mathscr{C}) : R\mathrm{Const}\,, \qquad L\mathrm{Const} : \mathrm{Ho}\,(\mathscr{C}^{\mathcal{B}}) \rightleftarrows \mathrm{Ho}\,(\mathscr{C}) : R\lim_{\mathcal{B}}\,.$$

Proof. See [17]. □

Definition 1.2.21. Let $F\colon \mathcal{B} \to \mathscr{C}$ be a functor. The *homotopy colimit* of F, denoted by Hocolim$F$, is the object $(L\operatorname{colim}_{\mathcal{B}})(F)$ of $\mathrm{Ho}(\mathscr{C})$.

Let $\mathscr{C}$ and $\mathcal{B}$ as before, and let $Q^{\mathcal{B}}$ be a cofibrant replacement of $\mathscr{C}^{\mathcal{B}}$. We write

$$\mathrm{hocolim}_{\mathcal{B}} := \operatorname{colim}_{\mathcal{B}} \circ Q^{\mathcal{B}} \colon \mathscr{C}^{\mathcal{B}} \longrightarrow \mathscr{C}\,.$$

Notice that for any functor $F\colon \mathcal{B} \to \mathscr{C}$, we have a canonical morphism

$$\mathrm{hocolim}_{\mathcal{B}} F \to \operatorname{colim}_{\mathcal{B}} F\,. \tag{1.5}$$

This morphism do not need to be an isomorphism, however if F is cofibrant in $\mathscr{C}^{\mathcal{B}}$, then the above morphism is a weak equivalence in $\mathscr{C}$, see [17, Theorem 11.6.8].

Borel construction

Definition 1.2.22 (Simplicial bar construction)**.** Let X and Y be a left and a right G-set respectively. The *simplicial bar construction* of X and Y is a simplicial set $B(X, G, Y)$ such that it has the Cartesian product $X \times G^n \times Y$ as the its of n-simplices for $n \in \mathbb{N}$, where G^0 is the trivial group $\{e\}$. Writing an element of $X \times G^n \times Y$ in the form $(x;\, g_1, \dots, g_n;\, y)$, the face and degeneracy morphisms are given by the formulae

$$d_i(x; g_1, \dots, g_n; y) = \begin{cases} (x \cdot g_1; g_2, \dots, g_n; y)\,, & \text{if } i = 0\,, \\ (x \cdot g_1; g_2, \dots, g_{i-1}, g_i \cdot g_{i+1}, g_{i+2}, \dots, g_n; y)\,, & \text{if } 0 < i < n\,, \\ (x; g_1, \dots, g_{n-1}; g_n \cdot y)\,, & \text{if } i = n\,, \end{cases}$$

$$s_i(x; g_1, \dots, g_n; y) = (x; g_1, \dots, g_i, e, g_{i+1}, \dots, g_n; y). \tag{1.6}$$

Definition 1.2.23. For a group G, we define two simplicial sets

$$BG := B(*, G, *) \quad \text{and} \quad EG := B(*, G, G)\,,$$

where $*$ is the singleton seen as G-set. The simplicial set BG is the *simplicial classifying space* of G and is the *G-universal principal bundle.*

The set of n-simplices $(EG)_n$ of EG is the nth fold product G^{n+1} and the group G acts on it by the action of G on its diagonal. The simplicial set EG is contractible, see [33, Example 4.5.5].

Let $\mathscr{C}$ be a pointed simplicial cofibrantly generated model category. Let us take $\mathcal{B}$ of the previous paragraphs to be a group G seen as a category, and let us consider the projective model structure on $\mathscr{C}^G$. In this case, the cofibrant replacement functor has the shape

$$Q^G = (EG)_+ \wedge - \colon \mathscr{C}^G \to \mathscr{C}^G\,.$$

For every G-object X in $\mathscr{C}$, we have

$$\mathrm{hocolim}_G(X) = (EG_+ \wedge X)/G\,.$$

As in (1.5), one has a canonical morphism

$$\mathrm{hocolim}_G(X) \to \mathrm{colim}\,_G(X) = X/G\,,$$

induced by the morphism $EG_+ \wedge X \to X$ which results by mapping EG to $*$.

Definition 1.2.24. If $\mathscr{C}$ is, in addition, a symmetric monoidal model category, then for any object X of $\mathscr{C}$ we define the nth fold *homotopy symmetric power* of X as

$$\mathrm{Sym}^n_h(X){:=}\mathrm{hocolim}_{\Sigma_n}(X^{\wedge n})\,,$$

where Σ_n acts on $X^{\wedge n}$ by permuting factors.

We get an endofunctor $\mathrm{Sym}^n_h : \mathscr{C} \to \mathscr{C}$ sending an object X of $\mathscr{C}$ to $\mathrm{Sym}^n_h(X)$.

Homotopy cocartesian and cartesian diagrams

Let $\mathscr{C}$ be a left proper model category (see [17, Definition 13.1.1]). A commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

in $\mathscr{C}$, is called *homotopy cocartesian*, if f has a factorization $A \xrightarrow{j} B' \xrightarrow{p} B$ such that j cofibration and p is a weak equivalence, and such that the universal morphism

$$B' \times_A X \to Y$$

is a weak equivalence in $\mathscr{C}$. Let $\mathscr{C}$ be a right proper model category (see *loc.cit.*). A commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & Y \end{array}$$

in $\mathscr{C}$, is called *homotopy cartesian*, if f has a factorization $X \xrightarrow{j} X' \xrightarrow{p} Y$ such that j is a weak equivalence and p is a fibration, and such that the universal morphism

$$A \to B \times_Y X'$$

is a weak equivalence in $\mathscr{C}$.

1.3 Triangulated structures on model categories

The main references for this section are [32] and [18].

1.3.1 Cofibre and fibre sequences

Definition 1.3.1. Let $f\colon X \to Y$ be a morphism in a category $\mathscr{C}$ with terminal object $*$.

(1) The *cofibre* of $f\colon X \to Y$ is defined to be the pushout, if it exists, of the diagram in $\mathscr{C}$,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ * & & \end{array}$$

it will be denoted by Y/X.

(2) The *fibre* of f is defined to be the pullback, if it exists, of the diagram diagram in $\mathscr{C}$,

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle f} \\ * & \longrightarrow & Y \end{array}$$

Definition 1.3.2. Let $\mathscr{C}$ be a pointed simplicial model category. Let X, Y be two objects in $\mathscr{C}$ and let $f\colon X \to Y$ be a morphism.

(1) The *cone* of X is the object

$$\operatorname{cone}(X) := X \wedge \Delta[1]_+ \,,$$

where $\Delta[1]_+ = \Delta[1] \amalg \Delta[0]$. Notice that the morphism $i_1\colon \Delta[0] \to \Delta[1]$, induced by the 0-face morphism, induces a morphism

$$X \to \operatorname{cone}(X)\,,$$

which is a trivial cofibration in $\mathscr{C}$.

(2) The *cone* of f, denoted by $\operatorname{cone}(f)$, is the homotopy colimit of the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ \operatorname{cone}(X) & & \end{array}$$

In the sequel, S^1 will denote the usual pointed simplicial circle.

Lemma 1.3.3. *Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$, as before. Then the quotient* $\operatorname{cone}(X)/X$ *is isomorphic to the smash product $X \wedge S^1$, and the quotient* $\operatorname{cone}(f)/Y$ *is also isomorphic to the smash product $X \wedge S^1$.*

Proof. It follows from the following cocartesian square

$$\begin{array}{ccc} X & \longrightarrow & X \wedge \Delta[1]_+ \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \wedge S^1 \end{array}$$

On the other hand, the second assertion follows since we have a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{cone}(X) \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \longrightarrow & \operatorname{cone}(f) \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \wedge S^1 \end{array}$$

where both squares are cocartesian. □

Definition 1.3.4. Let $\mathscr{C}$ be a pointed simplicial model category.

(1) The *suspension functor*

$$\Sigma\colon \operatorname{Ho}\mathscr{C} \to \operatorname{Ho}\mathscr{C}$$

is the functor defined by $X \mapsto X \wedge^L S^1$, see Definition 1.2.19.

(2) Dually, the *loop functor*

$$\Omega\colon \operatorname{Ho}\mathscr{C} \to \operatorname{Ho}\mathscr{C}$$

is the functor defined by $X \mapsto R\operatorname{Hom}_*(S^1, X)$.

Cofibre sequences in pointed simplicial model categories

In the homotopy category $\operatorname{Ho}\mathscr{C}$ of a pointed model category $\mathscr{C}$, there is a natural coaction of ΣA on the cofibre of a cofibration of cofibrant objects $A \to B$, and dually there is a natural action of ΩB on the fibre of a fibration of fibrant objects $E \to B$. We shall describe more precisely in the next paragraphs.

In the next paragraphs, $\mathscr{C}$ will be a pointed simplicial model category.

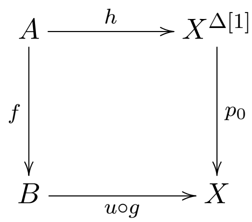
Coaction on ΣA

Let $f\colon A \to B$ be a cofibration of cofibrant objects in $\mathscr{C}$ and let $g\colon B \to C$ the cofibre of f. For any object X of $\mathscr{C}$, we define a right action

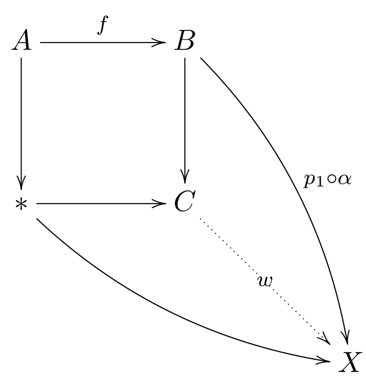
$$[C, X] \times [\Sigma A, X] \to [C, X]$$

as follows. Let us fix an object X of $\mathscr{C}$ and take two morphisms $h\colon A \to X^{\Delta[1]}$ and $u\colon C \to X$ representing elements in $[\Sigma A, X]$ and $[C, X]$ respectively.

We recall that the morphisms $i_0, i_1\colon \Delta[0] \to \Delta[1]$ induce two trivial fibrations $p_0, p_1\colon X^{\Delta[1]} \to X^{\Delta[0]} = X$, moreover, we have $p_0 \circ h = p_1 \circ h$ which is equal to the trivial morphism. Since the composition $g \circ f$ is the trivial morphism, we have a commutative diagram



which has a lifting $\alpha\colon B \to X^{\Delta[1]}$, as f is a cofibration and p_0 a trivial fibration. Since $p_1 \circ \alpha \circ f = p_1 \circ h$ is equal to the trivial morphism, we get a solid diagram



hence there is a unique morphism $w\colon C \to X$ such that $w \circ g = p_1 \circ \alpha$. We define a *coaction*

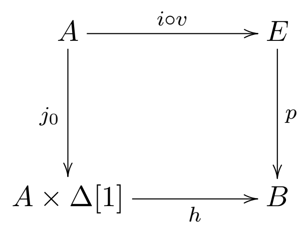
$$[u] \odot [h] := [w]\,.$$

Action on ΩB

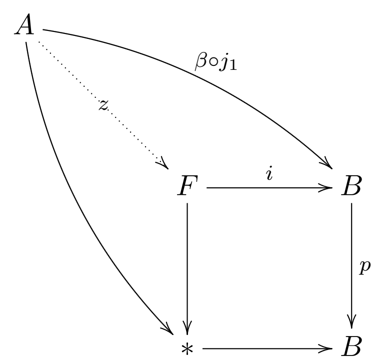
Let $p\colon E \to B$ be a fibration of fibrant objects in $\mathscr{C}$ and let $i\colon F \to B$ the fibre of p. For any object A of $\mathscr{C}$, we define a right action

$$[A, F] \times [A, \Omega B] \to [A, F]$$

as follows. Let us fix an object A of $\mathscr{C}$ and take two morphisms $h\colon A \times \Delta[1] \to X$ and $v\colon A \to F$ representing elements in $[A, \Omega B]$ and $[A, F]$ respectively. We recall that the morphisms $i_0, i_1 \colon \Delta[0] \to \Delta[1]$ induce trivial cofibrations $i_0, i_1 \colon A \to A \times \Delta[1]$ such that $h \circ j_0 = h \circ p_1$ is a trivial morphism. Since the composition $p \circ i$ is the trivial morphism, we have a commutative diagram



which has a lifting $\beta \colon A \times \Delta[1] \to E$, as j_0 is a trivial cofibration and p a fibration. Because $p \circ \beta \circ j_1 = p \circ h$ is equal to the trivial morphism, we get a solid diagram



hence there is a unique morphism $z \colon A \to F$ such that $w \circ g = p_1 \circ \alpha$. We define an *action*

$$[v] \odot [h] := [z] \, .$$

Theorem 1.3.5. *Let $\mathscr{C}$ be a model category as before.*

(a) *Suppose $f \colon A \to B$ is a cofibration of cofibrant objects in $\mathscr{C}$ with cofibre $g \colon B \to C$, and let X be a fibrant of $\mathscr{C}$ object. Then the function of sets*

$$[C, X] \times [\Sigma A, X] \to [C, X]$$

given by $([u], [h]) \mapsto [u] \odot [h]$ defines a right action of $[\Sigma A, X]$ on $[C, X]$.

(b) *Dually, suppose $p \colon E \to B$ is a fibration of fibrant objects in $\mathscr{C}$ with fibre $i \colon F \to E$, and let A be a cofibrant object of $\mathscr{C}$. Then the function of sets*

$$[A, F] \times [A, \Omega B] \to [A, F]$$

given by $([v], [h]) \mapsto [v] \odot [h]$ defines a right action of $[A, \Omega B]$ on $[A, F]$.

Proof. See [18, Theorem 6.2.1]. □

Definition 1.3.6. Let $\mathscr{C}$ be a pointed model category.

(1) If $f\colon A \to B$ is a cofibration of cofibrant objects in $\mathscr{C}$ with cofibre $g\colon B \to C$, then the sequence

$$A \xrightarrow{[f]} B \xrightarrow{[g]} C$$

in $\operatorname{Ho}\mathscr{C}$ is called *special cofibre sequence.*

(2) Dually, If $p\colon E \to B$ is a fibration of fibrant objects in $\mathscr{C}$ with fibre $i\colon F \to E$, then the sequence

$$F \xrightarrow{[i]} E \xrightarrow{[p]} B$$

in $\operatorname{Ho}\mathscr{C}$ is called *special fibre sequence.*

Proposition 1.3.7. *Let $\mathscr{C}$ is a pointed model category. Suppose that $A \xrightarrow{f} B \xrightarrow{g} C$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C'$ are two special cofibre sequences and there is a commutative square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A' & \xrightarrow[f']{} & B' \end{array}$$

Then the induced morphism $\gamma\colon C \to C'$ is $\Sigma\alpha$-coequivariant morphism of cogroups.

Proof. See [18, Proposition 6.2.5] □

Definition 1.3.8. Let $\mathscr{C}$ be a pointed model category.

(1) A *cofibre sequence* in $\operatorname{Ho}\mathscr{C}$ is a diagram

$$X \to Y \to Z$$

of morphisms in $\operatorname{Ho}\mathscr{C}$, together with a right coaction $Z \to Z \amalg \Sigma X$ of ΣX on Z, such that there is a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C \end{array}$$

where the vertical arrows are isomorphisms in $\operatorname{Ho}\mathscr{C}$, the horizontal line at the bottom is a special cofibre sequence, and in addition, the morphism w is coequivariant with respect to the isomorphism of cogroups $\Sigma u\colon \Sigma X \to \Sigma A$.

(2) Dually, a *fibre sequence* in $\operatorname{Ho}\mathscr{C}$ is a diagram

$$X \to Y \to Z$$

of morphisms in $\operatorname{Ho}\mathscr{C}$, together with a right action $X \times \Omega Z \to X$ of ΩZ on X, such that there is a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ F & \xrightarrow[f]{} & E & \xrightarrow[g]{} & B \end{array}$$

where the vertical arrows are isomorphisms in $\operatorname{Ho}\mathscr{C}$, the horizontal line at the bottom is a special fibre sequence, and in addition, the morphism u is equivariant with respect to the isomorphism of groups $\Omega w \colon \Omega Z \to \Omega B$.

Definition 1.3.9. Let $\mathscr{C}$ be a pointed simplicial model category.

(1) The *boundary morphism* of a cofibre sequence $X \to Y \to Z$ is a morphism $\partial \colon Z \to \Sigma X$ in $\operatorname{Ho}\mathscr{C}$ defined to be the composite

$$Z \to Z \amalg \Sigma X \overset{(*,\mathrm{id}_X)}{\longrightarrow} \Sigma X\,,$$

where the first arrow is the coaction of ΣX on Z.

(2) Dually, the *boundary morphism* of a fibre sequence $X \to Y \to Z$ is a morphism $\partial \colon \Omega Z \to X$ in $\operatorname{Ho}\mathscr{C}$ defined to be the composite

$$\Omega Z \overset{(*,\mathrm{id}_X)}{\longrightarrow} X \times \Omega Z \to X\,,$$

where the second arrow is the action of ΩZ on X.

Remark 1.3.10. Let $\mathscr{C}$ be a pointed model category. Every cofibre sequence in $\operatorname{Ho}\mathscr{C}$ of the form

$$X \to Y \to Z \to \Sigma X$$

is isomorphic to a cofibre sequence of the form

$$A \overset{f}{\to} B \overset{i_f}{\to} \operatorname{cone}(f) \overset{p_f}{\to} \Sigma\,,$$

where A and B are cofibrant objects of $\mathscr{C}$. That is, there is a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\ A & \xrightarrow[f]{} & B & \xrightarrow[i_f]{} & \operatorname{cone}(f) & \xrightarrow[p_f]{} & \Sigma A \end{array}$$

where the vertical arrows are isomorphisms in $\operatorname{Ho}\mathscr{C}$.

Remark 1.3.11. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a special cofibre sequence in the homotopy category of a pointed model category $\mathscr{C}$, and let $\partial\colon C \to \Sigma A$ be its boundary morphism. Then from the above definition, we deduce that for any fibrant object X in $\mathscr{C}$, the induced morphism

$$\partial^*\colon [\Sigma A, X] \to [C, X]$$

is defined by $[h] \mapsto [*] \odot [h]$, where $*$ is the trivial morphism$C \to X$. By definition of the coaction $\odot$, the morphism $[*] \odot [h]$ is represented by a morphism $c\colon C \to X$ in $\mathscr{C}$ such that

$$c \circ g = p_1 \circ \alpha\,,$$

where $\alpha\colon B \to X^{\Delta[1]}$ is a lifting of the square,

$$\begin{array}{ccc} A & \xrightarrow{h} & X^{\Delta[1]} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow[*]{} & X \end{array}$$

Thus, one has

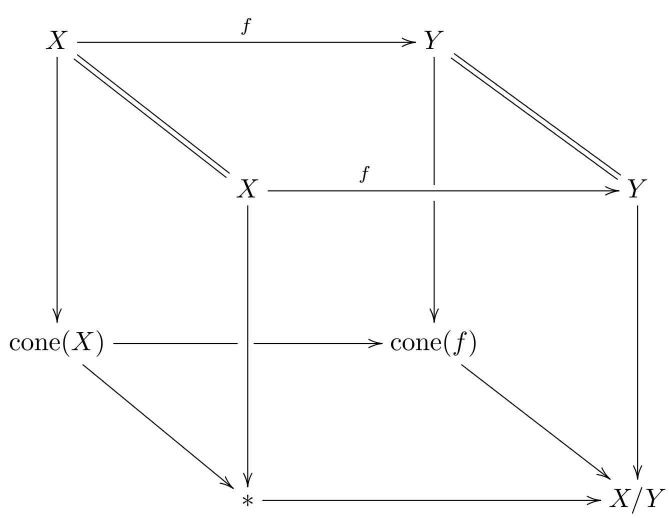
$$h \circ \partial = \partial^*(h) = [*] \odot [h] = [\alpha]\,.$$

Lemma 1.3.12. *Suppose $\mathscr{C}$ is a pointed model category. If $f\colon X \to Y$ is a cofibration in $\mathscr{C}$ between cofibrant objects, then the canonical morphism*

$$\mathrm{cone}(f) \to Y/X$$

is a weak equivalence.

Proof. Let us consider the following commutative cube of cofibrant objects



(1.7)

Since $\mathrm{cone}(X) \to *$ is a weak equivalence and $f\colon X \to Y$ is a cofibration, the cube lemma assures that the morphism $\mathrm{cone}(f) \to Y/X$ is a weak equivalence. □

1.3.2 Pre-triangulated structure on homotopy categories

Let $\mathscr{S}$ be a nontrivial right closed $\mathrm{Ho}\,(\Delta^{\mathrm{op}}\mathscr{S}ets_*)$-module, see [18, Definition 4.1.6]. A *pre-triangulation* on $\mathscr{S}$ is a collection of sequences in $\mathscr{S}$,

$$X \xrightarrow{f} Y \xrightarrow{g} Z\,,$$

provided of a coaction of the cogroup ΣX on Z, called *cofibre sequences*, together with a collection of sequences in $\mathscr{S}$,

$$X \xrightarrow{f} Y \xrightarrow{g} Z\,,$$

provided of an action of the group ΩZ on X, called *fibre sequences*, satisfying the following eight axioms:

(PT1)
- Every diagram isomorphic to a cofibre sequence is a cofibre sequence,
- Dually, every diagram isomorphic to a fibre sequence is a fibre sequence.

(PT2) For any object X in $\mathscr{S}$,

- the diagram $* \to X \xrightarrow{\mathrm{id}_X} X$ is a cofibre sequence,
- dually, the diagram $X \xrightarrow{\mathrm{id}_X} X \to *$ is a fibre sequence.

(PT3) For each morphism $f\colon X \to Y$ in $\mathscr{S}$,

- there is a cofibre sequence the diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$, where g is a morphism in $\mathscr{S}$,
- dually, there is a fibre sequence the diagram $W \xrightarrow{h} X \xrightarrow{f} Y$, where h is a morphism in $\mathscr{S}$.

(PT4) (*rotation*)

- If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofibre sequence, then the sequence

$$Y \xrightarrow{g} Z \xrightarrow{\partial} \Sigma X$$

is a cofibre sequence, where ∂ is the boundary morphism of the preceding cofibre sequence.

- Dually, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fibre sequence, then the sequence

$$\Omega Z \xrightarrow{\partial} X \xrightarrow{f} Y$$

is a fibre sequence, where ∂ is the boundary morphism of the preceding fibre sequence.

(PT5) Suppose we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

- If $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$ are two cofibre sequences, then there is a $\Sigma\alpha$-coequivariant morphism $\gamma\colon Z \to Z'$ such that the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \vdots{\scriptstyle\gamma} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' \end{array}$$

  commutes.

- Dually, if $W \xrightarrow{h} X \xrightarrow{f} Y$ and $W' \xrightarrow{h'} X' \xrightarrow{f'} Y'$ are two cofibre sequences, then there is a $\Omega\beta$-equivariant morphism $\xi\colon W \to W'$ such that the following diagram

$$\begin{array}{ccccc} W & \xrightarrow{h} & X & \xrightarrow{f} & Y \\ {\scriptstyle\xi}\vdots & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ W' & \xrightarrow[h']{} & X' & \xrightarrow[f']{} & Y' \end{array}$$

  commutes.

(PT6) (*octahedron*) Suppose we have a morphisms $X \xrightarrow{v} Y \xrightarrow{u} Z$.

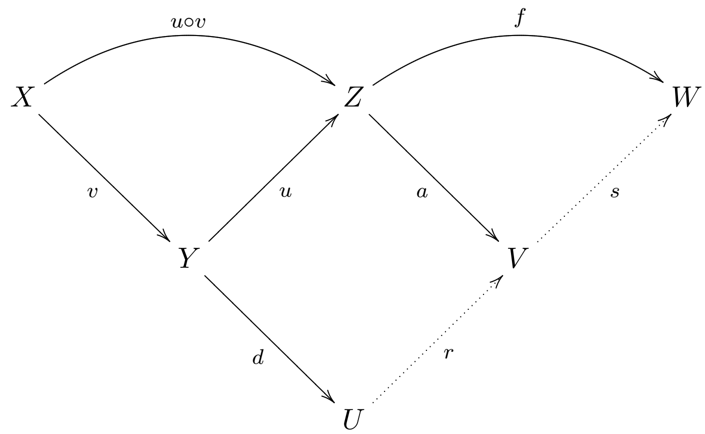
- If we have cofibre sequences

$$X \xrightarrow{v} Y \xrightarrow{d} U\,,$$

$$X \xrightarrow{u\circ v} Z \xrightarrow{a} V\,,$$

$$Y \xrightarrow{u} Z \xrightarrow{f} W$$

  then there is a cofibre sequence $U \xrightarrow{r} V \xrightarrow{s} W$ together with a commutative diagram



such that r is $\Sigma \mathrm{id}_X$-coinvariant and s is Σv-coinvariant.

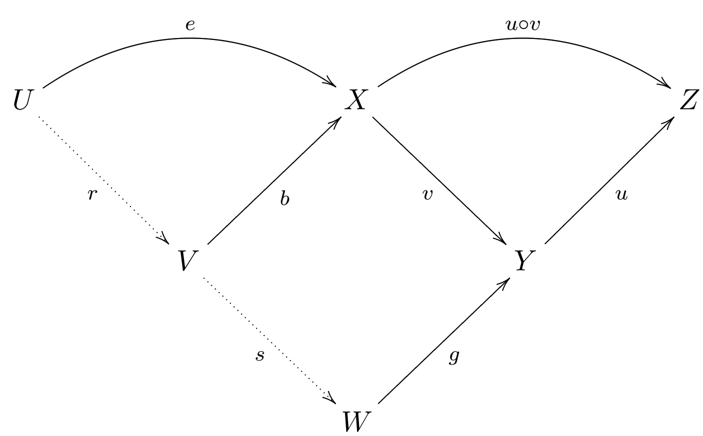
- Dually, If we have fibre sequences

$$U \xrightarrow{e} X \xrightarrow{v} Y\,,$$

$$V \xrightarrow{b} X \xrightarrow{u \circ v} Z\,,$$

$$W \xrightarrow{g} Y \xrightarrow{u} Z$$

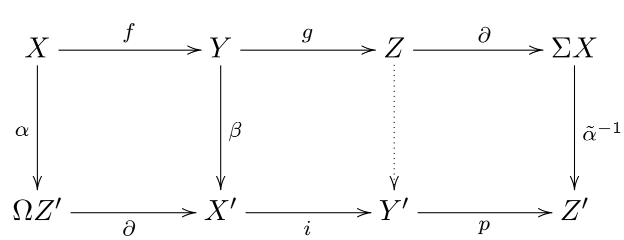
then there is a fibre sequence $U \xrightarrow{r} V \xrightarrow{s} W$ together with a commutative diagram



such that r is Ωu-invariant and s is $\Omega \mathrm{id}_Z$-invariant.

(PT7) (*compatibility of sequences*) Suppose we have a cofibre sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ and a fibre sequence $X' \xrightarrow{i} Y' \xrightarrow{p} Z'$.

- If we have a solid commutative diagram



where $\tilde{\alpha}^{-1}$ is the inverse of the adjoint of α as an element of the group $[\Sigma X, Z']$, then there is a morphism $\gamma\colon Z \to Y'$ making the diagram commutative.

- dually, if we have a solid commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{\partial} & \Sigma X \\ \downarrow{\scriptstyle \tilde{\delta}^{-1}} & & \vdots & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\ \Omega Z' & \xrightarrow[\partial]{} & X' & \xrightarrow[i]{} & Y' & \xrightarrow[p]{} & Z' \end{array}$$

  where $\tilde{\delta}^{-1}$ is the inverse of the adjoint of δ as an element of the group $[X, \Omega Z']$, then there is a morphism $\beta\colon Y \to X'$ making the diagram commutative.

(PT8) (*compatibility with the monoidal structure*)

- The functor $- \wedge^L -\colon \mathscr{S} \times \mathrm{Ho}\,(\Delta^{\mathrm{op}}\mathscr{S}ets_*) \to \mathscr{S}$ preserves cofibre sequences in each variable.
- The functor $R\mathrm{Hom}_*(-,-)\colon \mathscr{S} \times \mathrm{Ho}\,(\Delta^{\mathrm{op}}\mathscr{S}ets_*) \to \mathscr{S}$ preserves fibre sequences in the second variable and converts cofibre sequences into fibre sequences in the first variable.
- Similarly, the functor $\mathrm{Map}_*(-,-)\colon \mathscr{S}^{\mathrm{op}} \times \mathscr{S} \to \mathrm{Ho}\,(\Delta^{\mathrm{op}}\mathscr{S}ets_*)$ preserves fibre sequences in the second variable and converts cofibre sequences into fibre sequences in the first variable.

Definition 1.3.13. A *pre-triangulated category* $\mathscr{S}$ is a nontrivial right closed $\mathrm{Ho}\,(\Delta^{\mathrm{op}}\mathscr{S}ets_*)$-module with products and coproducts, together with a pre-triangulation on $\mathscr{S}$.

Theorem 1.3.14. *The homotopy category* $\mathrm{Ho}\,\mathscr{C}$ *of a pointed model category* $\mathscr{C}$ *is a pre-triangulated category.*

Proof. It is proven throughout Section 6.4 of [18]. □

1.3.3 Triangulated structure on homotopy categories

Our principal goal in this section is to see that a stable homotopy category $\mathrm{Ho}\,\mathscr{C}$ together with its cofibre sequences is a triangulated category.

Lemma 1.3.15. *Suppose the suspension functor* $\Sigma\colon \mathrm{Ho}\,\mathscr{C} \to \mathrm{Ho}\,\mathscr{C}$ *is an equivalence of categories. Then* $\mathrm{Ho}\,\mathscr{C}$ *is additive.*

Proof. Since any pre-additive[1] category admitting finite coproducts is additive, it it enough to show that $\operatorname{Ho}\mathscr{C}$ is pre-additive. In fact, since $\Sigma\colon \operatorname{Ho}\mathscr{C} \to \operatorname{Ho}\mathscr{C}$ is an equivalence of categories, $\Sigma^2\colon \operatorname{Ho}\mathscr{C} \to \operatorname{Ho}\mathscr{C}$ is also an equivalence of categories such that we have a functorial isomorphism

$$\Sigma^2(\Omega^2 X) \simeq X\,,$$

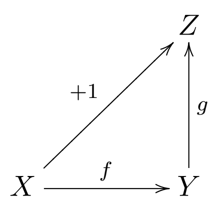
for any objet X in $\operatorname{Ho}\mathscr{C}$. Notice that $\Sigma^2(\Omega^2 X)$ is an Abelian cogroup object in $\operatorname{Ho}\mathscr{C}$. Then, any object of $\operatorname{Ho}\mathscr{C}$ is an Abelian cogroup object in $\operatorname{Ho}\mathscr{C}$. In particular for any two objects X, Y in $\operatorname{Ho}\mathscr{C}$, the set $\operatorname{Hom}_{\operatorname{Ho}\mathscr{C}}(X,Y)$ is endowed with a structure of an Abelian group. □

Triangulated categories

A *triangulated category* is a triplet $(\mathscr{S}, \Sigma, S)$, where $\mathscr{S}$ is an additive category, $\Sigma\colon \mathscr{S} \to \mathscr{S}$ is an auto-equivalence and S is a set of sequences of morphisms in $\mathscr{S}$,

$$X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} \Sigma X$$

called *distinguished triangles*, usually denoted by



,

satisfying the following axioms:

(TR1) If $D \in S$ and $D \simeq D'$, then $D' \in S$. Moreover, for any $X \in \mathscr{S}$, then

$$(X \overset{\mathrm{id}_X}{\to} X \overset{0}{\to} 0 \overset{0}{\to} \Sigma X) \in S\,.$$

(TR2) For each morphism $f\colon X \to Y$ in $\mathscr{S}$, there is a distinguished triangle

$$X \overset{f}{\to} Y \to Z \to \Sigma X$$

in $\mathcal{S}$.

(TR3) (*rotation*) The triangle $X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} \Sigma X$ belongs to S if and only if $Y \overset{g}{\to} Z \overset{h}{\to} \Sigma X \overset{-\Sigma f}{\to} \Sigma Y)$ belongs to S.

[1] A *pre-additive* category is a category $\mathscr{C}$ that the set of morphisms $\operatorname{Hom}_{\mathscr{C}}(X,Y)$ is endowed with a structure of an Abelian group and the composition $\circ$ is bilinear.

(TR4) Given two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X\,,$$
$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'\,,$$

and a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

there exists a morphism $w\colon Z \to Z'$ such that the triplet (u, v, w) is a morphism of triangles, that is, the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \vdots\,{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X' \end{array}$$

commutes.

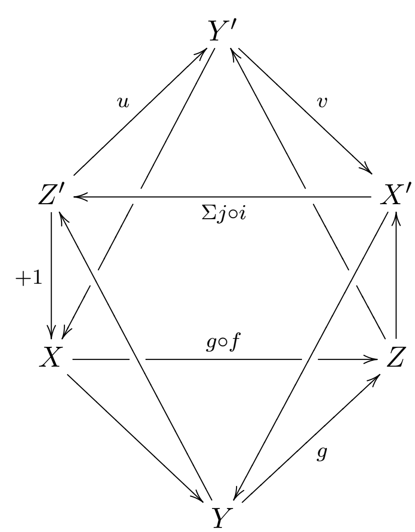
(TR5) (*octahedron*) Given

$$X \xrightarrow{f} Y \xrightarrow{j} Z' \to \Sigma X\,,$$
$$Y \xrightarrow{g} Z \to X' \xrightarrow{i} \Sigma Y\,,$$
$$X \xrightarrow{g\circ f} Z \to Y' \to \Sigma X\,,$$

in $\mathcal{S}$, there exist morphisms $u\colon Z' \to Y'$ and $v\colon Y' \to X'$ such that

$$Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{\Sigma j \circ i} \Sigma Z'$$

is distinguished, and in the diagram



(id_X, g, u) is a morphism from the triangle XYZ' to the triangle XZY' and (f, id_Z, v) is a morphism from the triangle XZY' to the triangle YZX'.

Theorem 1.3.16. *Let $\mathscr{C}$ be a pointed model category. If the suspension functor $\Sigma\colon \mathrm{Ho}\,\mathscr{C} \to \mathrm{Ho}\,\mathscr{C}$ is an equivalence of categories, then $\mathrm{Ho}\,\mathscr{C}$ is a triangulated category where its distinguished triangles are cofibre sequences.*

Proof. By Lemma 1.3.15, the homotopy category $\mathrm{Ho}\,\mathscr{C}$ is additive. Let us verify the axioms of a triangulated category:

Axiom TR1: We have $\mathrm{cone}(\mathrm{id}_X) = \mathrm{cone}(X)$ and the morphism $\mathrm{cone}(X) \to *$ is a weak equivalence, we have a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & \mathrm{cone}(X) & \longrightarrow & \Sigma X \\ \| & & \| & & \downarrow & & \| \\ X & \xrightarrow[f]{} & X & \longrightarrow & * & \longrightarrow & \Sigma A \end{array}$$

then the sequence at the bottom is a cofibre sequence.

Axiom TR2: Let $g\colon X \to Y$ be a morphism in $\mathrm{Ho}\,\mathscr{C}$. We choose a morphism $f\colon X \to Y$ in $\mathscr{C}$ which represents g. We consider the fibre sequence

$$X \xrightarrow{f} Y \xrightarrow{i_f} \mathrm{cone}(f) \xrightarrow{p_f} \Sigma X$$

in $\mathscr{C}$. This sequence is equal to the sequence

$$X \xrightarrow{g} Y \xrightarrow{i_f} \mathrm{cone}(f) \xrightarrow{p_f} \Sigma X$$

in $\mathrm{Ho}\,\mathscr{C}$.

Axiom TR3:

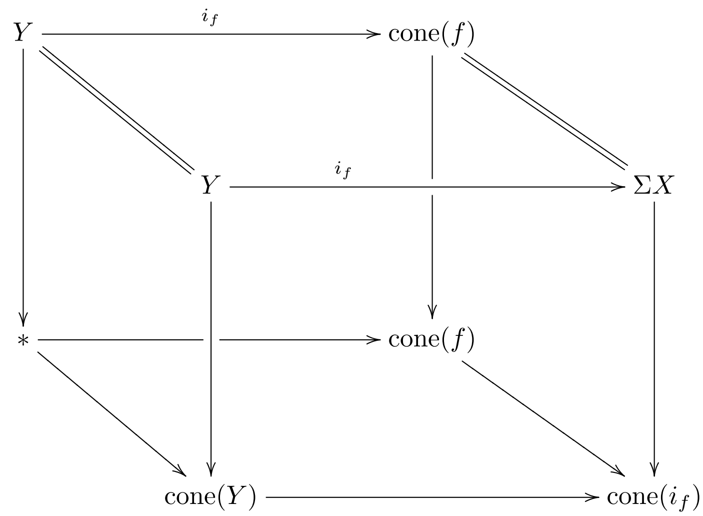
Suppose we have a cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow{i_f} \mathrm{cone}(f) \xrightarrow{p_f} \Sigma X\,.$$

Since ΣX is a colimit of the diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{cone}(f) \\ \downarrow & & \\ * & & \end{array}$$

then, there is an universal morphism $u\colon \Sigma X \to \mathrm{cone}(i_f)$ together with a commutative diagram



(1.8)

Since the morphism $i_f\colon Y \to \mathrm{cone}(f)$ is a cofibration and $* \to \mathrm{cone}(Y)$ is a weak equivalence, by the cube lemma, we get that u is a weak equivalence such that the following diagram

$$\begin{array}{ccccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i_f} & \mathrm{cone}(f) & \xrightarrow{p_f} & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \\ & & \| & & \| & & \downarrow{\scriptstyle u} & & \| \\ & & Y & \xrightarrow[i_f]{} & \mathrm{cone}(f) & \xrightarrow[i(i_f)]{} & \mathrm{cone}(i_f) & \xrightarrow[p(i_f)]{} & \Sigma Y \end{array}$$

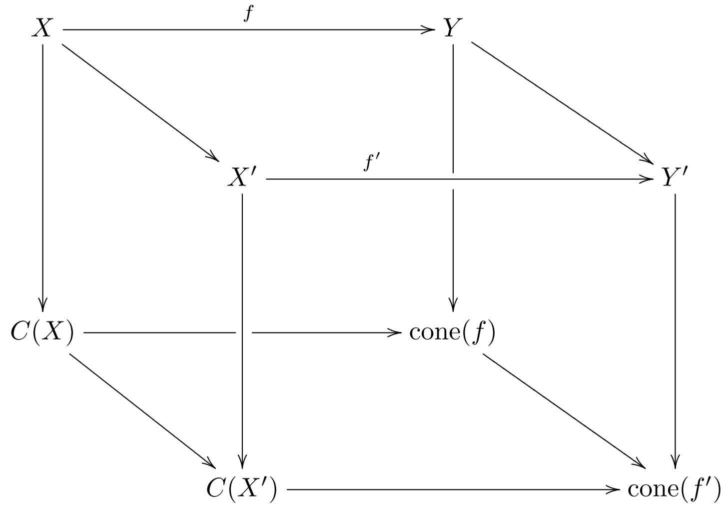
is commutative.

Axiom TR4:

Suppose we have a commutative diagram

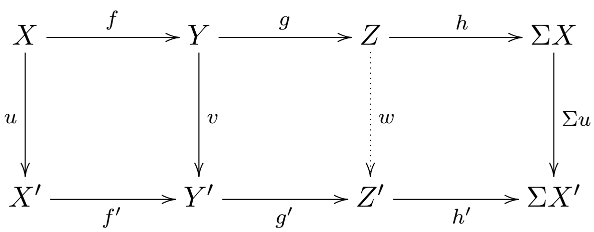
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

then we consider the following diagram



(1.9)

thus, we obtain a commutative diagram



Axiom TR5:

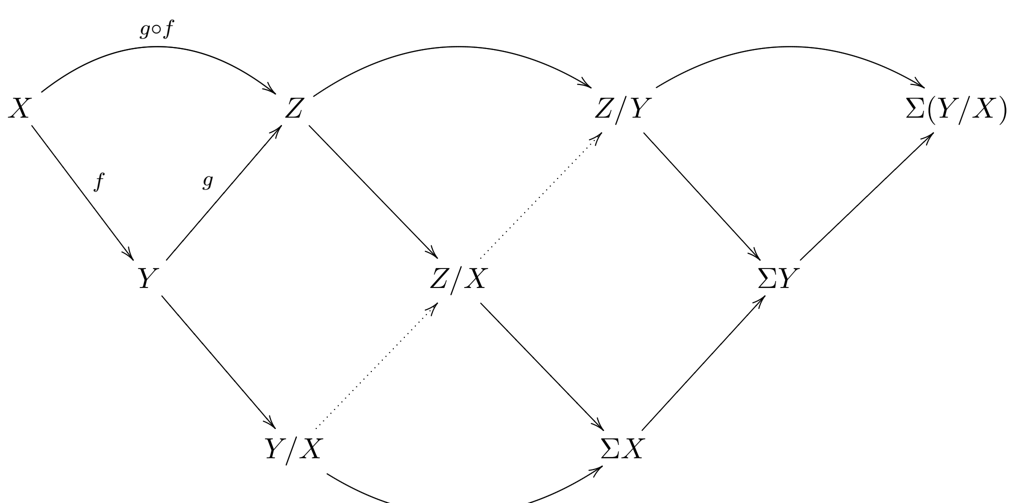
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two cofibrations. We have cofibre sequences

$$X \xrightarrow{f} Y \to Y/X \to \Sigma X\,,$$

$$Y \xrightarrow{g} Z \to Z/Y \to \Sigma Y\,,$$

$$X \xrightarrow{g\circ f} Z \to Z/X \to \Sigma X\,.$$

Then we get a diagram



where the sequence $Y/X \to Z/X \to Z/Y \to \Sigma(Y/X)$ is a cofibre sequence. This finishes the proof. $\square$

We recall that, if $\mathscr{C}$ is a pointed model category then the homotopy category $\mathrm{Ho}(\mathscr{C})$ is a closed-$\mathrm{Ho}(\Delta^{\mathrm{op}}\mathscr{S}ets_*)$-module, see [18].

Definition 1.3.17. A *stable model category* $\mathscr{C}$ is a pointed model such that the suspension functor $\Sigma\colon \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{C})$ is an equivalence of categories.

Theorem 1.3.16 says that the homotopy category of a stable model category is a triangulated category.

Stable homotopy category with weak generators

Definition 1.3.18. Let $\mathscr{T}$ be a triangulated category with arbitrary coproducts. An object X of $\mathscr{T}$ is called *compact*, if for any family $\{Y_i\}_{i\in I}$ of objects of $\mathscr{T}$, the canonical homomorphism of Abelian groups

$$\bigoplus_{i\in I} \mathrm{Hom}_{\mathscr{T}}(X, Y_i) \to \mathrm{Hom}_{\mathscr{T}}(X, \bigoplus_{i\in I} Y_i)$$

is an isomorphism.

Definition 1.3.19. Let $\mathscr{S}$ be a pre-triangulated category and let $\mathcal{G}$ be a set of objects of $\mathscr{S}$. The set $\mathcal{G}$ is called *set of weak generators* for $\mathscr{S}$, if $\mathrm{Hom}_{\mathscr{S}}(\Sigma^n G, X) = 0$ for all $G \in \mathcal{G}$ and all $n \geq 0$ implies $X \simeq *$.

Let $\mathscr{T}$ be a triangulated category. For any object G of $\mathscr{T}$, we set $\Sigma^n G = \Omega^{-n}$ for integers $n < 0$. Then, we say that a set $\mathcal{G}$ of objects of $\mathscr{T}$ is a *set of weak generators* for $\mathscr{S}$, if $\mathrm{Hom}_{\mathscr{S}}(\Sigma^n G, X) = 0$ for all $G \in \mathcal{G}$ and all $n \in \mathbb{Z}$ implies $X \simeq *$.

Example 1.3.20. If $\mathbb{S}$ is the sphere spectrum, then $\mathcal{G} = \{\mathbb{S}\}$ is a set of weak generators of the stable homotopy category of symmetric spectra of simplicial sets [20].

1.4 Symmetric spectra

Let us start this section with some preliminaries. The main reference for this section is [19]. Throughout all the text, a spectrum will be a symmetric spectrum.

1.4.1 Restriction and corestriction on categories

Let G be a group. We can consider G itself as a category with an object and G as set of morphisms. For a category $\mathscr{C}$, we denote by $\mathscr{C}^G$ the category of functors from G to $\mathscr{C}$. A *G-object* of $\mathscr{C}$ is a pair (X, ρ_X), where X is an object of $\mathscr{C}$ and $\rho_X : G \to \mathrm{Aut}_{\mathscr{C}}(X)$ is a homomorphism of groups. A morphism in $\mathscr{C}^G$ corresponds to a G-equivariant morphism $(X, \rho_X) \to (X', \rho_{X'})$ of G-objects of $\mathscr{C}$, that is, an endomorphism $\varphi \colon X \to X'$ such that

$$\varphi \circ \rho_X(g) = \rho_{X'}(g) \circ \varphi$$

for all $g \in G$. Note that the giving of a functor $G \to \mathscr{C}$ is the same as giving a G-object of $\mathscr{C}$.

Suppose that $\mathscr{C}$ is a category with coproducts and G is a finite group and let $n = |G|$ be the order of G. For any object X of $\mathscr{C}$, we define an object $G \times X$ of $\mathscr{C}^G$ to the functor $G \to \mathscr{C}$ associated to a pair $(X^{\amalg n}, \rho_{X^{\amalg n}})$ where $\rho_{X^{\amalg n}} : G \to \mathrm{Aut}_{\mathscr{C}}(X^{\amalg n})$ is defined by permuting the components of $X^{\amalg n}$. We have a functor $G \times - \colon \mathscr{C} \to \mathscr{C}^G$. For any object X of $\mathscr{C}^G$, we define an object X/G of $\mathscr{C}$ to the colimit $X/G := \mathrm{colim}\, X$, where X is viewed as a functor $G \to \mathscr{C}$. We have a functor $-/G \colon \mathscr{C}^G \to \mathscr{C}$.

Definition 1.4.1. Let H be a subgroup of G. The *restriction functor*

$$\mathrm{res}^G_H \colon \mathscr{C}^G \to \mathscr{C}^H$$

sends a functor $G \to \mathscr{C}$ to the composite $H \hookrightarrow G \to \mathscr{C}$. In terms of G-objects, res^G_H sends a G-object (X, ρ_X) to X, ρ'_X, where ρ'_X is a composition of the inclusion $H \hookrightarrow G$ and $\rho_X : G \to \mathrm{Aut}_{\mathscr{C}}(X)$.

The functor $G \times - \colon \mathscr{C} \to \mathscr{C}^G$ induces a functor $\Phi^G_H \colon \mathscr{C}^H \to (\mathscr{C}^G)^H$ which sends a functor $H \to \mathscr{C}$ to the composite

$$H \to \mathscr{C} \xrightarrow{G\times -} \mathscr{C}^G .$$

Definition 1.4.2. For any object X in $\mathscr{C}^H$, we define

$$\mathrm{cor}^G_H(X) := \mathrm{colim}\, \Phi^G_H(X) ,$$

where $\Phi^G_H(X)$ is a functor $H \to \mathscr{C}^G$. In other words, if X is an H-object of $\mathscr{C}$, then $G \times X$ is naturally an $H \times G$-object, thus $G \times X$ can be consider as an H-object and a G-object. We have,

$$\mathrm{cor}^G_H(X) = (G \times X)/H ,$$

which is naturally a G-object. The functor

$$\mathrm{cor}^G_H \colon \mathscr{C}^H \to \mathscr{C}^G$$

is called *corestriction functor*.

Remark 1.4.3. Suppose that G is a finite group. For any object G-object X, the restriction $\operatorname{res}_0^G(X)$ is the same object X, that is, the functor res_0^G is the forgetful functor. Moreover, for any object Y of $\mathscr{C}$, one has

$$\operatorname{cor}_0^G(Y) = G \times Y\,.$$

Lemma 1.4.4. (a) *The pair* $(G \times -, \operatorname{res}_0^G)$ *is an adjunction, that is, one has a bijection of sets*

$$\operatorname{Hom}_{\mathscr{C}^G}(G \times X, Y) \simeq \operatorname{Hom}_{\mathscr{C}}(X, Y)\,.$$

for any two objects $X \in \mathscr{C}$ *and* $Y \in \mathscr{C}^G$.

(b) *The pair* $(\operatorname{cor}_H^G, \operatorname{res}_H^G)$ *is an adjunction, that is, one has a bijection of sets*

$$\operatorname{Hom}_{\mathscr{C}^G}(\operatorname{cor}_H^G(X), Y) \simeq \operatorname{Hom}_{\mathscr{C}^H}(X, \operatorname{res}_H^G(Y))$$

for any two objects $X \in \mathscr{C}^H$ *and* $Y \in \mathscr{C}^G$.

Proof. (a). First of all we prove that there is a bijection

$$\operatorname{Hom}_{\mathscr{C}^G}(G \times X, Y) \simeq \operatorname{Hom}_{\mathscr{C}}(X, Y)\,.$$

Let $G = \{g_1, \ldots, g_n\}$ with $g_1 = e$. By definition we have $G \times X = X \amalg \cdots \amalg X$ (n copies of X). Suppose we have a morphism $\varphi\colon X \amalg \cdots \amalg X \to Y$ of G-objects. Let $i_1\colon X \to X \amalg \cdots \amalg X$ be the canonical morphism corresponding to the first component of $X \amalg \cdots \amalg X$. Then the composite

$$X \xrightarrow{i_1} X \amalg \cdots \amalg X \xrightarrow{\varphi} Y$$

gives a morphism $\psi\colon X \to Y$. Reciprocally, if we have a morphism $\psi\colon X \to Y$. By the universal property, the morphisms $X \xrightarrow{\psi} Y \xrightarrow{g_i} Y$ for $i = 1, \ldots, n$, induce a morphism $\varphi\colon X \amalg \cdots \amalg X \to Y$ such that the diagram

$$\begin{array}{ccc} X \amalg \cdots \amalg X & \xrightarrow{\varphi} & Y \\ \downarrow g & & \downarrow g \\ X \amalg \cdots \amalg X & \xrightarrow[\varphi]{} & Y \end{array}$$

is commutative. Thus $\varphi\colon G \times X \to Y$ is a morphism of G-objects of $\mathscr{C}$, moreover we have $\varphi \circ i_1 = \psi$. This proves the required bijection.

(b). It follows from (a) with the additional observation that for every $h \in H$, the set of commutative diagrams

$$\begin{array}{ccc} X \amalg \cdots \amalg X & \xrightarrow{\varphi} & Y \\ \downarrow h & & \downarrow h \\ X \amalg \cdots \amalg X & \xrightarrow[\varphi]{} & Y \end{array}$$

where the action of the left vertical arrow is induced by the action of H on X, is in bijection with the set of commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[\psi]{} & Y \end{array}$$

$\square$

Lemma 1.4.5. *Let $K \subset H$ be two subgroups of a finite group G. We have an isomorphism of functors*

$$\mathrm{cor}_H^G \circ \mathrm{cor}_K^H \simeq \mathrm{cor}_K^G .$$

Proof. It follows from Lemma 1.4.4. $\square$

1.4.2 Symmetric sequences

We denote by Σ the coproduct

$$\Sigma = \Sigma_0 \amalg \Sigma_1 \amalg \Sigma_2 \amalg \cdots \amalg \Sigma_n \cdots ,$$

i.e. the category whose objects are non-negative numbers and morphism are given by

$$\mathrm{Hom}_\Sigma(m,n) = \begin{cases} \Sigma_n\,, & \text{if } m = n\,, \\ 0\,, & \text{if } m \neq n\,. \end{cases}$$

Definition 1.4.6. Let $\mathscr{C}$ be a category. A functor $\Sigma \to \mathscr{C}$ is called *symmetric sequence*. The category of functors $\mathscr{C}^\Sigma$ is called *category of symmetric sequences* over $\mathscr{C}$.

Remark 1.4.7. Since $\Sigma = \coprod_{n\in\mathbb{N}} \Sigma_n$, to provide a symmetric sequence $\Sigma \to \mathscr{C}$ is the same as providing functors $\Sigma_n \to \mathscr{C}$ for all $n \in \mathbb{N}$, which is the same as giving a sequence

$$(X_0, X_1, X_2, \dots)\,,$$

where X_n is a Σ_n-equivariant object of $\mathscr{C}$ for $n \in \mathbb{N}$.

From the definition, one can deduce that if X and Y are two symmetric sequences over $\mathscr{C}$, then

$$\mathrm{Hom}_{\mathscr{C}^\Sigma}(X,Y) = \prod_{n\in\mathbb{N}} \mathrm{Hom}_{\mathscr{C}^{\Sigma_n}}(X_n, Y_n)\,.$$

For every $n \in \mathbb{N}$, we have an *evaluation functor*

$$\underline{\mathrm{Ev}}_n \colon \mathscr{C}^\Sigma \to \mathscr{C}^{\Sigma_n}\,,$$

which sends a symmetric sequence X to its n-slide X_n. We also have an *evaluation functor*

$$\mathrm{Ev}_n \colon \mathscr{C}^{\Sigma} \to \mathscr{C}\,,$$

which sends a symmetric sequence X to its n-slide X_n without the action of Σ_n, that is, $\mathrm{Ev}_n(X) = \mathrm{res}^{\Sigma_n}_{\Sigma_0}(X_n)$. The *free functor* $G_n \colon \mathscr{C} \to \mathscr{C}^{\Sigma}$ is the functor defined as

$$G_n(X) = \Sigma_n \times X$$

for all $n \geq 0$. The free functor $G_n \colon \mathscr{C} \to \mathscr{C}^{\Sigma}$ is left adjoint to the evaluation functor $\mathrm{Ev}_n \colon \mathscr{C}^{\Sigma} \to \mathscr{C}$. In fact, if X is an object in $\mathscr{C}$ and if Y is an symmetric sequence in $\mathscr{C}$, then to give a morphism $\Sigma_n \times X \to Y_n$ of Σ_n-objects, is the same as giving a morphism $X \to \mathrm{res}^{\Sigma_n}_{\Sigma_0}(Y_n) = \mathrm{Ev}_n(Y)$.

Remark 1.4.8. If $\mathscr{C}$ is a monoidal category, then $\mathscr{C}^{\Sigma}$ is naturally a monoidal $\mathscr{C}$-category. Indeed, we define a product $-\otimes - \colon \mathscr{C}^{\Sigma} \times \mathscr{C} \to \mathscr{C}^{\Sigma}$, as follows. For any object X in $\mathscr{C}^{\Sigma}$ and any object K in $\mathscr{C}$, we define a symmetric sequence $X \otimes K$ by setting

$$(X \otimes K)_n = X_n \otimes K$$

for all $n \geq 0$. If L is another object of $\mathscr{C}$, we have a natural isomorphism

$$(X \otimes K) \otimes L \simeq X \otimes (K \otimes L)\,,$$

and if $\mathbb{1}$ is the unit of $\mathscr{C}$, we have a natural isomorphism

$$X \otimes \mathbb{1} \simeq X\,.$$

Lemma 1.4.9. *Suppose $\mathscr{C}$ is a complete and cocomplete category. Then the category $\mathscr{C}^{\Sigma}$ of symmetric sequences is also complete and cocomplete.*

Proof. Let $\Phi \colon \mathscr{I} \to \mathscr{C}^{\Sigma}$ be a functor. We define the limit $\lim \Phi$ and colimit $\operatorname{colim} \Phi$ to be

$$(\lim \Phi)_n := \lim(\underline{\mathrm{Ev}}_n \circ \Phi)$$

and

$$(\operatorname{colim} \Phi)_n := \operatorname{colim}(\underline{\mathrm{Ev}}_n \circ \Phi)\,.$$

Since $\mathscr{C}^{\Sigma_n}$ is complete and cocomplete, $\lim(\underline{\mathrm{Ev}}_n \circ \Phi)$ and $\operatorname{colim}(\underline{\mathrm{Ev}}_n \circ \Phi)$ are objects of $\mathscr{C}^{\Sigma_n}$, hence $\lim \Phi$ and $\operatorname{colim} \Phi$ are objects of $\mathscr{C}^{\Sigma}$. □

Remark 1.4.10. Suppose $\mathscr{C}$ is a symmetric monoidal category with a monoidal product $\otimes$. Then for any couple of integers $m, n \geq 0$, we have a canonical functor

$$\otimes \colon \mathscr{C}^{\Sigma_m} \times \mathscr{C}^{\Sigma_n} \to \mathscr{C}^{\Sigma_m \times \Sigma_n}\,.$$

Indeed, suppose that we have objects $X \in \mathscr{C}^{\Sigma_m}$ and $Y \in \mathscr{C}^{\Sigma_n}$ for integers $m, n \geq 0$, and suppose that $\rho \colon \Sigma_m \to \mathrm{Aut}(X)$ and $\rho' \colon \Sigma_n \to \mathrm{Aut}(Y)$ are their corresponding representations. We define a homomorphism of groups $\rho \otimes \rho' \colon \Sigma_m \times \Sigma_n \to \mathrm{Aut}(X \otimes Y)$ as follows, for any element $(\sigma, \tau) \in \Sigma_m \times \Sigma_n$, we set

$$(\rho \otimes \rho')(\sigma, \tau) = \rho_\sigma \otimes \rho'_\tau \,.$$

Now, if X is an Σ_m-object and Y is an Σ_n-object of $\mathscr{C}$, then the product $X \otimes Y$ is an $\Sigma_m \times \Sigma_n$-object of $\mathscr{C}$. Thus we have a functor $\otimes \colon \mathscr{C}^{\Sigma_m} \times \mathscr{C}^{\Sigma_n} \to \mathscr{C}^{\Sigma_m \times \Sigma_n}$.

Suppose $\mathscr{C}$ is a symmetric monoidal category with coproducts. The *product* of the symmetric sequences $X \otimes Y$ of two symmetric sequences X and Y in $\mathscr{C}^\Sigma$ is defined as

$$(X \otimes Y)_n = \coprod_{i+j=n} \mathrm{cor}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (X_i \otimes Y_j)\,.$$

We obtain a bifunctor $- \otimes - \colon \mathscr{C}^\Sigma \times \mathscr{C}^\Sigma \to \mathscr{C}^\Sigma$ which sends a couple (X, Y) to $X \otimes Y$.

Lemma 1.4.11. *For any three symmetric sequences X, Y and Z on $\mathscr{C}$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathscr{C}^\Sigma}(X \otimes Y, Z) \simeq \prod_{(i,j)\in\mathbb{N}^2} \mathrm{Hom}_{\mathscr{C}^{\Sigma_i \times \Sigma_j}} \left(X_i \otimes Y_j, \mathrm{res}_{\Sigma_i \times \Sigma_j}^{\Sigma_{i+j}} (Z_{i+j})\right)\,.$$

Proof. We have,

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}^\Sigma}(X \otimes Y, Z) &\simeq \prod_{n\in\mathbb{N}} \mathrm{Hom}_{\mathscr{C}^{\Sigma_n}} \left(\coprod_{i+j=n} \mathrm{cor}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (X_i \otimes Y_j), Z_n \right) \\
&\simeq \prod_{n\in\mathbb{N}} \prod_{i+j=n} \mathrm{Hom}_{\mathscr{C}^{\Sigma_n}} \left(\mathrm{cor}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (X_i \otimes Y_j), Z_n \right) \\
&\simeq \prod_{n\in\mathbb{N}} \prod_{i+j=n} \mathrm{Hom}_{\mathscr{C}^{\Sigma_n}} \left(X_i \otimes Y_j, \mathrm{res}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} Z_n \right) \\
&\simeq \prod_{(i,j)\in\mathbb{N}^2} \mathrm{Hom}_{\mathscr{C}^{\Sigma_i \times \Sigma_j}} \left(X_i \otimes Y_j, \mathrm{res}_{\Sigma_i \times \Sigma_j}^{\Sigma_{i+j}} (Z_{i+j})\right)\,.
\end{aligned}$$

as required. □

Proposition 1.4.12. *If $\mathscr{C}$ is a closed symmetric monoidal category, then the operation $\otimes$ is a closed symmetric monoidal product on the category $\mathscr{C}^\Sigma$.*

Proof. The unit of $\mathscr{C}^\Sigma$ is the symmetric sequence

$$G_0(\mathbb{1}) = (\mathbb{1}, \emptyset, \emptyset, \cdots)\,,$$

where $\mathbb{1}$ is the unit of $\mathscr{C}$. Now, let us prove the associativity of $\otimes$ in $\mathscr{C}^{\Sigma}$. On one hand, we have

$$\begin{aligned}
((X \otimes Y) \otimes Z)_n &= \coprod_{l+k=n} \operatorname{cor}_{\Sigma_l \times \Sigma_k}^{\Sigma_n}((X \otimes Y)_l \otimes Z_k) \\
&= \coprod_{l+k=n} \operatorname{cor}_{\Sigma_l \times \Sigma_k}^{\Sigma_n} \left(\coprod_{i+j=l} \operatorname{cor}_{\Sigma_i \times \Sigma_j}^{\Sigma_l}(X_i \otimes Y_j) \otimes Z_k \right) \\
&\simeq \coprod_{l+k=n} \coprod_{i+j=l} \operatorname{cor}_{\Sigma_l \times \Sigma_k}^{\Sigma_n} \left(\operatorname{cor}_{\Sigma_i \times \Sigma_j}^{\Sigma_l}(X_i \otimes Y_j) \otimes Z_k \right) \\
&\simeq \coprod_{i+j+k=n} \operatorname{cor}_{\Sigma_l \times \Sigma_k}^{\Sigma_n} \left(\operatorname{cor}_{\Sigma_i \times \Sigma_j \times \Sigma_k}^{\Sigma_l \times \Sigma_k}((X_i \otimes Y_j) \otimes Z_k) \right) \\
&\simeq \coprod_{i+j+k=n} \operatorname{cor}_{\Sigma_i \times \Sigma_j \times \Sigma_k}^{\Sigma_n} ((X_i \otimes Y_j) \otimes Z_k) \ .
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(X \otimes (Y \otimes Z))_n &= \coprod_{i+l=n} \operatorname{cor}_{\Sigma_i \times \Sigma_l}^{\Sigma_n}(X_i \otimes (Y \otimes Z)_l \\
&= \coprod_{i+l=n} \operatorname{cor}_{\Sigma_i \times \Sigma_l}^{\Sigma_n} \left(X_i \otimes \coprod_{j+k=l} \operatorname{cor}_{\Sigma_j \times \Sigma_k}^{\Sigma_l}(Y_j \otimes Z_k) \right) \\
&\simeq \coprod_{l+k=n} \coprod_{j+k=l} \operatorname{cor}_{\Sigma_i \times \Sigma_l}^{\Sigma_n} \left(X_i \otimes \operatorname{cor}_{\Sigma_k \times \Sigma_k}^{\Sigma_l}(Y_j \otimes Z_k) \right) \\
&\simeq \coprod_{i+j+k=n} \operatorname{cor}_{\Sigma_i \times \Sigma_l}^{\Sigma_n} \left(\operatorname{cor}_{\Sigma_i \times \Sigma_j \times \Sigma_k}^{\Sigma_i \times \Sigma_l}(X_i \otimes (Y_j \otimes Z_k)) \right) \\
&\simeq \coprod_{i+j+k=n} \operatorname{cor}_{\Sigma_i \times \Sigma_j \times \Sigma_k}^{\Sigma_n} (X_i \otimes (Y_j \otimes Z_k)) \ .
\end{aligned}$$

Hence the isomorphisms $(X_i \otimes Y_j) \otimes Z_k \simeq X_i \otimes (Y_j \otimes Z_k)$ induces an isomorphism $((X \otimes Y) \otimes Z)_n \simeq (X \otimes (Y \otimes Z))_n$ as Σ_n-objects, therefore we get an isomorphism of symmetric sequences

$$(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z) \, .$$

Let us prove the commutativity of $\otimes$. We have,

$$\begin{aligned}
(X \otimes Y)_n &= \coprod_{i+j=n} \operatorname{cor}_{\Sigma_i \times \Sigma_j}^{\Sigma_n}(X_i \otimes Y_j) \\
&\simeq \coprod_{j+i=n} \operatorname{cor}_{\Sigma_j \times \Sigma_i}^{\Sigma_n}(Y_j \otimes X_i) \\
&= (Y \otimes X)_n \, ,
\end{aligned}$$

hence $(X \otimes Y)_n \simeq (Y \otimes X)_n$ for all $n \in \mathbb{N}$, then $X \otimes Y \simeq Y \otimes X$. □

Now, we give the definition of a monoid in a symmetric monoidal category (see also [25, Section 4.3]).

Definition 1.4.13. Let $\otimes$ be the symmetric monoidal product defined on a category $\mathscr{C}$ with unit $\mathbb{1}$. A *monoid* in $\mathscr{C}$ is a triplet (R, μ_R, η_R), where R is an object of $\mathscr{C}$, $\mu_R\colon R \otimes R \to R$ is a morphism called *multiplication* and $\eta_R\colon \mathbb{1} \to R$ is a morphism called *unit morphism*, such that they satisfy the conditions below:

(1) (*associativity*) The diagram

$$\begin{array}{ccc} R \otimes R \otimes R & \xrightarrow{\mu_R \otimes \mathrm{id}_R} & R \otimes R \\ {\scriptstyle \mathrm{id}_R \otimes \mu_R}\big\downarrow & & \big\downarrow{\scriptstyle \mu_R} \\ R \otimes R & \xrightarrow[\mu_R]{} & R \end{array}$$

is commutative.

(2) (*compatibility with the unit*) The composites

$$\mathbb{1} \otimes R \overset{\eta_R \otimes \mathrm{id}_R}{\longrightarrow} R \otimes R \overset{\mu_R}{\longrightarrow} R\,,$$
$$R \otimes \mathbb{1} \overset{\mathrm{id}_R \otimes \eta_R}{\longrightarrow} R \otimes R \overset{\mu_R}{\longrightarrow} R\,,$$

are the unit isomorphisms of the product $\otimes$.

In the sequel, we shall simply write R instead of (R, μ_R, η_R). A monoid R is *commutative* if it satisfies the following condition:

(3) (*commutativity*) The diagram

$$\begin{array}{ccc} R \otimes R & \xrightarrow{\mu_R} & R \\ {\scriptstyle \tau}\big\downarrow & & \big\| \\ R \otimes R & \xrightarrow[\mu_R]{} & R \end{array}$$

commutes, where $\tau\colon R \otimes R \to R \otimes R$ is the twist isomorphism of $\otimes$.

Definition 1.4.14. Let R be a (commutative) monoid in a symmetric monoidal category $(\mathscr{C}, \otimes)$ with unit $\mathbb{1}$. A *left R-module* in $\mathscr{C}$ is a pair (X, μ_X), where X is an object of $\mathscr{C}$, $\mu_X\colon R \otimes X \to X$ is a morphism called *left action* such that they satisfy the conditions below:

(1) (*associativity*), the diagram

$$\begin{array}{ccc} R \otimes R \otimes X & \xrightarrow{\mu_R \otimes \mathrm{id}_X} & R \otimes X \\ {\scriptstyle \mathrm{id}_X \otimes \mu_X}\big\downarrow & & \big\downarrow{\scriptstyle \mu_X} \\ R \otimes X & \xrightarrow[\mu_X]{} & X \end{array}$$

commutes.

(2) (*compatibility with the unit*), the composite

$$\mathbb{1} \otimes X \overset{\eta_R \otimes \mathrm{id}_X}{\to} R \otimes X \overset{\mu_X}{\to} X$$

is the unit isomorphisms of the product $\otimes$.

We shall simply write X instead of (X, μ_X).

Let $\mathscr{C}$ be a symmetric monoidal category that is cocomplete. Suppose that R is a commutative monoid in $\mathscr{C}$.

We define a new symmetric product on the category Mod_R of left R-modules in $\mathscr{C}$ as follows. We define a product

$$- \otimes_R - \colon \mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R$$

given by

$$(X, Y) \mapsto X \otimes_R Y := \mathrm{coeq}\left(X \otimes (R \otimes Y) \underset{\mathrm{id} \otimes m}{\overset{m \otimes \mathrm{id}}{\rightrightarrows}} X \otimes Y \right),$$

where the arrow at the top means the composite $X \otimes (R \otimes Y) \simeq (X \otimes R) \otimes Y \to X \otimes Y$ induced by the action $X \otimes R \to X$, and the arrow at the bottom is the morphism $X \otimes (R \otimes Y) \to X \otimes Y$ induced by the action $R \otimes Y \to Y$.

Lemma 1.4.15. *Suppose that $R \otimes - \colon \mathscr{C} \to \mathscr{C}$ preserves coequalizers. Then, for every pair of left R-modules X and Y in $\mathscr{C}$, the product $X \otimes_R Y$ is also a left R-module in $\mathscr{C}$.*

Proof. We define a morphism $R \otimes (X \otimes_R Y) \to (X \otimes_R Y)$. Since $R \otimes - \colon \mathscr{C} \to \mathscr{C}$ preserves coequalizers, $R \otimes (X \otimes_R Y)$ is the equalizer of the diagram

$$R \otimes (X \otimes (R \otimes Y)) \underset{\mathrm{id} \otimes m}{\overset{m \otimes \mathrm{id}}{\rightrightarrows}} R \otimes (X \otimes Y) \; .$$

We have a diagram of the form

$$\begin{array}{ccc} (R \otimes X) \otimes (R \otimes Y) & \rightrightarrows & R \otimes (X \otimes Y) \\ \downarrow & & \downarrow \\ X \otimes (R \otimes Y) & \rightrightarrows & X \otimes Y \end{array}$$

induced by the actions $X \otimes R \to X$ and $R \otimes Y \to Y$. Since we have an isomorphism $R \otimes (X \otimes (R \otimes Y)) \simeq (R \otimes X) \otimes (R \otimes Y)$, there is a universal morphism

$$\mathrm{coeq}\left(R \otimes \big(X \otimes (R \otimes Y)\big) \underset{\mathrm{id} \otimes m}{\overset{m \otimes \mathrm{id}}{\rightrightarrows}} R \otimes (X \otimes Y) \right) \longrightarrow \mathrm{coeq}\left(X \otimes (R \otimes Y) \underset{\mathrm{id} \otimes m}{\overset{m \otimes \mathrm{id}}{\rightrightarrows}} X \otimes Y \right),$$

i.e. a morphism $R \otimes (X \otimes_R Y) \to (X \otimes_R Y)$. This morphism defines an action for $X \otimes_R Y$ as required. □

1.4.3 Definition of symmetric spectra

In the sequel, $\mathscr{C}$ will be a symmetric monoidal model category, $\mathscr{D}$ will be a $\mathscr{C}$-model category (see [18]) and T will be an object of $\mathscr{C}$.

Definition 1.4.16. The category of *symmetric spectra* $\mathrm{Spt}_T(\mathscr{D})$ is defined as follows. A *symmetric spectrum* is an object $X = (X_0, X_1, \ldots, X_n \ldots)$ of $\mathscr{D}^\Sigma$ together with Σ_n-equivariant morphisms $X_n \otimes T \to X_{n+1}$, such that the composite

$$X_n \otimes T^{\otimes p} \to X_{n+1} \otimes T^{\otimes(p-1)} \to \cdots \to X_{n+p}$$

is $\Sigma_n \times \Sigma_p$-equivariant for all $n, p \geq 0$. A *morphism of symmetric spectra* is a collection of Σ_n-equivariant morphisms $\{f_n \colon X_n \to Y_n\}_{n\in\mathbb{N}}$ such that the following diagram

$$\begin{array}{ccc} X_n \otimes T & \xrightarrow{\sigma_X} & X_{n+1} \\ {\scriptstyle f_n\otimes T}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ Y_n \otimes T & \xrightarrow[\sigma_X]{} & Y_{n+1} \end{array}$$

is commutative for all $n \geq 0$.

Remark 1.4.17. A symmetric spectrum is an object $X = (X_0, X_1, \ldots, X_n \ldots)$ of $\mathscr{D}^\Sigma$ where X_n is an object of $\mathscr{D}^{\Sigma_n}$, together with morphisms $X_n \otimes T \to X_{n+1}$ in $\mathscr{D}^{\Sigma_n}$, such that the composite

$$X_n \otimes T^{\otimes p} \to X_{n+1} \otimes T^{\otimes(p-1)} \to \cdots \to X_{n+p}$$

is a morphism in $\mathscr{D}^{\Sigma_n \times \Sigma_p}$ for all $n, p \geq 0$.

Lemma 1.4.18. *The category of symmetric spectra* $\mathrm{Spt}_T(\mathscr{D})$ *is complete and cocomplete.*

Proof. If $\Phi \colon \mathscr{I} \to \mathrm{Spt}_T(\mathscr{D})$ is a functor, we define the limit $\lim \Phi$ and colimit $\operatorname{colim} \Phi$ to be

$$(\lim \Phi)_n := \lim(\mathrm{Ev}_n \circ \Phi)$$

and

$$(\operatorname{colim} \Phi)_n := \operatorname{colim}(\mathrm{Ev}_n \circ \Phi)\,.$$

Let $G = - \otimes T$. Since $\mathscr{C}^\Sigma$ is complete and cocomplete, $\lim \Phi$ and $\operatorname{colim} \Phi$ are objects of $\mathscr{C}^\Sigma$. To prove that they are object in $\mathrm{Spt}_T(\mathscr{D})$ we must define their structural morphisms. First of all, notice that for any functor $\Psi \colon \mathscr{I} \to \mathscr{C}$ there is a natural morphism $G(\lim \Psi) \to \lim G \circ \Psi$. In particular taking $\Psi = \mathrm{Ev}_n \circ \Phi$, we have a natural morphism

$$G(\lim \mathrm{Ev}_n \circ \Phi) \to \lim\left(G \circ \mathrm{Ev}_n \circ \Phi\right)\,.$$

On the other hand, the natural transformation $G \circ \mathrm{Ev}_n \to \mathrm{Ev}_{n+1}$ induces a natural transformation

$$G \circ \mathrm{Ev}_n \circ \Phi \to \mathrm{Ev}_{n+1} \circ \Phi \,,$$

hence a morphism $\lim G \circ \mathrm{Ev}_n \circ \Phi \to \lim \mathrm{Ev}_{n+1} \circ \Phi$. Then, we define the structure morphisms of $\lim \Phi$ to be the composite

$$G(\lim \mathrm{Ev}_n \circ \Phi) \to \lim\,(G \circ \mathrm{Ev}_n \circ \Phi) \to \lim(\mathrm{Ev}_{n+1} \circ \Phi)\,.$$

Since G is a left adjoint functor, it preserves colimit. Then the structure morphisms for $\operatorname{colim} \Phi$ is defined as the composite

$$G(\operatorname{colim} \mathrm{Ev}_n \circ \Phi) \simeq \operatorname{colim}\,(G \circ \mathrm{Ev}_n \circ \Phi) \xrightarrow{\operatorname{colim}\,(\sigma\circ\Phi)} \operatorname{colim}\,(G \circ \mathrm{Ev}_{n+1} \circ \Phi)\,.$$

Therefore, $\lim \Phi$ and $\operatorname{colim} \Phi$ are symmetric T-spectra. □

We have an endofunctor $- \otimes T \colon \mathscr{C} \to \mathscr{C}$ which sends an object X to $X \otimes T$. We set

$$\mathrm{sym}(T) := (\mathbb{1}, T, T^{\otimes 2}, T^{\otimes 3}, \dots)\,.$$

Fix an integer $n \geq 0$. For any pair (i,j) of non-negative integers such that $i + j = n$, we have a canonical morphism

$$T^{\otimes i} \otimes T^{\otimes j} \to \mathrm{res}^{\Sigma_n}_{\Sigma_i \times \Sigma_j}(T^{\otimes n})\,,$$

and this has an adjoint morphism

$$\mathrm{cor}^{\Sigma_n}_{\Sigma_i \times \Sigma_j}(T^{\otimes i} \otimes T^{\otimes j}) \to T^{\otimes n}$$

where $\mathrm{cor}^{\Sigma_n}_{\Sigma_i \times \Sigma_j}(T^{\otimes i} \otimes T^{\otimes j}) = \big(\mathrm{sym}(T) \otimes \mathrm{sym}(T)\big)_n$. Thus, we have a canonical morphism of symmetric sequences

$$m \colon \mathrm{sym}(T) \otimes \mathrm{sym}(T) \to \mathrm{sym}(T)\,.$$

Lemma 1.4.19. *The object* $\mathrm{sym}(T)$ *is a commutative monoid in* $\mathscr{C}^{\Sigma}$*.*

Proof. We prove the commutativity of m on $\mathrm{sym}(T)$. Notice that $\mathrm{cor}^{\Sigma_n}_{\Sigma_i \times \Sigma_j}(T^{\otimes i} \otimes T^{\otimes j})$ is the coproduct of $\binom{n}{i}$ copies of $T^{\otimes i} \otimes T^{\otimes j}$ and $\mathrm{cor}^{\Sigma_n}_{\Sigma_j \times \Sigma_i}(T^{\otimes j} \otimes T^{\otimes i})$ is the coproduct of $\binom{n}{i}$ copies of $T^{\otimes j} \otimes T^{\otimes i}$. Since we have an isomorphism

$$T^{\otimes i} \otimes T^{\otimes j} \simeq T^{\otimes j} \otimes T^{\otimes i}\,,$$

we get an isomorphism

$$\mathrm{cor}^{\Sigma_n}_{\Sigma_i \times \Sigma_j}(T^{\otimes i} \otimes T^{\otimes j}) \simeq \mathrm{cor}^{\Sigma_n}_{\Sigma_j \times \Sigma_i}(T^{\otimes j} \otimes T^{\otimes i})\,.$$

A similar computation shows the associativity of m and the compatibility with the unit. □

Proposition 1.4.20. *The category* $\mathrm{Spt}_T(\mathscr{D})$ *of symmetric spectra is equivalent to the category of left* $\mathrm{sym}(T)$*-modules in the category of symmetric sequences* $\mathscr{D}^\Sigma$*.*

Proof. It follows after noticing that the giving of a multiplication

$$m\colon X \otimes \mathrm{sym}(T) \to X$$

is the same as providing a collection of $\Sigma_i \times \Sigma_j$-equivariant morphisms

$$m_{i,j}\colon X_i \otimes T^{\otimes j} \to X_{i+j}\,,$$

for all $(i,j) \in \mathbb{N}$ the compatibility conditions of Definition 1.4.16. □

Definition 1.4.21. For each $n \in \mathbb{N}$, we define the *evaluation functor*

$$\mathrm{Ev}_n\colon \mathrm{Spt}_T(\mathscr{D}) \to \mathscr{D}$$

which sends a symmetric spectrum X to its n-slice X_n. For each $n \geq 0$, we define a functor $\tilde{F}_n\colon \mathscr{D} \to \mathscr{D}^\Sigma$ taking an object of A of $\mathscr{D}$ into the symmetric sequence

$$(0, \ldots, 0, \Sigma_n \times A, 0, 0, \ldots)\,,$$

where $\Sigma_n \times A$ lies in the n-th place. Hence, we set

$$F_n(A) := \tilde{F}_n(A) \otimes \mathrm{sym}(T)\,.$$

Remark 1.4.22. We have $F_0(A) = (A, A \otimes T, \ldots, A \otimes T^{\otimes n}, \ldots)$. In particular, one has

$$F_0(\mathbb{1}) = \mathrm{sym}(T)\,.$$

From the definition, we deduce the following formula:

$$(F_n A)_m = \begin{cases} \emptyset\,, & \text{if } m < n\,,\\ \Sigma_m \times_{\Sigma_{m-n}} (A \otimes T^{\otimes (m-n)})\,, & \text{if } m \geq n\,. \end{cases}$$

For each $n \geq 0$, we define a functor $\tilde{R}_n\colon \mathscr{D} \to \mathscr{D}^\Sigma$ to be the functor which sends an object of A of $\mathscr{D}$ to the symmetric sequence

$$(*, \ldots, *, \mathrm{Map}\,(\Sigma_n, A), *, *, \ldots)\,,$$

where $\mathrm{Map}\,(\Sigma_n, A)$ lies in the n-th place. Now, we define a functor $R_n\colon \mathscr{D} \to \mathscr{D}^\Sigma$ to be the functor

$$A \mapsto \mathrm{Hom}(\mathrm{sym}(T), \tilde{R}_n(A))\,.$$

Lemma 1.4.23. *For each* $n \geq 0$*, we have:*

(a) *The functor* $F_n\colon \mathscr{D} \to \mathrm{Spt}_T(\mathscr{D})$ *is a left adjoint to the evaluation functor* Ev_n*.*

(b) *The functor* $R_n \colon \mathscr{D} \to \mathrm{Spt}_T(\mathscr{D})$ *is a right adjoint to the evaluation functor* Ev_n.

Proof. It follows from the definitions. □

Lemma 1.4.24. *For any object* A *in* $\mathscr{D}$ *and* K *in* $\mathscr{C}$*, we have an isomorphism*

$$F_n(A) \otimes_{\mathrm{sym}(T)} F_m(K) \simeq F_{m+n}(A \otimes K)$$

for any pair $n, m \in \mathbb{N}$.

Proof. It follows from the definitions. □

1.4.4 Model structures on symmetric spectra

In this section, $\mathscr{C}$ will be a left proper cellular symmetric monoidal model category and $\mathscr{D}$ will be a left proper cellular $\mathscr{C}$-model category, and T will denote a cofibrant object of $\mathscr{C}$. See [17, Definition 12.1.1] for the definition of a cellular model category.

Definition 1.4.25. Let f be a morphism in $\mathrm{Spt}_T(\mathscr{D})$.

(1) f is a *level weak equivalence* if each morphism f_n is a weak equivalence in $\mathscr{D}$. Let W_T be the class of level weak equivalences.

(2) f is a *level fibration* if each morphism f_n is a fibration in $\mathscr{D}$.

(3) f is a *projective cofibration* if it has the left lifting property with respect to all level trivial fibrations.

Projective model structure on symmetric spectra

Let I be the generating cofibrations of $\mathscr{D}$ and J be the generating trivial cofibrations of $\mathscr{D}$. We denote

$$I_T := \bigcup_{n\in\mathbb{N}} F_n I \quad \text{and} \quad J_T := \bigcup_{n\in\mathbb{N}} F_n J\,.$$

Lemma 1.4.26. *If an object* A *of a model category* $\mathscr{D}$ *is small relative to the cofibrations (resp. trivial cofibrations) in* $\mathscr{D}$*, then for any* $n \geq 0$*, the spectrum* $F_n(A)$ *is small relative to level cofibrations (resp. level trivial cofibrations) in* $\mathrm{Spt}_T(\mathscr{D})$.

Proof. Let κ be a cardinal such that A is a κ-small relative to the cofibrations in $\mathscr{D}$. Let $X \colon \lambda \to \mathrm{Spt}_T(\mathscr{D})$ be a λ-sequence such that each morphism $X_\beta \to X_{\beta+1}$ is a level cofibration for $\beta+1 < \lambda$. In particular, the composition $\mathrm{Ev}_n \circ X \colon \lambda \to \mathscr{D}$ is a λ-sequence such that each morphism $\mathrm{Ev}_n(X_\beta) \to \mathrm{Ev}_n(X_{\beta+1})$ is a cofibration in $\mathscr{D}$ for $\beta + 1 < \lambda$. Then, we have

$$\mathrm{colim}_{\,\beta<\lambda} \mathrm{Hom}_{\mathscr{C}}(A, \mathrm{Ev}_n(X_\beta)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(A, \mathrm{colim}_{\,\beta<\lambda} \mathrm{Ev}_n(X_\beta))\,.$$

Since Ev_n commutes with colimits we have $\operatorname{colim}_{\beta<\lambda}\mathrm{Ev}_n(X_\beta) \simeq \mathrm{Ev}_n(\operatorname{colim}_{\beta<\lambda}X_\beta)$. Hence,

$$\begin{aligned}
\operatorname{colim}_{\beta<\lambda}\mathrm{Hom}_{\mathrm{Spt}_T(\mathscr{D})}(F_nA, X_\beta) &\simeq \operatorname{colim}_{\beta<\lambda}\mathrm{Hom}_{\mathscr{D}}(A, \mathrm{Ev}_n(X_\beta)) \\
&\simeq \mathrm{Hom}_{\mathscr{D}}(A, \operatorname{colim}_{\beta<\lambda}\mathrm{Ev}_n(X_\beta)) \\
&\simeq \mathrm{Hom}_{\mathscr{D}}(A, \mathrm{Ev}_n(\operatorname{colim}_{\beta<\lambda}X_\beta)) \\
&\simeq \mathrm{Hom}_{\mathrm{Spt}_T(\mathscr{D})}(F_nA, \operatorname{colim}_{\beta<\lambda}X_\beta)\,.
\end{aligned}$$

This proves that $F_n(A)$ is small relative to the level cofibrations in $\mathrm{Spt}_T(\mathscr{D})$. In a similar way, we prove that $F_n(A)$ is small relative to level trivial cofibrations in $\mathrm{Spt}_T(\mathscr{D})$ if A is small relative to trivial cofibrations in $\mathscr{D}$. □

Lemma 1.4.27. *We have the following statements:*

(a) *A morphism in* $\mathrm{Spt}_T(\mathscr{D})$ *is a level cofibration if and only if it is in* S_T-proj, *where* $S_T = \bigcup_{n\in\mathbb{N}} R_n(S)$ *and* S *is the class of trivial fibrations. Similarly, a morphism in* $\mathrm{Spt}_T(\mathscr{D})$ *is a level trivial cofibration if and only if it is in* S_T-proj, *where* $S_T = \bigcup_{n\in\mathbb{N}} R_n(S)$ *and* S *is the class of fibrations.*

(b) *Every morphism in* I_T-cof *is a level cofibration and every morphism in* J_T-cof *is a level trivial cofibration.*

Proof. (a). Let $f\colon X \to Y$ be a morphism in $\mathrm{Spt}_T(\mathscr{D})$ and let $g\colon A \to B$ be a morphism in $\mathscr{D}$. Since the functor R_n is right adjoint to Ev_n for $n \geq 0$, a diagram of the form

$$\begin{array}{ccc}
X & \xrightarrow{\varphi} & R_n(A) \\
{\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle R_n(g)} \\
Y & \xrightarrow[\psi]{} & R_n(B)
\end{array}$$

corresponds biunivocally to a diagram

$$\begin{array}{ccc}
\mathrm{Ev}_n(X) & \xrightarrow{\varphi} & A \\
{\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\
\mathrm{Ev}_n(Y) & \xrightarrow[\psi]{} & B
\end{array}$$

Then, we deduce that f is a level cofibration (resp. level trivial cofibration) if and only if f has the left lifting property with respect to $R_n(g)$ and all trivial fibration (resp. fibration) g in $\mathscr{C}$.

(b). Let $f\colon A \to B$ be a morphism in I. From the definition, we have

$$\mathrm{Ev}_m(F_n(f)) = \begin{cases} 0 \to 0\,, & \text{if } m < n\,, \\ \mathrm{cor}_{\Sigma_{m-n}}^{\Sigma_m}(f \otimes T^{m-n})\,, & \text{if } m \geq n\,. \end{cases}$$

Notice that, for $m \geq n$, one has that $\mathrm{cor}_{\Sigma_{m-n}}^{\Sigma_m}(f \otimes T^{m-n})$ is a coproduct of $m!/(m-n)!$ copies of $f \otimes T^{m-n}$. Since $- \otimes T$ is a left Quillen functor and f is a cofibration in $\mathscr{D}$, $f \otimes T^{m-n}$ is a cofibration in $\mathscr{D}$. Hence the morphism $\mathrm{Ev}_m(F_n(f))$ is a cofibration in $\mathscr{D}$. Then every morphism of $F_n(I)$ is a level cofibration, and every morphism of $I_T = \bigcup_{n \in \mathbb{N}} F_n(I)$ is a level cofibration. By (a), we deduce that $I_T \subset S_T$-proj, hence I_T-cof $\subset$ (S_T-proj)-cof, but (S_T-proj)-cof $= S_T$-proj, then I_T-cof $\subset S_T$-proj. Again by (a), we conclude that every morphism in I_T-cof is a level cofibration. The proof of the second case for J_T-cof is similar. □

Corollary 1.4.28. *The domains of the morphisms of I_T are small relative to I_T-cell and the domains of the morphisms of J_T are small relative to J_T-cell.*

Proof: From the definition of I_T, we get $\mathrm{dom}\,(I_T) = \bigcup_{n \in \mathbb{N}} F_n(\mathrm{dom}\,(I))$. If $X \in \mathrm{dom}\,(I_T)$, then X is equal to $F_n(A)$ for some $n \in \mathbb{N}$ and some object $A \in \mathrm{dom}\,(I)$. Since $\mathscr{D}$ is a cofibrantly generated model category and I is its set of generating cofibrations, the domains of I are small relative to the cofibrations of $\mathscr{D}$; in particular, A has this property. By Lemma 1.4.26, the symmetric spectrum $X = F_n(A)$ is small relative to level cofibrations in $\mathrm{Spt}_T(\mathscr{D})$. By Lemma 1.4.27, the class I_T-cof is contained in the class of level cofibrations. Since I_T-cell $\subset I_T$-cof, the class I_T-cell is contained in the class of level cofibrations. Hence, $X = F_n(A)$ is small relative to I_T-cell, as required. In a similar way, we prove that the domains of the morphisms of J_T are small relative to J-cell. □

Proposition 1.4.29. *We have the following assertions:*

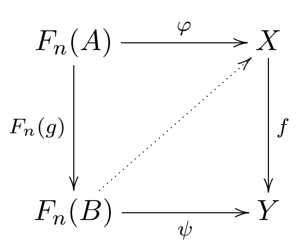
(a) *A morphism of symmetric spectra is a level trivial fibration if and only if it is in I_T-inj.*

(b) *A morphism of symmetric spectra is a projective cofibration if and only if it is in I_T-cof.*

(c) *A morphism of symmetric spectra is a level fibration if and only if it is in J_T-inj.*

(d) *A morphism of symmetric spectra is a projective cofibration and level weak equivalence if and only if it is in J_T-cof.*

Proof. (a). Let $f\colon X \to Y$ be a morphism in $\mathrm{Spt}_T(\mathscr{D})$ and let $g\colon A \to B$ be a morphism in I. Since the functor F_n is left adjoint to Ev_n for $n \geq 0$, a diagram



corresponds biunivocally to a diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \mathrm{Ev}_n(X) \\ {\scriptstyle g}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{Ev}_n(f)} \\ B & \xrightarrow[\psi]{} & \mathrm{Ev}_n(Y) \end{array}$$

We deduce that a morphism f is a level trivial fibration if and only if it is in I_T-inj.

(b). It follows immediately from (a).

(c). Let $f\colon X \to Y$ be a morphism in $\mathrm{Spt}_T(\mathscr{D})$ and let $g\colon A \to B$ be a morphism in J. Since the functor F_n is left adjoint to Ev_n for $n \geq 0$, a diagram

$$\begin{array}{ccc} F_n(A) & \xrightarrow{\varphi} & X \\ {\scriptstyle F_n(g)}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ F_n(B) & \xrightarrow[\psi]{} & Y \end{array}$$

corresponds biunivocally to a diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \mathrm{Ev}_n(X) \\ {\scriptstyle g}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{Ev}_n(f)} \\ B & \xrightarrow[\psi]{} & \mathrm{Ev}_n(Y) \end{array}$$

We deduce that a morphism f is a level fibration if and only if it is in J_T-inj.

(d). Since the class J_T-inj is equal to the class of level fibration, every morphism in J_T-cof has the left lifting property with respect to level fibrations, and in particular to level trivial fibrations. Thus every morphism in J_T-cof is a projective cofibration. One deduces that every morphism in J_T-cof is a level weak equivalence. Therefore, every morphism in J_T-cof is a projective cofibration and a level weak equivalence. Reciprocally, suppose that f is both a projective cofibration and a level weak equivalence. By the small object argument, we can decompose f into a composite $p \circ i$, where p is in J_T-inj and i is in J_T-cof. By what we said above, i is in particular a level weak equivalence. Hence, by the 2-out-of-3 axiom p is a level equivalence. Then, p is level trivial fibration, and f has the left lifting property with respect to p, so that f is retract of i. This allows us to conclude that f is in J_T-cof, as required. □

Theorem 1.4.30. *The projective cofibrations, level fibrations and level weak equivalence define a left proper cellular model structure on* $\mathrm{Spt}_T(\mathscr{D})$ *generated by the triplet*

$$(I_T, J_T, W_T)\,.$$

Proof of (MC1): By Lemma 1.4.18, the category $\mathrm{Spt}_T(\mathscr{D})$ is complete and cocomplete. □

Proof of (MC2): Let

$$\begin{array}{ccc} X & \xrightarrow{g\circ f} & Z \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & Y & \end{array}$$

be a commutative triangle of symmetric T-spectra, where two of f, g and $g \circ f$ are level weak equivalences. Then for any $n \geq 0$, we have commutative triangles

$$\begin{array}{ccc} X_n & \xrightarrow{g_n\circ f_n} & Z_n \\ & {\scriptstyle f_n}\searrow \quad \nearrow{\scriptstyle g_n} & \\ & Y_n & \end{array}$$

where two of f_n, g_n and $g_n \circ f_n$ are weak equivalences in $\mathscr{D}$. Since $\mathscr{D}$ is a model category, it satisfies the 2-out-of-3 axiom, then all the three morphisms f_n, g_n and $g_n \circ f_n$ are weak equivalences for all n. Thus, the three morphisms f, g and $g \circ f$ are level weak equivalences. □

Proof of (MC3): Let $f\colon X \to Y$ and $g\colon X' \to Y'$ are morphisms of spectra such that f is a retract of g. By definition, we have a commutative diagram of the form

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & X' & \xrightarrow{\varphi'} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\phi]{} & Y' & \xrightarrow[\psi']{} & Y \end{array}$$

where the horizontal composites are identities. Then, for $n \geq 0$ we have a commutative diagram

$$\begin{array}{ccccc} X_n & \xrightarrow{\varphi} & X'_n & \xrightarrow{\varphi'_n} & X_n \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle f_n} \\ Y_n & \xrightarrow[\phi_n]{} & Y'_n & \xrightarrow[\psi'_n]{} & Y_n \end{array}$$

where the horizontal composites are identities. By the retract axiom of $\mathscr{D}$, one deduces that, if g is a level weak equivalence or a level fibration, then f is so. On the other hand, if g is a projective cofibration, then f is projective cofibration, as the class of projective cofibrations are defined by using the left lifting property. □

Proof of (MC4): By definition, projective cofibrations have the left lifting property with respect with level trivial fibrations. Let

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[\psi]{} & Y \end{array}$$

be a commutative square where p is a level fibration and i is both a projective cofibration and a level weak equivalence. By Proposition 1.4.29 (c) and (d), the square above has a lifting, as required. □

Proof of (MC5): The class of morphisms I_T and J_T permits the small object argument, see [19]. Then, there are functorial factorizations α and β such that any morphism $f\colon X \to Y$ of symmetric T-spectra can be factored as

$$f = \beta(f) \circ \alpha(f) \text{ and } f = \delta(f) \circ \gamma(f)\,,$$

where

- $\beta(f)$ is in I_T-inj, $\alpha(f)$ is in I_T-cell,
- $\delta(f)$ is in J_T-inj, $\gamma(f)$ is in J_T-cell.

Since I_T-cell $\subset I_T$-cof, the Proposition 1.4.29 implies that:

- $\beta(f)$ is a level trivial fibration, $\alpha(f)$ is a projective cofibration, and
- $\delta(f)$ is a level fibration, $\gamma(f)$ is a projective cofibration and level weak equivalence.

This proves that $\mathrm{Spt}_T(\mathscr{D})$ is a model category with generating cofibrations I_T and generating trivial cofibrations J_T. Since colimits and pushouts in $\mathrm{Spt}_T(\mathscr{D})$ are taken level-wise and every projective cofibration is in particular a level cofibrations, the left properness on $\mathrm{Spt}_T(\mathscr{D})$ follows immediately. For the proof of the cellularity condition, see appendix of [19]. This completes the proof of the theorem. □

Stable model structure on symmetric spectra

In order to define the stable model structure on $\mathrm{Spt}_T(\mathscr{D})$, we shall use the Bousfield localization of the projective model structure on $\mathrm{Spt}_T(\mathscr{D})$ with respect to a certain set S, so that the functor $-\otimes T\colon \mathrm{Spt}_T(\mathscr{D}) \to \mathrm{Spt}_T(\mathscr{D})$ will be a Quillen equivalence. We shall define the S as follows. For each object X in $\mathscr{D}$ and integer $n \geq 0$, let

$$\zeta_n^X\colon F_{n+1}(X\otimes T) \to F_n(X)$$

the morphism corresponding by adjunction to the morphism

$$X\otimes T \to \mathrm{Ev}_{n+1}(F_n(X)) = \Sigma_{n+1}\times_{\Sigma_1}(X\otimes T)\,,$$

which induced by the canonical embedding of Σ_1 into Σ_n.

Definition 1.4.31. A symmetric spectrum X is called *U-spectrum* if X is level fibrant and the adjoint $\tilde{\sigma}\colon X_n \to UX_{n+1}$ of the structural morphism $\sigma\colon X_n\otimes T \to X_{n+1}$, is a weak equivalence for all $n \geq 0$.

Lemma 1.4.32. *Let Q be the cofibrant replacement functor of $\mathscr{D}$. Then, the following statements are equivalents:*

(a) *A symmetric spectrum X is an U-spectrum.*

(b) *For any object C in $\mathrm{dom}\,(I)$, the morphism ζ_n^{QC} from $F_{n+1}(QC\otimes T)$ to F_nQC induces an isomorphism*

$$\mathrm{map}\,(F_nQC, X) \xrightarrow{\sim} \mathrm{map}\,(F_{n+1}(QC\otimes T), X)\,,$$

where $\mathrm{map}\,(-,-)$ *is the homotopy function complex, see* [19, p. 74].

Proof. It follows from the definition of the morphisms ζ_n^{QC} for $n\in\mathbb{N}$. □

The previous lemma motivates the following definition to define S as the set

$$\left\{\zeta_n^{QC} \mid C\in \mathrm{dom}\,(I)\cup \mathrm{codom}\,(I),\, n\in\mathbb{N}\right\}.$$

Definition 1.4.33. We define the *stable model structure* on $\mathrm{Spt}_T(\mathscr{D})$ to be the localization of the projective model structure on $\mathrm{Spt}_T(\mathscr{D})$ with respect to S. We shall refer to the S-local weak equivalences as *stable weak equivalences* and to the S-local fibrations as *stable fibrations*.

The *stable model structure* on $\mathrm{Spt}_T(\mathscr{D})$ is the Bousfield localization, cf. [17], of the projective model structure on $\mathrm{Spt}_T(\mathscr{D})$ with respect to a certain set S. The stable model structure on $\mathrm{Spt}_T(\mathscr{D})$ is left proper and cellular generated by

$$(I_T, J_{T,S}, W_{S,T})\,.$$

Theorem 1.4.34. *Let $\mathscr{C}$ be a left proper cellular symmetric monoidal model category and let $\mathscr{D}$ be a left proper cellular $\mathscr{C}$-model category. Suppose the domains of the generating cofibrations of $\mathscr{C}$, $\mathscr{D}$ are cofibrant. If $f\colon T \to T'$ is a weak equivalence of cofibrant objects of $\mathscr{C}$, then f induces a natural Quillen equivalence*

$$(-) \otimes_{\mathrm{sym}(T)} \mathrm{sym}(T')\colon \mathrm{Spt}_T(\mathscr{D}) \to \mathrm{Spt}_{T'}(\mathscr{D})\,.$$

Proof. See [19]. □

Chapter 2

Motivic categories

In this chapter, we compile fundamental results of $\mathbb{A}^1$-homotopy theory of schemes developed by F. Morel and V. Voevodsky, [30].

2.1 Simplicial presheaves and sheaves

In this section we shall overview different model structures on the category of simplicial (pre-) sheaves on a small Grothendieck site.

2.1.1 Simplicial presheaves

Let $\mathscr{C}$ be a category. The category of presheaves $Pre(\mathscr{C})$ is by definition the category $\mathscr{S}ets^{\mathscr{C}^{\mathrm{op}}}$ of functors from $\mathscr{C}^{\mathrm{op}}$ to $\mathscr{S}ets$. The category of *simplicial presheaves* on $\mathscr{C}$ is the category of simplicial objects in $Pre(\mathscr{C})$ which is denoted by $\Delta^{\mathrm{op}}Pre(\mathscr{C})$. An object $\mathscr{X}$ of $\Delta^{\mathrm{op}}Pre(\mathscr{C})$ is determined by a sequence $\{\mathscr{X}_n\}_{n\geq 0}$ together with face morphisms $d_i^n : \mathscr{X}_n \to \mathscr{X}_{n-1}$ for $n \geq 1$ and $0 \leq i \leq n$; and degeneracy morphisms $s_j^n : \mathscr{X}_n \to \mathscr{X}_{n+1}$ for $n \geq 0$ and $0 \leq j \leq n$, satisfying the following simplicial relations:

$$
\begin{aligned}
d_i^n \circ d_j^{n+1} &= d_{j-1}^n \circ d_i^{n+1}\,, \qquad (i < j)\,,\\
s_i^{n+1} \circ s_j^n &= s_{j+1}^{n+1} \circ s_i^n\,, \qquad (i \leq j)\,,\\
d_i^n \circ s_j^{n-1} &= \begin{cases} s_{j-1}^{n-2} \circ d_i^{n-1}\,, & \text{if } j < i\,,\\ \mathrm{id}_{\mathscr{X}_{n-1}}\,, & \text{if } i = j \text{ or } i = j+1\,,\\ s_j^{n-2} \circ d_{i-1}^{n-1}\,, & \text{if } i > j+1\,. \end{cases}
\end{aligned}
\tag{2.1}
$$

Let $\mathscr{X}, \mathscr{Y}$ be two simplicial presheaves. The giving of a morphism of simplicial presheaves $f : \mathscr{X} \to \mathscr{Y}$ is the same as giving a sequence of morphisms of presheaves

$$\{f_n : \mathscr{X}_n \to \mathscr{Y}_n\}_{n\in\mathbb{N}}\,,$$

satisfying the equalities:

$$_{\mathscr{Y}}d_i^n \circ f_n = f_{n-1} \circ {}_{\mathscr{X}}d_i^n \quad (n \geq 1)\,, \qquad _{\mathscr{Y}}s_i^n \circ f_n = f_{n+1} \circ {}_{\mathscr{X}}s_i^n \quad (n \geq 0)\,,$$

for all $0 \leq i \leq n$, where ${}_{\mathscr{X}}d_i^n$, ${}_{\mathscr{X}}s_i^n$ (resp. ${}_{\mathscr{Y}}d_i^n$, ${}_{\mathscr{Y}}s_i^n$) are the face and degeneracy morphisms of $\mathscr{X}$ (resp. of $\mathscr{Y}$).

Remark 2.1.1. The category $\Delta^{\mathrm{op}}Pre(\mathscr{C})$ of simplicial presheaves is canonically isomorphic to the following categories:

(1) $\mathscr{S}ets^{(\Delta\times\mathscr{C})^{\mathrm{op}}}$,

(2) $\mathscr{S}ets^{\Delta^{\mathrm{op}}\times\mathscr{C}^{\mathrm{op}}}$,

(3) $(\mathscr{S}ets^{\Delta^{\mathrm{op}}})^{\mathscr{C}^{\mathrm{op}}} = (\Delta^{\mathrm{op}}\mathscr{S}ets)^{\mathscr{C}^{\mathrm{op}}}$,

(4) $(\mathscr{S}ets^{\mathscr{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}} = (Pre\mathscr{C})^{\Delta^{\mathrm{op}}}$.

Definition 2.1.2. Let $\mathscr{C}$ be a category. For every object $U \in \mathscr{C}$ and every integer $n \geq 0$, we shall denote by $\Delta_U[n]$ the representable functor

$$\mathrm{Hom}_{\mathscr{C}\times\Delta}\big(-,(U,[n])\big) : (\mathscr{C}\times\Delta)^{\mathrm{op}} \longrightarrow \mathscr{S}ets\,,$$

defined by

$$(X,[m]) \mapsto \mathrm{Hom}_{\mathscr{C}}(X,U) \times \mathrm{Hom}_{\Delta}([m],[n])\,.$$

Notice that $\Delta_U[n]$ is an object of $\mathscr{S}ets^{(\mathscr{C}\times\Delta)^{\mathrm{op}}}$, and by Remark 2.1.1, it can be seen as an object of $\Delta^{\mathrm{op}}Pre(\mathscr{C})$.

Lemma 2.1.3. *Let $\mathscr{C}$ be a category. The functor*

$$\mathscr{C} \to \Delta^{\mathrm{op}}Pre(\mathscr{C})$$

defined by $U \mapsto \Delta_U^0$ is fully faithful.

Proof. By Remark 2.1.1, the category $\Delta^{\mathrm{op}}Pre(\mathscr{C})$ is identified with $\mathscr{S}ets^{(\Delta\times\mathscr{C})^{\mathrm{op}}}$. By the Yoneda's lemma, the canonical functor $\mathscr{C}\times\Delta \to \mathscr{S}ets^{(\Delta\times\mathscr{C})^{\mathrm{op}}}$ is fully faithful. On the other hand, the functor $\mathscr{C} \to \mathscr{C}\times\Delta$, which sends an object U of $\mathscr{C}$ to the object $(U,[0])$ of $\mathscr{C}\times\Delta$, is also fully faithful. Therefore, the functor $\mathscr{C} \to \Delta^{\mathrm{op}}Pre(\mathscr{C})$ is fully faithful since it is the composition of two fully faithful functors. $\square$

Lemma 2.1.4. *Let $\mathscr{X}$ be an object of $\Delta^{\mathrm{op}}Pre(\mathscr{C})$ and let U be an object of $\mathscr{C}$. If x is an n-simplex of $\mathscr{X}(U)$, then x induces a canonical morphism of simplicial presheaves $\Delta_U[n] \to \mathscr{X}$.*

Proof. Let us fix an object $U \in \mathscr{C}$. For any object $V \in \mathscr{C}$, we have by definition,

$$\Delta_U[n](V) = \mathrm{Hom}_{\mathscr{C}}(V,U)\times\Delta[n] \simeq \coprod_{\varphi\in\mathrm{Hom}_{\mathscr{C}}(V,U)} \Delta[n]\,.$$

Then, for every object $V \in \mathscr{C}$, we define a morphism $\varphi_V : \Delta_U[n](V) \to \mathscr{X}(V)$ to be the morphism induced by the composite

$$\Delta[n] \xrightarrow{\tilde{x}} \mathscr{X}(U) \xrightarrow{\varphi^*} \mathscr{X}(V)\,,$$

where $\tilde{x}$ is the induced morphism by x. If $V \to V'$ is a morphism in $\mathscr{C}$, we naturally deduce a commutative diagram

$$\begin{array}{ccc} \Delta_U[n](V') & \xrightarrow{\varphi_{V'}} & \mathscr{X}(V') \\ \downarrow & & \downarrow \\ \Delta_U[n](V) & \xrightarrow{\varphi_V} & \mathscr{X}(V) \end{array}$$

where the vertical morphisms are the restriction morphisms. This shows that the morphisms φ_V give a morphism of simplicial presheaves $\Delta_U[n] \to \mathscr{X}$. □

2.1.2 Standard model structures on simplicial presheaves

Here we give a brief overview of several model structures of the category of simplicial presheaves on a small Grothendieck site.

We recall that a *Grothendieck site* is a category equipped with a Grothendieck topology, see [38]. We refer to [1] for an exhaustive treatment of the theory of sheaves and topos. Notice that a Grothendieck topology in [38] is called a Grothendieck pretopology in [1]. In the sequel, a Grothendieck site will always be a small Grothendieck site, i.e. the underlying category is small.

In the next paragraphs, $\mathscr{C}$ will be a Grothendieck site and $Shv(\mathscr{C})$ will denote the category of sheaves on $\mathscr{C}$. We have a *sheafification* functor $-^{\mathrm{a}}$ from $Pre(\mathscr{C})$ to $Shv(\mathscr{C})$ defined as the left adjoint,

$$-^{\mathrm{a}} : Pre(\mathscr{C}) \rightleftarrows Shv(\mathscr{C}) \ , \tag{2.2}$$

of the forgetful functor. A *point* x of the site $\mathscr{C}$ is a geometric morphism

$$x : \mathscr{S}ets \longrightarrow Shv(\mathscr{C}) \,,$$

that is, an adjunction (x^*, x_*) between $Shv(\mathscr{C})$ and $\mathscr{S}ets$, such that x^* preserves finite limits. The *stalk* of a sheaf F in $Shv(\mathscr{C})$ at x is the set $x^*(F)$, whereas the *stalk* of a presheaf G in $Pre(\mathscr{C})$ at x is the set $x^*(G^{\mathrm{a}})$.

The adjunction (2.2) and the adjunction (x^*, x_*) induce a composition of adjunctions

$$\Delta^{\mathrm{op}} Pre(\mathscr{C}) \rightleftarrows \Delta^{\mathrm{op}} Shv(\mathscr{C}) \rightleftarrows \Delta^{\mathrm{op}} \mathscr{S}ets \ .$$

If $\mathscr{X}$ is an object in $Shv(\mathscr{C})$ (resp. in $\Delta^{\mathrm{op}} Pre(\mathscr{C})$), then the *stalk* of $\mathscr{X}$ at x is the image of $\mathscr{X}$ trough the above functor from $\Delta^{\mathrm{op}} Shv(\mathscr{C})$ to $\Delta^{\mathrm{op}} \mathscr{S}ets$ (resp. from $\Delta^{\mathrm{op}} Pre(\mathscr{C})$ to $\Delta^{\mathrm{op}} \mathscr{S}ets$). In the next paragraphs, we shall suppose that $\mathscr{C}$ is a site with enough points, see [1].

Definition 2.1.5. A morphism $f : \mathscr{X} \to \mathscr{Y}$ in $Pre(\mathscr{C})$, or in $Shv(\mathscr{C})$, is a *sectionwise weak equivalence* (resp. a *sectionwise fibration*, or a *sectionwise cofibration*) if for every object U in $\mathscr{C}$, the morphism $f(U) : \mathscr{X}(U) \to \mathscr{Y}(U)$ is a weak equivalence (resp. a fibration, or a cofibration) of simplicial sets (see Example 1.1.11).

Definition 2.1.6. A morphism $f : \mathscr{X} \to \mathscr{Y}$ in $Pre(\mathscr{C})$, or in $Shv(\mathscr{C})$, is a *local weak equivalence* if f is a stalkwise weak equivalence of simplicial sets.

The following table shows the standard model structures on the category of simplicial presheaves.

Category	**Weak equivalences**	**Fibrations**	**Cofibrations**
$\Delta^{\text{op}} Pre(\mathscr{C})_{\text{inj}}$	sectionwise weak equiv.	RLP	sectionwise cof.
$\Delta^{\text{op}} Pre(\mathscr{C})_{\text{proj}}$	sectionwise weak equiv.	sectionwise fib.	LLP
$\Delta^{\text{op}} Pre(\mathscr{C})^{\text{loc}}_{\text{inj}}$	local weak equiv.	RLP	sectiontwise cof.
$\Delta^{\text{op}} Pre(\mathscr{C})^{\text{loc}}_{\text{proj}}$	local weak equiv.	sectionwise fib.	LLP

Here, RLP (resp. LLP) means that the class of fibrations (resp. cofibrations) is defined by using the right lifting property (resp. left lifting property). The abbreviation inj (resp. proj) means injective (resp. projective) model structure. We use the same notations for $Shv(\mathscr{C})$.

Theorem 2.1.7 (Heller)**.** *The category* $\Delta^{\text{op}} Pre(\mathscr{C})_{\text{inj}}$ *acquires a structure of a proper simplicial cofibrantly generated model category.*

Proof. See [16]. □

Theorem 2.1.8 (Bousfield-Kan)**.** *The category* $\Delta^{\text{op}} Pre(\mathscr{C})_{\text{proj}}$ *admits a structure of a proper simplicial cellular model category.*

Proof. More generally, see [17] for projective model structures for diagrams. □

Theorem 2.1.9 (Jardine)**.** *The category* $\Delta^{\text{op}} Pre(\mathscr{C})^{\text{loc}}_{\text{inj}}$ *is a proper simplicial cellular generated category.*

Proof. The idea of the proof consists in using Joyal's trick (Lemma 1.1.52), see [21]. □

Theorem 2.1.10 (Blander)**.** *The category* $\Delta^{\text{op}} Pre(\mathscr{C})^{\text{loc}}_{\text{proj}}$ *is a proper simplicial cellular model category.*

Proof. See [3]. □

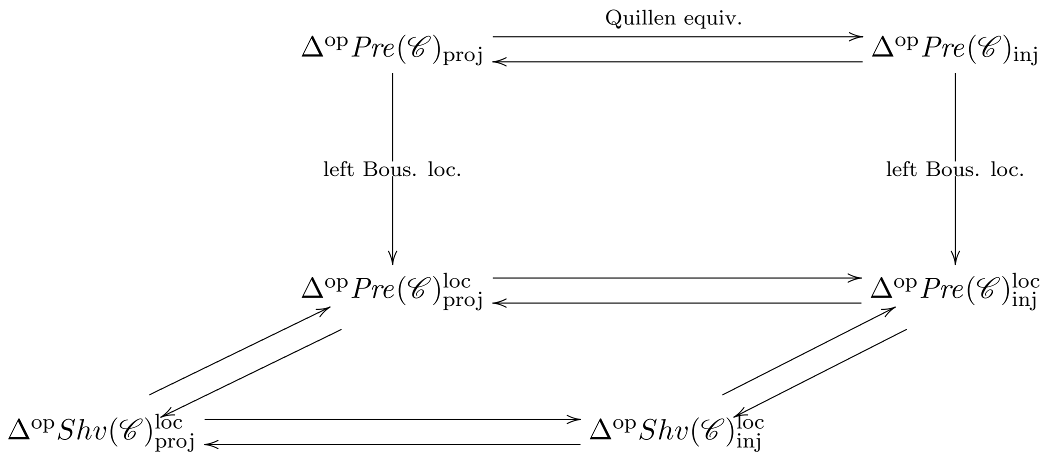
Theorem 2.1.11 (Joyal)**.** *The category* $\Delta^{\text{op}} Shv(\mathscr{C})^{\text{loc}}_{\text{inj}}$ *acquires a structure of a proper simplicial cofibrantly generated model category.*

Proof. We refer to [21]. □

Theorem 2.1.12 (Brown-Gersten)**.** *The category $\Delta^{\mathrm{op}} Shv(\mathscr{C})^{\mathrm{loc}}_{\mathrm{proj}}$ is a proper simplicial cellular model category.*

Proof. See [42]. □

The following diagram shows the relationship of standard model structures on simplicial (pre-) sheaves on a site $\mathscr{C}$,



where the double arrows mean Quillen adjunctions, see Definition 1.2.1.

2.2 Simplicial radditive functors

A radditive functor means a right additive functor, i.e. a functor that sends finite coproducts to finite products, see Definition 2.2.1. The main reference for this section is [41]. In this section, $\mathscr{C}$ will be a category closed under finite coproducts, unless otherwise mentioned.

2.2.1 Radditive functors

We start our discussion in this section giving the definition of radditive functors.

Definition 2.2.1. A functor

$$F : \mathscr{C}^{\mathrm{op}} \to \mathscr{S}ets$$

is called *radditive*[1] if it satisfies the following axioms:

(1) If $\emptyset$ is the initial object of $\mathscr{C}$, then $F(\emptyset) = \mathrm{pt}$

[1] Or right additive functor

(2) For any integer $n \geq 1$ and any finite collection $\{X_i\}_{i=1}^n$ of objects of $\mathscr{C}$, the canonical morphism of sets

$$F\left(\coprod_{i=1}^{n} X_i\right) \to \prod_{i=1}^{n} F(X_i)$$

is bijective.

We shall denote by $\mathrm{Rad}(\mathscr{C})$ the full subcategory of the category of presheaves $Pre(\mathscr{C})$ consisting of radditive functors.

By definition, we have a full embedding functor given by the forgetful functor,

$$\iota : \mathrm{Rad}(\mathscr{C}) \hookrightarrow Pre(\mathscr{C}) .$$

Example 2.2.2. If $\mathscr{C}$ is an additive category, then $\mathrm{Rad}(\mathscr{C})$ is equivalent to the Abelian category of functors from $\mathscr{C}^{\mathrm{op}}$ to the category of Abelian groups.

Remark 2.2.3. The coproduct of $\mathrm{Rad}(\mathscr{C})$ is not the coproduct of $Pre(\mathscr{C})$. For example, if X, Y are two objects of $\mathscr{C}$, then the coproduct $h_X \amalg h_Y$ in $Pre(\mathscr{C})$ is not a radditive functor, because it does not satisfy the conditions (1) and (2) of the Definition 2.2.1. In fact, if U and V are two objects of $\mathscr{C}$, then, on the one hand we have

$$\begin{aligned}(h_X \amalg h_Y)(U \amalg V) &= h_X(U \amalg V) \amalg h_Y(U \amalg V)\\ &= \mathrm{Hom}_{\mathscr{C}}(U \amalg V, X) \amalg \mathrm{Hom}_{\mathscr{C}}(U \amalg V, Y)\\ &= \Big(\mathrm{Hom}_{\mathscr{C}}(U, X) \times \mathrm{Hom}_{\mathscr{C}}(V, X)\Big) \amalg \Big(\mathrm{Hom}_{\mathscr{C}}(U, Y) \times \mathrm{Hom}_{\mathscr{C}}(V, Y)\Big),\end{aligned}$$

and, on the other hand we have

$$(h_X \amalg h_Y)(U) \times (h_X \amalg h_Y)(V) = \Big(h_X(U) \amalg h_Y(U)\Big) \times \Big(h_X(V) \amalg h_Y(V)\Big),$$

where the right-hand side is bijective to

$$\Big(h_X(U) \times h_X(V)\Big) \amalg \Big(h_X(U) \amalg h_Y(V)\Big) \amalg \Big(h_Y(U) \times h_X(V)\Big) \amalg \Big(h_Y(U) \amalg h_Y(V)\Big) .$$

Then $(h_X \amalg h_Y)(U \amalg V)$ is not canonically bijective to $(h_X \amalg h_Y)(U) \times (h_X \amalg h_Y)(V)$, thus $h_X \amalg h_Y$ fails condition (2) of Definition 2.2.1. In general, if F and G are two radditive functors, then coproduct $F \amalg G$ in $Pre(\mathscr{C})$ does not satisfies condition (1), since one has $(F \amalg G)(0) = F(0) \amalg G(0) = \mathrm{pt} \amalg \mathrm{pt}$ and $\mathrm{pt} \amalg \mathrm{pt}$ is not a final object in $\mathrm{Rad}(\mathscr{C})$.

Definition 2.2.4. Let $\mathscr{C}$ be a small category. We denote by $\mathscr{C}^{\amalg_{<\infty}}$ the full subcategory of $Pre(\mathscr{C})$ generated by finite coproducts of representable presheaves on $\mathscr{C}$.

The following lemma says that one can recover the category of presheaves from the category of radditive functors, see [41, Example 3.1].

Lemma 2.2.5. *Let $\mathscr{C}$ be a small category. Then, we have an isomorphism of categories*

$$\mathrm{Rad}(\mathscr{C}^{\amalg_{<\infty}}) \simeq \mathit{Pre}(\mathscr{C})\,.$$

Proof. Let F be a presheaf on $\mathscr{C}$. We define a contravariant functor $\overline{F} : \mathscr{C}^{\amalg_{<\infty}} \to \mathscr{S}ets$ defined by $\overline{F}(h_X){:=}F(X)$ for all object $X \in \mathscr{C}$. If $h_X \to h_Y$ is a morphism of representable functors determined by a morphism $f : X \to Y$ in $\mathscr{C}$, then we defined $\overline{F}(h_X \to h_Y)$ to be the morphism $F(f) : F(Y) \to F(X)$. If $\mathscr{C}$ has a initial object $\emptyset$, then $\overline{F}(\emptyset) = F(\emptyset) = \mathrm{pt}$, and

$$\overline{F}\left(\coprod_{i=1}^{n} h_{X_i}\right) := \prod_{i=1}^{n} F(X_i)\,,$$

for all finite collection $\{X_i\}_{i=1}^n$ of objects of $\mathscr{C}$. This defines a functor

$$\mathit{Pre}(\mathscr{C}) \to \mathrm{Rad}(\mathscr{C}^{\amalg_{<\infty}})\,.$$

On the other hand, if G is an object in $\mathrm{Rad}(\mathscr{C}^{\amalg_{<\infty}})$, we define a functor $\tilde{G} : \mathscr{C}^{\mathrm{op}} \to \mathscr{S}ets$ by $X \mapsto G(h_X)$. This defines the inverse of the above functor. □

Suppose that $\mathscr{C}$ has a final object $*$. We recall that $\mathscr{C}_+$ denotes the full category of the category of pointed objects in $\mathscr{C}$, generated by objects $X_+ = X \amalg *$ for objects X in $\mathscr{C}$, see 5.

Lemma 2.2.6. *Suppose that $\mathscr{C}$ has a final object $*$. The category $\mathrm{Rad}(\mathscr{C}_+)$ is equivalent to the category $\mathrm{Rad}(\mathscr{C})_*$.*

Proof. We define a functor $\Phi : \mathrm{Rad}(\mathscr{C}_+) \to \mathrm{Rad}(\mathscr{C})_*$ sending any F in $\mathrm{Rad}(\mathscr{C}_+)$ to a functor $\Phi(F)$ given by $\Phi(F)(X) = F(X_+)$ for every object X in $\mathscr{C}$. Notice that the canonical morphism $X_+ \to *$ induces a morphism $* = F(*) \to F(X_+)$, which makes of $\Phi(F)$ a pointed presheaf. Reciprocally, we define a functor $\Psi : \mathrm{Rad}(\mathscr{C})_* \to \mathrm{Rad}(\mathscr{C}_+)$ sending a pointed functor $(G, *)$ to a functor $\Psi(G, *)$ given by $\Psi(G, *)(X_+) = G(X)$ for every object X_+ in $\mathscr{C}_+$. For a morphism $f : X_+ \to Y_+$, we set $\Psi(G, *)(f)$ to be the composite

$$G(Y) \xrightarrow{(\mathrm{id},*)} G(Y) \times G(*) \simeq G(Y_+) \to G(X_+) \simeq G(X) \times G(*) \to G(X)\,,$$

where the last morphism of induced by the identity of $G(X)$ and the restriction morphism induced by the morphism $X \to *$. It is not difficult to verify that Φ and Ψ define an equivalence of categories. □

Lemma 2.2.7. *We have the following assertions:*

(*a*) *Any representable functor is radditive.*

(*b*) *The Yoneda embedding induces a functor*

$$h : \mathscr{C} \to \mathrm{Rad}(\mathscr{C})\,,$$

which commutes with finite products.

(*c*) *If X and Y are two objects of $\mathscr{C}$, then the coproduct of h_X and h_Y in $\mathrm{Rad}(\mathscr{C})$ is the representable functor $h_{X \amalg Y}$. In consequence, the Yoneda embedding h of $\mathscr{C}$ into $\mathrm{Rad}(\mathscr{C})$ preserves finite coproducts.*

(*d*) *The category $\mathrm{Rad}(\mathscr{C})$ is complete.*

(*e*) *If $F : J \to \mathrm{Rad}(\mathscr{C})$ is a filtered functor, then the colimit $\operatorname{colim} F$ in $\Delta^{\mathrm{op}}Pre(\mathscr{C})$ is radditive.*

(*f*) *The category $\mathrm{Rad}(\mathscr{C})$ is closed under arbitrary coproducts of representable functors.*

Proof. (a). Let X be an object of $\mathscr{C}$. We have $h_X(\emptyset) = \mathrm{Hom}_{\mathscr{C}}(\emptyset, X) = \mathrm{pt}$. If $\{U_i\}_{i\in I}$ is a finite collection of objects of $\mathscr{C}$, then we have,

$$h_X\left(\coprod_{i\in I} U_i\right) = \mathrm{Hom}_{\mathscr{C}}\left(\coprod_{i\in I} U_i, X\right) = \prod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(U_i, X) = \prod_{i\in I} h_X(U_i)\,,$$

therefore h_X is radditive.

(b). By (a), the Yoneda embedding $\mathscr{C} \to Pre(\mathscr{C})$ factors through $h : \mathscr{C} \to \mathrm{Rad}(\mathscr{C})$. Now, if X and Y be two objects of $\mathscr{C}$, then we have $h_{X\times Y} = h_X \times h_Y$; moreover, a finite product of radditive functors is radditive (see also proof of (d)).

(c). Let X and Y be two objects of $\mathscr{C}$, let F be a radditive functor on $\mathscr{C}$ and suppose that there are two morphisms $h_X \to F$ and $h_Y \to F$. Since F is radditive, we have $F(X \amalg Y) \simeq F(X) \times F(Y)$. By Yoneda's lemma the morphisms $h_X \to F$ and $h_Y \to F$ correspond to two elements $a \in F(X)$ and $b \in F(Y)$. Since $(a, b) \in F(X) \times F(Y)$ and $F(X) \times F(Y) \simeq F(X \amalg Y)$, the pair (a, b) corresponds, by the Yoneda's lemma, to a morphism $h_{X\amalg Y} \to F$ such that the following diagram

$$\begin{array}{ccc} h_X & & \\ & \searrow & \\ & h_{X\amalg Y} & \longrightarrow F \\ & \nearrow & \\ h_Y & & \end{array}$$

is commutative. This proves that $h_{X \amalg Y}$ is the coproduct of h_X and h_Y in $\mathrm{Rad}(\mathscr{C})$.

(d). Let $\Phi : J \to \mathrm{Rad}(\mathscr{C})$ be a functor and let $\iota : \mathrm{Rad}(\mathscr{C}) \to \mathit{Pre}(\mathscr{C})$ be the forgetful functor. Since arbitrary limits commute finite products, the limit $\lim(\iota \circ \Phi)$ is a radditive functor. Then we define $\lim \Phi$ to be the limit $\lim(\iota \circ \Phi)$.

(e). Let $F : J \to \mathrm{Rad}(\mathscr{C})$ be a filtered functor and let $\operatorname{colim} F$ be the colimit in $\Delta^{\mathrm{op}}\mathit{Pre}(\mathscr{C})$. For every object $j \in J$, we have $F(j)(\emptyset) = \mathrm{pt}$. Notice that the functor $J \to \mathscr{S}\!\mathit{ets}$ given by $j \mapsto F(j)(\emptyset) = \mathrm{pt}$ has colimit $(\operatorname{colim} F)(\emptyset)$. Since J is a filtered category, we get $(\operatorname{colim} F)(\emptyset) = \mathrm{pt}$. On the other hand, for each object $j \in J$ and for every two objects $X, Y \in \mathscr{C}$, we have $F(j)(X \amalg Y) = F(j)(X) \times F(j)(Y)$. Since filtered colimits commute with finite products in the category of sets, we have

$$(\operatorname{colim} F)(X \amalg Y) = (\operatorname{colim} F)(X) \times (\operatorname{colim} F)(Y)\,,$$

thus, $\operatorname{colim} F$ is a radditive functor.

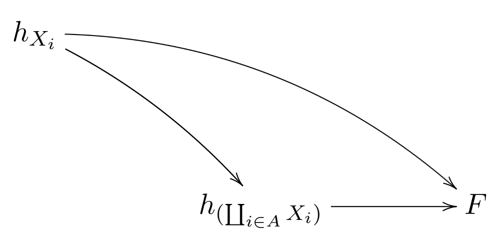
(f). Let I be a non-empty set of indices and let $\{X_i\}_{i \in I}$ be a family of objects of $\mathscr{C}$. Let us denote by $\mathscr{P}_f(I)$ the set of finite subsets of I. We order $\mathscr{P}_{fin}(I)$ with the inclusion of sets $\subseteq$. Thus $\mathscr{P}_{fin}(I)$ can be consider as a category, in which the morphism are determined by the partial order $\subseteq$. We define a functor

$$\Phi : \mathscr{P}_{fin}(I) \to \mathrm{Rad}(\mathscr{C})$$

given by

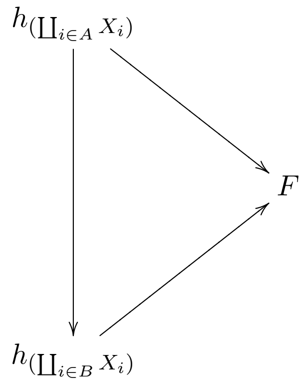
$$A \mapsto h_{(\coprod_{i \in A} X_i)}\,.$$

This is a functor; indeed, if $A \subseteq B$, then we have a canonical morphism $\coprod_{i \in A} X_i \to \coprod_{i \in B} X_i$ in $\mathscr{C}$, hence we have a morphism $h_{(\coprod_{i \in A} X_i)} \to h_{(\coprod_{i \in B} X_i)}$ in $\mathrm{Rad}(\mathscr{C})$. We claim that $\operatorname{colim} \Phi$ is the coproduct in $\mathrm{Rad}(\mathscr{C})$ of the collection $\{h_{X_i}\}_{\in I}$. Indeed, let F be an object of $\mathrm{Rad}(\mathscr{C})$ and suppose that we have a collection $\{h_{X_i} \to F\}_{i \in I}$ of morphism of simplicial radditive functors. For each $A \in \mathscr{P}_{fin}(I)$, the item (c) allows to deduce that $h_{(\coprod_{i \in A} X_i)}$ is the coproduct of the finite family $\{h_{X_i}\}_{i \in A}$. Hence, there exists a universal morphism $h_{(\coprod_{i \in A} X_i)} \to F$ such that we have a commutative diagram

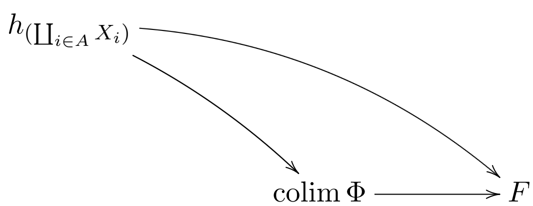


(2.3)

for all $i \in A$. Now, if $A \subseteq B$ is an inclusion of elements of $\mathscr{P}_{fin}(I)$, then we have a commutative diagram

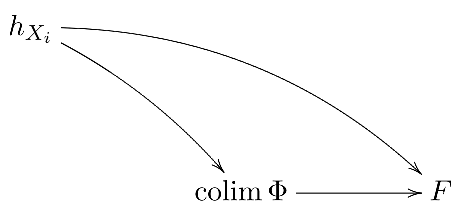


It follows that, there is a universal morphism $\operatorname{colim}\Phi \to F$ such that we have a commutative diagram



(2.4)

for all $A \in \mathscr{P}_{fin}(I)$. Combining diagrams (2.3) and (2.4), we get a commutative diagram



for all $i \in I$. This proves our claim. □

Reflexive coequalizers

Next, we shall recall the notion of reflexive coequalizer, and prove in Lemma 2.2.11, that the category of simplicial presheaves $\mathrm{Rad}(\mathscr{C})$ is closed under reflexive coequalizers. This result will be used in the proof of the existence of the radditivization functor (see Proposition 2.2.13).

Definition 2.2.8. Let $\mathscr{C}$ be an arbitrary category and let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \longrightarrow X$$

be a coequalizer in $\mathscr{C}$. We say that X is a *reflexive coequalizer*, if f and g have a common section, that is, there is a morphism $s : B \to A$ such that $f \circ s = g \circ s = \mathrm{id}_B$. In this case, the pair (f, g) is called *reflexive diagram.*

Lemma 2.2.9. *Let $\mathscr{C}$ be an arbitrary category. Suppose that we have a diagram*

$$\begin{array}{ccccc}
X_1 & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & X_2 & \xrightarrow{f_3} & X_3 \\
{\scriptstyle\alpha_1}\downarrow\downarrow{\scriptstyle\alpha_2} & & {\scriptstyle\beta_1}\downarrow\downarrow{\scriptstyle\beta_2} & & {\scriptstyle\gamma_1}\downarrow\downarrow{\scriptstyle\gamma_2} \\
Y_1 & \overset{g_1}{\underset{g_2}{\rightrightarrows}} & Y_2 & \xrightarrow{g_3} & Y_3 \\
{\scriptstyle\alpha_3}\downarrow & & \downarrow{\scriptstyle\beta_3} & & \downarrow{\scriptstyle\gamma_3} \\
Z_1 & \overset{h_1}{\underset{h_2}{\rightrightarrows}} & Z_2 & \xrightarrow{h_3} & Z_3
\end{array}$$

in $\mathscr{C}$, in which the rows and the columns are coequalizers, and the pairs (f_1, f_2) and (α_1, α_2) are reflexive, and the following diagram

$$\begin{array}{ccccc}
X_1 & \xrightarrow{f_i} & X_2 & \xrightarrow{f_3} & X_3 \\
\downarrow{\scriptstyle\alpha_j} & & \downarrow{\scriptstyle\beta_j} & & {\scriptstyle\gamma_j}\downarrow \\
Y_1 & \xrightarrow{g_i} & Y_2 & \xrightarrow{g_3} & Y_3 \\
{\scriptstyle\alpha_3}\downarrow & & \downarrow{\scriptstyle\beta_3} & & \downarrow{\scriptstyle\gamma_3} \\
Z_1 & \xrightarrow{h_i} & Z_2 & \xrightarrow{h_3} & Z_3
\end{array}$$

is commutative for $1 \leq i, j \leq 2$. Then the diagonal

$$X_1 \overset{\beta_1 \circ f_1}{\underset{\beta_2 \circ f_2}{\rightrightarrows}} Y_2 \xrightarrow{\gamma_3 \circ g_3} Z_3 \tag{2.5}$$

is a coequalizer.

Proof. First of all, we shall prove that the lower right-hand square is a pushout. Indeed, by hypothesis f_3 and α_3 are coequalizers, so, they are epimorphisms. Hence we have the following,

$$\gamma_3 = \operatorname{coeq}(\gamma_1, \gamma_2) = \operatorname{coeq}(\gamma_1 \circ f_3, \gamma_2 \circ f_3)\,,$$

$$h_3 = \operatorname{coeq}(h_1, h_2) = \operatorname{coeq}(h_1 \circ \alpha_3, h_2 \circ \alpha_3)\,.$$

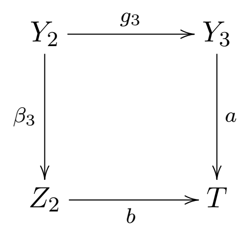
Since $\gamma_1 \circ f_3 = g_3 \circ \beta_1$ and $\gamma_1 \circ f_3 = g_3 \circ \beta_1$, we get

$$\gamma_3 = \operatorname{coeq}(g_3 \circ \beta_1, g_3 \circ \beta_2)\,.$$

Similarly, since $h_1 \circ \alpha_3 = \beta_3 \circ g_1$ and $h_2 \circ \alpha_3 = \beta_3 \circ g_2$, we get

$$h_3 = \operatorname{coeq}(\beta_3 \circ g_1, \beta_3 \circ g_2)\,.$$

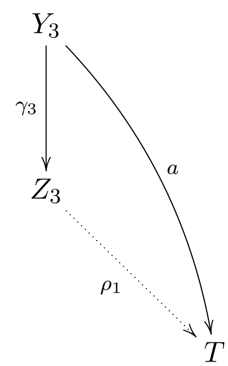
Now, let



be a commutative square in $\mathscr{C}$, so that $a \circ g_3 = b \circ \beta_3$. We have the equalities,

$$
\begin{aligned}
a \circ (g_3 \circ \beta_1) &= (a \circ g_3) \circ \beta_1 \\
&= (b \circ \beta_3) \circ \beta_1 \\
&= b \circ (\beta_3 \circ \beta_1) \\
&= b \circ (\beta_3 \circ \beta_2) \qquad \text{(because } \beta_3 = \operatorname{coeq}(\beta_1, \beta_2)\text{)} \\
&= (b \circ \beta_3) \circ \beta_2 \\
&= (a \circ g_3) \circ \beta_2 \\
&= a \circ (g_3 \circ g_2)\,,
\end{aligned}
$$

so that, we get $a \circ (g_3 \circ \beta_1) = a \circ (g_3 \circ \beta_2)$. Since $\gamma_3 = \operatorname{coeq}(g_3 \circ \beta_1, g_3 \circ \beta_2)$, there is a universal morphism $\rho_1 : T \to Z_3$ together with a commutative diagram

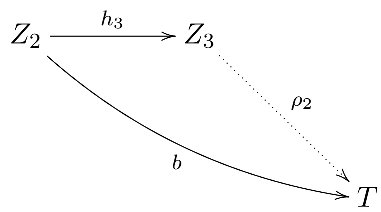


(2.6)

Similarly, we have the equalities,

$$
\begin{aligned}
b \circ (\beta_3 \circ g_1) &= (b \circ \beta_3) \circ g_1 \\
&= (a \circ g_3) \circ g_1 \\
&= a \circ (g_3 \circ g_1) \\
&= a \circ (g_3 \circ g_2) \qquad \text{(because } g_3 = \operatorname{coeq}(g_1, g_2)\text{)} \\
&= (a \circ g_3) \circ g_2 \\
&= (b \circ \beta_3) \circ g_2 \\
&= b \circ (\beta_3 \circ g_2)\,.
\end{aligned}
$$

Thus, we get $b \circ (\beta_3 \circ g_1) = b \circ (\beta_3 \circ g_2)$. Since $h_3 = \operatorname{coeq}(\beta_3 \circ g_1, \beta_3 \circ g_2)$, there is a universal morphism $\rho_2 : T \to Z_3$ together with a commutative diagram

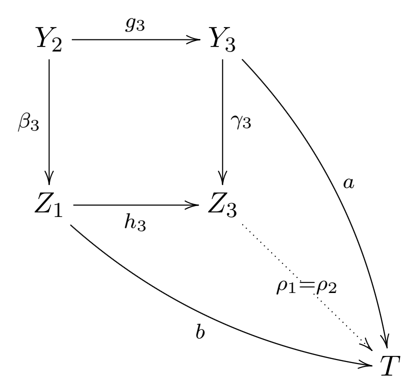


(2.7)

We claim that, $\rho_1 = \rho_2$. By the universal property of coequalizer, it is enough to show that $\rho_1 \circ h_3 = b$. In fact, we have the equalities,

$$\begin{aligned}(\rho_1 \circ h_3) \circ \beta_3 &= \rho_1 \circ (h_3 \circ \beta_3)\\ &= \rho_1 \circ (\gamma_3 \circ g_3)\\ &= (\rho_1 \circ \gamma_3) \circ g_3\\ &= a \circ g_3 \qquad\qquad \text{(by diagram (2.6))}\\ &= b \circ \beta_3\,,\end{aligned}$$

thus, $(\rho_1 \circ h_3) \circ \beta_3 = b \circ \beta_3$. Since β_3 is a coequalizer, it is an epimorphism, hence from the preceding equality, we get $\rho_1 \circ h_3 = b$, as required. Therefore, we ha a commutative diagram



which proves that the above square is, indeed, a pushout. Now, let $\theta : Y_2 \to W$ be a morphism in $\mathscr{C}$ such that

$$\theta \circ (\beta_1 \circ f_1) = \theta \circ (\beta_2 \circ f_2)\,.$$

We shall prove the following equalities

$$\theta \circ \beta_1 = \theta \circ \beta_2 \quad \text{and} \quad \theta \circ g_1 = \theta \circ g_2\,.$$

Indeed, by hypothesis the couples (f_1, f_2) and (α_1, α_2) are reflexive, then there are two morphisms $s : X_2 \to X_1$ and $t : Y_1 \to X_1$, such that

$$f_1 \circ s = f_2 \circ s = \mathrm{id} \quad \text{and} \quad \alpha_1 \circ t = \alpha_2 \circ t = \mathrm{id}\,.$$

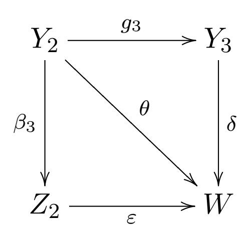
Hence, we have,

$$
\begin{aligned}
\theta \circ \beta_1 &= \theta \circ \beta_1 \circ (f_1 \circ s) \\
&= (\theta \circ \beta_1 \circ f_1) \circ s \\
&= (\theta \circ \beta_2 \circ f_2) \circ s \\
&= \theta \circ \beta_2 \circ (f_2 \circ s) \\
&= \theta \circ \beta_2 \,,
\end{aligned}
$$

thus $\theta \circ \beta_1 = \theta \circ \beta_2$. Similarly, one has,

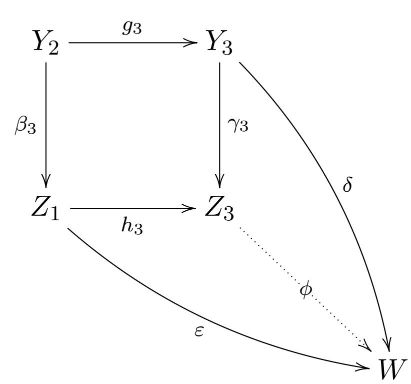
$$
\begin{aligned}
\theta \circ g_1 &= \theta \circ g_1 \circ (\alpha_1 \circ t) \\
&= (\theta \circ g_1 \circ \alpha_1) \circ t \\
&= (\theta \circ g_2 \circ \alpha_2) \circ t \\
&= \theta \circ g_2 \circ (\alpha_2 \circ t) \\
&= \theta \circ g_2 \,,
\end{aligned}
$$

so that $\theta \circ \beta_1 = \theta \circ \beta_2$. By the universal property of coequalizer, there are two morphisms $\delta : Y_3 \to W$ and $\varepsilon : 2 \to W$ such that the following diagram



(2.8)

is commutative. Hence by the universal property of pushout, there is a morphism $\phi : Z_3 \to W$ together with a commutative diagram



(2.9)

To conclude that (2.5) is a coequalizer, it remains to prove that $(\gamma_3 \circ g_3) \circ \phi = \theta$. Indeed, one has

$$
\begin{aligned}
\phi \circ (\gamma_3 \circ g_3) &= (\phi \circ \gamma_3) \circ g_3 \\
&= \delta \circ g_3 && \text{by diagram (2.9)} \\
&= \theta && \text{by diagram (2.8)}\,.
\end{aligned}
$$

This completes the proof. $\square$

Lemma 2.2.10. *In the category of sets, reflexive coequalizers commute with finite products.*

Proof. Let

$$A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \xrightarrow{\alpha} X \ ,$$

$$C \underset{g_2}{\overset{g_1}{\rightrightarrows}} D \xrightarrow{e_i} Y \ ,$$

be two reflexive coequalizers in $\mathscr{S}ets$. We shall prove that, induced diagram

$$A \times C \underset{f_2 \times g_2}{\overset{f_1 \times g_1}{\rightrightarrows}} B \times D \xrightarrow{\alpha \times \beta} X \times Y$$

is a coequalizer. In fact, for each index, let $s : B \to A$ and $t : D \to C$ be two common section of the pair (f_1, f_2) and (g_1, g_2) respectively. Then, $s \times \mathrm{id}_C : B \times C \to A \times C$ is a common section of pair $(f_1 \times \mathrm{id}_C, f_2 \times \mathrm{id}_C)$ and $\mathrm{id}_C \times t : A \times C \to A \times D$ is a common section of pair $(\mathrm{id}_A \times g_1, \mathrm{id}_A \times f_2)$. On the other hand, we have a commutative diagram

$$\begin{array}{ccccc}
A \times C & \xrightarrow{f_i \times \mathrm{id}_C} & B \times C & \xrightarrow{\alpha \times \mathrm{id}_C} & X \times C \\
\Big\downarrow{\scriptstyle \mathrm{id}_A \times g_j} & & \Big\downarrow{\scriptstyle \mathrm{id}_B \times g_j} & & \Big\downarrow{\scriptstyle \mathrm{id}_X \times g_j} \\
A \times D & \xrightarrow{f_i \times \mathrm{id}_D} & B \times D & \xrightarrow{\alpha \times \mathrm{id}_D} & X \times D \\
\Big\downarrow{\scriptstyle \mathrm{id}_A \times \beta} & & \Big\downarrow{\scriptstyle \mathrm{id}_C \times \beta} & & \Big\downarrow{\scriptstyle \mathrm{id}_X \times \beta} \\
A \times Y & \xrightarrow{f_i \times \mathrm{id}_Y} & B_1 \times Y & \xrightarrow{\alpha \times \mathrm{id}_Y} & X \times Y
\end{array}$$

for each index $i = 1, 2$. Moreover, since for any set Z, the functor $- \times Z$ is a left adjoint to the functor $\mathrm{Hom}_{\mathscr{S}ets}(Z, -)$. Then $- \times Z$ preserves colimits in $\mathscr{S}ets$, hence we get a diagram

$$\begin{array}{ccccc}
A \times C & \overset{f_1 \times \mathrm{id}_C}{\underset{f_2 \times \mathrm{id}_C}{\rightrightarrows}} & B \times C & \xrightarrow{\alpha \times \mathrm{id}_B} & X \times C \\
{\scriptstyle \mathrm{id}_A \times g_1} \downdownarrows {\scriptstyle \mathrm{id}_A \times g_2} & & {\scriptstyle \mathrm{id}_B \times g_1} \downdownarrows {\scriptstyle \mathrm{id}_B \times g_2} & & {\scriptstyle \mathrm{id}_X \times g_1} \downdownarrows {\scriptstyle \mathrm{id}_X \times g_2} \\
A \times D & \overset{f_1 \times \mathrm{id}_D}{\underset{f_2 \times \mathrm{id}_D}{\rightrightarrows}} & B \times D & \xrightarrow{\alpha \times \mathrm{id}_D} & X \times D \\
\downarrow {\scriptstyle \mathrm{id}_A \times \beta} & & \downarrow {\scriptstyle \mathrm{id}_C \times \beta} & & \downarrow {\scriptstyle \mathrm{id}_X \times \beta} \\
A \times Y & \overset{f_1 \times \mathrm{id}_Y}{\underset{f_2 \times \mathrm{id}_Y}{\rightrightarrows}} & B \times Y & \xrightarrow{\alpha \times \mathrm{id}_Y} & X \times Y
\end{array}$$

in which the rows and columns are coequalizers. By the previous lemma, the diagonal

$$A \times C \overset{f_1 \times g_1}{\underset{f_2 \times g_2}{\rightrightarrows}} B \times D \xrightarrow{\alpha \times \beta} X \times Y$$

is also a coequalizer. □

Lemma 2.2.11. *Suppose that $F \in Pre(\mathscr{C})$ is a coequalizer of a diagram $f, g : A \rightrightarrows B$ of radditive functors. If F is reflexive, then F is radditive.*

Proof. It follows since reflexive coequalizers in the category of sets commute with finite products (see previous lemma). □

The Proposition 2.2.7 (f), allows us to give the following provisional definition (see Definition 2.2.16 for a generalization of it).

Definition 2.2.12. Let $\{X_i\}_{i \in I}$ be a family of objects of $\mathscr{C}$. We denote by

$$\coprod_{i \in I}^{\mathrm{rad}} h_{X_i}$$

the coproduct in $\mathrm{Rad}(\mathscr{C})$.

In the following proposition we shall prove that the forgetful functor ι from $\mathrm{Rad}(\mathscr{C})$ to $Pre(\mathscr{C})$ has a left adjoint functor denoted by ℓ_{rad}, which plays the role of sheafification functor.

Proposition 2.2.13. *The forgetful functor* $\iota : \mathrm{Rad}(\mathscr{C}) \to \mathit{Pre}(\mathscr{C})$ *has a left adjoint functor*

$$\ell_{\mathrm{rad}} : \mathit{Pre}(\mathscr{C}) \to \mathrm{Rad}(\mathscr{C}) .$$

Moreover, for every radditive functor F *on* $\mathscr{C}$*, we have an isomorphism* $(\ell_{\mathrm{rad}} \circ \iota)(F) \simeq F$*, i.e. we have an isomorphism of functors*

$$\ell_{\mathrm{rad}} \circ \iota \simeq \mathrm{id}_{\mathrm{Rad}(\mathscr{C})} .$$

Proof. We define a functor $\ell_{\mathrm{rad}} : \mathit{Pre}(\mathscr{C}) \to \mathrm{Rad}(\mathscr{C})$, as follows. Let F be an object of $\mathit{Pre}(\mathscr{C})$. By Lemma 2.2.7 (f), $\mathrm{Rad}(\mathscr{C})$ is closed under arbitrary coproduct of representable functors; in particular, we consider the coproducts in $\mathrm{Rad}(\mathscr{C})$,

$$\coprod_{(p:U\to V)\in\mathscr{C}}^{\mathrm{rad}} \coprod_{F(V)}^{\mathrm{rad}} h_U \quad \text{and} \quad \coprod_{W\in\mathscr{C}}^{\mathrm{rad}} \coprod_{F(W)}^{\mathrm{rad}} h_W .$$

We shall define two morphisms ρ_1 and ρ_2,

$$\coprod_{(p:U\to V)\in\mathscr{C}}^{\mathrm{rad}} \coprod_{F(V)}^{\mathrm{rad}} h_U \underset{\rho_2}{\overset{\rho_1}{\rightrightarrows}} \coprod_{W\in\mathscr{C}}^{\mathrm{rad}} \coprod_{F(W)}^{\mathrm{rad}} h_W \ ,$$

as follows. For each morphism $p : U \to V$ of $\mathscr{C}$ and each element $f \in F(V)$, we have a morphism $p_* : h_U \to h_V$ and an element $F(p)(f) \in F(V)$; hence, we consider the composites

$$h_U \overset{p_*}{\to} h_V \to \coprod_{W\in\mathscr{C}}^{\mathrm{rad}} \coprod_{F(W)}^{\mathrm{rad}} h_W ,$$

then, we define ρ_1 as the universal morphism induced by these morphism as follows. On the other hand, we have the restriction morphism $F(p) : F(V) \to F(U)$, so $F(p)(f) \in F(U)$, hence, we consider the canonical morphisms

$$h_U \to \coprod_{W\in\mathscr{C}}^{\mathrm{rad}} \coprod_{F(W)}^{\mathrm{rad}} h_W ,$$

corresponding to the pair $(U, F(p)(f))$ in the set of indices of the above coproduct. Then, we define ρ_2 as the universal morphism induced by these morphisms. We define $\ell_{\mathrm{rad}}(F)$ to be the coequalizer

$$\ell_{\mathrm{rad}}(F) {:=} \mathrm{coeq}(\rho_1, \rho_2)$$

in $\mathit{Pre}(\mathscr{C})$. We claim that $\ell_{\mathrm{rad}}(F)$ is a reflexive coequalizer. First of all, we define a morphism

$$s : \coprod_{W\in\mathscr{C}}^{\mathrm{rad}} \coprod_{F(W)}^{\mathrm{rad}} h_W \to \coprod_{(p:U\to V)\in\mathscr{C}}^{\mathrm{rad}} \coprod_{F(V)}^{\mathrm{rad}} h_U ,$$

as follows. For every object W of $\mathscr{C}$ and every $f \in F(W)$, we have a morphism

$$h_W \to \coprod_{(p:U\to V)\in\mathscr{C}}^{\mathrm{rad}} \coprod_{F(V)}^{\mathrm{rad}} h_U$$

corresponding to the index $(\mathrm{id} : W \to W, f)$. Then we have $\rho_1 \circ s = \rho_2 \circ s = \mathrm{id}$, which proves our claim. Hence, by Lemma 2.2.11, $\ell_{\mathrm{rad}}(F)$ is radditive. It remains to show an isomorphism

$$\mathrm{Hom}_{\mathrm{Rad}(\mathscr{C})}(\ell_{\mathrm{rad}}(F), G) \simeq \mathrm{Hom}_{Pre(\mathscr{C})}(F, G)\,.$$

Indeed, by the universal property of coequalizer, to give a morphism $\ell_{\mathrm{rad}}(F) \to G$ in $\mathrm{Rad}(\mathscr{C})$ is the same as giving a diagram

$$\coprod_{(p:U\to V)\in\mathscr{C}}^{\mathrm{rad}} \coprod_{F(V)}^{\mathrm{rad}} h_U \underset{\rho_2}{\overset{\rho_1}{\rightrightarrows}} \coprod_{W\in\mathscr{C}}^{\mathrm{rad}} \coprod_{F(W)}^{\mathrm{rad}} h_W \xrightarrow{\quad\phi\quad} G\ ,$$

such that $\phi \circ \rho_1 = \phi \circ \rho_2$. Since G is an object of $\mathrm{Rad}(\mathscr{C})$, by the universal property of $\coprod^{\mathrm{rad}}$, the morphism ϕ corresponds to morphisms $\phi_W(f) : h_W \to G$, for all objects $W \in \mathscr{C}$ and all elements $f \in F(W)$. By the construction of ρ_1 and ρ_2, to give the above diagram is the same as giving, for every morphism $p : U \to V$ and every section $f \in F(V)$, a commutative diagram

$$\begin{array}{ccccc} & & h_V & & \\ & \overset{p_*}{\nearrow} & & \overset{\phi_V(f)}{\searrow} & \\ h_U & & & & G \\ & \underset{\mathrm{id}}{\searrow} & & \underset{\phi_U(F(p)(f))}{\nearrow} & \\ & & h_U & & \end{array}$$

For each object $W \in \mathscr{C}$ and each element $f \in F(W)$, let us denote by $\varphi_W(f)$ the element of $G(W)$ corresponding to morphism $\phi_W(f) : h_W \to G$ by Yoneda's lemma. Then, the commutativity of the previous square is paraphrased in the following equality

$$G(p)(\varphi_V(f)) = \varphi_U(F(p)(f))\,,$$

for every $p : U \to V$ and every $f \in F(V)$. In other words, it is the same as giving a collection of morphisms $\varphi_W : F(W) \to G(W)$ defined by $f \mapsto \varphi_W(f)$, for every object $W \in \mathscr{C}$, such that there is a commutative square

$$\begin{array}{ccc} F(V) & \xrightarrow{\varphi_V} & G(V) \\ {\scriptstyle F(p)}\downarrow & & \downarrow{\scriptstyle G(p)} \\ F(U) & \xrightarrow[\varphi_U]{} & G(U) \end{array}$$

for every morphism $p : U \to V$ in $\mathscr{C}$. But it means that, the collection of morphisms $\varphi_W : F(W) \to G(W)$, for $W \in \mathscr{C}$, defines a morphism of presheaves $\varphi : F \to G$. We have proved that, to give a morphism $\ell_{\text{rad}}(F) \to G$ in $\text{Rad}(\mathscr{C})$ is the same as giving a morphism $F \to G$ in *Pre*($\mathscr{C}$), which proves the required adjunction. Finally, if F is a radditive functor, then the canonical morphism $\coprod^{\text{rad}}_{W\in\mathscr{C}} \coprod^{\text{rad}}_{F(W)} h_W \to F$ induces a coequalizer diagram

$$\coprod^{\text{rad}}_{(p:U\to V)\in\mathscr{C}} \coprod^{\text{rad}}_{F(V)} h_U \underset{\rho_2}{\overset{\rho_1}{\rightrightarrows}} \coprod^{\text{rad}}_{W\in\mathscr{C}} \coprod^{\text{rad}}_{F(W)} h_W \longrightarrow F$$

in *Pre*($\mathscr{C}$). Therefore, we get a functorial isomorphism $(\ell_{\text{rad}} \circ \iota)(F) \simeq F$. □

Definition 2.2.14. The functor $\ell_{\text{rad}} : Pre(\mathscr{C}) \to \text{Rad}(\mathscr{C})$ is called *radditivization functor*. If F is a presheaf on $\mathscr{C}$, then $\ell_{\text{rad}}(F)$ is called *radditivization* of F.

Proposition 2.2.15. *The category* $\text{Rad}(\mathscr{C})$ *is complete and cocomplete.*

Proof. Let $\Phi : J \to \text{Rad}(\mathscr{C})$ be a functor and we recall that ι denotes the forgetful functor $\text{Rad}(\mathscr{C}) \to Pre(\mathscr{C})$. By Proposition 2.2.13, we have a isomorphism $\ell_{\text{rad}} \circ \iota \simeq \text{id}_{\text{Rad}(\mathscr{C})}$, hence, we get an isomorphism $\ell_{\text{rad}} \circ \iota \circ \Phi \simeq \Phi$. Moreover, since ℓ_{rad} is left adjoint, it commutes with colimits, then we have

$$\text{colim}\,(\ell_{\text{rad}} \circ \iota \circ \Phi) \simeq \ell_{\text{rad}}(\text{colim}\,(\iota \circ \Phi))\,.$$

Then, the isomorphism $\ell_{\text{rad}} \circ \iota \circ \Phi \simeq \Phi$ allows us to define $\text{colim}\,\Phi$ as the object $\ell_{\text{rad}}(\text{colim}\,(\iota \circ \Phi))$ of $\text{Rad}(\mathscr{C})$. □

If $\{X_i\}_{i\in I}$ is a family of objects of $\mathscr{C}$, then

$$\ell_{\text{rad}}(\coprod_{i\in I} h_{X_i})$$

is the coproduct in $\text{Rad}(\mathscr{C})$ of the objects $\ell_{\text{rad}}(h_{X_i}) \simeq h_{X_i}$ for all $i \in I$. Thus the following definition generalizes the Definition 2.2.12.

Definition 2.2.16. Let $\{F_i\}_{i\in I}$ be a family of objects in $\text{Rad}(\mathscr{C})$. We denote

$$\coprod^{\text{rad}}_{i\in I} F_i := \ell_{\text{rad}}\left(\coprod_{i\in I} F_i\right),$$

where $\coprod_{i\in I} F_i$ is the coproduct in *Pre*($\mathscr{C}$).

Simplicial structure

Here, we describe the simplicial structure of the category of simplicial radditive functors, see Proposition 2.2.25.

Definition 2.2.17. We say that a simplicial set K is *finite*, if for each $n \in \mathbb{N}$, the set K_n is finite.

Definition 2.2.18. For an object U of $\mathscr{C}$ and a finite simplicial set K, we define $U \otimes K$ to be the simplicial object in $\Delta^{\mathrm{op}}\mathscr{C}$ such that

$$(U \otimes K)_n := U \otimes K_n \,,$$

for all $n \in \mathbb{N}$, and the face and degeneracy morphism of $U \otimes K$ are induced by the face and degeneracy morphism of K. Observe that $U \otimes K$ is functorial in U and in K. Notice that this definition is weaker than the Definition 1.2.14.

Example 2.2.19. For every object X in a category $\mathscr{C}$ with finite coproducts, we have $X \otimes \Delta[0] = X$ and $X \otimes \partial\Delta[1] = X \amalg X$, as object in $\Delta^{\mathrm{op}}\mathscr{C}$.

Lemma 2.2.20. *Let X be an object in a category $\mathscr{C}$ with finite coproducts. For couple of termwise finite simplicial sets K and L, we have*

$$X \otimes (K \times L) = (X \otimes K) \otimes L \,.$$

Proof. For every $n \in \mathbb{N}$, we have

$$(X \otimes (K \times L))_n = \coprod_{K_n \times L_n} X = \coprod_{L_n} \left(\coprod_{K_n} X \right) = \coprod_{L_n} (X \otimes K)_n = ((X \otimes K) \otimes L)_n \,.$$

This proves that $(X \otimes (K \times L))_n = ((X \otimes K) \otimes L)_n$ for all $n \in \mathbb{N}$. □

Remark 2.2.21. Considering the embedding $h : \mathscr{C} \to \mathrm{Rad}(\mathscr{C})$, we get an embedding

$$\Delta^{\mathrm{op}} h : \Delta^{\mathrm{op}}\mathscr{C} \to \Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C}) \,.$$

Thus, for an object U of $\mathscr{C}$ and a finite simplicial set K, the product $U \otimes K$ can be considered as an object of $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$.

The following definition generalizes the previous definition.

Definition 2.2.22. For every object $\mathscr{X}$ of $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ and for every simplicial set K, we define the product $\mathscr{X} \otimes K$ to be the functor

$$\mathscr{X} \otimes K : \Delta^{\mathrm{op}} \to \mathrm{Rad}(\mathscr{C})$$

given by $[n] \mapsto \coprod_{K_n}^{\mathrm{rad}} \mathscr{X}_n$, where $\coprod^{\mathrm{rad}}$ is the coproduct in $\mathrm{Rad}(\mathscr{C})$ (see Definition 2.2.16).

We have a bifunctor

$$- \otimes - : \Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C}) \times \Delta^{\mathrm{op}}\mathscr{S}ets \to \Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$$

defined by $(\mathscr{X}, K) \mapsto \mathscr{X} \otimes K$.

Definition 2.2.23. Let $\mathscr{X}$ and $\mathscr{Y}$ be two objects in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$. We define a simplicial set $\mathrm{Map}_{\mathrm{rad}}(\mathscr{X}, \mathscr{Y})$ as the functor $\Delta^{\mathrm{op}} \to \mathscr{S}ets$ given by

$$[n] \mapsto \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})}(\mathscr{X} \otimes \Delta[n], \mathscr{Y}) .$$

Definition 2.2.24. For every object $\mathscr{X}$ of $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ and for every simplicial set K, we define a simplicial radditive functor $\mathrm{Hom}_{\otimes}(K, \mathscr{X})$ as the functor $\mathscr{C}^{\mathrm{op}} \to \Delta^{\mathrm{op}}\mathscr{S}ets$ given by

$$U \mapsto \mathrm{Map}\,(K, \mathscr{X}(U)) ,$$

where $\mathrm{Map}\,(-,-)$ is the function complex in $\Delta^{\mathrm{op}}\mathscr{S}ets$. We have a bifunctor

$$\mathrm{Hom}_{\otimes}(-,-) : \Delta^{\mathrm{op}}\mathscr{S}ets \times \Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C}) \to \Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$$

defined by $(K, \mathscr{X}) \mapsto \mathrm{Hom}_{\otimes}(K, \mathscr{X})$.

Proposition 2.2.25. *The category $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ together with the bifunctors $- \otimes -$ and $\mathrm{Map}_{\mathrm{rad}}(-,-)$ and $\mathrm{Hom}_{\otimes}(K, \mathscr{X})$ of definitions 2.2.22, 2.2.23 and 2.2.24, is a simplicial category.*

Proof. By Proposition 2.2.15, the category $\mathrm{Rad}(\mathscr{C})$ is complete and cocomplete. Since Definition 2.2.21 is a particular case of Definition 1.2.14, the proposition follows from Theorem 1.2.16. □

Corollary 2.2.26. *Let U be an object in $\mathscr{C}$, let K be a finite simplicial set and let $\mathscr{X}$ be a simplicial radditive functor on $\mathscr{C}$. Then there is a natural bijection of sets:*

$$\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})}(U \otimes K, \mathscr{X}) \simeq \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathscr{S}ets}(K, \mathscr{X}(U)) .$$

Proof. Let $f : U \otimes K \to \mathscr{X}$ be a morphism of simplicial radditive functors on $\mathscr{C}$. Let n be an integer. By definition, we have $(U \otimes K)_n = h_{\coprod_{K_n} U}$ which is a $|K_n|$ copies of U. By Yoneda's lemma, we have a functorial bijection of sets

$$\mathrm{Hom}_{\mathrm{Rad}(\mathscr{C})}(h_{(\coprod_{K_n} U)}, \mathscr{X}_n) \simeq \mathscr{X}_n(\coprod_{K_n} U) .$$

The morphism of radditive functors $f_n : (U \otimes K)_n \to \mathscr{X}_n$ corresponds to an element of $\mathscr{X}_n(\coprod_{K_n} U)$, but since $\mathscr{X}_n$ is an radditive functor and K_n is a finite set, we have $\mathscr{X}_n(\coprod_{K_n} U) = \prod_{K_n} \mathscr{X}_n(U)$. On the other hand, there is a bijection

$$\prod_{K_n} \mathscr{X}_n(U) \simeq \mathrm{Hom}_{\mathscr{S}ets}(K_n, \mathscr{X}_n(U)) .$$

Thus, a morphism $f_n : (U \otimes K)_n \to \mathscr{X}_n$ corresponds bijectively to an element of $\mathrm{Hom}_{\mathscr{S}ets}(K_n, \mathscr{X}_n(U))$, and they are compatible the the face and degeneracy morphisms. This gives the expected bijection. □

Corollary 2.2.27. *Let U be an object of $\mathscr{C}$. For every morphism $i : K \to L$ of finite simplicial sets and every morphism $p : \mathscr{X} \to \mathscr{Y}$ of simplicial radditive functors, a commutative diagram*

$$\begin{array}{ccc} U \otimes K & \longrightarrow & \mathscr{X} \\ {\scriptstyle U\otimes i}\downarrow & & \downarrow{\scriptstyle p} \\ U \otimes L & \longrightarrow & \mathscr{Y} \end{array} \tag{2.10}$$

in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ corresponds biunivocally to a commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & \mathscr{X}(U) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p(U)} \\ L & \longrightarrow & \mathscr{Y}(U) \end{array} \tag{2.11}$$

in $\Delta^{\mathrm{op}}\mathscr{S}ets$.

Proof. It follows from corollary 2.2.26. □

2.2.2 Δ-closed classes

If X is an object in $\Delta^{\mathrm{op}}\mathscr{C}$, then, by Definition 2.2.18, $X \otimes \Delta[1]$ is a simplicial object on $\mathscr{C}$. By the same definition, we have $X = X \otimes \Delta[0]$. If $i_0, i_1 : \Delta[0] \to \Delta[1]$ are the morphisms induced by the face morphisms $\partial_0, \partial_1 : [0] \to [1]$, then i_0, i_1 induce two canonical morphisms

$$\begin{aligned} \mathrm{id}_X \otimes i_0 &: X \to X \otimes \Delta[1]\,, \\ \mathrm{id}_X \otimes i_1 &: X \to X \otimes \Delta[1]\,. \end{aligned} \tag{2.12}$$

Definition 2.2.28. Let $f, g : X \to Y$ be two morphisms in $\Delta^{\mathrm{op}}\mathscr{C}$. A morphism $H : X \otimes \Delta[1] \to Y$ in $\Delta^{\mathrm{op}}\mathscr{C}$ is called a *homotopy* from f to g, if there is a commutative diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle \mathrm{id}_X\otimes i_0}\downarrow & \searrow^{f} & \\ X \otimes \Delta[1] & \xrightarrow{H} & Y \\ {\scriptstyle \mathrm{id}_X\otimes i_1}\uparrow & \nearrow_{g} & \\ X & & \end{array}$$

Two morphisms $f, g : X \to Y$ in $\Delta^{\mathrm{op}}\mathscr{C}$ are called *homotopic*, if there is a natural n and two families of morphisms

$$\{f_i : X \to Y \,|\, i = 0, \dots, n\} \quad \text{and} \quad \{H_i : X \otimes \Delta[1] \to Y \,|\, i = 1, \dots, n\}\,.$$

such that $f_0 = f$, $f_n = g$, and for each $i = 1, \dots, n$, the morphism H_i is a homotopy from f_{i-1} to f_i.

Definition 2.2.29. A morphism $f : X \to Y$ in $\Delta^{\mathrm{op}}\mathscr{C}$ is called a *homotopy equivalence*, if there exists a morphism $g : Y \to X$ in $\Delta^{\mathrm{op}}\mathscr{C}$ such that the compositions $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

Definition 2.2.30. A class of morphisms E of $\Delta^{\mathrm{op}}\mathscr{C}$ is called Δ-*closed*, if it it satisfies the following axioms:

(1) E contains all homotopy equivalences in $\Delta^{\mathrm{op}}\mathscr{C}$.

(2) E has the 2-out-of-3 property.

(3) If $f : X \to X'$ is a morphism of bisimplicial objects in $\Delta^{\mathrm{op}}\Delta^{\mathrm{op}}\mathscr{C}$, such that for every integer $n \geq 0$, either $f([n], -)$ or $f(-, [n])$ belongs to E, then the diagonal morphism $\Delta(f)$ belongs to E.

Definition 2.2.31. A Δ-closed class is called $(\Delta, \amalg_{<\infty})$-*closed* if it is closed under finite coproducts. It is called $\bar{\Delta}$-*closed* if it is closed under filtered colimits. For any class of morphisms S in $\Delta^{\mathrm{op}}\mathscr{C}$, we denote by $cl_{\Delta}(S)$ the smallest Δ-closed class containing S. Similarly, we denote by $cl_{\bar{\Delta}}(S)$ the smallest $\bar{\Delta}$-closed class containing a class S of morphisms of $\Delta^{\mathrm{op}}\mathscr{C}$.

Lemma 2.2.32. The class of weak equivalences in $\Delta^{\mathrm{op}}\mathscr{S}ets$ coincides with $\mathrm{cl}_{\Delta}(\emptyset)$. In particular, it is $\bar{\Delta}$-closed.

Proof. See [18] Lemma 5.3.1. □

Lemma 2.2.33. *Let $F : \mathscr{C} \to \mathscr{C}'$ be a functor preserving filtered colimits. Then for any class S of morphism in $\Delta^{\mathrm{op}}\mathscr{C}$, we have*

$$F(\mathrm{cl}_{\bar{\Delta}}(S)) \subset \mathrm{cl}_{\bar{\Delta}}(F(S))\,.$$

Proof. See Lemma 2.20 of [41]. □

Proposition 2.2.34. *The class of projective weak equivalences of simplicial radditive functors is $\bar{\Delta}$-closed, and it contains $\bar{\Delta}(\emptyset)$.*

Proof. It follows in view of Lemma 2.2.32 and Lemma and 2.2.33 applied to the functor of sections $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C}) \to \Delta^{\mathrm{op}}\mathscr{S}ets$ defined for every object U of $\mathscr{C}$ to be the functor $\mathscr{X} \mapsto \mathscr{X}(U)$. □

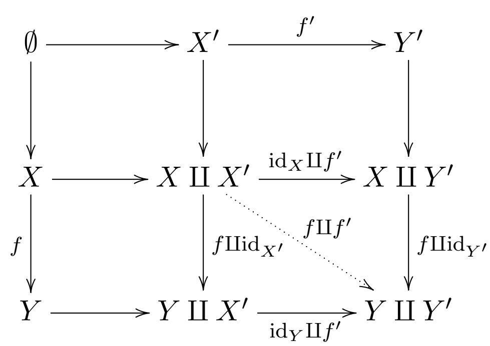
Corollary 2.2.35. *Let $\mathscr{X}$ be an object of $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$. Then, for every weak equivalence of simplicial sets $K \to L$, then induced morphism $\mathscr{X} \otimes K \to \mathscr{X} \otimes L$ is a projective equivalence.*

Proof. If follows from the previous proposition. □

Let S be a class of morphisms in $\Delta^{\mathrm{op}}\mathscr{C}$. We denote by $S \amalg \mathrm{id}_{\mathscr{C}}$ the class of morphisms of the form $f \amalg \mathrm{id}_X$, for $f \in S$ and $X \in \mathrm{ob}\,(\mathscr{C})$.

Proposition 2.2.36. *Let S be a class of morphisms in $\Delta^{\mathrm{op}}\mathscr{C}$. Then the class $\mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}})$ is closed under coproducts.*

Proof. Let $f : X \to Y$ and $f' : X' \to Y'$ be two morphism in $\Delta^{\mathrm{op}}\mathscr{C}$. From the following commutative diagram



one gets, in particular, the equality

$$f \amalg f' = (\mathrm{id}_Y \amalg f') \circ (f \amalg \mathrm{id}_{X'})\,.$$

Then, it is enough to verify that for a morphism f in $\mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}})$ and an object X in $\Delta^{\mathrm{op}}\mathscr{C}$, we have $f \amalg \mathrm{id}_X \in \mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}})$. We can simplify the problem even more, as follows. Notice that, if f in $\mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}})$ and if X is an object in $\Delta^{\mathrm{op}}\mathscr{C}$, then the coproduct $f \amalg \mathrm{id}_X$ is the diagonal of a morphism of bisimplicial objects in $\Delta^{\mathrm{op}}\Delta^{\mathrm{op}}\mathscr{C}$ whose arrows or columns are of the form $f \amalg \mathrm{id}_A$, where A is an object of $\mathscr{C}$ viewed as a constant simplicial object in Δ^{op}. Indeed, the morphism of bisimplicial objects given by $([i],[j]) \mapsto f_i \amalg \mathrm{id}_{X_j}$ has columns of the form $f \amalg \mathrm{id}_{X_j}$ and has $f \amalg \mathrm{id}_X$ as its diagonal, as we can observe it in the following diagram

$$\begin{array}{ccccccc}
f_0 \amalg \mathrm{id}_{X_0} & \rightleftarrows & f_0 \amalg \mathrm{id}_{X_1} & \rightleftarrows & f_0 \amalg \mathrm{id}_{X_2} & \rightleftarrows & \cdots \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & \\
f_1 \amalg \mathrm{id}_{X_0} & \rightleftarrows & f_1 \amalg \mathrm{id}_{X_1} & \rightleftarrows & f_1 \amalg \mathrm{id}_{X_2} & \rightleftarrows & \cdots \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & \\
f_2 \amalg \mathrm{id}_{X_0} & \rightleftarrows & f_2 \amalg \mathrm{id}_{X_1} & \rightleftarrows & f_2 \amalg \mathrm{id}_{X_2} & \rightleftarrows & \cdots \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & \\
\vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

where the vertical and horizontal arrows mean the face and degeneracy morphisms. Thus, it is enough to show that, for every morphism f in $\mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}})$ and every object A in $\mathscr{C}$, one has $f \amalg \mathrm{id} \in \mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}})$, but it follows from the inclusions

$$\mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}}) \amalg \mathrm{id}_A \subset \mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}}) \amalg \mathrm{id}_A \,,$$

for all object A in $\mathscr{C}$. □

Corollary 2.2.37. *For any class of morphisms S in $\Delta^{\mathrm{op}}\mathscr{C}$, we have*

$$\mathrm{cl}_{\Delta,\amalg_{<\infty}}(S) = \mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}}) \,.$$

In consequence, we have

$$\mathrm{cl}_{\Delta,\amalg_{<\infty}}(\emptyset) = \mathrm{cl}_\Delta(\emptyset) \,.$$

Proof. Let S be a class of morphisms in $\Delta^{\mathrm{op}}\mathscr{C}$. By definition, $\mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}})$ is contained in $\mathrm{cl}_{\Delta,\amalg_{<\infty}}(S \amalg \mathrm{id}_{\mathscr{C}})$. Reciprocally, by Proposition 2.2.36, the class $\mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}})$ is $(\Delta, \amalg_{<\infty})$-closed, then $\mathrm{cl}_{\Delta,\amalg_{<\infty}}(S \amalg \mathrm{id}_{\mathscr{C}})$ is contained in $\mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}})$. Hence, we get the following equality

$$\mathrm{cl}_{\Delta,\amalg_{<\infty}}(S \amalg \mathrm{id}_{\mathscr{C}}) = \mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}}) \,.$$

Since all identity morphisms are, in particular, homotopy equivalences, they are in $\mathrm{cl}_{\Delta,\amalg_{<\infty}}(S)$, then $S \amalg \mathrm{id}_{\mathscr{C}}$ is contained in $\mathrm{cl}_{\Delta,\amalg_{<\infty}}(S)$, hence we deduce the equality

$$\mathrm{cl}_{\Delta,\amalg_{<\infty}}(S \amalg \mathrm{id}_{\mathscr{C}}) = \mathrm{cl}_{\Delta,\amalg_{<\infty}}(S) \,.$$

Thus, we get $\mathrm{cl}_{\Delta,\amalg_{<\infty}}(S) = \mathrm{cl}_\Delta(S \amalg \mathrm{id}_{\mathscr{C}})$. In particular, we have

$$\mathrm{cl}_{\Delta,\amalg_{<\infty}}(\emptyset) = \mathrm{cl}_\Delta(\mathrm{id}_{\mathscr{C}}) \,,$$

and since $\mathrm{cl}_\Delta(\emptyset)$ contains all identity morphisms, we have $\mathrm{cl}_\Delta(\mathrm{id}_{\mathscr{C}}) = \mathrm{cl}_\Delta(\emptyset)$, which implies that $\mathrm{cl}_{\Delta,\amalg_{<\infty}}(\emptyset) = \mathrm{cl}_\Delta(\emptyset)$. □

Definition 2.2.38. Let $\mathscr{C}$ be a category with finite coproducts as before.

(1) A morphism $f : A \to X$ in $\mathscr{C}$ is called *coprojection*, if there exists an object Y of $\mathscr{C}$ such that f is isomorphic the canonical morphism $A \to A \amalg Y$.

(2) A morphism $f : A \to X$ in $\Delta^{\mathrm{op}}\mathscr{C}$ is called *termwise coprojection*, if for each $n \in \mathbb{N}$, the morphism $f_n : A_n \to X_n$ is a coprojection.

Lemma 2.2.39. *We have the following assertions:*

(a) *For every morphism $f : A \to B$ and object Y in $\mathscr{C}$, the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\;i_A\;} & A \amalg Y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g\amalg \mathrm{id}_Y} \\ B & \xrightarrow[\;i_B\;]{} & B \amalg Y \end{array}$$

where the horizontal morphisms are the canonical ones, is a cocartesian square. In consequence, coprojections are stable under pushout.

(b) *Let*

$$\begin{array}{ccc} A & \xrightarrow{\qquad} & X \\ {\scriptstyle f}\big\downarrow & & \\ B & & \end{array}$$

be a diagram in $\Delta^{\mathrm{op}}\mathscr{C}$, where f is a termwise coprojection. Then the pushout of this diagram exists. In consequence, termwise coprojections are stable under pushout.

(c) *The coproduct of a family of termwise coprojections in $\Delta^{\mathrm{op}}\mathscr{C}$, if it exists, is a termwise coprojection.*

Proof. We have a diagram

$$\begin{array}{ccc} \emptyset & \xrightarrow{\qquad} & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle i_Y} \\ A & \xrightarrow{\;i_A\;} & A \amalg Y \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g\amalg \mathrm{id}_Y} \\ B & \xrightarrow[\;i_B\;]{} & B \amalg Y \end{array}$$

in which the upper square and the big square are cocartesian, thus the lower square is cocartesian, thus we have (a). Item (b) follows from (a). Item (c) is an easy exercise. $\square$

Lemma 2.2.40. *Suppose that $\mathscr{C}$ has small coproducts. Then, the transfinite composition of termwise coprojections in $\Delta^{\mathrm{op}}\mathscr{C}$ is also a termwise coprojection.*

Proof. Notice that it is enough to show that the transfinite composition of coprojections in $\mathscr{C}$ is a coprojection. Indeed, let α be a limit ordinal an let

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots \quad (\beta < \alpha)\,,$$

be an α-sequence such that each $X_\beta \to X_{\beta+1}$ is a coprojection. By transfinite induction, we can express each X_β as a coproduct of the form $\coprod_{\gamma<\beta} X'_\gamma$ with $X'_0 = X_0$. We deduce that X_α is isomorphic to $\coprod_{\gamma<\alpha} X'_\gamma$ and the canonical morphism $X_0 \to \coprod_{\gamma<\alpha} X'_\gamma$ is the transfinite composition of this α-sequence. This proves the lemma. □

Definition 2.2.41. A commutative square in $\Delta^{\mathrm{op}}\mathscr{C}$ is called an *elementary pushout square*, if it is isomorphic to the pushout square of the form

$$\begin{array}{ccc} B & \xrightarrow{e} & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

in $\Delta^{\mathrm{op}}\mathscr{C}$, where e is a termwise coprojection.

Remark 2.2.42. Let

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \\ A & & \end{array}$$

be a diagram in $\mathscr{C}$. Since we have a canonical functor $\mathrm{Const}\colon \mathscr{C} \to \Delta^{\mathrm{op}}\mathscr{C}$, we can consider the above square as a square in $\Delta^{\mathrm{op}}\mathscr{C}$. Notice that the inclusion of simplicial sets $\partial\Delta[1] \hookrightarrow \Delta[1]$ induces a morphism

$$B \amalg B = B \otimes \partial\Delta[1] \hookrightarrow B \otimes \Delta[1]$$

in $\Delta^{\mathrm{op}}\mathscr{C}$. On the other hand, the morphisms $B \to A$ and $B \to Y$ viewed as a morphism in $\Delta^{\mathrm{op}}\mathscr{C}$ induce a canonical morphism

$$B \amalg B \to B \amalg Y$$

in $\Delta^{\mathrm{op}}\mathscr{C}$. Thus, we have a diagram

$$\begin{array}{ccc} B \amalg B & \longrightarrow & B \otimes \Delta[1] \\ \downarrow & & \\ A \amalg Y & & \end{array} \tag{2.13}$$

in $\Delta^{\mathrm{op}}\mathscr{C}$.

Remark 2.2.43. Let

$$\mathcal{Q}: \quad \begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \underset{j}{\longrightarrow} & X \end{array} \tag{2.14}$$

be a commutative in $\mathscr{C}$. The morphism $B \to X$ coming from the above diagram, induces, by the universal property of coproduct, a morphism

$$\coprod_{\Delta[1]_n} B \to X$$

in $\mathscr{C}$, for all $n \in \mathbb{N}$. Since $(B \otimes \Delta[1])_n$ is, by definition, equal to the coproduct $\coprod_{\Delta[1]_n} B$, we get a morphism

$$B \otimes \Delta[1] \to X$$

in $\Delta^{\mathrm{op}}\mathscr{C}$. On the other hand, the morphisms $A \to X$ and $Y \to X$ induces a morphism

$$A \amalg Y \to X$$

in $\Delta^{\mathrm{op}}\mathscr{C}$. Thus, we get a diagram

$$\begin{array}{ccc} B \amalg B & \longrightarrow & B \otimes \Delta[1] \\ \downarrow & & \downarrow \\ A \amalg Y & \longrightarrow & X \end{array} \tag{2.15}$$

Definition 2.2.44. For every commutative square $\mathcal{Q}$, as in (2.14), in $\mathscr{C}$, we shall denote by $K_{\mathcal{Q}}$ the pushout of the diagram (2.13). In view of the commutative square (2.15), we have a universal morphism $K_{\mathcal{Q}} \to X$.

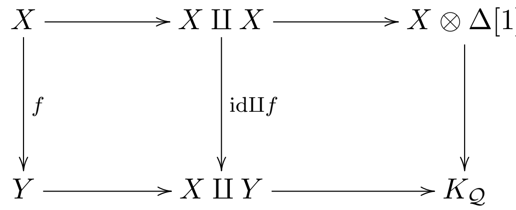
Example 2.2.45. Let $f \colon X \to Y$ be a morphism in $\mathscr{C}$. If we consider the diagram

$$\mathcal{Q}: \quad \begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

then $K_{\mathcal{Q}}$ is the cylinder $\mathrm{Cyl}(f)$ of f, where $\mathrm{Cyl}(f)$ is a pushout of the diagram

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ \downarrow & & \\ X \otimes \Delta[1] & & \end{array}$$

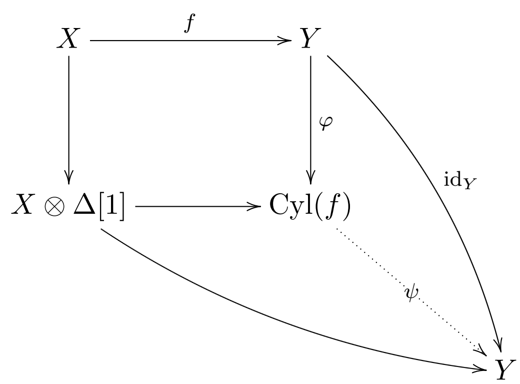
Indeed, it follows since from the following commutative diagram



in which each square is a pushout.

Lemma 2.2.46. *Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$. The canonical morphisms $\varphi\colon Y \to \mathrm{Cyl}(f)$ and $\psi\colon \mathrm{Cyl}(f) \to Y$ are each other inverses homotopy equivalences.*

Proof. We recall that $\psi\colon \mathrm{Cyl}(f) \to Y$ is defined to be the universal morphism in the following pushout diagram



In particular, we get that the composite $Y \xrightarrow{\psi} \mathrm{Cyl}(f) \xrightarrow{\varphi} Y$ is the identity. On the other hand, the composite $\mathrm{Cyl}(f) \xrightarrow{\varphi} Y \xrightarrow{\psi} \mathrm{Cyl}(f)$ is induced by the composite

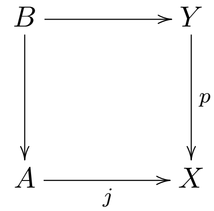
$$\Delta[1] \to \Delta[0] \xrightarrow{i_0} \Delta[1]\,,$$

which is homotopic to the identity $\Delta[1] \to \Delta[1]$. $\square$

Lemma 2.2.47. *We have the following:*

(a) For every diagram

$\mathcal{Q}$:



in $\mathscr{C}$, the morphism $A \amalg Y \to K_{\mathcal{Q}}$ is a termwise coprojection.

(b) For every morphism $f\colon X \to Y$ in $\mathscr{C}$, the canonical morphism $\varphi\colon Y \to \mathrm{Cyl}(f)$ is a termwise coprojection.

Proof. The proof follows without difficulty from the definitions. $\square$

Lemma 2.2.48. *Let*

$$\mathcal{Q}: \quad \begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[e]{} & X \end{array}$$

be an elementary pushout square in $\Delta^{\mathrm{op}}\mathscr{C}$*. Then the canonical morphism* $p_{\mathcal{Q}}\colon K_{\mathcal{Q}} \to X$ *is in* $\mathrm{cl}_{\Delta}(\emptyset)$*.*

Proof. For every $i \in \mathbb{N}$, let $\mathcal{Q}_i$ be the i-th term of $\mathcal{Q}$,

$$\mathcal{Q}_i: \quad \begin{array}{ccc} B_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow{\scriptstyle p_i} \\ A_i & \xrightarrow[e_i]{} & X_i \end{array}$$

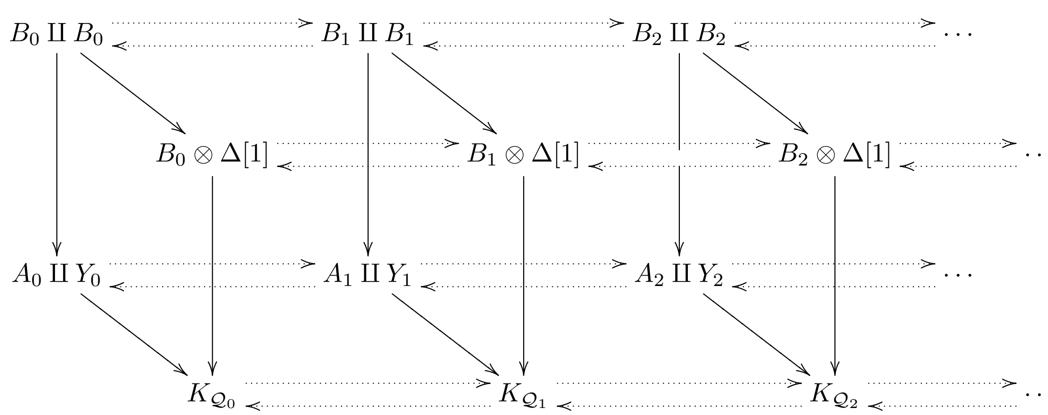
in $\mathscr{C}$. Then, for each $i \in \mathbb{N}$, we have a canonical morphism $p_{\mathcal{Q}_i}\colon K_{\mathcal{Q}_i} \to X_i$ in $\Delta^{\mathrm{op}}\mathscr{C}$, deduced from the pushout

$$\begin{array}{ccc} B_i \amalg B_i & \longrightarrow & B_i \otimes \Delta[1] \\ \downarrow & & \downarrow \\ A_i \amalg Y_i & \longrightarrow & K_{\mathcal{Q}_i} \end{array}$$

Let $B \boxtimes \Delta[1]$ the bisimplicial object given by $([i],[j]) \mapsto \coprod_{\Delta[1]_j} B_i$. Let us consider a cocartesian square,

$$\begin{array}{ccc} B \amalg B & \longrightarrow & B \boxtimes \Delta[1] \\ \downarrow & & \downarrow \\ A \amalg Y & \longrightarrow & K \end{array}$$

in $\Delta^{\mathrm{op}}\Delta^{\mathrm{op}}\mathscr{C}$. This square induces a diagram



where the horizontal arrows are the face and degeneracy morphisms. We can deduce that K has the simplicial objects $K_{\mathcal{Q}_i}$ for $i \in \mathbb{N}$, as its arrows (or columns). Since the simplicial object $B \otimes \Delta[1]$ is the diagonal of the bisimplicial object $B \boxtimes \Delta[1]$, we deduce that $K_{\mathcal{Q}}$ is also the diagonal of the bisimplicial object K, because a pushout in $\Delta^{\mathrm{op}}\mathscr{C}$ (if it exists) is a termwise pushout. Therefore, it is enough to prove the lemma for a square of the form

$$\mathcal{Q}: \quad \begin{array}{ccc} B & \xrightarrow{e_B} & B \amalg X \\ \downarrow & & \downarrow \\ A & \xrightarrow[e_A]{} & A \amalg X \end{array}$$

in $\mathscr{C}$. Notice that $\mathcal{Q}$ can be decompose as a coproduct $\mathcal{Q} = \mathcal{Q}_1 \amalg \mathcal{Q}_2$, where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are of the form

$$\mathcal{Q}_1: \quad \begin{array}{ccc} B & \xrightarrow{\mathrm{id}_B} & B \\ \downarrow & & \downarrow \\ A & \xrightarrow[\mathrm{id}_A]{} & A \end{array}, \qquad \mathcal{Q}_2: \quad \begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}_X} \\ \emptyset & \longrightarrow & X \end{array}$$

moreover, we have $K_{\mathcal{Q}} = K_{\mathcal{Q}_1 \amalg \mathcal{Q}_2} = K_{\mathcal{Q}_1} \amalg K_{\mathcal{Q}_2}$. Hence, by Corollary 2.2.37, $\mathrm{cl}_\Delta(\emptyset)$ is closed under finite coproducts, so it is enough enough to prove the lemma for squares of the form $\mathcal{Q}_1$ and $\mathcal{Q}_2$, but one can notice that they are both, up to transposition, of the form

$$\mathcal{Q}': \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

By Example 2.2.45, $K_{\mathcal{Q}'}$ coincide with the cone of f, hence by Lemma 2.2.46, it follows that $p_{\mathcal{Q}'}\colon \mathrm{cone}(f) \to Y$ is a simplicial homotopy equivalence, therefore $p_{\mathcal{Q}'}$ is in $\mathrm{cl}_\Delta(\emptyset)$. This finishes the proof. □

The following definition is a particular case of Definition 2.2.38.

Definition 2.2.49. A morphism $f\colon A \to X$ in $\mathrm{Rad}(\mathscr{C})$ is called *coprojection*, if there exists an object Y of $\mathrm{Rad}(\mathscr{C})$ such that f is isomorphic the canonical morphism from A to $A \overset{\mathrm{rad}}{\amalg} Y$. A morphism $f\colon \mathscr{A} \to \mathscr{X}$ in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ is called *termwise coprojection*, if for each integer $n \geq 0$, the morphism $f_n\colon \mathscr{A}_n \to \mathscr{X}_n$ is a coprojection.

Corollary 2.2.50. *Let I be a set of morphisms in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ consisting of termwise coprojections. Then any countable transfinite composition of pushouts of coproducts of elements of I, is a termwise coprojection.*

Proof. Let

$$\mathscr{X}_0 \to \mathscr{X}_1 \to \cdots \to \mathscr{X}_n \to \mathscr{X}_{n+1} \to \cdots \tag{2.16}$$

be a ω-sequence such that each morphism $\mathscr{X}_n \to \mathscr{X}_{n+1}$ is a pushout of coproducts of elements of I. Since I consists of termwise coprojections, by Lemma 2.2.39, the coproduct of elements of I is a termwise coprojection, hence by the same lemma (b), the morphism $\mathscr{X}_n \to \mathscr{X}_{n+1}$ is a termwise coprojection for each $n < \omega$. Finally, by Lemma 2.2.40, the we conclude the transfinite composition of (2.16) is a termwise coprojection. □

2.2.3 Model structure on simplicial radditive functors

In this section we shall prove that if $\mathscr{C}$ is a category with finite coproducts, then category of simplicial radditive functors $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ is provided of a projective model structure (see Theorem 2.2.59).

Definition 2.2.51. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$. The morphism f is:

(1) a *weak equivalence* in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$, if f is an object-wise weak equivalence, that is, for every object U in $\mathscr{C}$, the morphism of simplicial sets $f(U)\colon \mathscr{X}(U) \to \mathscr{Y}(U)$ is a weak equivalence in $\Delta^{\mathrm{op}}\mathscr{S}ets$. We denote by $\mathbf{W}_{\mathrm{rad}}$ the class of weak equivalences in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$.

(2) a *fibration* in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$, if f is an objectwise fibration, that is, for every object U in $\mathscr{C}$, the morphism of simplicial sets $f(U)\colon \mathscr{X}(U) \to \mathscr{Y}(U)$ is a fibration in $\Delta^{\mathrm{op}}\mathscr{S}ets$.

(3) a *cofibration* in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$, if f has the left lifting property with respect to weak equivalences and fibrations in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$.

In view of Remark 2.2.21, we define the following sets of morphisms in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$:

$$I_{\mathrm{rad}} := \{U \otimes \partial\Delta[n] \to U \otimes \Delta[n] \,|\, U \in \mathscr{C}, n \geq 0\}\ ,$$
$$J_{\mathrm{rad}} := \{U \otimes \Lambda^r[n] \to U \otimes \Delta[n] \,|\, U \in \mathscr{C}, n \geq 0, 0 \leq r \leq n\}\ .$$

In Theorem 2.2.59, we shall prove that $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ is a cofibrantly generated model category in which I_{rad} is the class of generating cofibrations and J_{rad} is the class of generating trivial cofibrations. Let I be the set of simplicial sets $\partial\Delta[n] \to \Delta[n]$ for $n \geq 0$. Let J be the set of simplicial sets $\Lambda^r[n] \to \Delta[n]$ for $n \geq 0$ and $0 \leq r \leq n$.

Lemma 2.2.52. *Every object in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ is small. In consequence, I_{rad} and J_{rad} permit the small object argument.*

Proof. Let $\mathscr{A}$ be an object of $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$. Let κ be the cardinal of the set,

$$S := \coprod_{(U,n)\in \mathrm{obj}\,(\mathscr{C})\times\mathbb{N}} \mathscr{A}_n(U)\,.$$

We shall prove that $\mathscr{A}$ is κ-small relative to the class of all morphisms in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$. Indeed, let λ be a κ-filtered ordinal and let $\mathscr{X} : \lambda \to \Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ be a λ-sequence. It is not difficult to see that the canonical function of sets

$$\theta\colon \mathrm{colim}_{\,\gamma<\lambda}\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathit{Pre}(\mathscr{C})}(\mathscr{A}, \mathscr{X}_\gamma) \to \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathit{Pre}(\mathscr{C})}\,(\mathscr{A}, \mathrm{colim}_{\,\gamma<\lambda}\mathscr{X}_\gamma)$$

is bijective. Considering that $\mathscr{X} : \lambda \to \Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ is a filtered functor, Lemma 2.2.7 (d) asserts that $\mathrm{colim}_{\,\gamma<\lambda}\mathscr{X}_\gamma$ is an object of $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$; then, we have

$$\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})}\,(\mathscr{A}, \mathrm{colim}_{\,\beta<\lambda}\mathscr{X}_\beta) = \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathit{Pre}(\mathscr{C})}\,(\mathscr{A}, \mathrm{colim}_{\,\beta<\lambda}\mathscr{X}_\beta)\ ,$$

because $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ is a full subcategory of $\Delta^{\mathrm{op}}\mathit{Pre}(\mathscr{C})$. Hence we have a commutative diagram,

$$\begin{array}{ccc}
\mathrm{colim}_{\,\beta<\lambda}\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})}(\mathscr{A}, \mathscr{X}_\beta) & \longrightarrow & \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})}\,(\mathscr{A}, \mathrm{colim}_{\,\beta<\lambda}\mathscr{X}_\beta) \\
\Vert & & \Vert \\
\mathrm{colim}_{\,\beta<\lambda}\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathit{Pre}(\mathscr{C})}(\mathscr{A}, \mathscr{X}_\beta) & \xrightarrow[\theta]{} & \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathit{Pre}(\mathscr{C})}\,(\mathscr{A}, \mathrm{colim}_{\,\beta<\lambda}(\mathscr{X}_\beta))
\end{array}$$

Since θ is bijective, the top arrow is bijective, as required. □

Lemma 2.2.53. *For any object $U \in \mathscr{C}$ and every finite simplicial set K, the object $U \otimes K$ of $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ is finite.*

Proof. Let us fix an object $U \in \mathscr{C}$ and a finite simplicial set K. Since K is finite, there is a finite cardinal such that K is κ-small relative to all morphisms of $\Delta^{\mathrm{op}}\mathscr{S}ets$. We claim that $U \otimes K$ is κ-small relative to all morphisms in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$. Indeed, let λ be a κ -filtered ordinal and let

$$\mathscr{X}_0 \to \mathscr{X}_1 \to \cdots \to \mathscr{X}_\beta \to \cdots (\beta < \lambda)$$

be a λ-sequence of simplicial radditive functors. By Lemma 2.2.7 (e), filtered colimits in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ comes from the colimits in $\Delta^{\mathrm{op}}\mathit{Pre}(\mathscr{C})$, hence we obtain a λ-sequence of simplicial sets,

$$\mathscr{X}_0(U) \to \mathscr{X}_1(U) \to \cdots \to \mathscr{X}_\beta(U) \to \cdots (\beta < \lambda)\,.$$

We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{colim}_{\,\beta<\lambda}\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})}(U\otimes K, \mathscr{X}_\beta) & \longrightarrow & \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})}\Big(U\otimes K, \mathrm{colim}_{\,\beta<\lambda}\mathscr{X}_\beta\Big) \\
\downarrow & & \downarrow \\
\mathrm{colim}_{\,\beta<\lambda}\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathscr{S}ets}(K, \mathscr{X}_\beta(U)) & \longrightarrow & \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathscr{S}ets}\Big(K, \mathrm{colim}_{\,\beta<\lambda}\mathscr{X}_\beta(U)\Big)
\end{array}$$

where the vertical arrows are bijections deduced by Corollary 2.2.26. Since K is κ-small relative to all morphisms of $\Delta^{\mathrm{op}}\mathscr{S}ets$, the horizontal arrow at the bottom of the preceding diagram is bijective, hence the top arrow is so, finishing thus the proof. $\square$

The following corollary is a strong version of the small object argument, as we get that every morphism has a functorial factorization in a morphism having the right lifting property and a morphism that is a countable transfinite composition of coproducts of certain morphisms.

Corollary 2.2.54. *There exist two functorial factorizations* (α, β) *and* (γ, δ) *on* $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ *such that for every morphism* f *in* $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C}$*, we can write*

$$f = \beta(f) \circ \alpha(f)\,,$$

where $\alpha(f)$ *is a countable transfinite composition of pushouts of coproducts of elements of* I_{rad} *and* $\beta(f)$ *in* I_{rad}-inj*, and*

$$f = \delta(f) \circ \gamma(f)\,,$$

where $\gamma(f)$ *is a countable transfinite composition of pushouts of coproducts of elements of* J_{rad} *and* $\delta(f)$ *in* J_{rad}-inj*,*

Proof. It is a consequence of the previous lemma. $\square$

Definition 2.2.55. We denote by $\bar{\mathscr{C}}$ the full subcategory of small coproducts of objects of the form h_X in $\mathrm{Rad}(\mathscr{C})$ for objects X in $\mathscr{C}$.

Corollary 2.2.56. *Let* Q *be a cofibrant replacement functor of category* $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ *with respect to the projective model structure. Then* Q *takes values in* $\Delta^{\mathrm{op}}\bar{\mathscr{C}}$*.*

Proof. It is a consequence of the previous corollary. $\square$

Lemma 2.2.57. *We have the following assertions:*

(a) *A morphism is a fibration in* $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ *if and only if it is in* J_{rad}-inj.

(b) *A morphism is a fibration and a weak equivalence in* $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ *if and only if it is in* I_{rad}-inj.

(c) *A morphism is a cofibration in* $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ *if and only if it is in* I_{rad}-cof.

Proof. (a). By Corollary 2.2.26, a commutative diagram

$$\begin{array}{ccc} U \otimes \Lambda^r[n] & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow{\scriptstyle p} \\ U \otimes \Delta[n] & \longrightarrow & \mathscr{Y} \end{array}$$

corresponds biunivocally to a commutative diagram

$$\begin{array}{ccc} \partial\Lambda^r[n] & \longrightarrow & \mathscr{X}(U) \\ \big\downarrow & & \big\downarrow{\scriptstyle p(U)} \\ \Delta[n] & \longrightarrow & \mathscr{Y}(U) \end{array}$$

Then, we observe that p in J_{rad}-inj if and only if the morphism $p(U)$ is in J-inj for every object $U \in \mathscr{C}$, i.e. p in J_{rad}-inj if and only if the morphism p is a object-wise fibration.

(b). Similarly, by Corollary 2.2.26, a commutative diagram

$$\begin{array}{ccc} U \otimes \partial\Delta[n] & \longrightarrow & \mathscr{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ U \otimes \Delta[n] & \longrightarrow & \mathscr{Y} \end{array}$$

corresponds biunivocally to a commutative diagram

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & \mathscr{X}(U) \\ \big\downarrow & & \big\downarrow{\scriptstyle p(U)} \\ \Delta[n] & \longrightarrow & \mathscr{Y}(U) \end{array}$$

Then, we observe that p in I_{rad}-inj if and only if the morphism $p(U)$ is in I-inj for every object $U \in \mathscr{C}$, i.e. p in I_{rad}-inj if and only if the morphism p is both an object-wise fibration and an object-wise weak equivalence.

(c). Since I_{rad}-cof $= (I_{\text{rad}}$-inj$)$-proj and cofibrations in $\Delta^{\text{op}}\text{Rad}(\mathscr{C})$ have the left lifting property with respect to both fibrations and weak equivalences, we deduce from (b), that a morphism is a cofibration in $\Delta^{\text{op}}\text{Rad}(\mathscr{C})$ if and only if it is in I_{rad}-cof. □

Lemma 2.2.58. *We have* J_{rad}-cell $\subset \mathbf{W}_{\text{rad}} \cap I$-cof*.*

Proof. It is an easy exercise to show that J_{rad}-cell is contained in I-cof. Hence, it is enough to show the inclusion J_{rad}-cell $\subset I$-cof, but it follows by applying Proposition 2.2.34 in a suitable way. □

Theorem 2.2.59. *The weak equivalences, fibrations and cofibrations given in Definition 2.2.51 provides a cofibrantly generated model structure on* $\Delta^{\text{op}}\text{Rad}(\mathscr{C})$ *in which* I_{rad} *is the class of generating cofibrations and* J_{rad} *is the class of generating trivial cofibrations.*

Proof. We shall verify that $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ satisfies the hypothesis of the Recognition theorem (Th. 1.1.51). Indeed, by Proposition 2.2.15, the category $\mathrm{Rad}(\mathscr{C})$ is complete and cocomplete, then the category $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ is so. Since weak equivalences and fibrations in $\mathrm{Rad}(\mathscr{C})$ are defined to be object-wise weak equivalences and fibrations respectively, the 2-out-of-3 and the retracts axioms for $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$ follow from the 2-out-of-3 and the retracts axioms for simplicial sets. Since cofibrations in $\mathrm{Rad}(\mathscr{C})$ is defined by using the left lifting property, the retracts axiom for cofibrations follows from Lemma 1.1.43 (b). By Lemma 2.2.52, the sets I_{rad} and I_{rad} permit the small object argument. We recall that $\mathbf{W}_{\mathrm{rad}}$ denotes the class of weak equivalences on $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$. By Lemma 2.2.57 (a) and (b), we deduce that I_{rad}-inj $= \mathbf{W}_{\mathrm{rad}} \cap J_{\mathrm{rad}}$-inj. Finally, by Lemma 2.2.58, we have J_{rad}-cell $\subset \mathbf{W}_{\mathrm{rad}} \cap I$-cof, which completes the hypothesis of the Recognition theorem (see Theorem 1.1.51). □

Proposition 2.2.60. *Suppose that $f\colon \mathscr{X} \to \mathscr{Y}$ be a projective cofibration in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$. Then there exist two morphisms $s\colon \mathscr{Y} \to \mathscr{A}$ and $p\colon \mathscr{A} \to \mathscr{Y}$ such that, for each index $n \geq 0$, the term $(s \circ f)_n$ has the form $\mathscr{X}_n \to \mathscr{X}_n \amalg F_n$, where F_n is a coproduct of representable radditive functors, and f is a retract of $s \circ f$ which fixes $\mathscr{X}$, that is, we have a commutative diagram*

$$\begin{array}{ccccc} \mathscr{X} & = & \mathscr{X} & = & \mathscr{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle s\circ f} & & \downarrow{\scriptstyle f} \\ \mathscr{Y} & \xrightarrow{s} & \mathscr{A} & \xrightarrow{p} & \mathscr{Y} \end{array}$$

where the horizontal composites are the identities.

Proof. Similarly as the Corollary 1.1.47, by Corollary 2.2.54, we get a factorization $f = p \circ g$, where g is a countable transfinite composition of coproducts of elements of I_{rad} and p in I_{rad}-inj. By Lemma 2.2.57, the morphism p is trivial fibration in $\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$. Then f has the left lifting property with respect to p, and so by the retract argument, there exists a morphism $s\colon \mathscr{Y} \to \mathscr{A}$ such that we have a commutative diagram

$$\begin{array}{ccccc} \mathscr{X} & = & \mathscr{X} & = & \mathscr{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \mathscr{Y} & \xrightarrow{s} & \mathscr{A} & \xrightarrow{p} & \mathscr{Y} \end{array}$$

such that $p \circ s = \mathrm{id}$. In particular, we have $g = s \circ f$. It remains to show that, for each integer $n \geq 0$, the term $(s \circ f)_n$ has the form $\mathscr{X}_n \to \mathscr{X}_n \amalg F_n$. Indeed, suppose that g is a transfinite composition of the ω-sequence

$$\mathscr{X}_0 \to \mathscr{X}_1 \to \cdots \to \mathscr{X}_n \to \cdots (n < \omega)\,,$$

such that for every $i \in \mathbb{N}$, the morphism $\mathscr{X}_i \to \mathscr{X}_{i+1}$ is a pushout

$$\begin{array}{ccc} \coprod^{\mathrm{rad}}_{D \in S} U \otimes \partial\Delta[m] & \longrightarrow & \mathscr{X}_i \\ \downarrow & & \downarrow \\ \coprod^{\mathrm{rad}}_{D \in S} U \otimes \partial\Delta[m] & \longrightarrow & \mathscr{X}_{i+1} \end{array}$$

and S is the set of diagrams

$$\begin{array}{ccc} U \otimes \partial\Delta[m] & \longrightarrow & \mathscr{X}_i \\ \downarrow & & \downarrow \\ U \otimes \partial\Delta[m] & \longrightarrow & \mathscr{X}_{i+1} \end{array}$$

for morphisms $U \otimes \partial\Delta[m] \to U \otimes \Delta[m]$ in I_{rad}. Notice that every morphism from $U \otimes \partial\Delta[m]$ to $U \otimes \Delta[m]$ in I_{rad} is a termwise coprojection. By corollary 2.2.50, the transfinite composition the above ω-sequence, which is g, is a termwise coprojection. □

2.3 Simplicial Nisnevich sheaves

In this section, we study simplicial Nisnevich sheaves defined on an admissible category of schemes [40, Appendix A].

2.3.1 Admissible categories

The category of smooth varieties is not good enough to study geometric symmetric powers, as symmetric powers of a higher dimensional smooth variety have singularities. This issue can be solved by considering admissible categories of schemes.

Let $\mathscr{S}ch/k$ be the category of schemes over k. For two k-schemes X and Y, we write $X \times Y$ to mean the Cartesian product $X \times_{\mathrm{Spec}(k)} Y$. We also denote by $X \amalg Y$ the disjoint union of X and Y, as schemes. We recall that the point $\mathrm{Spec}(k)$ is the terminal object of $\mathscr{S}ch/k$, whereas the empty scheme $\emptyset$ is its initial object. An *étale* morphism is a flat and unramified morphism of schemes, see [28].

Definition 2.3.1. Let k be a field. A small full subcategory $\mathscr{C}$ of $\mathscr{S}ch/k$ is called *admissible* [2], if it satisfies the following axioms:

[2] f-admissible in [40]

(1) $\mathrm{Spec}(k)$ and $\mathbb{A}^1$ are objects in $\mathscr{C}$,

(2) $\mathscr{C}$ is closed under the product $\times$, that is, for any two objects X and Y of $\mathscr{C}$, the product $X \times Y$ is in $\mathscr{C}$.

(3) $\mathscr{C}$ is closed under the coproduct $\amalg$, that is, for any two objects X and Y of $\mathscr{C}$, the coproduct $X \amalg Y$ is in $\mathscr{C}$.

(4) If U is a k-scheme such that there is an étale morphism $U \to X$ with X is in $\mathscr{C}$, then U is in $\mathscr{C}$.

(5) If G is finite group acting on an object X of $\mathscr{C}$, then the (categorical) quotient X/G is in $\mathscr{C}$.

Example 2.3.2. The following categories are admissible:

(1) The category of schemes of quasi-projective schemes over a field k.

(2) The category of normal quasi-projective schemes over a perfect field k.

(3) The category of normal quasi-affine schemes over a perfect field k.

Remark 2.3.3. By definition every admissible category of schemes over a field contains the affine line $\mathbb{A}^1$, but it is not true that all admissible categories contain the projective line $\mathbb{P}^1$ over a field. For example, the subcategory of normal quasi-affine schemes over a perfect field is admissible, but the projective line $\mathbb{P}^1$ is not quasi-affine.

Nisnevich sheaves

Unless otherwise mentioned, $\mathscr{C}$ will be an admissible category, see Definition 2.3.1.

Definition 2.3.4. An *elementary distinguished square* in $\mathscr{C}$ is a Cartesian square of the form

$$\mathcal{Q}: \quad \begin{array}{ccc} Y & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{\;j\;} & X \end{array} \tag{2.17}$$

where j is an open embedding and p is an étale morphism such that the induced morphism $p^{-1}(X-U)_{\mathrm{red}} \to (X-U)_{\mathrm{red}}$ of reduced schemes is an isomorphism.

Definition 2.3.5. A family of étale morphisms $\{f_i \colon U_i \to X\}_{i \in I}$ of $\mathscr{C}$ is a *Nisnevich covering* if for every point[3] $x \in X$ there exists an index $i \in I$ and a point $y \in U_i$ such that $f_i(y) = x$ and the corresponding morphism of residual fields $k(x) \to k(y)$ is an isomorphism.

[3] not necessarily a closed point.

The Nisnevich topology on $\mathscr{C}$ can be described as the smallest Grothendieck topology generated by families of the form $\{j: U \to X, p: V \to X\}$ associated to elementary distinguished squares of the form (2.17), see [42, page 1400]. We denote by $\mathscr{C}_{\mathrm{Nis}}$ the site consisting of $\mathscr{C}$ and the Nisnevich topology on it.

Proposition 2.3.6. *A presheaf F on $\mathscr{C}$ is a sheaf in the Nisnevich topology if and only if for each elementary distinguished square as* (2.17)*, the square of sets*

$$F(\mathcal{Q}): \qquad \begin{array}{ccc} F(X) & \xrightarrow{F(p)} & F(V) \\ {\scriptstyle F(j)}\downarrow & & \downarrow \\ F(U) & \longrightarrow & F(Y) \end{array}$$

is Cartesian.

Proof. see [30, Prop. 14, page 96]. □

Terminology. Unless otherwise specified, $\mathscr{S}$ will be the category of sheaves on the Nisnevich site $\mathscr{C}_{\mathrm{Nis}}$.

As representable functors are Nisnevich sheaves, we shall use the letter h to denote the full embedding of $\mathscr{C}$ into $\mathscr{S}$, so that we have a commutative triangle:

$$\begin{array}{ccc} \mathscr{C} & \longrightarrow & Pre(\mathscr{C}) \\ & {\scriptstyle h}\searrow \quad \nearrow & \\ & \mathscr{S} & \end{array}$$

The category $\mathscr{S}$ is complete and cocomplete, its terminal object is $h_{\mathrm{Spec}(k)}$, and filtered colimits of Nisnevich sheaves in the category of presheaves are Nisnevich sheaves. Let $\{F_i\}_{i\in I}$ be a family of objects in $\mathscr{S}$. The coproduct of this family in $\mathscr{S}$ is the sheafification $\mathrm{a}_{\mathrm{Nis}}\left(\coprod_{i\in I} F_i\right)$ of the coproduct $\coprod_{i\in I} F_i$ in $Pre(\mathscr{C})$. We abusively denote it by $\coprod_{i\in I} F_i$ if no confusion arises.

In the sequel, we shall consider the injective model structure on the category of simplicial sheaves $\Delta^{\mathrm{op}}\mathscr{S}$, see Theorem 2.1.11, where the class of cofibrations is the class of monomorphisms, a weak equivalence is a stalkwise weak equivalence and fibrations are morphisms having the right lifting property with respect to trivial cofibrations.

Simplicial structure

We shall describe the simplicial structure on the category $\Delta^{\mathrm{op}}\mathscr{S}$. For a simplicial sheaf $\mathscr{X}$ and a simplicial set K, we define the product $\mathscr{X} \times K$ to be the simplicial sheaf, such that for every $n \in \mathbb{N}$, its term $(\mathscr{X} \times K)_n$ is defined to be the coproduct $\coprod_{K_n} \mathscr{X}_n$ in $\mathscr{S}$.

For a couple of sheaves $(\mathscr{X},\mathscr{Y})$, the function complex $\mathrm{Map}\,(\mathscr{X},\mathscr{Y})$ is defined to be the simplicial set which assigns an object $[n]$ of Δ to the set $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathscr{S}}(\mathscr{X}\times\Delta[n],\mathscr{Y})$. Then, for every pair of simplicial sheaves $(\mathscr{X},\mathscr{Y})$ and every simplicial set K, one has a natural bijection,

$$\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathscr{S}}(\mathscr{X}\times K,\mathscr{Y})\simeq\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathscr{S}ets}\,(K,\mathrm{Map}\,(\mathscr{X},\mathscr{Y}))\ , \tag{2.18}$$

which is functorial in $\mathscr{X}$, $\mathscr{Y}$ and K.

For each object U of $\mathscr{C}$, we denote by $\Delta_U[0]$ the constant functor from Δ^{op} to $\mathscr{S}$ with value h_U. Sometimes, we shall simply write h_U instead of $\Delta_U[0]$ if no confusion arises. For each $n\in\mathbb{N}$ and each object U of $\mathscr{C}$, we denote by $\Delta_U[n]$ the simplicial sheaf $\Delta_U[0]\times\Delta[n]$. Similarly, we denote by $\partial\Delta_U[n]$ the simplicial sheaf $\Delta_U[0]\times\partial\Delta[n]$.

Notice that Yoneda lemma provides an isomorphism $\mathrm{Map}\,(\Delta_U[0],\mathscr{Y})\simeq\mathscr{Y}(U)$ for every object U of $\mathscr{C}$ and every simplicial sheaf $\mathscr{Y}$. Hence, replacing $\mathscr{X}$ by $\Delta_U[0]$ in (2.18), we obtain an isomorphism

$$\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathscr{S}}(\Delta_U[0]\times K,\mathscr{Y})\simeq\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathscr{S}ets}\,(K,\mathscr{Y}(U))\ . \tag{2.19}$$

Example 2.3.7. Let $\mathscr{X}$ be a simplicial sheaf on $\mathscr{C}_{\mathrm{Nis}}$. If $K\subset L$ is an inclusion of simplicial sets, then the induced morphism from $\mathscr{X}\times K$ to $\mathscr{X}\times L$ is a termwise coprojection (see Definition 2.2.38). Indeed, for each natural n, the n-simplex $(\mathscr{X}\times K)_n$ is equal to the coproduct of sheaves $\coprod_{K_n}\mathscr{X}_n$, similarly, $(\mathscr{X}\times L)_n$ is equal to $\coprod_{L_n}\mathscr{X}_n$. In view of the inclusion $K_n\subset L_n$, we have a canonical isomorphism

$$\coprod_{L_n}\mathscr{X}_n\simeq\left(\coprod_{K_n}\mathscr{X}_n\right)\amalg\left(\coprod_{L_n\backslash K_n}\mathscr{X}_n\right)\ ,$$

which allow us to deduce that $(\mathscr{X}\times K)_n\to(\mathscr{X}\times L)_n$ is a coprojection for all $n\in\mathbb{N}$.

We recall that $\mathscr{C}_+$ denotes the full subcategory of the pointed category $\mathscr{C}_*$ generated by objects of the form $X_+:=X\amalg\mathrm{Spec}(k)$, see page 5. We denote by $\mathscr{S}_*$ the pointed category of $\mathscr{S}$. The symbols $\vee$ and $\wedge$ denote, respectively, the coproduct and the smash product in $\mathscr{S}_*$. An elementary distinguished square in $\mathscr{C}_+$ is a square of the form

$$\mathscr{Q}_+:\quad\begin{array}{ccc} Y_+ & \longrightarrow & V_+ \\ \big\downarrow & & \big\downarrow{\scriptstyle p_+} \\ U_+ & \xrightarrow[j_+]{} & X_+ \end{array} \tag{2.20}$$

where $\mathscr{Q}$ is an elementary distinguished square in $\mathscr{C}$ of the form (2.17). We denote by $\mathscr{C}_{+,\mathrm{Nis}}$ the site consisting of $\mathscr{C}_+$ and the smallest Grothendieck topology generated by the families of the form $\{j_+\colon U_+\to X_+,\,p_+\colon V_+\to X_+\}$ which are associated to elementary distinguished squares of the form (2.20).

Lemma 2.3.8. *The category* $Shv(\mathscr{C}_{+,\mathrm{Nis}})$ *is equivalent to the pointed category* $\mathscr{S}_*$.

Proof. We consider the functor $\Phi\colon \mathrm{Rad}(\mathscr{C}_+) \to \mathrm{Rad}(\mathscr{C})_*$ defining an equivalence of categories between $\mathrm{Rad}(\mathscr{C}_+)$ and $\mathrm{Rad}(\mathscr{C})_*$, see proof of Lemma 2.2.6. The lemma follows after noticing that for a radditive functor F in $\mathrm{Rad}(\mathscr{C}_+)$, we have that F is in $Shv(\mathscr{C}_{+,\mathrm{Nis}})$ if and only if $\Phi(F)$ is in $\mathscr{S}_*$. □

Definition 2.3.9. We denote by $\mathscr{H}(\mathscr{C}_{\mathrm{Nis}})$ the homotopy category of $\Delta^{\mathrm{op}}\mathscr{S}$ localized with respect to weak equivalences of the injective model structure. We write $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}})$ for the homotopy category of $\Delta^{\mathrm{op}}\mathscr{S}_*$ localized with respect to weak equivalences, see 1.1.18.

Definition 2.3.10. A simplicial sheaf $\mathscr{X}$ in $\Delta^{\mathrm{op}}\mathscr{S}$ is called $\mathbb{A}^1$-*local* if for any simplicial sheaf $\mathscr{Y}$, the map

$$\mathrm{pr}_1^*\colon \mathrm{Hom}_{\mathscr{H}(\mathscr{C}_{\mathrm{Nis}})}(\mathscr{Y}, \mathscr{X}) \to \mathrm{Hom}_{\mathscr{H}(\mathscr{C}_{\mathrm{Nis}})}(\mathscr{Y} \times \mathbb{A}^1, \mathscr{X}),$$

induced by the projection $\mathrm{pr}_1\colon \mathscr{Y} \times \mathbb{A}^1 \to \mathscr{Y}$, is a bijection. A morphism of simplicial sheaves $f\colon \mathscr{X} \to \mathscr{Y}$ is an $\mathbb{A}^1$-*weak equivalence* if for any $\mathbb{A}^1$-local fibrant sheaf $\mathscr{Z}$, the morphism of simplicial sets

$$f^*\colon \mathrm{Map}\,(\mathscr{Y}, \mathscr{Z}) \to \mathrm{Map}\,(\mathscr{X}, \mathscr{Z})$$

is a weak equivalence.

Definition 2.3.11. We denote by $\mathscr{H}(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$ the homotopy category of $\Delta^{\mathrm{op}}\mathscr{S}$ localized with respect to $\mathbb{A}^1$-weak equivalences. We write $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$ for the homotopy category of $\Delta^{\mathrm{op}}\mathscr{S}_*$ localized with respect to $\mathbb{A}^1$-weak equivalences.

Example 2.3.12. The class of $\mathbb{A}^1$-weak equivalences in $\Delta^{\mathrm{op}}\mathscr{S}$ coincides with the $\bar{\Delta}$-class $cl_{\bar{\Delta}}(\mathbf{W}_{\mathrm{Nis}} \cup \mathcal{P}_{\mathbb{A}^1})$ (see Definition 2.2.30), where $\mathbf{W}_{\mathrm{Nis}}$ is the class of local equivalences with respect to the Nisnevich topology and $\mathcal{P}_{\mathbb{A}^1}$ is the class of projections from $\Delta_X[0] \times \Delta_{\mathbb{A}^1}[0]$ to $\Delta_X[0]$, for $X \in \mathscr{C}$ (see [7, Th. 4, page 378]). Similarly, the class of $\mathbb{A}^1$-weak equivalences in $\Delta^{\mathrm{op}}\mathscr{S}_*$ coincides with the class $cl_{\bar{\Delta}}(\mathbf{W}_{\mathrm{Nis},+} \cup \mathcal{P}_{\mathbb{A}^1,+})$, where $\mathbf{W}_{\mathrm{Nis},+}$ is the image of $\mathbf{W}_{\mathrm{Nis}}$ through the functor which sends a simplicial sheaf $\mathscr{X}$ to the pointed simplicial sheaf $\mathscr{X}_+$ and $\mathcal{P}_{\mathbb{A}^1,+}$ is the image of $\mathcal{P}_{\mathbb{A}^1}$ through the same functor.

Remark 2.3.13. The category $\mathscr{H}(\mathscr{C}_{+,\mathrm{Nis}})$ is equivalent to the pointed homotopy category $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}})$. Similarly, the category $\mathscr{H}(\mathscr{C}_{+,\mathrm{Nis}}, \mathbb{A}^1)$ is equivalent to the pointed homotopy category $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$.

Definition 2.3.14. We denote by $\bar{\mathscr{C}}$ the full subcategory of small coproducts of objects h_X in $\mathscr{S}$ for objects X in $\mathscr{C}$. Similarly, we denote by $\bar{\mathscr{C}}$ the full subcategory of small coproducts of objects h_X in $\mathscr{S}_*$ for objects X in $\mathscr{C}$.

We define the following sets of morphisms of simplicial sheaves

$$I_{\text{proj}} := \{\partial\Delta_U[n] \to \Delta_U[n] \,|\, U \in \mathscr{C}, n \in \mathbb{N}\} \ . \tag{2.21}$$

Notice that, by Example 2.3.7, the morphisms $\partial\Delta_U[n] \to \Delta_U[n]$ are termwise coprojections in $\Delta^{\text{op}}\bar{\mathscr{C}}$ for all $U \in \mathscr{C}$ and $n \in \mathbb{N}$. We define the following sets of morphisms of pointed simplicial sheaves

$$I^{+}_{\text{proj}} := \{\partial\Delta_U[n]_+ \to \Delta_U[n]_+ \,|\, U \in \mathscr{C}, n \in \mathbb{N}\} \ . \tag{2.22}$$

The morphisms $\partial\Delta_U[n]_+ \to \Delta_U[n]_+$ are termwise coprojections in $\Delta^{\text{op}}\bar{\mathscr{C}}_+$ for all $U \in \mathscr{C}$ and $n \in \mathbb{N}$

Lemma 2.3.15. *For any object $U \in \mathscr{C}$ and every finite simplicial set K (see Definition 2.2.17), the object $\Delta_U[0] \times K$ is finite relative to $\Delta^{\text{op}}\mathscr{S}$ in the sense of Definition 2.1.4 of [18].*

Proof. Let us fix an object $U \in \mathscr{C}$ and a finite simplicial set K. Since K is finite, there is a finite cardinal κ such that K is κ-small relative to all morphisms of $\Delta^{\text{op}}\mathscr{S}ets$. We claim that $\Delta_U[0] \times K$ is κ-small relative to all morphisms in $\Delta^{\text{op}}\mathscr{S}$. Indeed, let λ be a κ -filtered ordinal and let

$$\mathscr{X}_0 \to \mathscr{X}_1 \to \cdots \to \mathscr{X}_\beta \to \cdots (\beta < \lambda)$$

be a λ-sequence of simplicial sheaves on $\mathscr{C}_{\text{Nis}}$. Since filtered colimits of Nisnevich sheaves (computed in the category of presheaves) are sheaves, we obtain a λ-sequence of simplicial sets,

$$\mathscr{X}_0(U) \to \mathscr{X}_1(U) \to \cdots \to \mathscr{X}_\beta(U) \to \cdots (\beta < \lambda) \,.$$

Then, we have a commutative diagram

$$\begin{array}{ccc}
\operatorname{colim}_{\beta<\lambda}\operatorname{Hom}_{\Delta^{\text{op}}\mathscr{S}}(\Delta_U[0] \times K, \mathscr{X}_\beta) & \longrightarrow & \operatorname{Hom}_{\Delta^{\text{op}}\mathscr{S}}\Big(\Delta_U[0] \times K, \operatorname{colim}_{\beta<\lambda}\mathscr{X}_\beta\Big) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{\beta<\lambda}\operatorname{Hom}_{\Delta^{\text{op}}\mathscr{S}ets}(K, \mathscr{X}_\beta(U)) & \longrightarrow & \operatorname{Hom}_{\Delta^{\text{op}}\mathscr{S}ets}\Big(K, \operatorname{colim}_{\beta<\lambda}\mathscr{X}_\beta(U)\Big)
\end{array}$$

where the vertical arrows are bijections. Since K is κ-small relative to all morphisms of $\Delta^{\text{op}}\mathscr{S}ets$, the horizontal arrow at the bottom of the preceding diagram is bijective, hence the top arrow is so. This completes the proof. □

Definition 2.3.16. Let $\mathscr{D}$ be a category admitting filtered colimits. An object X of $\mathscr{D}$ is called *compact* if the corepresentable functor $\operatorname{Hom}_{\mathscr{D}}(X, -)$ preserves filtered colimits.

Example 2.3.17. Representable presheaves are compact objects in the category of presheaves. In consequence, representable sheaves are compact objects in the category of Nisnevich sheaves.

Remark 2.3.18. Let us consider the hypothesis of Lemma 2.3.15. The $\Delta_U[0] \times K$ is compact in $\Delta^{\mathrm{op}}\mathscr{S}$ in the sense of Definition 2.3.16. Indeed, it follows from the fact that K is a compact object in $\Delta^{\mathrm{op}}\mathscr{S}ets$ and a representable sheaf is a compact object in $\mathscr{S}$ (Example 2.3.17).

Lemma 2.3.19. *Every morphism in* (I_{proj})-inj *is a sectionwise trivial fibration.*

Proof. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism in (I_{proj})-inj and let us fix an object U of $\mathscr{C}$. By the naturality of the isomorphism (2.19), a commutative diagram

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & \mathscr{X}(U) \\ \downarrow & & \downarrow \\ \Delta[n] & \longrightarrow & \mathscr{Y}(U) \end{array} \tag{2.23}$$

in $\Delta^{\mathrm{op}}\mathscr{S}ets$, corresponds biunivocally to a diagram

$$\begin{array}{ccc} \partial\Delta_U[n] & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \Delta_U[n] & \longrightarrow & \mathscr{Y} \end{array}$$

in $\Delta^{\mathrm{op}}\mathscr{S}$. As the left vertical arrow is an element of I_{proj}, the above diagram has a lifting. Therefore, the bijection (2.19) induces a lifting of (2.23). □

The following corollary is a consequence of the small object argument. It will be useful to show that the cofibrant resolution takes its values in the category $\Delta^{\mathrm{op}}\bar{\mathscr{C}}$.

Corollary 2.3.20. *There exists a functorial factorization* (α, β) *on* $\Delta^{\mathrm{op}}\mathscr{S}$ *such that for every morphism* f *is factored as* $f = \beta(f) \circ \alpha(f)$*, where* $\beta(f)$ *is sectionwise trivial fibration and* $\alpha(f)$ *is a termwise coprojection with terms form* $\mathscr{X}_n \to \mathscr{X}_n \amalg \mathscr{Y}_n$*, where* $\mathscr{Y}_n$ *is an object of* $\bar{\mathscr{C}}$.

Proof. By Lemma 2.3.15, the objects $\partial\Delta_U[n]$ and $\Delta_U[n]$ are finite relative to $\Delta^{\mathrm{op}}\mathscr{S}$. Since the countable ordinal ω is κ-filtered, the small object argument provides a factorization such that $\beta(f)$ in (I_{proj})-inj and $\alpha(f)$ is a countable transfinite composition of pushouts of coproducts of elements of I_{proj}. By Example 2.3.7, every morphism $\partial\Delta_U[n] \to \Delta_U[n]$ of I_{proj} is a termwise coprojection in $\Delta^{\mathrm{op}}\bar{\mathscr{C}}$. Therefore, Corollary 2.2.50 provides the desired factorization. □

We denote by Q^{proj} the endofunctor of $\Delta^{\mathrm{op}}\mathscr{S}$ which sends a simplicial sheaf $\mathscr{X}$ to the codomain of the morphism $\alpha(\emptyset \to \mathscr{X})$, where $\emptyset$ is the initial object of $\Delta^{\mathrm{op}}\mathscr{S}$. The endofunctor Q^{proj} will be called *cofibrant resolution*. In particular, for every object $\mathscr{X}$ of $\Delta^{\mathrm{op}}\mathscr{S}$, the canonical morphism from $Q^{\mathrm{proj}}(\mathscr{X})$ to $\mathscr{X}$ is a sectionwise trivial fibration.

Corollary 2.3.21. *The functor Q^{proj} takes values in $\Delta^{\mathrm{op}}\bar{\mathscr{C}}$.*

Proof. Let $\mathscr{X}$ be a simplicial sheaf in $\Delta^{\mathrm{op}}\mathscr{S}$. By Corollary 2.3.20, the morphism of simplicial sheaves $\emptyset \to \mathscr{X}$, where $\emptyset$ is the initial object of $\Delta^{\mathrm{op}}\mathscr{S}$, factors into $\emptyset \to Q^{\mathrm{proj}}(\mathscr{X}) \to \mathscr{X}$ such that the terms of $Q^{\mathrm{proj}}(\mathscr{X})$ are in $\bar{\mathscr{C}}$, that is, $Q^{\mathrm{proj}}(\mathscr{X})$ is in $\Delta^{\mathrm{op}}\bar{\mathscr{C}}$. □

Remark 2.3.22. As in Corollary 2.3.21, we also have a pointed cofibrant resolution $\Delta^{\mathrm{op}}\mathscr{S}_* \to \Delta^{\mathrm{op}}\bar{\mathscr{C}}_+$. We shall denote it by the same symbol Q^{proj} if no confusion arises.

Lemma 2.3.23. *The class of $\mathbb{A}^1$-weak equivalences in $\Delta^{\mathrm{op}}\mathscr{S}_*$ is closed under finite coproducts and smash products.*

Proof. By Example 2.3.12, the class of $\mathbb{A}^1$-weak equivalences in $\Delta^{\mathrm{op}}\mathscr{S}_*$ is $\bar{\Delta}$-closed. Then, it is closed under finite coproducts. Next, let us prove that this class is closed under smash products. By the cube lemma (see [18, Lemma 5.2.6]), one reduces the problem to the unpointed case, i.e. for products in $\Delta^{\mathrm{op}}\mathscr{S}$. Using standard simplicial methods, the problem is reduced to show that: for every $\mathbb{A}^1$-weak equivalence and every simplicial sheaf $\mathscr{Z}$ of the $\Delta_U[0]$ for U in $\mathscr{C}$, the product $f \times \mathrm{id}_{\mathscr{Z}}$ is an $\mathbb{A}^1$-weak equivalence. But it follows from Example 2.3.12 and Lemma 2.2.33 applied to the functor $(-) \times \mathrm{id}_{\mathscr{Z}}$. □

2.3.2 Simplicial sheaves on Σ_n-schemes

In this section, we shall define geometric symmetric powers of (simplicial) Nisnevich sheaves as left Kan extensions. The smallness condition on an admissible category will allow us to express a geometric symmetric power in terms of colimits. We follow the ideas of Voevodsky [40] in order to prove that geometric symmetric powers preserve $\mathbb{A}^1$-weak equivalences between simplicial Nisnevich sheaves which termwise are coproducts of representable sheaves. We also prove the existence of the left derived functors associated to geometric symmetric powers.

Let $\mathscr{C}$ be an admissible category of schemes over a field k. For an integer $n \geq 1$, the category $\mathscr{C}^{\Sigma_n}$ denotes the category of functors $\Sigma_n \to \mathscr{C}$, where Σ_n is viewed as a category. We recall that $\mathscr{C}^{\Sigma_n}$ can be viewed as the category of Σ_n-objects of $\mathscr{C}$.

Definition 2.3.24. Let X be an Σ_n-object on $\mathscr{C}$ and let $x \in X$. The *stabilizer* of x is the subgroup $\mathrm{stab}(x) \subset \Sigma_n$ consisting of elements $\sigma \in \Sigma_n$ such that $\sigma.x = x$.

Definition 2.3.25. A family of morphisms $\{f_i \colon U_i \to X\}_{i \in I}$ in $\mathscr{C}^{\Sigma_n}$ is called Σ_n-*equivariant Nisnevich covering* if each morphism f_i, viewed as a morphism of $\mathscr{C}$, is étale and we have the following property: for each point $x \in X$, viewed as an object of $\mathscr{C}$, there exist an index $i \in I$ and a point $y \in U_i$ such that: $f_i(y) = x$, the canonical homomorphism of residual fields $k(x) \to k(y)$ is an isomorphism, and the induced homomorphisms of groups $\mathrm{stab}(y) \to \mathrm{stab}(x)$ is an isomorphism.

Let $\mathscr{C}^{\Sigma_n}_{\mathrm{Nis}}$ be the site consisting of $\mathscr{C}^{\Sigma_n}$ and the Grothendieck topology formed by the Σ_n-equivariant Nisnevich coverings. We denote by $\mathscr{S}^{\Sigma_n}$ the category of sheaves on $\mathscr{C}^{\Sigma_n}_{\mathrm{Nis}}$. We point out that $\mathscr{S}^{\Sigma_n}$ is not the category of Σ_n-objects in $\mathscr{S}$.

Remark 2.3.26. For $n = 1$, a Σ_n-equivariant Nisnevich covering is a usual Nisnevich covering in $\mathscr{C}$.

Definition 2.3.27. A Cartesian square in $\mathscr{C}^{\Sigma_n}$ of the form (2.17) is an *elementary distinguished square* if p is an étale morphism and j is an open embedding when we forget the action of Σ_n, such that the induced morphism of reduced schemes

$$p|_{p^{-1}(X-U)_{\mathrm{red}}} \colon p^{-1}(X-U)_{\mathrm{red}} \to (X-U)_{\mathrm{red}}$$

is an isomorphism.

Remark 2.3.28. Notice that when $n = 1$, the above definition coincide with the usual definition of an elementary distinguished square.

Let us keep the considerations of Definition 2.3.27. An elementary square in $\mathscr{C}^{\Sigma_n}$ of the form (2.17) induces a diagram

$$\begin{array}{ccc} \Delta_Y[0]_+ \vee \Delta_Y[0]_+ & \longrightarrow & \Delta_Y[0]_+ \wedge \Delta[1]_+ \\ \big\downarrow & & \\ \Delta_U[0]_+ \vee \Delta_V[0]_+ & & \end{array}$$

Definition 2.3.29. We denote by $K_{\mathcal{Q}}$ the pushout in $\Delta^{\mathrm{op}}\mathscr{S}^{\Sigma_n}_*$ of the above diagram and denote by $\mathcal{G}_{\Sigma_n,\mathrm{Nis}}$ the set of morphisms in $\mathscr{C}^{\Sigma_n}$ of canonical morphisms from $K_{\mathcal{Q}}$ to $\Delta_X[0]_+$. The set $\mathcal{G}_{\Sigma_n,\mathrm{Nis}}$ is called *set of generating Nisnevich equivalences.*

On the other hand, we denote by $\mathcal{P}_{\Sigma_n,\mathbb{A}^1}$ the set of morphisms in $\mathscr{C}^{\Sigma_n}$ which is isomorphic to the projection from $\Delta_X[0]_+ \wedge \Delta_{\mathbb{A}^1}[0]_+$ to $\Delta_X[0]_+$, for X in $\mathscr{C}^{\Sigma_n}$. By Lemma 13 [7, page 392], the class of $\mathbb{A}^1$-weak equivalences in $\Delta^{\mathrm{op}}\mathscr{S}^{\Sigma_n}$ coincides with the class

$$\mathrm{cl}_{\bar{\Delta}}(\mathcal{G}_{\Sigma_n,\mathrm{Nis}} \cup \mathcal{P}_{\Sigma_n,\mathbb{A}^1})\,. \tag{2.24}$$

We denote by $\mathrm{Const}\colon \mathscr{C} \to \mathscr{C}^{\Sigma_n}$ the functor which sends X to the Σ_n-object X, where Σ_n acts on X trivially. Let $\operatorname{colim}_{\Sigma_n}\colon \mathscr{C}^{\Sigma_n} \to \mathscr{C}$ be the functor which sends X to $\operatorname{colim}_{\Sigma_n} X = X/\Sigma_n$. By definition of colimit, the functor $\operatorname{colim}_{\Sigma_n}$ is left adjoint to the functor Const. It turns out that the functor Const preserves finite limits and it sends Nisnevich coverings to Σ_n-equivariant Nisnevich coverings. In consequence, the functor Const is continuous and the functor $\operatorname{colim}_{\Sigma_n}$ is cocontinuous.

Let $\Lambda_n\colon \mathscr{C} \to \mathscr{C}^{\Sigma_n}$ be the functor which sends X to the nth fold product $X^{\times n}$. Then, the endofunctor Sym^n of $\mathscr{C}$ is nothing but the composition of $\operatorname{colim}_{\Sigma_n}$ with Λ_n.

Proposition 2.3.30. *The cocontinuous functor* $\operatorname{colim}_{\Sigma_n}\colon \mathscr{C}^{\Sigma_n}_{\mathrm{Nis}} \to \mathscr{C}_{\mathrm{Nis}}$ *is also continuous. In consequence, it is a morphism of sites.*

Proof. See [7, Prop. 43]. □

The previous proposition says that the functor $\operatorname{colim}_{\Sigma_n}$ is a morphism of sites, then it induces an adjunction between the inverse and direct image functors,

$$(\operatorname{colim}_{\Sigma_n})_* : \mathscr{S} \rightleftarrows \mathscr{S}^{\Sigma_n} : (\operatorname{colim}_{\Sigma_n})^* \,.$$

Hence, one has a commutative diagram up to isomorphisms

$$\begin{array}{ccc} \mathscr{C}^{\Sigma_n}_{\mathrm{Nis}} & \xrightarrow{\operatorname{colim}_{\Sigma_n}} & \mathscr{C}_{\mathrm{Nis}} \\ \downarrow h & & \downarrow h \\ \mathscr{S}^{\Sigma_n} & \xrightarrow[(\operatorname{colim}_{\Sigma_n})^*]{} & \mathscr{S} \end{array} \tag{2.25}$$

where h is the Yoneda embedding. We denote by

$$\gamma_n\colon \Delta^{\mathrm{op}}\mathscr{S}^{\Sigma_n} \longrightarrow \Delta^{\mathrm{op}}\mathscr{S}$$

the functor induced by $(\operatorname{colim}_{\Sigma_n})^*$ defined termwise. From the diagram (2.25), we deduce that γ_n preserve terminal object, then it induces a functor

$$\gamma_{n,+}\colon \Delta^{\mathrm{op}}\mathscr{S}^{\Sigma_n}_* \longrightarrow \Delta^{\mathrm{op}}\mathscr{S}_* \,.$$

We write $\tilde{\Lambda}_n$ for the left Kan extension of the composite $\mathscr{C} \xrightarrow{\Lambda_n} \mathscr{C}^{\Sigma_n} \xrightarrow{h} \mathscr{S}^{\Sigma_n}$ along the Yoneda embedding $h\colon \mathscr{C} \longrightarrow \mathscr{S}$. Denote by

$$\lambda_n\colon \Delta^{\mathrm{op}}\mathscr{S} \longrightarrow \Delta^{\mathrm{op}}\mathscr{S}^{\Sigma_n}$$

the functor induced by $\tilde{\Lambda}_n$ defined termwise. Since $\tilde{\Lambda}_n$ preserves terminal objects, the functor λ_n does so, hence it induces a functor

$$\lambda_{n,+}\colon \Delta^{\mathrm{op}}\mathscr{S}_* \longrightarrow \Delta^{\mathrm{op}}\mathscr{S}^{\Sigma_n}_* \,.$$

2.3.3 Geometric symmetric powers

Let $\mathscr{C} \subset \mathscr{S}ch/k$ be an admissible category. Fix an object X of $\mathscr{C}$ and an integer $n \geq 1$. By definition of an admissible category, $\mathscr{C}$ is closed under finite products and quotients under finite groups. Then nth fold product $X^{\times n}$ is an object of $\mathscr{C}$, hence, the quotient $X^{\times n}/\Sigma_n$ is also in $\mathscr{C}$. Denote this quotient by $\mathrm{Sym}^n(X)$. Then, we have a functor $\mathrm{Sym}^n \colon \mathscr{C} \to \mathscr{C}$. It is immediate to observe that $\mathrm{Sym}^n(\mathrm{Spec}(k))$ is isomorphic to the point $\mathrm{Spec}(k)$ for $n \geq 1$. By convention, Sym^0 will be the constant endofunctor of $\mathscr{C}$ which sends an object X of $\mathscr{C}$ to the point $\mathrm{Spec}(k)$.

Let us fix $n \in \mathbb{N}$. Since $\mathscr{C}$ is a small category and $\Delta^{\mathrm{op}}\mathscr{S}$ is cocomplete, Theorem 3.7.2 of [4] asserts the existence of the left Kan extension of the composite

$$\mathscr{C} \xrightarrow{\mathrm{Sym}^n} \mathscr{C} \xrightarrow{h} \mathscr{S}$$

along the Yoneda embedding h.

Definition 2.3.31. We denote by Sym^n_g the above left Kan extension, and call it the nth-fold *geometric symmetric power* of Nisnevich sheaves.

Explicitly, Sym^n_g is described as follows. For a sheaf $\mathscr{X}$ in $\mathscr{S}$, we denote by $(h \downarrow \mathscr{X})$ the comma category whose objects are arrows of the form $h_U \to \mathscr{X}$ for $U \in \mathrm{ob}(\mathscr{C})$. Let $F_{\mathscr{X}} \colon (h \downarrow \mathscr{X}) \to \mathscr{S}$ be the functor which sends a morphism $h_U \to \mathscr{X}$ to the representable sheaf $h_{\mathrm{Sym}^n U}$. Then, $\mathrm{Sym}^n_g(\mathscr{X})$ is nothing but the colimit of the functor $F_{\mathscr{X}}$.

Definition 2.3.32. The endofunctor Sym^n_g of Definition 2.3.31 induces an endofunctor of $\Delta^{\mathrm{op}}\mathscr{S}$. We call it the nth-fold *geometric symmetric power* of simplicial Nisnevich sheaves. By abuse of notation, we denote this endofunctor by the same symbol Sym^n_g if no confusion arises.

Example 2.3.33. Fix a natural number n. For each k-scheme X in $\mathscr{C}$, the nth fold geometric symmetric power $\mathrm{Sym}^n_g(h_X)$ of the representable functor h_X coincides with the representable functor $h_{\mathrm{Sym}^n X}$. The section $\mathrm{Sym}^n_g(h_X)(\mathrm{Spec}(k))$ is nothing but the set of effective zero cycles of degree n on X.

Remark 2.3.34. Since $\mathrm{Sym}^n_g : \mathscr{S} \to \mathscr{S}$ preserves the point $\mathrm{Spec}(k)$, it induces an endofunctor of $\mathscr{S}_*$, and hence an endofunctor of $\Delta^{\mathrm{op}}\mathscr{S}_*$.

Warning 2.3.35. As many statements hold similarly for pointed and unpointed (simplicial) sheaves, we shall use the same symbol Sym^n_g to denote the nth fold geometric symmetric power both pointed and unpointed (simplicial) sheaves if no confusion arises.

Lemma 2.3.36. *Left adjoint functors preserves left Kan extensions, in the following sense. Let $L \colon \mathscr{E} \to \mathscr{E}'$ be a left adjoint functor. If $\mathrm{Lan}_G F$ is the left Kan extension of a functor $F \colon \mathscr{C} \to \mathscr{E}$ along a functor $G \colon \mathscr{C} \to \mathscr{D}$, then the composite $L \circ \mathrm{Lan}_G F$ is the left Kan extension of the composite $L \circ F$ along G.*

Proof. See [33, Lemma 1.3.3]. □

Lemma 2.3.37. *For every natural n, the endofunctor* Sym_g^n *of* $\Delta^{\mathrm{op}}\mathscr{S}$ *is isomorphic to the composition* $\gamma_n \circ \lambda_n$*. Similarly,* Sym_g^n *as an endofunctor of* $\Delta^{\mathrm{op}}\mathscr{S}_*$ *is isomorphic to the composition* $\gamma_{n,+} \circ \lambda_{n,+}$*.*

Proof. Since the functors Sym_g^n, γ_n and λ_n are termwise, it is enough to show that Sym_g^n, as a endofunctor of $\mathscr{S}$, is isomorphic to the composition of $\tilde{\Lambda}_n$ with $(\operatorname{colim}_{\Sigma_n})^*$. Indeed, as the functor $(\operatorname{colim}_{\Sigma_n})^*$ is left adjoint, Lemma 2.3.36 implies that the composite

$$\mathscr{S} \xrightarrow{\tilde{\Lambda}_n} \mathscr{S}^{\Sigma_n} \xrightarrow{(\operatorname{colim}_{\Sigma_n})^*} \mathscr{S} \tag{2.26}$$

is the left Kan extension of the composite

$$\mathscr{C} \xrightarrow{\Lambda_n} \mathscr{C}^{\Sigma_n} \xrightarrow{h} \mathscr{S}^{\Sigma_n} \xrightarrow{(\operatorname{colim}_{\Sigma_n})^*} \mathscr{S}$$

along the embedding $h\colon \mathscr{C} \to \mathscr{S}$. Now, in view of the commutativity of diagram (2.25), the preceding composite is isomorphic to the composite

$$\mathscr{C} \xrightarrow{\Lambda_n} \mathscr{C}^{\Sigma_n} \xrightarrow{\operatorname{colim}_{\Sigma_n}} \mathscr{C} \xrightarrow{h} \mathscr{S} \ ,$$

but it is isomorphic to the composite $\mathscr{C} \xrightarrow{\mathrm{Sym}^n} \mathscr{C} \xrightarrow{h} \mathscr{S}$, which implies that the composite (2.26) is isomorphic to Sym_g^n, as required. □

We denote by $\bar{\mathscr{C}}_+$ the full subcategory of coproducts of pointed objects of the form $(h_X)_+$ in $\mathscr{S}_*$ for objects X in $\mathscr{C}$. For every object X in $\mathscr{C}$, the pointed sheaf $(h_X)_+$ is isomorphic to $h_{(X_+)}$. Indeed, $(h_X)_+$ is by definition equal to the coproduct $h_X \amalg h_{\mathrm{Spec}(k)}$ and this coproduct is isomorphic to the representable functor $h_{X \amalg \mathrm{Spec}(k)}$ which is equal to $h_{(X_+)}$.

Similarly, we denote by $\bar{\mathscr{C}}_+^{\Sigma_n}$ the full subcategory of coproducts of pointed objects $(h_X)_+$ in $\mathscr{S}_*^{\Sigma_n}$ for objects X in $\mathscr{C}^{\Sigma_n}$.

Theorem 2.3.38 (Voevodsky)**.** *Let* $f\colon \mathscr{X} \to \mathscr{Y}$ *be a morphism in* $\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+$*. If* f *is an* $\mathbb{A}^1$*-weak equivalence in* $\Delta^{\mathrm{op}}\mathscr{S}_*$*, then* $\mathrm{Sym}_g^n(f)$ *is an* $\mathbb{A}^1$*-weak equivalence.*

Proof. By Lemma 2.3.37, Sym_g^n is the composition $\gamma_{n,+} \circ \lambda_{n,+}$. The idea of the proof is to show that $\gamma_{n,+}$ and $\lambda_{n,+}$ preserve $\mathbb{A}^1$-weak equivalences between objects which termwise are coproducts of representable sheaves. The functor $\lambda_{n,+}$ sends morphisms of $\mathbf{W}_{\mathrm{Nis},+} \cup \mathcal{P}_{\mathrm{Nis},+}$ between objects in $\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+$ to $\mathbb{A}^1$-weak equivalences between objects in $\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+^{\Sigma_n}$. Since $\lambda_{n,+}$ preserves filtered colimits, Lemma 2.20 of [41] implies that $\lambda_{n,+}$ preserves $\mathbb{A}^1$-weak equivalence as claimed. Similarly, in view of the class given in (2.24), we use again Lemma 2.20 of *loc.cit.* to prove that $\gamma_{n,+}$ sends $\mathbb{A}^1$-weak equivalences between objects in $\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+^{\Sigma_n}$ to $\mathbb{A}^1$-weak equivalences, as required □

We define the functor $\Phi\colon \Delta^{\mathrm{op}}\bar{\mathscr{C}}_+ \to \mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$ as the composite

$$\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+ \hookrightarrow \Delta^{\mathrm{op}}\mathscr{S}_* \to \mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)\,,$$

where the first arrow is the inclusion functor and the second arrow is the localization functor with respect to the $\mathbb{A}^1$-weak equivalences.

Lemma 2.3.39. *Let $\mathscr{C}$ be an admissible category. The functor*

$$\Phi\colon \Delta^{\mathrm{op}}\bar{\mathscr{C}}_+ \to \mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$$

is a strict localization, that is, for every morphism f in $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$, there is a morphism g of $\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+$ such that the image $\Phi(g)$ is isomorphic to f.

Proof. By Theorem 2.5 of [30, page 71], the category $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$ is the localization of the category $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}})$ with respect to the image of $\mathbb{A}^1$-weak equivalences trough the canonical functor. Then, it is enough to prove that the canonical functor from $\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+$ to $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}})$ is a strict localization. Indeed, let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of pointed simplicial sheaves on the site $\mathscr{C}_{\mathrm{Nis}}$ representing a morphism in $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}})$. The functorial resolution Q^{proj} gives a commutative square

$$\begin{CD}
Q^{\mathrm{proj}}(\mathscr{X}) @>{Q^{\mathrm{proj}}(f)}>> Q^{\mathrm{proj}}(\mathscr{Y}) \\
@VVV @VVV \\
\mathscr{X} @>>{f}> \mathscr{Y}
\end{CD}$$

where the vertical arrows are object-wise weak equivalences. Since the object-wise weak equivalences are local weak equivalences, the vertical arrows of the above diagram are weak equivalences. This implies that f is isomorphic to $Q^{\mathrm{proj}}(f)$ in $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}})$. Moreover, by Corollary 2.3.21, the morphism $Q^{\mathrm{proj}}(f)$ is in $\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+$. $\square$

Corollary 2.3.40. *For each integer $n \geq 1$, there exists the left derived functor $L\mathrm{Sym}_g^n$ from $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$ to itself such that we have a commutative diagram up to isomorphism*

$$\begin{CD}
\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+ @>{\mathrm{Sym}_g^n}>> \Delta^{\mathrm{op}}\mathscr{S}_* \\
@V{\Phi}VV @VVV \\
\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1) @>>{L\mathrm{Sym}_g^n}> \mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)
\end{CD} \tag{2.27}$$

where the right arrow is the localization functor.

Proof. By Theorem 2.3.38, the functor Sym^n_g preserves $\mathbb{A}^1$-weak equivalences between objects in $\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+$. Hence, the composite

$$\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+ \xrightarrow{\mathrm{Sym}^n_g} \Delta^{\mathrm{op}}\mathscr{S}_* \longrightarrow \mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$$

sends $\mathbb{A}^1$-weak equivalences to isomorphisms. Then, by Lemma 2.3.39 there exists a functor $L\mathrm{Sym}^n_g$ such the diagram (2.27) commutes and for every simplicial sheaf $\mathscr{X}$, the object $L\mathrm{Sym}^n_g(\mathscr{X})$ is isomorphic to $\mathrm{Sym}^n_g(Q^{\mathrm{proj}}(\mathscr{X}))$ in $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$. □

2.4 Stable motivic category

In this section $\mathscr{C}$ will denote a small admissible category contained in the category of quasi-projective schemes over a field k of arbitrary characteristic. The letter $\mathscr{S}$ to denote the category of Nisnevish sheaves and the category $\Delta^{\mathrm{op}}\mathscr{S}_*$ is the category of pointed simplicial sheaves studied in the previous sections. We write S^1 for pointed simplicial circle, i.e. the cokernel of the morphism $\partial\Delta[1]_+ \to \Delta[1]_+$ in $\Delta^{\mathrm{op}}\mathscr{S}ets_*$. We shall denote by T the smash product $S^1 \wedge (\mathbb{G}_m, 1)$. There is an isomorphism $T \simeq (\mathbb{P}^1, \infty)$ in $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$, cf. [30, Lemma 3.2.15].

Generalities

We denote by $\mathrm{Spt}_T(k)$ the category of symmetric T-spectra on the category $\Delta^{\mathrm{op}}\mathscr{S}_*$. The category $\mathrm{Spt}_T(k)$ is naturally equivalent to the category of left modules over the commutative monoid $\mathrm{sym}(T) := (\mathrm{Spec}(k)_+, T, T^{\wedge 2}, T^{\wedge 3}, \dots)$. For each $n \in \mathbb{N}$, there is an evaluation functor Ev_n from $\mathrm{Spt}_T(k)$ to $\Delta^{\mathrm{op}}\mathscr{S}_*$ which takes a symmetric T-spectrum $\mathscr{X}$ to its nth slice $\mathscr{X}_n$. The evaluation functor Ev_n has a left adjoint functor denoted by F_n. The functor F_0 is called *suspension functor*, and it is usually denoted by Σ^∞_T. This functor takes simplicial sheaf $\mathscr{X}$ to the symmetric T-spectrum

$$(\mathscr{X}, \mathscr{X} \wedge T, \mathscr{X} \wedge T^{\wedge 2}, \dots).$$

For a scheme X in $\mathscr{C}$, we write $\Sigma^\infty_T(X_+)$ instead of $\Sigma^\infty_T(\Delta_X[0]_+)$. A morphism of T-spectra $f\colon \mathscr{X} \to \mathscr{Y}$ is a level $\mathbb{A}^1$-weak equivalence (a level fibration) if each term f_n is an $\mathbb{A}^1$-weak equivalence (a fibration) in $\Delta^{\mathrm{op}}\mathscr{S}_*$ for all $n \in \mathbb{N}$. We say that f is a projective cofibration if it has the left lifting property with respect to both level $\mathbb{A}^1$-equivalences and level fibrations. The class of level $\mathbb{A}^1$-weak equivalences, the class of the level fibrations and the class of projective cofibrations define a left proper cellular model structure on $\mathrm{Spt}_T(k)$ called projective model structure, see [19]. Let I (resp. J) be the set of generating (resp. trivial) cofibrations of the injective model structure of $\Delta^{\mathrm{op}}\mathscr{S}_*$. The set $I_T := \bigcup_{n\geq 0} F_n(I)$ (resp. $J_T := \bigcup_{n\geq 0} F_n(J)$) is the set of generating cofibrations (resp. trivial cofibrations) of the projective model structure of $\mathrm{Spt}_T(k)$, cf. [19].

In order to define the stable model structure on $\mathrm{Spt}_T(k)$, one uses the Bousfield localization of its projective model structure with respect to a certain set of morphisms of symmetric T-spectra, so that the functor $-\wedge T\colon \mathrm{Spt}_T(k)\to \mathrm{Spt}_T(k)$ becomes a Quillen equivalence. We shall define this set as follows. For every simplicial sheaf $\mathscr{X}$ in $\Delta^{\mathrm{op}}\mathscr{S}_*$ and every $n\in\mathbb{N}$, we denote by $\zeta_n^{\mathscr{X}}: F_{n+1}(\mathscr{X}\wedge T)\to F_n(\mathscr{X})$ the morphism which is adjoint to the morphism

$$\mathscr{X}\wedge T\longrightarrow \mathrm{Ev}_{n+1}(F_n(\mathscr{X}))=\Sigma_{n+1}\times_{\Sigma_1}(\mathscr{X}\wedge T)\,,$$

induced by the canonical embedding of Σ_1 into Σ_n. We set

$$S{:=}\Big\{\zeta_n^{\mathscr{X}}\mid \mathscr{X}\in \mathrm{dom}\,(I)\cup \mathrm{codom}\,(I), n\in\mathbb{N}\Big\}\,.$$

The *stable model structure* on $\mathrm{Spt}_T(k)$ is the Bousfield localization of the projective model structure on $\mathrm{Spt}_T(k)$ with respect to S, cf. [19]. A S-local weak equivalence will be called a *stable weak equivalence*. The stable model structure on $\mathrm{Spt}_T(k)$ is left proper and cellular. The functor $\Sigma_T^\infty\colon \Delta^{\mathrm{op}}\mathscr{S}_*\to \mathrm{Spt}_T(k)$ is a left Quillen functor, see *loc.cit.* For any two symmetric T-spectra $\mathscr{X}$ and $\mathscr{Y}$, its *smash* product $\mathscr{X}\wedge_{\mathrm{sym}(T)}\mathscr{Y}$ is defined to be the coequalizer of the diagram

$$\mathscr{X}\wedge \mathrm{sym}(T)\wedge\mathscr{Y}\rightrightarrows \mathscr{X}\wedge\mathscr{Y}\,,$$

induced by the canonical morphisms $\mathscr{X}\wedge\mathrm{sym}(T)\to\mathscr{X}$ and $\mathrm{sym}(T)\wedge\mathscr{Y}\to\mathscr{Y}$. The smash product of spectra defines a symmetric monoidal structure on $\mathrm{Spt}_T(k)$. We denote by $\mathcal{SH}_T(k)$ the homotopy category of the category $\mathrm{Spt}_T(k)$ with respect to stable $\mathbb{A}^1$-weak equivalences.

Chain complexes

Let $\mathcal{A}b$ be the category of Abelian groups. The classical *Dold-Kan correspondence* establishes a Quillen equivalence

$$N:\Delta^{\mathrm{op}}\mathcal{A}b\rightleftarrows \mathrm{ch}_+(\mathcal{A}b):\Gamma\,,$$

between the category of simplicial Abelian groups and the category of $\mathbb{N}$-graded chain complexes of Abelian groups. Let $\mathscr{A}$ be an Abelian Grothendieck category. We write $\mathrm{ch}_+(\mathscr{A})$ for the category of $\mathbb{N}$-graded chain complexes on $\mathscr{A}$. The above adjunction induces an adjunction

$$N:\Delta^{\mathrm{op}}\mathscr{A}\rightleftarrows \mathrm{ch}_+(\mathscr{A}):\Gamma\,. \tag{2.28}$$

The category $\mathrm{ch}_+(\mathscr{A})$ has a monoidal proper closed simplicial model category such that the class of weak equivalences are quasi-isomorphisms and such that the adjunction

2.28 becomes a Quillen equivalence [23, Lemma 2.5]. For any $n \in \mathbb{Z}$, we have the translation functor $\mathrm{ch}_+(\mathscr{A}) \to \mathrm{ch}_+(\mathscr{A})$ which sends a chain complex C to $C[n]$ defined by $(C[n])_i := C_{n+i}$ for $i \geq 0$. For each $n \geq 0$, we denote by $\mathbb{Z}[n]$ the chain complex

$$\cdots \to 0 \to \mathbb{Z} \to 0 \to \cdots \to 0$$

concentrated in degree n. If the symbol $\otimes$ denotes the tensor product of $\mathbb{N}$-graded chain complexes of Abelian groups, then, for $n \in \mathbb{N}$, we have $\mathbb{Z}[n] = \mathbb{Z}[1]^{\otimes n}$. Hence, the symmetric group Σ_n acts naturally on $\mathbb{Z}[n]$, and we have the symmetric sequence

$$\mathrm{sym}(\mathbb{Z}[1]) = (\mathbb{Z}[0], \mathbb{Z}[1], \mathbb{Z}[2], \cdots)$$

in $\mathrm{ch}_+(\mathcal{A}b)$. For any chain complex C_* in $\mathrm{ch}_-(\mathscr{A})$, we have

$$C_* \otimes \mathbb{Z}[n] = C_*[-n]\,.$$

Let $\mathrm{Spt}_{\mathbb{Z}[1]}(\mathrm{ch}_+(\mathcal{A}))$ be the category of symmetric $\mathbb{Z}[1]$-spectra. Its objects are symmetric sequences $(C_0, C_1, \ldots, C_n \ldots)$ where each C_n is a chain complex in $\mathrm{ch}_+(\mathcal{A})$ together with an action of the symmetric group Σ_n on it. For a symmetric $\mathbb{Z}[1]$-spectrum C_*, we have structural morphisms of the form $C_n \otimes \mathbb{Z}[1] \to C_{n+1}$ for $n \in \mathbb{N}$.

2.4.1 Rational stable homotopy category of schemes

In the next paragraphs, we shall recall some results on rational stable homotopy categories of schemes over a field. Here, $\mathcal{SH}_T(k)$ will be the stable $\mathbb{A}^1$-homotopy category of smooth schemes over a field k constructed in [22]. One result that is very important is a theorem due to Morel which asserts an equivalence of categories between the rational stable homotopy category $\mathcal{SH}_T(k)_\mathbb{Q}$ and the rational big Voevodsky's category $\mathsf{DM}(k)_\mathbb{Q}$. This will allows us to show the existence of transfers of some morphisms in $\mathcal{SH}_T(k)_\mathbb{Q}$ that will be studied in Section 4.3.1 and 4.3.2.

Let $\mathscr{T}$ be a triangulated category with small sums and with a small set of compact generators [31]. An object T in $\mathscr{T}$ is said to be *torsion* (resp. *uniquely divisible*) if for every compact generator X in $\mathscr{T}$, the canonical morphism from $\mathrm{Hom}_\mathscr{T}(X,T)$ to $\mathrm{Hom}_\mathscr{T}(X,T) \otimes_\mathbb{Z} \mathbb{Q}$ is the zero morphism (resp. an isomorphism). Let $\mathscr{T}_{\mathrm{tor}}$ (resp. $\mathscr{T}_\mathbb{Q}$) be the triangulated subcategory of $\mathscr{T}$ generated by the torsion objects (resp. uniquely divisible objects). The full embedding functor $\mathscr{T}_\mathbb{Q} \hookrightarrow \mathscr{T}$ has a left adjoint $L_\mathbb{Q} \colon \mathscr{T} \to \mathscr{T}_\mathbb{Q}$ and its kernel is nothing but $\mathscr{T}_{\mathrm{tor}}$. Then, $\mathscr{T}_\mathbb{Q}$ is equivalent to the Verdier quotient $\mathscr{T}/\mathscr{T}_{\mathrm{tor}}$ (see [34, Annexe A]). We denote by $\mathcal{SH}_T(k)_\mathbb{Q}$ the Verdier quotient of $\mathcal{SH}_T(k)$ by the full-subcategory $\mathcal{SH}_T(k)_{\mathrm{tor}}$ generated by compact torsion objects. We recall that a morphism of symmetric T-spectra $f \colon \mathscr{X} \to \mathscr{Y}$ is a stable $\mathbb{A}^1$-weak equivalence if and only if the induced morphism

$$f_* \colon \mathrm{Hom}_{\mathcal{SH}_T(k)}\Big(\Sigma^\infty_T(S^r \wedge \mathbb{G}^s_m \wedge U_+), \mathscr{X}\Big) \longrightarrow \mathrm{Hom}_{\mathcal{SH}_T(k)}\Big(\Sigma^\infty_T(S^r \wedge \mathbb{G}^s_m \wedge U_+), \mathscr{Y}\Big)$$

is an isomorphism of Abelian groups for all couples $(r,s)\in\mathbb{N}^2$ and all smooth schemes U over k (see [18, Th. 1.2.10(iv)].

A morphism of T-spectra $f\colon \mathscr{X}\to\mathscr{Y}$ is called *rational stable* $\mathbb{A}^1$*-weak equivalence* if the induced morphism $f_*\otimes\mathbb{Q}$ is an isomorphism of $\mathbb{Q}$-vector spaces for all couples $(r,s)\in\mathbb{N}^2$ and all smooth schemes U over k. The localization of $\mathcal{SH}_T(k)$ with respect to the rational stable $\mathbb{A}^1$-weak equivalences coincides with $\mathcal{SH}_T(k)_{\mathbb{Q}}$.

2.4.2 The motivic Hurewicz functor

Let $\mathcal{A}b^{\mathrm{tr}}_{\mathrm{Nis}}$ be the category of Nisnevich Abelian sheaves with transfers on the category of smooth schemes $\mathscr{S}m/k$ over a field k c.f. [27]. Let τ be either the h-topology or qfh-topology on the category of k-schemes of finite type. We write $\underline{\mathcal{A}b}^{\mathrm{tr}}_{\tau}$ for the category of τ-Abelian sheaves with transfers on the category of k-schemes of finite type. We consider the $\mathbb{A}^1$-localized model category of the projective model structure on $\mathrm{ch}_+(\mathcal{A}b^{\mathrm{tr}}_{\mathrm{Nis}})$ and in $\mathrm{ch}_+(\underline{\mathcal{A}b}^{\mathrm{tr}}_{\tau})$. Let $\mathsf{DM}(k)$ be the homotopy category of the category of symmetric T-spectra $\mathrm{Spt}_T(\mathrm{ch}_+(\mathcal{A}b^{\mathrm{tr}}_{\mathrm{Nis}}))$ with respect to stable $\mathbb{A}^1$-weak equivalences. If the characteristic of k is zero, then $\mathsf{DM}(k)$ is equivalent to the homotopy category of the category of modules over the motivic Eilenberg-MacLane spectrum [35]. We denote by $\underline{\mathsf{DM}}(k)_\tau$ the homotopy category of the category of symmetric T-spectra $\mathrm{Spt}_T(\mathrm{ch}_+(\underline{\mathcal{A}b}^{\mathrm{tr}}_{\tau}))$ with respect to stable $\mathbb{A}^1$-weak equivalences. We write $\mathsf{DM}_\tau(k)$ for the localizing subcategory of $\underline{\mathsf{DM}}_\tau(k)$ generated by the objects of the form $\Sigma^\infty_T\mathbb{Z}_\tau(X)(m)[n]$ for k-smooth schemes of finite type X and for all couples $(m,n)\in\mathbb{Z}$, see [6]. One has an adjunction of triangulated categories

$$Hu : \mathcal{SH}_T(k) \rightleftarrows \mathsf{DM}(k) : H\,,$$

where Hu is the *motivic Hurewicz functor* and H is the *Eilenberg-MacLane spectrum functor* [29, 6]. This adjunction induces an adjunction of triangulated categories with rational coefficients

$$Hu_{\mathbb{Q}} : \mathcal{SH}_T(k)_{\mathbb{Q}} \rightleftarrows \mathsf{DM}(k)_{\mathbb{Q}} : H_{\mathbb{Q}}\,.$$

We write $\mathbb{S}^0$ for the sphere T-spectrum. Let $\epsilon:\mathbb{S}^0\to\mathbb{S}^0$ be the morphism of spectra induced by the morphism $\mathbb{G}_m\to\mathbb{G}_m$ which comes from the homomorphism of k-algebras $k[x,x^{-1}]\to k[x,x^{-1}]$ given by $x\mapsto x^{-1}$. Notice that $\epsilon^{\circ 2}=\mathrm{id}$. We set $e_+{:=}(\epsilon^{\circ 2}-1)/2$ and $e_-{:=}(\epsilon^{\circ 2}+1)/2$. Notice that e_+ and e_- are both idempotent. Since $\mathcal{SH}_T(k)_{\mathbb{Q}}$ has small coproducts (see [31]), the triangulated category $\mathcal{SH}_T(k)_{\mathbb{Q}}$ is pseudo-abelian, hence the morphisms e_+ and e_- have image. We put $\mathbb{S}^0_{\mathbb{Q},+}{:=}\mathrm{im}\,e_+$ and $\mathbb{S}^0_{\mathbb{Q},-}{:=}\mathrm{im}\,e_-$. Then, they induce two functors

$$\mathcal{SH}_T(k)_{\mathbb{Q}} \longrightarrow \mathcal{SH}_T(k)_{\mathbb{Q},+}\,,$$

$$\mathcal{SH}_T(k)_{\mathbb{Q}} \longrightarrow \mathcal{SH}_T(k)_{\mathbb{Q},-}\,,$$

defined by $\mathscr{X} \mapsto \mathscr{X} \wedge^L \mathbb{S}^0_{\mathbb{Q},+}$ and $\mathscr{X} \mapsto \mathscr{X} \wedge^L \mathbb{S}^0_{\mathbb{Q},-}$ respectively. Since $\mathbb{S}^0_{\mathbb{Q}} = \mathbb{S}^0_{\mathbb{Q},+} \oplus \mathbb{S}^0_{\mathbb{Q},-}$, it induces a decomposition

$$\mathcal{SH}_T(k)_{\mathbb{Q}} = \mathcal{SH}_T(k)_{\mathbb{Q},+} \times \mathcal{SH}_T(k)_{\mathbb{Q},-}\,.$$

Remark 2.4.1. For the existence of the above decomposition of $\mathcal{SH}_T(k)_{\mathbb{Q}}$, we have only used the fact that 2 is invertible in $\mathbb{Q}$. In fact, this decomposition is true for triangulated category $\mathcal{SH}_T(k)_{\mathbb{Z}[\frac{1}{2}]}$ with $\mathbb{Z}[\frac{1}{2}]$-coefficients.

The following theorem was predicted by F. Morel.

Theorem 2.4.2. *Suppose that -1 is a sum of squares in k. Then we have an equivalence of categories $\mathcal{SH}_T(k)_{\mathbb{Q}} \simeq \mathsf{DM}(k)_{\mathbb{Q}}$.*

Proof. The fact that -1 is a sum of squares in k implies that the category $\mathcal{SH}_T(k)_{\mathbb{Q},+}$ coincides with $\mathcal{SH}_T(k)_{\mathbb{Q}}$. Hence, the theorem follows from Theorem 16.1.4 and Theorem 16.2.13 in [6]. □

Let $D_{\mathbb{A}^1}(k)$ be the homotopy category of the category of symmetric T-spectra $\mathrm{Spt}_T(\mathrm{ch}_+(\mathcal{A}b_{\mathrm{Nis}}))$ with respect to stable $\mathbb{A}^1$-weak equivalences. The category of *Beilinson motives* $\mathsf{DM}_{\mathrm{B}}(k)$ is the Verdier quotient of $D_{\mathbb{A}^1}(k)_{\mathbb{Q}}$ by the localizing subcategory generated by H_{B}-acyclic objects, where H_{B} is the Beilison motivic spectrum, see [6, 34]. If -1 is a sum of squares in k, then we have a diagram of equivalences of categories:

$$\begin{array}{ccccccc}
\mathcal{SH}_T(k)_{\mathbb{Q}} & = & D_{\mathbb{A}^1}(k)_{\mathbb{Q}} & = & \mathsf{DM}_{\mathrm{B}}(k) & = & \mathsf{DM}(k)_{\mathbb{Q}} \\
 & & & & \| & & \\
 & & & & \mathsf{DM}_h(k)_{\mathbb{Q}} & & \\
 & & & & \| & & \\
 & & & & \mathsf{DM}_{\mathrm{qfh}}(k)_{\mathbb{Q}} & &
\end{array}$$

For the proof of these equivalences see [6, 29]. In consequence, we obtain the following corollary.

Corollary 2.4.3. *If -1 is a sum of squares in k, then we have an equivalence of categories $\mathcal{SH}_T(k)_{\mathbb{Q}} \simeq \mathsf{DM}_{\mathrm{qfh}}(k)_{\mathbb{Q}}$.*

Proof. See [6]. □

Chapter 3

Geometric symmetric powers in motivic categories

In this chapter, we study the Künneth towers of geometric symmetric powers of motivic spaces defined in Section 2.3.3. We also study geometric symmetric powers for motivic symmetric spectra, see Section 3.3. Finally, we study the differences between the categoric and geometric symmetric powers of presheaves represented by particular schemes, such as, finite Galois extensions, the double point, affine line and affine plane, see Section 3.4. We shall start this chapter giving some preliminaries on categoric symmetric powers, see [13].

3.1 Categoric symmetric powers

Symmetric powers appear in many areas of mathematics as an important tool, for instance the singular homology of a CW-complex can be understood as a homotopy group of infinite symmetric powers. Let us give some ideas. If (X, x) is a pointed topological space, then for each integer $n \geq 0$, we have the n-fold symmetric power $\mathrm{Sym}^n(X, x)$. We have a sequence of embeddings

$$\mathrm{Sym}^1(X, x) \hookrightarrow \mathrm{Sym}^2(X, x) \hookrightarrow \cdots \hookrightarrow \mathrm{Sym}^n(X, x) \hookrightarrow \cdots$$

and it induces an infinite symmetric power

$$\mathrm{Sym}^\infty(X, x) := \mathrm{colim}_{n \in \mathbb{N}} \mathrm{Sym}^n(X, x) ,$$

which plays an important role in the Dold-Thom theorem.

For a set X, let $\mathbb{N}[X]$ (resp. $\mathbb{Z}[X]$) be the free commutative monoid (resp. free Abelian group) generated by X. If x is an element of X, we write $\mathbb{N}[x]$ instead of $\mathbb{N}[\{x\}]$, similarly for $\mathbb{Z}[x]$. Notice that the elements of $\mathbb{N}[x]$ have the form $m \cdot x$ with $m \in \mathbb{N}$. Let n be a positive integer. The nth fold symmetric power $\mathrm{Sym}^n(X) := X^n/\Sigma_n$ can be seen as the set of linear combinations $\sum_{i=1}^n x_i \in \mathbb{N}[X]$, where each x_i is an element of X.

Lemma 3.1.1. *Suppose that X is a finite set, say it has a cardinality equal to $r \geq 1$. Then $\mathrm{Sym}^n(X)$ has a cardinality equal to $\binom{r+n-1}{n}$.*

Proof. It follows after noticing that $\mathrm{Sym}^n(X)$ is bijective to the set of all combination with repetition of r elements choose n. □

Example 3.1.2. If $X = \{a, b, c\}$ is a set with three elements, then $\mathrm{Sym}^2(X)$ is the set

$$\left\{a+a,\ a+b,\ b+b,\ b+c,\ c+c,\ c+a\right\},$$

which has $\binom{3+2-1}{2} = 6$ elements.

For a pointed set (X, x), there is an isomorphism of monoids

$$\mathrm{Sym}^\infty(X, x) \simeq \mathbb{N}[X]/\mathbb{N}[x]\,,$$

hence, we have an isomorphism of Abelian groups

$$\mathrm{Sym}^\infty(X, x)^+ \simeq \mathbb{Z}[X]/\mathbb{Z}[x]\,.$$

where the left-hand side is the group completion of $\mathrm{Sym}^\infty(X, x)$.

The Dold-Thom theorem asserts that for any pointed connected CW complex $(X, *)$, there is a weak equivalence

$$\mathrm{Sym}^\infty(X, *) \to \prod_{n \geq 1} K(H_n(X, \mathbb{Z}), n)\,,$$

where $H_n(X, \mathbb{Z})$ is the singular homology of X; or alternatively, an isomorphism

$$\pi_n(\mathrm{Sym}^\infty(X, *)) \simeq H_n(X, \mathbb{Z})\,,$$

for all $n \geq 1$. Removing the connectedness assumption on X, the Dold-Thom theorem can be reformulated by stating an isomorphism

$$\pi_n(\mathrm{Sym}^\infty(X, *)^+) \simeq \tilde{H}_n(X, \mathbb{Z})\,,$$

for all $n \geq 0$, where $\tilde{H}_n(X, \mathbb{Z})$ is the reduced singular homology of X.

3.1.1 Pushout-products

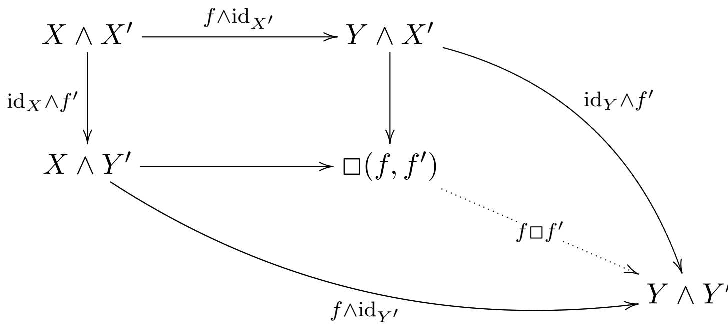
Assumption 3.1.3. Unless otherwise specified, we shall assume that the monoidal product $\wedge$ of a symmetric monoidal category with pushouts preserves pushouts on both sides, i.e. for any two objects X and Y, the functors $X \wedge -$ and $- \wedge Y$ preserve pushouts. Similarly, the monoidal product $\wedge$ of a symmetric monoidal category with finite colimits will always preserve finite colimits on both sides.

For example, these assumptions are satisfied when the monoidal category in question is closed, [26, p. 180].

Definition 3.1.4. Let $\mathscr{C}$ be a symmetric monoidal category with pushouts. We denote by $\wedge$ its monoidal product. We recall that, for any two morphisms $f : X \to Y$ and $f' : X' \to Y'$ in $\mathscr{C}$, the *box operation* of f and f' is the pushout

$$\square(f, f') := (X \wedge Y') \vee_{X \wedge X'} (Y \wedge X') \,.$$

The universal morphism $f \square f' : \square(f, f') \to Y \wedge Y'$ is called *pushout-product* of f and f', which fits into the following pushout diagram:



(3.1)

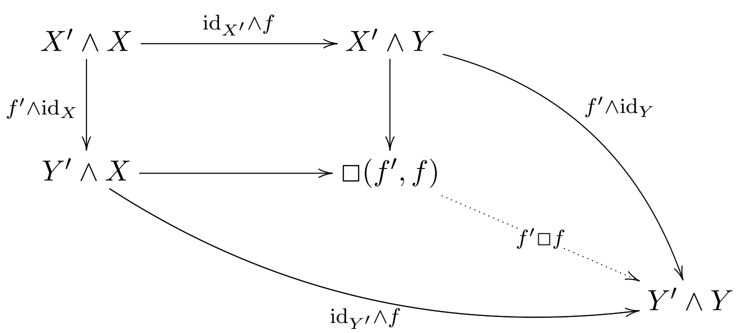
Proposition 3.1.5. *The pushout-product $\square$ is commutative and associative. More precisely, if $f : X \to Y$, $f' : X' \to Y'$ and $f'' : X'' \to Y''$ are three morphisms in $\mathscr{C}$, then there exist a canonical isomorphism of commutativity*

$$f \square f' \simeq f' \square f \,, \tag{3.2}$$

and a canonical isomorphism of associativity

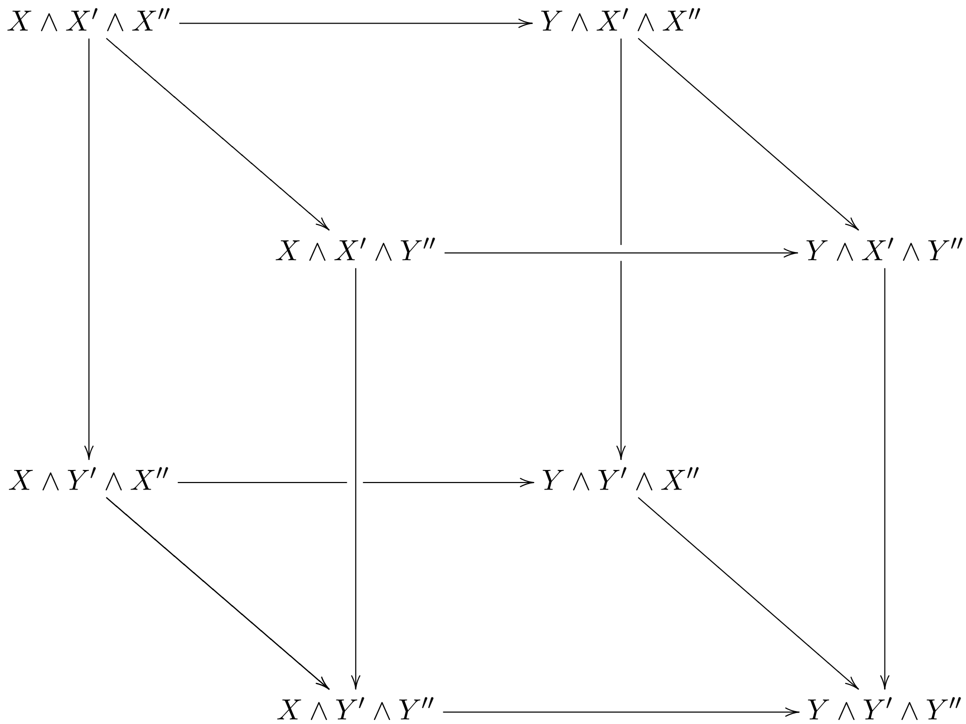
$$(f \square f') \square f'' \simeq f \square (f' \square f'') \,. \tag{3.3}$$

Proof. Let $f : X \to Y$, $f' : X' \to Y'$ and $f'' : X'' \to Y''$ be three morphisms in $\mathscr{C}$. Since the monoidal product $\wedge$ is symmetric, the diagram 3.1 is isomorphic to the following diagram



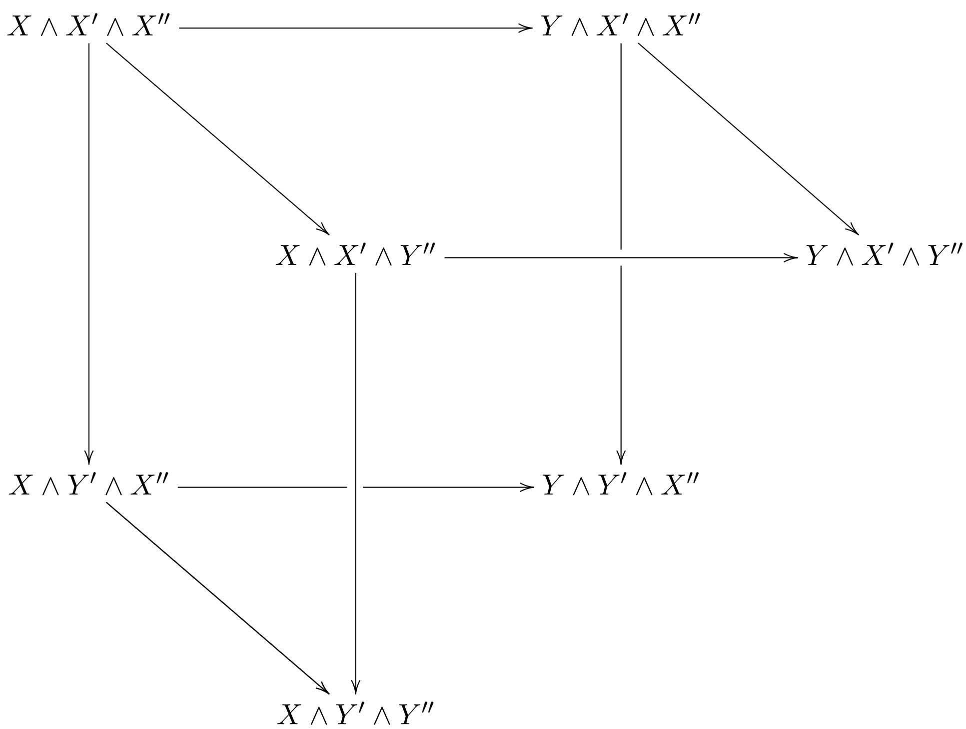
(3.4)

Then, we get the isomorphism (3.2), proving thus the commutativity of $\square$. Let us prove that associativity of $\square$. Indeed, the morphisms f, f' and f'' induce a commutative diagram



(3.5)

The colimit of the diagram



(3.6)

can be computed by means of pushouts. For instance, considering the vertex $Y \wedge Y' \wedge X''$

of diagram (3.6), we deduce a diagram

$$\begin{array}{ccc} (Y \wedge X' \wedge X'') \bigvee_{X\wedge X'\wedge X''} (X \wedge Y' \wedge X'') & \longrightarrow & (Y \wedge X' \wedge Y'') \bigvee_{X\wedge X'\wedge Y''} (X \wedge Y' \wedge Y'') \\ \downarrow & & \\ Y \wedge Y' \wedge X'' & & \end{array} \tag{3.7}$$

whose pushout is isomorphic to the colimit of diagram (3.6). Similarly, considering the vertex $X \wedge Y' \wedge Y''$, we obtain a diagram

$$\begin{array}{ccc} (X \wedge Y' \wedge X'') \bigvee_{X\wedge X'\wedge X''} (X \wedge X' \wedge Y'') & \longrightarrow & (Y \wedge Y' \wedge X'') \bigvee_{Y\wedge X'\wedge X''} (Y \wedge X' \wedge Y'') \\ \downarrow & & \\ X \wedge Y' \wedge Y'' & & \end{array} \tag{3.8}$$

whose pushout is isomorphic to the colimit of the same diagram. Since the monoidal product $\wedge$ commutes with pushouts, we get the following canonical isomorphisms

$$(Y \wedge X' \wedge X'') \bigvee_{X\wedge X'\wedge X''} (X \wedge Y' \wedge X'') \simeq \square(f, f') \wedge X'',$$

$$(Y \wedge X' \wedge Y'') \bigvee_{X\wedge X'\wedge Y''} (X \wedge Y' \wedge Y'') \simeq \square(f, f') \wedge Y'',$$

$$(X \wedge Y' \wedge X'') \bigvee_{X\wedge X'\wedge X''} (X \wedge X' \wedge Y'') \simeq X \wedge \square(f', f''),$$

$$(Y \wedge Y' \wedge X'') \bigvee_{Y\wedge X'\wedge X''} (Y \wedge X' \wedge Y'') \simeq Y \wedge \square(f', f'').$$

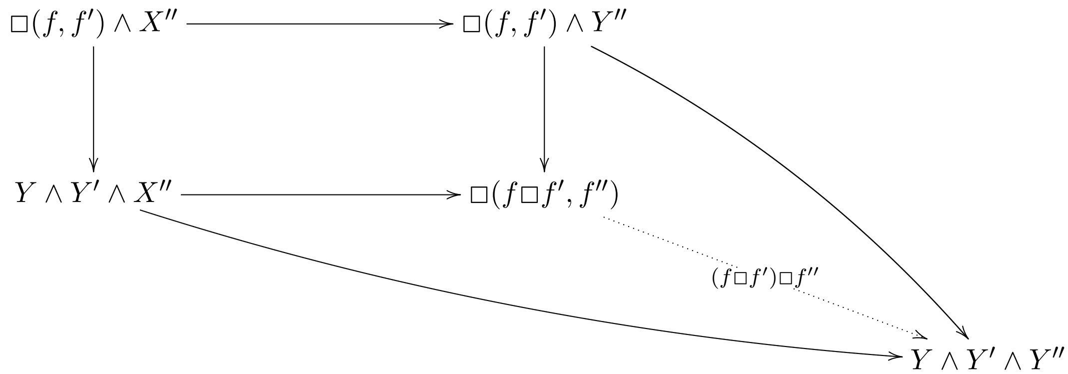
Then, the diagram (3.7) is isomorphic to the diagram

$$\begin{array}{ccc} \square(f, f') \wedge X'' & \longrightarrow & \square(f, f') \wedge Y'' \\ \downarrow & & \\ Y \wedge Y' \wedge X'' & & \end{array} \tag{3.9}$$

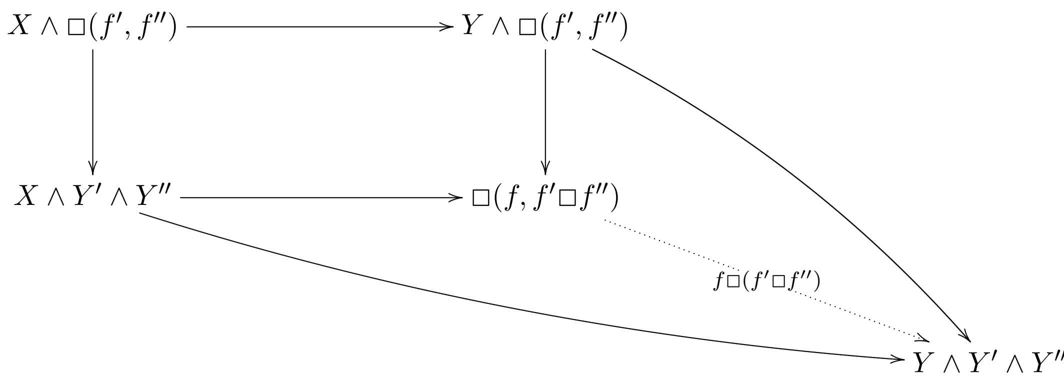
and the diagram (3.8) is isomorphic to the diagram

$$\begin{array}{ccc} X \wedge \square(f', f'') & \longrightarrow & Y \wedge \square(f', f'') \\ \downarrow & & \\ X \wedge Y' \wedge Y'' & & \end{array} \tag{3.10}$$

Finally, from diagram (3.5), we deduce that the diagram (3.9) fits into a pushout diagram



whereas the diagram (3.10) fits into a pushout diagram



Thus, we obtain an isomorphism $\square(f \square f', f'') \simeq f \square (f' \square f'')$. Therefore, we have an isomorphism (3.3), as required. $\square$

Corollary 3.1.6. *Let $\mathscr{C}$ be a symmetric monoidal category with pushouts. Then, the pushout-product $\square$ is a symmetric monoidal product in the category of morphisms* $\mathrm{Map}\,(\mathscr{C})$.

Proof. Since the monoidal product $\wedge$ of $\mathscr{C}$ preserves pushouts, for every object X of $\mathscr{C}$, we have canonical isomorphisms

$$X \wedge \emptyset \simeq \emptyset \simeq \emptyset \wedge X\,,$$

where $\emptyset$ is the initial object of $\mathscr{C}$. Then, the canonical morphism $\emptyset \to \mathbb{1}$ is the unit object for the category $\mathrm{Map}\,(\mathscr{C})$, where $\mathbb{1}$ denotes the unit object of $\mathscr{C}$. Indeed, let $f : X \to Y$ be a morphism in $\mathscr{C}$. Replacing $\emptyset \to \mathbb{1}$ by f' in diagram (3.1), we deduce that $\square(f, \emptyset \to \mathbb{1})$ is isomorphic to X, and $f \square (\emptyset \to \mathbb{1})$ is isomorphic to f. Hence, the corollary follows from Proposition 3.1.5. Notice that the pentagon and the coherence axioms follows from the axioms of the monoidal structure on $\wedge$ and the universal property of pushout. $\square$

By virtue of Proposition 3.1.5, for finite collection $\{f_i : X_i \to Y_i \,|\, i = 1, \ldots, n\}$ of morphisms in $\mathscr{C}$, we can omit the parentheses on the product

$$(\cdots((f_1 \square f_2)\square f_3)\square \cdots \square f_{n-1})\square f_n$$

and write simply

$$f_1 \square \cdots \square f_n : \square(f_1, \ldots, f_n) \to Y_1 \wedge \cdots \wedge Y_n \,.$$

For a morphism $f : X \to Y$ in $\mathscr{C}$ and integer $n \geq 2$, we shall write $\square^n(f) = \square(f, \cdots, f)$ and $f^{\square n} = f \square \cdots \square f$. By convention, we write $\square^1(f) = X$ and $f^{\square 1} = f$.

3.1.2 Künneth towers

Let $\mathbf{2} = \{0, 1\}$ be the category with two objects and one non-identity morphism $0 \to 1$. We denote by $\mathbf{2}^n$ the n-fold cartesian product of categories of $\mathbf{2}$ with itself. Observe that the objects of $\mathbf{2}^n$ are n-tuples $(a_1, \ldots, a_n)$, where each a_i is 0 or 1, and a morphism from $(a_1, \ldots, a_n)$ to another n-tuple $(a'_1, \ldots, a'_n)$ is determined by the condition $a_i \leq a'_i$ for all $i = 1, \ldots, n$.

Remark 3.1.7. Let $\mathscr{C}$ be a category. The giving of a functor $K : \mathbf{2} \to \mathscr{C}$ is the same as giving two objects $K(0) = X$, $K(1) = Y$ and a morphism $K(0 \to 1) = f$ from X to Y. We shall denote K by $K(f)$.

Definition 3.1.8. Let $\mathscr{C}$ be a category. For any morphism $f : X \to Y$ in $\mathscr{C}$ and any integer $n \geq 1$, let $K^n(f)$ be the composition

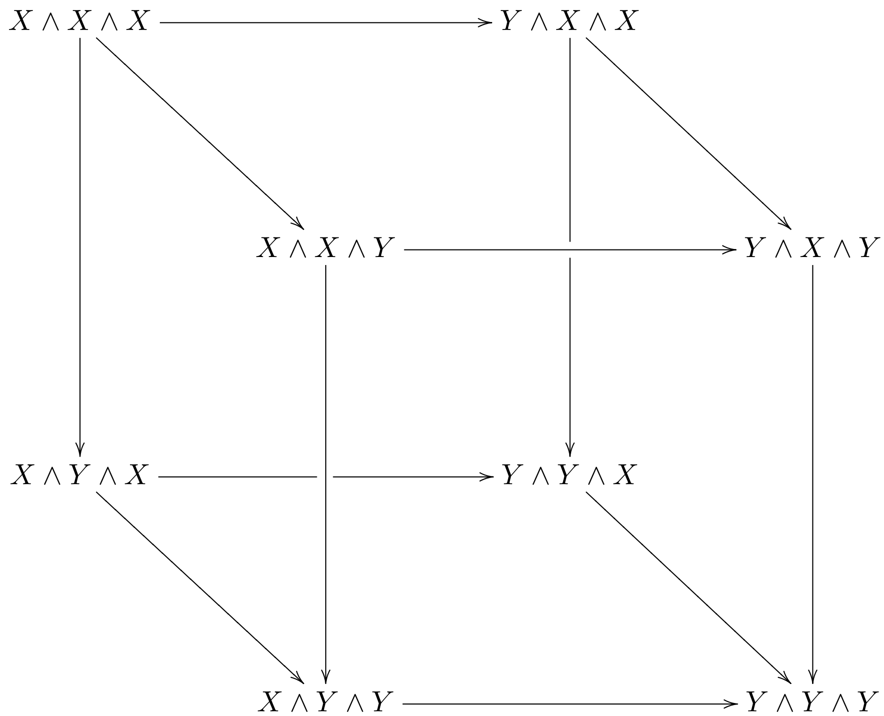
$$\mathbf{2}^n \to \mathscr{C}^n \xrightarrow{\wedge} \mathscr{C}$$

of the n-fold cartesian product of the functor $K(f) : \mathbf{2} \to \mathscr{C}$ and the functor $\wedge : \mathscr{C}^n \to \mathscr{C}$ sending an object $(X_1, \ldots, X_n)$ to the product $X_1 \wedge \cdots \wedge X_n$.

Example 3.1.9. For a morphism $f : X \to Y$ in a category $\mathscr{C}$, the functor $K^2(f)$ can be seen as a commutative square

$$\begin{array}{ccc} X \wedge X & \longrightarrow & Y \wedge X \\ \downarrow & & \downarrow \\ X \wedge Y & \longrightarrow & Y \wedge Y \end{array}$$

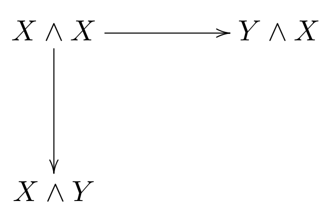
induced by f, and the functor $K^3(f)$ can be thought as a commutative cube:



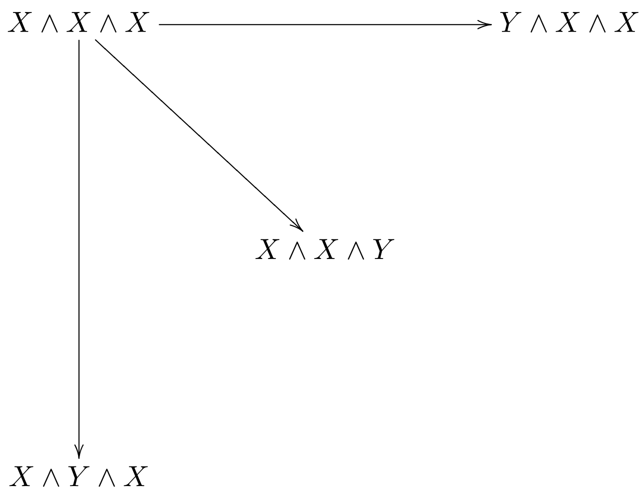
(3.11)

Definition 3.1.10. For any $0 \leq i \leq n$, we denote by $\mathbf{2}_i^n$ the full subcategory of $\mathbf{2}^n$ generated by n-tuples $(a_1, \ldots, a_n)$ such that $a_1 + \cdots + a_n \leq i$. We shall denote by $K_i^n(f)$, the restriction of $K^n(f)$ to $\mathbf{2}_i^n$.

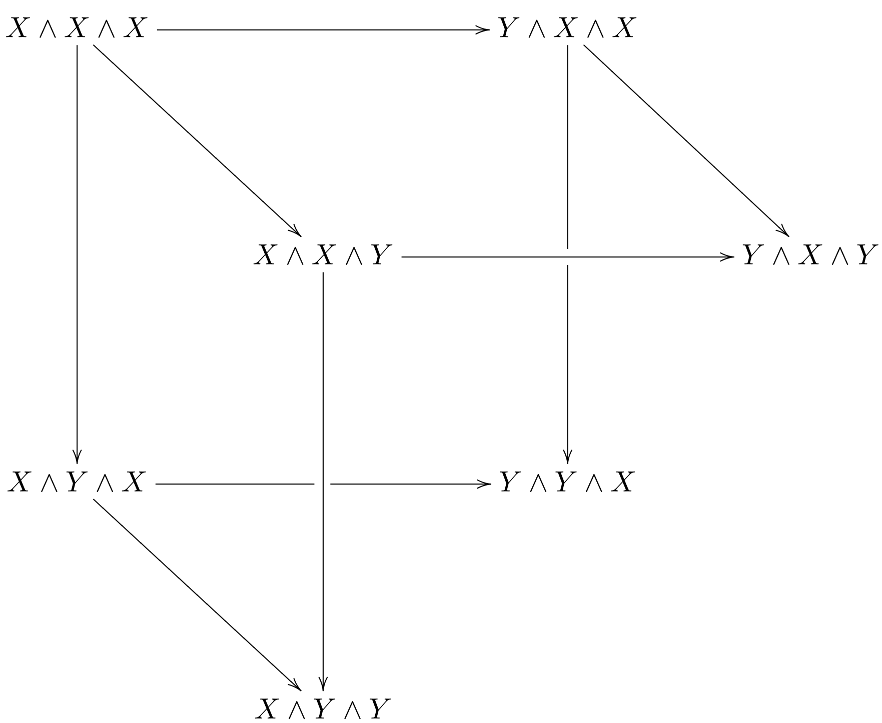
Example 3.1.11. Let $f : X \to Y$ be a morphism. If $n = 2$, then $K_0^2(f)$ consists of the object $X \wedge X$, $K_1^2(f)$ is the diagram



and $K_2^2(f) = K^2(f)$. If $n = 3$, then $K_0^3(f)$ is $X^{\wedge 3}$, $K_1^3(f)$ is the diagram:



$K_2^3(f)$ is the diagram:



and $K_3^3(f) = K^3(f)$, see diagram (3.11).

Remark 3.1.12. Let $0 \leq i \leq n$ be two indices. The category $\mathbf{2}$ can be seen as a poset with 2 elements. Then, the category $\mathbf{2}^n$ is a poset with the product order, and the category $\mathbf{2}_i^n$ is a subposet with the restricted partial order of $\mathbf{2}^n$.

Lemma 3.1.13. *For every positive integer n, the symmetric group Σ_n acts naturally on $\mathbf{2}_i^n$ for all $i = 1, \ldots, n$.*

Proof. Let us fix an index $0 \leq i \leq n$. Any permutation $\sigma \in \Sigma_n$ induces an automorphism $\sigma : \mathbf{2}^n \to \mathbf{2}^n$ taking an n-tuple $(a_1, \ldots, a_n)$ to $(a_{\sigma(1)}, \ldots, a_{\sigma(1)})$. Notice that if $a_1 + \cdots + a_n \leq i$, then one has $a_{\sigma(1)} + \cdots + a_{\sigma(n)} = a_1 + \cdots + a_n \leq i$, so the subcategory $\mathbf{2}^n_i$ is invariant under the action of Σ_n. Thus, every automorphism $\sigma : \mathbf{2}^n \to \mathbf{2}^n$ induces an automorphism $\sigma : \mathbf{2}^n_i \to \mathbf{2}^n_i$ for $1 \leq i \leq n$. □

Proposition 3.1.14. *Let $\mathscr{C}$ be a symmetric monoidal category and let $f : X \to Y$ be a morphism in $\mathscr{C}$. Suppose that the for every index $0 \leq i \leq n$, the ith fold pushout-product $f^{\square i}$ of f exists. Then, for every $0 \leq i \leq n$, the colimit of the diagram $K^n_i(f)$ exists.*

Proof. The idea is to use induction on n. Notice that the case when n is equal to 1 or 2, the colimit of each $K^n_i(f)$ exists. Now, suppose that $n > 2$ and the statement is true for positive integers strictly less than n. Let $r_{n,i} = \binom{n}{i}$ and let us choose r permutations $\sigma_1, \ldots, \sigma_{r_{n,i}}$ of Σ_n that represent the elements of the quotient $\Sigma_n/(\Sigma_{n-i} \times \Sigma_i)$. Let $0 \leq j \leq i$ be Notice that identifying an object $(a_1 \ldots, a_i)$ of $\mathbf{2}^i_j$ with an object of the form $(0, \ldots, 0, a_1, \ldots, a_i)$ in $\mathbf{2}^n_j$, we get an inclusion $\{0\}^{n-i} \times \mathbf{2}^i_j \hookrightarrow \mathbf{2}^n_j$. Let $\xi^n_{(j,i)}$ be the universal morphism of posets $\amalg^{r_{n,i}}_{k=1}(\{0\}^{n-i}\mathbf{2}^i_j) \to \mathbf{2}^n_j$ induced by the composites

$$\{0\}^{n-i} \times \mathbf{2}^i_j \hookrightarrow \mathbf{2}^n_j \xrightarrow{\sigma_k} \mathbf{2}^n_j \ ,$$

for $k = 1, \cdots, r_{n,i}$. The commutative square

$$\begin{array}{ccc} \coprod^{r_{n,i}}_{k=1}\left(\{0\}^{n-i} \times \mathbf{2}^i_{i-1}\right) & \hookrightarrow & \coprod^{r_{n,i}}_{k=1}\left(\{0\}^{n-i} \times \mathbf{2}^i_i\right) \\ \downarrow{\scriptstyle \xi^n_{i-1,i}} & & \downarrow{\scriptstyle \xi^n_{i,i}} \\ \mathbf{2}^n_{i-1} & \hookrightarrow & \mathbf{2}^n_i \end{array}$$

is a pushout in the category of posets. Therefore, the above square allows one to construct inductively a cocartesian square

$$\begin{array}{ccc} \coprod^{r_{n,i}}_{k=1}\left(X^{\wedge(n-i)} \wedge \operatorname{colim} K^i_{i-1}(f)\right) & \longrightarrow & \coprod^{r_{n,i}}_{k=1}\left(X^{\wedge(n-i)} \wedge Y^{\wedge i}\right) \\ \downarrow & & \downarrow \\ \operatorname{colim} K^n_{i-1}(f) & \longrightarrow & \operatorname{colim} K^n_i(f) \end{array} \tag{3.12}$$

as required. □

Let $f : X \to Y$ be a morphism in a symmetric monoidal category with pushouts. For each index $0 \leq i \leq n$, we set

$$\square^n_i(f) = \operatorname{colim} K^n_i(f)\,,$$

Since K_0^n is the diagram consisting of one object $X^{\wedge n}$, we have $\square_0^n = X^{\wedge n}$. On the other hand, the nth tuple $(1, 1, \ldots, 1)$ is the terminal object of $\mathbf{2}^n$, and $K_n^n(f) = K^n(f)$; hence we have $\square_n^n(f) = Y^{\wedge n}$. Then, the sequence of subdiagrams

$$K_0^n(f) \subset K_1^n(f) \subset \cdots \subset K_n^n(f)\,,$$

induce a sequence of morphisms in $\mathscr{C}$,

$$X^{\wedge n} = \square_0^n(f) \to \square_1^n(f) \to \cdots \to \square_n^n(f) = Y^{\wedge n}\,,$$

whose composite is nothing but the n-fold product $f^{\wedge n} : X^{\wedge n} \to Y^{\wedge n}$ of f. The above sequence will be called *Künneth tower* of $f^{\wedge n}$.

Corollary 3.1.15. *Let $\mathscr{C}$ be a symmetric monoidal category with pushouts. Then, for every morphism f in $\mathscr{C}$. The symmetric group Σ_n acts naturally on each object $\square_i^n(f)$ for all $i = 1, \ldots, n$.*

Proof. Let us fix an index $0 \leq i \leq n$. By Lemma 3.1.13, the symmetric group Σ_n acts on the poset $\mathbf{2}_i^n$, hence this action induces an action on $K_i^n(f)$. For any morphism $(a_1, \ldots, a_n) \to (a'_1, \ldots, a'_n)$ in $\{0, 1\}^n$, we have a commutative square

$$\begin{array}{ccc}
K_i^n(f)(a_1, \ldots, a_n) & \xrightarrow{\sigma} & K_i^n(f)(a_{\sigma(1)}, \ldots, a_{\sigma(1)}) \\
\downarrow & & \downarrow \\
K_i^n(f)(a'_1, \ldots, a'_n) & \xrightarrow[\sigma]{} & K_i^n(f)(a'_{\sigma(1)}, \ldots, a'_{\sigma(1)})
\end{array}$$

Then, by the universal property of colimit, there is a unique automorphism ϕ_σ of $\square_i^n$ such that we have a commutative diagram

$$\begin{array}{ccc}
K_i^n(f)(a_1, \ldots, a_n) & \xrightarrow{\sigma} & K_i^n(f)(a_{\sigma(1)}, \ldots, a_{\sigma(1)}) \\
\downarrow & & \downarrow \\
\square_i^n(f) & \xrightarrow[\phi_\sigma]{} & \square_i^n(f)
\end{array}$$

where the vertical morphisms are the canonical morphism. Moreover, the map $\phi : \Sigma_n \to \mathrm{Aut}(\square_i^n(f))$ given by $\sigma \mapsto \phi_\sigma$ is a homomorphism of groups. This gives an action of Σ_n on $\square_i^n(f)$. $\square$

Definition 3.1.16. Let $(\mathscr{C}, \wedge)$ be a symmetric monoidal category. For an object X of $\mathscr{C}$, we shall write $\mathrm{Sym}^n(X)$ for the quotient $X^{\wedge n}/\Sigma_n$, if it exists, and call it the nth fold *(categoric) symmetric power* of X.

Let $\mathscr{C}$ be a symmetric monoidal category with finite colimits. The previous Corollary allows to take the quotient of $\square_i^n(f)$ by the symmetric group Σ_n for $0 \leq i \leq n$. We write

$$\tilde{\square}_i^n(f) := \square_i^n(f)/\Sigma_n \,.$$

In particular, we have $\tilde{\square}_0^n(f) = X^n/\Sigma_n = \mathrm{Sym}^n X$ and $\tilde{\square}_n^n(f) = Y^n/\Sigma_n = \mathrm{Sym}^n Y$. Thus we have a following commutative diagram,

$$\begin{array}{ccccccccc}
& & & & \xrightarrow{\quad f^{\wedge n} \quad} & & & & \\
X^n = \square_0^n(f) & \longrightarrow & \square_1^n(f) & \longrightarrow & \cdots \quad \cdots & \longrightarrow & \square_{n-1}^n(f) & \longrightarrow & \square_n^n(f) = Y^n \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\mathrm{Sym}^n X = \tilde{\square}_0^n(f) & \longrightarrow & \tilde{\square}_1^n(f) & \longrightarrow & \cdots \quad \cdots & \longrightarrow & \tilde{\square}_{n-1}^n(f) & \longrightarrow & \tilde{\square}_n^n(f) = \mathrm{Sym}^n Y \\
& & & & \xrightarrow[\quad \mathrm{Sym}^n f \quad]{} & & & &
\end{array}$$

The filtration

$$\mathrm{Sym}^n(X) = \tilde{\square}_0^n(f) \to \tilde{\square}_1^n(f) \to \cdots \to \tilde{\square}_n^n(f) = \mathrm{Sym}^n(Y)$$

of $\mathrm{Sym}^n(f)$ will be called *Künneth tower* of $\mathrm{Sym}^n(f)$.

Example 3.1.17. For any morphism $f : X \to Y$ in a model category, we have $\square_1^2(f) = \square(f, f)$. If f is a cofibration, then

$$f^{\square 2} : \square(f, f) \to Y \times Y$$

is a cofibration, see [18].

Proposition 3.1.18. *Let*

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow[f']{} & Y'
\end{array}$$

be a pushout in a symmetric monoidal category with finite colimits. It induces a diagram

$$\begin{array}{ccc}
K_{n-1}^n(f) & \longrightarrow & Y^{\wedge n} \\
\downarrow & & \\
K_{n-1}^n(f') & &
\end{array}$$

whose colimit is $Y'^{\wedge n}$. Consequently, we have a cocartesian square

$$\begin{array}{ccc} \square^n_{n-1}(f) & \xrightarrow{f^{\square n}} & Y \\ \downarrow & & \downarrow \\ \square^n_{n-1}(f') & \xrightarrow[f'^{\square n}]{} & Y' \end{array}$$

Proof. See [13]. □

We recall that the cofibre of a morphism $X \to Y$ in a category with terminal object is denoted by Y/X, see Definition 1.3.1. In the rest of this section, we shall assume that all categories are pointed.

Corollary 3.1.19. *Let $\mathscr{C}$ be a symmetric monoidal category with finite colimits. Let $f : X \to Y$ be a morphism in $\mathscr{C}$ and put $Z = Y/X$. Then for any integer $n \geq 1$, we have two natural isomorphisms*

$$Y^{\wedge n}/\square^n_{n-1}(f) \simeq Z^{\wedge n} \,,$$

$$\mathrm{Sym}^n Y/\tilde{\square}^n_{n-1}(f) \simeq \mathrm{Sym}^n Z \,.$$

Proof. In Proposition 3.1.18, we take f' to be the morphism $* \to Z$. The corollary follows from the preceding proposition, after noticing that $\square^n_{n-1}(f') = *$. □

Lemma 3.1.20. *Let $1 \leq i \leq n$ be two integers. For every morphism $f : X \to Y$ in $\mathscr{C}$, we have a cocartesian square*

$$\begin{array}{ccc} \mathrm{Sym}^{n-i} X \wedge \tilde{\square}^i_{i-1}(f) & \longrightarrow & \mathrm{Sym}^{n-i} X \wedge \mathrm{Sym}^i Y \\ \downarrow & & \downarrow \\ \tilde{\square}^n_{i-1}(f) & \longrightarrow & \tilde{\square}^n_i(f) \end{array} \qquad (3.13)$$

Proof. Let us fix $n \in \mathbb{N}$. For any $1 \leq i \leq n$, the diagram $X^{\wedge(n-i)} \wedge K^i_{i-1}(f)$ is a subdiagram of $K^n_{i-1}(f)$. Then, we have a universal morphism

$$\mathrm{colim}\left(X^{(n-i)} \wedge K^i_{i-1}(f)\right) \to \mathrm{colim}\, K^n_{i-1}(f)\,.$$

Notice that $\mathrm{colim}\left(X^{(n-i)} \wedge K^i_{i-1}(f)\right) = X^{(n-i)} \wedge \mathrm{colim}\, K^i_{i-1}(f)$, and by definition $\square^i_{i-1}(f) = \mathrm{colim}\, K^i_{i-1}(f)$, $\square^n_{i-1}(f) = \mathrm{colim}\, K^n_{i-1}(f)$. Thus, we get a morphism

$$X^{(n-i)} \wedge \square^i_{i-1}(f) \to \square^n_{i-1}(f)\,,$$

together with a commutative diagram

$$\begin{array}{ccc} X^{\wedge(n-i)} \wedge \square^i_{i-1}(f) & \longrightarrow & X^{\wedge(n-i)} \wedge Y^{\wedge i} \\ \downarrow & & \downarrow \\ \square^n_{i-1}(f) & \longrightarrow & \square^n_i(f) \end{array}$$

This induces a commutative diagram of Σ_n-objects

$$\begin{array}{ccc} \mathrm{cor}^{\Sigma_n}_{\Sigma_{n-i}\times\Sigma_i}\left(X^{\wedge(n-i)} \wedge \square^i_{i-1}(f)\right) & \longrightarrow & \mathrm{cor}^{\Sigma_n}_{\Sigma_{n-i}\times\Sigma_i}\left(X^{\wedge(n-i)} \wedge Y^{\wedge i}\right) \\ \downarrow & & \downarrow \\ \square^n_{i-1}(f) & \longrightarrow & \square^n_i(f) \end{array}$$

In view of diagram (3.12), this square is cocartesian. Finally, taking colimit $\mathrm{colim}_{\Sigma_n}$, we get the cocartesian square (3.13). $\square$

Proposition 3.1.21. *Let $\mathscr{C}$ be a symmetric monoidal category and let $f : X \to Y$ be a morphism in $\mathscr{C}$ with cofibre $Z = Y/X$. Fix a positive integer n and assume that the colimit of $K^n_i(f)$ exists for all $0 \leq i \leq n$. We have the following assertions:*

(a) *If*

$$X^{\wedge n} = \square^n_0(f) \to \square^n_1(f) \to \cdots \to \square^n_n(f) = Y^{\wedge n}$$

is the Künneth tower of $f^{\wedge n}$, then for each index $1 \leq i \leq n$, we have a Σ_n-equivariant isomorphism

$$\square^n_i(f)/\square^n_{i-1}(f) \simeq \mathrm{cor}^{\Sigma_n}_{\Sigma_{n-i}\times\Sigma_i}(X^{\wedge(n-i)} \wedge Z^{\wedge i}) .$$

(b) *If*

$$\mathrm{Sym}^n(X) = \tilde{\square}^n_0(f) \to \tilde{\square}^n_1(f) \to \cdots \to \tilde{\square}^n_n(f) = \mathrm{Sym}^n(Y)$$

is the Künneth tower of $\mathrm{Sym}^n(f)$, then for each index $1 \leq i \leq n$, we have an isomorphism

$$\tilde{\square}^n_i(f)/\tilde{\square}^n_{i-1}(f) \simeq \mathrm{Sym}^{n-i}X \wedge \mathrm{Sym}^i Z .$$

Proof. By Corollary 3.1.19, we have

$$Y^{\wedge i}/\square^i_{i-1}(f) \simeq Z^{\wedge i} .$$

Hence, we obtain a cocartesian square

$$\begin{array}{ccc} X^{\wedge(n-i)} \wedge \square^i_{i-1}(f) & \longrightarrow & X^{\wedge(n-i)} \wedge Y^{\wedge i} \\ \downarrow & & \downarrow \\ * & \longrightarrow & X^{\wedge(n-i)} \wedge Z^{\wedge i} \end{array}$$

which induces a cocartesian square

$$\begin{CD}
\operatorname{cor}_{\Sigma_{n-i}\times\Sigma_i}^{\Sigma_n}\left(X^{\wedge(n-i)}\wedge\square_{i-1}^{i}(f)\right) @>>> \operatorname{cor}_{\Sigma_{n-i}\times\Sigma_i}^{\Sigma_n}\left(X^{\wedge(n-i)}\wedge Y^{\wedge i}\right)\\
@VVV @VVV\\
* @>>> \operatorname{cor}_{\Sigma_{n-i}\times\Sigma_i}^{\Sigma_n}\left(X^{\wedge(n-i)}\wedge Z^{\wedge i}\right)
\end{CD}$$

Then, we wet a commutative diagram

$$\begin{CD}
\operatorname{cor}_{\Sigma_{n-i}\times\Sigma_i}^{\Sigma_n}\left(X^{\wedge(n-i)}\wedge\square_{i-1}^{i}(f)\right) @>>> \operatorname{cor}_{\Sigma_{n-i}\times\Sigma_i}^{\Sigma_n}\left(X^{\wedge(n-i)}\wedge Y^{\wedge i}\right)\\
@VVV @VVV\\
\square_{i-1}^{n}(f) @>>> \square_{i}^{n}(f)\\
@VVV @VVV\\
* @>>> \operatorname{cor}_{\Sigma_{n-i}\times\Sigma_i}^{\Sigma_n}\left(X^{\wedge(n-i)}\wedge Z^{\wedge i}\right)
\end{CD}$$

This allows to deduce an isomorphism

$$\square_{i}^{n}(f)/\square_{i-1}^{n}(f)\simeq\operatorname{cor}_{\Sigma_{n-i}\times\Sigma_i}^{\Sigma_n}\left(X^{\wedge(n-i)}\wedge Z^{\wedge i}\right).$$

This proves item (a). On the other hand, by Lemma 3.1.20, we have a cocartesian square

$$\begin{CD}
\operatorname{Sym}^{n-i}X\wedge\tilde{\square}_{i-1}^{i}(f) @>>> \operatorname{Sym}^{n-i}X\wedge\operatorname{Sym}^{i}Y\\
@VVV @VVV\\
\tilde{\square}_{i-1}^{n}(f) @>>> \tilde{\square}_{i}^{n}(f)
\end{CD}$$

Then, one has

$$\begin{aligned}
\tilde{\square}_{i}^{n}(f)/\tilde{\square}_{i-1}^{n}(f) &\simeq\left(\operatorname{Sym}^{n-i}X\wedge\operatorname{Sym}^{i}Y\right)\Big/\left(\operatorname{Sym}^{n-i}X\wedge\tilde{\square}_{i-1}^{i}(f)\right)\\
&\simeq\operatorname{Sym}^{n-i}X\wedge\left(\operatorname{Sym}^{i}Y/\tilde{\square}_{i-1}^{i}(f)\right).
\end{aligned}$$

Thus, we get an isomorphism

$$\tilde{\square}_{i}^{n}(f)/\tilde{\square}_{i-1}^{n}(f)\simeq\operatorname{Sym}^{n-i}X\wedge\operatorname{Sym}^{i}Z\,.$$

This proves item (b). $\square$

3.1.3 Symmetrizable cofibrations

Let $\mathscr{C}$ be a (pointed) closed symmetric monoidal model category.

Definition 3.1.22. A morphism $f : X \to Y$ in $\mathscr{C}$ is called *symmetrizable (trivial) cofibration* if the corresponding morphism

$$f^{\tilde{\square} n} : \tilde{\square}^n_{n-1}(f) \to \mathrm{Sym}^n Y$$

is a (trivial) cofibration for all integers $n \geq 1$.

Notice that the morphism $f^{\tilde{\square} 1} : \tilde{\square}^1_0(f) \to \mathrm{Sym}^1 Y$ is $f : X \to Y$ itself. Hence, every symmetrizable (trivial) cofibration is a (trivial) cofibration.

Definition 3.1.23. A morphism $f : X \to Y$ in $\mathscr{C}$ is called *strongly symmetrizable (trivial) cofibration* if the corresponding morphism

$$f^{\square n} : \square^n_{n-1}(f) \to Y^n$$

is a (trivial) cofibration for all integers $n \geq 1$.

Theorem 3.1.24. *Let $\mathscr{C}$ be a category as before. The class of (strongly) symmetrizable (trivial) cofibrations in $\mathscr{C}$ is closed under pushouts, retracts and transfinite compositions.*

Proof. See [13]. □

Corollary 3.1.25. *Suppose that $\mathscr{C}$ is also a cofibrantly generated model category with a set of generating cofibrations I, and suppose that every morphism in I is symmetrizable. Then, for any integer $n \geq 1$ and any cofibrant object X in $\mathscr{C}$, the symmetric power* $\mathrm{Sym}^n(X)$ *is also cofibrant.*

Proof. See [13]. □

Theorem 3.1.26 (Gorchinskiy-Guletskiĭ)**.** *Suppose that $\mathscr{C}$ is a closed symmetric monoidal model category. Let*

$$X \xrightarrow{f} Y \to Z$$

be a cofibre sequence in $\mathscr{C}$ with X and Y being cofibrant, and let

$$\mathrm{Sym}^n(X) = \tilde{\square}^n_0(f) \to \tilde{\square}^n_1(f) \to \cdots \to \tilde{\square}^n_n(f) = \mathrm{Sym}^n(Y)$$

be the Künneth tower of $\mathrm{Sym}^n(f)$*. We have the following assertions:*

(*a*) *If f is a symmetrizable cofibration, then for every index $i \leq n$ the canonical morphism $\tilde{\square}^n_{i-1}(f) \to \tilde{\square}^n_i(f)$ is a cofibration.*

(*b*) *If f is a symmetrizable trivial cofibration, then for every index $i \leq n$ the canonical morphism $\tilde{\square}^n_{i-1}(f) \to \tilde{\square}^n_i(f)$ is a trivial cofibration.*

Proof. We refer the reader to [13]. □

Corollary 3.1.27. *Let f be a trivial cofibration between cofibrant objects which is also symmetrizable as a cofibration in a category $\mathscr{C}$, as before. Then f is a symmetrizable trivial cofibration if and only if* $\mathrm{Sym}^n(f)$ *is a trivial cofibration for all* $n \in \mathbb{N}$.

Proof. See [13]. □

Theorem 3.1.28 (Gorchinskiy-Guletskiĭ)**.** *Let $\mathscr{C}$ be a closed symmetric monoidal model category and suppose that it is also cofibrantly generated. Assume that the set of generating cofibrations and the set of generating trivial cofibrations are both symmetrizable. Then the symmetric powers* $\mathrm{Sym}^n : \mathscr{C} \to \mathscr{C}$ *take weak equivalences between cofibrant objects to weak equivalences. Consequently, there exist the left derived symmetric powers* $L\mathrm{Sym}^n$ *defined on* $\mathrm{Ho}(\mathscr{C})$ *for* $n \in \mathbb{N}$.

Proof. Let us fix a natural number n. By the Ken Brown's lemma (See Lemma 1.1.15), it is enough to show that the functor $\mathrm{Sym}^n : \mathscr{C} \to \mathscr{C}$ takes trivial cofibrantion between cofibrant objects to weak equivalences. Suppose that $f : X \to Y$ is a trivial cofibration between cofibrant objects in $\mathscr{C}$. By virtue of Theorem 3.1.24, one deduces that all cofibrations and all trivial cofibrations are symmetrizable in $\mathscr{C}$. In particular f is a symmetrizable trivial cofibration. Hence by Corollary 3.1.27, $\mathrm{Sym}^n(f)$ is a trivial cofibration, in particular $\mathrm{Sym}^n(f)$ is a weak equivalence as wanted. See [13] for more details. □

3.2 Geometric symmetric powers in the unstable set-up

In the sequel, k will denote a field of arbitrary characteristic, $\mathscr{C} \subset \mathscr{S}ch/k$ will be an admissible category and $\mathscr{S}$ will be the category of Nisnevich sheaves on $\mathscr{C}$, as in Section2.3.

We recall that the nth fold geometric symmetric power $\mathrm{Sym}^n_g(\mathscr{X})$ of a sheaf $\mathscr{X}$ in $\mathscr{S}$ is the colimit of the functor $F_{\mathscr{X}} : (h \downarrow \mathscr{X}) \to \mathscr{S}$ which sends a morphism $h_U \to \mathscr{X}$ to the representable sheaf $h_{\mathrm{Sym}^n U}$, see Section 2.3.3. Sometimes, we shall write $\mathrm{colim}_{h_X \to \mathscr{X}} h_{\mathrm{Sym}^n X}$ to mean the colimit of the functor $F_{\mathscr{X}}$. On the other hand, if $\mathscr{X}$ is a pointed sheaf, then the nth fold geometric symmetric power of $\mathscr{X}$ is a colimit of the form $\mathrm{colim}_{h_{X_+} \to \mathscr{X}} h_{\mathrm{Sym}^n X_+}$, where the colimit is computed in $\mathscr{S}_*$.

3.2.1 Künneth rules

Here, we study the Künneth rules for geometric symmetric powers (see Corollary 3.2.17).

Let X be an object of $\Delta^{\mathrm{op}}\mathscr{C}$. The nth fold symmetric power $\mathrm{Sym}^n(X)$ is the simplicial object on $\mathscr{C}$ whose terms are $\mathrm{Sym}^n(X)_i{:=}\mathrm{Sym}^n(X_i)$ for all $i \in \mathbb{N}$. Thus, Sym^n induces a endofunctor of $\Delta^{\mathrm{op}}\mathscr{C}$.

Lemma 3.2.1. *For each $n \in \mathbb{N}$, Sym^n_g is isomorphic to the left Kan extension of the composite*

$$\Delta^{\mathrm{op}}\mathscr{C} \xrightarrow{\mathrm{Sym}^n} \Delta^{\mathrm{op}}\mathscr{C} \xrightarrow{\Delta^{\mathrm{op}}h} \Delta^{\mathrm{op}}\mathscr{S}$$

along $\Delta^{\mathrm{op}}h$.

Proof. Notice that $\Delta^{\mathrm{op}}\mathscr{C}$ is a small category. Let $\mathscr{X}$ be a simplicial sheaf and fix a natural number i. Let us consider the functor $F_{\mathscr{X}_i}$ such that $\mathrm{Sym}^n_g(\mathscr{X}_i) = \operatorname{colim} F_{\mathscr{X}_i}$, as defined in page 2.3.3. Let us consider the functor

$$J_{\mathscr{X},i} : (\Delta^{\mathrm{op}}\mathscr{C} \downarrow \mathscr{X}) \to (\mathscr{C} \downarrow \mathscr{X}_i)\,,$$

given by $(\Delta^{\mathrm{op}}h_U \to \mathscr{X}) \mapsto (h_{U_i} \to \mathscr{X}_i)$. Let $\varphi : h_V \to \mathscr{X}_i$ be a morphism of sheaves. The morphism φ induces a morphism $\tilde{\varphi} : \Delta_V[i] \to \mathscr{X}$. Notice that $\Delta_V[i]$ coincides with $\Delta^{\mathrm{op}}h_{V\otimes\Delta[i]}$, and $J_{\mathscr{X},i}$ sends the morphism $\tilde{\varphi}$ to a morphism $h_{(V\otimes\Delta[i])_i} \to \mathscr{X}_i$ such that we have a commutative diagram

$$\begin{array}{ccc} h_V & \xrightarrow{\quad u_i \quad} & h_{(V\otimes\Delta[i])_i} \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle J_{\mathscr{X},i}(\tilde{\varphi})} & \\ & \mathscr{X}_i & \end{array}$$

where u_i is the morphism induced by the canonical morphism

$$V \longrightarrow \coprod_{\Delta[i]_i} V = (V \otimes \Delta[i])_i\,.$$

Then, the composites of the form

$$F_{\mathscr{X}_i}(\varphi) \xrightarrow{F_{\mathscr{X}_i}(u_i)} (F_{\mathscr{X}_i} \circ J_{\mathscr{X},i})(\tilde{\varphi}) \longrightarrow \operatorname{colim}\,(F_{\mathscr{X}_i} \circ J_{\mathscr{X},i})$$

define a cocone with base $F_{\mathscr{X}_i}$ and vertex $\operatorname{colim}\,(F_{\mathscr{X}_i} \circ J_{\mathscr{X},i})$. By a simple computation, one sees that this cocone is universal, so that we have a canonical isomorphism

$$\operatorname{colim}\,(F_{\mathscr{X}_i} \circ J_{\mathscr{X},i}) \simeq \operatorname{colim} F_{\mathscr{X}_i}\,.$$

We observe that the colimit of $F_{\mathscr{X}_i} \circ J_{\mathscr{X},i}$ is nothing but the ith term of the simplicial sheaf $\mathrm{Lan}_{\Delta^{\mathrm{op}}h}(\Delta^{\mathrm{op}}h \circ \mathrm{Sym}^n)(\mathscr{X})$. Thus, we get a canonical isomorphism

$$\mathrm{Lan}_{\Delta^{\mathrm{op}}h}(\Delta^{\mathrm{op}}h \circ \mathrm{Sym}^n)(\mathscr{X}) \simeq \mathrm{Sym}^n_g(\mathscr{X})\,,$$

for every object $\mathscr{X}$ in $\Delta^{\mathrm{op}}\mathscr{S}$. □

We denote by h^+ the canonical functor from $\mathscr{C}_+$ to $\mathscr{S}_*$.

Corollary 3.2.2. *For each $n \in \mathbb{N}$, the nth fold geometric symmetric power* Sym^n_g *on* $\Delta^{\mathrm{op}}\mathscr{S}_*$ *is isomorphic to the left Kan extension of the composite*

$$\Delta^{\mathrm{op}}\mathscr{C}_+ \xrightarrow{\mathrm{Sym}^n} \Delta^{\mathrm{op}}\mathscr{C}_+ \xrightarrow{\Delta^{\mathrm{op}}h^+} \Delta^{\mathrm{op}}\mathscr{S}_*$$

along $\Delta^{\mathrm{op}}h^+$.

Proof. It follows from the previous lemma in view that the canonical functor from $\Delta^{\mathrm{op}}\mathscr{S}$ to $\Delta^{\mathrm{op}}\mathscr{S}_*$ is left adjoint. □

We provide Lemmas 3.2.3, 3.2.4 and Proposition 3.2.5 in order to prove the Künneth rule for symmetric for schemes (Corollary 3.2.7), that is, for a natural number n and for two schemes X and Y on an admissible category of schemes, the nth fold symmetric power $\mathrm{Sym}^n(X \amalg Y)$ is isomorphic to the coproduct $\coprod_{i+j=n}(\mathrm{Sym}^i X \times \mathrm{Sym}^j Y)$. We recall that for a category $\mathscr{C}$ and a finite group G, the category $\mathscr{C}^G$ is the category of functors $G \to \mathscr{C}$, where G is viewed as a category. A functor $G \to \mathscr{C}$ is identified with a G-object of $\mathscr{C}$. If H is a subgroup of G, then we the *restriction* functor $\mathrm{res}^G_H : \mathscr{C}^G \to \mathscr{C}^H$ sends a functor $G \to \mathscr{C}$ to the composite $H \hookrightarrow G \to \mathscr{C}$. If $\mathscr{C}$ has finite coproducts and quotients under finite groups, then res^G_H has left adjoint. The left adjoint of res^G_H is called *corestriction* functor, we denoted it by cor^G_H.

Lemma 3.2.3. *Let $\mathscr{C}$ be a category with finite coproducts and quotients under finite groups. Let G be a finite group and let H be a subgroup of G. If X is an H-object of $\mathscr{C}$, then*

$$\mathrm{cor}^G_H(X)/G \simeq X/H\,.$$

Proof. Suppose X is an H-object of $\mathscr{C}$. We recall that $\mathrm{cor}^G_0(X)$ coincides with the coproduct of $|G|$-copies of X, it is usually denoted by $G \times X$ in the literature. Observe that the group $G \times H$ acts canonically on $G \times X$. By definition, $\mathrm{cor}^G_H(X)$ is equal to $\mathrm{colim}_H(G \times X)$. One can also notice that $\mathrm{colim}_G(G \times X) = X$. Then, we have,

$$\begin{aligned}
\mathrm{cor}^G_H(X)/G &= \mathrm{colim}_G\,\mathrm{cor}^G_H(X)\\
&= \mathrm{colim}_G\,\mathrm{colim}_H(G \times X)\\
&= \mathrm{colim}_H\,\mathrm{colim}_G(G \times X) && \text{(change of colimits)}\\
&= \mathrm{colim}_H X\,.
\end{aligned}$$

By definition, X/H is equal to $\mathrm{colim}_H X$, thus we obtain that $\mathrm{cor}^G_H(X)/G$ is isomorphic to X/H. □

Lemma 3.2.4. *Let $\mathscr{C}$ be a symmetric monoidal category with finite coproducts and quotients under finite groups. Let n, i, j be three natural numbers such that $i, j \leq n$ and $i + j = n$, and let X_0, X_1 be two objects of $\mathscr{C}$. Then, the symmetric group Σ_n acts on the coproduct $\bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n}$ by permuting the indices of the factors, and one has an isomorphism*

$$\left(\bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n}\right) /\Sigma_n \simeq \mathrm{Sym}^i X_0 \wedge \mathrm{Sym}^j X_1 \,.$$

Proof. After reordering of factors in a suitable way, we can notice that the coproduct $\bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n}$ is isomorphic to the coproduct of $\binom{n}{i}$-copies of the term $X_0^{\wedge i} \wedge X_1^{\wedge j}$, in other words, it is isomorphic to $\mathrm{cor}_{\Sigma_i \times \Sigma_j}^{\Sigma_n}(X_0^{\wedge i} \wedge X_1^{\wedge j})$ which is a Σ_n-object. By Lemma 3.2.3, we have an isomorphism

$$\left(\mathrm{cor}_{\Sigma_i \times \Sigma_j}^{\Sigma_n}(X_0^{\wedge i} \wedge X_1^{\wedge j})\right) / \Sigma_n \simeq (X_0^{\wedge i} \wedge X_1^{\wedge j}) \Big/ (\Sigma_i \times \Sigma_j) \,,$$

and the right-hand side is isomorphic to $\mathrm{Sym}^i X_0 \wedge \mathrm{Sym}^j X_1$, which implies the expected isomorphism. □

Proposition 3.2.5. *Suppose $\mathscr{C}$ is a category as in Lemma 3.2.4. Let X_0, X_1 be two objects of $\mathscr{C}$. For any integer $n \geq 1$, there is an isomorphism*

$$\mathrm{Sym}^n(X_0 \vee X_1) \simeq \bigvee_{i+j=n} (\mathrm{Sym}^i X_0 \wedge \mathrm{Sym}^j X_1) \,. \tag{3.14}$$

Proof. Let us fix an integer $n \geq 1$. We have the following isomorphism,

$$(X_0 \vee X_1)^{\wedge n} \simeq \bigvee_{0 \leq j \leq n} \left(\bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n} \right) ,$$

and for each index $0 \leq j \leq n$, the symmetric group Σ_n acts by permuting factors on the coproduct

$$\coprod_{j=0}^{n} \left(\bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n} \right) .$$

Hence, we deduce that $(X_0 \vee X_1)^{\wedge n} /\Sigma_n$ is isomorphic to the coproduct

$$\coprod_{j=0}^{n} \left(\left(\bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n}\right) /\Sigma_n \right) .$$

Finally, by Lemma 3.2.4, we obtain that $\mathrm{Sym}^n(X_0 \vee X_1)$ is isomorphic to the coproduct $\bigvee_{0 \leq j \leq n}(\mathrm{Sym}^{n-j} X_0 \wedge \mathrm{Sym}^j X_1)$, thus we have the isomorphism (3.14). □

Definition 3.2.6. Suppose that $\mathscr{C}$ is an admissible category. Let $X = A_+$ and $Y = B_+$ be two objects of $\mathscr{C}_+$. We denote by $X \vee Y$ the object $(X \amalg Y)_+$ and by $X \wedge Y$ the object $(X \times Y)_+$. Notice that the category $\mathscr{C}_+$ with the product $\wedge$ is a symmetric monoidal category.

Corollary 3.2.7. Let $\mathscr{C} \subset \mathscr{S}ch/k$ be an admissible category. Then, for every integer $n \geq 1$ and for any two objects X, Y of $\mathscr{C}_+$, we have an isomorphism

$$\mathrm{Sym}^n(X \vee Y) \simeq \bigvee_{i+j=n} (\mathrm{Sym}^i X \wedge \mathrm{Sym}^j Y)\,.$$

Proof. It follows from the previous proposition in view that $\mathscr{C}_+$ is symmetric monoidal and has quotients by finite groups. □

Remark 3.2.8. Let f be a morphism of the form $X \to X \vee Y$ in $\mathscr{C}_+$. Then, for every integer $n \geq 1$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^n X & \xrightarrow{\mathrm{Sym}^n(f)} & \mathrm{Sym}^n(X \vee Y) \\ \| & & \downarrow \\ \mathrm{Sym}^n X & \longrightarrow & \bigvee_{i+j=n}(\mathrm{Sym}^i X \wedge \mathrm{Sym}^j Y) \end{array}$$

where the right vertical arrow is the isomorphism given in Corollary 3.2.7 and the bottom arrow is the canonical morphism.

In the following lemma we consider the notations used in Section 3.1.

Lemma 3.2.9. *Let $\mathscr{C}$ be an admissible category and let $\varphi : X \to X \vee Y$ be a coprojection in $\mathscr{C}_+$. Then, for every positive integer n, the colimit of the diagram $K_i^n(\varphi)$ exists, and one has a filtration*

$$X^{\wedge n} = \square_0^n(\varphi) \to \square_1^n(\varphi) \to \cdots \to \square_n^n(\varphi) = (X \vee Y)^{\wedge n}\,,$$

where $\square_i^n(\varphi)$ is isomorphic to $\bigvee_{n-i \leq j \leq n} X^j \wedge Y^{n-j}$ for all indices $0 \leq i \leq n$. Moreover, this filtration induces a filtration

$$\mathrm{Sym}^n(X) = \tilde{\square}_0^n(\varphi) \to \tilde{\square}_1^n(\varphi) \to \cdots \to \tilde{\square}_n^n(\varphi) = \mathrm{Sym}^n(X \vee Y)\,,$$

where each $\tilde{\square}_i^n(\varphi)$ is isomorphic to $\bigvee_{n-i \leq j \leq n}(\mathrm{Sym}^j X \wedge \mathrm{Sym}^{n-j} Y)$.

Proof. Since φ is a coprojection, Lemma 2.2.39 implies that the ith fold pushout-product of φ exists for all indices i. Hence, by virtue of Proposition 3.1.14 the diagrams $K_i^n(\varphi)$ exist. Then, by Proposition 3.1.21, the morphisms $\varphi^{\wedge n}$ and $\mathrm{Sym}^n(\varphi)$ have the above filtration. Finally, by Corollary 3.2.7, we deduce that the each morphism from $\tilde{\square}_{i-1}^n(\varphi)$ to $\tilde{\square}_i^n(\varphi)$ is isomorphic to the canonical morphism

$$\coprod_{n-(i-1) \leq j \leq n} (\mathrm{Sym}^j X \times \mathrm{Sym}^{n-j} Y) \to \coprod_{n-i \leq j \leq n} (\mathrm{Sym}^j X \times \mathrm{Sym}^{n-j} Y)\,,$$

as required. □

Example 3.2.10. If we take X to be the point $*$ in the previous lemma, then the morphism $\varphi : * \to Y = * \vee Y$ induces a filtration

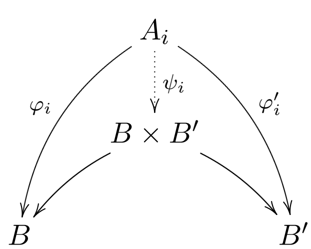
$$* = \tilde{\square}^n_0(\varphi) \to \tilde{\square}^n_1(\varphi) \to \cdots \to \tilde{\square}^n_n(\varphi) = \mathrm{Sym}^n(Y)\,,$$

where each $\tilde{\square}^n_i(\varphi)$ is isomorphic to $\mathrm{Sym}^i(Y)$. In consequence, the morphism $* \to \Delta^{\mathrm{op}} h^+_Y$ induces a Künneth filtration of pointed simplicial sheaves

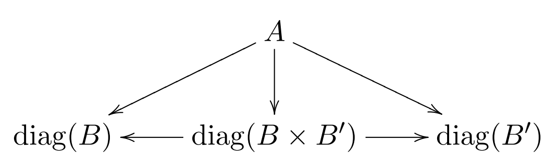
$$* = \mathrm{Sym}^0_g(\Delta^{\mathrm{op}} h^+_Y) \to \mathrm{Sym}^1_g(\Delta^{\mathrm{op}} h^+_Y) \to \cdots \to \mathrm{Sym}^n_g(\Delta^{\mathrm{op}} h^+_Y)\,.$$

Lemma 3.2.11. *Let $\mathscr{J}$ be a category with finite coproducts and Cartesian products. Then, for every integer $n \geq 1$, the diagonal functor* diag: $\mathscr{J} \to \mathscr{J}^{\times n}$ *is final (see [26, page 213]).*

Proof. Let $A = (A_1, \ldots, A_n)$ be an object of $\mathscr{J}^{\times n}$. We shall prove that the comma category $A \downarrow \mathrm{diag}$, whose objects has the form $A \to \mathrm{diag}(B)$ for B in $\mathscr{J}$, is nonempty and connected. We set $B{:=}A_1 \amalg \cdots \amalg A_n$. For every index $0 \leq i \leq n$, we have a canonical morphism $A_i \to B$, then we get a morphism from A to $\mathrm{diag}(B)$. Thus, the comma category $A \downarrow \mathrm{diag}$ is nonempty. Let B and B' be two objects of $\mathscr{J}$ and let $A = (A_1, \ldots, A_n)$ be an object of $\mathscr{J}^{\times n}$. Suppose that we have two morphisms: $(\varphi_1, \ldots, \varphi_n)$ from A to $\mathrm{diag}(B)$ and $(\varphi'_1, \ldots, \varphi'_n)$ from A to $\mathrm{diag}(B')$. For every index $0 \leq i \leq n$, we have a commutative diagram



where the dotted arrow exists by the universal property of product. Notice that we get a morphism $(\psi_1, \ldots, \psi_n)$ from A to $\mathrm{diag}(B \times B')$ and a commutative diagram



Thus, the comma category $A \downarrow \mathrm{diag}$ is connected. □

Lemma 3.2.12. *Let $\mathscr{X}$ be a sheaf in $\mathscr{S}$. For every integer $n \geq 1$, we have an isomorphism*

$$\mathscr{X}^{\times n} \simeq \mathrm{colim}_{\,h_X \to \mathscr{X}}\, h_{X^n}\,, \tag{3.15}$$

Proof. Let $(h \downarrow \mathscr{X})$ be a comma category and let us consider the functor $F_{\mathscr{X},n}$ from $(h \downarrow \mathscr{X})^{\times n}$ to $\Delta^{\mathrm{op}}\mathscr{S}$ defined by

$$\Big(h_{X_1} \to \mathscr{X}, \ldots, h_{X_n} \to \mathscr{X}\Big) \mapsto h_{X_1} \times \cdots \times h_{X_n} \,.$$

For every integer $n \geq 1$, let us consider the diagonal functor diag from $(h \downarrow \mathscr{X})$ to $(h \downarrow \mathscr{X})^{\times n}$. We recall that we have an isomorphism

$$\mathscr{X} \simeq \operatorname{colim}_{h_X \to \mathscr{X}} h_X \,, \tag{3.16}$$

where the colimit is taken from the comma category with objects $h_X \to \mathscr{X}$, for $X \in \mathscr{C}$ to the category of sheaves. Hence, we deduce an isomorphism

$$\mathscr{X}^{\times n} \simeq \operatorname{colim} F_{\mathscr{X},n} \,.$$

Next, we shall prove that the canonical morphism $\operatorname{colim}(F_{\mathscr{X},n} \circ \mathrm{diag}) \to \operatorname{colim} F_{\mathscr{X},n}$ is an isomorphism. Let us write $\mathscr{Y} := \operatorname{colim}(F_{\mathscr{X},n} \circ \mathrm{diag})$ and let $\mu \colon F_{\mathscr{X},n} \circ \mathrm{diag} \to \Delta_{\mathscr{Y}}$ be the universal cocone, where $\Delta_{\mathscr{Y}}$ denotes the constant functor with value $\mathscr{Y}$. We would like to find a universal cocone $\tau \colon F_{\mathscr{X},n} \to \Delta_{\mathscr{Y}}$. The canonical morphisms $h_{X_i} \to h_{X_1} \amalg \cdots \amalg h_{X_n}$, for $1 \leq i \leq n$, induce a morphism

$$\Big(h_{X_1} \to \mathscr{X}, \ldots, h_{X_n} \to \mathscr{X}\Big) \longrightarrow \mathrm{diag}\Big(h_{X_1} \amalg \cdots \amalg h_{X_n} \to \mathscr{X}\Big) \,.$$

Hence, the composite

$$F_{\mathscr{X},n}\Big(h_{X_1} \to \mathscr{X}, \ldots, h_{X_n} \to \mathscr{X}\Big) \longrightarrow (F_{\mathscr{X},n} \circ \mathrm{diag})\Big(h_{X_1} \amalg \cdots \amalg h_{X_n} \to \mathscr{X}\Big) \longrightarrow \mathscr{Y} \,, \tag{3.17}$$

where the object in the middle is equal to the nth fold product $(h_{X_1} \amalg \cdots \amalg h_{X_n})^{\times n}$, and the arrow on the right-hand side is the morphism induced by the universal cocone μ. Now, any morphism from $(h_{X_1} \to \mathscr{X}, \ldots, h_{X_n} \to \mathscr{X}) \to (h_{X'_1} \to \mathscr{X}, \ldots, h_{X'_n} \to \mathscr{X})$ is induced by a collection of morphisms $X_i \to X'_i$ for $i = 1, \ldots, n$; and they provide the following diagram

$$\begin{array}{ccccc}
h_{X_1} \times \cdots \times h_{X_n} & \longrightarrow & (h_{X_1} \amalg \cdots \amalg h_{X_n})^{\times n} & \longrightarrow & \mathscr{Y} \\
\downarrow & & \downarrow & & \| \\
h_{X'_1} \times \cdots \times h_{X'_n} & \longrightarrow & (h_{X'_1} \amalg \cdots \amalg h_{X'_n})^{\times n} & \longrightarrow & \mathscr{Y}
\end{array}$$

making the composite (3.17) functorial. Thus, we obtain a cocone $\tau \colon F_{\mathscr{X},n} \to \Delta_{\mathscr{Y}}$. It remains to prove that this cocone is universal. Indeed, let $\lambda \colon F_{\mathscr{X},n} \to \Delta_{\mathscr{Z}}$ be another cocone. Then, the composite $\lambda \circ \mathrm{diag} \colon F_{\mathscr{X},n} \circ \mathrm{diag} \to \Delta_{\mathscr{Z}}$ is also a cocone. By the universal property of $\mathscr{Y}$, there exists a morphism $f \colon \mathscr{Y} \to \mathscr{Z}$ such that $\Delta_f \circ \mu = \lambda \circ \mathrm{diag}$. Hence, we get $\Delta_f \circ \tau = \lambda$. This proves that τ is a universal cocone. Notice that the composite functor $F_{\mathscr{X},n} \circ \mathrm{diag}$ is given by $(h_X \to \mathscr{X}) \mapsto h_{X^n}$. Finally, composing the following isomorphisms

$$\mathscr{X}^{\times n} \simeq \operatorname{colim} F_{\mathscr{X},n} \simeq \operatorname{colim} (F_{\mathscr{X},n} \circ \mathrm{diag}) = \operatorname{colim}_{h_X \to \mathscr{X}} h_{X^n}\,, \tag{3.18}$$

we get the required isomorphism. □

Lemma 3.2.13. *Let F, G be two objects in $\mathscr{S}$. For any integer $n \geq 1$, there is an isomorphism*

$$\mathrm{Sym}^n_g(F \amalg G) \simeq \coprod_{i+j=n} (\mathrm{Sym}^i_g F \times \mathrm{Sym}^j_g G)\,.$$

Proof. Let us fix an integer $n \geq 1$. By Corollary 3.2.7, for any two objects X and Y of $\mathscr{C}$, we have an isomorphism

$$\mathrm{Sym}^n(X \amalg Y) \simeq \coprod_{i+j=n} (\mathrm{Sym}^i X \times \mathrm{Sym}^j Y)\,.$$

Since the Yoneda embedding $h\colon \mathscr{C} \to \mathscr{S}$ preserves finite product and coproduct, we get an isomorphism

$$h_{\mathrm{Sym}^n(X \amalg Y)} \simeq \coprod_{i+j=n} \left(h_{\mathrm{Sym}^i X} \times h_{\mathrm{Sym}^j Y} \right)\,. \tag{3.19}$$

By definition, we have $\mathrm{Sym}^n_g(h_X) = h_{\mathrm{Sym}^n(X)}$, $\mathrm{Sym}^n_g(h_Y) = h_{\mathrm{Sym}^n(Y)}$ and $\mathrm{Sym}^n_g(h_{X \amalg Y})$ is equal to $h_{\mathrm{Sym}^n(X \amalg Y)}$. Replacing all these in (3.19), we get an isomorphism

$$\mathrm{Sym}^n_g(h_X \amalg h_Y) \simeq \coprod_{i+j=n} \left(\mathrm{Sym}^i_g(h_X) \times \mathrm{Sym}^j_g(h_Y) \right)\,. \tag{3.20}$$

Let us consider the functor $\Phi_1\colon \mathscr{C} \times \mathscr{C} \to \mathscr{S}$ which sends a pair (X, Y) to $\mathrm{Sym}^n_g(h_X \amalg h_Y)$ and the functor $\Phi_2\colon \mathscr{C} \times \mathscr{C} \to \mathscr{S}$ which sends a pair (X, Y) to $\coprod_{i+j=n}(\mathrm{Sym}^i_g h_X \times \mathrm{Sym}^j_g h_Y)$. Let

$$\mathrm{Lan}\Phi_1, \mathrm{Lan}\Phi_2\colon \mathscr{S} \times \mathscr{S} \to \mathscr{S}$$

be the left Kan extension of Φ_1 and Φ_2, respectively, along the embedding $h \times h$ from $\mathscr{C} \times \mathscr{C}$ into $\mathscr{S} \times \mathscr{S}$. Since $\mathscr{S}$ is an extensive category, it follows that the coproduct functor $(\mathscr{C} \downarrow F) \times (\mathscr{C} \downarrow F) \to (\mathscr{C} \downarrow F \amalg G)$ is an equivalence of categories; hence, one deduces that the functor $\mathrm{Lan}\Phi_1$ is nothing but the functor that sends a pair (F, G) to $\mathrm{Sym}^n_g(F \amalg G)$. By [5, Prop. 3.4.17] $\mathscr{S}$ is a Cartesian closed, hence, one deduces that $\mathrm{Lan}\Phi_2$ sends a pair (F, G) to $\coprod_{i+j=n}(\mathrm{Sym}^i_g F \times \mathrm{Sym}^j_g G)$. Finally, from the isomorphism (3.20), we have $\Phi_1 \simeq \Phi_2$, which implies that $\mathrm{Lan}\Phi_1$ is isomorphic to $\mathrm{Lan}\Phi_2$. This proves the lemma. □

Corollary 3.2.14 (Künneth rule)**.** *Let $\mathscr{X}$, $\mathscr{Y}$ be two objects in $\Delta^{\mathrm{op}}\mathscr{S}$. For any integer $n \geq 1$, there is an isomorphism*

$$\mathrm{Sym}^n_g(\mathscr{X} \amalg \mathscr{Y}) \simeq \coprod_{i+j=n} (\mathrm{Sym}^i_g \mathscr{X} \times \mathrm{Sym}^j_g \mathscr{Y})\,.$$

Proof. It follows from Lemma 3.2.13. □

Remark 3.2.15. Let $f\colon \mathscr{X} \to \mathscr{X} \vee \mathscr{Y}$ be a coprojection in $\Delta^{\mathrm{op}}\mathscr{S}_*$. Using left Kan extensions, we deduce from Remark 3.2.8 that for every integer $n \geq 1$, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Sym}^n_g \mathscr{X} & \xrightarrow{\mathrm{Sym}^n(f)} & \mathrm{Sym}^n_g(\mathscr{X} \vee \mathscr{Y}) \\
\Vert & & \downarrow \\
\mathrm{Sym}^n_g \mathscr{X} & \longrightarrow & \bigvee_{i+j=n}(\mathrm{Sym}^i_g \mathscr{X} \wedge \mathrm{Sym}^j_g \mathscr{Y})
\end{array}$$

where the right vertical arrow is the isomorphism given in Corollary 3.2.14 and the bottom arrow is the canonical morphism.

We recall that $\Delta_{\mathrm{Spec}(k)}[0]$ is the terminal object of $\Delta^{\mathrm{op}}\mathscr{S}$. From the definition, we observe that the functor Sym^n_g preserves terminal object $\Delta_{\mathrm{Spec}(k)}[0]$, for $n \in \mathbb{N}$. Hence the endofunctor Sym^n_g of $\Delta^{\mathrm{op}}\mathscr{S}$ extends to an endofunctor of $\Delta^{\mathrm{op}}\mathscr{S}_*$, denoted by the same symbol Sym^n_g if no confusion arises.

Lemma 3.2.16. *Let F, G be two objects in $\mathscr{S}_*$. For any integer $n \geq 1$, there is an isomorphism*

$$\mathrm{Sym}^n_g(F \vee G) \simeq \bigvee_{i+j=n} (\mathrm{Sym}^i_g F \wedge \mathrm{Sym}^j_g G)\,.$$

Proof. The proof is similar to proof of Lemma 3.2.13. In this case we define two functor Φ_1 and Φ_2 from $\mathscr{C}_+ \times \mathscr{C}_+$ to $\mathscr{S}_*$ such that Φ_1 takes a pair (X_+, Y_+) to $\mathrm{Sym}^n_g(h_{X_+} \vee h_{Y_+})$ and Φ_2 takes a pair (X_+, Y_+) to $\bigvee_{i+j=n}(\mathrm{Sym}^i_g h_{X_+} \wedge \mathrm{Sym}^j_g h_{Y_+})$. Hence we prove that the left Kan extensions of Φ_1 and Φ_2, along the canonical functor $\mathscr{C}_+ \times \mathscr{C}_+ \to \mathscr{S}_*$, are isomorphic. □

Corollary 3.2.17 (Pointed version of Künneth rule)**.** *Let $\mathscr{X}$, $\mathscr{Y}$ be two objects in $\Delta^{\mathrm{op}}\mathscr{S}_*$. For any integer $n \geq 1$, there is an isomorphism*

$$\mathrm{Sym}^n_g(\mathscr{X} \vee \mathscr{Y}) \simeq \bigvee_{i+j=n} (\mathrm{Sym}^i_g \mathscr{X} \wedge \mathrm{Sym}^j_g \mathscr{Y})\,.$$

Proof. It is a consequence of Lemma 3.2.16. □

Proposition 3.2.18. *For each $n \in \mathbb{N}$, the functor Sym^n_g preserves termwise coprojections.*

Proof. It follows from Lemma 3.2.13 for the unpointed case and from Lemma 3.2.16 for the pointed case. □

3.2.2 Künneth towers

Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of pointed simplicial sheaves. A filtration of $\mathrm{Sym}^n_g(f)$ in $\Delta^{\mathrm{op}}\mathscr{S}_*$,

$$\mathrm{Sym}^n_g(\mathscr{X}) = \mathscr{L}^n_0(f) \to \mathscr{L}^n_1(f) \to \cdots \to \mathscr{L}^n_n(f) = \mathrm{Sym}^n_g(\mathscr{Y})\,,$$

is called (*geometric*) *Künneth tower* of $\mathrm{Sym}^n_g(f)$, if for every index $1 \leq i \leq n$, there is an isomorphism

$$\mathrm{cone}\Big(\mathscr{L}^n_{i-1}(f) \to \mathscr{L}^n_i(f)\Big) \simeq \mathrm{Sym}^{n-i}_g(\mathscr{X}) \wedge \mathrm{Sym}^i_g(\mathscr{X})$$

in $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$.

Later, we shall prove that the nth fold geometric symmetric power of an I^+_{proj}-cell complex has canonical Künneth towers, see Proposition 4.1.2. In the next paragraphs, $\Delta^{\mathrm{op}}h^+$ will denote the canonical functor from $\Delta^{\mathrm{op}}\mathscr{C}_+$ to $\Delta^{\mathrm{op}}\mathscr{S}_*$.

Definition 3.2.19. A pointed simplicial sheaf is called *representable*, if it is isomorphic to a simplicial sheaf of the form $\Delta^{\mathrm{op}}h^+_X$, where X is a simplicial object on $\mathscr{C}$.

Example 3.2.20. For any object U in $\mathscr{C}$ and $n \in \mathbb{N}$, the simplicial sheaves $\Delta_U[n]_+$ and $\partial\Delta_U[n]_+$ are both representable.

Proposition 3.2.21. *For every $n \in \mathbb{N}$, the nth fold geometric symmetric power of a morphism of representable simplicial sheaves induced by a termwise coprojection has a canonical Künneth tower.*

Proof. Let $\varphi\colon X \to Y$ be a termwise coprojection in $\Delta^{\mathrm{op}}\mathscr{C}_+$ and denote by Z the cofibre Y/X. By Lemma 3.2.9, there is a filtration

$$X^{\wedge n} = \square^n_0(\varphi) \to \square^n_1(\varphi) \to \cdots \to \square^n_n(\varphi) = Y^{\wedge n}\,.$$

Since $\mathscr{C}$ is admissible, $\Delta^{\mathrm{op}}\mathscr{C}_+$ allows quotients by finite groups. Then, the above filtration induces a filtration

$$\mathrm{Sym}^n(X) = \tilde{\square}^n_0(\varphi) \to \tilde{\square}^n_1(\varphi) \to \cdots \to \tilde{\square}^n_n(\varphi) = \mathrm{Sym}^n(Y)\,, \tag{3.21}$$

such that, for every index $1 \leq i \leq n$, there is an isomorphism

$$\tilde{\square}^n_i/\tilde{\square}^n_{i-1} \simeq \mathrm{Sym}^{n-i}(X) \wedge \mathrm{Sym}^i(Z)$$

Since h preserves finite coproducts and products, the filtration (3.21) induces a filtration of $\mathrm{Sym}^n_g(\Delta^{\mathrm{op}}h^+_\varphi)$,

$$\Delta^{\mathrm{op}}h^+_{\tilde{\square}^n_0(\varphi)} \longrightarrow \Delta^{\mathrm{op}}h^+_{\tilde{\square}^n_1(\varphi)} \longrightarrow \cdots \longrightarrow \Delta^{\mathrm{op}}h^+_{\tilde{\square}^n_n(\varphi)}\,,$$

which is a Künneth tower of $\mathrm{Sym}^n_g(\Delta^{\mathrm{op}}h^+_\varphi)$. □

Directed colimits of representable simplicial sheaves

We recall that a *directed colimit* is the colimit of a directed diagram, i.e. a functor whose source is a directed set (see Definition 1.1.20).

Definition 3.2.22. We shall denote by $(\Delta^{\mathrm{op}}\mathscr{C}_+)^{\#}$ the full subcategory of $\Delta^{\mathrm{op}}\mathscr{S}_*$ generated by directed colimits of representable simplicial sheaves (Definition 3.2.19).

Proposition 3.2.23. *Let $f\colon \mathscr{X} \to \mathscr{X} \vee \mathscr{Y}$ be a coprojection of simplicial sheaves, where $\mathscr{X}$ and $\mathscr{Y}$ are in $(\Delta^{\mathrm{op}}\mathscr{C}_+)^{\#}$. Then, for every $n \in \mathbb{N}$, the Künneth tower of $\mathrm{Sym}^n_g(f)$ is a sequence*

$$\mathscr{L}^n_0(f) \longrightarrow \mathscr{L}^n_1(f) \longrightarrow \cdots \longrightarrow \mathscr{L}^n_n(f)\,,$$

such that each term $\mathscr{L}^n_i(f)$ is isomorphic to the coproduct

$$\bigvee_{(n-i)\leq l\leq n} (\mathrm{Sym}^l_g \mathscr{X} \wedge \mathrm{Sym}^{n-l}_g \mathscr{Y})\,. \tag{3.22}$$

Proof. Let us write $\mathscr{X} := \mathrm{colim}_{\,d\in D}\Delta^{\mathrm{op}}h^+_{X_d}$ and $\mathscr{Y} := \mathrm{colim}_{\,e\in E}\Delta^{\mathrm{op}}h^+_{Y_e}$, where X_d and Y_e are in $\Delta^{\mathrm{op}}\mathscr{C}_+$. Then, the coproduct $\mathscr{X} \vee \mathscr{Y}$ is isomorphic to the colimit

$$\mathrm{colim}_{\,(d,e)\in D\times E}(\Delta^{\mathrm{op}}h^+_{X_d} \vee \Delta^{\mathrm{op}}h^+_{Y_e})\,,$$

and f is the colimit of the coprojections $\Delta^{\mathrm{op}}h^+_{X_d} \to \Delta^{\mathrm{op}}h^+_{X_d} \vee \Delta^{\mathrm{op}}h^+_{Y_e}$ over all pairs (d,e) in $D \times E$. Let us write $\varphi_{d,e}$ for the coprojection $X_d \to X_d \vee Y_e$. By Lemma 3.2.9, the morphism $\mathrm{Sym}^n(\varphi_{d,e})$ has a Künneth tower whose ith term has the form

$$\tilde{\square}_i(\varphi_{d,e}) \simeq \bigvee_{(n-i)\leq l\leq n} (\mathrm{Sym}^l X_d \wedge \mathrm{Sym}^{n-l} Y_e)\,.$$

Hence, we have an isomorphism

$$\Delta^{\mathrm{op}}h^+\tilde{\square}^n_i(\varphi_{d,e}) \simeq \bigvee_{(n-i)\leq l\leq n} \left(\mathrm{Sym}^l_g \Delta^{\mathrm{op}}h^+_{X_d} \wedge \mathrm{Sym}^{n-l}_g \Delta^{\mathrm{op}}h^+_{Y_e}\right).$$

Taking colimit over $D \times E$, we get that

$$\mathscr{L}^n_i(f) := \mathrm{colim}_{\,(d,e)\in D\times E}\Delta^{\mathrm{op}}h^+_{\tilde{\square}^n_i(\varphi_{d,e})}$$

is isomorphic to the coproduct (3.22), as required. □

Lemma 3.2.24. *The subcategory $(\Delta^{\mathrm{op}}\mathscr{C}_+)^{\#}$ is closed under directed colimits.*

Proof. Let $\mathscr{X} : I \to (\Delta^{\mathrm{op}}\mathscr{C}_+)^{\#}$ be a directed functor. We aim to prove that $\mathrm{colim}\,\mathscr{X}$ is an object of $(\Delta^{\mathrm{op}}\mathscr{C}_+)^{\#}$. Indeed, there exists a collection of directed sets $\{J_i \mid i \in I\}$, such that, each object $\mathscr{X}(i)$ is the colimit of a directed diagram $\mathscr{X}_i\colon J_i \to \Delta^{\mathrm{op}}\mathscr{S}_*$

whose values are in the image of the functor $\Delta^{\mathrm{op}}h^{+}$. We set $J := \cup_{i\in I} J_i$, note that it is also a directed set. Let L be the set consisting of pairs $(j,i) \in J \times I$ such that $j \in J_i$. The preorder on $J \times I$, induces a preorder on L, so it is also an directed set. We define a diagram $\mathscr{Y}\colon L \to \Delta^{\mathrm{op}}\mathscr{S}_*$ that assigns an index $(j,i) \in L$ to the object $\mathscr{X}_i(j)$. We have

$$\operatorname{colim} \mathscr{X} = \operatorname{colim}_{i\in I} \mathscr{X}(i) \simeq \operatorname{colim}_{i\in I} \operatorname{colim}_{j\in J_i} \mathscr{X}_i(j) \simeq \operatorname{colim}_{(j,i)\in L} \mathscr{X}_i(j) = \operatorname{colim} \mathscr{Y}.$$

Therefore, colim $\mathscr{X}$ is a directed colimit of representable simplicial sheaves. $\square$

Lemma 3.2.25. *A morphism between representable simplicial sheaves has the form* $\Delta^{\mathrm{op}}h^{+}_{\varphi}$*, where* φ *is a morphism in* $\Delta^{\mathrm{op}}\mathscr{C}_{+}$*.*

Proof. Suppose that $\mathscr{X} = \Delta^{\mathrm{op}}h^{+}_{X}$ and $\mathscr{Y} = \Delta^{\mathrm{op}}h^{+}_{Y}$, where X and Y are two objects of $\Delta^{\mathrm{op}}\mathscr{C}_{+}$. For every $n \in \mathbb{N}$, the morphism $f_n\colon \mathscr{X}_n \to \mathscr{Y}_n$ is a morphism of the form $h^{+}_{X_n} \to h^{+}_{Y_n}$, and by Yoneda's lemma this morphism is canonically isomorphic to morphism of the form $h^{+}_{\varphi_n}$, where $\varphi_n\colon X_n \to Y_n$ is a morphism in $\mathscr{C}_{+}$. Now, let $\theta\colon [m] \to [n]$ be a morphism in Δ. We have a commutative square

$$\begin{array}{ccc} h^{+}_{X_n} & \xrightarrow{h^{+}_{\varphi_n}} & h^{+}_{Y_n} \\ \downarrow{\scriptstyle \theta^*_{\mathscr{X}}} & & \downarrow{\scriptstyle \theta^*_{\mathscr{Y}}} \\ h^{+}_{X_m} & \xrightarrow[h^{+}_{\varphi_m}]{} & h^{+}_{Y_m} \end{array}$$

where the vertical morphisms are the morphisms induced by θ. By Yoneda's lemma, the morphism $\theta^*_{\mathscr{X}}$ is canonically isomorphic to a morphism of the form $h_{\theta^*_X}$, where $\theta^*_X\colon X_n \to X_m$ is a morphism in $\mathscr{C}$. By the same reason, $\theta^*_{\mathscr{Y}}$ is canonically isomorphic to a morphism of the form $h_{\theta^*_Y}$, where $\theta^*_Y\colon Y_n \to Y_m$ is a morphism in $\mathscr{C}$. Moreover, we have a commutative diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\varphi_n} & Y_n \\ \downarrow{\scriptstyle \theta^*_X} & & \downarrow{\scriptstyle \theta^*_Y} \\ X_m & \xrightarrow[\varphi_m]{} & Y_m \end{array}$$

This shows that the morphisms φ_n, for $n \in \mathbb{N}$, define a morphism $\varphi\colon X \to Y$ in $\Delta^{\mathrm{op}}\mathscr{C}_{+}$ such that f is canonically isomorphic to $\Delta^{\mathrm{op}}h^{+}_{\varphi}$. $\square$

Lemma 3.2.26. *Let*

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{g} & \mathscr{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathscr{B} & \xrightarrow[g']{} & \mathscr{Y} \end{array} \tag{3.23}$$

be a cocartesian in $\Delta^{\mathrm{op}}\mathscr{S}_*$, where f is the image of a termwise coprojection in $\Delta^{\mathrm{op}}\mathscr{C}_+$ through the functor $\Delta^{\mathrm{op}}h^+$. One has the following assertions:

(a) If $\mathscr{X}$ is a representable simplicial sheaf, then $\mathscr{Y}$ is so, and f' is the image of a termwise coprojection in $\Delta^{\mathrm{op}}\mathscr{C}_+$ through the functor $\Delta^{\mathrm{op}}h^+$.

(b) Suppose that $\mathscr{A}$ and $\mathscr{B}$ are compact objects. If $\mathscr{X}$ is in $(\Delta^{\mathrm{op}}\mathscr{C}_+)^{\#}$, then so is $\mathscr{Y}$. Moreover, if $\mathscr{X}$ is a directed colimit of representable simplicial sheaves which are compact, then so is $\mathscr{Y}$.

Proof. (a). By hypothesis, there are a termwise coprojection $\varphi\colon A \to B$ and a morphism $\psi\colon A \to X$ in $\Delta^{\mathrm{op}}\mathscr{C}_+$ such that $f = \Delta^{\mathrm{op}}h^+_\varphi$ and $g = \Delta^{\mathrm{op}}h^+_\psi$. Since φ is a termwise coprojection, we have a cocartesian square

$$\begin{array}{ccc} A & \xrightarrow{\psi} & X \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi'} \\ B & \xrightarrow[\psi']{} & Y \end{array}$$

in $\Delta^{\mathrm{op}}\mathscr{C}_+$, where φ' is a termwise coprojection. As h preserves finite coproducts, we deduce that $\mathscr{Y}$ is isomorphic to $\Delta^{\mathrm{op}}h^+_Y$ and $f' = \Delta^{\mathrm{op}}h^+_{\varphi'}$.

(b). Suppose that $\mathscr{X}$ is the colimit of a directed diagram $\{\mathscr{X}_d\}_{d\in D}$, where $\mathscr{X}_e$ is a representable simplicial sheaf. Since $\mathscr{A}$ is compact, there exists an element $e \in D$ such that the morphism g factors through an object $\mathscr{X}_e$. For every ordinal $d \in D$ with $e \leq d$, we consider the following cocartesian square

$$\begin{array}{ccc} \mathscr{A} & \longrightarrow & \mathscr{X}_d \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathscr{B} & \longrightarrow & \mathscr{B} \vee_{\mathscr{A}} \mathscr{X}_d \end{array}$$

By item (a), the simplicial sheaf $\mathscr{B} \amalg_{\mathscr{A}} \mathscr{X}_d$ is representable. Therefore, we get a cocartesian square

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{g} & \operatorname{colim}_{d\in D}\mathscr{X}_d \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathscr{B} & \longrightarrow & \operatorname{colim}_{\substack{e\leq d \\ d\in D}}(\mathscr{B}\vee_{\mathscr{A}}\mathscr{X}_d) \end{array}$$

as required. □

Lemma 3.2.27. *Every I^+_{proj}-cell complex of $\Delta^{\mathrm{op}}\mathscr{S}_*$ is the colimit of a directed diagram of the form $\{\mathscr{X}_d\}_{d\in D}$ such that, for $d \leq d'$ in D, the corresponding morphism from $\mathscr{X}_d$ to $\mathscr{X}_{d'}$ is a termwise coprojection of compact representable simplicial sheaves. In particular, every I^+_{proj}-cell complex of $\Delta^{\mathrm{op}}\mathscr{S}_*$ is in $(\Delta^{\mathrm{op}}\mathscr{C}_+)^{\#}$.*

Proof. Notice that the domain and codomain of the elements of I^+_{proj} are compact. Since an element of I^+_{proj}-cell is a transfinite composition of pushouts of element of I^+_{proj}, the lemma follows by transfinite induction in view of Lemma 3.2.26 (b). □

Infinite geometric symmetric powers

Let $\mathscr{X}$ be a pointed simplicial sheaf in $(\Delta^{\mathrm{op}}\mathscr{C}_+)^{\#}$. Then, in view of Example 3.2.10, we deduce a sequence,

$$* \longrightarrow \mathrm{Sym}^1_g(\mathscr{X}) \longrightarrow \mathrm{Sym}^2_g(\mathscr{X}) \longrightarrow \cdots\cdots \longrightarrow \mathrm{Sym}^n_g(\mathscr{X}) \longrightarrow \cdots .$$

We define

$$\mathrm{Sym}^{\infty}_g(\mathscr{X}) := \operatorname{colim}_{n\in\mathbb{N}}\mathrm{Sym}^n_g(\mathscr{X}) .$$

Proposition 3.2.28. *We have an isomorphism*

$$\mathrm{Sym}^{\infty}_g(\mathscr{X}\vee\mathscr{Y}) \simeq \mathrm{Sym}^{\infty}_g(\mathscr{X})\wedge\mathrm{Sym}^{\infty}_g(\mathscr{Y}) .$$

Proof. Since $\mathbb{N}$ is filtered, the product $\mathrm{Sym}^{\infty}_g(\mathscr{X})\wedge\mathrm{Sym}^{\infty}_g(\mathscr{Y})$ can be computed as the colimit

$$\operatorname{colim}_{i\in\mathbb{N},j\in\mathbb{N}}\mathrm{Sym}^i_g(\mathscr{X})\wedge\mathrm{Sym}^j_g(\mathscr{X}) .$$

By Corollary 3.2.17, for every $n\in\mathbb{N}$, the geometric symmetric power $\mathrm{Sym}^n_g(\mathscr{X}\vee\mathscr{Y})$ is isomorphic to the coproduct $\bigvee_{i+j=n}(\mathrm{Sym}^i_g\mathscr{X}\wedge\mathrm{Sym}^j_g\mathscr{Y})$. Hence, the composites

$$\mathrm{Sym}^i_g(\mathscr{X})\wedge\mathrm{Sym}^j_g(\mathscr{X}) \longrightarrow \bigvee_{i+j=n}(\mathrm{Sym}^i_g\mathscr{X}\wedge\mathrm{Sym}^j_g\mathscr{Y}) \xrightarrow{\sim} \mathrm{Sym}^n_g(\mathscr{X}\vee\mathscr{Y}) ,$$

for $(i,j)\in\mathbb{N}^2$, induce a morphism $\alpha\colon \mathrm{Sym}^{\infty}_g(\mathscr{X})\wedge\mathrm{Sym}^{\infty}_g(\mathscr{Y}) \to \mathrm{Sym}^{\infty}_g(\mathscr{X}\vee\mathscr{Y})$. On the other hand, for a pair of indices $p,q\geq n$, we have a canonical morphism from the coproduct $\bigvee_{i+j=n}(\mathrm{Sym}^i_g\mathscr{X}\wedge\mathrm{Sym}^j_g\mathscr{Y})$ to $\mathrm{Sym}^p_g\mathscr{X}\wedge\mathrm{Sym}^q_g\mathscr{Y}$. Hence, the composite

$$\mathrm{Sym}^n_g(\mathscr{X}\vee\mathscr{Y}) \xrightarrow{\sim} \bigvee_{i+j=n}(\mathrm{Sym}^i_g\mathscr{X}\wedge\mathrm{Sym}^j_g\mathscr{Y}) \longrightarrow \mathrm{Sym}^p_g(\mathscr{X})\wedge\mathrm{Sym}^q_g(\mathscr{X})$$

induce a morphism $\beta\colon \mathrm{Sym}_g^\infty(\mathscr{X}\vee\mathscr{Y})\to\mathrm{Sym}_g^\infty(\mathscr{X})$. From the constructions of α and β, one observes that they are mutually inverses. □

3.2.3 Geometric symmetric powers of radditive functors

Let k be a field and let $\mathscr{C}\subset\mathscr{S}ch/k$ be an admissible category (Definition 2.3.1). As in Definition 2.3.31, for any integer $n\geq 1$, the left Kan extension induces a functor

$$\mathrm{Sym}^n_{\mathrm{rad},g}\colon \mathrm{Rad}(\mathscr{C})\to\mathrm{Rad}(\mathscr{C})$$

such that there is a diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\mathrm{Sym}^n} & \mathscr{C} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathrm{Rad}(\mathscr{C}) & \xrightarrow[\mathrm{Sym}^n_{\mathrm{rad},g}]{} & \mathrm{Rad}(\mathscr{C}) \end{array} \tag{3.24}$$

where h is the functor is the Yoneda embedding. That is, the left Kan extension $\mathrm{Lan}_h(h\circ\mathrm{Sym}^n)\colon \mathrm{Rad}(\mathscr{C})\to\mathrm{Rad}(\mathscr{C})$ of $h\circ\mathrm{Sym}^n$ along h along, as shown in the following diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{h\circ\mathrm{Sym}^n} & \mathrm{Rad}(\mathscr{C}) \\ {\scriptstyle h}\downarrow & \nearrow{\scriptstyle \mathrm{Lan}_h(h\circ\mathrm{Sym}^n)} & \\ \mathrm{Rad}(\mathscr{C}) & & \end{array}$$

More explicitly, for a radditive functor $\mathscr{X}$, $\mathrm{Sym}^n_{\mathrm{rad},g}(\mathscr{X})$ is defined as follows. If $h\downarrow\mathscr{X}$ is the comma category with objects $h_U\to\mathscr{X}$ for $U\in\mathscr{C}$, and if $F_{\mathscr{X}}\colon h\downarrow\mathscr{X}\to\mathrm{Rad}(\mathscr{C})$ is the functor defined by

$$(h_U\to\mathscr{X})\mapsto h_{\mathrm{Sym}^n U}\ ,$$

then, we have

$$\mathrm{Sym}^n_{\mathrm{rad},g}(\mathscr{X})=\mathrm{colim}\,F_{\mathscr{X}}\ .$$

Definition 3.2.29. The above left Kan extension induces a functor

$$\mathrm{Sym}^n_{\mathrm{rad},g}\colon \Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})\to\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$$

and we called the nth *geometric symmetric power* of radditive functors.

Let $\mathrm{a_{Nis}}$ be the left adjoint of the forgetful functor $\Delta^{\mathrm{op}}\mathscr{S}\to\Delta^{\mathrm{op}}\mathrm{Rad}(\mathscr{C})$. For a radditive functor we write $\mathscr{X}^{\mathrm{a_{Nis}}}$ instead of $\mathrm{a_{Nis}}(\mathscr{X})$.

The following proposition shows the connection between geometric symmetric powers of simplicial Nisnevich sheaves and geometric symmetric powers of simplicial radditive functors defined in [40].

Proposition 3.2.30. *For every simplicial radditive functor $\mathscr{X}$, we have an isomorphism*

$$(\mathrm{Sym}^n_{\mathrm{rad},g}(\mathscr{X}))^{\mathrm{a_{Nis}}} \simeq \mathrm{Sym}^n_g(\mathscr{X}^{\mathrm{a_{Nis}}})\,.$$

Proof. It is enough to prove for radditive functors and Nisnevich sheaves, but it follows since geometric symmetric powers are expressed in terms of colimits and they commute with a left adjoint functor. □

Lemma 3.2.31. *Let $\mathscr{X}$, $\mathscr{Y}$ be two objects in* $\mathrm{Rad}(\mathscr{C})$*. For any integer $n \geq 1$, there is an isomorphism*

$$\mathrm{Sym}^n_g(\mathscr{X} \overset{\mathrm{rad}}{\amalg} \mathscr{Y}) \simeq \coprod_{i+j=n} (\mathrm{Sym}^i_g \mathscr{X} \times \mathrm{Sym}^j_g \mathscr{Y})\,.$$

Proof. The proof is similar to the proof of Lemma 3.2.13. □

Corollary 3.2.32. *Let $\mathscr{X}$, $\mathscr{Y}$ be two objects in* $\mathrm{Rad}(\mathscr{C})_*$*. For any integer $n \geq 1$, there is an isomorphism*

$$\mathrm{Sym}^n_g(\mathscr{X} \vee \mathscr{Y}) \simeq \bigvee_{i+j=n} (\mathrm{Sym}^i_g \mathscr{X} \wedge \mathrm{Sym}^j_g \mathscr{Y})\,.$$

Proof. It follows from the previous lemma, see also [40, Lemma 2.15]. □

3.3 Geometric symmetric powers of motivic spectra

In this section, we define a stable version of the unstable geometric symmetric powers of motivic spaces defined in Section 2.3.3. We show that the stable geometric symmetric powers extend naturally the unstable ones, see Proposition 3.3.4 and Corollary 3.3.10.

3.3.1 Constructions

Let $\mathscr{C} \subset \mathscr{S}ch/k$ be an admissible category as in the previous sections. The category $\Delta^{\mathrm{op}}\mathscr{C}_+$ is symmetric monoidal. For two simplicial objects X and Y, the product $X \wedge Y$ is the simplicial object such that each term $(X \wedge Y)_n$ is given by the product $X_n \wedge Y_n$, see Definition 3.2.6. If $X = (X_0, X_1, X_2, \dots)$ and $Y = (Y_0, Y_1, Y_2, \dots)$ are two symmetric sequences on $\Delta^{\mathrm{op}}\mathscr{C}_+$, then we have a product $X \otimes Y$ which given by the formula

$$(X \otimes Y)_n = \bigvee_{i+j=n} \mathrm{cor}^{\Sigma_n}_{\Sigma_i \times \Sigma_j}(X_i \wedge Y_j)\,.$$

For every symmetric sequence $X = (X_0, X_1, X_2, \dots)$ on the category $\Delta^{\mathrm{op}}\mathscr{C}_+$ and for every $n \in \mathbb{N}$, there exists the quotient $X^{\otimes n}/\Sigma_n$ in the category of symmetric sequences on $\Delta^{\mathrm{op}}\mathscr{C}_+$. For every $p \in \mathbb{N}$, we have

$$(X^{\otimes n})_p = \bigvee_{i_1+\dots+i_n=p} \mathrm{cor}^{\Sigma_p}_{\Sigma_{i_1} \times \dots \times \Sigma_{i_n}}(X_{i_1} \wedge \dots \wedge X_{i_n})\,,$$

and the symmetric group Σ_n acts on $(X^{\otimes n})_p$ by permutation of factors. As $\mathscr{C}_+$ allows quotients under finite groups, the quotient $(X^{\otimes n})_p/\Sigma_n$ is an object of $\mathscr{C}_+$ for all $p \in \mathbb{N}$. Notice that the 0th slice of $X^{\otimes n}/\Sigma_n$ is nothing but the usual nth symmetric power $\mathrm{Sym}^n(X_0) = X_0^{\wedge n}/\Sigma_n$ in $\mathscr{C}_+$.

Let us fix an object S of $\Delta^{\mathrm{op}}\mathscr{C}_+$. A symmetric S-spectrum on $\Delta^{\mathrm{op}}\mathscr{C}_+$ is a sequence of Σ_n-objects X_n in $\Delta^{\mathrm{op}}\mathscr{C}_+$ together with Σ_n-equivariant morphisms $X_n \wedge S \to X_{n+1}$ for $n \in \mathbb{N}$, such that the composite

$$X_m \wedge S^{\wedge n} \to X_{m+1} \wedge S^{\wedge(n-1)} \to \cdots \to X_{m+n}$$

is Σ_{m+n}-equivariant for couples $(m,n) \in \mathbb{N}^2$.

Terminology. We denote by $\mathrm{Spt}_S(\Delta^{\mathrm{op}}\mathscr{C}_+)$ the category of symmetric S-spectra on the category $\Delta^{\mathrm{op}}\mathscr{C}_+$.

We have a functor F_0 from $\Delta^{\mathrm{op}}\mathscr{C}_+$ to $\mathrm{Spt}_S(\Delta^{\mathrm{op}}\mathscr{C}_+)$ that takes an object X of $\Delta^{\mathrm{op}}\mathscr{C}_+$ to the symmetric S-spectrum of the form $(X, X \wedge S, X \wedge S^{\wedge 2}, \dots)$. We have a commutative diagram up to isomorphisms

$$\begin{array}{ccc}
\mathscr{C}_+ & \xrightarrow{\mathrm{Sym}^n} & \mathscr{C}_+ \\
\downarrow{\scriptstyle \mathrm{Const}} & & \downarrow{\scriptstyle \mathrm{Const}} \\
\Delta^{\mathrm{op}}\mathscr{C}_+ & \xrightarrow{\mathrm{Sym}^n} & \Delta^{\mathrm{op}}\mathscr{C}_+ \\
\downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle F_0} \\
\mathrm{Spt}_S(\Delta^{\mathrm{op}}\mathscr{C}_+) & \xrightarrow[\mathrm{Sym}^n_S]{} & \mathrm{Spt}_S(\Delta^{\mathrm{op}}\mathscr{C}_+)
\end{array}$$

Let T be the pointed simplicial sheaf $(\mathbb{P}^1, \infty)$ and let T' be the pointed simplicial sheaf $\mathbb{P}^1_+$ in $\Delta^{\mathrm{op}}\mathscr{S}_*$. We recall that $\mathrm{Spt}_T(k)$ denotes the category of symmetric T-spectra and $\mathrm{Spt}_{T'}(k)$ denotes the category of symmetric T'-spectra on the category $\Delta^{\mathrm{op}}\mathscr{S}_*$. The canonical functor $\Delta^{\mathrm{op}}h^+ \colon \Delta^{\mathrm{op}}\mathscr{C}_+ \to \Delta^{\mathrm{op}}\mathscr{S}_*$ induces a functor

$$H' \colon \mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+) \to \mathrm{Spt}_{T'}(k)\,,$$

that takes a symmetric $\mathbb{P}^1_+$-spectrum $(X_0, X_1, \dots)$ to the symmetric T'-spectrum

$$(\Delta^{\mathrm{op}}h^+_{X_0}, \Delta^{\mathrm{op}}h^+_{X_1}, \dots)\,.$$

Since $\mathscr{C}$ is a small category, the category $\Delta^{\mathrm{op}}\mathscr{C}_+$ is also small. Hence, the category $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$ is so.

Let $f \colon T' \to T$ be the canonical morphism of simplicial sheaves. This morphism induces a morphism of commutative monoids $\mathrm{sym}(T') \to \mathrm{sym}(T)$. In particular, $\mathrm{sym}(T)$ can be seen as a symmetric T'-spectrum.

For any two symmetric T'-spectra $\mathscr{X}$ and $\mathscr{Y}$, we write $\mathscr{X} \wedge_{\mathrm{sym}(T')} \mathscr{Y}$ for the coequalizer of the diagram

$$\mathscr{X} \wedge \mathrm{sym}(T') \wedge \mathscr{Y} \rightrightarrows \mathscr{X} \wedge \mathscr{Y}$$

induced by the canonical morphisms $\mathscr{X} \wedge \mathrm{sym}(T') \to \mathscr{X}$ and $\mathrm{sym}(T') \wedge \mathscr{Y} \to \mathscr{Y}$. For every symmetric T'-spectrum $\mathscr{X}$, the symmetric sequence $\mathscr{X} \wedge_{\mathrm{sym}(T')} \mathrm{sym}(T)$ is a symmetric T-spectrum. We have a functor

$$(-) \wedge_{\mathrm{sym}(T')} \mathrm{sym}(T) \colon \mathrm{Spt}_{T'}(k) \longrightarrow \mathrm{Spt}_T(k)\,.$$

Its right adjoint is the restriction functor $\mathrm{res}_{T/T'}$ that sends a symmetric T-spectrum $\mathscr{X}$ to $\mathscr{X}$ itself thought as a symmetric T'-spectrum via the morphism $f\colon T' \to T$. Let

$$H \colon \mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+) \to \mathrm{Spt}_T(k)$$

be the composition of H' with the functor $(-) \wedge_{\mathrm{sym}(T')} \mathrm{sym}(T)$. We have a diagram

$$\begin{array}{ccc}
\Delta^{\mathrm{op}}\mathscr{S}_* & = & \Delta^{\mathrm{op}}\mathscr{S}_* \\
{\scriptstyle \mathrm{Ev}_n}\uparrow\downarrow{\scriptstyle F_n} & & {\scriptstyle \mathrm{Ev}_n}\uparrow\downarrow{\scriptstyle F_n} \\
\mathrm{Spt}_{T'}(k) & \longrightarrow & \mathrm{Spt}_T(k)
\end{array}$$

Let U be an object in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$ and let n be a positive integer. The canonical morphisms $U \otimes \mathrm{sym}(\mathbb{P}^1_+) \to U$ and $\mathrm{sym}(\mathbb{P}^1_+) \otimes U \to U$ induce a diagram of the form

$$U \otimes \mathrm{sym}(\mathbb{P}^1_+) \otimes U \otimes \cdots \otimes \mathrm{sym}(\mathbb{P}^1_+) \otimes U \;\overset{\cdots\cdots}{\rightrightarrows}\; U^{\otimes n}\;.$$

On the product of left-hand side, U appears n times. The symmetric group acts on this product by permuting of factors of U. Hence, we obtain a diagram

$$H\Big(\big(U \otimes \mathrm{sym}(\mathbb{P}^1_+) \otimes U \otimes \cdots \otimes \mathrm{sym}(\mathbb{P}^1_+) \otimes U\big)/\Sigma_n\Big) \;\overset{\cdots\cdots}{\rightrightarrows}\; H(U^{\otimes n}/\Sigma_n)\;. \tag{3.25}$$

This diagram can be seen as a functor from the category $\{0,1\}$, with two objects and n non trivial arrows $0 \to 1$, to the category $\mathrm{Spt}_T(k)$. For instance, when $n = 2$, one can think of this diagram as a coequalizer diagram.

Stable geometric symmetric powers

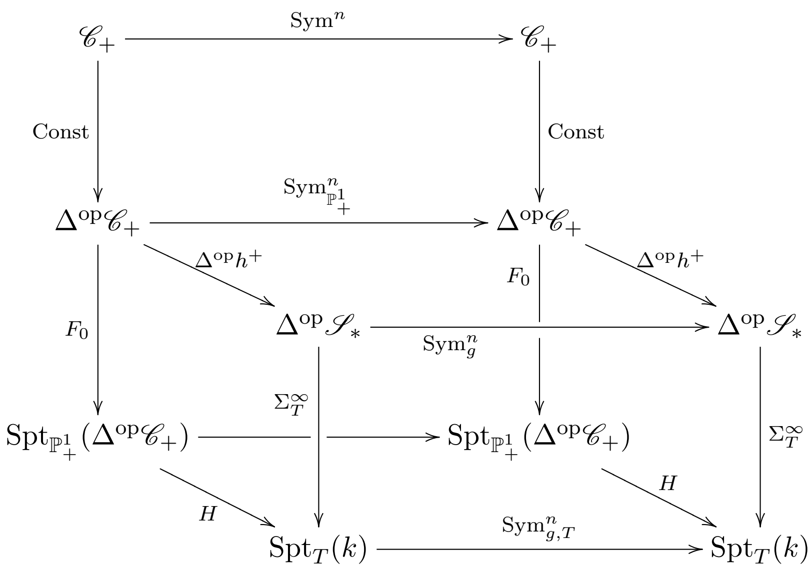
For a spectrum $\mathscr{X}$, we denote by $(H \downarrow \mathscr{X})$ the comma category whose objects are arrows of the form $H(U) \to \mathscr{X}$ for all U in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$. Let

$$F_{\mathscr{X}} \colon (H \downarrow \mathscr{X}) \to \mathrm{Spt}_T(k)$$

be the functor which sends a morphism $H(U) \to \mathscr{X}$ to the symmetric T-spectrum colimit of the diagram (3.25).

Definition 3.3.1. We define $\mathrm{Sym}^n_{g,T}(\mathscr{X})$ to be the colimit of the functor $F_{\mathscr{X}}$. The functor $\mathrm{Sym}^n_{g,T}$ is called the nth-fold *(stable) geometric symmetric power* of symmetric T-spectra.

Our constructions above are summarized in the following diagram:



(3.26)

Next, we shall study the essential properties of geometric symmetric powers.

Lemma 3.3.2. *Let U be an object in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$ and let n be a positive integer. We have a canonical morphism $\vartheta^n_U \colon \mathrm{Sym}^n_T H(U) \to \mathrm{Sym}^n_{g,T} H(U)$.*

Proof. The diagram (3.25) yields into a commutative diagram

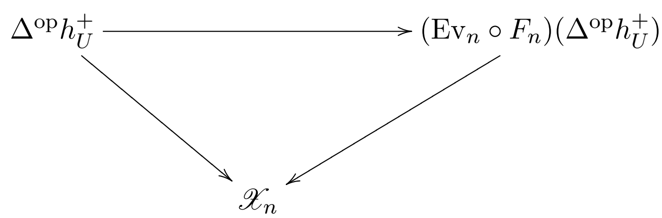
$$\begin{array}{ccc} \Big(H(U)\wedge \mathrm{sym}(T)\wedge H(U)\wedge\cdots\wedge \mathrm{sym}(T)\wedge H(U)\Big)/\Sigma_n & \overset{\cdots\cdots}{\rightrightarrows} & H(U)^{\wedge n}/\Sigma_n \\ \downarrow & & \downarrow \\ H\Big(\big(U\otimes \mathrm{sym}(\mathbb{P}^1_+)\otimes U\otimes\cdots\otimes \mathrm{sym}(\mathbb{P}^1_+)\otimes U\big)/\Sigma_n\Big) & \overset{\cdots\cdots}{\rightrightarrows} & H(U^{\otimes n}/\Sigma_n) \end{array} \tag{3.27}$$

where the vertical morphisms are the canonical morphisms. By taking colimit on the above diagram, we obtain a morphism from $\mathrm{Sym}^n_T H(U)$ to $\mathrm{Sym}^n_{g,T} H(U)$. □

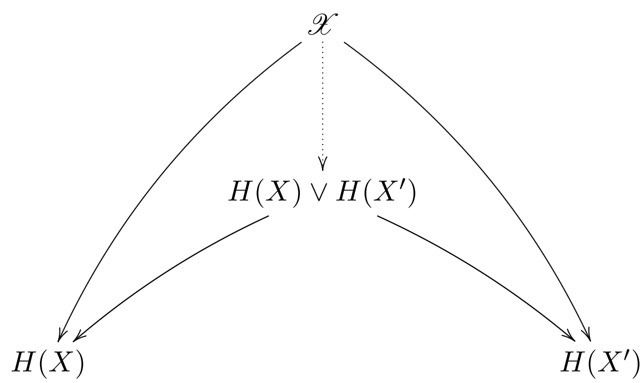
We recall that $\Delta^{\mathrm{op}}h^+$ denotes the canonical functor from $\Delta^{\mathrm{op}}\mathscr{C}_+$ to $\Delta^{\mathrm{op}}\mathscr{S}_*$.

Lemma 3.3.3. *Let $\mathscr{X} = (\mathscr{X}_0, \mathscr{X}_1, \dots)$ be a symmetric T-spectrum in $\mathrm{Spt}_T(k)$. Then, the functor $\mathrm{Ev}_n \colon (H \downarrow \mathscr{X}) \to (\Delta^{\mathrm{op}}h^+ \downarrow \mathscr{X}_n)$ is final.*

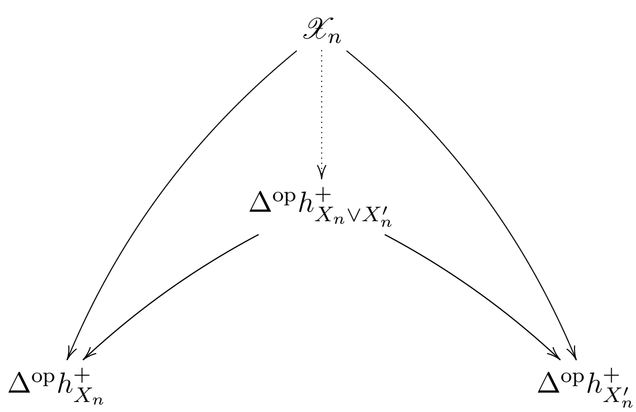
Proof. Suppose that it is given a morphism $\Delta^{\mathrm{op}} h_U^+ \to \mathscr{X}_n$, where U is an object of $\Delta^{\mathrm{op}}\mathscr{C}$. By adjunction, this morphism corresponds to a morphism of T-spectra $F_n(\Delta^{\mathrm{op}} h_U^+) \to \mathscr{X}$. Since $F_n \circ \Delta^{\mathrm{op}} h^+ = H \circ F_n$, we have a morphism $H(F_n(U)) \to \mathscr{X}$. The unit morphism $\Delta^{\mathrm{op}} h_U^+ \to (\mathrm{Ev}_n \circ F_n)(\Delta^{\mathrm{op}} h_U^+)$ gives a commutative diagram



where $(\mathrm{Ev}_n \circ F_n)(\Delta^{\mathrm{op}} h_U^+) = \Delta^{\mathrm{op}} h^+_{\mathrm{cor}^{\Sigma_n}_{\Sigma_0}(U)}$. Now, suppose that there are two morphisms $H(X) \to \mathscr{X}$ and $H(X') \to \mathscr{X}$ where X and X' are in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$. We have a commutative diagram



where the dotted arrow exists by the universal property of coproduct. As $H(X \vee X')$ is isomorphic to $H(X) \vee H(X')$, the above diagram induces a commutative diagram



This proves that the required functor is final [26, page 213]. $\square$

Proposition 3.3.4. *Let n be a natural number. For every symmetric T-spectrum $\mathscr{X}$ in $\mathrm{Spt}_T(k)$, we have a canonical isomorphism*

$$\mathrm{Ev}_0 \circ \mathrm{Sym}^n_{g,T}(\mathscr{X}) \simeq \mathrm{Sym}^n_g \circ \mathrm{Ev}_0(\mathscr{X})\,.$$

Proof. Let U be an object in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$. Since $\mathrm{Ev}_0(\mathrm{sym}(\mathbb{P}^1_+)) = \mathrm{Spec}(k)$. Applying functor Ev_0 to the diagram (3.25), we obtain diagram consisting of identity morphisms of $\Delta^{\mathrm{op}}h^+_{U_0^{\wedge n}/\Sigma_n} \to \Delta^{\mathrm{op}}h^+_{U_0^{\wedge n}/\Sigma_n}$. Hence, the colimit of this diagram is $\Delta^{\mathrm{op}}h^+_{U_0^{\wedge n}/\Sigma_n}$ itself. Thus, we have

$$\mathrm{Ev}_0 \circ \mathrm{Sym}^n_{g,T}(\mathscr{X}) = \mathrm{colim}_{\,H(U)\to\mathscr{X}}\, \Delta^{\mathrm{op}}h^+_{U_0^{\wedge n}/\Sigma_n}\,.$$

By Lemma 3.3.3, the right-hand side is isomorphic to

$$\mathrm{colim}_{\,\Delta^{\mathrm{op}}h^+_{U_0}\to Ev_0(\mathscr{X})}\Delta^{\mathrm{op}}h^+_{U_0^{\wedge n}/\Sigma_n}\,,$$

and by Corollary 3.2.2, the latter is isomorphic to $\mathrm{Sym}^n_g \circ \mathrm{Ev}_0(\mathscr{X})$. □

Corollary 3.3.5. *For every simplicial sheaf $\mathscr{X}$ in $\Delta^{\mathrm{op}}\mathscr{S}_*$, there is a canonical isomorphism*

$$\mathrm{Ev}_0(\mathrm{Sym}^n_{g,T}(\Sigma^\infty_T\mathscr{X})) \simeq \mathrm{Sym}^n_g(\mathscr{X})\,.$$

Proof. It follows from the precedent proposition in view that $\mathrm{Ev}_0(\Sigma^\infty_T\mathscr{X})$ is equal to $\mathscr{X}$. □

We denote by Sym^n_T the categoric symmetric power in $\mathrm{Spt}_T(k)$, that is, for a symmetric T-spectrum $\mathscr{X}$, $\mathrm{Sym}^n_T(\mathscr{X})$ is the quotient of the nth fold product $\mathscr{X}^{\wedge n}$ by the symmetric group Σ_n.

Lemma 3.3.6. *We have a commutative diagram*

$$\begin{array}{ccc}
\Delta^{\mathrm{op}}\mathscr{S}_* & \xrightarrow{\mathrm{Sym}^n} & \Delta^{\mathrm{op}}\mathscr{S}_* \\
\Big\downarrow{\scriptstyle \Sigma^\infty_T} & & \Big\downarrow{\scriptstyle \Sigma^\infty_T} \\
\mathrm{Spt}_T(k) & \xrightarrow[\mathrm{Sym}^n_T]{} & \mathrm{Spt}_T(k)
\end{array}$$

Proof. Let $\mathscr{X}$ be a pointed simplicial sheaf in $\Delta^{\mathrm{op}}\mathscr{S}_*$. By [18, Th. 6.3], the functor $\Sigma^\infty_T \colon \Delta^{\mathrm{op}}\mathscr{S}_* \to \mathrm{Spt}_T(k)$ is a monoidal Quillen functor. Hence, for $n \in \mathbb{N}$, the suspension $\Sigma^\infty_T(\mathscr{X}^{\wedge n})$ is isomorphic to the product $\Sigma^\infty_T(\mathscr{X})^{\wedge n}$. Since Σ^∞_T commutes with colimits, we have

$$\begin{aligned}
\Sigma^\infty_T(\mathrm{Sym}^n\mathscr{X}) &= \Sigma^\infty_T(\mathscr{X}^{\wedge n}/\Sigma_n) \\
&\simeq \Sigma^\infty_T(\mathscr{X}^{\wedge n})/\Sigma_n \\
&\simeq \Sigma^\infty_T(\mathscr{X})^{\wedge n}/\Sigma_n \\
&\simeq \mathrm{Sym}^n_T(\Sigma^\infty_T\mathscr{X})\,.
\end{aligned}$$

This proves the lemma. □

Corollary 3.3.7. *For every simplicial sheaf $\mathscr{X}$ in $\Delta^{\mathrm{op}}\mathscr{S}_*$, we have an isomorphism*

$$(\mathrm{Ev}_0 \circ \mathrm{Sym}^n_T \circ \Sigma^\infty_T)(\mathscr{X}) \simeq \mathrm{Sym}^n(\mathscr{X})\,.$$

Proof. It follows from the previous lemma in view that $\mathrm{Ev}_0(\Sigma^\infty_T \mathscr{Y}) = \mathscr{Y}$ for a pointed simplicial sheaf $\mathscr{Y}$. □

For a symmetric T-spectrum $\mathscr{X}$, we shall write $\vartheta^n(\mathscr{X})$ for $\vartheta^n_{\mathscr{X}}$.

Corollary 3.3.8. *Let $\mathscr{X}$ be a pointed simplicial sheaf in $\Delta^{\mathrm{op}}\mathscr{S}_*$. If the natural morphism $\vartheta^n_T(\Sigma^\infty_T \mathscr{X})\colon \mathrm{Sym}^n_T(\Sigma^\infty_T \mathscr{X}) \to \mathrm{Sym}^n_{g,T}(\Sigma^\infty_T \mathscr{X})$ is a stable $\mathbb{A}^1$-weak equivalence, then the natural morphism $\mathrm{Sym}^n(\mathscr{X}) \to \mathrm{Sym}^n_g(\mathscr{X})$ is an $\mathbb{A}^1$-weak equivalence.*

Proof. In virtue of Corollary 3.3.7 and Proposition 3.3.4, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^n(\mathscr{X}) & \xrightarrow{\vartheta^n(\mathscr{X})} & \mathrm{Sym}^n_g(\mathscr{X}) \\ \downarrow & & \downarrow \\ \mathrm{Ev}_0(\mathrm{Sym}^n_T(\Sigma^\infty_T \mathscr{X})) & \xrightarrow[\mathrm{Ev}_0(\vartheta^n_T(\Sigma^\infty_T \mathscr{X}))]{} & \mathrm{Ev}_0(\mathrm{Sym}^n_{g,T}(\Sigma^\infty_T \mathscr{X})) \end{array} \qquad (3.28)$$

where the vertical morphisms are isomorphisms. Since $\vartheta^n_T(\Sigma^\infty_T \mathscr{X})$ is a stable $\mathbb{A}^1$-weak equivalence, the morphism $\mathrm{Ev}_0(\vartheta^n_T(\Sigma^\infty_T \mathscr{X}))$ is an $\mathbb{A}^1$-weak equivalence. Therefore, $\vartheta^n_{\mathscr{X}}$ is an $\mathbb{A}^n$-weak equivalence. □

Proposition 3.3.9. *Let X and object of $\Delta^{\mathrm{op}}\mathscr{C}_+$. We have an isomorphism*

$$\mathrm{Sym}^n_{g,T}\Sigma^\infty_T(\Delta^{\mathrm{op}}h^+_X) \simeq \Sigma^\infty_T \mathrm{Sym}^n_g(\Delta^{\mathrm{op}}h^+_X)\,.$$

Proof. We have that $\Sigma^\infty_T(\Delta^{\mathrm{op}}h^+_X) = H(F_0(X))$, hence

$$\mathrm{Sym}^n_{g,T}\Sigma^\infty_T(\Delta^{\mathrm{op}}h^+_X) = \mathrm{Sym}^n_{g,T}H(F_0(X))\,.$$

By definition, $\mathrm{Sym}^n_{g,T}H(F_0(X))$ is the coequalizer of the diagram (3.25) in which $U = F_0(X)$. One has,

$$H\Big(F_0(X)^{\otimes n}/\Sigma_n\Big) \simeq H\Big(F_0(X^{\wedge n}/\Sigma_n)\Big) = \Sigma^\infty_T\big(\Delta^{\mathrm{op}}h^+_{\mathrm{Sym}^n X}\big) = \Sigma^\infty_T \mathrm{Sym}^n_g(\Delta^{\mathrm{op}}h^+_X)\,.$$

Since $\mathrm{sym}(\mathbb{P}^1_+) = F_0(\mathrm{Spec}(k)_+)$, the object on left-hand side of diagram (3.25) is nothing but $H\big(F_0(X)^{\otimes n}/\Sigma_n\big)$ and the arrows are the identities. Therefore, the colimit of this diagram is $H\big(F_0(X)^{\otimes n}/\Sigma_n\big)$ which is isomorphic to $\Sigma^\infty_T \mathrm{Sym}^n_g(\Delta^{\mathrm{op}}h^+_X)$. □

Corollary 3.3.10. *For any simplicial sheaf $\mathscr{X}$ in $\mathscr{S}_*$, one has an isomorphism*

$$\mathrm{Sym}^n_{g,T}(\Sigma^\infty_T \mathscr{X}) \simeq \Sigma^\infty_T \mathrm{Sym}^n_g(\mathscr{X})\,.$$

Proof. It is a consequence of the previous proposition in view of Corollary 3.2.2 and Lemma 3.3.3. □

Let $n \in \mathbb{N}$. For a symmetric sequence $\mathscr{X} = (\mathscr{X}_0, \mathscr{X}_1, \dots)$, we define $\mathrm{Sym}^n_{\ell,T'}(\mathscr{X})$ to be the symmetric sequence $\Big(\mathrm{Sym}^n_g(\mathscr{X}_0), \mathrm{Sym}^n_g(\mathscr{X}_1), \dots\Big)$, and call it the nth fold *level geometric symmetric powers* of $\mathscr{X}$. From the definition, we have

$$\mathrm{Ev}_i(\mathrm{Sym}^n_{\ell,T}(\mathscr{X})) = \mathrm{Sym}^n_g(\mathrm{Ev}_i(\mathscr{X}))\,,$$

for $i \in \mathbb{N}$.

Lemma 3.3.11. *For any symmetric T'-spectrum $\mathscr{X} = (\mathscr{X}_0, \mathscr{X}_1, \dots)$, the nth level geometric symmetric power of $\mathscr{X}$ is a symmetric T'-spectrum.*

Proof. Let us consider a symmetric T-spectrum $\mathscr{X} = (\mathscr{X}_0, \mathscr{X}_1, \mathscr{X}_2, \dots)$. For a k-scheme U in $\mathscr{C}$, we define a morphism of k-schemes from $U^n \times \mathbb{P}^1$ to $(U \times \mathbb{P}^1)^n$ as the composite

$$U^n \times \mathbb{P}^1 \xrightarrow{\mathrm{id}\times\Delta_{\mathbb{P}^1}} U^n \times (\mathbb{P}^1)^n \longrightarrow (U \times \mathbb{P}^1)^n$$

where $\Delta_{\mathbb{P}^1}$ is the diagonal morphism and the second arrow is the canonical isomorphism. This morphism induces a morphism from $\mathrm{Sym}^n(U) \times \mathbb{P}^1$ to $\mathrm{Sym}^n(U \times \mathbb{P}^1)$. Let us fix a natural number i. To construct a natural morphism $\mathrm{Sym}^n_g(\mathscr{X}_i) \wedge \mathbb{P}^1_+ \to \mathrm{Sym}^n_g(\mathscr{X}_{i+1})$, it is enough to construct a morphism $\mathrm{Sym}^n_g(\mathscr{X}_i) \times \mathbb{P}^1 \to \mathrm{Sym}^n_g(\mathscr{X}_{i+1})$ considered as unpointed sheaves. Any morphism $h_U \to \mathscr{X}_i$ induces a morphism $h_{U\times\mathbb{P}^1} \to \mathscr{X}_i \times h_{\mathbb{P}^1}$. Composing with the preceding morphism, we obtain a morphism $h_{U\times\mathbb{P}^1} \to \mathscr{X}_{i+1}$. Hence, in view of the above morphism $\mathrm{Sym}^n(U) \times \mathbb{P}^1$ to $\mathrm{Sym}^n(U \times \mathbb{P}^1)$, we deduce a morphism from $\mathrm{colim}_{\,h_U\to\mathscr{X}_i} h_{\mathrm{Sym}^n(U)\times\mathbb{P}^1}$ to $\mathrm{colim}_{\,h_V\to\mathscr{X}_{i+1}} h_{\mathrm{Sym}^n(V)}$. This gives a morphism from $\mathrm{Sym}^n_g(\mathscr{X}_i) \times \mathbb{P}^1$ to $\mathrm{Sym}^n_g(\mathscr{X}_{i+1})$. Since this morphism was constructed in a natural way for all index i, we get structural morphisms for $\mathrm{Sym}^n_{\ell,T}(\mathscr{X})$. □

Proposition 3.3.12. *For each $n \in \mathbb{N}$, the functor $\mathrm{Sym}^n_{\ell,T'}$ preserves levelwise $\mathbb{A}^1$-weak equivalences between symmetric T'-spectra whose slices are termwise coproduct of representable sheaves, i.e. objects in $\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+$.*

Proof. Let f be a morphism of symmetric T'-spectra. From the definition we have an equality $\mathrm{Ev}_i(\mathrm{Sym}^n_{\ell,T'}(f)) = \mathrm{Sym}^n_g(\mathrm{Ev}_i(f))$ for every $i \in \mathbb{N}$. Hence the proposition follows from Theorem 2.3.38. □

Remark 3.3.13. The left Kan extension of the composite $\mathscr{C}_+ \xrightarrow{\mathrm{Sym}^n} \mathscr{C}_+ \xrightarrow{\Sigma^\infty_T} \mathrm{Spt}_T(k)$ along the functor Σ^∞_T is not a good candidate for a (geometric) symmetric power, as this Kan extension is not isomorphic to the identity functor of $\mathrm{Spt}_T(k)$ when $n = 1$.

Remark 3.3.14. For a symmetric T-spectrum, the canonical morphism $\vartheta^n_{\mathscr{X}}$ from the categoric symmetric power $\mathrm{Sym}^n_T(\mathscr{X})$ to geometric symmetric power $\mathrm{Sym}^n_g(\mathscr{X})$ is not always a stable $\mathbb{A}^1$-weak equivalence. For instance when $\mathscr{X}$ is represented by the affine space $\mathbb{A}^2$, Corollary 3.3.8 implies that the canonical morphism from $\mathrm{Sym}^n_T(\Sigma^\infty_T \mathbb{A}^2_+)$ to $\mathrm{Sym}^n_{g,T}(\Sigma^\infty_T \mathbb{A}^2_+)$ is not a stable $\mathbb{A}^1$-weak equivalence.

3.3.2 Künneth towers

Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of symmetric spectra in $\mathrm{Spt}_T(k)$. A filtration of $\mathrm{Sym}^n_{g,T}(f)$ of the form

$$\mathrm{Sym}^n_{g,T}(\mathscr{X}) = \mathscr{L}^n_0(f) \to \mathscr{L}^n_1(f) \to \cdots \to \mathscr{L}^n_n(f) = \mathrm{Sym}^n_{g,T}(\mathscr{Y})$$

is called (*geometric*) *Künneth tower* of $\mathrm{Sym}^n_{g,T}(f)$, if for each index $1 \le i \le n$, there is an isomorphism

$$\mathrm{cone}\Big(\mathscr{L}^n_{i-1}(f) \to \mathscr{L}^n_{i-1}(f)\Big) \simeq \mathrm{Sym}^{n-i}_{g,T}(\mathscr{X}) \wedge \mathrm{Sym}^i_{g,T}(\mathscr{X})\,.$$

in $\mathcal{SH}_T(k)$.

Definition 3.3.15. A symmetric T-spectrum is called *representable*, if it is isomorphic to a T-spectrum of the form $H(U)$, where U is an object on $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$. A symmetric T'-spectrum is called *representable*, if it is isomorphic to a T'-spectrum of the form $H'(U)$, where U is an object on $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$.

Definition 3.3.16. Denote by $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)^\#$ the full subcategory of $\mathrm{Spt}_{T'}(k)$ generated by directed colimits of representable spectra.

Definition 3.3.17. Let $\mathscr{D}$ be a symmetric monoidal model category and let S be an object of $\mathscr{D}$. Let $K\colon \mathbf{2} \to \mathscr{C}$ be a functor, where $\mathbf{2}$ is the category with two objects and one nontrivial morphism. Let $\phi_S\colon \mathscr{D}^n \to \mathscr{C}^{2n-1}$ the functor that sends an n-tuple $(X_1, \dots, X_n)$ to a $(2n-1)$-tuple $(X_1, S, X_2, S, \dots, X_{n-1}, S, X_n)$. For any morphism $f\colon X \to Y$ in $\mathscr{C}$ and any integer $n \ge 1$, let $K^n_S(f)$ be the composite

$$\mathbf{2}^n \to \mathscr{C}^n \xrightarrow{\psi_S} \mathscr{C}^{2n-1} \xrightarrow{\wedge} \mathscr{C}\,.$$

For each index $0 \le i \le n$, we denote by $K^n_{S,i}(f)$ the restriction of $K^n_S(f)$ to $\mathbf{2}^n_i$, see 126. We denote

$$\square^n_{S,i}(f) := \mathrm{colim}\, K^n_{S,i}(f)\,,$$

if this colimit exists. Since the symmetric group Σ_n acts on $\mathbf{2}^n_i$, one deduces that Σ_n acts on $\square_{S,i}(f)$. We denote

$$\tilde{\square}^n_{S,i}(f) := \square^n_{S,i}(f)/\Sigma_n\,,$$

if this quotient exists.

Lemma 3.3.18. *Let $\mathscr{D}$ be a symmetric monoidal model category, let S be a monoid and let $f\colon X \to Y$ be a morphism of S-modules. Suppose that for $0 \le i \le n$, the objects $\square_{S,i}(f)$ and $\square_i(f)$ exist. Then, there are n canonical morphisms*

$$\square^n_{S,i}(f) \overset{\textstyle\rightrightarrows}{\underset{\textstyle\rightrightarrows}{\cdots\cdots}} \square^n_i(f)\ ,$$

for $0 \le i \le n$, induced by the actions of S-modules. Moreover, if $\mathscr{D}$ allows quotients by finite groups, then they induce n canonical morphisms

$$\tilde{\square}^n_{S,i}(f) \overset{\textstyle\rightrightarrows}{\underset{\textstyle\rightrightarrows}{\cdots\cdots}} \tilde{\square}^n_i(f)\ .$$

Proof. These morphisms are constructed from the actions of S-modules. □

Definition 3.3.19. A morphism φ of $\mathbb{P}^1_+$-spectra is called *level-termwise coprojection* if for every $n \in \mathbb{N}$, its nth slice φ_n is a termwise coprojection in $\Delta^{\mathrm{op}}\mathscr{C}_+$. Similarly, a morphism f of T-spectra (or T'-spectra) is called *level-termwise coprojection* if for every $n \in \mathbb{N}$, its nth slice f_n is a termwise coprojection in $\Delta^{\mathrm{op}}\mathscr{S}_*$.

Proposition 3.3.20. *For every $n \in \mathbb{N}$, the nth fold geometric symmetric symmetric power of a morphism of representable T'-spectra (resp. T-spectra), induced by a level-termwise coprojection in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$, has a canonical Künneth tower.*

Proof. Let $\varphi\colon U \to V$ be a level-termwise coprojection in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$. Let us write $K^n_{\mathbb{P}^1,i}(\varphi)$ instead of $K^n_{\mathrm{sym}(\mathbb{P}^1_+),i}(\varphi)$. Since $\varphi\colon U \to V$ is a level-termwise coprojection the colimit $\square^n_{\mathbb{P}^1,i}(\varphi)$ of $K^n_{\mathbb{P}^1,i}(\varphi)$ exist in $(\Delta^{\mathrm{op}}\mathscr{C}_+)^\Sigma$ for every $0 \le i \le n$. Moreover, since $\mathscr{C}$ is admissible, the objects $\tilde{\square}^n_{\mathbb{P}^1,i}(\varphi)$ exists in $(\Delta^{\mathrm{op}}\mathscr{C}_+)^\Sigma$. For similar reason, the objects $\tilde{\square}^n_i(\varphi)$ also exist. By Lemma 3.3.18, we have n canonical morphisms

$$\tilde{\square}^n_{S,i}(\varphi) \overset{\textstyle\rightrightarrows}{\underset{\textstyle\rightrightarrows}{\cdots\cdots}} \tilde{\square}^n_i(\varphi)\ ,$$

induced by the action of $\mathrm{sym}(\mathbb{P}^1_+)$-modules. These morphism induce a diagram

$$\begin{array}{ccccccccc}
\tilde{\square}^n_{\mathbb{P}^1,0}(\varphi) & \longrightarrow & \tilde{\square}^n_{\mathbb{P}^1,1}(\varphi) & \longrightarrow & \cdots & \longrightarrow & \tilde{\square}^n_{\mathbb{P}^1,n-1}(\varphi) & \longrightarrow & \tilde{\square}^n_{\mathbb{P}^1,n}(\varphi) \\
\downdownarrows\cdots\downdownarrows & & \downdownarrows\cdots\downdownarrows & & & & \downdownarrows\cdots\downdownarrows & & \downdownarrows\cdots\downdownarrows \\
\tilde{\square}^n_0(\varphi) & \longrightarrow & \tilde{\square}^n_1(\varphi) & \longrightarrow & \cdots & \longrightarrow & \tilde{\square}^n_{n-1}(\varphi) & \longrightarrow & \tilde{\square}^n_n(\varphi)
\end{array}$$

Notice that

$$\tilde{\square}^n_i(\varphi)/\tilde{\square}^n_{i-1}(\varphi) \simeq U^{\otimes(n-i)}/\Sigma_{n-i} \otimes (V/U)^{\otimes i}/\Sigma_i\,,$$

and $\tilde{\square}^n_{\mathbb{P}^1,i}(\varphi)/\tilde{\square}^n_{\mathbb{P}^1,i-1}(\varphi)$ is isomorphic to the product of

$$\Big(U \otimes \mathrm{sym}(\mathbb{P}^1_+) \otimes U \otimes \cdots \otimes \mathrm{sym}(\mathbb{P}^1_+) \otimes U\Big) \Big/ \Sigma_{n-i}$$

with $\Big(U/V \otimes \mathrm{sym}(\mathbb{P}^1_+) \otimes U/V \otimes \cdots \otimes \mathrm{sym}(\mathbb{P}^1_+) \otimes U/V\Big)\,/\Sigma_i$ for every $0 \leq i \leq n$. Hence, we get a diagram

$$\begin{array}{ccccccc} H'(\tilde{\square}^n_{\mathbb{P}^1,0}(\varphi)) & \longrightarrow & H'(\tilde{\square}^n_{\mathbb{P}^1,1}(\varphi)) & \longrightarrow \cdots \longrightarrow & H'(\tilde{\square}^n_{\mathbb{P}^1,n-1}(\varphi)) & \longrightarrow & H'(\tilde{\square}^n_{\mathbb{P}^1,n}(\varphi)) \\ \Downarrow\Downarrow & & \Downarrow\Downarrow & & \Downarrow\Downarrow & & \Downarrow\Downarrow \\ H'(\tilde{\square}^n_0(\varphi)) & \longrightarrow & H'(\tilde{\square}^n_1(\varphi)) & \longrightarrow \cdots \longrightarrow & \tilde{\square}^n_{n-1}(\varphi) & \longrightarrow & H'(\tilde{\square}^n_n(\varphi)) \end{array}$$

Taking colimit, the above diagram induces a sequence

$$\mathscr{L}^n_0 \to \mathscr{L}^n_1 \to \cdots \to \mathscr{L}^n_n \, .$$

By definition of geometric symmetric powers, we deduce that $\mathscr{L}^n_0 = \mathrm{Sym}^n_{g,T'}(H'(U))$ and $\mathscr{L}^n_n = \mathrm{Sym}^n_{g,T'}(H'(V))$. Moreover, from the above one has

$$\mathscr{L}^n_i/\mathscr{L}^n_{i-1} \simeq \mathrm{Sym}^n_{g,T'}(H'(V)) \wedge \mathrm{Sym}^n_{g,T'}\Big(H'(V)/H'(U)\Big) \, .$$

Thus, the above sequence is a Künneth tower of the morphism $\mathrm{Sym}^n_{g,T'}(H'(\varphi))$. □

3.4 Special symmetric powers

Let k be a field, and suppose that A and B are two k-algebras. We shall denote by $\mathrm{Hom}_k(A, B)$ the set of morphisms of k-algebras, that is, the set of ring homomorphisms $f\colon A \to B$ such that there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & \nwarrow \quad \nearrow & \\ & k & \end{array}$$

Notice that for any k-algebra B, the set $\mathrm{Hom}_k(k, B)$ consists of one single element, in other words, k is the initial object in the category of k-algebras. In this section, h will be understood as the Yoneda embedding of $\mathscr{S}ch/k$ into $Pre(\mathscr{S}ch/k)$. Sometimes, we shall write $\mathrm{Hom}_k(-,-)$ instead of $\mathrm{Hom}_{\mathscr{S}ch/k}(-,-)$. If X is a k-scheme, then $\mathrm{Sym}^n_g(h_X)$ is the representable functor $h_{\mathrm{Sym}^n X}$ for $n \in \mathbb{N}$.

3.4.1 Symmetric powers of a point: Galois extensions

Let L/k be a finite Galois field extension and set $X = \mathrm{Spec}(L)$. Let K be an algebraically closed field containing L and let $U = \mathrm{Spec}(K)$. In the following paragraphs, we shall prove that for any $n \in \mathbb{N}$, the canonical morphism of sets

$$\vartheta^n_X(U)\colon (\mathrm{Sym}^n h_X)(U) \to (\mathrm{Sym}^n_g h_X)(U) \, .$$

is an isomorphism (see Proposition 3.4.4).

Lemma 3.4.1. *Let L/k be a finite Galois extension of degree $r \geq 1$ and let n be an integer $n \geq 1$. The k-algebra $(L^{\otimes_k n})^{\Sigma_n}$ has dimension $\binom{r+n-1}{n}$ as k-vector space.*

Proof. Since L is a Galois extension over k of degree r, the tensor product $L^{\otimes_k n}$ is isomorphic to $L^{\times r^{n-1}}$ as vector spaces over k. Let $\{v_1, v_2, \ldots, v_r\}$ be a k-basis of L. Then the family $\{v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_n}\}_{0 \leq i_1, i_2, \ldots, i_n \leq r}$ is a k-basis of $L^{\otimes_k n}$. An element of $(L^{\otimes_k n})^{\Sigma_n}$ is a linear combination

$$\sum_{0 \leq i_1, \ldots, i_n \leq r} a_{i_1, \ldots, i_n} \cdot v_{i_1} \otimes \ldots \otimes v_{i_n} ,$$

such that

$$\sum_{0 \leq i_1, \ldots, i_n \leq r} a_{i_1, \ldots, i_n} . v_{i_{\sigma(1)}} \otimes \ldots \otimes v_{i_{\sigma(n)}} = \sum_{0 \leq i_1, \ldots, i_n \leq r} a_{i_1, \ldots, i_n} . v_{i_1} \otimes \ldots \otimes v_{i_n} ,$$

for all $\sigma \in \Sigma_n$. From the above equality, we deduce the following relations

$$a_{i_1, \ldots, i_n} = a_{i_{\sigma(1)}, \ldots, i_{\sigma(n)}} , \tag{3.29}$$

for all $\sigma \in \Sigma_n$ and for all indices $i_1, \ldots, i_n$. We recall that a combination of $\{1, 2, \ldots, r\}$ choosing n elements is an unordered n-tuple $\{i_1, \ldots, i_n\}$ allowing repetition of the elements $i_1, \ldots, i_n$ in $\{1, 2, \ldots, r\}$. Let us denote by $C(r, n)$ the set of all repetitions of $\{1, 2, \ldots, r\}$ choosing n elements, and fix $I = \{i_1, \ldots, i_n\}$ in $C(r, n)$. Suppose I has p different elements $j_1, \ldots, j_p$, where $1 \leq p \leq n$, such that each there are k_l repetitions of the element j_l in I for $1 \leq l \leq p$. In particular, one has $\sum_{j=1}^{l} k_l = n$. Let us denote by $P(I) = P(i_1, \ldots, i_n)$ the set of permutations with repetitions of $\{i_1, \ldots, i_n\}$. By an elementary computation in combinatorics, $P(I)$ has a cardinality equal to $\frac{n!}{k_1! . \cdots . k_p!}$ elements. Then we have

$$\sum_{\{i'_1, \ldots, i'_n\} \in P(i_1, \ldots, i_n)} v_{i'_1} \otimes \ldots \otimes v_{i'_n} = \frac{k_1! . \cdots . k_p!}{n!} \cdot \sum_{\sigma \in \Sigma_n} v_{\sigma(i_1)} \otimes \ldots \otimes v_{\sigma(i_n)} .$$

and, from (3.29), we deduce that $a_{i'_1, \ldots, i'_n} = a_{i_1, \ldots, i_n}$ for all $\{i'_1, \ldots, i'_n\} \in P(i_1, \ldots, i_n)$. Hence

$$\sum_{0 \leq i_1, \ldots, i_n \leq r} a_{i_1, \ldots, i_n} . v_{i_1} \otimes \ldots \otimes v_{i_n} = \sum_{\{i_1, \ldots, i_n\} \in C(r,n)} \ \sum_{\{i'_1, \ldots, i'_n\} \in P(i_1, \ldots, i_n)} a_{i_1, \ldots, i_n} \cdot v_{i'_1} \otimes \ldots \otimes v_{i'_n} .$$

Observe that the set

$$\left\{ \sum_{\{i'_1, \ldots, i'_n\} \in P(i_1, \ldots, i_n)} v_{i'_1} \otimes \ldots \otimes v_{i'_n} \right\}_{\{i_1, \ldots, i_n\} \in C(r,n)}$$

is formed by linearly independent vectors in the k-vector space $L^{\otimes_k n}$. Hence, it is a basis of $(L^{\otimes_k n})^{\Sigma_n}$. Then the dimension of $(L^{\otimes_k n})^{\Sigma_n}$ is determined by the cardinality of $C(r, n)$, thus $(L^{\otimes_k n})^{\Sigma_n}$ has dimension $|C(r, n)| = \binom{r+n-1}{n}$. □

Example 3.4.2. In the previous lemma, if L/k is a cubic extension, i.e. $r = 3$ with a k-basis $\{v_1, v_2, v_3\}$ and $n = 2$, then the k-algebra $(L \otimes L)^{\Sigma_2}$ has dimension 6 as k-vector space and its canonical basis consists of six vectors

$$\begin{aligned}
&v_1 \otimes v_1\,,\\
&v_2 \otimes v_2\,,\\
&v_3 \otimes v_3\,,\\
&v_1 \otimes v_2 + v_2 \otimes v_1\,,\\
&v_1 \otimes v_3 + v_3 \otimes v_1\,,\\
&v_2 \otimes v_3 + v_3 \otimes v_2\,.
\end{aligned}$$

Lemma 3.4.3. *Let L/k be a finite Galois extension of degree $r \geq 1$ and set $X = \mathrm{Spec}(L)$. Let K be an algebraically closed field containing L and let $U = \mathrm{Spec}(K)$. Then, for any integer $n \geq 1$ the set $h_{\mathrm{Sym}^n X}(U)$ is a finite set with $\binom{r+n-1}{n}$ elements.*

Proof. Since $(L^{\otimes_k n})^{\Sigma_n}$ is a sub-algebra of $L^{\otimes_k n} \simeq L^{\times r^{n-1}}$, the k-algebra $(L^{\otimes_k n})^{\Sigma_n}$ is isomorphic to a product $\prod_{j=1}^{r^{n-1}} L_j$, where each L_j is a field extension of k contained in L. By the previous lemma, we have that the sum $\sum_{j=1}^{r^{n-1}} \dim_k L_j$ is equal to $\binom{r+n-1}{n}$. Let K be an algebraically closed field containing L. One has,

$$\begin{aligned}
\mathrm{Hom}_k\big((L^{\otimes_k n})^{\Sigma_n}, K\big) &= \mathrm{Hom}_k\left(\prod_{1 \leq j \leq r^{n-1}} L_j, K\right)\\
&\simeq \coprod_{1 \leq j \leq r^{n-1}} \mathrm{Hom}_k(L_j, K)\,.
\end{aligned}$$

Since L_j/k is a finite separable extension and K is algebraically closed, $\mathrm{Hom}_k(L_j, K)$ is a finite set with cardinality equal to $\dim_k L_j$ for all $j = 1, \ldots, r^{n-1}$. Hence, the set $\mathrm{Hom}_k\left((L^{\otimes_k n})^{\Sigma_n}, K\right)$ is finite and has a cardinality equal to $\sum_{j=1}^{r^{n-1}} \dim_k L_j = \binom{r+n-1}{n}$. Let $U = \mathrm{Spec}(K)$. We have,

$$\begin{aligned}
h_{\mathrm{Sym}^n X}(U) &= \mathrm{Hom}_k(U, \mathrm{Sym}^n X)\\
&= \mathrm{Hom}_{\mathrm{Spec}(k)}\Big(\mathrm{Spec}(K), \mathrm{Spec}\big((L^{\otimes_k n})^{\Sigma_n}\big)\Big)\\
&= \mathrm{Hom}_k\Big((L^{\otimes_k n})^{\Sigma_n}, K\Big)\,.
\end{aligned}$$

Thus, we conclude that $h_{\mathrm{Sym}^n X}(U)$ is a finite set with $\binom{r+n-1}{n}$ elements. □

Proposition 3.4.4. *Let L/k be a finite Galois extension and set $X = \mathrm{Spec}(L)$. Let K be an algebraically closed field containing L and let $U = \mathrm{Spec}(K)$. Then, for any integer $n \geq 0$, the canonical morphism of sets*

$$\vartheta_X^n(U)\colon (\mathrm{Sym}^n h_X)(U) \to (\mathrm{Sym}_g^n h_X)(U)$$

is an isomorphism.

Proof. It is trivial if $n = 0$, assume that $n \geq 1$. Suppose that $L = k(\alpha)$ where α is a root of an irreducible polynomial $P(t)$ of degree $r \geq 1$. Notice that $(\mathrm{Sym}^n h_X)(U) = \mathrm{Hom}_k(L, K)^n/\Sigma_n$ is a finite set with $\binom{r+n-1}{n}$ elements. On the other hand, by Lemma 3.4.3, $(\mathrm{Sym}^n h_X)(U)$ is also a finite set with $\binom{r+n-1}{n}$ elements, then it is enough to prove the injectivity of the canonical morphism of sets from $\mathrm{Hom}_k(L, K)^n/\Sigma_n$ to $\mathrm{Hom}_k\left((L^{\otimes_k n})^{\Sigma_n}, K\right)$, defined by $\{f_1, \ldots, f_n\} \mapsto (f_1 \otimes \cdots \otimes f_n)|_{(L^{\otimes_k n})^{\Sigma_n}}$. Indeed, let $\{f_1, \ldots, f_n\}$ and $\{f'_1, \ldots, f'_n\}$ be two unordered n-tuple in $\mathrm{Hom}_k(L, K)^n/\Sigma_n$ such that

$$(f_1 \otimes \cdots \otimes f_n)|_{(L^{\otimes_k n})^{\Sigma_n}} = (f'_1 \otimes \cdots \otimes f'_n)|_{(L^{\otimes_k n})^{\Sigma_n}} \tag{3.30}$$

We put $\alpha_1 = f_1(\alpha), \ldots, \alpha_n = f_n(\alpha)$ and $\alpha'_1 = f'_1(\alpha), \ldots, \alpha'_r = f'_n(\alpha)$. Then $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\alpha'_1, \ldots, \alpha'_n\}$ are two unordered n-tuples formed by roots of $P(t)$ non necessarily distinct from each other. Notice that to prove that the set $\{f_1, \ldots, f_n\}$ is equal to $\{f'_1, \ldots, f'_n\}$, it will be enough to prove that the set $\{\alpha_1, \ldots, \alpha_n\}$ is equal to $\{\alpha'_1, \ldots, \alpha'_n\}$, as a homomorphism of k-algebras $L \to K$ is uniquely determined by a root of $P(t)$. Indeed, observe that the elements

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \Big(1 \otimes \cdots \otimes 1 \otimes \underbrace{\alpha}_{i\text{th position}} \otimes 1 \otimes \cdots \otimes 1\Big), \\ \displaystyle\sum_{1 \leq i < j \leq n} \Big(1 \otimes \cdots \otimes 1 \otimes \underbrace{\alpha}_{i\text{th position}} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\alpha}_{j\text{th position}} \otimes 1 \otimes \cdots \otimes 1\Big), \\ \quad \cdots\cdots \\ \quad \cdots\cdots \\ \alpha \otimes \alpha \otimes \cdots \otimes \alpha, \end{cases}$$

lie in $(L^{\otimes n})^{\Sigma_n}$. In view of the equality $(f_1 \otimes \cdots \otimes f_n)(a_1 \otimes \cdots \otimes a_n) = a_1 \cdot \cdots \cdot a_n$ for all elements $a_1, \ldots, a_n$ in L, we deduce the following equalities,

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i &= (f_1 \otimes \cdots \otimes f_n)\left(\sum_{i=1}^{n} 1 \otimes \cdots \otimes \alpha \otimes \cdots \otimes 1\right), \\ \sum_{1 \leq i < j \leq n} \alpha_i \cdot \alpha_j &= (f_1 \otimes \cdots \otimes f_n)\left(\sum_{1 \leq i < j \leq n} 1 \otimes \cdots \otimes \alpha \otimes \cdots \otimes \alpha \otimes \cdots \otimes 1\right), \\ &\cdots\cdots \\ &\cdots\cdots \\ \alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n &= (f_1 \otimes \cdots \otimes f_n)(\alpha \otimes \alpha \otimes \cdots \otimes \alpha). \end{aligned}$$

Using (3.30), these equalities allow us to deduce the following,

$$
\begin{aligned}
\sum_{i=1}^{n} \alpha_i &= \sum_{i=1}^{n} \alpha'_i\,, \\
\sum_{1\le i<j\le n}^{n} \alpha_i \cdot \alpha_j &= \sum_{1\le i<j\le n}^{n} \alpha'_i \cdot \alpha'_j\,, \\
&\cdots\cdots \\
&\cdots\cdots \\
\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n &= \alpha'_1 \cdot \alpha'_2 \cdot \cdots \cdot \alpha'_n\,.
\end{aligned}
$$

Notice also that these elements are in k, because they are invariants under $\mathrm{Gal}(L/k)$. Now, observe that $\alpha_1, \dots, \alpha_n$ are all the solutions of the polynomial

$$P(t) := t^n - \left(\sum_{i=1}^{n} \alpha_i\right) \cdot t^{n-1} + \left(\sum_{1\le i<j\le n}^{n} \alpha_i \cdot \alpha_j\right) \cdot t^{n-2} + \cdots + (-1)^n \cdot \alpha_1 \cdot \cdots \cdot \alpha_n$$

in $k[t]$, whereas $\alpha'_1, \dots, \alpha'_n$ are all the solutions of the polynomial

$$P'(t) := t^n - \left(\sum_{i=1}^{n} \alpha'_i\right) \cdot t^{n-1} + \left(\sum_{1\le i<j\le n}^{n} \alpha'_i \cdot \alpha'_j\right) \cdot t^{n-2} + \cdots + (-1)^n \cdot \alpha'_1 \cdot \cdots \cdot \alpha'_n$$

which is also in $k[t]$. Since $P(t) = P'(t)$, we conclude that $\{\alpha_1, \dots, \alpha_n\} = \{\alpha'_1, \dots, \alpha'_n\}$, as required. $\square$

3.4.2 Symmetric powers of a double point

Here, we shall study the square symmetric power of $X = \mathrm{Spec}\big(k[x]/(x^2)\big)$. Our goal in the next paragraphs is Proposition 3.4.6.

Notice that there is a natural isomorphism of k-algebras $k[x] \otimes_k k[x] \simeq k[x, y]$ defined by $x \otimes 1 \mapsto x$ and $1 \otimes x \mapsto y$. The universal property of tensor product provides an isomorphism of k-algebras,

$$\Big(k[x]/(x^2)\Big) \otimes_k \Big(k[x]/(x^2)\Big) \simeq k[x, y]/(x^2, y^2)\,.$$

Let τ be the transposition of Σ_2. The symmetric group Σ_2 acts on $k[x] \otimes_k k[x]$ by $\tau(x \otimes 1) = 1 \otimes x$ and $\tau(1 \otimes x) = x \otimes 1$. Then τ acts on $k[x, y]$ by setting $\tau(x) = y$ and $\tau(y) = x$. Thus we have an isomorphism of k-algebras $\tau\colon k[x, y] \to k[x, y]$. Since $\tau\left((x^2, y^2)\right) = (x^2, y^2)$, the permutation τ induces an isomorphism of k-algebras $\tau\colon k[x, y]/(x^2, y^2) \to k[x, y]/(x^2, y^2)$ such that the following diagram

$$
\begin{array}{ccc}
k[x,y] & \xrightarrow{\ \tau\ } & k[x,y] \\
\downarrow & & \downarrow \\
k[x,y]/(x^2,y^2) & \xrightarrow[\ \tau\ ]{} & k[x,y]/(x^2,y^2)
\end{array}
$$

is commutative, where the vertical diagrams are the canonical homomorphisms. From the above diagram, Σ_2 acts on $k[x,y]/(x^2,y^2)$ by $\tau(\overline{x}) = \overline{y}$ and $\tau(\overline{y}) = \overline{x}$. On the other hand, we know that there is an isomorphism of k-algebras $k[x,y]^{\Sigma_2} \simeq k[u,v]$, where $u = x + y$ and $v = xy$.

Lemma 3.4.5. *There is an isomorphism of k-algebras,*

$$\left(\frac{k[x,y]}{(x^2,y^2)}\right)^{\Sigma_2} \simeq k[u,v]/(u^2 - 2v, v^2, uv)\,,$$

such that we have a commutative diagram

$$\begin{array}{ccc} k[x,y]^{\Sigma_2} & \longrightarrow & k[u,v] \\ \downarrow & & \downarrow \\ \left(\frac{k[x,y]}{(x^2,y^2)}\right)^{\Sigma_2} & \longrightarrow & \frac{k[u,v]}{(u^2-2v,v^2,uv)} \end{array}$$

Proof. Indeed, every element of $k[x,y]/(x^2,y^2)$ has the form

$$f(\overline{x},\overline{y}) = a + b\cdot\overline{x} + c\cdot\overline{y} + d\cdot\overline{x}\cdot\overline{y}\,,$$

where a,b,c and d are elements of k. Now, if $f(\overline{x},\overline{y}) \in \left(\frac{k[x,y]}{(x^2,y^2)}\right)^{\Sigma_2}$ then we have $\tau(f(\overline{x},\overline{y})) = f(\overline{x},\overline{y})$. Hence,

$$a + b\cdot\overline{y} + c\cdot\overline{x} + d\cdot\overline{y}\cdot\overline{x} = a + b\cdot\overline{x} + c\cdot\overline{y} + d\cdot\overline{x}\cdot\overline{y}\,,$$

then $b = c$. Thus, any element of $\left(\frac{k[x,y]}{(x^2,y^2)}\right)^{\Sigma_2}$ can uniquely be written as

$$f(\overline{x},\overline{y}) = a + b\cdot(\overline{x}+\overline{y}) + d\cdot\overline{x}\cdot\overline{y}\,,$$

where a,b and d are elements of k. Since $u = x + y$, $v = xy$, we have

$$(x^2,y^2)\cap k[x,y]^{\Sigma_2} = (u^2 - 2v, v^2, uv)\,.$$

In fact, to prove this equality, one uses the following relations $u^2 - 2v = x^2 + y^2$, $v^2 = x^2y^2$ and $uv = x^2y + xy^2$. Any element of $\left(\frac{k[x,y]}{(x^2,y^2)}\right)^{\Sigma_2}$ can uniquely be written as

$$f(\overline{x},\overline{y}) = a + b\cdot\overline{u} + d\cdot\overline{v}\,,$$

but the right-hand side is an element of $\frac{k[u,v]}{(u^2-2v,v^2,uv)}$. Reciprocally, any element of $\frac{k[u,v]}{(u^2-2v,v^2,uv)}$ can uniquely be written as $a + b\cdot\overline{u} + d\cdot\overline{v}$ with $a,b,d\in k$. This show that the isomorphism $k[x,y]^{\Sigma_2} \simeq k[u,v]$ induces an isomorphism $\left(\frac{k[x,y]}{(x^2,y^2)}\right)^{\Sigma_2} \simeq \frac{k[u,v]}{(u^2-2v,v^2,uv)}$ such that the above diagram is commutative. Therefore,

$$\left(\frac{k[x]}{(x^2)}\otimes_k\frac{k[x]}{(x^2)}\right)^{\Sigma_2} \simeq \left(\frac{k[x,y]}{(x^2,y^2)}\right)^{\Sigma_2} \simeq k[u,v]/(u^2 - 2v, v^2, uv)\,.$$

$\square$

Proposition 3.4.6. *Let $X = \operatorname{Spec}(k[x]/(x^2))$ and let $U = \operatorname{Spec}(A)$, where A is a k-algebra. Then the canonical morphism of sets*

$$\vartheta^2_X(U)\colon (\operatorname{Sym}^2 h_X)(U) \to (\operatorname{Sym}^2_g h_X)(U)$$

is injective. Moreover, if A is a reduced algebra, then $\vartheta^2_X(U)$ is bijective.

Proof. We have

$$h_X = \operatorname{Hom}_k(U, X) \simeq \operatorname{Hom}_k(k[x]/(x^2), A) \simeq \{a \in A | a^2 = 0\},$$

and

$$\begin{aligned} h_{\operatorname{Sym}^2(X)}(U) &= \operatorname{Hom}_k\left(U, \operatorname{Sym}^2(X)\right) \\ &\simeq \operatorname{Hom}_k\left(\operatorname{Spec}(A), \operatorname{Spec}\left(\frac{k[u,v]}{(u^2-2v, v^2, uv)}\right)\right) \\ &= \operatorname{Hom}_k\left(\frac{k[u,v]}{(u^2-2v, v^2, uv)}, A\right) \\ &\simeq \left\{(c,d) \in A^2 | c^2 - 2d = d^2 = c \cdot d = 0\right\}. \end{aligned}$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc} \left(\operatorname{Sym}^2 h_X\right)(U) & \longrightarrow & h_{\operatorname{Sym}^2(X)}(U) \\ \downarrow & & \downarrow \\ \left\{a \in A | a^2 = 0\right\}^{\times 2}/\Sigma_2 & \xrightarrow[\xi]{} & \left\{(c,d) \in A^2 | c^2 - 2d = d^2 = c \cdot d = 0\right\} \end{array}$$

where the vertical arrows are bijections and the morphism of sets

$$\xi\colon \left\{a \in A | a^2 = 0\right\}^{\times 2}/\Sigma_2 \longrightarrow \left\{(c,d) \in A^2 | c^2 - 2d = d^2 = c \cdot d = 0\right\}$$

is defined by

$$\{a, b\} \mapsto (a + b, a \cdot b)$$

By the Vieta's formulae, two elements a and b in A are roots of the quadratic polynomial

$$t^2 - (a+b) \cdot t + a \cdot b = 0$$

in $A[t]$. Then we deduce that ξ is injective. Now, if A is a reduced algebra, then ξ is a map of sets with one element. Therefore, $\vartheta^2_X(U)$ is bijective. □

3.4.3 Symmetric powers of the affine line

I learnt the following proposition from V. Guletskiĭ, though he attributes this result to S. Gorchinskiy.

Proposition 3.4.7. *Let K be a field extension over a ground field k, and put $X = \mathrm{Spec}(\mathbb{A}^1)$ and $U = \mathrm{Spec}(K)$. Fix an integer $n \geq 2$. Then the canonical morphism of sets*

$$\vartheta_X^n(U)\colon (\mathrm{Sym}^n h_X)(U) \to (\mathrm{Sym}_g^n h_X)(U)$$

is injective and has cofiber $H^1_{\acute{e}t}(U, \Sigma_n)$.

Proof. If $k[x_1, x_2, \cdots, x_n]$ is the ring of polynomial with n-variables, then we have an isomorphism of k-algebras $k[x_1, x_2, \cdots, x_n]^{\Sigma_n} \simeq k[u_1, u_2, \cdots, u_n]$, where

$$\begin{aligned} u_1 &= \sum_{i=1}^{n} x_i\,, \\ u_2 &= \sum_{1\leq i<j\leq n}^{n} x_i \cdot x_j\,, \\ &\cdots \\ u_n &= x_1 \cdots x_n\,. \end{aligned}$$

Hence, we have

$$\mathrm{Sym}^n(\mathbb{A}^1) = \mathrm{Spec}(k[x_1, x_2, \cdots, x_n]^{\Sigma_n}) \simeq \mathrm{Spec}(k[u_1, u_2, \cdots, u_n]) \simeq \mathbb{A}^n\,.$$

Then,

$$h_{\mathrm{Sym}^n(\mathbb{A}^1)}(U) \simeq h_{\mathbb{A}^n}(U) \simeq K^n\,.$$

We have a commutative diagram

$$\begin{array}{ccc} (\mathrm{Sym}^n h_X)(U) & \longrightarrow & h_{\mathrm{Sym}^n(X)}(U) \\ \downarrow & & \downarrow \\ K^n/\Sigma_n & \longrightarrow & K^n \end{array}$$

where the vertical arrows are bijections and $K^n/\Sigma_n \to K^n$ is the morphism of sets which sends an unordered n-tuple $\{a_1, \ldots, a_n\}$ to the ordered n-tuple

$$\left(\sum_{i=1}^{n} a_i, \sum_{1\leq i<j\leq n}^{n} a_i \cdot a_j, \ldots, a_1 \cdots a_n\right)\,.$$

For any element $(c_1, \ldots, c_n)$ of K^n, we denote the monic polynomial in $K[t]$

$$P_{c_1,\ldots,c_n}(t) := t^n - c_1 \cdot t^{n-1} + c_2 \cdot t^{n-2} + \cdots + c_n\,.$$

Observe that, by the Vieta's formulae, any unordered n-tuple $\{a_1, \dots, a_n\}$ of elements of K is a set of solutions of the polynomial

$$t^n - \left(\sum_{i=1}^{n} a_i\right) \cdot t^{n-1} + \left(\sum_{1 \le i < j \le n}^{n} a_i \cdot a_j\right) \cdot t^{n-2} + \cdots + (-1)^n \cdot a_1 \cdots a_n \,.$$

We claim that $K^n/\Sigma_n \to K^n$ is injective. In fact, if $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are two unordered n-tuples such that

$$\left(\sum_{i=1}^{n} a_i, \sum_{1 \le i < j \le n}^{n} a_i \cdot a_j, \dots, a_1 \cdots a_n\right) = \left(\sum_{i=1}^{n} b_i, \sum_{1 \le i < j \le n}^{n} b_i \cdot b_j, \dots, b_1 \cdots b_n\right) .$$

Then, $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are both the set of solutions of the equation

$$P_{c_1, \dots, c_n}(t) = 0 \,,$$

therefore, we have $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$, showing the injectivity of the above morphism of sets. We define a morphism of sets $\beta_n \colon K^n \to H^1_{\text{ét}}(U, \Sigma_n)$ as follows. If $(c_1, \dots, c_n) \in K^n$, we denote by $E = E_{c_1, \dots, c_n}$ the splitting field of the polynomial $P_{c_1, \dots, c_n}(t) \in K[t]$. Then $\mathrm{Gal}(E/K) \subset \Sigma_n$. We define $\beta_n(c_1, \dots, c_n)$ to be the composite

$$\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(E/K) \hookrightarrow \Sigma_n \,.$$

Next, we shall prove that the following diagram of sets

$$\begin{array}{ccc} K^n/\Sigma_n & \longrightarrow & K^n \\ \big\downarrow & & \big\downarrow {\scriptstyle \beta_n} \\ \mathrm{pt} & \longrightarrow & H^1_{\text{ét}}(U, \Sigma_n) \end{array}$$

is a pushout, in other words, we a bijection of sets

$$K^n/(K^n/\Sigma_n) \simeq H^1_{\text{ét}}(U, \Sigma_n)$$

induced by β_n. To see this bijection it is enough to prove that if $f_0 \colon \mathrm{Gal}(\overline{K}/K) \to \Sigma_n$ is the trivial homomorphism we have

$$\beta_n^{-1}(f_0) = K^n/\Sigma_n \,.$$

In fact, if $(c_1, \dots, c_n)$ is in K^n/Σ_n if and only if the solutions of the polynomial $P_{c_1, \dots, c_n}(t)$ are all in K, if and only if the splitting field $E = E_{c_1, \dots, c_n}$ of $P_{c_1, \dots, c_n}(t)$ is equal to K, if and only if the composite

$$\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(E/K) \hookrightarrow \Sigma_n$$

is the trivial homomorphism f_0, that is, $\beta_n(c_1, \dots, c_n) = f_0$. $\square$

3.4.4 Symmetric powers of the affine 2-dimensional space

Proposition 3.4.8. *We have an isomorphism*

$$\mathrm{Sym}^2(\mathbb{A}^2) \simeq \mathbb{A}^2 \times \mathscr{Q}$$

where $\mathscr{Q}$ is the quadratic cone $\{uw - v^2 = 0\}$ over a field k.

Proof. Let x_1, y_1, x_2, y_2 be the coordinates of $\mathbb{A}^2$ with coefficients over a field k, that is, $\mathbb{A}^2 = \mathrm{Spec}(k[x_1, y_1, x_2, y_2])$. We set $x := x_1 - x_2$, $y := y_1 - y_2$, $x' := x_1 + x_2$, $y' := y_1 + y_2$. We have

$$\mathrm{Sym}^2(\mathbb{A}^2) \simeq \mathrm{Spec}\left(k[x_1, y_1, x_2, y_2]^{\Sigma_2}\right) ,$$

and

$$\mathbb{A}^2 \times \mathscr{Q} \simeq \mathrm{Spec}(k[x', y']) \times_k \mathrm{Spec}\left(\frac{k[u, v, w]}{(uw - v^2)}\right) \simeq \mathrm{Spec}\left(k[x', y'] \otimes_k \frac{k[u, v, w]}{(uw - v^2)}\right) .$$

Let τ the transposition of Σ_2. Notice that $\tau(x) = -x$, $\tau(y) = -y$, $\tau(x') = x'$ and $\tau(y') = y'$. The transposition τ induces a morphism of k-algebras $\tau\colon k[x, y] \to k[x, y]$. Then, we have

$$k[x', y', x, y]^{\Sigma_2} = (k[x, y]^{\Sigma_2})[x', y'] .$$

Hence, all we need is to show the following isomorphism

$$k[x, y]^{\Sigma_2} \simeq k[u, v, w]/(uw - v^2) .$$

We define a morphism of k-algebras $\varphi\colon k[u, v, w] \to k[x, y]^{\Sigma_2}$ given by $\varphi(u) = x^2$, $\varphi(v) = xy$ and $\varphi(w) = y^2$. Notice that $\tau(x^2) = (-x)^2 = x^2$, similarly $\tau(xy) = xy$ and $\tau(y^2) = y^2$, then φ is well defined. We shall show now that $\ker(\varphi) = \langle uw - v^2 \rangle$. Note that the inclusion $\langle uw - v^2 \rangle \subset \ker(\varphi)$ is immediate to see. To show that the other inclusion, notice that set $\{x^2, xy, y^2\}$ is algebraically dependent over k and if $f(u, v, w)$ is a polynomial in $k[u, v, w]$ of minimal absolute degree such that $f(x^2, xy, y^2) = 0$, then $f(u, v, w)$ is equal to $uw - v^2$ up to a multiplication by an element in $k^\times$. This shows the required inclusion. □

Claim 3.4.9. *Let $X = \mathbb{A}^2$ the affine plane over a field k and let A be a k-algebra. Then the canonical morphism of sets*

$$\vartheta^n_X(U)\colon (\mathrm{Sym}^n h_X)(U) \to (\mathrm{Sym}^n_g h_X)(U)$$

is not always surjective for $n > 1$.

Proof. Let us consider $n = 2$. We have

$$h_X(U) = \mathrm{Hom}_k(U, X) \simeq \mathrm{Hom}_k(k[x, y], A) \simeq A^2 .$$

In view of the previous proposition, we have

$$\begin{aligned} h_{\mathrm{Sym}^2(X)}(U) &= \mathrm{Hom}_k(U, \mathbb{A}^2 \times \mathscr{Q}) \\ &\simeq \mathrm{Hom}_k(\mathrm{Spec}(A), \mathrm{Spec}(k[x,y,u,v,w]/(uw - v^2)) \\ &= \mathrm{Hom}_k(k[x,y,u,v,w]/(uw - v^2), A) \\ &\simeq A^2 \times \mathscr{Q}(A)\,, \end{aligned}$$

where $\mathscr{Q}(A)$ is the set of elements $(a,b,c) \in A^3$ such that $ac = b^2$. The morphism of sets $A^2/\Sigma_2 \to A^2 \times \mathscr{Q}(A)$ sends a unordered pair $\{(x_1,y_1),(x_2,y_2)\}$ to the 5-tuple

$$\Big(x_1 + x_2, y_1 + y_2, (x_1 - x_2)^2, (x_1 - x_2)\cdot(y_1 - y_2), (y_1 - y_2)^2\Big)\,.$$

Notice that this application is well-defined. Now, take for example $k = \mathbb{Q}$ and $A = \mathbb{Q}$. The morphism of sets $\psi\colon A^2/\Sigma_2 \to A^2 \times \mathscr{Q}(A)$ is not surjective, for instance the element $(0,0,2,1,1/2) \in A^2 \times \mathscr{Q}(A)$ does not lie in the image of ψ because 2 is not a square in $\mathbb{Q}$. □

Chapter 4

Lambda structures in motivic categories

In algebraic geometry, the theory of λ-structures on rings has allowed to develop systematically a formalism of the Riemann-Roch algebra on Grothendieck groups of algebraic varieties, [10]. Let R be a commutative ring with unit 1. A λ-structure on R is a sequence

$$\{\Lambda^n \colon R \to R\}_{n\in\mathbb{N}}$$

of endomorphisms of R such that one has the following axioms:

(i) $\Lambda^0(a) = 1$, $\Lambda^1(a) = a$, for every $a \in R$,

(ii) $\Lambda^n(a+b) = \sum_{i+j=n} \Lambda^i(a)\cdot\Lambda^j(b)$, for every $a, b \in R$.

See *loc.cit.* Let us give an illustrate example. Let X be an algebraic variety and let us denote by $\mathscr{V}_X$ the category of locally free sheaves on X. The Grothendieck group $K(X)$ of X is the free Abelian group $\mathbb{Z}[\mathscr{V}_X]$, generated by classes of isomorphisms of objects in $\mathscr{V}_X$, modulo the following relations

$$[\mathscr{F}] - [\mathscr{E}] - [\mathscr{G}]\,,$$

whenever one has an exact sequence

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0\,. \tag{4.1}$$

The tensor product $\otimes$ on $\mathscr{V}_X$ induces a multiplication on $K(X)$ by setting

$$[\mathscr{E}] \otimes [\mathscr{F}] := [\mathscr{E} \otimes \mathscr{F}]\,,$$

for objects $\mathscr{E}$ and $\mathscr{F}$ in $\mathscr{V}_X$. The unit of $K(X)$ is $[\mathscr{O}_X]$, where $\mathscr{O}_X$ is the structural sheaf of X. By definition, the exact sequence

$$0 \to \mathscr{E} \to \mathscr{E} \oplus \mathscr{F} \to \mathscr{F} \to 0$$

gives an equality $[\mathscr{E} \oplus \mathscr{F}] = [\mathscr{E}] + [\mathscr{F}]$. The nth fold symmetric power Sym^n of sheaves in $\mathscr{V}_X$ induces an endomorphism $\underline{\mathrm{Sym}}^n$ of $K(X)$. It turns out that the sequence

$$\left\{\underline{\mathrm{Sym}}^n \colon K(X) \to K(X)\right\}_{n\in\mathbb{N}}$$

is a λ-structure on $K(X)$. Indeed, we have $\mathrm{Sym}^0\mathscr{E} \simeq \mathscr{O}_X$ and $\mathrm{Sym}^1\mathscr{E} \simeq \mathscr{E}$ for all $\mathscr{E}$ in $\mathscr{V}_X$. Suppose we have an exact sequence (4.1) and fix an positive integer n. For each index $0 \le i \le n$, let us write L_i^n for the image of the canonical morphism $\mathrm{Sym}^{n-i}\mathscr{E} \otimes \mathrm{Sym}^i\mathscr{F} \to \mathrm{Sym}^n\mathscr{F}$. Then the induced morphism $\mathrm{Sym}^n\mathscr{E} \to \mathrm{Sym}^n\mathscr{F}$ has a filtration

$$\mathrm{Sym}^n\mathscr{E} = L_0^n \subset L_1^n \subset \cdots \subset L_n^n = \mathrm{Sym}^n\mathscr{F}\,,$$

such that there is an isomorphism

$$L_i^n/L_{i-1}^n \simeq \mathrm{Sym}^{n-i}\mathscr{E} \otimes \mathrm{Sym}^i\mathscr{G}\,,$$

for $1 \le i \le n$. The important point is that we have a filtration in the category $\mathscr{V}_X$, i.e. a filtration before taking isomorphism classes. This suggests the possibility of study a global or categoric theory of λ-structures on categories with short sequences, or more generally on categories with cofibre sequences studied in homotopical algebra. The idea of λ-structure on symmetric monoidal model categories was introduced in [13]. It allows one to study systematically various sorts of symmetric powers in such model categories and in their homotopy categories.

4.0.5 Lambda-structures

Let us give a precise definition of a λ-structure.

Definition 4.0.10. Let $\mathscr{C}$ be a closed symmetric monoidal model category with unit $\mathbb{1}$. A *λ-structure* on $\mathscr{C}$ is a sequence $\Lambda^* = (\Lambda^0, \Lambda^1, \Lambda^2, \dots)$ consisting of endofunctors $\Lambda^n \colon \mathscr{C} \to \mathscr{C}$ for $n \in \mathbb{N}$, satisfying the following:

(i) $\Lambda^0 = \mathbb{1}$, $\Lambda^1 = \mathrm{id}$,

(ii) *(Künneth towers).* For any special cofibre sequence $X \xrightarrow{f} Y \to Z$ in $\mathscr{C}$, and any $n \in \mathbb{N}$, there is a unique sequence of cofibrations between cofibrant objects

$$\Lambda^n(X) = L_0^n \to L_1^n \to \cdots \to L_i^n \to \cdots \to L_n^n = \Lambda^n(Y)\,,$$

called *Künneth tower*, such that for any index $0 \le i \le n$, there is an isomorphism

$$L_i^n/L_{i-1}^n \simeq \Lambda^{n-i}(X) \wedge \Lambda^i(Z)\,.$$

(iii) *(Functoriality)*. For any commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array} \tag{4.2}$$

in which the horizontal lines are special cofibre sequences, there is a commutative diagram

$$\begin{array}{ccccccccccc} \Lambda^n(X) = L_0^n & \longrightarrow & L_1^n & \longrightarrow & L_2^n & \longrightarrow \cdots & & \cdots \longrightarrow & L_{n-1}^n & \longrightarrow & L_n^n = \Lambda^n(Y) \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \Lambda^n(X') = L'^n_0 & \longrightarrow & L'^n_1 & \longrightarrow & L'^n_2 & \longrightarrow \cdots & & \cdots \longrightarrow & L'^n_{n-1} & \longrightarrow & L'^n_n = \Lambda^n(Y') \end{array} \tag{4.3}$$

in $\mathscr{C}$.

Example 4.0.11. Let $\mathscr{C}$ be a closed symmetric monoidal model category such that cofibrations in $\mathscr{C}$ are symmetrizable (see Definition 3.1.22). Then Theorem 3.1.26 implies that the categoric symmetric powers $\mathrm{Sym}^n \colon \mathscr{C} \to \mathscr{C}$, for $n \in \mathbb{N}$, define a λ-structure on $\mathscr{C}$.

Similarly, we give the definition of λ-structure on the homotopy category of a symmetric monoidal model category.

Definition 4.0.12. Let $\mathscr{C}$ be a closed symmetric monoidal model category. A λ-*structure* on $\mathrm{Ho}(\mathscr{C})$ is a sequence $\Lambda^* = (\Lambda^0, \Lambda^1, \Lambda^2, \dots)$ consisting of endofunctors Λ^n of $\mathrm{Ho}(\mathscr{C})$ for $n \in \mathbb{N}$, satisfying the following axioms:

(i) $\Lambda^0 = \mathbb{1}$, $\Lambda^1 = \mathrm{id}$,

(ii) *(Künneth tower axiom)*. For any cofibre sequence $X \xrightarrow{f} Y \to Z$ in $\mathrm{Ho}(\mathscr{C})$, and any $n \in \mathbb{N}$, there is a unique sequence

$$\Lambda^n(X) = L_0^n \to L_1^n \to \cdots \to L_i^n \to \cdots \to L_n^n = \Lambda^n(Y)\,.$$

called *Künneth tower*, such that for any index $0 \le i \le n$, the quotient L_i^n / L_{i-1}^n in $\mathscr{C}$ is weak equivalent to the product $\Lambda^{n-i}(X) \wedge \Lambda^i(Z)$.

(iii) *(Functoriality axiom)*. For any morphism of cofibre sequences in $\mathrm{Ho}(\mathscr{C})$ of the (4.2), there is a commutative diagram of the form (4.3) in $\mathrm{Ho}(\mathscr{C})$, in which the horizontal sequences are the respective Künneth towers.

Example 4.0.13. Let Sym^n be the categoric nth fold symmetric power defined on $\Delta^{\mathrm{op}}\mathscr{S}_*$, for $n \in \mathbb{N}$. The left derived functors $L\mathrm{Sym}^n$, for $n \in \mathbb{N}$, provide a λ-structure on $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$ (see [13, Theorem 57] for the proof in the context Nisnevich sheaves on the category of smooth schemes). Indeed, the morphism $\Delta_{\mathbb{A}^1}[0] \to \Delta_{\mathrm{Spec}(k)}[0]$ is a diagonalizable interval, meaning that $\Delta_{\mathbb{A}^1}[0]$ has a structure of symmetric co-algebra in the category $\Delta^{\mathrm{op}}\mathscr{S}$. We claim that the class of cofibrations and the class of trivial cofibrations in $\Delta^{\mathrm{op}}\mathscr{S}$ are symmetrizable. Since cofibrations in $\Delta^{\mathrm{op}}\mathscr{S}$ are section-wise cofibrations of simplicial sets, it follows from Proposition 55 of [13] that cofibrations are symmetrizable. Let f be a trivial cofibration in $\Delta^{\mathrm{op}}\mathscr{S}$. As f is a cofibration, it is a symmetrizable cofibration. For every point P of the site $\mathscr{C}_{\mathrm{Nis}}$, the induced morphism f_P is a weak equivalence of simplicial sets. By [13, Lemma 54], the nth fold symmetric power $\mathrm{Sym}^n(f_P)$ is also a weak equivalence. Since the morphism $\mathrm{Sym}^n(f)_P$ coincide with $\mathrm{Sym}^n(f_P)$, we deduce that the nth fold symmetric power $\mathrm{Sym}^n(f)$ is a weak equivalence too. Hence, by [13, Corollary 54], f is a symmetrizable trivial cofibration. Finally, Theorem 38 and Theorem 22 of [13] imply the existence of left derived functors $L\mathrm{Sym}^n$, for $n \in \mathbb{N}$, and they provide a λ-structure on $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$.

Example 4.0.14. Let $\mathscr{D}$ be a simplicial symmetric monoidal $\mathbb{Q}$-linear stable model category [6]. The projector symmetric powers $\mathrm{Sym}^n_{\mathrm{pr}}$ of Definition 4.3.7, for all $n \in \mathbb{N}$, induce a λ-structure on $\mathrm{Ho}\,(\mathscr{D})$, see Proposition 4.3.9.

Example 4.0.15. The endofunctors $L\mathrm{Sym}^n_g$, for $n \in \mathbb{N}$, provides a λ-structure on the category $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$, see Theorem 4.1.4.

Morphisms of lambda-structures

Next, we define a morphism between two λ-structures as a sequence of natural transformations which are compatible with their Künneth towers.

Definition 4.0.16. Let $\mathscr{C}$ be a closed symmetric monoidal model category with unit $\mathbb{1}$ and let Λ^* and Λ'^* be two λ-structures on $\mathscr{C}$. A *morphism* of λ-structures from Λ^* to Λ'^* consists of a sequence $\Phi^* = (\Phi^0, \Phi^1, \Phi^2, \dots)$ of natural transformations Φ^n from Λ^n to Λ'^n for $n \in \mathbb{N}$, such that for any cofibre sequence $X \to Y \to Z$ in $\mathscr{C}$ and any $n \in \mathbb{N}$, there a commutative diagram

$$
\begin{array}{ccccccccccc}
\Lambda^n(X) = L^n_0 & \longrightarrow & L^n_1 & \longrightarrow & L^n_2 & \longrightarrow \cdots & & \cdots \longrightarrow & L^n_{n-1} & \longrightarrow & L^n_n = \Lambda^n(Y) \\
\downarrow{\scriptstyle \Phi^n(X)} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \Phi^n(Y)} \\
\Lambda'^n(X) = L'^n_0 & \longrightarrow & L'^n_1 & \longrightarrow & L'^n_2 & \longrightarrow \cdots & & \cdots \longrightarrow & L'^n_{n-1} & \longrightarrow & L'^n_n = \Lambda'^n(Y)
\end{array}
\tag{4.4}
$$

Example 4.0.17. Let $\mathscr{C}$ be a closed symmetric monoidal model category. The natural transformations $\mathrm{Sym}^n_h \to \mathrm{Sym}^n$, for $n \in \mathbb{N}$, from the homotopy to the categoric symmetric powers, define a morphism of λ-structures on $\mathscr{C}$, c.f. [12].

Definition 4.0.18. Let $\mathscr{C}$ be a closed symmetric monoidal model category with unit $\mathbb{1}$ and let Λ^* and Λ'^* be two λ-structures on $\mathrm{Ho}(\mathscr{C})$. A *morphism* of λ-structures from Λ^* to Λ'^* consists of a sequence $\Phi^* = (\Phi^0, \Phi^1, \Phi^2, \dots)$ of natural transformations Φ^n from Λ^n to Λ'^n for $n \in \mathbb{N}$, such that for any cofibre sequence $X \to Y \to Z$ in $\mathrm{Ho}(\mathscr{C})$ and any $n \in \mathbb{N}$, there a commutative diagram of the form (4.4) in $\mathrm{Ho}(\mathscr{C})$.

Example 4.0.19. The natural transformations $\vartheta^n \colon \mathrm{Sym}^n \to \mathrm{Sym}^n_g$, for $n \in \mathbb{N}$, induce a morphism of λ-structures from the left derived categoric symmetric powers to the left derived geometric powers on $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$, see Theorem 4.1.10.

Example 4.0.20. Suppose that, for every $n \in \mathbb{N}$, the left derived functor of $\mathrm{Sym}^n_{g,T}$ exists on $\mathcal{SH}_T(k)$. Then, the natural transformations $\vartheta^n \colon \mathrm{Sym}^n_T \to \mathrm{Sym}^n_{g,T}$, for $n \in \mathbb{N}$, induce a morphism of λ-structures from the left derived categoric symmetric powers to the left derived geometric powers on $\mathcal{SH}_T(k)$, see Theorem 4.2.13.

4.1 Lambda-structures in the unstable set-up

Our goal in this section is to prove the main result, Theorem 4.1.4, which asserts that the left derived geometric symmetric powers $L\mathrm{Sym}^n_g$, for $n \in \mathbb{N}$ (see Corollary 2.3.40), induce a λ-structure on the pointed motivic homotopy category $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$.

Proposition 4.1.1. *Let $\mathscr{C}$ be an admissible category. Every cofibre sequence in the homotopy category $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$ is isomorphic to a cofibre sequence of the form*

$$\mathscr{A} \to \mathscr{B} \to \mathscr{B}/\mathscr{A}\ ,$$

where $\mathscr{A} \to \mathscr{B}$ is in I^+_{proj}-cell *and $\mathscr{A}$ is an I^+_{proj}-cell complex. In particular, $\mathscr{A} \to \mathscr{B}$ is a morphism in $\Delta^{\mathrm{op}}\bar{\mathscr{C}}_+$.*

Proof. Let $\mathscr{X} \to \mathscr{Y} \to \mathscr{Z}$ be a cofibre sequence in $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$, where f is a cofibration from $\mathscr{X}$ to $\mathscr{Y}$ in $\Delta^{\mathrm{op}}\mathscr{S}_*$, such that $\mathscr{Z} = \mathscr{Y}/\mathscr{X}$. We write $\mathscr{A} := Q^{\mathrm{proj}}(\mathscr{X})$ and consider the induced morphism $\mathscr{A} \to \mathscr{X}$. By Corollary 2.3.20 and Remark 2.3.22, the composition of $\mathscr{A} \to \mathscr{X}$ with f induces a commutative diagram

$$\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\alpha(f)} & \mathscr{B} \\
\downarrow & & \downarrow{\scriptstyle \beta(f)} \\
\mathscr{X} & \xrightarrow[f]{} & \mathscr{Y}
\end{array}$$

where $\beta(f)$ is a sectionwise trivial fibration and $\alpha(f)$ is in I^+_{proj}-cell. By [18, Prop. 6.2.5], the cofibre sequence $\mathscr{A} \to \mathscr{B} \to \mathscr{B}/\mathscr{A}$ is isomorphic to the cofibre sequence $\mathscr{X} \stackrel{[f]}{\to} \mathscr{Y} \to \mathscr{Z}$ in $\mathscr{H}_*(\mathscr{C}_{\text{Nis}}, \mathbb{A}^1)$.

□

Proposition 4.1.2. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism in I^+_{proj}-cell, where $\mathscr{X}$ is an I^+_{proj}-cell complex. Then, for each $n \in \mathbb{N}$, $\text{Sym}^n_g(f)$ has a functorial Künneth tower.*

Proof. By virtue of Lemma 3.2.27, the morphism f can be expressed as the colimit of a directed diagram $\{f_d\}_{d\in D}$ of termwise coprojections of representable simplicial sheaves. Let us write $f_d\colon \Delta^{\text{op}} h^+_{X_d} \to \Delta^{\text{op}} h^+_{Y_d}$, where X and Y are simplicial objects on $\mathscr{C}$ for every $d \in D$, . Hence, by Proposition 3.2.21, the nth fold geometric symmetric power $\text{Sym}^n_g(f_d)$ has a Künneth tower

$$\mathscr{L}^n_0(f_d) \longrightarrow \mathscr{L}^n_1(f_d) \longrightarrow \cdots \longrightarrow \mathscr{L}^n_n(f_d) \ . \tag{4.5}$$

For each index $0 \leq i \leq n$, we define

$$\mathscr{L}^n_i(f) := \text{colim}_{\, d\in D} \mathscr{L}^n_i(f_d) \, .$$

Thus, we get a sequence

$$\mathscr{L}^n_0(f) \longrightarrow \mathscr{L}^n_1(f) \longrightarrow \cdots \longrightarrow \mathscr{L}^n_n(f) \, . \tag{4.6}$$

Let us show that this gives a Künneth tower of $\text{Sym}^n_g(f)$ that is functorial in f. Since the sequence (4.5) is a Künneth tower of $\text{Sym}^n_g(f_d)$, we have an isomorphism

$$\mathscr{L}^n_i(f_d)/\mathscr{L}^n_{i-1}(f_d) \simeq \text{Sym}^{n-i}_g(\Delta^{\text{op}} h^+_{X_d}) \wedge \text{Sym}^i_g\Big(\Delta^{\text{op}} h^+_{Y_d}/\Delta^{\text{op}} h^+_{X_d}\Big) \, .$$

Hence, taking the colimit on the indices $d \in D$, we get an isomorphism

$$\mathscr{L}^n_i(f)/\mathscr{L}^n_{i-1}(f) \simeq \text{Sym}^{n-i}_g(\mathscr{X}) \wedge \text{Sym}^i_g(\mathscr{Y}/\mathscr{X}) \, . \tag{4.7}$$

□

Lemma 4.1.3. *The endofunctor $L\text{Sym}^0_g$ of $\mathscr{H}_*(\mathscr{C}_{\text{Nis}}, \mathbb{A}^1)$ is the constant functor with value $\mathbb{1}$, where $\mathbb{1}$ is the object $\Delta_{\text{Spec}(k)}[0]_+$ in $\mathscr{H}_*(\mathscr{C}_{\text{Nis}}, \mathbb{A}^1)$, and the endofunctor $L\text{Sym}^1_g$ is the identity functor on $\mathscr{H}_*(\mathscr{C}_{\text{Nis}}, \mathbb{A}^1)$.*

Proof. Since $\text{Sym}^0 X = \text{Spec}(k)_+$ for every object X in $\mathscr{C}_+$, the endofunctor Sym^0 of $\mathscr{C}_+$ is constant with value $\text{Spec}(k)_+$. By the lef Kan extension, we deduce that Sym^0 extends to an endofunctor Sym^0_g of $\Delta^{\text{op}}\mathscr{S}_*$ given by $\mathscr{X} \mapsto \Delta_{\text{Spec}(k)}[0]_+$. Hence, we deduce that $L\text{Sym}^0_g$ is the endofunctor of $\mathscr{H}_*(\mathscr{C}_{\text{Nis}}, \mathbb{A}^1)$ given by $\mathscr{X} \mapsto \mathbb{1}$. On the other hand, for every object X in $\mathscr{C}_+$, we have $\text{Sym}^1 X = X$. By the left Kan extension, we deduce that the endofunctor Sym^1_g of $\Delta^{\text{op}}\mathscr{S}_*$ is the identity functor, then $L\text{Sym}^0_g$ is the identity functor on $\mathscr{H}_*(\mathscr{C}_{\text{Nis}}, \mathbb{A}^1)$. □

Now, we are ready to state and prove our main theorem in this section.

Theorem 4.1.4. *The endofunctors* $L\mathrm{Sym}_g^n$*, for* $n \in \mathbb{N}$*, provides a* λ*-structure on* $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$.

Proof. By Lemma 4.1.3, $L\mathrm{Sym}_g^0$ is the constant functor with value $\mathbb{1}$, and $L\mathrm{Sym}_g^1$ is the identity functor on $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$. Let $\mathscr{X} \to \mathscr{Y} \to \mathscr{Z}$ be a cofibre sequence in $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$ induced by a cofibration $f\colon \mathscr{X} \to \mathscr{Y}$ in the injective model structure of $\Delta^{\mathrm{op}}\mathscr{S}_*$. By Proposition 4.1.1, we can assume that f is in I^+_{proj}-cell and $\mathscr{X}$ is an I^+_{proj}-cell complex. Hence, by Proposition 4.1.2, for each index $n \in \mathbb{N}$, $\mathrm{Sym}_g^n(f)$ has a Künneth tower,

$$\mathrm{Sym}_g^n(\mathscr{X}) = \mathscr{L}_0^n(f) \to \mathscr{L}_1^n(f) \to \cdots \to \mathscr{L}_n^n(f) = \mathrm{Sym}_g^n(\mathscr{Y})\,, \tag{4.8}$$

which induces a Künneth tower,

$$L\mathrm{Sym}_g^n(\mathscr{X}) = L\mathscr{L}_0^n(f) \to L\mathscr{L}_1^n(f) \to \cdots \to L\mathscr{L}_n^n(f) = L\mathrm{Sym}_g^n(\mathscr{Y})\,,$$

of $L\mathrm{Sym}_g^n(f)$ in $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$. Finally, the functoriality axiom follows from the functionality of Künneth towers of the form (4.8), see Proposition 4.1.2. □

4.1.1 A morphism of lambda-structures

In this section, we show the existence of a morphism of λ-structures from left derived categoric symmetric powers to the left derived geometric symmetric powers, see Theorem 4.1.10.

Let us consider the smash product $\wedge$ on $\Delta^{\mathrm{op}}\mathscr{S}_*$. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism in $\Delta^{\mathrm{op}}\mathscr{S}_*$. We recall from Section 3.1.2 that one has a sequence of subdiagrams

$$K_0^n(f) \subset K_1^n(f) \subset \cdots \subset K_n^n(f)\,.$$

This induces a sequence of morphisms in $\Delta^{\mathrm{op}}\mathscr{S}_*$,

$$\mathscr{X}^{\wedge n} = \Box_0^n(f) \to \Box_1^n(f) \to \cdots \to \Box_n^n(f) = \mathscr{Y}^{\wedge n}\,,$$

and its composite is nothing but the n-fold smash product $f^{\wedge n}\colon \mathscr{X}^{\wedge n} \to \mathscr{Y}^{\wedge n}$ of f. For every $0 \le i \le n$, we denote

$$L_i^n(f) = \Box_i^n(f)/\Sigma_n\,.$$

In particular, we have $L_0^n = \mathscr{X}^{\wedge n}/\Sigma_n = \mathrm{Sym}^n(\mathscr{X})$ and $L_n^n = \mathscr{Y}^{\wedge n}/\Sigma_n = \mathrm{Sym}^n(\mathscr{Y})$.

One has the following commutative diagram,

$$\begin{array}{ccccccccc} & & & & f^{\wedge n} & & & & \\ \mathscr{X}^{\wedge n} = \square_0^n(f) & \longrightarrow & \square_1^n(f) & \longrightarrow & \cdots & \longrightarrow & \square_{n-1}^n(f) & \longrightarrow & \square_n^n(f) = \mathscr{Y}^{\wedge n} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \mathrm{Sym}^n \mathscr{X} = L_0^n(f) & \longrightarrow & L_1^n(f) & \longrightarrow & \cdots & \longrightarrow & L_{n-1}^n(f) & \longrightarrow & L_n^n(f) = \mathrm{Sym}^n \mathscr{Y} \\ & & & & \mathrm{Sym}^n f & & & & \end{array}$$

A functorial morphism

For every simplicial sheaf $\mathscr{X}$, we want to construct a natural morphism $\vartheta^n_{\mathscr{X}}$ from $\mathrm{Sym}^n(\mathscr{X})$ to $\mathrm{Sym}^n_g(\mathscr{X})$. First of all, let us consider the case when $\mathscr{X}$ is a representable simplicial sheaf h_X for X in $\mathscr{C}$. In this case, $\mathrm{Sym}^n_g(h_X)$ is nothing but $h_{\mathrm{Sym}^n X}$. In view of the isomorphism $(h_X)^{\times n} \simeq h_{X^n}$, the canonical morphism $h_{X^n} \to h_{\mathrm{Sym}^n}$ induces a morphism $(h_X)^{\times n}/\Sigma_n \to h_{\mathrm{Sym}^n}$, that is, a morphism $\mathrm{Sym}^n(h_X) \to \mathrm{Sym}^n_g(h_X)$. We denote this morphism by $\vartheta^n_{h_X}$ or simply by ϑ^n_X.

Proposition 4.1.5. *For every simplicial sheaf $\mathscr{X}$, there is a functorial morphism*

$$\vartheta^n_{\mathscr{X}} \colon \mathrm{Sym}^n(\mathscr{X}) \to \mathrm{Sym}^n_g(\mathscr{X}) .$$

Proof. It is enough to show for a sheaf $\mathscr{X}$. Indeed, in view of Lemma 3.2.12, we have an isomorphism $\mathscr{X}^{\times n} \simeq \mathrm{colim}_{\, h_X \to \mathscr{X}} h_{X^n}$. Hence, one has

$$\begin{aligned} \mathrm{Sym}^n(\mathscr{X}) &= (\mathscr{X}^{\times n})/\Sigma_n \\ &\simeq (\mathrm{colim}_{\, h_X \to \mathscr{X}} h_{X^n})/\Sigma_n \\ &\simeq \mathrm{colim}_{\, h_X \to \mathscr{X}} \, (h_{X^n}/\Sigma_n) \\ &= \mathrm{colim}_{\, h_X \to \mathscr{X}} \mathrm{Sym}^n(h_X) . \end{aligned}$$

Taking colimit to the canonical morphisms $\vartheta^n_X \colon \mathrm{Sym}^n h_X \to \mathrm{Sym}^n_g h_X$, for X in $\mathscr{C}$, we get a morphism

$$\mathrm{colim}_{\, h_X \to \mathscr{X}} \vartheta^n_X \colon \mathrm{colim}_{\, h_X \to \mathscr{X}} \mathrm{Sym}^n h_X \to \mathrm{colim}_{\, h_X \to \mathscr{X}} \mathrm{Sym}^n_g h_X .$$

On the one hand, we have seen above that $\mathrm{colim}_{\, h_X \to \mathscr{X}} \mathrm{Sym}^n h_X$ is isomorphic to $\mathrm{Sym}^n(\mathscr{X})$, and on the other hand, $\mathrm{colim}_{\, h_X \to \mathscr{X}} \mathrm{Sym}^n_g h_X$ is by definition equal to $\mathrm{Sym}^n_g \mathscr{X}$. Thus, we get a functorial morphism from $\mathrm{Sym}^n(\mathscr{X})$ to $\mathrm{Sym}^n_g(\mathscr{X})$ which we denote it by $\vartheta^n_{\mathscr{X}}$. □

Corollary 4.1.6. *For every pointed simplicial sheaf $\mathscr{X}$, there is a functorial morphism*

$$\vartheta^n_{\mathscr{X}} \colon \mathrm{Sym}^n(\mathscr{X}) \to \mathrm{Sym}^n_g(\mathscr{X}) .$$

Proof. It follows from the previous Proposition 4.1.5. □

For each $n \in \mathbb{N}$, we denote by $\vartheta^n \colon \mathrm{Sym}^n \to \mathrm{Sym}^n_g$ the natural transformation defined for every pointed simplicial sheaf $\mathscr{X}$ to be the functorial morphism $\vartheta^n(\mathscr{X}) := \vartheta^n_{\mathscr{X}}$ of Corollary 4.1.6.

Lemma 4.1.7. *Let $\varphi \colon X \to Y$ be termwise coprojection in $\Delta^{\mathrm{op}}\mathscr{C}_+$ and let us write $f := \Delta^{\mathrm{op}} h^+_{\varphi}$. Then for every pair of numbers $(n, i) \in \mathbb{N}^2$ with $0 \leq i \leq n$, there exists a canonical morphism*

$$\vartheta^n_i(f) \colon L^n_i(f) \to \mathscr{L}^n_i(f)\,,$$

such that one has a commutative diagram

$$\begin{array}{ccccccccc} L^n_0(f) & \longrightarrow & L^n_1(f) & \longrightarrow \cdots & & \cdots \longrightarrow & L^n_{n-1}(f) & \longrightarrow & L^n_n(f) \\ \downarrow {\scriptstyle \vartheta^n_0(f)} & & \downarrow {\scriptstyle \vartheta^n_1(f)} & & & & \downarrow {\scriptstyle \vartheta^n_{n-1}(f)} & & \downarrow {\scriptstyle \vartheta^n_n(f)} \\ \mathscr{L}^n_0(f) & \longrightarrow & \mathscr{L}^n_1(f) & \longrightarrow \cdots & & \cdots \longrightarrow & \mathscr{L}^n_{n-1}(f) & \longrightarrow & \mathscr{L}^n_n(f) \end{array} \tag{4.9}$$

Proof. Let us fix a natural number n. For each index $0 \leq i \leq n$, $\mathscr{L}^n_i(f)$ is nothing but the object $\Delta^{\mathrm{op}} h^+_{\bar{\square}^n_i(\varphi)}$, see Proposition 3.2.21. Since the functor $h^+ \colon \mathscr{C}_+ \to \mathscr{S}_*$ is monoidal, $\square^n_i(f)$ is canonically isomorphic to $\Delta^{\mathrm{op}} h^+_{\square^n_i(\varphi)}$. Thus, we have a canonical morphism $\square^n_i(f) \to \mathscr{L}^n_i(f)$, and this morphism induces a morphism

$$\vartheta^n_i(f) \colon L^n_i(f) \to \mathscr{L}^n_i(f)\,.$$

Since $\vartheta^n_i(f)$ is constructed canonically, we get a commutative diagram (4.9). □

Example 4.1.8. Let us consider a coprojection $X \to X \vee Y$ in $\Delta^{\mathrm{op}}\mathscr{C}_+$ and let f be the morphism $\Delta^{\mathrm{op}} h^+_{\varphi}$. We have a commutative diagram

$$\begin{array}{ccc} \Delta^{\mathrm{op}} h^+_X \wedge \Delta^{\mathrm{op}} h^+_X & \longrightarrow & \left(\Delta^{\mathrm{op}} h^+_X \vee \Delta^{\mathrm{op}} h^+_Y\right) \wedge \Delta^{\mathrm{op}} h^+_X \\ \downarrow & & \downarrow \\ \Delta^{\mathrm{op}} h^+_X \wedge \left(\Delta^{\mathrm{op}} h^+_X \vee \Delta^{\mathrm{op}} h^+_Y\right) & \longrightarrow & \left(\Delta^{\mathrm{op}} h^+_X \vee \Delta^{\mathrm{op}} h^+_Y\right) \wedge \left(\Delta^{\mathrm{op}} h^+_X \vee \Delta^{\mathrm{op}} h^+_Y\right) \end{array} \tag{4.10}$$

which is induced by a diagram

$$\begin{array}{ccc} X \wedge X & \longrightarrow & (X \vee Y) \wedge X \\ \downarrow & & \downarrow \\ X \wedge (X \vee Y) & \longrightarrow & (X \vee Y) \wedge (X \vee Y) \end{array}$$

Then, one gets canonical morphisms

$$\begin{aligned}\vartheta^2_0(f)\colon L^2_0(f) &\longrightarrow \mathscr{L}^2_0(f)\,,\\ \vartheta^2_1(f)\colon L^2_1(f) &\longrightarrow \mathscr{L}^2_1(f)\,,\\ \vartheta^2_2(f)\colon L^2_2(f) &\longrightarrow \mathscr{L}^2_2(f)\,,\end{aligned}$$

where their domains have the form

$$\begin{aligned}\Box^2_0(f) &= \Delta^{\mathrm{op}} h^+_{X\wedge X}\,,\\ \Box^2_1(f) &= \Delta^{\mathrm{op}} h^+_{X\wedge(X\vee Y)} \wedge_{\Delta^{\mathrm{op}} h^+_{X\wedge X}} \Delta^{\mathrm{op}} h^+_{(X\vee Y)\wedge X}\,,\\ \Box^2_2(f) &= \Delta^{\mathrm{op}} h^+_{X\wedge Y} \wedge \Delta^{\mathrm{op}} h^+_{X\wedge Y}\,,\end{aligned}$$

and their codomains have the shape

$$\begin{aligned}\mathscr{L}^2_0(f) &= \Delta^{\mathrm{op}} h^+_{\mathrm{Sym}^2 X}\,,\\ \mathscr{L}^2_1(f) &= \Delta^{\mathrm{op}} h^+_{\mathrm{Sym}^2 X} \wedge \left(\Delta^{\mathrm{op}} h^+_{\mathrm{Sym}^1 X} \vee \Delta^{\mathrm{op}} h^+_{\mathrm{Sym}^1 Y}\right),\\ \mathscr{L}^2_2(f) &= \Delta^{\mathrm{op}} h^+_{\mathrm{Sym}^2 X} \wedge \left(\Delta^{\mathrm{op}} h^+_{\mathrm{Sym}^1 X} \vee \Delta^{\mathrm{op}} h^+_{\mathrm{Sym}^1 Y}\right) \wedge \Delta^{\mathrm{op}} h^+_{\mathrm{Sym}^2 Y}\,.\end{aligned}$$

Proposition 4.1.9. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of pointed simplicial sheaves in I^+_{proj} such that $\mathscr{X}$ is an I^+_{proj}-cell complex. Then for every index $0 \le i \le n$, there exists a canonical morphism*

$$\vartheta^n_i(f)\colon L^n_i(f) \to \mathscr{L}^n_i(f)\,,$$

such that one has a commutative diagram

$$\begin{array}{ccccccccc} L^n_0(f) & \longrightarrow & L^n_1(f) & \longrightarrow \cdots & & \cdots \longrightarrow & L^n_{n-1}(f) & \longrightarrow & L^n_n(f) \\ \downarrow{\scriptstyle \vartheta^n_0(f)} & & \downarrow{\scriptstyle \vartheta^n_1(f)} & & & & \downarrow{\scriptstyle \vartheta^n_{n-1}(f)} & & \downarrow{\scriptstyle \vartheta^n_n(f)} \\ \mathscr{L}^n_0(f) & \longrightarrow & \mathscr{L}^n_1(f) & \longrightarrow \cdots & & \cdots \longrightarrow & \mathscr{L}^n_{n-1}(f) & \longrightarrow & \mathscr{L}^n_n(f) \end{array} \qquad (4.11)$$

where $\vartheta^n_0(f) = \vartheta^n_{\mathscr{X}}$ and $\vartheta^n_n(f) = \vartheta^n_{\mathscr{Y}}$.

Proof. By virtue of Lemma 3.2.27, the morphism f can be expressed as the colimit of a directed diagram $\{f_d\}_{d\in D}$ of termwise coprojections of representable simplicial sheaves. Let us fix an index $0 \le i \le n$. By Lemma 4.1.7, we have canonical morphisms $\vartheta^n_i(f_d)\colon L^n_i(f_d) \to \mathscr{L}^n_i(f_d)$ for $d \in D$. Hence, taking colimit we get a morphism

$$\operatorname{colim}_{d\in D}\vartheta^n_i(f_d)\colon \operatorname{colim}_{d\in D} L^n_i(f_d) \to \operatorname{colim}_{d\in D}\mathscr{L}^n_i(f_d)\,,$$

This morphism gives a morphism from $L^n_i(f)$ to $\mathscr{L}^n_i(f)$, and we denote it by $\vartheta^n_i(f)$. Finally, the diagrams of the form (4.9) induce a commutative diagram of the form (4.11). $\square$

By virtue of Proposition 4.1.5, for each $n \in \mathbb{N}$, we get a natural transformation

$$\vartheta^n \colon \mathrm{Sym}^n \to \mathrm{Sym}^n_g \,.$$

Theorem 4.1.10. *The natural transformations $\vartheta^n \colon \mathrm{Sym}^n \to \mathrm{Sym}^n_g$, for $n \in \mathbb{N}$ induce a morphism of λ-structures from the left derived categoric symmetric powers to the left derived geometric powers on $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$.*

Proof. The natural transformations $\vartheta^n \colon \mathrm{Sym}^n \to \mathrm{Sym}^n_g$, for $n \in \mathbb{N}$, induce a natural transformation of derived functors $L\vartheta^n \colon L\mathrm{Sym}^n \to L\mathrm{Sym}^n_g$ on $\mathscr{H}_*(\mathscr{C}_{\mathrm{Nis}}, \mathbb{A}^1)$. Hence, by Proposition 4.1.9, the endofunctors $L\vartheta^n$ defines a morphism of λ-structures. □

Geometric versus categoric symmetric powers

Let $\mathscr{C}$ be the category of quasi-projective schemes over a field k. It turns out that, if X is the 2-dimensional affine space $\mathbb{A}^2$ over k, then the canonical morphism ϑ^n_X from $\mathrm{Sym}^n h_X$ to $\mathrm{Sym}^n_g h_X$ is not an $\mathbb{A}^1$-weak equivalence in $\Delta^{\mathrm{op}}\mathscr{S}$, see Proposition 4.1.12,.

Lemma 4.1.11. *Let X be a scheme in $\mathscr{C}$. The morphism of simplicial presheaf $\vartheta^n_X \colon \mathrm{Sym}^n h_X \to \mathrm{Sym}^n_g h_X$ is an $\mathbb{A}^1$-weak equivalence if and only if for every $\mathbb{A}^1$-local simplicial presheaf $\mathscr{Z}$ the induced morphism $(\vartheta^n_X)^* \colon \mathscr{Z}(\mathrm{Sym}^n X) \to \mathscr{Z}(X^n)^{\Sigma_n}$ is a weak equivalence of simplicial sets.*

Proof. By definition of $\mathbb{A}^1$-weak equivalence, ϑ^n_X is an $\mathbb{A}^1$-weak equivalence if and only if for every $\mathbb{A}^1$-local simplicial presheaf the induced morphism

$$(\vartheta^n_X)^* \colon \mathrm{Map}\,(\mathrm{Sym}^n_g h_X, \mathscr{Z}) \longrightarrow \mathrm{Map}\,(\mathrm{Sym}^n h_X, \mathscr{Z})$$

is a weak equivalence of simplicial sets. On one side, we have

$$\mathrm{Map}\,(\mathrm{Sym}^n_g h_X, \mathscr{Z}) = \mathrm{Map}\,(h_{\mathrm{Sym}^n X}, \mathscr{Z}) \simeq \mathscr{Z}(\mathrm{Sym}^n X)\,,$$

where the above isomorphism follows from the Yoneda's lemma. On the other hand, the functor $\mathrm{Map}\,(-, \mathscr{Z})$ sends colimits to limits, in particular, we have

$$\mathrm{Map}\,((h_X^{\times n})/\Sigma_n, \mathscr{Z}) \simeq \mathrm{Map}\,(h_X^{\times n}, \mathscr{Z})^{\Sigma_n}\,.$$

Then, we have

$$\mathrm{Map}\,(\mathrm{Sym}^n h_X, \mathscr{Z}) \simeq \mathrm{Map}\,(h_X^{\times n}, \mathscr{Z})^{\Sigma_n} \simeq \mathrm{Map}\,(h_{X^n}, \mathscr{Z})^{\Sigma_n} \simeq \mathscr{Z}(X^n)^{\Sigma_n}\,.$$

Thus, the lemma follows. □

Proposition 4.1.12. *Let $X = \mathbb{A}^2$ be the 2-dimensional affine space over a field k. Then, the natural morphism ϑ^n_X is not an $\mathbb{A}^1$-weak equivalence.*

Proof. We recall that Chow groups $CH^i(-)$, for $i \in \mathbb{N}$, are $\mathbb{A}^1$-homotopy invariant (see [9]). Then $CH^i(-)$ is $\mathbb{A}^1$-local as a constant simplicial presheaf. We take $\mathscr{Z} = CH^1(-)$ in the previous lemma. On one side, we have $X^2 = \mathbb{A}^4$, hence $CH^1(X^2) = CH^1(\mathbb{A}^4)$ is zero, see [9, p. 23]. On the other hand, $\mathrm{Sym}^2(\mathbb{A}^2)$ is isomorphic to the product of $\mathbb{A}^2$ with the quadric cone $\mathcal{Q}$ defined by the equation $uw - v^2 = 0$ in $\mathbb{A}^3$. By the $\mathbb{A}^1$-homotopy invariance, $CH^1(\mathbb{A}^2 \times \mathcal{Q})$ is isomorphic to $CH^1(\mathcal{Q})$. By Example 2.1.3 of [9], $CH^1(\mathcal{Q}) = CH_1(\mathcal{Q})$ it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Then $(\vartheta^2_{\mathbb{A}^2})^*$ is the morphism of constant simplicial sets induced by a morphism of sets $\mathbb{Z}/2\mathbb{Z} \to 0$. Since $\mathbb{Z}/2\mathbb{Z}$ consists of two points, the morphism $(\vartheta^2_{\mathbb{A}^2})^*$ cannot be a weak equivalence. We conclude that $\vartheta^2_{\mathbb{A}^2}$ is not an isomorphism in the motivic $\mathbb{A}^1$-homotopy category. □

4.2 Geometric symmetric powers in the stable set-up

The main result in this section is Theorem 4.2.9 which says that geometric symmetric powers induce a λ-structure on the stable motivic homotopy category, under the assumption of the existence of their left derived functors.

We set $I_{T,\mathrm{proj}} := \bigcup_{n\geq 0} F_n(I^+_{\mathrm{proj}})$, where I^+_{proj} is the set of morphisms defined in page 106. Similarly, we define a set $I_{T',\mathrm{proj}}$, but in this case F_n is seen as a functor from $\Delta^{\mathrm{op}}\mathscr{S}_*$ to $\mathrm{Spt}_{T'}(k)$.

Our next goal is to study Künneth towers associated to relative $I_{T,\mathrm{proj}}$-cell complexes, see Proposition 4.2.3.

Lemma 4.2.1. *One has the following assertions:*

(*a*) *A morphism of representable T'-spectra is isomorphic to the image of a morphism of $\mathbb{P}^1_+$-spectra through the functor H'.*

(*b*) *Let*

$$\begin{array}{ccc} \mathscr{A} & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathscr{B} & \longrightarrow & \mathscr{Y} \end{array} \tag{4.12}$$

be a cocartesian square of T'-spectra, such that the morphism $\mathscr{A} \to \mathscr{B}$ is the image of a level-termwise coprojection in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$ through the functor H'. Then, if $\mathscr{X}$ is a representable T'-spectrum, then so is $\mathscr{Y}$.

(*c*) *Consider the diagram (4.12). Suppose that $\mathscr{A}$ and $\mathscr{B}$ are compact objects. If $\mathscr{X}$ is in $\mathrm{Spt}_{T'}(\Delta^{\mathrm{op}}\mathscr{C})^{\#}$, then so is $\mathscr{Y}$. Moreover, if $\mathscr{X}$ is a directed colimit of representable T'-spectra that are compact, then so is $\mathscr{Y}$.*

Proof. (a). It is a termwise verification.

(b). Let us write $\mathscr{A} = H'(A)$, $\mathscr{B} = H'(B)$ and $\mathscr{X} = H'(X)$, where A, B and X are objects of $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$. Suppose that $\mathscr{A} \to \mathscr{B}$ is a morphism of the form $H'(\varphi)$, where $\varphi\colon A \to B$ is a level-termwise coprojection in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$. By item (a), the morphism $\mathscr{A} \to \mathscr{X}$ is canonically isomorphic to a morphism of the form $H'(\psi)$, where $\psi\colon A \to X$ is a morphism in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$. Since φ is a level-termwise coprojection, there exists an object Y in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$ such that there is a cocartesian square

$$\begin{array}{ccc} A & \xrightarrow{\psi} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

Hence, $\mathscr{Y}$ is isomorphic to $H'(Y)$. This proves (b).

(c). It is immediate from item (b) and the fact that finite colimits of compact objects are compact. □

Lemma 4.2.2. *Every $I_{T,\mathrm{proj}}$-cell complex of* $\mathrm{Spt}_T(k)$ *is the colimit of a directed diagram of the form $\{\mathscr{X}_d\}_{d\in D}$ such that, for $d \leq d'$ in D, the corresponding morphism from $\mathscr{X}_d$ to $\mathscr{X}_{d'}$ is a level-termwise coprojection of compact representable T-spectra. Every $I_{T,\mathrm{proj}}$-cell complex of* $\mathrm{Spt}_T(k)$ *is in* $\mathrm{Spt}_T(\Delta^{\mathrm{op}}\mathscr{C})^{\#}$.

Proof. We reduce the problem in showing that every $I_{T',\mathrm{proj}}$-cell complex of $\mathrm{Spt}_{T'}(k)$ is in $\mathrm{Spt}_{T'}(\Delta^{\mathrm{op}}\mathscr{C})^{\#}$. Since an element of $I_{T',\mathrm{proj}}$-cell is a transfinite composition of pushouts of element of $I_{T',\mathrm{proj}}$, this follows by transfinite induction in view of Lemma 4.2.1 and the fact that the domain and codomain of the elements of $I_{T',\mathrm{proj}}$ are compact. □

Proposition 4.2.3. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism in $I_{T,\mathrm{proj}}$-*cell*, where $\mathscr{X}$ is an $I_{T,\mathrm{proj}}$-cell complex. Then, for each $n \in \mathbb{N}$, $\mathrm{Sym}^n_{g,T}(f)$ has a functorial Künneth tower.*

Proof. By virtue of Lemma 4.2.2, one deduces that the morphism f can be expressed as the colimit of a directed diagram $\{f_d\}_{d\in D}$ of level-termwise coprojections of representable T-spectra. Hence, by Proposition 3.3.20, the nth fold geometric symmetric power $\mathrm{Sym}^n_{g,T}(f_d)$ has a canonical Künneth tower

$$\mathscr{L}^n_0(f_d) \longrightarrow \mathscr{L}^n_1(f_d) \longrightarrow \cdots \longrightarrow \mathscr{L}^n_n(f_d)\ . \tag{4.13}$$

For each index $0 \leq i \leq n$, we define

$$\mathscr{L}^n_i(f) := \mathrm{colim}_{\,d\in D}\mathscr{L}^n_i(f_d)\,.$$

Then, we get a sequence

$$\mathscr{L}^n_0(f) \longrightarrow \mathscr{L}^n_1(f) \longrightarrow \cdots \longrightarrow \mathscr{L}^n_n(f)\,. \tag{4.14}$$

which is a Künneth tower of $\mathrm{Sym}^n_{g,T}(f)$. □

Lemma 4.2.4. *The set $I_{T,\mathrm{proj}}$ permits the small object argument.*

Proof. Notice that one has to prove that for every pair $(n,m) \in \mathbb{N}^2$ and every object U of $\mathscr{C}$, the object $F_m(\partial\Delta_U[n]_+)$ is compact relative to $I_{T,\mathrm{proj}}$, see [17] for the definition of a compact relative object. Since the category $\Delta^{\mathrm{op}}\mathscr{S}_*$ is a cellular model category with respect to the projective-local model structure (Theorem 2.1.12) having $I^+_{T,\mathrm{proj}}$ as its set of generating cofibrations, we can follow the arguments of the proof of Proposition A.8 in [19]. □

Corollary 4.2.5. *There exist a functorial factorization (α, β) on $\mathrm{Spt}_T(k)$ such that for every morphism f is factored as $f = \beta(f) \circ \alpha(f)$, where $\alpha(f)$ is in $I_{T,\mathrm{proj}}$-cell and $\beta(f)$ is in $I_{T,\mathrm{proj}}$-inj.*

Proof. It is a consequence of Lemma 4.2.4. □

Proposition 4.2.6. *Every cofibre sequence in $\mathcal{SH}_T(k)$ is isomorphic to a cofibre sequence of the form*

$$\mathscr{A} \to \mathscr{B} \to \mathscr{B}/\mathscr{A}\,,$$

where $\mathscr{A} \to \mathscr{B}$ is in $I_{T,\mathrm{proj}}$-cell and $\mathscr{A}$ is an $I_{T,\mathrm{proj}}$-cell complex.

Proof. Let $\mathscr{X} \to \mathscr{Y} \to \mathscr{Z}$ be a cofibre sequence in $\mathcal{SH}_T(k)$, where f is a projective cofibration from $\mathscr{X}$ to $\mathscr{Y}$ in $\mathrm{Spt}_T(k)$, such that $\mathscr{Z} = \mathscr{Y}/\mathscr{X}$. By Corollary 4.2.5, the morphism $* \to \mathscr{X}$ factors into $* \to \mathscr{A} \to \mathscr{X}$. Again, by Corollary 4.2.5, the composition of $\mathscr{A} \to \mathscr{X}$ with f induces a commutative diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\alpha(f)} & \mathscr{B} \\ \downarrow & & \downarrow{\scriptstyle \beta(f)} \\ \mathscr{X} & \xrightarrow[f]{} & \mathscr{Y} \end{array}$$

where $\beta(f)$ is a sectionwise trivial fibration and $\alpha(f)$ is in $I_{T,\mathrm{proj}}$-cell. By [18, Prop. 6.2.5], the cofibre sequence $\mathscr{A} \to \mathscr{B} \to \mathscr{B}/\mathscr{A}$ is isomorphic to the cofibre sequence $\mathscr{X} \stackrel{[f]}{\to} \mathscr{Y} \to \mathscr{Z}$ in $\mathcal{SH}_T(k)$.

□

Lemma 4.2.7. *For any T-spectrum $\mathscr{X}$, there is an isomorphism*

$$\operatorname{colim}_{H(U)\to\mathscr{X}} H(U) \simeq \mathscr{X}\,.$$

Proof. Notice that for a symmetric $\mathbb{P}^1_+$-spectrum U, we have that $\mathrm{Ev}_n(H(U))$ coincides with $\Delta^{\mathrm{op}} h^+_{U_n}$. By virtue of Lemma 3.3.3, we get canonical isomorphisms

$$\mathrm{Ev}_n\Big(\operatorname{colim}_{H(U)\to \mathscr{X}} H(U)\Big) = \operatorname{colim}_{H(U)\to \mathscr{X}} \Delta^{\mathrm{op}} h^+_{U_n} \simeq \operatorname{colim}_{\Delta^{\mathrm{op}} h^+_V \to \mathscr{X}_n} \Delta^{\mathrm{op}} h^+_V = \mathscr{X}_n\,,$$

which allow us to deduce the expected isomorphism. □

Corollary 4.2.8. *For any T-spectrum $\mathscr{X}$, there is an isomorphism $\mathrm{Sym}^1_{g,T}(\mathscr{X}) \simeq \mathscr{X}$.*

Proof. For $n = 1$, the equalizer of diagram (3.25) is $H(U)$. Hence, we are in the case of Lemma 4.2.7. □

Now, we are ready to state and prove our main theorem.

Theorem 4.2.9. *Suppose that, for every $n \in \mathbb{N}$, the left derived functor $L\mathrm{Sym}^n_{T,g}$ exists on $\mathscr{SH}_T(k)$. Then, the endofunctors $L\mathrm{Sym}^n_{T,g}$, for $n \in \mathbb{N}$, provides a λ-structure on $\mathscr{SH}_T(k)$.*

Proof. We have evidently that $L\mathrm{Sym}^0_{g,T}$ is the constant functor with value $\mathbb{1}$. By Corollary 4.2.8, $L\mathrm{Sym}^1_{g,T}$ is the identity functor on $\mathscr{SH}_T(k)$. Let $\mathscr{X} \to \mathscr{Y} \to \mathscr{Z}$ be a cofibre sequence in $\mathscr{SH}_T(k)$ induced by a cofibration $f\colon \mathscr{X} \to \mathscr{Y}$ in $\mathrm{Spt}_T(k)$. By Proposition 4.2.6, we can assume that f is in $I_{T,\mathrm{proj}}$-cell and $\mathscr{X}$ is an $I_{T,\mathrm{proj}}$-cell complex. Hence, by Proposition 4.2.3, for each index $n \in \mathbb{N}$, $\mathrm{Sym}^n_{g,T}(f)$ has a Künneth tower,

$$\mathrm{Sym}^n_{g,T}(\mathscr{X}) = \mathscr{L}^n_0(f) \to \mathscr{L}^n_1(f) \to \cdots \to \mathscr{L}^n_n(f) = \mathrm{Sym}^n_{g,T}(\mathscr{Y})\,, \tag{4.15}$$

which induces a Künneth tower,

$$L\mathrm{Sym}^n_{g,T}(\mathscr{X}) = L\mathscr{L}^n_0(f) \to L\mathscr{L}^n_1(f) \to \cdots \to L\mathscr{L}^n_n(f) = L\mathrm{Sym}^n_{g,T}(\mathscr{Y})\,,$$

of $L\mathrm{Sym}^n_{g,T}(f)$. The functoriality axiom follows from the functionality of Künneth towers of the form (4.15). □

4.2.1 A morphism of lambda-structures

For a symmetric T-spectrum $\mathscr{X}$, we shall construct a natural morphism $\vartheta^n_{\mathscr{X}}$ from $\mathrm{Sym}^n_T(\mathscr{X})$ to $\mathrm{Sym}^n_{g,T}(\mathscr{X})$. The main result is Theorem 4.2.13.

Proposition 4.2.10. *Let $\mathscr{X}$ be an object in $\mathrm{Spt}_T(k)$ and let $n \in \mathbb{N}$. Then, we have a canonical morphism $\vartheta^n_{\mathscr{X}}\colon \mathrm{Sym}^n_T(\mathscr{X}) \to \mathrm{Sym}^n_{g,T}(\mathscr{X})$.*

Proof. We define $\vartheta^n_{\mathscr{X}}$ to be the colimit of the morphisms $\vartheta^n_{H(U)}$ of Lemma 3.3.2, where $H(U) \to \mathscr{X}$ runs on the objects of the comma category $(H \downarrow \mathscr{X})$. By definition $\mathrm{Sym}^n_{g,T}\mathscr{X} = \mathrm{colim}_{\,H(U)\to\mathscr{X}}\mathrm{Sym}^n_{g,T}H(U)$. It remains to show that there is a canonical isomorphism $\mathrm{Sym}^n_T\mathscr{X} = \mathrm{colim}_{\,H(U)\to\mathscr{X}}\mathrm{Sym}^n_T H(U)$. Notice the Cartesian product of $\Delta^{\mathrm{op}}\mathscr{C}$ induces a Cartesian product on category $(H \downarrow \mathscr{X})$. By Lemma 3.2.11 and Lemma 4.2.7, we deduce an isomorphism $\mathscr{X}^{\wedge n} \simeq \mathrm{colim}_{\,H(U)\to\mathscr{X}} H(U)^{\wedge n}$. By the same argument, we deduce that the product $\mathscr{X} \wedge \mathrm{sym}(T) \wedge \mathscr{X} \wedge \cdots \wedge \mathrm{sym}(T) \wedge \mathscr{X}$, in which the object $\mathscr{X}$ appears n times, is isomorphic to the colimit

$$\mathrm{colim}_{\,H(U)\to\mathscr{X}}\Big(H(U) \wedge \mathrm{sym}(T) \wedge H(U) \wedge \cdots \wedge \mathrm{sym}(T) \wedge H(U)\Big)\,.$$

By change of colimits and by the above considerations, we deduce that the colimit of the diagram

$$\Big(\mathscr{X} \wedge \mathrm{sym}(T) \wedge \mathscr{X} \wedge \cdots \wedge \mathrm{sym}(T) \wedge \mathscr{X}\Big)\,/\Sigma_n \;\overset{\textstyle\rightrightarrows}{\underset{\textstyle\rightrightarrows}{\cdots\cdots}}\; \mathscr{X}^{\wedge n}/\Sigma_n$$

is a double colimit, that is, the colimit of the colimits of diagrams of the form

$$\Big(H(U) \wedge \mathrm{sym}(T) \wedge H(U) \wedge \cdots \wedge \mathrm{sym}(T) \wedge H(U)\Big)\,/\Sigma_n \;\overset{\textstyle\rightrightarrows}{\underset{\textstyle\rightrightarrows}{\cdots\cdots}}\; H(U)^{\wedge n}/\Sigma_n\;,$$

where $H(U) \to \mathscr{X}$ runs on the objects of $(H \downarrow \mathscr{X})$. This implies that $\mathrm{Sym}^n_T\mathscr{X}$ is isomorphic to $\mathrm{colim}_{\,H(U)\to\mathscr{X}}\mathrm{Sym}^n_T H(U)$. □

For each $n \in \mathbb{N}$, we denote by $\vartheta^n \colon \mathrm{Sym}^n_T \to \mathrm{Sym}^n_{g,T}$ the natural transformation defined for every pointed simplicial sheaf $\mathscr{X}$ to be the functorial morphism $\vartheta^n(\mathscr{X}) := \vartheta^n_{\mathscr{X}}$.

Lemma 4.2.11. *Let $\varphi\colon X \to Y$ be a level-termwise coprojection in $\mathrm{Spt}_{\mathbb{P}^1_+}(\Delta^{\mathrm{op}}\mathscr{C}_+)$ and let us write $f := H(\varphi)$. Then, for every pair of numbers $(n,i) \in \mathbb{N}^2$ with $0 \le i \le n$, there exists a canonical morphism*

$$\vartheta^n_i(f)\colon L^n_i(f) \to \mathscr{L}^n_i(f)\,,$$

such that one has a commutative diagram

$$\begin{array}{ccccccccc}
L^n_0(f) & \longrightarrow & L^n_1(f) & \longrightarrow \cdots & & \cdots \longrightarrow & L^n_{n-1}(f) & \longrightarrow & L^n_n(f) \\
\Big\downarrow{\scriptstyle \vartheta^n_0(f)} & & \Big\downarrow{\scriptstyle \vartheta^n_1(f)} & & & & \Big\downarrow{\scriptstyle \vartheta^n_{n-1}(f)} & & \Big\downarrow{\scriptstyle \vartheta^n_n(f)} \\
\mathscr{L}^n_0(f) & \longrightarrow & \mathscr{L}^n_1(f) & \longrightarrow \cdots & & \cdots \longrightarrow & \mathscr{L}^n_{n-1}(f) & \longrightarrow & \mathscr{L}^n_n(f)
\end{array} \tag{4.16}$$

Proof. Let us fix a natural number n. For each index $0 \le i \le n$, $\mathscr{L}^n_i(f)$ is nothing but the object $H(\tilde{\square}^n_i(\varphi))$, see Proposition 3.3.20. Since the functor H is monoidal, $\square^n_i(f)$ is canonically isomorphic to $H(\square^n_i(\varphi))$. Thus, we have a canonical morphism $\square^n_i(f) \to \mathscr{L}^n_i(f)$, and this morphism induces a morphism $\vartheta^n_i(f)\colon L^n_i(f) \to \mathscr{L}^n_i(f)$. Since $\vartheta^n_i(f)$ is constructed canonically, we get a commutative diagram (4.16). □

Proposition 4.2.12. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of T-spectra in $I_{T,\mathrm{proj}}$ such that $\mathscr{X}$ is an $I_{T,\mathrm{proj}}$-cell complex. Then, for every index $0 \le i \le n$, there exists a canonical morphism*

$$\vartheta_i^n(f)\colon L_i^n(f) \to \mathscr{L}_i^n(f)\,,$$

such that one has a commutative diagram

$$\begin{array}{ccccccccc} L_0^n(f) & \longrightarrow & L_1^n(f) & \longrightarrow \cdots & & \cdots \longrightarrow & L_{n-1}^n(f) & \longrightarrow & L_n^n(f) \\ \downarrow{\scriptstyle \vartheta_0^n(f)} & & \downarrow{\scriptstyle \vartheta_1^n(f)} & & & & \downarrow{\scriptstyle \vartheta_{n-1}^n(f)} & & \downarrow{\scriptstyle \vartheta_n^n(f)} \\ \mathscr{L}_0^n(f) & \longrightarrow & \mathscr{L}_1^n(f) & \longrightarrow \cdots & & \cdots \longrightarrow & \mathscr{L}_{n-1}^n(f) & \longrightarrow & \mathscr{L}_n^n(f) \end{array} \tag{4.17}$$

where $\vartheta_0^n(f) = \vartheta_{\mathscr{X}}^n$ and $\vartheta_n^n(f) = \vartheta_{\mathscr{Y}}^n$.

Proof. As in Proposition 4.2.3, the morphism f can be expressed as the colimit of a directed diagram $\{f_d\}_{d\in D}$ of morphisms of representable T-spectra. Let us fix an index $0 \le i \le n$. By Lemma 4.2.11, we have canonical morphisms $\vartheta_i^n(f_d)\colon L_i^n(f_d) \to \mathscr{L}_i^n(f_d)$ for $d \in D$. Hence, taking colimit we get a morphism

$$\operatorname{colim}_{d\in D}\vartheta_i^n(f_d)\colon \operatorname{colim}_{d\in D} L_i^n(f_d) \to \operatorname{colim}_{d\in D}\mathscr{L}_i^n(f_d)\,,$$

This morphism gives a morphism from $L_i^n(f)$ to $\mathscr{L}_i^n(f)$, and we denote it by $\vartheta_i^n(f)$. Finally, the diagrams of the form (4.16) induce a commutative diagram of the form (4.17). □

Theorem 4.2.13. *Suppose that, for every $n \in \mathbb{N}$, the left derived functor of $\mathrm{Sym}_{g,T}^n$ exists on $\mathcal{SH}_T(k)$. Then, the natural transformations $\vartheta^n\colon \mathrm{Sym}_T^n \to \mathrm{Sym}_{g,T}^n$, for $n \in \mathbb{N}$, induce a morphism of λ-structures from the left derived categoric symmetric powers to the left derived geometric powers on $\mathcal{SH}_T(k)$.*

Proof. It follows from Proposition 4.2.12 and Proposition 4.2.6. □

4.3 Comparison of symmetric powers

The main result in this section is Theorem 4.3.20, which asserts that if -1 is a sum of squares, then the categoric, geometric and projector symmetric powers of a quasi-projective scheme are isomorphic in $\mathcal{SH}_T(k)_{\mathbb{Q}}$.

4.3.1 Formalism of transfers

The purpose of this section is to study the notion of transfer of morphisms in a categorical context involving the transfers that appear in topology, in homotopy theory, and in the theory of pure motives and Voevodsky's motives.

In the next paragraphs $(\mathscr{D}, \wedge)$ and $(\mathscr{E}, \otimes)$ will be two symmetric monoidal categories, where $\mathscr{E}$ is an additive category. Let

$$E : (\mathscr{D}, \wedge) \to (\mathscr{E}, \otimes)$$

be a monoidal functor. Let us fix a finite group G and suppose that X is a G-object in $\mathscr{D}$ with a representation $\rho_X \colon G \to \mathrm{Aut}(X)$ of G on X. The functor E induces an homomorphism of groups $\mathrm{Aut}(X) \to \mathrm{Aut}E(X)$. Notice that the composition of this homomorphism with ρ_X gives an homomorphism of groups $G \to \mathrm{Aut}E(X)$, hence G acts on $E(X)$. This homomorphism induces an homomorphism of Abelian groups $\mathbb{Z}[G] \to \mathrm{End}E(X)$.

Definition 4.3.1. The *norm* $\mathrm{Nm}E(X)$ of $E(X)$ is the image of the element $\sum_{g\in G} g$ under this map. Explicitly, it is given by the formula

$$\mathrm{Nm}E(X) = \sum_{g\in G} E(\rho_X(g))\,.$$

Now, suppose that the quotient X/G exists in $\mathscr{D}$ and let $\pi \colon X \to X/G$ be the canonical morphism.

Definition 4.3.2. The *transfer morphism*, or simply, the *transfer* of $E(\pi)$ is a morphism

$$\mathrm{tr}^E(\pi) \colon E(X/G) \to E(X)\,,$$

such that $E(\pi) \circ \mathrm{tr}^E(\pi) = n \cdot \mathrm{id}_{E(X/G)}$ and $\mathrm{tr}^E(\pi) \circ E(\pi) = \mathrm{Nm}E(X)$.

Example 4.3.3. Consider $(\mathscr{D}, \wedge)$ to be the category of quasi-projective schemes over a field k together with the Cartesian product of schemes over k, and consider $(\mathscr{E}, \otimes)$ to be the category of qfh-sheaves together with the Cartesian product of sheaves. For every $n \in \mathbb{N}$ and for every quasi-projective k-scheme X, the canonical morphism from $\mathbb{Z}_{\mathrm{qfh}}(X^n)$ to $\mathbb{Z}_{\mathrm{qfh}}(\mathrm{Sym}^n X)$ has transfer, see Proposition 4.3.11.

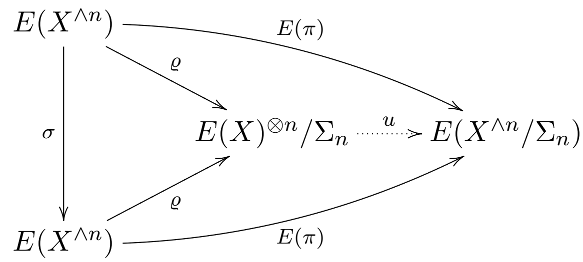
The following example is a consequence of the previous one.

Example 4.3.4. If $(\mathscr{D}, \wedge)$ is the same category as in the previous example, and if $(\mathscr{E}, \otimes)$ is the category of qfh-motives together with the monoidal product of qfh-motives [39], then the canonical morphism of qfh-motives $M_{\mathrm{qfh}}(X^n) \to M_{\mathrm{qfh}}(\mathrm{Sym}^n X)$ has transfer.

Let us study the case when G is the symmetric group Σ_n acting of the nth fold product $X^{\wedge n}$ of an object X of $\mathscr{D}$. Since E is monoidal we have an isomorphism

$$E(X^{\wedge n}) \simeq E(X)^{\otimes n}.$$

Assume that the quotient $E(X)^{\otimes n}/\Sigma_n$ exists in $\mathscr{E}$ and let $\varrho\colon E(X^{\wedge n}) \to E(X)^{\otimes n}/\Sigma_n$ be the composition of the isomorphism $E(X^{\wedge n}) \simeq E(X)^{\otimes n}$ with the canonical morphism $E(X)^{\otimes n} \to E(X)^{\otimes n}/\Sigma_n$. One has a commutative diagram



(4.18)

where the dotted arrow exists by the universal property of quotient by Σ_n. Let us keep these considerations for the proof of Proposition 4.3.5.

A $\mathbb{Q}$-linear category is a category enriched over the category of $\mathbb{Q}$-vector spaces.

Proposition 4.3.5. *Suppose $E\colon (\mathscr{D}, \wedge) \to (\mathscr{E}, \otimes)$ is a monoidal functor of symmetric monoidal categories, where $\mathscr{E}$ is also a $\mathbb{Q}$-linear category. Let X be an object of $\mathscr{D}$, and assume that $X^{\wedge n}/\Sigma_n$ exists in $\mathscr{D}$ and $E(X)^{\otimes n}/\Sigma_n$ exists in $\mathscr{E}$. Let $\pi\colon X^{\wedge n} \to X^{\wedge n}/\Sigma_n$ be the canonical morphism, and suppose that $E(\pi)$ is an epimorphism and has a transfer $\operatorname{tr}^E(\pi)$. Then, the universal morphism*

$$u\colon E(X)^{\otimes n}/\Sigma_n \to E(X^{\wedge n}/\Sigma_n)$$

is an isomorphism.

Proof. Let consider the diagram (4.18). Set $\xi := \varrho \circ \operatorname{tr}^E(\pi)$. We have

$$\begin{aligned} \xi \circ u \circ \varrho &= \varrho \circ \operatorname{tr}^E(\pi) \circ u \circ \varrho \\ &= \varrho \circ \operatorname{tr}^E(\pi) \circ E(\pi) \\ &= \varrho \circ \operatorname{Nm} E(X) \\ &= n! \cdot \varrho \end{aligned} \tag{4.19}$$

Hence, $\xi \circ u \circ \varrho = n! \cdot \varrho$. From the universal property of $E(X)^{\otimes n}/\Sigma_n$, one deduces that ϱ is an epimorphism. This implies the equality $\xi \circ u = n! \cdot \mathrm{id}$. On the other hand, we have

$$\begin{aligned} u \circ \left(\frac{1}{n!} \cdot \xi\right) \circ E(\pi) &= u \circ \Big(\frac{1}{n!} \cdot \varrho \circ \operatorname{tr}^E(\pi)\Big) \circ E(\pi) \\ &= \frac{1}{n!} \cdot \Big(E(\pi) \circ \operatorname{tr}^E(\pi) \circ E(\pi)\Big) \\ &= \frac{1}{n!} \cdot \big(n! \cdot E(\pi)\big) \\ &= E(\pi) \end{aligned} \tag{4.20}$$

It follows that $u \circ (1/n! \cdot \xi) \circ E(\pi) = E(\pi)$. By assumption $E(\pi)$ is an epimorphism. Therefore, we get $u \circ (1/n! \cdot \xi) = \mathrm{id}$ and conclude that u is an isomorphism with inverse $1/n! \cdot \xi$. □

Remark 4.3.6. In the previous proposition it is enough to assume that $\mathscr{E}$ is a $\mathbb{Z}[\frac{1}{n!}]$-linear category.

Projector symmetric powers

Let $(\mathscr{T}, \otimes)$ be a $\mathbb{Q}$-linear symmetric monoidal triangulated category. We fix an object X of $\mathscr{T}$. For a positive integer n, we have a representation $\rho_{X^{\otimes n}} \colon \Sigma_n \to \mathrm{Aut}(X^{\otimes n})$ of Σ_n on $X^{\otimes n}$ induced by permutation of factors. Set

$$d_n := \frac{1}{n!} \cdot \mathrm{Nm}(X^{\otimes n}) = \frac{1}{n!} \cdot \sum_{\sigma \in \Sigma_n} \rho_{X^{\otimes n}}(\sigma)\,.$$

This endomorphism is nothing but that the image of the symmetrization projector $1/n! \cdot \sum_{\sigma \in \Sigma_n} \sigma$ under the induced $\mathbb{Q}$-linear map $\mathbb{Q}[\Sigma_n] \to \mathrm{End}(X^{\otimes n})$. Since the category $\mathscr{T}$ is a $\mathbb{Q}$-linear triangulated category with small coproducts, it is a pseudo-abelian category, see [31]. As d_n is idempotent, i.e. $d_n \circ d_n = d_n$, it splits in $\mathscr{T}$. This implies that p has an image in $\mathscr{T}$.

Definition 4.3.7. We write

$$\mathrm{Sym}^n_{\mathrm{pr}}(X) := \mathrm{im}\, d_n\,,$$

and call it the nth *fold projector symmetric power* of X.

By convention, for $n = 0$, $\mathrm{Sym}^n_{\mathrm{pr}}(X)$ will be the unit object $\mathscr{T}$.

Example 4.3.8. Let $\mathsf{DM}^-(k, \mathbb{Q})$ be the Voevodsky's category with rational coefficients over a field k [27]. A k-rational point of smooth projective curve C induces a decomposition of the motive $M(C)$ into $\mathbb{Q} \oplus M^1(C) \oplus \mathbb{Q}(1)[2]$ in $\mathsf{DM}^-(k, \mathbb{Q})$. The nth fold projector symmetric power $\mathrm{Sym}^n_{\mathrm{pr}}(M^1(C))$ vanishes for n sufficiently bigger that $2g$, where g is the genus of C.

We recall that a stable model category (Definition 1.3.17) is called $\mathbb{Q}$-*linear*, if its homotopy category is a $\mathbb{Q}$-linear triangulated category.

Proposition 4.3.9. *Let $\mathscr{D}$ be a simplicial symmetric monoidal $\mathbb{Q}$-linear stable model category [6]. Then, the projector symmetric powers* $\mathrm{Sym}^n_{\mathrm{pr}}$*, for all $n \in \mathbb{N}$, induce a λ-structure on* $\mathrm{Ho}\,(\mathscr{D})$.

Proof. By convention $\mathrm{Sym}^0_{\mathrm{pr}}$ is the constant endofunctor whose value is the unit object of $\mathrm{Ho}(\mathscr{C})$. From the definition, the endofunctor $\mathrm{Sym}^1_{\mathrm{pr}}$ is the identity on $\mathrm{Ho}(\mathscr{C})$. Let $X \to Y \to Z$ be a cofibre sequence in $\mathrm{Ho}(\mathscr{C})$. By [14, Proposition 15], there exists a sequence

$$\mathrm{Sym}^n_{\mathrm{pr}}(X) = A_0 \to A_1 \to \cdots \to A_n = \mathrm{Sym}^n_{\mathrm{pr}}(Y) \tag{4.21}$$

in $\mathrm{Ho}(\mathscr{C})$, such that for each $0 \le i \le n$, we have

$$\mathrm{cone}(A_{i-1} \to A_i) = \mathrm{Sym}^{n-i}_{\mathrm{pr}}(X) \otimes \mathrm{Sym}^i_{\mathrm{pr}}(Z)\,,$$

where $A_{-1} = 0$. Thus, the Künneth tower axiom is satisfied. The functorial axiom on cofibre sequences follows from the functorial construction of the sequences of the form(4.21), see *loc.cit.*

□

Let τ be a Grothendieck topology on an admissible category $\mathscr{C}$. We denote by

$$\mathbb{Z}_\tau(-)\colon \mathit{Shv}_\tau(\mathscr{C}) \to \mathcal{A}b_\tau(\mathscr{C})$$

the functor which sends a sheaf F in $\mathit{Shv}_\tau(\mathscr{C})$ to the Abelian sheaf $\mathbb{Z}_\tau(F)$ freely generated by F . Denote by

$$\mathbb{Q}_\tau(-)\colon \mathit{Shv}_\tau(\mathscr{C}) \to \mathcal{A}b_\tau(\mathscr{C}) \otimes \mathbb{Q}$$

the composition of the functor $\mathbb{Z}_\tau(-)$ with the canonical functor $\mathcal{A}b_\tau(\mathscr{C}) \to \mathcal{A}b_\tau(\mathscr{C}) \otimes \mathbb{Q}$. Notice that $\mathcal{A}b_\tau(\mathscr{C}) \otimes \mathbb{Q}$ is identified with the category of sheaves of $\mathbb{Q}$-vector spaces. For an object X of $\mathscr{C}$, we shall often write $\mathbb{Z}_\tau(X)$ instead of $\mathbb{Z}_\tau(h_X)$. Similarly, we write $\mathbb{Q}_\tau(X)$ instead of $\mathbb{Q}_\tau(h_X)$.

Lemma 4.3.10 (Voevodsky)**.** *Let X be a quasi-projective k-scheme and let π be the canonical morphism from X^n onto* $\mathrm{Sym}^n(X)$*. Suppose F is a* qfh*-sheaf of Abelian monoids on the category of k-schemes of finite type, and let*

$$\pi^*\colon F(\mathrm{Sym}^n(X)) \to F(X^n)$$

be the restriction morphism induced by π. Then the image of π^* coincides with $F(X^n)^{\Sigma_n}$.

Proof. As the morphism π forms a qfh-covering of $\mathrm{Sym}^n(X)$, we follow the arguments of the proof of [39, Prop. 3.3.2] or [37, Lemma 5.16]. □

Proposition 4.3.11. *Let X be a quasi-projective k-scheme and let $\pi\colon X^n \to \mathrm{Sym}^n(X)$ be the canonical morphism for an integer $n \ge 1$. Then, the induced morphism*

$$\mathbb{Z}_{\mathrm{qfh}}(\pi)\colon \mathbb{Z}_{\mathrm{qfh}}(X^n) \to \mathbb{Z}_{\mathrm{qfh}}(\mathrm{Sym}^n X)$$

has transfer, i.e. there exists a morphism $\mathrm{tr}(\pi)$ *such that*

$$\mathbb{Z}_{\mathrm{qfh}}(\pi)\circ \mathrm{tr}(\pi)=\sum_{\sigma}\mathbb{Z}_{\mathrm{qfh}}(\sigma), \qquad \textit{and} \tag{4.22}$$

$$\mathbb{Z}_{\mathrm{qfh}}(\pi)\circ \mathrm{tr}(\pi)=n!\cdot \mathrm{id}_{\mathbb{Z}_{\mathrm{qfh}}(\mathrm{Sym}^n X)}\,. \tag{4.23}$$

Proof. Let us consider the representable qfh-sheaf $F=\mathbb{Z}_{\mathrm{qfh}}(X^n)$. Every permutation σ in Σ_n induces an automorphism $\sigma\colon X^n\to X^n$ by permuting factors, σ corresponds to an element of $F(X^n)$, denoted by the same letter. Notice that the element $\theta_n:=\sum_{\sigma\in\Sigma_n}\sigma$ is an element of $F(X^n)$ which is Σ_n-invariant, i.e. $\sigma(\theta_n)=\theta$ for all permutation $\sigma\in\Sigma_n$. By Lemma 4.3.10, there exists an element t_n of $F(\mathrm{Sym}^n X)$ such that $t_n\circ\pi^*=\theta$. We denote by $\mathrm{tr}(\pi)\colon \mathbb{Z}_{\mathrm{qfh}}(\mathrm{Sym}^n)\to\mathbb{Z}_{\mathrm{qfh}}(X^n)$ the morphism of qfh-sheaves corresponding to the section t_n. Then the equality $t_n\circ\pi^*=\theta$ gives the equality (4.22). Now, from (4.22), we have

$$\begin{aligned}\mathbb{Z}_{\mathrm{qfh}}(\pi)\circ \mathrm{tr}(\pi)\circ\mathbb{Z}_{\mathrm{qfh}}(\pi)&=\left(\sum_{\sigma}\mathbb{Z}_{\mathrm{qfh}}(\sigma)\right)\circ\mathbb{Z}_{\mathrm{qfh}}(\pi)\\&=\sum_{\sigma}\mathbb{Z}_{\mathrm{qfh}}(\sigma)\circ\mathbb{Z}_{\mathrm{qfh}}(\pi)\\&=\sum_{\sigma}\mathbb{Z}_{\mathrm{qfh}}(\pi)\\&=n!\cdot\mathbb{Z}_{\mathrm{qfh}}(\pi)\,.\end{aligned}$$

hence, $\mathbb{Z}_{\mathrm{qfh}}(\pi)\circ\mathrm{tr}(\pi)\circ\mathbb{Z}_{\mathrm{qfh}}(\pi)=n!\cdot\mathbb{Z}_{\mathrm{qfh}}(\pi)$. This induces the equality (4.23). ☐

Lemma 4.3.12. *For every object X object in an admissible category, we have canonical isomorphisms*

$$\mathbb{Z}_{\mathrm{qfh}}(X)^{\otimes n}/\Sigma_n\simeq\mathbb{Z}(\mathrm{Sym}^n h_X)\,,$$

$$\mathbb{Q}_{\mathrm{qfh}}(X)^{\otimes n}/\Sigma_n\simeq\mathbb{Q}(\mathrm{Sym}^n h_X)\,.$$

Proof. These equalities follow since both $\mathbb{Z}_{\mathrm{qfh}}(-)$ and $\mathbb{Q}_{\mathrm{qfh}}(-)$ are monoidal and left adjoint functors. ☐

Corollary 4.3.13. *Let X be a quasi-projective k-scheme. Then, the canonical morphism* $\mathbb{Q}_{\mathrm{qfh}}(\mathrm{Sym}^n h_X)\to\mathbb{Q}_{\mathrm{qfh}}(\mathrm{Sym}^n_g h_X)$ *is an isomorphism of* qfh*-sheaves of* $\mathbb{Q}$*-vector spaces.*

Proof. Let $\pi\colon X^n\to\mathrm{Sym}^n(X)$ be the canonical morphism. By Proposition 4.3.11, the morphism $\mathbb{Z}_{\mathrm{qfh}}(\pi)\colon\mathbb{Z}_{\mathrm{qfh}}(X)^{\otimes n}\to\mathbb{Z}_{\mathrm{qfh}}(\mathrm{Sym}^n X)$ has transfer, then the morphism $\mathbb{Q}_{\mathrm{qfh}}(\pi)\colon\mathbb{Q}_{\mathrm{qfh}}(X)^{\otimes n}\to\mathbb{Q}_{\mathrm{qfh}}(\mathrm{Sym}^n X)$ has also transfer. Notice that $\mathbb{Q}_{\mathrm{qfh}}(\pi)$ is an epimorphism. Hence, by Proposition 4.3.5, the morphism $\mathbb{Q}_{\mathrm{qfh}}(\pi)$ induces an isomorphism

$$\mathbb{Q}_{\mathrm{qfh}}(X)^{\otimes n}/\Sigma_n\to\mathbb{Q}_{\mathrm{qfh}}(\mathrm{Sym}^n X)\,.$$

Finally, by Lemma 4.3.12, $\mathbb{Q}_{\mathrm{qfh}}(X)^{\otimes n}/\Sigma_n$ is isomorphic to $\mathbb{Q}_{\mathrm{qfh}}(\mathrm{Sym}^n h_X)$, and by definition, $\mathbb{Q}_{\mathrm{qfh}}(\mathrm{Sym}^n X)$ is equal to $\mathbb{Q}_{\mathrm{qfh}}(\mathrm{Sym}^n_g h_X)$. ☐

Corollary 4.3.14. *Let X be a quasi-projective k-scheme. Then the morphism from $\mathbb{Q}_{\mathrm{qfh}}(\mathrm{Sym}^n h_X)$ to $\mathbb{Q}_{\mathrm{qfh}}(\mathrm{Sym}^n_g h_X)$ is an isomorphism in $\mathsf{DM}_{\mathrm{qfh}}(k)_{\mathbb{Q}}$.*

Proof. It follows from Corollary 4.3.13 and [6, Prop. 5.3.37]. □

Let

$$M_{\mathrm{qfh},\mathbb{Q}} \colon \mathscr{S}ch/k \to \mathsf{DM}_{\mathrm{qfh}}(k)_{\mathbb{Q}}$$

be be the canonical functor from the category of k-schemes $\mathscr{S}ch/k$ of finite type to $\mathsf{DM}_{\mathrm{qfh}}(k)_{\mathbb{Q}}$.

Corollary 4.3.15. *Let X be a quasi-projective k-scheme and let $\pi\colon X^n \to \mathrm{Sym}^n(X)$ be the canonical morphism. Then the morphism $M_{\mathrm{qfh},\mathbb{Q}}(\pi)$ has transfer.*

Proof. It follows from Proposition 4.3.11. □

Let $E_{\mathbb{Q}}$ be the canonical functor from the category of k-schemes of finite type to $\mathcal{SH}_T(k)_{\mathbb{Q}}$.

Corollary 4.3.16. *Suppose that -1 is a sum of squares in a field k. For a quasi-projective k-scheme X, the induced morphism $E_{\mathbb{Q}}(\pi)$ from $E_{\mathbb{Q}}(X^n)$ to $E_{\mathbb{Q}}(\mathrm{Sym}^n X)$ has transfer.*

Proof. It follows from Corollary 4.3.15 and Corollary 2.4.3.

□

Proposition 4.3.17. *Assume -1 is a sum of squares in a field k. For a quasi-projective k-scheme X, one has an isomorphism*

$$\mathrm{Sym}^n_{\mathrm{pr}} E_{\mathbb{Q}}(X) \simeq E_{\mathbb{Q}}(\mathrm{Sym}^n X)\,.$$

Proof. By Corollary 4.3.16, the morphism $E_{\mathbb{Q}}(\pi)$ has transfer, say $\mathrm{tr}_{\mathbb{Q}}(\pi)$. From the equality $\mathrm{tr}_{\mathbb{Q}}(\pi) \circ E_{\mathbb{Q}}(\pi) = \mathrm{Nm}(E_{\mathbb{Q}}(X))$, we obtain that the projector d_n is equal to $1/n! \cdot \mathrm{tr}_{\mathbb{Q}}(\pi) \circ E_{\mathbb{Q}}(\pi)$. Hence, from the equality $E(\pi) \circ \mathrm{tr}_{\mathbb{Q}}(\pi) = n!.\mathrm{id}$, we deduce that $\mathrm{im}\, d_n \simeq E_{\mathbb{Q}}(\mathrm{Sym}^n X)$. □

Remark 4.3.18. All the results of this section are also valid in the stable motivic homotopy category with $\mathbb{Z}[\frac{1}{n!}]$-coefficients for a fixed natural number n.

4.3.2 Main theorem

In the next paragraphs we shall prove our main theorem which states that for a k-scheme in $\mathscr{C}$, the canonical morphism from $L\mathrm{Sym}^n_T E_{\mathbb{Q}}(X)$ to $L\mathrm{Sym}^n_{g,T} E_{\mathbb{Q}}(X)$ is an isomorphism in the stable $\mathbb{A}^1$-homotopy category on $\mathscr{C}$. We recall that $\mathcal{SH}_T(k)$ is the stable homotopy category of schemes over a field k constructed in [22].

Proposition 4.3.19. *Suppose that -1 is a sum of squares in k. For every quasi-projective k-scheme X, the canonical morphism*

$$\mathrm{Sym}^n_T(\Sigma^\infty_T X_+) \to \Sigma^\infty_T(\mathrm{Sym}^n X)_+$$

is a stable rational $\mathbb{A}^1$-weak equivalence.

Proof. By Lemma 3.3.6, the morphism $\mathrm{Sym}^n_T(\Sigma^\infty_T X_+) \to \Sigma^\infty_T(\mathrm{Sym}^n X)_+$ is isomorphic to the T-suspension of the canonical morphism $\mathrm{Sym}^n_T(h_{X_+}) \to \mathrm{Sym}^n_{g,T}(h_{X_+})$ of pointed simplicial sheaves. Hence the proposition follows from Corollary 4.3.14 and Corolary 2.4.3. □

Next, we compare the three types of symmetric powers in the stable rational homotopy category of schemes over a field. More precisely, the left derived functors of the categoric, geometric and homotopy symmetric powers of a suspension of a representable sheaf coincide. We recall that $E_{\mathbb{Q}}$ is the canonical functor from the category of k-schemes of finite type to $\mathcal{SH}_T(k)_{\mathbb{Q}}$.

Theorem 4.3.20. *Suppose that -1 is a sum of squares in a field k. For any quasi-projective k-scheme X, we have the following isomorphisms*

$$L\mathrm{Sym}^n_T E_{\mathbb{Q}}(X) \simeq E_{\mathbb{Q}}(\mathrm{Sym}^n X) \simeq \mathrm{Sym}^n_{\mathrm{pr}} E_{\mathbb{Q}}(X)\,.$$

Proof. The isomorphism on the left-hand side follows from Proposition 4.3.19. The second isomorphism follows from Proposition 4.3.17. □

Let us consider the sets $I^+_T = \bigcup_{n>0} F_n(I)$, $J^+_T = \bigcup_{n>0} F_n(J)$, where I (resp. J) is the class of generating (resp. trivial) cofibrations of the injective model structure of $\Delta^{\mathrm{op}}\mathscr{S}_*$. Denote by W^+_T the class of morphisms of symmetric T-spectra $f\colon \mathscr{X} \to \mathscr{Y}$ such that each term $f_n\colon \mathscr{X}_n \to \mathscr{Y}_n$ is an $\mathbb{A}^1$-weak equivalence for $n > 0$. The sets I^+_T, J^+_T and the class W^+_T define on $\mathrm{Spt}_T(k)$ a cofibrantly generated model structure called *positive projective model structure*, see [13]. The positive projective cofibrations are projective cofibrations that are isomorphisms in the level zero.

For a T-spectrum $\mathscr{X}$ in $\mathrm{Spt}_T(k)$, the nth fold *homotopy symmetric power* $\mathrm{Sym}^n_{h,T}(\mathscr{X})$ is defined as the homotopy colimit $\mathrm{hocolim}_{\Sigma_n} \mathscr{X}^{\wedge n}$. The Borel construction allows one to express $\mathrm{Sym}^n_{h,T}(\mathscr{X})$ as the homotopy quotient $(E\Sigma_n)_+ \wedge_{\Sigma_n} \mathscr{X}^{\wedge n}$, where $E\Sigma_n$ is the Σ_n-universal principal bundle, see Definition 1.2.24. The canonical morphism from $(E\Sigma_n)_+ \wedge \mathscr{X}^{\wedge n}$ to $\mathscr{X}^{\wedge n}$ induces a morphism

$$\mathrm{Sym}^n_{h,T}(\mathscr{X}) \to \mathrm{Sym}^n_T(\mathscr{X})\,,$$

which is a stable $\mathbb{A}^1$-weak equivalence when $\mathscr{X}$ is a cofibrant T-spectrum with respect to the positive projective model structure. This implies the existence of an isomorphism of endofunctors

$$\mathrm{Sym}^n_{h,T}(\mathscr{X}) \xrightarrow{\sim} L\mathrm{Sym}^n_T \tag{4.24}$$

on stable $\mathbb{A}^1$-homotopy category $\mathcal{SH}_T(k)$, see [12].

Remark 4.3.21. *By Theorem 4.3.20 and* (4.24)*, we get the following isomorphisms*

$$\mathrm{Sym}^n_{h,T}E_{\mathbb{Q}}(X) \simeq L\mathrm{Sym}^n_T E_{\mathbb{Q}}(X) \simeq E_{\mathbb{Q}}(\mathrm{Sym}^n X) \simeq \mathrm{Sym}^n_{\mathrm{pr}} E_{\mathbb{Q}}(X)$$

for any quasi-projective k-scheme X.

Example 4.3.22. Let X be the 2-dimensional affine space $\mathbb{A}^2$ over k. Then, by Proposition 4.1.12, the canonical morphism $L\vartheta_X \colon L\mathrm{Sym}^n h_X \simeq L\mathrm{Sym}^n_g h_X$ is not an isomorphism in the unstable motivic category over k. However, by Theorem 4.3.20, ϑ_X induces an isomorphism $L\mathrm{Sym}^n_T E_{\mathbb{Q}}(X) \simeq E_{\mathbb{Q}}(\mathrm{Sym}^n_g X)$.

Appendices

Appendix A

Transfers

The notion of transfer appears in several contexts in mathematics. For instance, in topology one has the notion of transfer associated to a finite covering of topological spaces $p : X \to S$, that is, if $p_* : H_*(X, \mathbb{Z}) \to H_*(Y, \mathbb{Z})$ is the corresponding homomorphism of singular homologies, then the transfer of p is a homomorphism $\mathrm{tr}(p) : H_*(Y, \mathbb{Z}) \to H_*(X, \mathbb{Z})$ such that the composition $p_* \circ \mathrm{tr}(p)$ is the multiplication map by the degree of p.

In [39], Voevodsky proves the existence of transfers in the category of qfh-sheaves and in the category of triangulated motives. More precisely, if $p : X \to S$ is a finite morphism of separable degree d, where S is a normal connected scheme, then there is a morphism of sheaves in the qfh-topology, called *transfer*,

$$\mathrm{tr}(p) : \mathbb{Z}_{\mathrm{qfh}}(X) \to \mathbb{Z}_{\mathrm{qfh}}(S)\,,$$

such that $\mathbb{Z}_{\mathrm{qfh}}(p) \circ \mathrm{tr}(p) = d \cdot \mathrm{id}_{\mathbb{Z}_{\mathrm{qfh}}(X)}$. A generalization of this result says that, if $\mathscr{F}$ is any qfh-sheaf and p is the same as before, then there exists a transfer morphism $\mathrm{tr}(p) : \mathscr{F}(X) \to \mathscr{F}(S)$ satisfying the equality $\mathrm{tr}(p) \circ p^* = d \cdot \mathrm{id}_{\mathscr{F}(S)}$, see [37].

qfh-**Topologies**

Definition A.0.23. We recall that a morphism of schemes $p : X \to Y$ is called a *topological epimorphism* if p is surjective and a subset A is Zariski open in Y if and only if $p^{-1}(A)$ is Zariski open in X. A topological epimorphism $p : X \to Y$ is *universal* if for any morphism $Y' \to Y$ the projection $Y' \times_Y X \to Y$ is a topological epimorphism. An *h-covering* of a scheme X is a finite family $\{p_i : X_i \to X\}_{i \in I}$ of morphisms of finite type such that the induced morphism $\amalg_{i \in I} p_i : \coprod_{i \in I} X_i \to X$ is a universal topological epimorphism. A qfh-*covering* of X is a h-covering $\{p_i : X_i \to X\}_{i \in I}$ such that p_i is quasi-finite for all $i \in I$ (see [39]).

Example A.0.24. Let $p : X \to Y$ be a morphism of schemes. The family with one element $\{p : X \to Y\}$ is a qfh-covering of Y for instance if:

(1) p is a surjective proper morphism of finite type, or

(2) Y is the quotient scheme X/G, where G is a finite group acting on X, and $p : X \to Y$ is the canonical morphism.

In the next paragraphs, all qfh-sheaves are defined on the category of schemes of finite type over a field k.

Definition A.0.25. Let X be an integral scheme and let $E/k(X)$ be a field extension. We say that X is *integrally closed* in E, if the local rings of X are integrally closed in E at every point of X.

Proposition A.0.26. *Let X be an integral scheme and let $E/k(X)$ be a finite field extension. Then there exists a scheme X' and a morphism $X' \to X$ with the following universal property: For any dominant morphism $f : Z \to X$, where Z is integrally closed in E, the morphism f factors uniquely through X'.*

Proof. Ones uses gluing of schemes to construct X'. □

Definition A.0.27. The scheme X' in the previous proposition is called *normalization* of X in E.

Lemma A.0.28. *Let $q : X \to S$ be a finite morphism and let G be a finite group acting on X/S. The following statements are equivalent:*

(*a*) *For any point $s \in S$, the action of G on the fibre $q^{-1}(s)$ is transitive. Moreover, for any point $x \in q^{-1}(s)$ the field extension $k(x)/k(s)$ is normal and the natural homomorphism*

$$\mathrm{stab}_G(x) \to \mathrm{Gal}(k(x)/k(s))$$

is surjective.

(*b*) *For any algebraically closed field Ω and for any geometric point $\eta : \mathrm{Spec}(\Omega) \to S$, the action of G on the geometric fibre $X_\eta = X \times_S \mathrm{Spec}(\Omega)$ is transitive.*

Proof. See [37, Lemma 5.1]. □

Pseudo- Galois coverings

For a scheme X/S, we write $\mathrm{Aut}_S(X)$ to denote the group of automorphisms of X over S.

Definition A.0.29. Let $p : X \to S$ be a finite surjective morphism of integral schemes. We say that p is a *pseudo-Galois covering* if its associated field extension $k(X)/k(S)$ is normal and canonical homomorphism of groups

$$\mathrm{Aut}_S(X) \to \mathrm{Gal}(k(X)/k(S))$$

is an isomorphism.

Lemma A.0.30. *If S is an integral scheme and $Y \to S$ is the normalization of S in a finite normal extension of the field $k(S)$, then $Y \to S$ is a pseudo-Galois covering.*

Proof. See [37]. $\square$

Lemma A.0.31. *Let $q : Y \to S$ be a pseudo-Galois covering of an integral normal scheme S, and put $G = \mathrm{Aut}_S(Y)$.*

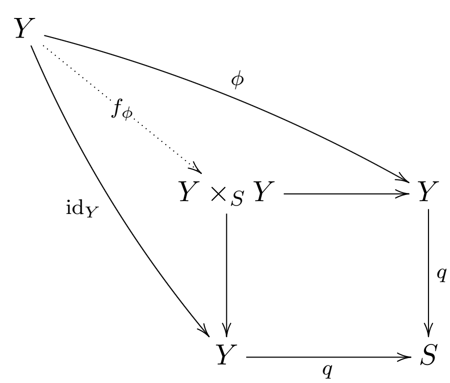
(*a*) *If $\mathscr{F}$ is a* qfh*-sheaf of Abelian groups, then the restriction morphism q^* from $\mathscr{F}(S)$ to $\mathscr{F}(Y)$ induces an isomorphism $\mathscr{F}(S) \xrightarrow{\sim} \mathscr{F}(Y)^G$.*

(*b*) *If $f : \mathscr{F} \to \mathscr{F}'$ is a morphism of* qfh*-sheaves, then we have a commutative diagram*

$$\begin{array}{ccc} \mathscr{F}(S) & \longrightarrow & \mathscr{F}(Y)^G \\ \downarrow & & \downarrow \\ \mathscr{F}'(S) & \longrightarrow & \mathscr{F}'(Y)^G \end{array}$$

where the horizontal arrows are isomorphisms.

Proof. (a). For every $\phi \in G$, we consider the universal morphism $f_\phi : Y \to Y \times_S Y$ coming from the following pullback diagram



Hence the morphisms f_ϕ, for $\phi \in G$, induce a morphism $f : \coprod_{\phi \in G} Y \to Y \times_S Y$. Observe that the hypothesis implies that f is finite and surjective, hence $\{f\}$ a qfh-covering. Since the sheaf $\mathscr{F}$ is, in particular, separated and $\{f\}$ a qfh-covering, the restriction homomorphism

$$f^* : \mathscr{F}(Y \times_S Y) \to \mathscr{F}\left(\coprod_{\phi \in G} Y\right) = \mathscr{F}(Y) \times G$$

is injective. Since $\{q\}$ is a qfh-covering, we have an equalizer diagram

$$\mathscr{F}(S) \xrightarrow{q^*} \mathscr{F}(Y) \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \mathscr{F}(Y \times_S Y) \ .$$

Notice that $\mathscr{F}(Y)^G$ is the equalizer of the diagram

$$\mathscr{F}(Y) \underset{f^*\circ \mathrm{pr}_2^*}{\overset{f^*\circ \mathrm{pr}_1^*}{\rightrightarrows}} \mathscr{F}(Y)\times G \ .$$

On the other hand, as f^* is injective, the $\mathscr{F}(S)$ is also the equalizer of this diagram. Therefore, we have an isomorphism $\mathscr{F}(S) \xrightarrow{\sim} \mathscr{F}(Y)^G$.

(b). It follows from the universal property of equalizer. □

Transfers

Here, we review some results from [37] and [39] on transfers of qfh-sheaves.

Theorem A.0.32. *Let $p : X \to S$ be a finite morphism of separable degree n, where Y is a normal connected scheme and let $\mathscr{F}$ be a* qfh*-sheaf of abelian groups. Then there is a morphism*

$$\mathrm{tr}(p) : \mathscr{F}(X) \to \mathscr{F}(S)\,,$$

such that $\mathrm{tr}(p)\circ p^* = \mathrm{id}_{\mathscr{F}(S)}$.

Proof. We choose a normalization $q : Y \to S$ in a finite normal extension of the field $k(S)$. We set $G = \mathrm{Aut}_S(Y)$. By Lemma A.0.30, $q : Y \to S$ is a pseudo-Galois covering, hence by Lemma A.0.31, restriction morphism $q^* : \mathscr{F}(S) \to \mathscr{F}(X)$ induces an isomorphism $q^* : \mathscr{F}(S) \xrightarrow{\sim} \mathscr{F}(X)^G$. On the other hand, we consider a morphism

$$\sum_{\psi\in \mathrm{Hom}_S(Y,X)} \psi^* : \mathscr{F}(X) \to \mathscr{F}(Y)$$

which will denoted simply by $\sum \psi^*$. Notice that, any $\phi \in G$ induces a bijection

$$\mathrm{Hom}_S(Y,X) \to \mathrm{Hom}_S(Y,X)$$

given by $\psi \mapsto \phi\circ\psi$, then the morphism $\sum\psi^*$ is G-invariant; indeed,

$$\left(\sum_{\psi\in \mathrm{Hom}_S(Y,X)} \psi^*\right)\circ\phi^* = \sum_{\psi\in \mathrm{Hom}_S(Y,X)} (\phi\circ\psi)^* = \sum_{\psi\in \mathrm{Hom}_S(Y,X)} \psi^*$$

for all $\phi\in G$. Hence, the morphism $\sum\psi^* : \mathscr{F}(X) \to \mathscr{F}(Y)$ factors through $\mathscr{F}(Y)^G$. Then we define $\mathrm{tr}(p) : \mathscr{F}(X) \to \mathscr{F}(S)$ to be the composite

$$\mathscr{F}(X) \xrightarrow{\sum\psi^*} \mathscr{F}(Y)^G \xrightarrow{(p^*)^{-1}} \mathscr{F}(S)\,.$$

It remains to verify that $\mathrm{tr}(p)\circ p^* = \mathrm{id}_{\mathscr{F}(S)}$. Notice that it is enough to see that $(\sum\psi^*)\circ p^* = n\cdot p^*$. Indeed, one has the equalities

$$\left(\sum_{\psi\in \mathrm{Hom}_S(Y,X)} \psi^*\right)\circ p^* = \sum_{\psi\in \mathrm{Hom}_S(Y,X)} (p\circ\psi)^* = \sum_{\psi\in \mathrm{Hom}_S(Y,X)} q^* = n\cdot q^*\,,$$

as required.

□

Theorem A.0.33. *Let $p : X \to S$ be a finite morphism of separable degree n, where Y is a normal connected scheme. Then there is a morphism of sheaves in the* qfh*-topology*

$$\operatorname{tr}(p) : \mathbb{Z}_{\mathrm{qfh}}(S) \to \mathbb{Z}_{\mathrm{qfh}}(X)\,,$$

such that $\mathbb{Z}_{\mathrm{qfh}}(p) \circ \operatorname{tr}(p) = n \cdot \mathrm{id}_{\mathbb{Z}_{\mathrm{qfh}}(S)}$.

Proof. First of all, notice that for any $g \in G$, the map $\operatorname{Hom}_S(Y, X) \to \operatorname{Hom}_S(Y, X)$ defined by $\psi \mapsto \psi \circ g$, is bijective, and the element

$$\sum_{\phi \in \operatorname{Hom}_S(Y,X)} \mathbb{Z}_{\mathrm{qfh}}(\phi)$$

of $\mathbb{Z}_{\mathrm{qfh}}(X)(Y)$ is G-invariant. We have

$$\begin{aligned} \mathbb{Z}_{\mathrm{qfh}}(p) \circ \left(\sum_{\phi \in \operatorname{Hom}_S(Y,X)} \mathbb{Z}_{\mathrm{qfh}}(\phi) \right) &= \sum_{\phi \in \operatorname{Hom}_S(Y,X)} \mathbb{Z}_{\mathrm{qfh}}(p \circ \phi) \\ &= \sum_{\phi \in \operatorname{Hom}_S(Y,X)} \mathbb{Z}_{\mathrm{qfh}}(q) \\ &= n \cdot \mathbb{Z}_{\mathrm{qfh}}(q) \end{aligned}$$

Now, let us consider the morphism $\mathbb{Z}_{\mathrm{qfh}}(p) : \mathbb{Z}_{\mathrm{qfh}}(X) \to \mathbb{Z}_{\mathrm{qfh}}(S)$. By Lemma A.0.31(b) applied to the morphism $q : Y \to S$, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_{\mathrm{qfh}}(X)(S) & \longrightarrow & \mathbb{Z}_{\mathrm{qfh}}(X)(Y)^G \\ \Big\downarrow{\scriptstyle \mathbb{Z}_{\mathrm{qfh}}(p)(S)} & & \Big\downarrow{\scriptstyle \mathbb{Z}_{\mathrm{qfh}}(p)(Y)^G} \\ \mathbb{Z}_{\mathrm{qfh}}(S)(S) & \longrightarrow & \mathbb{Z}_{\mathrm{qfh}}(S)(Y)^G \end{array}$$

where the horizontal arrows are isomorphisms by Theorem A.0.32. Notice that $\mathbb{Z}_{\mathrm{qfh}}(p)(Y)$ sends $\sum_{\phi \in \operatorname{Hom}_S(Y,X)} \mathbb{Z}_{\mathrm{qfh}}(\phi)$ to $\mathbb{Z}_{\mathrm{qfh}}(p) \circ \left(\sum_{\phi \in \operatorname{Hom}_S(Y,X)} \mathbb{Z}_{\mathrm{qfh}}(\phi)\right)$ which is equal to $n \cdot \mathbb{Z}_{\mathrm{qfh}}(q)$. By the above commutative diagram we deduce the equality

$$\mathbb{Z}_{\mathrm{qfh}}(p) \circ \operatorname{tr}(p) = n \cdot \mathrm{id}_{\mathbb{Z}_{\mathrm{qfh}}(S)}$$

as required. □

Let $\mathsf{DM}_{\mathrm{qfh}}(S)$ be the category of motives with respect to the qfh-topology and let

$$M_{\mathrm{qfh}} : \mathscr{S}ch/S \to \mathsf{DM}_{\mathrm{qfh}}(S)$$

be the canonical functor.

Corollary A.0.34. *Let $p : Y \to X$ be a finite surjective morphism of normal connected schemes of separable degree $n > 0$. Then there is a morphism*

$$\operatorname{tr}(p) : M_{\mathrm{qfh}}(X) \to M_{\mathrm{qfh}}(Y)$$

such that $M_{\mathrm{qfh}}(p) \circ \operatorname{tr}(p) = n \cdot \mathrm{id}_{M_{\mathrm{qfh}}(X)}$.

Proof. See [39, Proposition 4.1.4]. □

Appendix B

Further research

A fascinating future research project is to investigate what would be an appropriate motivic version of the celebrated Barrat-Priddy-Quillen theorem, see [2]. This idea was suggested by Vladimir Guletskiĭ.

In topology, the Barratt-Priddy-Quillen theorem establishes a weak equivalence

$$B\Sigma_{\infty}^{\wedge} \simeq QS^0 ,$$

where the left hand side is the homotopy completion of the classifying space of the infinite symmetric group Σ_{∞}, and

$$QS^0 = \mathrm{hocolim}_n \Omega^n \Sigma^n S^0$$

is the space representing stable homotopy groups of spheres. It can be also reformulated by saying that QS^0 is homotopy equivalent to $\mathbb{Z} \times B\Sigma_{\infty}^{+}$, where $+$ denotes the Quillen plus construction. If π_n^s is the nth stable homotopy group of spheres, see [15, page 384], then the Barrat-Priddy-Quillen theorem implies an isomorphism,

$$\pi_n(B\Sigma_{\infty}^{\wedge}) \simeq \pi_n^s .$$

On the hand, Schlichtkrull proved in [36] a theorem related to the Barratt-Priddy-Quillen theorem. His result asserts that for any based CW-complex X, there is a chain of homotopy equivalences between the group completion of the infinite homotopy symmetric power $\mathrm{Sym}_h^{\infty}(X)$ and the space $Q(X) = \mathrm{hocolim}_n \Omega^n \Sigma^n X$, see Theorem 1.3 in *loc.cit.*

Now, let us consider the Schlichtkrull's method in the context of the $\mathbb{A}^1$-homotopy theory of schemes. For a pointed motivic space $\mathscr{X}$, let $Q_s(\mathscr{X})$ be the homotopy colimit

$$Q_s(\mathscr{X}) = \mathrm{hocolim}_n \Omega_s^n \Sigma_s^n \mathscr{X} ,$$

where Ω_s and Σ_s are the simplicial loop and suspension functors of motivic spaces, see [30]. We denote by $\mathrm{Sym}_h^{\infty}(\mathscr{X})$ the colimit of nth fold homotopy symmetric powers $\mathrm{Sym}_h^n(\mathscr{X})$ for $n \in \mathbb{N}$. A possible statement of a motivic Barratt-Priddy-Quillen theorem might read as follows:

Let $\mathscr{X}$ be a pointed motivic space. Then the group completion of the infinite symmetric power $\mathrm{Sym}_h^\infty(\mathscr{X})$ is $\mathbb{A}^1$-weak equivalent to the space $Q_s(\mathscr{X})$ in the unstable motivic category of schemes over a field.

Let $B\,\mathrm{Sym}_h^\infty(\mathscr{X})$ be the classifying space of $\mathrm{Sym}_h^\infty(\mathscr{X})$, see [30]. Schlichtkrull's method suggests that the above statement might follow from three independent $\mathbb{A}^1$-weak equivalences of the form:

(A) $\quad \mathrm{Sym}_h^\infty(\Sigma_s\mathscr{X}) \simeq B\,\mathrm{Sym}_h^\infty(\mathscr{X})$,

(B) $\quad \Omega_s\mathrm{Sym}_h^\infty(\Sigma_s\mathscr{X}) \simeq \mathrm{hocolim}_n\Omega_s^n\,\mathrm{Sym}_h^\infty(\Sigma_s^n\mathscr{X})$,

(C) $\quad Q_s(\mathscr{X}) \simeq \mathrm{hocolim}_n\Omega_s^n\,\mathrm{Sym}_h^\infty(\Sigma_s^n\mathscr{X})$.

We leave these questions for a future work.

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