$\mathbb{A}^1$-CONNECTIVITY ON CHOW MONOIDS V.S. RATIONAL EQUIVALENCE OF ALGEBRAIC CYCLES

VLADIMIR GULETSKIĬ

ABSTRACT. Let k be a field of characteristic zero, and let X be a projective variety embedded into a projective space over k. For two natural numbers r and d let $C_{r,d}(X)$ be the Chow scheme parametrizing effective cycles of dimension r and degree d on the variety X. Suppose there exists an effective r-cycle of degree 1 on X. It gives the chain of embeddings of $C_{r,d}(X)$ into $C_{r,d+1}(X)$, whose colimit is the connective Chow monoid $C_r^\infty(X)$ of r-cycles on X. Let $BC_r^\infty(X)$ be the classifying space of this monoid. In the paper we establish an isomorphism between the Chow group $CH_r(X)_0$ of degree 0 dimension r algebraic cycles modulo rational equivalence on X, and the group of sections of the sheaf of $\mathbb{A}^1$-path connected components of the loop space of $BC_r^\infty(X)$ at $\mathrm{Spec}(k)$. Equivalently, $CH_r(X)_0$ is isomorphic to the group of sections of the $S^1 \wedge \mathbb{A}^1$-fundamental group $\Pi_1^{S^1 \wedge \mathbb{A}^1}(BC_r^\infty(X))$ at $\mathrm{Spec}(k)$.

1. INTRODUCTION

Algebraic cycles are linear combinations of closed irreducible subvarieties in algebraic varieties. Two algebraic cycles A and B on the same variety X are said to be rationally equivalent if there exists an algebraic cycle Z on $X \times \mathbb{P}^1$, such that, for two fundamental points 0 and ∞ on $\mathbb{P}^1$, the cycle-theoretic fibres $Z(0)$ and $Z(\infty)$ are A and B respectively. Rational equivalence is a fundamental notion which substantially depends on the intersection multiplicities involved into the definition above. Intersection multiplicities are well controlled in cycles which are cascade intersections of cycles starting from codimension one. This is not always the case, of course. For example, if X is a $K3$-surface, the Chow group of 0-cycles modulo rational equivalence on X is large, in the sense that it cannot be geometrically described by an abelian variety over the ground field, [18]. On the other hand, its subgroup generated by divisorial intersections on X is just $\mathbb{Z}$, see [3]. This example tells us that intersection multiplicities are geometrically manageable only for a small fraction of all algebraic cycles appearing in nature.

Another difficulty with algebraic cycles is that they are originally given in terms of groups, i.e. positive and negative multiplicities can appear in a cycle simultaneously. The use of negative numbers was questionable for mathematicians dealing with algebraic equations in sixteenth century. In modern terms, the concern can be expressed by saying that the completion of a monoid is a too formal construction. The problem might seem to be not that funny when passing to the completions of Chow monoids, i.e. gatherings of Chow varieties parametrizing effective cycles on projective varieties embedded into projective spaces. The Chow monoids themselves are geometrically given in terms of Cayley forms, whereas their completions are less visible.

These two things have an effect that, in contrast to rational connectivity, rational equivalence is difficult to deform in a smooth projective family over a base, cf. [13]. As a consequence, the deep conjectures on rational equivalence are hard to approach, and by now they are solved in a small number of cases (see, for example, [23]). The state of things would be possibly better if we could recode rational equivalence into more effective (i.e. positive) data, appropriate for deformation of rational equivalence in smooth projective families over a base. The purpose of the present paper is to investigate whether the $\mathbb{A}^1$-homotopy type can help in finding such data.

Let X be a projective variety over a field k, and fix an embedding of X into the projective space $\mathbb{P}^m$. To avoid the troubles with representability of Chow sheaves in positive characteristic, we must assume that k is of characteristic zero. Effective algebraic cycles of dimension r and degree d on X, considered with regard to the embedding $X \subset \mathbb{P}^m$, are represented by the Chow scheme $C_{r,d}(X)$ over k. Suppose there exists a dimension r degree 1 cycle Z_0 on X. If $r = 0$ then Z_0 can be a single point, or, if $r = 1$ then Z_0 can be a line on X. The cycle Z_0 gives the chain of embeddings $C_{r,d}(X) \subset C_{r,d+1}(X)$, for all degrees d. Let $C_r^\infty(X)$ bet the colimit of that chain. We shall call $C_r^\infty(X)$ the *connective Chow monoid* of effective r-cycles on X. Let $C_r^\infty(X)^+$ be the completion of $C_r^\infty(X)$ in the category of set-valued simplicial sheaves on the smooth Nisnevich site over k. Let also $\Pi_0^{\mathbb{A}^1}$ be the functor of $\mathbb{A}^1$-connected components and $\Pi_1^{\mathbb{A}^1}$ be the functor of the $\mathbb{A}^1$-fundamental group on simplicial Nisnevich sheaves, see [17] or [1]. Let L be a finitely generated field extension of the ground field k. In the paper we establish a canonical isomorphism

$$CH_r(X_L)_0 \simeq \Pi_0^{\mathbb{A}^1}(C_r^\infty(X)^+)(\mathrm{Spec}(L)) \, ,$$

computing the Chow group $CH_r(X_L)_0$ in terms of L-points of the $\mathbb{A}^1$-connected components of the completed monoid $C_r^\infty(X)^+$ (Theorem 8). Let, furthermore, $BC_r^\infty(X)$ be the (motivic) classifying space of the connective Chow monoid $C_r^\infty(X)$, see page 123 in [17]. Using Lemma 1.2 on pages 123 - 124 in [17], we also prove that

$$CH_r(X_L)_0 \simeq \Pi_0^{\mathbb{A}^1}(\Omega Ex BC_r^\infty(X))(\mathrm{Spec}(L)) \, ,$$

where Ω is right adjoint to the simplicial suspension Σ in the pointed category of simplicial Nisnevich sheaves, and Ex is the simplicial fibrant replacement functor. Another reformulation of the main result is in terms of $S^1 \wedge \mathbb{A}^1$-fundamental groups, where S^1 is the simplicial circle. Namely,

$$CH_r(X_L)_0 \simeq \Pi_1^{S^1 \wedge \mathbb{A}^1}(BC_r^\infty(X))(\mathrm{Spec}(L)) \, ,$$

i.e. the Chow group of r-cycles of degree zero modulo rational equivalence on X is isomorphic to the group of L-points of the $S^1 \wedge \mathbb{A}^1$-fundamental group of the classifying space of the monoid $C_r^\infty(X)$ (Theorem 14). The meaning of the smashing by the circle S^1 is not yet understood and requires further investigation. All the above isomorphisms are canonical up to the choice of the embedding $X \subset \mathbb{P}^m$ and the cycle Z_0.

We think that the use of the second isomorphism is that rational equivalence on r-cycles on X is now fully encoded into $\mathbb{A}^1$-path connectedness on the motivic space $L_{\mathbb{A}^1} \Omega Ex BC_r^\infty(X)$. The localization functor $L_{\mathbb{A}^1}$ is a transfinite machine,

of course. Still, it can be described in terms of section wise fibrant replacement, the Godement resolution, homotopy limit of the corresponding cosimplicial simplicial sheaves and the Suslin-Voevodsky's singularization functor. All these operations are well understood and this gives a hope that the motivic space $L_{\mathbb{A}^1}\Omega ExBC_r^\infty(X)$ is actually quite manageable. The quadruple operation $L_{\mathbb{A}^1}\Omega ExB$ is thus a bigger machine recoding rational equivalence into $\mathbb{A}^1$-path connectivity, with some costs, of course.

The proof of the main result is basically a gathering of facts in $\mathbb{A}^1$-homotopy theory of schemes and Chow sheaves, collected in the right way. The substantial arguments are Lemma 5, which holds in the Nisnevich topology but possibly not in general, and the use of Proposition 6.2.6 from Asok-Morel paper [1]. In Section 2 we introduce the needed tools from homotopy algebra of simplicial sheaves on a small site. Section 3 is devoted to the localization of simplicial sheaves by an interval and the localization of the simplicial functor of connected components Π_0. In Section 4 we pass to Nisnevich sheaves on schemes and prove the needed lemmas on the localization functor $L_{\mathbb{A}^1}$. In Section 5 we recall Chow sheaves, following [22], and construct the Chow monoids. The new results appear in Section 6, where we prove the existence of the above isomorphisms between the Chow groups and the sections of the corresponding $\mathbb{A}^1$ and $S^1 \wedge \mathbb{A}^1$-homotopy groups of $C_r^\infty(X)^+$ and $BC_r^\infty(X)$ respectively. In Appendix we collect the needed basics from homotopical algebra, just to make the text more self-contained and to give more references on technicalities.

2. Π_0 AND MONOIDS IN SIMPLICIAL SHEAVES

Let Δ be the simplex category, i.e. the category whose objects are finite sets $[n] = \{0, 1, \ldots, n\}$, for all $n \in \mathbb{N}$, and morphisms $[m] \to [n]$ are order-preserving functions from $[m]$ to $[n]$. Let $\mathscr{S}$ be a cartesian monoidal category with a terminal object $*$. The category $\Delta^{op}\mathscr{S}$ of simplicial objects in $\mathscr{S}$ is the category of contravariant functors from Δ to $\mathscr{S}$. Since $[0]$ is the terminal object in Δ, the functor $\Gamma : \Delta^{op}\mathscr{S} \to \mathscr{S}$, sending $\mathscr{X}$ to $\mathscr{X}_0$, is the functor of global sections on simplicial objects in $\mathscr{S}$ considered as presheaves on Δ. The functor Γ admits left adjoint $\mathrm{Const} : \mathscr{S} \to \Delta^{op}\mathscr{S}$ sending an object $\mathscr{X}$ in $\mathscr{S}$ to the constant presheaf on Δ determined by $\mathscr{X}$.

Assume, moreover, that $\mathscr{S}$ is cocomplete. For any object $\mathscr{X}$ in $\Delta^{op}\mathscr{S}$, let $\Pi_0(\mathscr{X})$ be the coequalizer of the morphisms $\mathscr{X}_1 \rightrightarrows \mathscr{X}_0$ induced by the two morphisms from $\Delta[0]$ to $\Delta[1]$. This gives a functor $\Pi_0 : \Delta^{op}\mathscr{S} \to \mathscr{S}$ and the canonical epimorphism $\Psi : \Gamma \to \Pi_0$. If $\mathscr{Y}$ is an object in $\mathscr{S}$, and $f : \mathscr{X} \to \mathrm{Const}(\mathscr{Y})$ is a morphism in $\Delta^{op}\mathscr{S}$, the precompositions of $f_0 : \mathscr{X}_0 \to \mathscr{Y}$ with the two morphisms from $\mathscr{X}_1$ to $\mathscr{X}_0$ coincide. By universality of the coequalizer, we obtain the morphism $f' : \Pi_0(\mathscr{X}) \to \mathscr{Y}$. The correspondence $f \mapsto f'$ is one-to-one and natural in $\mathscr{X}$ and $\mathscr{Y}$. In other words, Π_0 is left adjoint to Const. Since products in $\Delta^{op}\mathscr{S}$ are objectwise, the functor Π_0 preserves finite products.

Let $\mathscr{C}$ be an essentially small category and let τ be a subcanonical topology on it. Assume also that $\mathscr{C}$ contains all finite products and let $*$ be the terminal object in it. Let $\mathscr{P}$ be the category of presheaves of sets on $\mathscr{C}$ and let $\mathscr{S}$ be the full subcategory of set valued sheaves on $\mathscr{C}$ in the topology τ. If h is the Yoneda embedding of $\mathscr{C}$ into $\mathscr{S}$, then h_* is the terminal object in $\mathscr{P}$ and $\mathscr{S}$. Limits in $\mathscr{S}$ are limits in $\mathscr{P}$. In particular, we have objectwise finite products in $\mathscr{S}$ and the category $\mathscr{S}$ is Cartesian monoidal.

For a presheaf $\mathscr{X}$, let $\mathscr{X}^{\mathrm{a}}$ be the sheaf associated to $\mathscr{X}$ in τ. Since $\mathscr{P}$ is complete, the sheafification of colimits in $\mathscr{P}$ shows that $\mathscr{S}$ is cocomplete too. In order to make a difference between Π_0 in $\Delta^{op}\mathscr{S}$ and Π_0 in $\Delta^{op}\mathscr{P}$, we shall denote the latter functor by π_0, so that, for a simplicial sheaf $\mathscr{X}$, one has $\Pi_0(\mathscr{X}) = \pi_0(\mathscr{X})^{\mathrm{a}}$. As the coequalizer π_0 is sectionwise, $\Pi_0(\mathscr{X})$ is the sheafififcation of the presheaf sending U to $\pi_0(\mathscr{X}(U))$.

Let $\Delta^{op}\mathscr{S}ets$ be the category of simplicial sets. For a natural number n let $\Delta[n]$ be the representable functor $\mathrm{Hom}_\Delta(-,[n])$. For any sheaf $\mathscr{F}$ on $\mathscr{C}$ let $\Delta_{\mathscr{F}}[n]$ be the simplicial sheaf defined by the formula

$$(\Delta_{\mathscr{F}}[n])_m(U) = \mathscr{F}(U) \times \mathrm{Hom}_\Delta([m],[n]) \ ,$$

for any $U \in \mathrm{Ob}(\mathscr{C})$ and any natural number m. This gives the full and faithful embeddings $\Delta_?[n] : \mathscr{S} \to \Delta^{op}\mathscr{S}$ and $\Delta_{\mathscr{F}}[?] : \Delta \to \Delta^{op}\mathscr{S}$. If $\mathscr{F}$ is h_X, for some object X in $\mathscr{C}$, then we write $\Delta_X[n]$ instead of $\Delta_{\mathscr{F}}[n]$, and use $\Delta[n]$ instead of $\Delta_*[n]$. To simplify notation further, we shall identify $\mathscr{C}$ with its image in $\Delta^{op}\mathscr{S}$ under the embedding $\Delta_?[0] = \mathrm{Const}$. For example, $X = \Delta_X[0] = \mathrm{Const}(h_X)$, and the same on morphisms in $\mathscr{C}$. The cosimplicial object $\Delta[?] : \Delta \to \Delta^{op}\mathscr{S}$ determines the embedding of simplicial sets into $\Delta^{op}\mathscr{S}$, so that we may also identify $\Delta^{op}\mathscr{S}ets$ with its image in $\Delta^{op}\mathscr{S}$. This gives the structure of a simplicial category on $\Delta^{op}\mathscr{S}$, such that, for any two simplicial sheaves $\mathscr{X}$ and $\mathscr{Y}$,

$$\mathbf{Hom}(\mathscr{X},\mathscr{Y}) = \mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathscr{X} \times \Delta[?],\mathscr{Y}) \ .$$

The corresponding (right) action of $\Delta^{op}\mathscr{S}ets$ on $\Delta^{op}\mathscr{S}$ is given by the formula

$$(\mathscr{X} \times K)_n(U) = \mathscr{X}_n(U) \times K_n \ ,$$

for any simplicial sheaf $\mathscr{X}$ and simplicial set K. For simplicity of notation, we shall write $\Delta[n]$ instead of $\Delta[n]$. Then $\Delta_X[n]$ is the product of $\Delta_X[0]$ and $\Delta[n]$.

Looking at $\Delta^{op}\mathscr{S}$ as a symmetric monoidal category with regard to the categorical product in it, one sees that it is closed symmetric monoidal. The internal Hom, bringing right adjoint to the Cartesian products, is given by the formula

$$\mathscr{H}om(\mathscr{X},\mathscr{Y})_n(U) = \mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathscr{X} \times \Delta_U[n],\mathscr{Y}) \ .$$

Throughout the paper we will be working with monoids in $\Delta^{op}\mathscr{S}$. All monoids and groups will be commutative by default. If $\mathscr{X}$ is a monoid in $\Delta^{op}\mathscr{S}$, let $\mathscr{X}^+$ be the completion of $\mathscr{X}$ in $\mathscr{S}$, i.e. the sheaf associated with the completion of $\mathscr{X}$ in the category $\Delta^{op}\mathscr{P}$. The latter is sectionwise and termwise completion and, for simplicity of notation, will be denoted by the same symbol $\mathscr{X}^+$. One has a morphism from $\mathscr{X} \times \mathscr{X}$ to $\mathscr{X}^+$, which is an epimorphism in $\Delta^{op}\mathscr{S}$.

Monoids form a subcategory in $\mathscr{P}$. The corresponding forgetful functor has left adjoint sending presheaves to free monoids with concatenation as monoidal operation. The notion of a cancellation monoid in $\mathscr{P}$ is standard and sectionwise. A free monoid in $\mathscr{P}$ is a cancellation monoid. As limits and colimits in $\Delta^{op}\mathscr{S}$ are

termwise, the functors Γ and Const preserve monoids and groups and $\Gamma(\mathscr{X}^+)$ is the same as $\Gamma(\mathscr{X})^+$. Since Π_0 commutes with finite products, it follows that Π_0 also preserves monoids and groups.

The monoid of natural numbers $\mathbb{N}$ is a simplicial sheaf on $\mathscr{C}$. A pointed monoid in $\Delta^{op}\mathscr{S}$ is a pair $(\mathscr{X}, \iota)$, where $\mathscr{X}$ is a monoid in $\Delta^{op}\mathscr{S}$ and ι is a morphism of monoids from $\mathbb{N}$ to $\mathscr{X}$. A graded pointed monoid is a triple $(\mathscr{X}, \iota, \sigma)$, where $(\mathscr{X}, \iota)$ is a pointed monoid and σ is a morphism of monoids from $\mathscr{X}$ to $\mathbb{N}$, such that $\sigma \circ \iota = \mathrm{id}_{\mathbb{N}}$, see page 126 in [17].

Let $(\mathscr{X}, \iota, \sigma)$ be a pointed graded monoid in $\Delta^{op}\mathscr{S}$. Since $\sigma \circ \iota = \mathrm{id}_{\mathbb{N}}$, it follows that, for any natural n and any object U in $\mathscr{C}$, we have two maps $\iota_{U,n} : \mathbb{N} \to \mathscr{X}_n(U)$ and $\sigma_{U,n} : \mathscr{X}_n(U) \to \mathbb{N}$. It implies that $\mathscr{X}_n(U)$ is the coproduct of the sets $\sigma_{U,n}^{-1}(d)$, for all $d \geq 0$. The sets $\sigma_{U,n}^{-1}(d)$ give rise to the simplicial sheaf which we denote by $\mathscr{X}^d$. Then $\mathscr{X}$ is the coproduct of $\mathscr{X}^d$ for all $d \geq 0$. The addition of $\iota(1)$ in $\mathscr{X}$ induces morphisms of simplicial sheaves $\mathscr{X}^d \to \mathscr{X}^{d+1}$ for all $d \geq 0$. Let $\mathscr{X}^\infty$ be the colimit

$$\mathscr{X}^\infty = \mathrm{colim}\,(\mathscr{X}^0 \to \mathscr{X}^1 \to \mathscr{X}^2 \to \dots)$$

in $\Delta^{op}\mathscr{S}$.

Since now we shall assume that the site $\mathscr{S}$ has enough points, and the category $\mathscr{C}$ is Noetherian. Since filtered colimits commute with finite products, $\mathscr{X}^\infty$ is the colimit taken in the category of simplicial presheaves, i.e. there is no need to take its sheafification. The commutativity of filtered colimits with finite products also yields the canonical isomorphism between the colimit of the obvious diagram composed by the objects $\mathscr{X}^d \times \mathscr{X}^{d'}$, for all $d, d' \geq 0$, and the product $\mathscr{X}^\infty \times \mathscr{X}^\infty$. Since the colimit of that diagram is the colimit of its diagonal, this gives the canonical morphism from $\mathscr{X}^\infty \times \mathscr{X}^\infty$ to $\mathscr{X}^\infty$. The latter defines the structure of a monoid on $\mathscr{X}^\infty$, such that the canonical morphism

$$\pi : \mathscr{X} = \coprod_{d \geq 0} \mathscr{X}^d \to \mathscr{X}^\infty$$

is a homomorphism of monoids in $\Delta^{op}\mathscr{S}$. We call $\mathscr{X}^\infty$ the *connective* monoid associated to the pointed graded monoid $\mathscr{X}$.

Notice that the category of simplicial sheaves is exhaustive. In particular, if all the morphisms $\mathscr{X}^d \to \mathscr{X}^{d+1}$ are monomorphisms, the transfinite compositions $\mathscr{X}^d \to \mathscr{X}^\infty$ are monomorphisms too. This happens if $\mathscr{X}$ is a termwise sectionwise cancelation monoid, in which case $\mathscr{X}^\infty$ is a termwise sectionwise cancelation monoid too.

The above homomorphisms π and σ give the homomorphism (π, σ) from $\mathscr{X}$ to $\mathscr{X}^\infty \times \mathbb{N}$. Passing to completions we obtain the homomorphism (π^+, σ^+) from $\mathscr{X}^+$ to $(\mathscr{X}^\infty)^+ \times \mathbb{Z}$.

Lemma 1. *Assume $\mathscr{X}$ is a sectionwise cancelation monoid. Then*

$$(\pi^+, \sigma^+) : \mathscr{X}^+ \to (\mathscr{X}^\infty)^+ \times \mathbb{Z}$$

is an isomorphism.

Proof. Since the site $\mathscr{S}$ has enough points, it suffices to prove the lemma sectionwise and termwise. Then, without loss of generality, we may assume that $\mathscr{X}$ is a set-theoretical pointed graded cancelation monoid. Clearly, ι^+ is an injection,

π^+ is a surjection, and $\pi^+\iota^+ = 0$. Since $\mathscr{X}$ is a cancelation monoid, $\mathscr{X}^+$ is the quotient-set of the set $\mathscr{X} \times \mathscr{X}$ modulo an equivalence relation

$$(x_1, x_2) \sim (x_1', x_2') \Leftrightarrow x_1 + x_2' = x_2 + x_1' \ .$$

For any element (x_1, x_2) in $\mathscr{X} \times \mathscr{X}$ let $[x_1, x_2]$ be the corresponding equivalence class. Since $\mathscr{X}$ is a cancelation monoid, so is the monoid $\mathscr{X}^\infty$ too. If $\pi^+[x_1, x_2]$ is zero, that is $[\pi(x_1), \pi(x_2)] = [0, 0]$ in $(\mathscr{X}^\infty)^+$, it is equivalent to say that $\pi(x_1) = \pi(x_2)$. The latter equality means that there exists a positive integer n, such that $x_2 = x_1 + n\iota(1)$, i.e. $[x_1, x_2] = [0, n\iota(1)]$ in $\mathscr{X}^+$. The element $[0, n\iota(1)]$ sits in the image of ι^+. □

3. Homotopy completion and localization of Π_0

All the above considerations were categorical. Let us now consider the injective model structures on $\Delta^{op}\mathscr{S}$. A morphism of simplicial sheaves $f : \mathscr{X} \to \mathscr{Y}$ is a weak equivalence if and only if for any point $P^* : \mathscr{S} \to \mathscr{S}ets$ the induced morphism $\Delta^{op}P^*(f)$ is a weak equivalence of simplicial sets. Cofibrations are monomorphisms, and fibrations are defined by the right lifting property in the standard way. The pair $(\mathscr{S}, \mathscr{M})$ is then a model category of simplicial sheaves on $\mathscr{C}$ in τ. Notice that the model structure $\mathscr{M}$ is left proper, see Remark 1.5 on page 49 in [17]. One can also show that it is cellular. Let H be the homotopy category $Ho(\Delta^{op}\mathscr{S})$ of the category $\Delta^{op}\mathscr{S}$ with regard to $\mathscr{M}$. For any two simplicial sheaves $\mathscr{X}$ and $\mathscr{Y}$ the set of morphisms from $\mathscr{X}$ to $\mathscr{Y}$ in H will be denoted by $[\mathscr{X}, \mathscr{Y}]$.

The simplicial structure on $\Delta^{op}\mathscr{S}$ is compatible with the model one, so that $\mathscr{S}$ is a simplicial model category. Since

$$[\mathscr{X}, \mathscr{Y}] \simeq \pi_0\mathbf{Hom}(\mathscr{X}, \mathscr{Y})$$

and

$$\mathbf{Hom}(\Delta_U[0], \mathscr{X}) \simeq \mathscr{X}(U) \ ,$$

$\Pi_0(\mathscr{X})$ is the sheafifification of the presheaf

$$\pi_0(\mathscr{X}) : U \mapsto \pi_0\mathbf{Hom}(\Delta_U[0], \mathscr{X}) = [\Delta_U[0], \mathscr{X}] = [\mathrm{Const}(h_U), \mathscr{X}]$$

on $\mathscr{C}$ in the topology τ. The multiplication of simplicial sheaves and their morphisms by a simplicial set admits right adjoint, so that it commutes with colimits. In particular, $\Pi_0(\Delta_X[n]) \simeq \Delta_X[0]$.

A pointed simplicial sheaf $(\mathscr{X}, x)$ is a pair consisting of a simplicial sheaf $\mathscr{X}$ and a morphism x from $*$ to $\mathscr{X}$. The definition of a morphism of pointed simplicial sheaves is obvious. Let $\Delta^{op}\mathscr{S}_*$ be the category of pointed simplicial sheaves. The corresponding forgetful functor has the standard left adjoint sending $\mathscr{X}$ to the coproduct $\mathscr{X}_+$ of $\mathscr{X}$ and $*$. The model structure $\mathscr{M}$ induces the corresponding model structure on $\Delta^{op}\mathscr{S}_*$, such that the above adjunction is a Quillen adjunction. Having two pointed simplicial sheaves $(\mathscr{X}, x)$ and $(\mathscr{Y}, y)$, their wedge product $(\mathscr{X}, x) \vee (\mathscr{Y}, y)$ is the coproduct in $\Delta^{op}\mathscr{S}_*$, and the smash product $(\mathscr{X}, x) \wedge (\mathscr{Y}, y)$ is the contraction of the wedge product in $(\mathscr{X} \times \mathscr{Y}, x \times y)$.

Let now S^1 be the simplicial circle $\Delta[1]/\partial\Delta[1]$ pointed by the image of the boundary $\partial\Delta[1]$ in then quotient simplicial set, and let S^1 be its image in $\Delta^{op}\mathscr{S}_*$. Define the simplicial suspension endofunctor Σ on $\Delta^{op}\mathscr{S}_*$ sending $(\mathscr{X}, x)$ to

$S^1 \wedge (\mathscr{X}, x)$. Its left adjoint is the simplicial loop functor Ω sending $(\mathscr{X}, x)$ to $\mathscr{H}om_*(S^1, (\mathscr{X}, x))$, where $\mathscr{H}om_*(-, -)$ is the obvious internal Hom in $\Delta^{op}\mathscr{S}_*$.

Let $\mathscr{X}$ be a monoid in $\Delta^{op}\mathscr{S}$. For any object U in $\mathscr{C}$ and any positive integer n let $N(\mathscr{X}_n(U))$ be the nerve of $\mathscr{X}_n(U)$. Let $B\mathscr{X}$ be the diagonal of the bisimplicial sheaf $\Delta^{op} \times \Delta^{op} \to \mathscr{S}$ sending $[m] \times [n]$ to the sheaf $U \mapsto N(\mathscr{X}_n(U))_m$. Then $(B\mathscr{X})_n$ is $\mathscr{X}_n^{\times n}$ for $n > 0$ and, by convention, $(B\mathscr{X})_0$ is the terminal object $*$ in $\mathscr{S}$, see page 123 in [17]. Just as in topology, there exists a canonical morphism $\mathscr{X} \to \Omega B(\mathscr{X})$, which is a weak equivalence if $\mathscr{X}$ is a group.

Following [17], we say that a monoid $\mathscr{X}$ in $\Delta^{op}\mathscr{S}$ is termwise free if for any natural n the sheaf $\mathscr{X}_n$ is a free monoid in the category of sheaves on $\mathscr{C}$ in τ. This is equivalent to say that $\mathscr{X}_n(U)$ is a free monoid on a set, for all $n \geq 0$ and all objects U in $\mathscr{C}$. The monoid $\mathscr{X}$ is termwise good, if all its terms are good in the sense of Section Q5 in [20]. If $\mathscr{X}$ is termwise free, then it is termwise good.

Let Ex be the fibrant replacement in $\mathscr{M}$. The right derived functor of Ω can be computed by precomposing it with Ex.

Lemma 2. *If $\mathscr{X}$ is termwise good, there is an isomorphism*

$$\mathscr{X}^+ \simeq \Omega Ex B(\mathscr{X})$$

in the homotopy category H.

Proof. If $\mathscr{X}$ is termwise free, the morphism $B(\nu_X)$ from $B(\mathscr{X})$ to $B(\mathscr{X}^+)$ is a weak equivalence in the pre-localized model structure $\mathscr{M}$, see the proof of Lemma 1.2 on page 123 in [17]. The same reasoning as in loc.cit. applies when $\mathscr{X}$ is termwise good. Applying ΩEx to $B(\nu_X)$, we obtain a new weak equivalence between $\Omega Ex B(\mathscr{X})$ and $\Omega Ex B(\mathscr{X}^+)$. Since the latter simplicial sheaf is weak equivalent to $\mathscr{X}^+$, we get a pre-localized, i.e. simplicial, weak equivalence between $\mathscr{X}^+$ and $\Omega Ex B(\mathscr{X})$, as required. $\qquad\square$

Next, let A be an object of $\mathscr{C}$, and let $\mathscr{A}$ be the corresponding constant simplicial sheaf $\Delta_A[0] = \mathrm{Const}(h_A)$ in $\Delta^{op}\mathscr{S}$. As in Appendix below, let

$$S = \{\mathscr{X} \wedge \mathscr{A} \to \mathscr{X} \mid \mathscr{X} \in \mathrm{dom}(I) \cup \mathrm{codom}(I)\}$$

be the set of morphisms induced by the morphism from $\mathscr{A}$ to $*$, where $\mathrm{dom}(I)$ and $\mathrm{codom}(I)$ are the sets of domains and codomains of the generating cofibrations in $\mathscr{M}$ on $\mathscr{S}$. As $\Delta^{op}\mathscr{S}$ is left proper simplicial cellular model category, there exists the left Bousfield localization of $\mathscr{M}$ by S in the sense of Hirschhorne, see [10]. Denote the localized model structure by $\mathscr{M}_A$, and let L_A be the corresponding S-localization functor, which is a fibrant approximation in $\mathscr{M}_A$ on $\Delta^{op}\mathscr{S}$, see Section 4.3 in [10]. Let

$$l : \mathrm{Id}_{\Delta^{op}\mathscr{S}} \to L_A$$

be the corresponding natural transformation. For any simplicial sheaf $\mathscr{X}$ the morphism $l_{\mathscr{X}} : \mathscr{X} \to L_A(\mathscr{X})$ is a weak cofibration and $L_A(\mathscr{X})$ is A-local, i.e. fibrant in $\mathscr{M}_A$. The basics on localization functors see Section 4.3 in Hirschhorn's book [10] and Appendix below.

Let H_A be the homotopy category of simplicial sheaves converting weak equivalences in $\mathscr{M}_S$ into isomorphisms. As simplicial sheaves with respect to $\mathscr{M}$ form a simplicial closed cartesian monoidal model category, so is the category of simplicial sheaves with respect to $\mathscr{M}_A$. All simplicial sheaves are cofibrant, in $\mathscr{M}$ and in $\mathscr{M}_A$. It follows that the canonical functors from simplicial sheaves to H

and H_A are monoidal. See Appendix for more details on all such things. For any two simplicial sheaves $\mathscr{X}$ and $\mathscr{Y}$ let $[\mathscr{X}, \mathscr{Y}]_A$ be the set of morphisms from $\mathscr{X}$ to $\mathscr{Y}$ in H_A. Define the A-localized functor Π_0^A from $\Delta^{op}\mathscr{S}$ to $\mathscr{S}$ by setting $\Pi_0^A(\mathscr{X})$ to be the sheaf associated to the presheaf

$$U \mapsto [\mathrm{Const}(h_U), \mathscr{X}]_A .$$

Then $\Pi_0^A(\mathscr{X})$ is canonically isomorphic to $\Pi_0(L_A(\mathscr{X}))$, and the morphism l induces the epimorphism $\Pi_0 \to \Pi_0^A$, see Corollary 3.22 on page 94 in [17].

Recall that an object I of a category $\mathscr{D}$ with a terminal object $*$ is called an interval if there exists a morphism

$$\mu : I \wedge I \to I$$

and two morphisms $i_0, i_1 : * \rightrightarrows I$, such that

$$\mu \circ (\mathrm{id}_I \wedge i_0) = i_0 \circ p \quad \text{and} \quad \mu \circ (\mathrm{id}_I \wedge i_1) = \mathrm{id}_I ,$$

where p is the unique morphism from I to $*$, and $i_0 \coprod i_1 : * \coprod * \to I$ is a monomorphism in $\mathscr{D}$, see [17]. Certainly, the object A is an interval in $\mathscr{C}$ if and only if the object $\mathscr{A}$ is an interval in $\Delta^{op}\mathscr{S}$. Since now we shall assume that A is an interval in $\mathscr{C}$.

In that case the A-localizing functor L_A can be chosen to be more explicit than the construction given in [10]. Following [17], see page 88, we consider the cosimplicial sheaf

$$\Delta_{A\bullet}[0] : \Delta \to \mathscr{S}$$

sending $[n]$ to the n-product

$$(\Delta_A[0])^n = \Delta_{A^n}[0]$$

and acting on morphisms as follows. For any morphism $f : [m] \to [n]$ define a morphism of sets

$$f' : \{1, \ldots, n\} \to \{0, 1, \ldots, m+1\}$$

setting

$$f'(i) = \begin{cases} \min\{l \in \{0, \ldots, m\} \mid f(l) \geq i\} , & \text{if this set is nonempty} \\ m+1 & \text{otherwise.} \end{cases}$$

If now $\mathrm{pr}_k : A^n \to A$ is the k-th projection and $p : A^n \to *$ the unique morphism to the terminal object, where A^n is the n-fold product of A, then

$$\mathrm{pr}_k \circ \Delta_{A\bullet}[0](f) = \begin{cases} \mathrm{pr}_{f'(k)} , & \text{if } f'(k) \in \{1, \ldots, m\} \\ i_0 \circ p , & \text{if } f'(k) = m+1 \\ i_1 \circ p , & \text{if } f'(k) = 0 . \end{cases}$$

For any $\mathscr{X}$ let $Sing_A(\mathscr{X})$ be the Suslin-Voevodsky simplicial sheaf

$$[n] \mapsto \mathscr{H}om(\Delta_{A^n}[0], \Delta_{\mathscr{X}_n}[0]) .$$

It is functorial in $\mathscr{X}$ and $p : A^n \to *$ induces the morphism

$$s : \mathrm{Id}_{\Delta^{op}\mathscr{S}} \to Sing_A .$$

Each morphism $s_{\mathscr{X}}$ from $\mathscr{X}$ to $Sing_A(\mathscr{X})$ is an A-weak equivalence, i.e. a weak equivalence in $\mathscr{M}_A$.

As it is shown in [17], there exists a sufficiently large ordinal ω, such that L_A can be taken to be the composition

$$L_A = (Ex \circ Sing_A)^\omega \circ Ex .$$

The functor Ex can be chosen to be the composition of the sectionwise fibrant replacement for simplicial sets, the Godement resolution and the homotopy limit, as it is constructed in Section 2.1 in loc.cit. Such defined fibrant replacement will be denoted by $Ex^{\mathcal{G}}$. If the localization functor L_A is constructed in terms of $Ex^{\mathcal{G}}$ and the Suslin-Voevodsky's singularization functor $Sing_A$, for the interval A, then L_A is getting to be quite explicit, and, respectively, this gives clearer picture of what are the functors π_0^A and Π_0^A.

The canonical functor from $\Delta^{op}\mathcal{S}$ to H preserves products. In other words, if $\mathscr{X} \times \mathscr{Y}$ is the product of two simplicial sheaves, the same object $\mathscr{X} \times \mathscr{Y}$, with the homotopy classes of the same projections, is the product of $\mathscr{X}$ and $\mathscr{Y}$ in H and in H_A (see Appendix). The advantage of the above explicit L_A is that it commutes with finite products. Most likely, the general Hirschhorne's construction (see Section 4.3 in [10]) also satisfies this property, but we could not find the proof in the literature. Since now the localization functor L_A will be always the above explicit one.

As L_A is monoidal,

$$
\begin{aligned}
\Pi_0^A(\mathscr{X} \times \mathscr{Y}) &= \Pi_0(L_A(\mathscr{X} \times \mathscr{Y})) \\
&= \Pi_0(L_A(\mathscr{X}) \times L_A(\mathscr{Y})) \\
&= \Pi_0(L_A(\mathscr{X})) \times \Pi_0(L_A(\mathscr{Y})) \\
&= \Pi_0^A(\mathscr{X}) \times \Pi_0^A(\mathscr{Y}) \, .
\end{aligned}
$$

This gives that Π_0^A preserves monoids and groups.

4. Monoids in simplicial Nisnevich sheaves

Now we turn from homotopy algebra to algebraic geometry. Throughout all schemes will be separated by default. Let k be a field, $\mathscr{S}m$ the category of smooth schemes of finite type over k, and let $\mathfrak{N}$ be the category of all noetherian schemes over k, not necessarily of finite type. We are going to specialize the abstract material of the previous sections to the case when $\mathscr{C}$ is $\mathscr{S}m$, the topology τ is the Nisnevich topology on $\mathscr{C}$, and A is the affine line $\mathbb{A}^1$ over k.

The standard Yoneda construction gives the functor h sending any scheme X from $\mathfrak{N}$ to the functor $\mathrm{Hom}_{\mathfrak{N}}(-, X)$, and the same on morphisms. This is a functor to the category of sheaves in étale topology, and so in the Nisnevich one, see [8], page 347, i.e. the Nisnevich topology is subcanonical. Composing h with the constant functor $\mathrm{Const} = \Delta_?[0]$ from $\mathscr{S}$ to $\Delta^{op}\mathscr{S}$ we obtain the embedding of $\mathfrak{N}$ into $\Delta^{op}\mathscr{S}$. We identify the categories $\mathfrak{N}$ and $\Delta^{op}\mathscr{S}ets$ with their images under the corresponding embeddings into $\Delta^{op}\mathscr{S}$.

The scheme $\mathrm{Spec}(k)$ is the terminal object in $\mathscr{C}$. The affine line $\mathbb{A}^1$ over k is an interval in $\Delta^{op}\mathscr{S}$ with two obvious morphisms i_0 and i_1 from $\mathrm{Spec}(k)$ to $\mathbb{A}^1$. As above, the interval $\mathbb{A}^1$ gives the natural cylinder and the corresponding notion of left homotopy on morphisms in $\Delta^{op}\mathscr{S}$. The set of points on $\mathscr{X}$ is the set $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X})$ of k-points on $\mathscr{X}$. The set of $\mathbb{A}^1$-path connected components on k-points is denoted by $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X})_{\mathbb{A}^1}$. If $\mathscr{X}$ is fibrant in $\mathscr{M}_{\mathbb{A}^1}$ then $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X})_{\mathbb{A}^1}$ is isomorphic to the set $[\mathrm{Spec}(k), \mathscr{X}]_{\mathbb{A}^1}$.

Let $\mathscr{X}$ be a monoid in $\Delta^{op}\mathscr{S}$. Its completion $\mathscr{X}^+$ is a group object, so that $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(*, \mathscr{X}^+)$ is a group in $\Delta^{op}\mathscr{S}$. The morphism $\mathscr{X} \to \mathscr{X}^+$ induces a map from $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X})$ to $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X}^+)$. By the universality of

group completion, there exists a unique map from $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X})^+$ to $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X}^+)$ with the obvious commutativity.

Lemma 3. *For a simplicial Nisnevich sheaf monoid $\mathscr{X}$, the canonical map from $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X})^+$ to $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X}^+)$ is bijective, and, repspectively, the map from $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X})_{\mathbb{A}^1}{}^+$ to $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X}^+)_{\mathbb{A}^1}$ is a surjection.*

Proof. Since $\mathrm{Spec}(k)$ is Henselian, the set $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X}^+)$ is the quotient of the Cartesian square $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X})^2$. The set $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X})^+$ is also the quotient of the same Cartesian square. The maps from Hom-sets to the sets of $\mathbb{A}^1$-homotopy classes are surjective. $\square$

The morphism $l_{\mathscr{X}}$ from $\mathscr{X}$ to $L_{\mathbb{A}^1}(\mathscr{X})$ induces the following commutative diagram

$$
\begin{array}{ccc}
\mathscr{X}^+ & \xrightarrow{\;l_{\mathscr{X}}^+\;} & L_{\mathbb{A}^1}(\mathscr{X})^+ \\
{\scriptstyle l_{\mathscr{X}^+}}\Big\downarrow & \swarrow & \Big\downarrow{\scriptstyle l_{L_{\mathbb{A}^1}(\mathscr{X})^+}} \\
L_{\mathbb{A}^1}(\mathscr{X}^+) & \xrightarrow{\;L_{\mathbb{A}^1}(l_{\mathscr{X}}^+)\;} & L_{\mathbb{A}^1}(L_{\mathbb{A}^1}(\mathscr{X})^+)
\end{array}
$$

in the category $\Delta^{op}\mathscr{S}$.

Lemma 4. *For any simplicial sheaf monoid $\mathscr{X}$, the above morphism*

$$L_{\mathbb{A}^1}(\mathscr{X})^+ \to L_{\mathbb{A}^1}(\mathscr{X}^+)$$

is an isomorphism in $\Delta^{op}\mathscr{S}$.

Proof. Since the right vertical morphism is an $\mathbb{A}^1$-weak cofibration, it is a monomorphism. Then the canonical morphism from $L_{\mathbb{A}^1}(\mathscr{X})^+$ to $L_{\mathbb{A}^1}(\mathscr{X}^+)$ is a monomorphism too. It is well known that, if $\mathscr{F}$ is a presheaf on $\mathscr{C}$ and $\mathscr{F}_{\mathrm{Nis}}$ is the Nisnevich sheafification, then $\mathscr{F}_{\mathrm{Nis}}(\mathrm{Spec}(\mathscr{O}_{Y,y}^h))$ is the same as $\mathscr{F}(\mathrm{Spec}(\mathscr{O}_{Y,y}^h))$, where $\mathscr{O}_{Y,y}^h$ is the Henselization of the local ring $\mathscr{O}_{Y,y}$, for any smooth scheme Y over k and a point y on Y. Due to this property, and the fact that $L_{\mathbb{A}^1}$ preserves products, the epimorphicity can be straightforwardly verified on the points of the Nisnevich site $\mathscr{C}$. $\square$

Due to Lemma 4, we may simply write $L_{\mathbb{A}^1}\mathscr{X}^+$ for either of the two canonically isomorphic simplicial sheaves. The next lemma is a motivic version of Lemma 2.3 in [15].

Lemma 5. *For any commutative monoid $\mathscr{X}$ in $\Delta^{op}\mathscr{S}$, canonically,*

$$\Pi_0^{\mathbb{A}^1}(\mathscr{X})^+ \simeq \Pi_0^{\mathbb{A}^1}(\mathscr{X}^+)\,.$$

Proof. Since $\Gamma(\mathscr{X}^+) = \Gamma(\mathscr{X})^+$ and $\Pi_0(\mathscr{X})^+$ are completions, one has the universal morphisms γ from $\Gamma(\mathscr{X}^+)$ to $\Pi_0(\mathscr{X})^+$ and δ from $\Pi_0(\mathscr{X})^+$ to $\Pi_0(\mathscr{X}^+)$. Since $\Gamma(\mathscr{X}) = \mathscr{X}_0$, $\Gamma(\mathscr{X}^+) = \mathscr{X}_0^+$ and $\gamma \circ \Gamma(\nu_{\mathscr{X}}) = \nu_{\Pi_0(\mathscr{X})} \circ \Psi$, where ν stays for the corresponding canonical morphisms from the monoids to their completions, the two compositions $\mathscr{X}_1^+ \rightrightarrows \mathscr{X}_0^+ \xrightarrow{\gamma} \Pi_0(\mathscr{X})^+$ coincide, which gives the universal morphism ε from $\Pi_0(\mathscr{X}^+)$ to $\Pi_0(\mathscr{X})^+$. Since Ψ is an epimorphism, and using the uniqueness of the appropriate universal morphisms, we show that

δ and ε are mutually inverse isomorphisms of groups in $\mathscr{S}$. Since $L_{\mathbb{A}^1}$ preserves completions of monoids by Lemma 4,

$$\begin{aligned}
\Pi_0^{\mathbb{A}^1}(\mathscr{X})^+ &= \Pi_0(L_{\mathbb{A}^1}(\mathscr{X}))^+ \\
&= \Pi_0(L_{\mathbb{A}^1}\mathscr{X}^+) \\
&= \Pi_0^{\mathbb{A}^1}(\mathscr{X}^+) .
\end{aligned}$$

$\square$

Let $f, g : \mathscr{X} \rightrightarrows \mathscr{Y}$ be two morphisms from $\mathscr{X}$ to $\mathscr{Y}$ in $\Delta^{op}\mathscr{S}$. An A-homotopy from f to g is a morphism $H : \mathscr{X} \times \mathscr{A} \to \mathscr{Y}$, such that $H \circ (\mathrm{id}_{\mathscr{X}} \times i_0) = f$ and $H \circ (\mathrm{id}_{\mathscr{X}} \times i_1) = g$. Since all simplicial sheaves are cofibrant in both model structures $\mathscr{M}$ and $\mathscr{M}_A$, the A-path connectedness is an equivalence relation on the set of points $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathscr{X}, \mathscr{Y})$, see [9], Proposition 1.2.5 (iii). Let $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathscr{X}, \mathscr{Y})_A$ be the set of equivalence classes modulo this equivalence relation. A point of a simplicial sheaf $\mathscr{X}$ is, by definition, a morphism from the terminal simplicial sheaf $*$ to $\mathscr{X}$. Two points on $\mathscr{X}$ are said to be A-path connected if and only if they are left homotopic with respect to $\mathscr{A}$.

Lemma 6. *For any simplicial sheaf $\mathscr{X}$ the canonical map*

$$\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X})_{\mathbb{A}^1} \to \mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), L_{\mathbb{A}^1}(\mathscr{X}))_{\mathbb{A}^1}$$

is surjective.

Proof. We know that l induces the epimorphism $\Pi_0 \to \Pi_0^{\mathbb{A}^1}$ by Corollary 3.22 in [17]. The morphism $\Psi : \Gamma \to \Pi_0$ is an epimorphism too. This gives that the map

$$\mathrm{Hom}_{\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X}_0) \to \mathrm{Hom}_{\mathscr{S}}(\mathrm{Spec}(k), \Pi_0(\mathscr{X})) \to \mathrm{Hom}_{\mathscr{S}}(\mathrm{Spec}(k), \Pi_0^A(\mathscr{X}))$$

is surjective. By adjunction, $\mathrm{Hom}_{\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X}_0) \simeq \mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), \mathscr{X})$, and since $L_{\mathbb{A}^1}(\mathscr{X})$ is $\mathbb{A}^1$-local, i.e. fibrant in $\mathscr{M}_{\mathbb{A}^1}$, $\mathrm{Hom}_{\mathscr{S}}(\mathrm{Spec}(k), \Pi_0^{\mathbb{A}^1}(\mathscr{X}))$ is isomorphic to $\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), L_{\mathbb{A}^1}(\mathscr{X}))_{\mathbb{A}^1}$. $\square$

Next, let L be a field extension of the ground field k, and let $\mathscr{S}_L$ be the category of set valued Nisnevich sheaves on the category $\mathscr{C}_L$ of smooth schemes over L. Let $\mathscr{M}_L$ be the injective model structure on $\Delta^{op}\mathscr{S}_L$, obtained in the same way as the model structure $\mathscr{M}$ has been obtained for $\Delta^{op}\mathscr{S}$ over $\mathrm{Spec}(k)$. Let $f : \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ be the morphism induced by the extension $k \subset L$, and let $f^* : \Delta^{op}\mathscr{S} \to \Delta^{op}\mathscr{S}_L$ be the scalar extension functor induced by sending schemes over k to their fibred products with $\mathrm{Spec}(L)$ over $\mathrm{Spec}(k)$, and then using the fact that any sheaf is a colimit of representable ones. As the morphism f is smooth, there are two standard adjunctions

$$f_\# \dashv f^* \dashv f_*$$

for the functor f^*, see, for example, [16].

Lemma 7. *If L is finitely generated over k, one can choose the localization functors $L_{\mathbb{A}^1}$ and $L_{\mathbb{A}^1_L}$ in $\Delta^{op}\mathscr{S}$ and $\Delta^{op}\mathscr{S}_L$ respectively, to have a canonical isomorphism*

$$f^* L_{\mathbb{A}^1} \simeq L_{\mathbb{A}^1_L} f^* .$$

Proof. Let $Ex_L^{\mathscr{G}}$ be the fibrant replacement in $\Delta^{op}\mathscr{S}_L$ obtained in the same way as $Ex^{\mathscr{G}}$ has been constructed for $\Delta^{op}\mathscr{S}$. Let also $Sing_L$ be the Suslin-Voevodsky endofunctor on $\Delta^{op}\mathscr{S}_L$. Straightforward verifications show that $f^* Ex^{\mathscr{G}} \simeq Ex_L^{\mathscr{G}} f^*$

and $f^* \circ Sing \simeq Sing_L \circ f^*$. Choose $L_{\mathbb{A}^1}$ (respectively, $L_{\mathbb{A}^1_L}$) to be the transfinite compositions of $Ex^{\mathscr{G}}$ and $Sing$ (respectively, $Ex^{\mathscr{G}}_L$ and $Sing_L$). $\qquad\square$

5. Connective Chow monoids in Nisnevich sheaves

We need to refresh some material from [22]. For a scheme X let $\mathscr{C}(X)$ be the free abelian monoid generated by points of X, and let $\mathscr{Z}(X)$ be the group completion of $\mathscr{C}(X)$. An algebraic cycle ζ is an element in $\mathscr{Z}(X)$. As such, ζ is a finite linear combination $\sum m_i \zeta_i$ of points ζ_i on X with integral coefficients m_i. The cycle ζ is said to be effective if and only if $m_i \geq 0$ for all i. This is equivalent to say that ζ is an element of $\mathscr{C}(X)$.

The support $\mathrm{Supp}(\zeta)$ of ζ is the union of the Zariski closures of the points ζ_i with the induced reduced structures on them. The correspondence between points on X and the reduced irreducible closed subschemes of X allows to consider algebraic cycles as linear combinations $Z = \sum m_i Z_i$, where Z_i is the Zariski closure of the point ζ_i, for each i. Then $\mathrm{Supp}(Z)$ is the same thing as $\mathrm{Supp}(\zeta)$. The points ζ_i, or the corresponding reduced closed subschemes Z_i, are prime cycles on X. The dimension of a point in X is the dimension of its Zariski closure in X. Let then $\mathscr{C}_r(X)$ be the submonoid in $\mathscr{C}(X)$ generated by points of dimension r, and, respectively, let $\mathscr{Z}_r(X)$ be the subgroup in the abelian group $\mathscr{Z}(X)$ generated by points of dimension r in X.

Let S be a Noetherian scheme, let k be a field, and let

$$P : \mathrm{Spec}(k) \to S$$

be a k-point of S. Recall that a fat point of S over P is an ordered pair (P_0, P_1) of two morphisms of schemes

$$P_0 : \mathrm{Spec}(k) \to \mathrm{Spec}(R) \quad \text{and} \quad P_1 : \mathrm{Spec}(R) \to S ,$$

where R is a discrete valuation ring, such that $P_1 \circ P_0 = P$, the image of P_0 is the closed point of $\mathrm{Spec}(R)$, and P_1 sends the generic point $\mathrm{Spec}(F)$ of $\mathrm{Spec}(R)$ into the generic point η of S.

Let $X \to S$ be a scheme of finite type over S, and let

$$Z \to X$$

be a closed subscheme in X. Let, furthermore, R be a discrete valuation ring, $D = \mathrm{Spec}(R)$, and let

$$f : D \to S$$

be an arbitrary morphism of schemes from D to S. Let also

$$\eta = \mathrm{Spec}(R_{(0)})$$

be the generic point of D,

$$X_D = X \times_S D , \quad Z_D = Z \times_S D \quad \text{and} \quad Z_\eta = Z \times_S \eta .$$

Then there exists a unique closed embedding

$$Z'_D \to Z_D ,$$

such that its pull-back $Z'_\eta \to Z_\eta$, with respect to the morphism $Z_\eta \to Z_D$, is an isomorphism, and the composition

$$Z'_D \to Z_D \to D$$

is a flat morphism of schemes, see Proposition 2.8.5 in [7].

In particular, we have such a "platification" if (P_0, P_1) is a fat point over the k-point P and $f = P_1$.

Let X_P be the fibre of the morphism $X_D \to D$ over the point P_0. Let also

$$Z_P = Z_D \times_{X_D} X_P \quad \text{and} \quad Z'_P = Z'_D \times_{Z_D} Z_P .$$

Since the closed subscheme Z'_D of X_D is flat over D, one can define the pull-back $(P_0, P_1)^*(Z)$ of the closed subscheme Z in X to the fibre X_P of the structural morphism $X \to S$ with respect to the fat point P as the cycle on X_P associated to the closed embedding $Z'_P \to X_P$ in the standard way, [6, 1.5].

In particular, if Z is a prime cycle on X, then we have the pull-back cycle $(P_0, P_1)^*(\zeta)$ on X_P. Extending by linearity we obtain a pull-back homomorphism

$$(P_0, P_1)^* : \mathscr{Z}(X) \to \mathscr{Z}(X_P) .$$

Following [22], we say that an algebraic cycle $\zeta = \sum m_i \zeta_i$ on X is a relative cycle on X over S if and only if the images of the points ζ_i under the structural morphism $X \to S$ are the generic points of the scheme S, and for any two fat points on S the pull-backs of the cycle $Z = \bar{\zeta}$ to X_P with respect to these two points coincide, see Definition 3.1.3 in loc.cit. Let $\mathscr{Z}_r(X/S)$ be the free abelian group generated by relative cycles on X/S of relative dimension r.

Now, fix a Noetherian reduced scheme T, and let $\mathfrak{N}$ be the category of Noetherian schemes over T. Let $X \to T$ be a scheme of finite type over T. For any object $S \to T$ in $\mathfrak{N}$ let

$$\mathscr{Z}_r(X/T)(S) = \mathscr{Z}_r(X \times_T S/S) .$$

If $f : S' \to S$ is a flat morphism of Noetherian schemes over T, the induced morphism $\mathrm{id}_X \times_T f : X \times_T S' \to X \times_T S$ is also flat, and one has the standard flat pull-back homomorphism

$$(\mathrm{id}_X \times_T f)^* : \mathscr{Z}_r(X \times_T S/S) \to \mathscr{Z}_r(X \times_T S'/S') .$$

If f is not flat, then the situation is more difficult. However, if T is a regular scheme, due to the above definition of relative cycles, the correct pull-back exists for any morphism f, see Proposition 3.3.15 in [22].

This all aggregates into the presheaf $\mathscr{Z}_r(X/T)$ on $\mathfrak{N}$. Let $\mathscr{C}_r(X/T)$ be a subpresheaf in $\mathscr{Z}_r(X/T)$ generated by relative effective cycles on $X \times_T S/S$, for all S in $\mathfrak{N}$. Then $\mathscr{C}_r(X/T)$ is the Chow-presheaf of monoids on $\mathfrak{N}$.

Suppose that T is regular of characteristic 0. Let $X \to T$ be a projective scheme over T, fix a closed embedding $X \to \mathbb{P}^m_T$ over T, and consider a subpresheaf $\mathscr{C}_{r,d}(X/T)$ of $\mathscr{C}_r(X/T)$, whose sections are generated by relative cycles of degree d, computed with regard to the embedding of X into $\mathbb{P}^m_T$ over T. Notice that d is a non-negative integer, and there is only one cycle in $\mathscr{C}_{r,0}(X/T)(S)$, namely the cycle 0 whose coefficients are all zeros. The grading by degrees gives the obvious structure of a graded monoid on the presheaf $\mathscr{C}_r(X/T)$ whose neutral element is 0 in $\mathscr{C}_{r,0}(X/T)(S)$.

It follows from the results in [22] (see also [12]) that the presheaves $\mathscr{C}_{r,d}(X/T)$ are representable by a scheme $C_{r,d}(X/T)$, the so-called *Chow scheme* of effective relative cycles of relative dimension r and degree d over T. This Chow scheme is projective over T. In other words, there exist a structural morphism from

$C_{r,d}(X/T)$ to T and a closed embedding of $C_{r,d}(X/T)$ into $\mathbb{P}_T^m$ over T, naturally arising from the above representability.

Notice that the above representability holds because T is a regular scheme of characteristic zero. If T would be of positive characteristic then only h-representability takes place, see [22].

For the purposes of this paper, it is actually enough to consider only the case when T is the spectrum of a field k. Then we must assume that the characteristic of k is zero. Since T is $\mathrm{Spec}(k)$, we will systematically drop the symbol $/T$ from the notation.

According to our rules to simplify notation, we write $C_{r,d}(X)$ instead of $h_{C_{r,d}(X)}$. The latter sheaf is isomorphic to the sheaf $\mathscr{C}_{r,d}(X)$. Let

$$C_r(X) = \coprod_{d \geq 0} C_{r,d}(X) \; ,$$

where the coproduct is taken in the category $\mathscr{S}$, not in $\mathfrak{N}$. Such defined $C_r(X)$ is noting but the Nisnevich sheafication of the corresponding coproduct in $\mathscr{P}$. If we would consider the coproduct of all Chow schemes $C_{r,d}(X)$ in $\mathfrak{N}$ and then embed it into $\mathscr{S}$ by the Yoneda embedding, that would be a priori a different sheaf, as Yoneda embedding in general does not commute with coproducts. However, the canonical morphism from the above sheafification to this second sheaf is an isomorphism on the Henselizations of the local rings at points of varieties over k. Therefore, the two constructions are actually isomorphic in $\mathscr{S}$. This also gives that the coproduct of $C_{r,d}(X)$, for all $d \geq 0$, in $\mathfrak{N}$ represents $\mathscr{C}_r(X)$.

The completion $C_r(X)^+$ of $C_r(X)$ in $\mathscr{S}$ is the sheafification of the completion of $C_r(X)$ as a presheaf. The latter is sectionwise. By its definition, $C_r(X)$ is a sectionwise cancelation monoid on points of the Nisnevich site, and the canonical morphism from $C_r(X)$ to $C_r(X)^+$ is thus a monomorphism.

Identifying $\mathscr{S}$ with its image in $\Delta^{op}\mathscr{S}$ under the functor Const, we may also look at $C_r(X)$ as the graded *Chow monoid* in the category of simplicial sheaves on the smooth Nisnevich site over $\mathrm{Spec}(k)$.

Let L be a finitely generated field extension of k. Since $\mathrm{Spec}(L)$ is Henselian, $C_r(X)^+(\mathrm{Spec}(L))$ is the same as $C_r(X)(\mathrm{Spec}(L))^+$. By the same reason, the latter group is isomorphic to the group $\mathscr{C}_r(X)(\mathrm{Spec}(L))^+$ of effective r-cycles on X. On the other hand, the same group $C_r(X)^+(\mathrm{Spec}(L))$ can be also identified with the group of morphisms from $\mathrm{Spec}(L)$ to $C_r(X)^+$, in the category of simplicial Nisnevich sheaves $\Delta^{op}\mathscr{S}$.

Suppose that there exists an effective dimension r and degree 1 algebraic cycle on X, with regard to the fixed embedding of X into $\mathbb{P}^m$. Denote this cycle by Z_0. For any natural number d the d-multiple dZ_0 is an effective dimension r degree d cycle on X. This gives a morphism α from $\mathbb{N}$ to $C_r(X)$. Since $C_r(X)$ is the coproduct of $C_{r,d}(X)$ for all $d \geq 0$, we also have the obvious morphism f from $C_r(X)$ to $\mathbb{N}$, such that $f \circ \alpha = \mathrm{id}_{\mathbb{N}}$. In other words, Z_0 gives the structure of a pointed graded monoid on $C_r(X)$. Automatically, we obtain the connective Chow monoid $C_r^{\infty}(X)$ associated to $C_r(X)$. By Lemma 1, we also have the canonical isomorphism of group objects

$$C_r(X)^+ \simeq C_r^{\infty}(X)^+ \times \mathbb{Z}$$

in $\Delta^{op}\mathscr{S}$.

As we have already mentioned above, in the general situation, the colimit $\mathscr{X}^\infty$ can be taken in the category of (simplicial) presheaves, as it is filtered and the category $\mathscr{S}m$ is Noetherian. By the same reasons, since all sheaves $C_{r,d}(X)$ are representable, $C_r^\infty(X)$ is the sheaf represented by the colimit of the filtered chain of Chow schemes $C_{r,d}(X)$ in the category $\mathfrak{N}$. Since $\mathscr{C}$ is Noetherian, $\mathscr{S}$ is exhaustive and $C_r(X)$ is a termwise sectionwise cancellation monoid, so is the monoid $C_r^\infty(X)$.

6. RATIONAL EQUIVALENCE AS $\mathbb{A}^1$-PATH CONNECTIVITY

For any algebraic scheme X over k let $CH_r(X)$ be the Chow group of r-dimensional algebraic cycles modulo rational equivalence on X. In this section we prove our main theorem and deduce three corollaries, which give something close to the desired effective interpretation of Chow groups in terms of $\mathbb{A}^1$-path connectivity on loop spaces of classifying spaces of the Chow monoid $C_r^\infty(X)$. We leave it for the reader to decide which of the obtained three isomorphisms is more useful for understanding of Chow groups (if any at all).

Theorem 8. *Let X be a projective algebraic variety with a fixed embedding into a projective space over k. For any finitely generated field extension L of the ground field k, there is a canonical isomorphism*

$$CH_r(X_L) \simeq \Pi_0^{\mathbb{A}^1}(C_r(X)^+)(\mathrm{Spec}(L)) \ .$$

Proof. Consider the obvious commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), C_r(X)^+)_{\mathbb{A}^1} & \xrightarrow{(l^+)_*} & \mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), L_{\mathbb{A}^1}C_r(X)^+)_{\mathbb{A}^1} \\
\uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\
\mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), C_r(X))_{\mathbb{A}^1}{}^+ & \xrightarrow{(l_*)^+} & \mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), L_{\mathbb{A}^1}C_r(X))_{\mathbb{A}^1}{}^+
\end{array}
$$

where $l = l_{C_r(X)}$. Since $L_{\mathbb{A}^1}C_r(X)^+$ is $\mathbb{A}^1$-local, the group in the top right corner is canonically isomorphic to the group $\Pi_0^{\mathbb{A}^1}(C_r(X)^+)(\mathrm{Spec}(k))$. By the same reason, the group in the bottom right corner is canonically isomorphic to the group $\Pi_0^{\mathbb{A}^1}(C_r(X))(\mathrm{Spec}(k))^+$. Since $\mathrm{Spec}(k)$ is Henselian, the latter group is nothing but the group $\Pi_0^{\mathbb{A}^1}(C_r(X))^+(\mathrm{Spec}(k))$. Then Lemma 5 gives that β is an isomorphism.

Let $q_0 : \mathrm{Spec}(k) \to C_r(X)$ and $q_1 : \mathrm{Spec}(k) \to C_r(X)$ be two k-points on $C_r(X)$, and suppose q_0 is connected to q_1 by an $\mathbb{A}^1$-path $H : \mathbb{A}^1 \to L_{\mathbb{A}^1}C_r(X)$ on $L_{\mathbb{A}^1}C_r(X)$. Since $\mathbb{A}^1$ is connected, it follows that H factorizes through $L_{\mathbb{A}^1}(Y)$, for some connected component Y in $C_{r,d}(X)$, containing the points q_0 and q_1. Since Y is proper and of finite type over the ground field k, Proposition 6.2.6 in [1] gives that the points q_0 and q_1 are $\mathbb{A}^1$-chain connected, and so $\mathbb{A}^1$-path connected, on Y, hence on $C_{r,d}(X)$. It means that the map

$$l_* : \mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), C_r(X))_{\mathbb{A}^1} \to \mathrm{Hom}_{\Delta^{op}\mathscr{S}}(\mathrm{Spec}(k), L_{\mathbb{A}^1}C_r(X))_{\mathbb{A}^1}$$

is a injective. Since l_* is surjective by Lemma 6, it is bijective. Then $(l^+)_*$ is an isomorphism too.

Since β and $(l^+)_*$ are isomorphisms, and α is an epimorphism by Lemma 3, we see that α is an isomorphism, and then all the maps in the commutative square above are isomorphisms.

Let now A and A' be two r-dimensional algebraic cycles on X. If A is rationally equivalent to A' on X, there exists an effective relative cycle Z on the scheme $X \times_{\mathrm{Spec}(k)} \mathbb{A}^1 / \mathbb{A}^1$ of relative dimension r, and an effective dimension r algebraic cycle B on X, such that

$$Z(0) = A + B \quad \text{and} \quad Z(1) = A' + B$$

on X. Let h_Z and $h_{B \times \mathbb{A}^1}$ be two regular morphisms from $\mathbb{A}^1$ to the Chow scheme $C_r(X)$ over $\mathrm{Spec}(k)$ corresponding to the relative cycles Z and $B \times \mathbb{A}^1$ on $X \times_{\mathrm{Spec}(k)} \mathbb{A}^1 / \mathbb{A}^1$ respectively. Let

$$h : \mathbb{A}^1 \to C_r(X) \times C_r(X)$$

be the product of h_Z and $h_{B \times \mathbb{A}^1}$ in the category $\Delta^{op} \mathscr{S}$. Let

$$H : \mathbb{A}^1 \to C_r(X) \times C_r(X) \to C_r(X)^+ \ ,$$

be the composition of h and the morphism from $C_r(X) \times C_r(X)$ to the completion $C_r(X)^+$, in $\Delta^{op} \mathscr{S}$. Then $H_0 = A$ and $H_1 = A'$, where H_0 and H_1 are the precompositions of H with i_0 and i_1 respectively. It means that the cycles A and A' are $\mathbb{A}^1$-path connected on $C_r(X)^+$.

Vice versa, suppose we have a morphism

$$H : \mathbb{A}^1 \to C_r(X)^+$$

in $\mathscr{S}$, and let H_0 and H_1 be the compositions of H with i_0 and i_1 respectively. Since $\mathrm{Spec}(k)$ is Henselian, H_0 is represented by two morphisms $H_{0,1}$ and $H_{0,2}$ from $\mathrm{Spec}(k)$ to $C_r(X)$. Similarly, H_1 is represented by two morphisms $H_{1,1}$ and $H_{1,2}$ from $\mathrm{Spec}(k)$ to $C_r(X)$. Since α is an isomorphism and H_0 is $\mathbb{A}^1$-path connected to H_1, it follows that there exist two morphisms f and G from $\mathrm{Spec}(k)$ to $C_r(X)$, such that $H_{0,1} + F$ is $\mathbb{A}^1$-path connected to $H_{0,2} + G$ and $H_{1,1} + F$ is $\mathbb{A}^1$-path connected to $H_{1,2} + G$ on $C_r(X)$. In terms of algebraic cycles on X, it means that the effective r-cycle $H_{0,1} + F$ is rationally equivalent to the effective r-cycle $H_{0,2} + G$, and, similarly, the cycle $H_{1,1} + F$ is rationally equivalent to $H_{1,2} + G$ on $C_r(X)$. Then the cycle $H_0 = H_{0,1} - H_{0,2}$ is rationally equivalent to the cycle $H_1 = H_{1,1} - H_{1,2}$ on $C_r(X)$.

Thus, the Chow group $CH_r(X)$ is isomorphic to $\mathrm{Hom}_{\Delta^{op} \mathscr{S}}(\mathrm{Spec}(k), C_r(X)^+)_{\mathbb{A}^1}$, i.e. the group in the top left corner of the diagram above. Since, moreover, $(l^+)_*$ is an isomorphism, and the group in the top right corner is canonically isomorphic to $\Pi_0^{\mathbb{A}^1}(C_r(X)^+)(\mathrm{Spec}(k))$, we obtain the required isomorphism in case when L is the ground field k.

To prove the theorem for an arbitrary L, we observe that $f^* C_r(X)$ is $C_r(X_L)$, whence

$$f^* L_{\mathbb{A}^1} C_r(X)^+ = L_{\mathbb{A}^1_L} f^* C_r(X)^+ = L_{\mathbb{A}^1_L} C_r(X_L)^+$$

by Lemma 7. Therefore,

$$
\begin{aligned}
\Pi_0^{\mathbb{A}^1}(C_r(X)^+)(\mathrm{Spec}(L)) &= \Pi_0(L_{\mathbb{A}^1}C_r(X)^+)(\mathrm{Spec}(L)) \\
&= f^*\Pi_0(L_{\mathbb{A}^1}C_r(X)^+)(\mathrm{Spec}(L)) \\
&= \Pi_0(f^*L_{\mathbb{A}^1}C_r(X)^+)(\mathrm{Spec}(L)) \\
&= \Pi_0(L_{\mathbb{A}^1_L}C_r(X_L)^+)(\mathrm{Spec}(L)) \\
&= \Pi_0^{\mathbb{A}^1_L}(C_r(X_L)^+)(\mathrm{Spec}(L)) \\
&\simeq CH_r(X_L)_0 \, .
\end{aligned}
$$

$\square$

Remark 9. Lemma 5 provides that the monoidal completion in Theorem 8 can be taken before or after computing the $\mathbb{A}^1$-connected component functor. Since the monoidal completion is sectionwise on stalks, we obtain the canonical isomorphisms

$$
\begin{aligned}
CH_r(X) &\simeq \Pi_0^{\mathbb{A}^1}(C_r(X)^+)(\mathrm{Spec}(k)) \\
&\simeq \Pi_0^{\mathbb{A}^1}(C_r(X))^+(\mathrm{Spec}(k)) \\
&\simeq \Pi_0^{\mathbb{A}^1}(C_r(X))(\mathrm{Spec}(k))^+ \, .
\end{aligned}
$$

The embedding $X \hookrightarrow \mathbb{P}^m$ gives the degree homomorphism from $CH_r(X)$ to $\mathbb{Z}$. Let $CH_r(X)_0$ be its kernel, i.e. the Chow group of degree 0 cycles of dimension r modulo rational equivalence on X. Then,

$$
CH_r(X) \simeq CH_r(X)_0 \times \mathbb{Z} \, .
$$

Assume that there exists a degree 1 effective r-cycle on X. As we have seen above, this gives the structure of a pointed graded cancellation monoid on $C_r(X)$, and $C_r^\infty(X)$ is a cancelation monoid too.

Corollary 10. *In the above terms,*

$$
CH_r(X_L)_0 \simeq \Pi_0^{\mathbb{A}^1}(C_r^\infty(X)^+)(\mathrm{Spec}(L)) \, .
$$

Proof. By Lemma 1,

$$
C_r(X)^+ \simeq C_r^\infty(X)^+ \times \mathbb{Z}
$$

Since the functor $\Pi_0^{\mathbb{A}^1}$ is monoidal and $\Pi_0^{\mathbb{A}^1}(\mathbb{Z}) = \mathbb{Z}$, we get the formula

$$
\Pi_0^{\mathbb{A}^1}(C_r(X)^+) \simeq \Pi_0^{\mathbb{A}^1}(C_r^\infty(X)^+) \times \mathbb{Z} \, .
$$

Then apply Theorem 8 and the isomorphism $CH_r(X) \simeq CH_r(X)_0 \times \mathbb{Z}$. $\square$

Warning 11. If $CH_r(X_L)_0 \simeq \Pi_0^{\mathbb{A}^1}(C_r^\infty(X))(\mathrm{Spec}(L))^+ = 0$, it does not imply that the monoid $\Pi_0^{\mathbb{A}^1}(C_r^\infty(X))(\mathrm{Spec}(L))$ vanishes, as this monoid is by no means a cancellation monoid. One of the reasons for that is that the Chow schemes $C_{r,d}(X)$ can have many components over k.

Corollary 12. *In the above terms,*

$$
CH_r(X_L)_0 \simeq \Pi_0^{\mathbb{A}^1}(\Omega ExB C_r^\infty(X))(\mathrm{Spec}(L)) \, .
$$

Proof. The Chow monoid $C_r(X)$ is termwise free, so that the connective Chow monoid $C_r^\infty(X)$ is good. Lemma 2 gives an isomorphism

$$
C_r^\infty(X)^+ \simeq \Omega ExB C_r^\infty(X)
$$

in H, whence

$$
\Pi_0^{\mathbb{A}^1}(C_r^\infty(X)^+) \simeq \Pi_0^{\mathbb{A}^1}(\Omega ExB C_r^\infty(X)) \, .
$$

Corollary 10 completes the proof. $\square$

Recall that, for a pointed simplicial Nisnevich sheaf $(\mathscr{X}, x)$, its motivic, i.e.
$\mathbb{A}^1$-fundamental group $\Pi_1^{\mathbb{A}^1}(\mathscr{X}, x)$ is, by definition, the Nisnevich sheaf associated to the presheaf sending a smooth scheme U to the set $[S^1 \wedge U_+, (\mathscr{X}, x)]_{\mathbb{A}^1}$,
where the symbol $[-, -]_{\mathbb{A}^1}$ stays now for the sets of morphisms in the pointed
homotopy category $\mathsf{H}_{\mathbb{A}^1}$, see [17] or [1]. Similarly, one can define, for a pointed
simplicial Nisnevich sheaf $(\mathscr{X}, x)$, the fundamental group $\Pi_1^{S^1 \wedge \mathbb{A}^1}(\mathscr{X}, x)$, where
$\mathbb{A}^1$ is pointed at any k-rational point on it. This is the Nisnevich sheaf associated
to the presheaf sending a smooth scheme U to the set

$$[S^1 \wedge U_+, (\mathscr{X}, x)]_{S^1 \wedge \mathbb{A}^1} ,$$

where the symbol $[-, -]_{S^1 \wedge \mathbb{A}^1}$ stays for the sets of morphisms in the pointed
homotopy category $\mathsf{H}_{S^1 \wedge \mathbb{A}^1}$.

Lemma 13. *Let $\mathscr{X}$ be a termwise good simplicial sheaf monoid. Then, for a
scheme U,*
$$\Pi_0^{\mathbb{A}^1}(\mathscr{X}^+)(U) \simeq \Pi_1^{S^1 \wedge \mathbb{A}^1}(B\mathscr{X})(U) .$$

Proof. Since $\mathscr{X}$ is termwise good, there is a isomorphism between $\mathscr{X}^+$ and
$\Omega ExB\mathscr{X}$ in the homotopy category H, by Lemma 1.2 on page 123 - 124 in [17].
Since the classifying space $B\mathscr{X}$ is pointed connected, the canonical morphism

$$L_{\mathbb{A}^1} \Omega ExB\mathscr{X} \to \Omega ExL_{S^1 \wedge \mathbb{A}^1} B\mathscr{X}$$

is a simplicial weak equivalence and

$$\Omega ExL_{S^1 \wedge \mathbb{A}^1} B\mathscr{X}$$

is $\mathbb{A}^1$-local by Theorem 2.34 on page 84 in loc.cit. This allows us to make the
following identifications:

$$\begin{aligned}
\Pi_0^{\mathbb{A}^1}(\mathscr{X}^+)(U) &\simeq [U, \mathscr{X}^+]_{\mathbb{A}^1} \\
&\simeq [U, \Omega ExB\mathscr{X}]_{\mathbb{A}^1} \\
&\simeq [U, L_{\mathbb{A}^1} \Omega ExB\mathscr{X}]_{\mathbb{A}^1} \\
&\simeq [U, \Omega ExL_{S^1 \wedge \mathbb{A}^1} B\mathscr{X}]_{\mathbb{A}^1} \\
&\simeq [U, \Omega ExL_{S^1 \wedge \mathbb{A}^1} B\mathscr{X}] \\
&\simeq [S^1 \wedge U_+, L_{S^1 \wedge \mathbb{A}^1} B\mathscr{X}] \\
&\simeq [S^1 \wedge U_+, B\mathscr{X}]_{S^1 \wedge \mathbb{A}^1} \\
&\simeq \Pi_1^{S^1 \wedge \mathbb{A}^1}(B\mathscr{X})(U) .
\end{aligned}$$

$\square$

Corollary 14. *In the above terms,*
$$CH_r(X_L)_0 \simeq \Pi_1^{S^1 \wedge \mathbb{A}^1}(BC_r^\infty(X))(\mathrm{Spec}(L)) .$$

Proof. This is a straightforward consequence of Corollary 10 and Lemma 13. $\square$

Remark 15. Let X be a nonsingular projective surface over k, where k is algebraically closed of characteristic zero. Assume that X is of general type and
has no transcendental second cohomology group, i.e. the cycle class map from
$CH^1(X)$ to the second Weil cohomology group $H^2(X)$ is surjective. In that case
the irregularity of X is zero. Bloch's conjecture predicts that $CH_0(X) = \mathbb{Z}$. In
other words, any two closed points on X are rationally equivalent to each other.

Fixing a point on the surface X yields the monoid $C_0^\infty(X)$, which is nothing else but the the infinite symmetric power $\mathrm{Sym}^\infty(X)$ of the smooth projective surface X. By Corollary 12, Bloch's conjecture holds for X if and only if all k-points on the motivic space $L_{\mathbb{A}^1}\Omega ExB\mathrm{Sym}^\infty(X)$ are $\mathbb{A}^1$-path connected. Bloch's conjecture holds, for example, for the classical Godeaux surfaces, [23], and for the Catanese and Barlow surfaces, see [2] and [24].

Remark 16. The above vision of Chow groups should be compared with the results of Friedlander, Lawson, Lima-Filho, Mazur and others, who considered topological (i.e. not motivic) homotopy completions of Chow monoids working over $\mathbb{C}$, see [5] and [15]. A nice survey of this topic, containing many useful references, is the article [14]. The topological homotopy completions of Chow monoids are helpful to understand algebraic cycles modulo algebraic equivalence relation, i.e. the groups $A_r(X)$ of algebraically trivial r-cycles cannot be catched by the topological methods. In contrast, the motivic, i.e. $\mathbb{A}^1$-homotopy completions of Chow monoids can give the description of $A_r(X)$, working over an arbitrary ground field of characteristic zero, as the previous examples show. Theorem 8 also suggests the following motivic analog of the Lawson homology groups given by the formula

$$L_r H_n(X) = \Pi_{n-2r}^{\mathbb{A}^1}(C_r(X)^+) \ .$$

7. APPENDIX: HOMOTOPICAL ALGEBRA

For the convenience of the reader we collect here the needed extractions from homotopical algebra. Let first $\mathscr{C}$ be a symmetric monoidal category with product $\otimes$ and unit $\mathbb{1}$. The monoidal product $\otimes$ is called to be closed, and the category $\mathscr{C}$ is called closed symmetric monoidal, if the product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is so-called adjunction of two variables, i.e. there is bifunctor $\mathscr{H}om$ and two functorial in X, Y, Z bijections

$$\mathrm{Hom}_{\mathscr{C}}(X, \mathscr{H}om(Y, Z)) \simeq \mathrm{Hom}_{\mathscr{C}}(X \otimes Y, Z) \simeq \mathrm{Hom}_{\mathscr{C}}(Y, \mathscr{H}om(X, Z)) \ .$$

If $\mathscr{C}$ has a model structure $\mathscr{M}$ in it, an adjunction of two variables on $\mathscr{C}$ is called Quillen adjunction of two variables, or Quillen bifunctor, if, for any two cofibrations $f : X \to Y$ and $f' : X' \to Y'$ in $\mathscr{M}$ the push-out product

$$f \Box f' : (X \otimes Y') \coprod_{X \otimes X'} (Y \otimes X') \to Y \wedge Y'$$

is also a cofibration in $\mathscr{M}$, which is trivial if either f and f' is. The model category $(\mathscr{C}, \mathscr{M})$ is called closed symmetric monoidal model category if $\otimes$ is a Quillen bifunctor and the following extra axiom holds. If $q : Q\mathbb{1} \to \mathbb{1}$ is a cofibrant replacement for the unit object $\mathbb{1}$, then the morphisms $q \wedge \mathrm{id} : Q\mathbb{1} \wedge X \to \mathbb{1} \wedge X$ and $\mathrm{id} \wedge q : X \wedge Q\mathbb{1} \to X \wedge \mathbb{1}$ are weak equivalences for all cofibrant objects X. If we consider the cartesian product $\mathscr{M} \times \mathscr{M}$ of the model structure $\mathscr{M}$ as a model structure on the cartesian product $\mathscr{C} \times \mathscr{C}$, then $\otimes$ and $\mathscr{H}om$ induce left derived functor $\otimes^L$ from $Ho(\mathscr{C} \times \mathscr{C})$ to $Ho(\mathscr{C})$, and right derived functor $R\mathscr{H}om$ from $Ho(\mathscr{C} \times \mathscr{C})$ to $Ho(\mathscr{C})$. It is well known that passing to localization commutes with products of categories, so that we have the equivalence between $Ho(\mathscr{C} \times \mathscr{C})$ and $Ho(\mathscr{C}) \times Ho(\mathscr{C})$. This gives the left derived functor

$$\otimes^L : Ho(\mathscr{C}) \times Ho(\mathscr{C}) \to Ho(\mathscr{C})$$

and the right derived functor

$$R\mathscr{H}om : Ho(\mathscr{C}) \times Ho(\mathscr{C}) \to Ho(\mathscr{C}) \ .$$

As it was shown in [9], the left derived $\otimes^L$ and the right derived $R\mathscr{H}om$ give the structure of a closed symmetric monoidal category on the homotopy category $Ho(\mathscr{C})$. Since we assume that all objects in $\mathscr{C}$ are cofibrant in $\mathscr{M}$, it is easy to see that the canonical functor from $\mathscr{C}$ to $Ho(\mathscr{C})$ is monoidal.

An important particular case is when the symmetric monoidal product $\otimes$ is given by the categorical product in $\mathscr{C}$, i.e. when $\mathscr{C}$ is the cartesian symmetric monoidal category. Since $Ho(\mathscr{C})$ admits products, and products in $\mathscr{C}$ are preserved in $Ho(\mathscr{C})$, for any three objects X, Y and Z in $\mathscr{C}$ one has the canonical isomorphism

$$[X, Y] \times [X, Z] \simeq [X, Y \times Z] \ .$$

Let now $\mathscr{C}$ be a left proper cellular simplicial model category with model structure $\mathscr{M} = (W, C, F)$ in it, let I and J be the sets of, respectively, generating cofibrations and generating trivial cofibrations in $\mathscr{C}$, and let S be a set of morphisms in $\mathscr{C}$. For simplicity we will also be assuming that all objects in $\mathscr{C}$ are cofibrant, which will always be the case in applications. An object Z in $\mathscr{C}$ is called S-local if it is fibrant, in the sense of the model structure $\mathscr{M}$, and for any morphism $g : A \to B$ between cofibrant objects in S the induced morphism from $\mathbf{Hom}(B, Z)$ to $\mathbf{Hom}(A, Z)$ is a weak equivalence in $\Delta^{op}\mathscr{S}ets$. A morphism $f : X \to Y$ in $\mathscr{C}$ is an S-local equivalence if the induced morphism from $\mathbf{Hom}(Y, Z)$ to $\mathbf{Hom}(X, Z)$ is a weak equivalence in $\Delta^{op}\mathscr{S}ets$ for any S-local object Z in $\mathscr{C}$. Then there exists a new left proper cellular model structure $\mathscr{M}_S = (W_S, C_S, F_S)$ on the same category $\mathscr{C}$, such that $C_S = C$, W_S consists of S-local equivalences in $\mathscr{C}$, so contains W, and F_S is standardly defined by the right lifting property and so is contained in F. The model structure $\mathscr{M}_S$ is again left proper and cellular with the same set of generating cofibrations I and the new set of generating trivial cofibrations J_S. The model category $(\mathscr{C}, \mathscr{M}_S)$ is called the (left) Bousfield localization of $(\mathscr{C}, \mathscr{M}_S)$ with respect to S. This all can be found in [10].

Notice that the identity adjunction on $\mathscr{C}$ is a Quillen adjunction and induces the derived adjunction $L\mathrm{Id} : Ho(\mathscr{C}) \dashv Ho(\mathscr{C}_S) : R\mathrm{Id}$, where $Ho(\mathscr{C}_S)$ is the homotopy category of $\mathscr{C}$ with respect to the model structure $\mathscr{M}_S$. Since cofibrations remain the same and, according to our assumption, all objects are cofibrant, the functor $L\mathrm{Id}$ is the identity on objects and surjective on Hom-sets. To describe $R\mathrm{Id}$ we observe the following. Since F_S is smaller than F, the fibre replacement functor in $(\mathscr{C}, \mathscr{M})$ is different from the fibre replacement functor in $(\mathscr{C}, \mathscr{M}_S)$. Taking into account that $\mathscr{C}$ is left proper and cellular, one can show that there exists a fibrant replacement $\mathrm{Id}_{\mathscr{C}} \to L_S$ in $\mathscr{M}_S$, such that, if X is already fibrant in $\mathscr{M}$, then $L_S(X)$ can be more or less visibly constructed from X and S, see Section 4.3 in [10] (or less abstract presentation in [4]). The right derived functor $R\mathrm{Id}$, being the composition of $Ho(L_S)$ and the functor induced by the embedding of S-local, i.e. cofibrant in $\mathscr{M}_S$, objects into $\mathscr{C}$, identifies $Ho(\mathscr{C}_S)$ with the full subcategory in $Ho(\mathscr{C})$ generated by S-local objects of $\mathscr{C}$.

Since $(\mathscr{C}, \mathscr{M})$ is a simplicial model category, then so is $(\mathscr{C}, \mathscr{M}_S)$, see Theorem 4.1.1 (4) in [10]. Suppose that $\mathscr{C}$ is, moreover, closed symmetric monoidal with product $\otimes$, and that the monoidal structure is compatible with the model one in

the standard sense, i.e. $\mathscr{C}$ is a symmetric monoidal model category (see above). The new model category $(\mathscr{C}, \mathscr{M}_S)$ is monoidal model, i.e. the model structure $\mathscr{M}_S$ is compatible with the existing monoidal product $\otimes$, if and only if for each f in S and any object X in the union of the domains $\mathrm{dom}(I)$ and codomains $\mathrm{codom}(I)$ of generating cofibrations I in $\mathscr{C}$ the product $\mathrm{id}_X \otimes f$ is in W_S.

This is exactly the case when the set S is generated by a morphism $p : A \to \mathbb{1}$, where A is an object in $\mathscr{C}$ and $\mathbb{1}$ is the unit object for the monoidal product $\otimes$, i.e.

$$S = \{X \wedge A \overset{\mathrm{id}_X \wedge p}{\longrightarrow} X \mid X \in \mathrm{dom}(I) \cup \mathrm{codom}(I)\} .$$

In that case the model structure $\mathscr{M}_S$ is compatible with the monoidal one, so that $(\mathscr{C}, \mathscr{M}_S)$ is a simplicial closed symmetric monoidal model category, which is left proper cellular.

Let us write $\mathscr{M}_A$ and L_A instead of, respectively, $\mathscr{M}_A$ and L_S when S is generated by A in the above sense. One of the fundamental properties of the localization functor LA is that, for any two objects X and Y in $\mathscr{C}$, the object $L_A(X \times Y)$ is weak equivalent to the object $L_A(X) \times L_A(Y)$, in the sense of the model structure $\mathscr{M}_A$. The proof of this fact in topology is given on page 36 of the book [4], and it can be verbally transported to abstract setting. All we need is the Quillen adjunction in two variables in $\mathscr{C}$, and the fact saying that if Y is S-local, then $\mathscr{H}om(X, Y)$ is S-local for any X in $\mathscr{C}$, which is also the consequence of adjunction.

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Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool L69 7ZL, England, UK

E-mail address: vladimir.guletskii@liverpool.ac.uk