UNIVERSITY OF LIVERPOOL

Real Forms of Higher Spin Structures on Riemann Orbifolds

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor of Philosophy by

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Dedication

To my parents and family for their love and support.

For the members of my family who are sadly missed.

Acknowledgements

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Abstract

In this thesis we study the space of m-spin structures on hyperbolic Klein orbifolds. A hyperbolic Klein orbifold is a hyperbolic 2-dimensional orbifold with a maximal atlas whose transition maps are either holomorphic or anti holomorphic. Hyperbolic Klein orbifolds can be described as pairs (P, τ), where P is a quotient $\mathbb{H}/\Gamma$ of the hyperbolic plane $\mathbb{H}$ by a Fuchsian group Γ and τ an anti-holomorphic involution on P. An m-spin structure on a hyperbolic Klein orbifold P is a complex line bundle L such that the m-th tensor power of L is isomorphic to the cotangent bundle of P and L is invariant under the involution τ. We only consider a certain class of hyperbolic Klein orbifolds which we call nice Klein orbifolds, namely those where no fixed points of the involution τ are fixed by any elements of the Fuchsian group Γ. We describe topological invariants of m-spin structures on nice Klein orbifolds and determine the conditions under which such m-spin structures exist. We describe all connected components of the space of m-spin structures on nice Klein orbifolds and prove that any connected component is homeomorphic to a quotient of $\mathbb{R}^d$ by a discrete group.

Contents

Chapter 1

Introduction

The classical spin structures on compact Riemann surfaces play an important role in algebraic geometry since Riemann [21]. Their modern interpretation and classification as complex line bundles such that the tensor square is isomorphic to the cotangent bundle of the surface was given by Atiyah [2] and Mumford [11]. Classification of classical spin structures on non-compact Riemann surfaces and the corresponding moduli spaces were studied in [12] and [14]. The moduli spaces of m-spin structures on Riemann surfaces play an important role in mathematical physics [24], [22] and singularity theory [5].

An m-spin bundle on a Riemann surface is a complex line bundle such that its m-th tensor power is isomorphic to the cotangent bundle of the surface. A hyperbolic Riemann surface is a quotient $P = \mathbb{H}/\Gamma$ of the hyperbolic plane $\mathbb{H}$ by a torsion-free Fuchsian group Γ. In the case where Γ contains elliptic elements we will refer to the quotient $\mathbb{H}/\Gamma$ as a Riemann orbifold. The fixed points of elliptic elements of Γ correspond to special points on the orbifold $\mathbb{H}/\Gamma$, called the orbifold points or marking points. Paper [17] studied m-spin structures on $P = \mathbb{H}/\Gamma$ which are invariant under an anti-holomorphic involution, where Γ does not contain elliptic elements. The aim of this project is to study the case where the Fuchsian group Γ contains elliptic elements, hence extending the results of paper [17].

A normal isolated singularity of dimension n is Gorenstein if and only if there is a nowhere vanishing n-form on a punctured neighbourhood of the singular point. According to the work of Dolgachev [5] hyperbolic Gorenstein quasi-homogeneous surface singularities of level m are in 1-1 correspondence with m-spin structures on Riemann orbifolds.

To study m-spin structures, we assign to each m-spin structure on P an associated m-Arf function, a certain function on the space of homotopy classes of simple contours on the orbifold P with values in $\mathbb{Z}/m\mathbb{Z}$, described by simple geometric properties. To do this we first establish a connection with lifts of Fuchsian groups and use properties of $\mathrm{Isom}(\mathbb{H})$, the group of isometries of the hyperbolic plane.

Definition 1.0.1 Let P be a Riemann orbifold. Let $p \in P$. Let $\pi(P, p)$ be the orbifold fundamental group of P (for details see section 5.3.2). We denote by $\pi^0(P, p)$ the set of all non-trivial elements of $\pi(P, p)$ that can be represented by simple contours. An *m-Arf function* is a function

$$\sigma : \pi^0(P, p) \to \mathbb{Z}/m\mathbb{Z}$$

satisfying the following conditions

1. $\sigma(bab^{-1}) = \sigma(a)$ for any elements $a, b \in \pi^0(P, p)$,
2. $\sigma(a^{-1}) = -\sigma(a)$ for any element $a \in \pi^0(P, p)$ that is not of order 2,
3. $\sigma(ab) = \sigma(a) + \sigma(b)$ for any elements a and b which can be represented by a pair of simple contours in P intersecting at exactly one point p with intersection number $\langle a, b\rangle \neq 0$,
4. $\sigma(ab) = \sigma(a) + \sigma(b) - 1$ for any elements $a, b \in \pi^0(P, p)$ such that the element ab is in $\pi^0(P, p)$ and the elements a and b can be represented by a pair of simple contours in P intersecting at exactly one point p with intersection number $\langle a, b\rangle = 0$ and placed in a neighbourhood of the point p as shown in Figure 5.3,
5. for any elliptic element c of order q we have $q \cdot \sigma(c) + 1 \equiv 0 \mod m$.

Our aim is to understand what restrictions arise if we assume that the m-Arf function is invariant under an anti-holomorphic involution of P.

The first step was to classify Arf functions on orbifolds with holes and punctures. Special cases of this classification problem have been considered previously, such as in the paper [15] which looked at classifying Arf functions on surfaces with holes and punctures, but no orbifold points. Paper [16] on the other hand looked at orbifolds with only orbifold points, no holes or punctures. To combine the results of these two papers it was necessary to return to the proofs. Section 5.5.3 outlines the main results.

A definition of a standard basis of $\pi(P,p)$ can be found in section 5.3.2.

We define the *Arf invariant* δ of σ as follows:

1. If $g > 1$ and m is even then we set $\delta = 0$ if there is a standard basis $\{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n\}$ of $\pi(P,p)$ such that:

$$\sum_{i=1}^{g}(1-\sigma(a_i))(1-\sigma(b_i)) \equiv 0 \mod 2$$

   and we set $\delta = 1$ otherwise.

2. If $g > 1$ and m is odd then we set $\delta = 0$.

3. If $g = 0$ then we set $\delta = 0$.

4. If $g = 1$ and there is a standard basis $\{a_1, b_1, c_1, \dots, c_n\}$ of $\pi(P,p)$ with $c_1, \dots, c_{l_h}$ holes, $c_{l_h+1}, \dots, c_{l_h+l_p}$ punctures and $c_{l_h+l_p+1}, \dots, c_{l_h+l_p+l_e}$ elliptics then we set:

   $$\delta = \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1)+1, \dots, \sigma(c_{l_h+l_p})+1, p_1-1, \dots, p_{l_e}-1),$$

   where p_i is the order of $c_{l_h+l_p+i}$.

Definition 1.0.2 Let σ be an m-Arf function on P and let:

1. n_j^h be the number of holes c_i with $\sigma(c_i) = j$,

2. n_j^p be the number of punctures c_i with $\sigma(c_i) = j$,

3. n_j^e be the number of elliptics c_i with $\sigma(c_i) = j$.

Then the *topological type* of σ is a tuple

$$t = (g, \delta, n_0^h, \dots, n_{m-1}^h, n_0^p, \dots, n_{m-1}^p, p_1, \dots, p_{l_e}),$$

where g is the genus of P, δ the Arf invariant of σ, $\sum_{j=0}^{m-1} n_j^h = l_h$, $\sum_{j=0}^{m-1} n_j^p = l_p$ and $\sum_{j=0}^{m-1} n_j^e = l_e$ and $p_1, \dots, p_{l_e}$ are the orders of elliptic elements.

Our first main result is the following theorem (Theorem 5.5.11), which gives conditions under which an m-Arf function and hence an m-spin structure exists on a surface with holes, punctures and elliptic points.

Theorem 1.0.3 *A tuple* $t = (g, \delta, n_0^h, \dots, n_{m-1}^h, n_0^p, \dots, n_{m-1}^p, p_1, \dots, p_{l_e})$ *is the topological type of a hyperbolic* m*-Arf function on a Riemann orbifold of genus* g *with* l_h *holes,* l_p *punctures and* l_e *orbifold points of orders* $p_1, \dots, p_{l_e}$*, if and only if it has the following properties:*

(a) If $g > 1$ *and* m *is odd, then* $\delta = 0$.

(b) If $g > 1$*,* m *is even and* $n_j^h + n_j^p \neq 0$ *for some even* j*, then* $\delta = 0$.

(c) If $g = 1$ *then* δ *is a divisor of* m*,* $\gcd\{j + 1 | n_j^h + n_j^p \neq 0\}$ *and* $\gcd(p_1 - 1, \dots, p_{l_e} - 1)$.

(d) The following degree conditions are satisfied: $(p_i, m) = 1$ *for* $i = 1, \dots, l_e$ *and*

$$\sum_{j=0}^{m-1} j(n_j^h + n_j^p) - \sum_{i=1}^{l_e} \frac{1}{p_i} = (2 - 2g) - (l_h + l_p + l_e) \mod m.$$

The next step was to use these results and paper [17] to understand the invariants and the deformation space of such m-Arf functions invariant under an involution. Sections 8.9, 8.10, 8.11 and 8.12 outline the main results.

A *Klein surface* is a topological surface with a maximal atlas whose transition maps are either holomorphic or anti-holomorphic. All Klein surfaces are equivalent to pairs (P, τ), where P is a Riemann surface and τ an anti-holomorphic involution on P. A *Klein orbifold* is a pair (P, τ), where P is a compact Riemann orbifold and τ is an anti-holomorphic involution on P.

There are two kinds of contours which are invariant under the involution τ:

- An *oval* is a simple closed smooth contour consisting of fixed points of τ.
- A *twist* is a simple contour which is invariant under the involution τ but does not contain any fixed points of τ.

We can decompose P into two surfaces P_1 and P_2 by removing some invariant contours.

Definition 1.0.4 Let P^τ be the fixed point set of τ. We say that a Klein orbifold (P, τ) is *separating* if the set $P \backslash P^\tau$ is not connected. Otherwise we say it is *non-separating.*

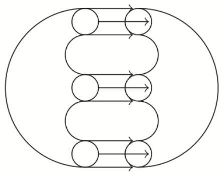

Figure 1.1: Separating Klein surface.

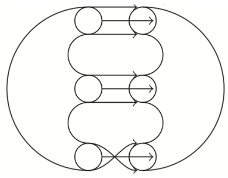

Figure 1.2: Non-Separating Klein surface.

The following figures illustrate the general method to construct Klein orbifolds described in more detail in Chapter 7.

Definition 1.0.5 We say (P, τ) is a *nice Klein orbifold* if the fixed point set P^τ does not contain any orbifold points of P.

Definition 1.0.6 Let (P, τ) be a nice Klein orbifold. The *topological type* of (P, τ) is the tuple

$$(g, 2r, k, \epsilon, p_1, \dots, p_r),$$

where g is the genus of the Riemann orbifold P, $2r$ the number of marking points on P, $p_1, p_1, p_2, p_2, \dots, p_r, p_r$ their orders, k is the number of connected components of P^τ, $\epsilon = 0$ if (P, τ) is non-separating and $\epsilon = 1$ otherwise.

The involution $\tau : P \to P$ acts on all structures related to the Riemann orbifold P, for example on m-spin structures. So the question is: given an m-spin structure on P and an anti-holomorphic involution on P, what properties does the Arf function have to have? All Klein surfaces can be constructed from discrete subgroups of G =Aut($\mathbb{H}$) called real Fuchsian groups. A lift of a real Fuchsian group into the m-fold cover G_m of G induces an m-Arf function. We study the properties of these Arf functions. In section **??** we show that for an Arf function σ to be invariant under the involution τ it must satisfy:

$$\sigma(\tau c) = -\sigma(c) \text{ for any } c \in \pi(P, p)$$

We show that for odd m, σ vanishes on all ovals, while for even m we have $\sigma = 0$ or $m/2$ on all ovals and σ vanishes on all twists.

Definition 1.0.7 Let (P, τ) be a non-separating nice Klein surface of type $(g, 2r, k, 0, p_1, \ldots, p_r)$. The *topological type* of a real m-Arf function σ on P is:

(i) If m is odd, then the topological type is a tuple $(g, k, p_1, \ldots, p_r)$, where k is the number of ovals of (P, τ).

(ii) If m is even, then the topological type is a tuple $(g, \delta, k_0, k_1, p_1, \ldots, p_r)$, where δ is the m-Arf invariant of σ and k_j is the number of ovals of (P, τ) with value of σ equal to $m \cdot j/2$.

Theorem 1.0.8 *The space of m-spin bundles on non-separating nice Klein orbifolds with $g \geq k+2$, marking points of orders ≥ 3 and even m decomposes into the connected components*

$$S(g, \delta, k_0, k_1, p_1, \ldots, p_r),$$

where $t = (g, \delta, k_0, k_1, , p_1, \ldots, p_r)$ satisfies the condition

$$\frac{m}{2} \cdot k_1 - \sum_{i=1}^{r} \frac{1}{p_i} = (1 - g - r) \mod m$$

and $S(t)$ is the set of all real m-spin bundles such that the associated m-Arf function is of type t. Each of the components $S(g, \delta, k_0, k_1, p_1, \ldots, p_r)$ is a branched covering of the moduli space of Klein surfaces of topological type $(g, 2r, k_0 + k_1, 0, p_1, \ldots, p_r)$ and is diffeomorphic to

$$\mathbb{R}^{3g-3+2r}/\mathrm{Mod}_{g,\delta,k_0,k_1,p_1,\ldots,p_r},$$

where $\mathrm{Mod}_{g,\delta,k,0,k_1,p_1,\ldots,p_r}$ is a discrete group of diffeomorphisms.

Theorem 1.0.9 *The space of m-spin bundles on non-separating nice Klein orbifolds with $g \geq k+2$, marking points of orders ≥ 3 and odd m decomposes into the connected components $S(g, k, p_1, \ldots, p_r)$, where $t = (g, k, p_1, \ldots, p_r)$ satisfies the condition*

$$-\sum_{i=1}^{r} \frac{1}{p_i} = (1 - g - r) \mod m$$

and $S(t)$ is the set of all real m-spin bundles such that the associated m-Arf function is of type t. Each of the components $S(g, k, p_1, \ldots, p_r)$ is a *branched covering of the moduli spaces of Klein surfaces of topological type* $(g, 2r, k, 0, p_1, \dots, p_r)$ *and is diffeomorphic to*

$$\mathbb{R}^{3g-3+2r}/\mathrm{Mod}_{g,k,p_1,\dots,p_r},$$

where $\mathrm{Mod}_{g,k,p_1,\dots,p_r}$ *is a discrete group of diffeomorphisms.*

Definition 1.0.10 Let (P, τ) be a separating nice Klein orbifold of type $(g, 2r, k, 1, p_1, \dots, p_r)$. Let P_1 and P_2 be the connected components of $P \backslash P^\tau$. If m is odd, then the *topological type* of a real m-Arf function σ on P is a tuple $(g, \tilde{\delta}, k, p_1, \dots, p_r)$, where $\tilde{\delta}$ is the m-Arf invariant of $\sigma|_{P_1}$ and k is the number of ovals of (P, τ).

Real m-Arf functions with even m on separating Klein orbifolds have additional topological invariants. We can define an equivalence relation on the set of ovals of P called similarity (see section 8.5).

Definition 1.0.11 *Let c be an oval in P^τ. If m is even, then the* topological type *of a real m-Arf function σ on P for even m is a tuple*

$$(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r),$$

where $\tilde{\delta}$ is the m-Arf invariant of $\sigma|_{P_1}$, k_j^0 is the number of ovals similar to c with value of σ equal to $j \cdot m/2$ and k_j^1 is the number of ovals not similar to c with value of σ equal to $j \cdot m/2$ on (P, τ).

Theorem 1.0.12 *The space of m-spin bundles on separating nice Klein orbifolds with $g \geq k+1$, orders of marking points ≥ 3 and even m decomposes into the connected components*

$$S(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r),$$

where

$$t = (g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r)$$

satisfies the conditions

(a) If $g > k+1$ and $k_0^0 + k_0^1 \neq 0$ then $\tilde{\delta} = 0$,

(b) If $g > k+1$ and $m \equiv 0 \mod 4$ then $\tilde{\delta} = 0$,

(c) If $g = k+1$ and $k_0^0 + k_0^1 \neq 0$ then $\tilde{\delta} = 1$,

(d) *If* $g = k+1$ *and* $k_0^0 + k_0^1 = 0$ *and* $m \equiv 0 \mod 4$ *then* $\tilde{\delta} = 1$,

(e) *If* $g = k+1$ *and* $k_0^0 + k_0^1 = 0$ *and* $m \equiv 2 \mod 4$ *then* $\tilde{\delta} \in \{1, 2\}$,

(f) *The following degree condition is satisfied*

$$\frac{m}{2} \cdot k_1 - \sum_{i=1}^{r} \frac{1}{p_i} = 1 - g - r \mod m,$$

and $S(t)$ *is the set of all real* m*-spin bundles such that the associated* m*-Arf function is of type* t*. Each of the components* $S(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \ldots, p_r)$ *is a branched covering of the moduli space of Klein surfaces of topological type* $(g, 2r, k_0^0 + k_1^0 + k_0^1 + k_1^1, 1, p_1, \ldots, p_r)$ *and is diffeomorphic to*

$$\mathbb{R}^{3g-3+2r}/\mathrm{Mod}_{g,\tilde{\delta},k_0^0,k_1^0,k_0^1,k_1^1,p_1,\ldots,p_r},$$

where $\mathrm{Mod}_{g,\tilde{\delta},k_0^0,k_1^0,k_0^1,k_1^1,p_1,\ldots,p_r}$ *is a discrete group of diffeomorphisms.*

Theorem 1.0.13 *The space of* m*-spin bundles on separating nice Klein orbifolds with* $g \geq k+1$*, orders of marking points* ≥ 3 *and odd* m *decomposes into the connected components* $S(g, \tilde{\delta}, k, p_1, \ldots, p_r)$*, where* $t = (g, \tilde{\delta}, k, p_1, \ldots, p_r)$ *satisfies the conditions*

(a) *If* $g > k+1$ *then* $\tilde{\delta} = 0$,

(b) *If* $g = k+1$ *then* $\tilde{\delta} = 1$,

(c) $-\sum_{i=1}^{r} \frac{1}{p_i} = (1 - g - r) \mod m$,

and $S(t)$ *is the set of all real* m*-spin bundles such that the associated* m*-Arf function is of type* t*. Each of the components* $S(g, \tilde{\delta}, k, p_1, \ldots, p_r)$ *is a branched covering of the moduli space of Klein surfaces of topological type* $(g, 2r, k, 1, p_1, \ldots, p_r)$ *and is diffeomorphic to*

$$\mathbb{R}^{3g-3+2r}/\mathrm{Mod}_{g,\tilde{\delta},k,p_1,\ldots,p_r},$$

where $\mathrm{Mod}_{g,\tilde{\delta},k,p_1,\ldots,p_r}$ *is a discrete group of diffeomorphisms.*

The thesis is organised as follows:

In Chapter 2 we study the isometries of $\mathbb{H}$, in particular the orientation preserving isometries. These isometries form a group $G^+ = \mathrm{PSL}(2, \mathbb{R})$. We then describe a covering G^+_m of G^+.

Chapter 3 is an introduction to spin structures with general facts on Riemann surfaces from [6] and [7]. In Chapter 4 we study two detailed examples of spin structures from [3].

In Chapter 5 we first establish a connection between m-spin structures on Riemann orbifolds and lifts of Fuchsian groups into the m-fold cover of $\mathrm{PSL}(2, \mathbb{R})$. We study lifts of Fuchsian groups in some detail. Next we introduce the level function, which takes an element of $\mathrm{PSL}(2, \mathbb{R})$ and assigns a number in $\mathbb{Z}/m\mathbb{Z}$. In this way we can define an Arf function $\sigma : \pi^0(P, p) \to \mathbb{Z}/m\mathbb{Z}$. We discuss the definition of m-Arf functions and their topological classification. The main result (Theorem 5.5.11) of this chapter gives conditions for there to be an m-Arf function and hence an m-spin structure on an orbifold with holes, punctures and orbifold points.

In Chapter 6 we discuss Gorenstein quasi-homogeneous surface singularities and establish a correspondence to spin structures.

In Chapter 7 we review general facts on Klein surfaces, including topological classification and their construction from real Fuchsian groups.

Chapter 8 contains the main results. We study higher spin bundles on nice Klein orbifolds, extending the results of paper [17] to surfaces with orbifold points.

Chapter 2

Basics

2.1 Isometries of $\mathbb{H}$ and Aut($\mathbb{H}$) = PSL($2, \mathbb{R}$)

The content of this section is taken from [16].

One model of the hyperbolic plane is the upper half plane model,

$$\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\},$$

equipped with the metric $ds^2 = \frac{|dz|^2}{(Im(z))^2}$.

An *isometry* of the hyperbolic plane is a transformation from $\mathbb{H}$ to itself which preserves distance. Such transformations form a group, we denote it by G. The group has two connected components, G^+ consisting of all orientation-preserving isometries of $\mathbb{H}$ and G^- consisting of all orientation-reversing isometries of $\mathbb{H}$.

Let $j \in G^-$ be the reflection in the imaginary axis, $j(z) = -\overline{z}$, let $h \in G^-$. Their product $g = j \cdot h$ is an element of G^+. This implies $h = j^{-1} \cdot g = j \cdot g \in j \cdot G^+$. (Since j is of order 2, $j^2 = 1$). So we can write the set of orientation reversing isometries as $G^- = j \cdot G^+$ and $G = G^+ \cup G^- = G^+ \cup j \cdot G^+$.

The special linear group $SL(2, \mathbb{R})$ is the group of all real 2×2 matrices with determinant one:

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}$$

The group $G^+ = PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}$ is the group of orientation-preserving isometries of $\mathbb{H}$. We describe G^+ by fractional linear transformations. That is, the action of an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2,\mathbb{R})$ on $\mathbb{H}$ is by $z \mapsto \frac{az+b}{cz+d}$. Elements of G^+ can be classified with respect to the fixed point behaviour of their action on $\mathbb{H}$. An element is called *hyperbolic* if it has 2 fixed points which lie on the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. A hyperbolic element with fixed points α, β in $\mathbb{R}$ is of the form

$$\tau_{\alpha,\beta}(\lambda) = \left[\frac{1}{(\alpha-\beta)\sqrt{\lambda}} \cdot \begin{pmatrix} \lambda\alpha - \beta & -(\lambda-1)\alpha\beta \\ \lambda - 1 & \alpha - \lambda\beta \end{pmatrix}\right],$$

where $\lambda > 0$. The map $\lambda \mapsto \tau_{\alpha,\beta}(\lambda)$ defines a homomorphism $\mathbb{R}_+ \to G^+$ (with respect to the multiplicative structure on $\mathbb{R}_+$).

One of the fixed points of a hyperbolic element is attracting, the other is repelling. The *axis* $l(g)$ of a hyperbolic element g is the geodesic between the fixed points of g, oriented from the repelling fixed point to the attracting fixed point. The element g preserves the geodesic $l(g)$. We call a hyperbolic element with attracting fixed point α and repelling fixed point β *positive* if $\alpha < \beta$. The *shift parameter* of a hyperbolic element g is the minimal displacement $\inf_{x\in\mathbb{H}} d(x, g(x))$.

An element is called *parabolic* if it has one fixed point, which is on the boundary $\partial\mathbb{H}$. A parabolic element with real fixed point α is of the form

$$\pi_\alpha(\lambda) = \left[\begin{pmatrix} 1 - \lambda\alpha & \lambda\alpha^2 \\ -\lambda & 1 + \lambda\alpha \end{pmatrix}\right].$$

The map $\lambda \mapsto \pi_\alpha(\lambda)$ defines a homomorphism $\mathbb{R} \to G^+$ (with respect to the additive structure on $\mathbb{R}$). We call a parabolic element g with fixed point α *positive* if $g(x) > x$ for all $x \in \mathbb{R}\backslash\{\alpha\}$.

A non-identity element that is neither hyperbolic nor parabolic is called *elliptic*. It has one fixed point in $\mathbb{H}$. Given a base point $x \in \mathbb{H}$ and a real number ϕ, let $\rho_x(\phi) \in G^+$ denote the rotation through angle ϕ counter clockwise about x. Any elliptic element is of the form $\rho_x(\phi)$, where x is the fixed point. Thus we obtain a 2π-periodic homomorphism $\rho_x : \mathbb{R} \to G^+$ (with respect to the additive structure on $\mathbb{R}$). We call an elliptic element of the form $\rho_x(\phi)$ with $\phi \in (0, \pi)$ *positive*.

2.2 Coverings of the Group $G = \mathbf{Isom}(\mathbb{H})$

The material of this section follows [17], but the exposition of the proofs is more detailed.

We introduce the following description of the Lie group coverings of $G = \mathrm{Isom}(\mathbb{H})$, compare with definition 2.1 in [17].

Proposition 2.2.1 *The m-fold covering group of $G^+ = PSL(2,\mathbb{R})$ can be described as*

$$G_m^+ = \{(g,\delta) \in G^+ \times Hol(\mathbb{H},\mathbb{C}^*) : \delta^m(z) = g'(z) \quad \forall z \in \mathbb{H}\}$$

with multiplication $(g_2,\delta_2)\cdot(g_1,\delta_1) = (g_2\cdot g_1, (\delta_2\circ g_1)\cdot\delta_1)$.

Proof Topologically G^+ is an open solid torus, $\pi(G^+) \cong \mathbb{Z}$, hence there exists an m-fold covering of G^+ which is unique up to a homeomorphism. One can check that the space G_m^+ is connected and that the map $G_m^+ \to G^+$ given by $(\gamma,\delta) \mapsto \gamma$ is an m-fold covering of G^+. Hence $G_m^+ \to G^+$ is the topological m-fold covering of G^+. Since G^+ is a Lie group there is a unique group structure G_m^+ Let us now check that the multiplication formula above indeed describes a group structure on G_m^+.

Let (g_1,δ_1), $(g_2,\delta_2) \in X$. We have $(g_2,\delta_2)\cdot(g_1,\delta_1) = (g_2\cdot g_1,(\delta_2\circ g_1)\cdot\delta_1)$ which belongs to X if $(g_2\cdot g_1)' = ((\delta_2\circ g_1)\cdot\delta_1)^m$. Using that $\delta_1^m = g_1'$ and $\delta_2^m = g_2'$ we can write the left hand side as

$$(g_2\circ g_1)' = g_2'(g_1)\cdot g_1' = \delta_2^m(g_1)\cdot\delta_1^m = ((\delta_2\circ g_1)\cdot\delta_1)^m,$$

so X is closed under multiplication.

To see the operation is associative, let (g_1,δ_1), (g_2,δ_2), $(g_3,\delta_3) \in X$. We have

$$\begin{aligned}
((g_1,\delta_1)\cdot(g_2,\delta_2))\cdot(g_3,\delta_3) &= (g_1\cdot g_2,(\delta_1\circ g_2)\cdot\delta_2)\cdot(g_3,\delta_3)\\
&= ((g_1\cdot g_2)\cdot g_3,[((\delta_1\circ g_2)\cdot\delta_2)\circ g_3]\circ\delta_3).\\
(g_1,\delta_1)\cdot((g_2,\delta_2)\cdot(g_3,\delta_3)) &= (g_1,\delta_1)\cdot(g_2\cdot g_3,(\delta_2\circ g_3)\cdot\delta_3)\\
&= (g_1\cdot(g_2\cdot g_3),(\delta_1\circ(g_2\cdot g_3))\cdot((\delta_2\circ g_3)\cdot\delta_3)).
\end{aligned}$$

Since $g_1, g_2, g_3 \in G^+$ and G^+ is a group, associativity is satisfied for the first component in the brackets. For the second component we see

$$\begin{aligned}([((\delta_1 \circ g_2) \cdot \delta_2) \circ g_3] \cdot \delta_3)(z) &= ([(\delta_1(g_2) \cdot \delta_2) \circ g_3] \cdot \delta_3)(z) \\ &= [\delta_1(g_2(g_3(z))) \cdot \delta_2(g_3(z))] \cdot \delta_3(z) \\ &= ((\delta_1 \circ (g_2 \cdot g_3)) \cdot (\delta_2 \circ g_3) \cdot \delta_3)(z).\end{aligned}$$

For the identity element of G^+ we want

$$(e, \delta_e) \cdot (g, \delta) = (g, \delta) \cdot (e, \delta_e) = (g, \delta) \text{ for all } (g, \delta) \in X.$$

That is $(g, \delta) = (e \cdot g, (\delta_e \circ g) \cdot \delta) = (g \cdot e, (\delta \circ e) \cdot \delta_e)$.

1. In the first component $e \cdot g = g \cdot e = g$ for all $g \in G^+ = PSL(2, \mathbb{R})$. This implies e is the identity of $PSL(2, \mathbb{R})$. So $e(z) = z$.

   Since $(e, \delta_e) \in X$ we must have $\delta_e^m = \frac{d}{dz}e$. Since $e(z) = z$ we have $\frac{de}{dz} = 1$. Hence $(\delta_e(z))^m = 1$ for all $z \in \mathbb{H}$. We have m choices for δ_e, that is $\delta_e(z) = 1\ \forall z$, $\delta_e(z) = \exp\left(\frac{2\pi i}{m}\right)\ \forall z$, $\delta_e(z) = \exp\left(\frac{4\pi i}{m}\right)\ \forall z$, $\delta_e(z) = \exp\left(\frac{6\pi i}{m}\right)\ \forall z$ etc.

2. $(\delta_e \circ g) \cdot \delta = (\delta \circ e) \cdot \delta_e = \delta$ for all $(g, \delta) \in X$. If $\delta_e(z) = 1\ \forall z$ then we have $(\delta_e \circ g)(z) = 1$ and $(\delta_e \circ g)(z) \cdot \delta(z) = 1 \cdot \delta(z) = \delta(z)$. If $\delta_e(z) = \exp\left(\frac{2\pi k i}{m}\right)\ \forall z$ with $k = 1, \ldots, m-1$, then $(\delta_e \circ g)(z) \cdot \delta(z) = \exp\left(\frac{2\pi k i}{m}\right) \delta(z) \neq \delta(z)$.

Therefore the identity in G^+ is (e, δ_e), where $e : \mathbb{H} \to \mathbb{H}$ is the identity map and $\delta_e(z) = 1$ for all $z \in \mathbb{H}$.

For $(g, \delta) \in X$ take the inverse to be $(g^{-1}, \delta^{-1} \circ g^{-1})$. Then we have

$$(g, \delta) \cdot (g^{-1}, \delta^{-1} \circ g^{-1}) = (g \cdot g^{-1}, (\delta \circ g^{-1}) \cdot (\delta^{-1} \circ g^{-1})) = (e, \delta(g^{-1}) \cdot \delta^{-1}(g^{-1})) = (e, 1),$$

and similarly $(g^{-1}, \delta^{-1} \circ g^{-1}) \cdot (g, \delta) = (e, 1)$. $\square$

The Lie group covering of the full isometry group $G = \mathrm{Isom}(\mathbb{H}) = G^+ \cup G^-$ can be described as follows:

Definition 2.2.2 Let $\pi : G_m \to G$ be the Lie group m-fold covering of G given by $G_m = G_m^+ \cup G_m^-$ with:

$$G_m^+ = \left\{ (g, \delta) \in G^+ \times \mathrm{Hol}(\mathbb{H}, \mathbb{C}^*) | \delta^m = \frac{d}{dz} g \right\},$$

$$G_m^- = \left\{ (g, \delta) \in G^- \times \overline{\mathrm{Hol}}(\mathbb{H}, \mathbb{C}^*) | \delta^m = \frac{d}{d\bar{z}} g \right\},$$

and the product of elements in G_m given by

$$(g_2, \delta_2) \cdot (g_1, \delta_1) = \begin{cases} (g_2 \circ g_1, (\delta_2 \circ g_1) \cdot \delta_1) & \text{if} \quad (g_2, \delta_2) \in G_m^+ \\ (g_2 \circ g_1, (\delta_2 \circ g_1) \cdot \overline{\delta_1}) & \text{if} \quad (g_2, \delta_2) \in G_m^-. \end{cases}$$

The identity element of G_m is $e_{G_m} = (e_G, 1)$, where e_G is the identity in G and 1 is the constant function $\delta(z) = 1$.

Let $J \in G_m^-$ be one of the m pre-images of the reflection in the imaginary axis j, then $G_m^+ = J \cdot G_m^-$.

Proposition 2.2.3 *For a pre-image $J \in G_m$ of j we have $J^2 = e_{G_m}$.*

Proof The element J must be of the form $J = (j, \delta)$ with $\delta^m = \frac{d}{d\bar{z}} j = -1$, i.e $\delta : \mathbb{H} \to \mathbb{C}^*$ is a constant function with $\delta^m = -1$. Hence $J^2 = (j, \delta) \cdot (j, \delta) = (j \circ j, (\delta \circ j) \cdot \overline{\delta}) = (e_G, |\delta|^2) = (e_G, 1) = e_{G_m}$. Note here we use that $\delta \cdot \overline{\delta} = |\delta|^2$. $\square$

Elements of G_m^+ can be classified with respect to the fixed point behaviour of the action on $\mathbb{H}$ of their image in G^+. We say that an element of G_m^+ is hyperbolic, parabolic, elliptic if its image in G^+ has this property.

The homomorphisms

$$\tau_{\alpha,\beta} : \mathbb{R}_+ \to G^+, \quad \pi_\alpha : \mathbb{R} \to G^+, \quad \rho_x : \mathbb{R} \to G^+$$

define one-parameter subgroups in the group G^+. Each of these homomorphisms lifts to a unique homomorphism in the m-fold cover:

$$T_{\alpha,\beta} : \mathbb{R}_+ \to G_m^+, \quad P_\alpha : \mathbb{R} \to G_m^+, \quad R_x : \mathbb{R} \to G_m^+.$$

The elements $T_{\alpha,\beta}(\lambda)$, $P_\alpha(\lambda)$ and $R_x(\xi)$ are called hyperbolic, parabolic and elliptic respectively.

Proposition 2.2.4 *We have* $j\tau_{\alpha,\beta}(\lambda)j^{-1} = \tau_{-\alpha,-\beta}(\lambda)$, $j\pi_\alpha(\lambda)j^{-1} = \pi_{-\alpha}(-\lambda)$, $j\rho_x(t)j^{-1} = \rho_{-\overline{x}}(-t)$. *In particular* $j\tau_{0,\infty}(\lambda)j^{-1} = \tau_{0,\infty}(\lambda)$, $j\pi_0(\lambda)j^{-1} = \pi_0(-\lambda)$, $j\rho_i(t)j^{-1} = \rho_i(-t)$.

Proof The statement can be easily shown geometrically by looking at the fixed points of the isometries of the form $j\phi j^{-1}$. Alternatively, we can use explicit formulas for different types of isometries. For the first identity, take a point $z \in \mathbb{H}$. For the left hand side we have

$$\begin{aligned}(j\tau_{\alpha,\beta}(\lambda)j^{-1})(z) &= j\tau_{\alpha,\beta}(\lambda)(-\overline{z}) \\ &= j\left(\frac{(\lambda\alpha-\beta)(-\overline{z})-(\lambda-1)\alpha\beta}{(\lambda-1)(-\overline{z})+\alpha-\lambda\beta}\right) \\ &= -\overline{\left(\frac{(\lambda\alpha-\beta)(-\overline{z})-(\lambda-1)\alpha\beta}{(\lambda-1)(-\overline{z})+\alpha-\lambda\beta}\right)}.\end{aligned}$$

Since $\alpha, \beta, \lambda \in \mathbb{R}$ we have $\overline{\alpha} = \alpha$, $\overline{\beta} = \beta$ and $\overline{\lambda} = \lambda$, so we have

$$\begin{aligned}(j\tau_{\alpha,\beta}(\lambda)j^{-1})(z) &= -\frac{(\lambda\alpha-\beta)(-z)-(\lambda-1)\alpha\beta}{(\lambda-1)(-z)+\alpha-\lambda\beta} \\ &= -\frac{(\lambda\alpha-\beta)z+(\lambda-1)\alpha\beta}{(\lambda-1)z-\alpha+\lambda\beta}.\end{aligned}$$

For the right hand side we have

$$\begin{aligned}\tau_{-\alpha,-\beta}(\lambda)(z) &= \frac{(\lambda(-\alpha)-(-\beta))z-(\lambda-1)(-\alpha)(-\beta)}{(\lambda-1)z+(-\alpha)-\lambda(-\beta)} \\ &= -\frac{(\lambda\alpha-\beta)z+(\lambda-1)\alpha\beta}{(\lambda-1)z-\alpha+\lambda\beta}.\end{aligned}$$

Similarly for $\pi_\alpha(\lambda)$ and $\rho_x(t)$. $\square$

2.3 Centres

The material on the centre of a group is basic algebra. The details of the examples are own calculations.

Let $(G, *)$ be a group. The centre $Z(G)$ is defined by

$$Z(G) = \{h \in G | h * g = g * h\, \forall g \in G\}$$

and is a commutative subgroup of G.

1. $G = SL(2,\mathbb{R})$. For the centre $Z(G)$ we need $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \forall \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in G$$

   That is we need $a, b, c, d \in \mathbb{R}$ such that

$$\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}.$$

   We have $ae+bg = ea+fc$, $af+bh = eb+fd$, $ce+dg = ga+hc$ and $cf + dh = gb + hd$. The first equation implies $bg = fc$, which must hold for all $f, g \in \mathbb{R}$ implying $c = b = 0$. Putting $b = 0$ in the second equation we get that $a = d$. So elements of the centre are of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and we require $a^2 = 1$. Thus $Z(G) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$. Clearly these two matrices commute with any other matrix.

2. $G = PSL(2,\mathbb{R}), Z(G) = \left\{ \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \right\}$.

3. Consider the m-fold covering G_m^+ of G^+. For $(g,\delta) \in Z(G_m^+)$ we need $(g,\delta)\cdot(g_1,\delta_1) = (g_1,\delta_1)\cdot(g,\delta)$ for any $(g_1,\delta_1) \in G_m^+$.

   That is we need $(g \cdot g_1, (\delta \circ g_1)\cdot \delta_1) = (g_1 \cdot g, (\delta_1 \circ g) \cdot \delta)$.

   So we need $g \in PSL(2,\mathbb{R})$ to commute with every $g_1 \in PSL(2,\mathbb{R})$. That is, we require g such that $g \cdot g_1 = g_1 \cdot g$ for any $g_1 \in PSL(2,\mathbb{R})$. The centre of $PSL(2,\mathbb{R})$ consists of the identity, therefore g is the identity map $g(z) = z$.

   Now $g(z) = z$ and $g'(z) = 1$ and we require $g'(z) = \delta^m(z)\ \forall z \in \mathbb{H}$ since $(g,\delta) \in G_m^+$. So $\delta^m(z) = 1$ and we have m possibilities: $\delta(z) = 1\ \forall z$, $\delta(z) = exp(\frac{2\pi i}{m})\ \forall z$, $\delta(z) = \exp(\frac{2\pi i}{m}\cdot 2)\ \forall z$ etc.

We need $(\delta\circ g_1)\cdot\delta_1 = (\delta_1\circ g)\cdot\delta$. If $\delta(z)=1\ \forall z$ then $(\delta\circ g_1)\cdot\delta_1=\delta_1$ and $((\delta_1\circ g)\cdot\delta)(z)=(\delta_1(g(z)))\cdot\delta(z)=(\delta_1(z))\cdot 1=\delta_1(z)$. So $(e,1)\in Z(G_m)$.

If $\delta(z)=\exp(\frac{2\pi i}{m})^k\ \forall z$ then $\delta(g_1(z))\cdot\delta_1(z)=\exp(\frac{2\pi i}{m})^k\cdot\delta_1(z)$ and $\delta_1(g(z))\cdot\delta(z)=\delta_1(z)\cdot\exp(\frac{2\pi i}{m})^k$. So $(e,\exp(\frac{2\pi i}{m})^k)\in Z(G_m)$.

Hence $Z(G_m)=\{(e,1),(e,\exp(\frac{2\pi i}{m})),(e,\exp(\frac{4\pi i}{m})),\cdots(e,\exp\left(\frac{2\pi i(m-1)}{m}\right))\}$.

We have $|Z(G_m)|=m$. Note that $Z(G_m)=\pi^{-1}(e)$, the preimage of the identity $e\in G^+$.

Let $U=(e,\exp(\frac{2\pi i}{m}))$. We proved that U is a generator of $Z(G_m)$.

Recall that the homomorphism $\rho_x:\mathbb{R}\to G^+$ of elliptic elements with the fixed point $x\in\mathbb{H}$ lifts to a homomorphism $R_x:\mathbb{R}\to G^+_m$. For $x=i$ it is easy to check that $R_i(2\pi)=(e_g,\exp(\frac{2\pi i}{m}))=U$ and $R_i(2\pi l)=(R_i(2\pi))^l=U^l\in Z(G^+_m)=\pi^{-1}(e_G)$. Note that for any $x\in\mathbb{H}$ we have $\pi(R_x(2\pi l))=\rho_x(2\pi l)=e_G$, hence $R_x(2\pi l)\in\pi^{-1}(e_G)=Z(G_m)$.

Note that the element $R_x(2\pi l)$ depends continuously on x, but the centre of G_m is discrete, so the element $R_x(2\pi l)$ must remain constant. Thus $R_x(2\pi l)$ does not depend on x. We obtain that$R_x(2\pi l)=U^l$ and

$$Z(G_m)=\{R_x(2\pi l)|l=0,\ldots,m-1\}$$

for any $x\in\mathbb{H}$.

Lifting the identities in Proposition 2.2.4 into G_m we have, following [17]:

Proposition 2.3.1

$$JT_{\alpha,\beta}(\lambda)J^{-1}=J^{-1}T_{\alpha,\beta}(\lambda)J=T_{-\alpha,-\beta}(\lambda),$$

$$JP_\alpha(\lambda)J^{-1}=J^{-1}P_\alpha(\lambda)J=P_{-\alpha}(-\lambda),$$

$$JR_x(t)J^{-1}=J^{-1}R_x(t)J=R_{-\overline{x}}(-t).$$

In particular

$$JT_{0,\infty}(\lambda)J^{-1}=J^{-1}T_{0,\infty}(\lambda)J=T_{0,\infty}(\lambda),$$

$$JP_0(\lambda)J^{-1} = J^{-1}P_0(\lambda)J = P_0(-\lambda),$$
$$JR_i(t)J^{-1} = J^{-1}R_i(t)J = R_i(-t),$$
$$JUJ^{-1} = J^{-1}UJ = U^{-1}.$$

Chapter 3

Introduction to Spin Bundles

The discussion of Riemann surfaces and vector bundles on them in this chapter is mostly based on [6] and [8].

3.1 Riemann Surfaces

Definition 3.1.1 A *Riemann surface* is a connected complex analytic manifold of complex dimension one, that is, a connected manifold M of real dimension two with a maximal set of charts $\{U_\alpha, z_\alpha\}_{\alpha \in A}$ on M (that is, $\{U_\alpha\}_{\alpha \in A}$ constitutes an open cover of M and

$$z_\alpha : U_\alpha \to \mathbb{C}$$

is a homeomorphism onto an open subset of the complex plane $\mathbb{C}$) such that the *transition functions*

$$f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic whenever $U_\alpha \cap U_\beta \neq \emptyset$.

Definition 3.1.2 A continuous mapping $f : M \to N$ between Riemann surfaces M, N is called *holomorphic* if for every local coordinate $\{U, z\}$ on M and every local coordinate $\{V, \zeta\}$ on N with $U \cap f^{-1}(V) \neq \emptyset$, the mapping $\zeta \circ f \circ z^{-1} : z(U \cap f^{-1}(V)) \to \zeta(V)$ is holomorphic (as a mapping from $\mathbb{C}$ to $\mathbb{C}$).

Definition 3.1.3 A *holomorphic function* on a Riemann surface M is a holomorphic mapping from M to $\mathbb{C}$.

Definition 3.1.4 A *meromorphic function* on a Riemann surface M is a holomorphic mapping from M to the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Theorem 3.1.5 (Uniformization Theorem) *Let M be a simply connected Riemann surface. Then M is conformally equivalent to one and only one of the following:*

1. *The Riemann sphere $\mathbb{C} \cup \{\infty\}$,*
2. *The complex plane $\mathbb{C}$,*
3. *The hyperbolic plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$.*

The surface is then called elliptic, parabolic or hyperbolic respectively.

We are interested in the hyperbolic case.

Definition 3.1.6 A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

Using the uniformization theorem we can write a hyperbolic Riemann surface as a quotient of the hyperbolic plane by a Fuchsian group.
Topologically, each compact connected surface is a sphere with g handles. We call g the genus of the surface. If the genus of the surface M is known we can calculate the Euler characteristic, $\chi(M) = 2 - 2g$. For example the Euler characteristic of a sphere (genus 0) is 2, the Euler characteristic of a torus (genus 1) is 0 and for surfaces of genus greater than or equal to 2 the Euler characteristic is negative.

Given a triangulation of a surface M we can calculate $\chi(M)$ according to the following formula:

$$\chi(M) = \#\text{verticies} - \#\text{edges} + \#\text{triangles}.$$

3.1.1 Construction by Gluing

We can construct hyperbolic Riemann surfaces by considering a fundamental domain for a Fuchsian group and gluing together the edges according to the group action. We consider an example from [3].

Example 3.1.7 *The surface M_4 is constructed as follows. Take a polygon in the hyperbolic plane consisting of 24 isometric regular triangles whose angles are $\frac{\pi}{6}$, see Figure 3.1. The boundary identifications are given by* $1-16$, $3-18$, $5-20$, $7-22$, $9-24$, $11-2$, $13-4$, $15-6$, $17-8$, $19-10$, $21-12$, $23-14$.

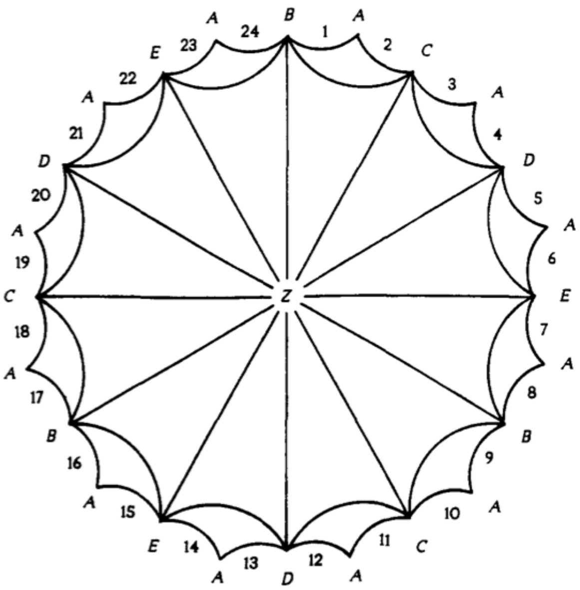

Figure 3.1: Fundamental domain in the hyperbolic plane for surface M_4.

Identifying given edges in our example M_4 (see Figure 3.1) we have 24 triangles, 36 edges and 6 vertices A, B, C, D, E, Z. We have

$$\chi(M) = 6 - 36 + 24 = -6 = 2 - 2g,$$

giving $g = 4$.

3.2 Bundles

Definition 3.2.1 An *n-dimensional vector bundle* over a field $\mathbb{F}$ is a map $\rho : E \to B$ together with an $\mathbb{F}$-vector space structure on $\rho^{-1}(b)$, for each $b \in B$, such that the following local triviality condition is satisfied: There is a cover of B by open sets U_α for each of which there exists a homeomorphism:

$$h_\alpha : \rho^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F}^n,$$

taking $\rho^{-1}(b)$ to $\{b\} \times \mathbb{F}^n$ by a vector space isomorphism for each $b \in U_\alpha$. Such a homeomorphism h_α is called a *local trivialization* of the vector bundle.

Here B is called the base space, E the total space and the vector spaces $\rho^{-1}(b)$ are called the fibres.

We refer to n as the *rank* of the bundle E over $\mathbb{F}$.

For example taking $\mathbb{F} = \mathbb{C}$ we get a complex bundle, or taking $\mathbb{F} = \mathbb{R}$ we get a real bundle.

Definition 3.2.2 A *line bundle* is a vector bundle of rank one.

Example 3.2.3 *The trivial bundle $E = B \times \mathbb{F}^N$ with $p : E \to B$ the projection onto the first factor, $p : (b, v) \to b$.*

We are interested in complex line bundles on Riemann surfaces.

Definition 3.2.4 An *isomorphism* between vector bundles $\rho_1 : E_1 \to B$ and $\rho_2 : E_2 \to B$ over the base space B is a homeomorphism $h : E_1 \to E_2$ taking each fibre $\rho_1^{-1}(b)$ to the corresponding fibre $\rho_2^{-1}(b)$ by a linear isomorphism.

3.2.1 Gluing Maps/ Cocycles

We will describe a method for constructing vector bundles. Given a vector bundle $\rho : E \to B$ and an open cover $\{U_\alpha\}$ of B with local trivializations $h_\alpha : \rho^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F}^n$, we can reconstruct E as the quotient space of the disjoint union $\amalg_\alpha (U_\alpha \times \mathbb{F}^n)$ obtained by identifying $(x, v) \in U_\alpha \times \mathbb{F}^n$ with $h_\beta h_\alpha^{-1}(x, v) \in U_\beta \times \mathbb{F}^n$ whenever $x \in U_\alpha \cap U_\beta$. The functions $h_\beta h_\alpha^{-1}$ can be viewed as maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_n(\mathbb{F})$. These satisfy the cocycle condition $g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Any collection of 'gluing functions' $g_{\beta\alpha}$ (also called a cocycle $(g_{\beta\alpha})$) satisfying this condition can be used to construct a vector bundle.

3.2.2 Cotangent Bundle

Let X be a Riemann surface and $\{U_i\}$ an open covering, $z_i : U_i \to \mathbb{C}$. On $U_i \cap U_j$ the function $g_{ij} = \frac{dz_j}{dz_i}$ is holomorphic and does not vanish. The functions satisfy the cocycle relation:

$$g_{jk} g_{ki} = \frac{dz_k}{dz_j} \frac{dz_i}{dz_k} = \frac{dz_i}{dz_j} = g_{ji}.$$

The line bundle associated with the cocycle (g_{ij}) is the cotangent bundle.

Remark For higher dimensional complex varieties, the canonical bundle is the n-th tensor power of the cotangent bundle, where n is the complex dimension of the variety. For Riemann surfaces $n = \dim_{\mathbb{C}} = 1$, hence the cotangent bundle is equivalent to the canonical bundle.

3.2.3 Operations with Bundles: Tensor Product

Suppose we have two line bundles $E = (U_i, g_{ij})$ and $F = (U_i, f_{ij})$, over the same base, where (U_i) is an open cover and g_{ij}, f_{ij} are gluing functions. Then we can define the *tensor product* in the following way:

$$E \otimes F = (U_i, g_{ij} \cdot f_{ij}).$$

3.2.4 Divisors

Definition 3.2.5 A *divisor* $\mathcal{U}$ on M is a mapping

$$\alpha : M \to \mathbb{Z}$$

which takes on non zero values for only finitely many points on M.

We can write the divisor $\mathcal{U}$ as:

$$\mathcal{U} = \prod_{j=1}^{m} P_j^{\alpha_j},$$

with $P_j \in M$, $\alpha_j \in \mathbb{Z}$.

If $\mathcal{U} = \prod_{j=1}^{m} P_j^{\alpha_j}$ and $\mathcal{B} = \prod_{j=1}^{m} P_j^{\beta_j}$, then we have:

$$\mathcal{U}\mathcal{B} = \prod_{j=1}^{m} P_j^{\alpha_j+\beta_j}$$

and

$$\mathcal{U}^{-1} = \prod_{j=1}^{m} P_j^{-\alpha_j}.$$

To any meromorphic function f on a compact Riemann surface M we can associate a divisor, denoted by (f). To each zero we associate the order of the zero and to each pole we associate minus the order of the pole, to all other points we associate zero.

Definition 3.2.6 The *degree* of the divisor $\mathcal{U} = \prod_{j=1}^{m} P_j^{\alpha_j}$ is the sum $\sum_{j=1}^{m} \alpha_j$.

Definition 3.2.7 A divisor that is the divisor of a meromorphic function is called *principal*. The degree of a principal divisor is 0 (since a meromorphic function has as many poles as zeros).

Definition 3.2.8 Two divisors that differ by a principal divisor are called *linearly equivalent*. That is, two divisors D_1, D_2 are linearly equivalent if $D_1 - D_2 = (f)$ for some meromorphic function f.

3.2.5 Connection between Line Bundles and Divisors

Let D be a divisor on a Riemann surface X. There exists an open covering $\{U_i\}$ of X and meromorphic functions ψ_i on U_i with $(\psi_i) = D$ on U_i. On $U_i \cap U_j$ we can take

$$g_{ij} := \frac{\psi_i}{\psi_j}.$$

The function $g_{ij} : U_i \cap U_j \to \mathbb{C}$ is holomorphic since ψ_i and ψ_j have the same zeros and poles with the same orders on $U_i \cap U_j$. The family (g_{ij}) forms a cocycle. The cocycle condition is satisfied:

$$g_{ik}g_{kj} = \frac{\psi_i}{\psi_k}\frac{\psi_k}{\psi_j} = \frac{\psi_i}{\psi_j} = g_{ij}.$$

The line bundle determined by the cocycle (g_{ij}) is denoted by (D).

Remark If we have a bundle E corresponding to the divisor D, then the mth tensor power of E corresponds to the divisor mD.

Definition 3.2.9 The *canonical divisor* is the divisor of the cotangent bundle. Any divisor linearly equivalent to the divisor of the cotangent bundle is also called canonical.

3.2.6 Spin Bundles

Definition 3.2.10 A *spin bundle* on a Riemann surface is a complex line bundle E such that the tensor square of E is isomorphic to the cotangent bundle of the surface. Equivalently we can require that twice the divisor of E corresponds to the canonical divisor.

Definition 3.2.11 Let m be an integer, $m \geq 2$. An *m-spin structure* on a Riemann surface is a complex line bundle E such that the m-th tensor power of E is isomorphic to the cotangent bundle of the surface. Equivalently m times the divisor of E corresponds to the canonical divisor.

Chapter 4

Examples of Spin Bundles

This chapter looks at two examples of spin structures. The general facts on Riemann surfaces are from [6]. Here we study examples from [3] in more detail.

4.1 Weierstrass Gap Theorem and Hyperelliptic Surfaces

Theorem 4.1.1 (The Weierstrass "gap" Theorem [6]) *Let M be a Riemann surface of positive genus g, and let $p \in M$ be arbitrary. There are precisely g integers*

$$1 = n_1 < n_2 < \cdots < n_g < 2g$$

such that there does not exist a meromorphic function f on M holomorphic on $M \backslash \{p\}$ with a pole of order n_j at p.

Remark The numbers n_j are called the gaps at p. Their complement in the positive integers are called the non-gaps.

To proceed we note that if m_1 and m_2 are non-gaps this means that we can have a function f_i, $i = 1, 2$, with a pole of order m_i at the point p; hence $m_1 + m_2$ is also a non-gap as $f_1 \cdot f_2$ has a pole of order $m_1 + m_2$ at p. So the non-gaps form an additive semi-group.

Definition 4.1.2 (Semi-Group) A *semi-group* is a set S and a binary operation * satisfying:

1. Closure: $\forall a, b \in S, a * b \in S$
2. Associativity: $\forall a, b, c \in S,\ (a * b) * c = a * (b * c)$

There are precisely g non-gaps in $\{2, \ldots 2g\}$ with $2g$ always a non-gap. These are the first g non-gaps in the semi-group of non-gaps.

Definition 4.1.3 The *weight* w of a point p on a Riemann surface is defined by $w = \sum_{i=1}^{g}(\rho_i - i)$, where $1 = \rho_1 < \rho_2 < \rho_3 < \ldots < \rho_g < 2g$ is the gap sequence of p. The weight of a point is always non-negative, the *Weierstrass points* are defined as exactly those points with positive weights.

Theorem 4.1.4 *On a Riemann surface of genus g the total weight is*

$$\sum_{p \in M} w(p) = (g-1)g(g+1).$$

Theorem 4.1.5 *If a non trivial automorphism of a Riemann surface of genus ≥ 2 has at least 5 fixed points, then all fixed points are Weierstrass points. [6]*

Definition 4.1.6 A Riemann surface M is called *hyperelliptic* provided there exists a non-constant meromorphic function on M with precisely 2 poles.

Proposition 4.1.7 *A hyperelliptic Riemann surface has precisely $2g + 2$ Weierstrass points. (See [6] page 94).*

Proposition 4.1.8 *For a point p on a Riemann surface M, $\nu \in \mathbb{N}$, $h_0(\nu p) = \dim_{\mathbb{C}}(M; \mathcal{O}(\nu p))$ corresponds to the number of linearly independent holomorphic sections in the bundle corresponding to the divisor νp. We can calculate h_0 using the formula: $h_0(\nu p) = \nu + 1 - \#\{gaps \leq \nu$ in the Weierstrass gap sequence of $p\}$. (See [8]).*

Remark If a line bundle L satisfies $\deg(L) = 2g - 2$ and $h_0(L) = g$ then L is canonical.

4.2 Non-Hyperelliptic Example of Genus 4

We will study m-spin bundles on the surface M_4 we constructed in Example 3.1.7.

4.2.1 Weierstrass points on M_4

The rotation around Z by $\frac{2\pi}{3}$ is an automorphism with A, B, C, D, E and Z as fixed points; hence Theorem 4.1.5 implies that they are Weierstrass points.

A reflection at the centre of an edge is an automorphism with 6 fixed points; hence all 36 edge centre points are Weierstrass points.

According to Proposition 4.1.7 on a hyperelliptic surface of genus $g = 4$ there are $2g+2 = 10$ Weierstrass points. Since we have already identified 6 vertices and 36 centres of the edges to be Weierstrass points on M_4, our surface M_4 can not be hyperelliptic.

M_4 has genus 4 so the total weight is $(4-1)4(4+1) = 3\cdot4\cdot5 = 60$, according to Theorem 4.1.4. Hence there are at most 60 Weierstrass points on M_4.

Definition 4.2.1 A group action $G \times X \to X$ is *transitive* if it possesses only a single orbit. That is, $\forall x, y \in X \ \exists\, g \in G$ such that $gx = y$.

The isometry group acts transitively on the set of triangles, so if we have any Weierstrass points in the interior of a triangle then there would be Weierstrass points in the interior of every triangle giving 24 more Weierstrass points. But then we would have $42 + 24 = 66$ Weierstrass points which can not happen since it would exceed the total weight of 60. Similarly we have no other Weierstrass points on the edges. So we have 6 vertices and 36 centre points of the edges with total weight 60. All vertices have the same weight, say x, all centres of the edges have the same weight, say y. Hence:

$$6x + 36y = 60.$$

Note that x and y are positive integers. Hence the vertices must have weight 4 and the edge centre points weight 1.

4.2.2 Gap Sequence of the point Z

We now look at the gap sequence of the vertex Z. Our surface has genus 4 thus the gap sequence is as follows:

$$1 = n_1 < n_2 < n_3 < n_4 < 2g = 8.$$

We know Z is a vertex and so has weight 4, hence using Definition 4.1.3 we have:

$$\sum_{i=1}^{g}(n_i - i) = 4.$$

If 2 was a non-gap, then 4, 6 would be non-gaps, then the gap sequence could only be $1 < 3 < 5 < 7$. For this gap sequence the weight is $(1-1)+(3-2)+(5-3)+(7-4) = 6$, but we know that the weight of Z is 4. Thus 2 must be a gap. Hence we have $n_2 = 2$ and $3 \leq n_3 < n_4 \leq 7$.

Since $n_1 = 1$ and $n_2 = 2$, the first two terms of the weight evaluate to zero and we have:

$$(n_3 - 3) + (n_4 - 4) = 4 \Rightarrow n_3 + n_4 = 11.$$

If $n_3 = 3$ then we have $n_4 = 8$, which is not possible since we require $n_4 \leq 7$. Similarly, if $n_3 = 6$ we have $n_4 = 5$ which can not be the case since we need $n_3 < n_4$. We are left with two possibilities:

$$n_3 = 4 \Rightarrow n_4 = 7,$$

$$n_3 = 5 \Rightarrow n_4 = 6.$$

So the gap sequence for Z is either:

$$1 < 2 < 4 < 7 \text{ or } 1 < 2 < 5 < 6.$$

If $1 < 2 < 5 < 6$ is our gap sequence, then 3, 4 and 7 are non-gaps. Since 3 is a non-gap we have that 6 is also a non-gap, however 6 is in the gap sequence. Contradiction. We are left with $1 < 2 < 4 < 7$ as our gap sequence. Note here the non-gaps 3, 5 and 6 are contained in an additive semi-group.

4.2.3 Higher Spin Bundles on M_4

The number of holomorphic sections in the bundle corresponding to the divisor $6Z$ is $h_0(6Z) = 6 + 1 - \#\{\text{gaps} \leq 6 \text{ in the Weierstrass gap sequence of } Z\} = 4 = g$. Hence $6Z$ is canonical. Therefore $3Z$ defines a 2-spin structure, $2Z$ defines a 3-spin structure and Z defines a 6-spin structure.

Remark Bär -Schmutz use this example in [3] to construct examples of non-hyperelliptic Riemann surfaces with largest possible dimension of the space of positive harmonic spinors.

4.3 Non-Hyperelliptic Example of Genus 6

4.3.1 Construction of M_6

We construct the surface M_6 as follows. The fundamental domain in the hyperbolic plane consists of 50 isometric regular triangles whose angles are

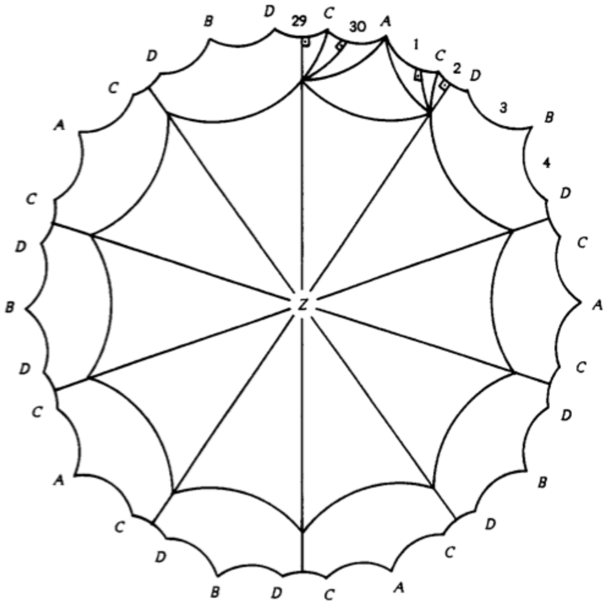

Figure 4.1: Fundamental domain in the hyperbolic plane for surface M_6.

$\frac{\pi}{5}$, see Figure 4.1. The boundary does not consist of the edges of theses triangles but of the geodesics dividing the triangles into smaller triangles. They go from one vertex to the centre point of the opposite edge. The boundary identifications are as follows: $2-17$, $5-20$, $8-23$, $11-26$, $14-29$, $1-24$, $4-27$, $7-30$, $10-3$, $13-6$, $16-9$, $19-12$, $22-15$, $25-18$, $28-21$. As in the previous section we calculate $\chi(M_6) = 15 - 75 + 50 = -10 = 2 - 2g$, giving $g = 6$. So M_6 is a surface of genus 6.

4.3.2 Weierstrass points on M_6

Some isometries of M_6 are the rotation around a vertex with angle $\frac{\pi}{5}$, a reflection at the centre point of an edge, the rotation around the centre point of a triangle with angle $\frac{2\pi}{3}$.

The isometry group acts transitively on the set of vertices, on the set of triangles and on the set of centre points of edges.

Using Theorem 4.1.4 we calculate the total weight to be 210. Hence there are at most 210 Weierstrass points.

The reflection at Z has 6 fixed points (Z and the 5 centre points of edges from C to D). Hence by Theorem 4.1.5, all these 6 fixed points are Weierstrass points.

If there are further Weierstrass points they either lie in the interior of a triangle (50) or appear twice on each edge ($75 \times 2 = 150$) so their number must be divisible by 50. The total weight is 210, which is divisible by 3. Since 50 is not divisible by 3 the total weight of further Weierstrass points would have to be at least 150. But then the total weight would be too large. Hence the vertices and the centre points of edges are the only Weierstrass points. As in the previous example, all vertices have the same weight, say x, and all centre points of edges have the same weight, say y, so we have:

$$15x + 75y = 210,$$

so either the vertices have weight 9 and the centre points of edges weight 1, or the vertices have weight 4 and the centre points weight 2.

4.3.3 Gap sequence of the point Z

We consider the gap sequence of Z:

$$1 = n_1 < n_2 < n_3 < n_4 < n_5 < n_6 < 2g = 12,$$

$$2 \leq n_2 < n_3 < n_4 < n_5 < n_6 \leq 11.$$

We need the following version of the Rieman-Hurwitz formula:

Theorem 4.3.1 (Riemann-Hurwitz Formula) *Let M be a Riemann surface of genus $g \geq 2$, $\sigma : M \to M$ an isometry with prime order* $\operatorname{ord}\sigma$. *Let γ be the genus of the quotient $M/\langle\sigma\rangle$. Then the number of fixed points of σ is equal to*

$$2 + \frac{2g - 2\gamma \cdot \operatorname{ord}\sigma}{\operatorname{ord}\sigma - 1}.$$

On M_6 the rotation ρ around Z with angle $\frac{2\pi}{5}$ has order 5 and 5 fixed points (A, B, C, D, Z). Using Theorem 4.3.1 we can calculate γ, the genus of the quotient $M_6/\langle\rho\rangle$:

$$5 = 2 + \frac{2 \cdot 6 - 2 \cdot \gamma \cdot 5}{5 - 1},$$

giving $\gamma = 0$. So $M_6/\langle\rho\rangle \cong \mathbb{C}P^1$.

By definition of the gap sequence (see Theorem 4.1.1) we have that 5 is a gap at Z if there is no meromorphic function f on M_6 such that f is holomorphic on $M_6 - \{Z\}$ and has a pole of order 5 at Z. So 5 is a non-gap at Z if there is a function f meromorphic on M_6 such that f is holomorphic on $M_6 - \{Z\}$ and has a pole of order 5 at Z.

The canonical projection map $f : M_6 \to M_6/\langle\rho\rangle \cong \mathbb{C}P^1$ is a meromorphic function on M_6, f is holomorphic on $M_6 - \mathrm{fix}(\rho^k) = M_6 - \{Z\}$.

Since ρ is the rotation through $\frac{2\pi}{5}$, $\mathrm{ord}(\rho) = 5$, we have:

$$\begin{aligned} f : M_6 &\to M_6/\langle\rho\rangle \\ z &\mapsto z^5 \end{aligned}$$

So f has a pole of order 5 at Z; f is a ramified covering of M_6 over $\mathbb{C}P^1$ of order 5 with Z as a branch point. So 5 is a non-gap of Z.

Since 5 is a non-gap, 10 is also a non-gap since the non-gaps form an additive semi-group. So we have $2 \leq n_2 < n_3 < n_4 < n_5 < n_6 \leq 11$ as our gap sequence, with $n_j \in \{2, 3, 4, 6, 7, 8, 9, 11\}$. If 2 were a non-gap then 4, 6 and 8 would be non-gaps, as well as $4 + 5 = 9$, $5 + 6 = 11$, leaving only 3 and 7 as possible values for $n_2, \ldots, n_6$. If 3 were a non-gap then 6 and 9 would also be non-gaps, as well as $3 + 5 = 8$, $3 + 6 = 11$ leaving only 2, 4 and 7 as possible values for $n_2, \ldots, n_6$. So 2 and 3 must be gaps, otherwise we'd have too many non-gaps. So we have $1 < 2 < 3 < n_4 < n_5 < n_6$ with $n_j \in \{4, 6, 7, 8, 9, 11\}$.

We know since Z is a vertex it has either weight 9 or weight 4. Using Definition 4.1.3 we have that the weight is equal to

$$\begin{aligned} \sum_{i=1}^{g}(n_i - i) &= (n_1 - 1) + (n_2 - 2) + (n_3 - 3) + (n_4 - 4) + (n_5 - 5) + (n_6 - 6) \\ &= (n_4 - 4) + (n_5 - 5) + (n_6 - 6) = n_4 + n_5 + n_6 - 15. \end{aligned}$$

1. To get weight 4 we need $n_4 + n_5 + n_6 = 19$.

   Suppose $n_4 = 4 \Rightarrow n_5 + n_6 = 15$. Then if $n_5 = 6 \Rightarrow n_6 = 9$. If $n_5 = 7 \Rightarrow n_6 = 8$. If $n_5 = 8 \Rightarrow n_6 = 7$, which can not be the case since we require $n_5 < n_6$. So we get two possibilities for the gap sequence:

$$1 < 2 < 3 < 4 < 6 < 9,$$

$$1 < 2 < 3 < 4 < 7 < 8.$$

Now if $n_4 = 6 \Rightarrow n_5 + n_6 = 13$. If $n_5 = 7 \Rightarrow n_6 = 6$, which can not be the case. Similarly for $n_5 = 9, 11$. (Note that in this case 4 is a non-gap so we have that 8 is also a non-gap).

If $n_4 = 7 \Rightarrow n_5 + n_6 = 12$. Then if $n_5 = 9 \Rightarrow n_6 = 3$, which can not happen. Similarly for $n_5 = 11$. (Note that here 4, 6 and 8 are non-gaps, leaving only 9 and 11 as choices for gaps).

$n_4 = 8$ is impossible since 4 and therefore 8 would be a non-gap in this case.

$n_4 = 9$ is impossible since we would have too many non-gaps.

2. To get weight 9 we need $n_4 + n_5 + n_6 = 24$. Similar considerations to the above show that the only possibilities for the gap sequence to have weight 9 are:

$$1 < 2 < 3 < 4 < 9 < 11,$$

$$1 < 2 < 3 < 6 < 7 < 11.$$

So we have four possibilities for the gap sequence of Z.

Let σ be the reflection at Z. The order of σ is 2 and the reflection at Z has 6 fixed points. So by Theorem 4.3.1 we calculate the genus of the quotient $M_6/\langle\sigma\rangle$ to be $\gamma = 2$.

Let Z' be the image of Z under the projection $M_6 \to M_6/\langle\sigma\rangle = M'$. The gap sequence of Z' in M' is $1 = n_1 < n_2 < 2\gamma = 4$. So we have two options for the gap sequence: $1 < 3$ if Z' is a Weierstrass point and $1 < 2$ if not. In every case 4 is a non-gap. This means there is a meromorphic function f on M' with a pole Z' of order 4, holomorphic on $M' - \{Z'\}$.

Since σ is a reflection in Z (order 2), the projection $g : M_6 \to M'$ has a pole of order 2 at Z. Hence the composition $f \circ g : M_6 \xrightarrow{g} M' \xrightarrow{f} \mathbb{C}P^1$ is a meromorphic function on M_6 with a pole of order $2 \cdot 4 = 8$ at Z. Hence 8 is a non-gap of Z in M_6 ruling out $1 < 2 < 3 < 4 < 7 < 8$ as the gap sequence.

The gap sequence of Z' is either $1 < 2$ or $1 < 3$, hence either 2 or 3 is a non-gap of Z', so there is a meromorphic function $f : M' \to \mathbb{C}P^1$ on M' with a pole Z' of order either 2 or 3 and holomorphic on $M' - \{Z'\}$. The composition $f \circ g : M_6 \to M' \to \mathbb{C}P^1$ is a meromorphic function on M_6 with Z a pole of order either $2 \cdot 2 = 4$ or $2 \cdot 3 = 6$, hence at least of 4 and 6 is a non-gap for Z. This rules out $1 < 2 < 3 < 4 < 6 < 9$.

Thus the gap sequence of Z must be either $1 < 2 < 3 < 6 < 7 < 11$ or $1 < 2 < 3 < 4 < 9 < 11$.

4.3.4 Higher Spin Bundles on M_6

Now the argument is as in section 4.2.3. According to Proposition 4.1.8 the number of linearly independent holomorphic sections in the bundle corresponding to the divisor $10Z$ is $10 + 1 - \#\{\text{gaps} \leq 10\} = 6 = g$. Hence $10Z$ is canonical. Therefore $5Z$ defines a 2-spin structure and $2Z$ defines a 5-spin structure.

Chapter 5

Higher Spin Bundles

Higher spin bundles were studied by Natanzon and Pratoussevitch. In this chapter we describe their results from [15], [16] and [17]. The following results are new: Lemma 5.2.5 and Proposition 5.2.6 extend Propositions 2.5 and 2.6 in [17] to include the case of elliptic elements. Lemma 5.3.21 is new. Section 5.5.3 combines the ideas from [15] and [16] to understand the case where we have holes, punctures and marked points.

We assign to each m-spin bundle on a Riemann orbifold an associated m-Arf function, a certain function on the space of homotopy classes of simple contours on the orbifold P with values in $\mathbb{Z}/m\mathbb{Z}$, described by simple geometric properties. To do this we first establish a connection between m-spin bundles on Riemann orbifolds and lifts of Fuchsian groups into the m-fold cover of $\mathrm{PSL}(2,\mathbb{R})$.

5.1 Higher Spin Bundles on Riemann Surfaces and Lifts of Fuchsian Groups

In this section we follow [16].

Recall that $G^+ = \mathrm{Isom}^+(\mathbb{H}) = \mathrm{PSL}(2,\mathbb{R})$ and a *Fuchsian group* is a discrete subgroup of $G^+ =\mathrm{PSL}(2,\mathbb{R})$. Recall that G^+_m is the m-fold covering Lie group of G^+.

Definition 5.1.1 Let Γ be a Fuchsian group. A *lift of the Fuchsian group* Γ *into* G^+_m is a subgroup Γ^* of G^+_m such that the restriction of the covering map $\pi : G^+_m \to G^+$ to Γ^* is an isomorphism $\Gamma^* \to \Gamma$.

Each element $g \in \Gamma$ has m pre-images in G_m^+, so when we lift Γ into G_m^+ we have m choices for each element. To lift the group Γ we need to lift the elements in such a way that the lifted elements form a subgroup of G_m^+. That is, for each $g \in \Gamma$ choose $g^* \in \pi^{-1}(g)$ in G_m^+ such that

1. $g_1^* \cdot g_2^* = (g_1 \cdot g_2)^*$ for any g_1, g_2 in Γ,

2. $e_G^* = e_{G_m}$,

3. $(g^{-1})^* = (g^*)^{-1}$ for any $g \in \Gamma$.

If we choose g^* in this way, the map $\pi|_{\Gamma^*} : \Gamma^* \to \Gamma$ is bijective because of the way in which we choose one pre-image for each element of Γ. The covering map π preserves mulitplication, $\pi(g_1^* \cdot g_2^*) = \pi(g_1^*) \cdot \pi(g_2^*) = g_1 \cdot g_2$.

It is actually sufficient to choose g^* for all elements of a set of generators in such a way that the relations of Γ are satisfied in Γ^*.

The following result was proved in [16]:

Theorem 5.1.2 *There is a* 1-1-*correspondence between m-spin bundles on Riemann orbifolds and lifts of Fuchsian groups into the m-fold cover of* $Aut_+(\mathbb{H})$.

5.2 Level Function

5.2.1 Definition of a level function

The material in this section is from section 3 in [16].

Let Δ be the set of all elliptic elements of order 2 in G^+. Let Ξ be the complement in G^+ of the set Δ. The subset Ξ is simply connected. The pre-image $\tilde{\Xi}$ of the subset Ξ in G_m^+ consists of m connected components, each of which is homeomorphic to Ξ. The connected components of Ξ are separated from each other by connected components of the preimage $\tilde{\Delta}$ of Δ. Each connected component of $\tilde{\Xi}$ contains one and only one pre-image of the identity element of G, i.e. one and only one element of the centre of G_m^+.

Definition 5.2.1 If an element of G_m^+ is contained in the same connected component of the set $\tilde{\Xi}$ as the central element $U^k, k \in \mathbb{Z}$, we say that the element is *at the level* k and set the *level function* s_m on this element to be equal to $k \mod m$. For pre-images of elliptic elements of order 2 we set $s_m(R_x(t)) = k \mod m$ for $t = \pi + 2\pi k$.

Any hyperbolic or parabolic element in G_m^+ is of the form $T_{\alpha,\beta}(\lambda) \cdot U^k$ or $P_\alpha(\lambda) \cdot U^k$. For elements written in this form we have

$$s_m(T_{\alpha,\beta}(\lambda) \cdot U^k) = k, \quad s_m(P_\alpha(\lambda) \cdot U^k) = k.$$

Any elliptic element in G_m^+ is of the form $R_x(t)$. For elements written in this form we have

$$s_m(R_x(t)) = k$$

if and only if $t \in (-\pi + 2\pi k, \pi + 2\pi k]$.

Definition 5.2.2 The *canonical lift* of an element g in G^+ into G_m^+ is an element g^* in G_m^+ such that $\pi(g^*) = g$ and $s_m(g^*) = 0$.

5.2.2 Properties of the level function

We look at the behaviour of the level function s_m under inversion (Lemma 5.2.3), conjugation (Lemmas 5.2.4, 5.2.5, 5.2.6) and multiplication in some special cases (Lemma 5.2.7).

Lemma 5.2.3 *(Lemma* 3.1 *in [16]) The equation $s_m(A^{-1}) = -s_m(A)$ is satisfied for any element A in G_m^+ with exception of pre-images of elliptic elements of order* 2*.*

Proof Let $A \in G_m^+$ and let $k = s_m(A)$, then A is in the same connected component of $\tilde{\Xi}$ as U^k. Let γ be the path in $\tilde{\Xi}$ that connects A with U^k. Consider the path $\delta(t) = (\gamma(t))^{-1}$. The path δ connects A^{-1} with U^{-k}. Since the path γ remains in the same connected component of $\tilde{\Xi}$, it avoids $\tilde{\Delta}$. Consequently, the path δ also avoids $\tilde{\Delta}$, i.e. it remains in the same component of $\tilde{\Xi}$. Thus the element A^{-1} is in the same connected component of $\tilde{\Xi}$ as U^{-k}, i.e. $s_m(A^{-1}) = -k = -s_m(A)$. □

Lemma 5.2.4 *(Lemma* 3.2 *in [16]) For any elements A and B in G_m we have $s_m(BAB^{-1}) = s_m(A)$.*

Proof An element $B \in G_m^+$ can be connected to the unit element in G_m^+ via a path β. The path γ given by $\gamma(t) = \beta(t) \cdot A \cdot (\beta(t))^{-1}$ connects the elements A and $B \cdot A \cdot B^{-1}$. If A is not in $\tilde{\Delta}$ then any conjugate $\gamma(t)$ of A is not in $\tilde{\Delta}$, hence the path γ remains in the same component of the set $\tilde{\Xi}$. If A is in $\tilde{\Delta}$ then any conjugate $\gamma(t)$ of A is also in $\tilde{\Delta}$, hence the path γ remains in the same component of the set $\tilde{\Delta}$. In both cases s_m is constant along γ, in particular $s_m(B \cdot A \cdot B^{-1}) = s_m(A)$. □

Conjugation of an element with $J \in G_m^-$. Here we generalise Proposition 2.5 in [17] to include the case of elliptic elements.

Lemma 5.2.5 *We have $s_m(JCJ) = -s_m(C)$ for any element C in G_m^+, apart from pre-images of elliptic elements of order 2.*

Proof Hyperbolic and parabolic elements of G_m^+ are of the form $T_{\alpha,\beta}(\lambda) \cdot U^k$ and $P_\alpha(\lambda) \cdot U^k$. By definition 5.2.1 we have $s_m(T_{\alpha,\beta}(\lambda) \cdot U^k) = k$ and $s_m(P_\alpha(\lambda) \cdot U^k) = k$. By proposition 2.3.1 we have $JT_{\alpha,\beta}(\lambda)J = T_{-\alpha,-\beta}(\lambda)$, $JP_\alpha(\lambda)J = P_{-\alpha}(-\lambda)$ and $JUJ = U^{-1}$. Hence

$$J(T_{\alpha,\beta}(\lambda) \cdot U^k)J = T_{-\alpha,-\beta}(\lambda) \cdot U^{-k}$$

and $J(P_\alpha(\lambda) \cdot U^k)J = P_{-\alpha}(-\lambda) \cdot U^{-k}$. So $s_m(T_{-\alpha,-\beta}(\lambda) \cdot U^{-k}) = -k = -s_m(T_{\alpha,\beta}(\lambda) \cdot U^k)$ and $s_m(P_{-\alpha}(-\lambda) \cdot U^{-k}) = -k = -s_m(P_\alpha(\lambda) \cdot U^k)$.

Let C be an elliptic element (not of order 2). Any elliptic element C in G_m^+ is of the form $R_x(t)$ for some x and t. By proposition 2.3.1 we have $JR_x(t)J = R_{-\bar{x}}(-t)$. (Note $-\bar{x} = j(x)$, where j is the reflection in the imaginary axis). If $s_m(R_x(t)) = k$ then $t \in (-\pi + 2\pi k, \pi + 2\pi k)$. So $-t \in (-\pi - 2\pi k, \pi - 2\pi k) = (-\pi + 2\pi(-k), \pi + 2\pi(-k))$, so $s_m(R_{j(x)}(-t)) = -k$. Hence $s_m(JR_x(t)J) = -s_m(R_x(t))$. □

Here we generalise Proposition 2.6 in [17] to include the case of elliptic elements.

Proposition 5.2.6 *We have $s_m(FCF^{-1}) = -s_m(C)$ for any element F in G_m^- and any C in G_m^+ apart from pre-images of elliptic elements of order 2.*

Proof We can write an element $F \in G_m^-$ as $F = A \cdot J$ for some $A \in G_m^+$, hence $FCF^{-1} = (AJ)C(AJ)^{-1} = A(JCJ^{-1})A^{-1} = A(JCJ)A^{-1}$. By Lemma 5.2.4 we have $s_m(A(JCJ)A^{-1}) = s_m(JCJ)$ and by lemma 5.2.5 we have $s_m(JCJ) = -s_m(C)$. □

Lemma 5.2.7 *(Lemma 3.3 in [16]) If the axes of two hyperbolic elements A and B in G_m^+ intersect then $s_m(AB) = s_m(A) + s_m(B)$.*

Proof Let l_A resp. l_B be the axes of A resp. B. Let x be the intersection point of l_A and l_B. Any hyperbolic transformation with the axis l_A is the product of a rotation by π at some point $y \neq x$ on l_A and a rotation by π at the point x. Similarly, any hyperbolic transformation with the axis l_B is a product of a rotation by π at the point x and a rotation by π at some point $z \neq x$ on l_B. Hence the product of any hyperbolic transformation with the axis l_A and any hyperbolic transformation with the axis l_B is a product of a rotation by π at a point $y \neq x$ on l_A and a rotation by π at a point $z \neq x$ on l_B, i.e. it is a hyperbolic transformation with an axis going through the points y and z. Thus the product of two hyperbolic elements with distinct but intersecting axes is always a hyperbolic element.

Assume without loss of generality that the elements $A, B \in G_m^+$ satisfy the conditions $s_m(A) = s_m(B) = 0$. We want to show $s_m(AB) = 0$. Let us deform the elements A and B. If we are decreasing the shift parameters while keeping the same axes, then the product tends to the identity element. On the other hand we have just explained that the product remains hyperbolic, i.e. it stays in $\tilde{\Xi}$. Therefore the value of s_m on the product remains constant, i.e. $s_m(AB) = s_m(id) = 0$. $\square$

5.3 Level functions on lifts of Fuchsian groups

5.3.1 Lifting elliptic cyclic subgroups

Here we give a more detailed proof of Lemma 4.1 in [16].

Lemma 5.3.1 *Let Γ be an elliptic cyclic Fuchsian group of order p. The group Γ is generated by an element $\gamma = \rho_x(2\pi/p)$ for some $x \in \mathbb{H}$.*

1. *Let us assume that p and m are relatively prime. Then the lift Γ^* of Γ into G_m^+ exists and is unique. There is a unique element $n \in \mathbb{Z}/m\mathbb{Z}$ such that $p \cdot n + 1 \equiv 0$ modulo m. The lift Γ^* is generated by the pre-image $\tilde{\gamma}$ of $\gamma = \rho_x(2\pi/p)$ in G_m^+ such that $s_m(\tilde{\gamma}) = n$.*

2. *If p and m are not relatively prime, then the group Γ cannot be lifted into G_m^+.*

Proof Finite order elements are of the form $\rho_x(\phi)$, for this element to be of order p we need

$$(\rho_x(\phi))^p = \rho_x(p \cdot \phi) = id.$$

For this element to be the identity we need $p \cdot \phi = 2\pi k, k \in \mathbb{Z}$, that is $\phi = \frac{2\pi}{p} \cdot k$. We have $(\rho_x(2\pi/p))^p = \rho_x(\frac{2\pi}{p} \cdot p) = \rho_x(2\pi) = id$. For p to be the order we require p to be the smallest such that $(\rho_x(2\pi/p))^p = id$. Suppose $0 < k < p$, $0 < \frac{k}{p} < 1$ then $0 < 2\pi\frac{k}{p} < 2\pi$. Then we have:

$$(\rho_x(2\pi/p))^k = \rho_x\left(2\pi\frac{k}{p}\right) \neq id$$

since $2\pi\frac{k}{p} \notin 2\pi\mathbb{Z}$.

To lift Γ into G_m^+ we have to find an element $\tilde{\gamma}$ in the pre-image of γ in G_m^+ such that $\tilde{\gamma}^p = 1$. The pre-image of γ in G_m can be described as the coset $\{U^n \cdot R_x(2\pi/p) | n \in \mathbb{Z}/m\mathbb{Z}\}$. For the element $R_x(2\pi/p)$ we obtain:

$$(R_x(2\pi/p))^p = R_x(2\pi) = U.$$

Hence for an element $U^n \cdot R_x(2\pi/p)$ we have:

$$(U^n \cdot R_x(2\pi/p))^p = U^{np}(R_x(2\pi/p))^p = U^{np}R_x(2\pi) = U^{np}U = U^{np+1}$$

and $U^{np+1} = \tilde{e}$ if and only if $np + 1 \equiv 0 \mod m$.

If $(m, p) \neq 1$, say they have a common divisor d, then np is divisible by d, $np+1$ is not divisible by d so $np+1$ is not divisible by m. Hence there is no n such that $np + 1 \equiv 0 \mod m$.

If $(m, p) = 1$ then by the Euclidian algorithm there are integers x, y such that $1 = mx + py$, re-writing as $1 + p(-y) = mx$ we have $1 + p(-y) = 0$ mod m. Take $n = -y \mod m$. Such an n is unique modulo m. Assume we have n_1 and n_2 such that $1 + n_1 p \equiv 0 \mod m$ and $1 + n_2 p \equiv 0 \mod m$. Taking the second equation from the first we get $(n_1 - n_2)p \equiv 0 \mod m$. Since $(m, p) = 1$ we have $n_1 - n_2 \equiv 0 \mod m$, i.e. $n_1 \equiv n_2 \mod m$.

Hence for not relatively prime p and m it is impossible to lift Γ into G_m^+. For relatively prime p and m there is a unique lift of Γ into G_m^+ generated by $U^n \cdot R_x(2\pi/p)$ with $np + 1 \equiv 0 \mod m$. □

5.3.2 Finitely generated Fuchsian groups

The following definitions follow [25] and section 4.2 in [16].

Definition 5.3.2 A *Riemann orbifold* (P, Q) of signature

$$(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$$

is a topological surface P of genus g with l_h holes and l_p punctures and a set $Q = \{(x_1, p_1), \ldots, (x_{l_e}, p_{l_e})\}$ of points x_i in P equipped with orders p_i such that $p_i \in \mathbb{Z}$, $p_i \geq 2$ and $x_i \neq x_j$ for $i \neq j$. The set Q is called the *marking* of the Riemann orbifold (P, Q).

Definition 5.3.3 Let $(P, Q = \{(x_1, p_1), \ldots, (x_{l_e}, p_{l_e})\})$ be a Riemann orbifold. Two curves γ_0 and γ_1 on P which do not pass through exceptional points $x_i \in Q$ are called *Q-homotopic* if γ_0 can be deformed into γ_1 by a finite sequence of the following processes:

1. Homotopic deformations with fixed starting points such that during the deformation no exceptional point is encountered.

2. Omitting a subcurve of γ_i which does not contain the starting point of γ_i and is of the form $\delta^{\pm p_i}$, where δ is a curve on P which bounds a disk that contains exactly one exceptional point x_i in the interior.

3. Inserting into γ_i a subcurve which does not contain the starting point of γ_i and is of the form $\delta^{\pm p_i}$, where δ is a curve on P which bounds a disk that contains exactly one exceptional point x_i in the interior.

Two curves γ_0 and γ_1 which do not pass through exceptional points $x_i \in Q$ are called *freely Q-homotopic* if γ_0 can be deformed into γ_1 by a finite sequence of deformations as above where the base point may be moved during the deformations.

Definition 5.3.4 Let $(P, Q = \{(x_1, p_1), \ldots, (x_{l_e}, p_{l_e})\})$ be a Riemann orbifold and let p be in $P \backslash Q$. Then the set of Q-homotopy classes of curves starting and ending in p forms a group. This group is called the *Q-fundamental group* or the *orbifold fundamental group* and denoted by $\pi^Q(P, p)$ or $\pi^{\mathrm{orb}}(P, p)$ or simply $\pi(P, p)$.

Definition 5.3.5 Let Γ be a Fuchsian group. The quotient $P = \mathbb{H}/\Gamma$ is a surface and the projection $\Psi : \mathbb{H} \to P$ is a branched cover. Let Q consist of the branching points and the corresponding orders. Then (P, Q) is a Riemann orbifold. We call the Riemann orbifold (P, Q) hyperbolic and say that it is defined by Γ.

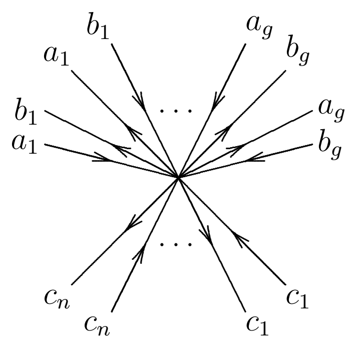

Figure 5.1: Canonical system of curves

Proposition 5.3.6 *Let Γ be a Fuchsian group, (P, Q) the corresponding Riemann orbifold and $p \in P \backslash Q$. Then $\pi(P, p) \cong \Gamma$.*

The following definitions are taken from section 4.2 in [16].

Definition 5.3.7 A *canonical system of curves* on a Riemann orbifold $(P, Q = \{(x_1, p_1), \ldots, (x_{l_e}, p_{l_e})\})$ of signature $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ is a set of simple closed curves

$$(a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n)$$

based at a point $p \in P$, where $n = l_h + l_p + l_e$, with the following properties:

1. The curve c_i encloses a hole in P for $i = 1, \ldots, l_h$, a puncture for $i = l_h+1, \ldots, l_h+l_p$ and the marking point $x_{i-l_h-l_p}$ for $i = l_h+l_p+1, \ldots, n$.

2. Any two curves only intersect at the point p.

3. In a neighbourhood of the point p, the curves are places as shown in Figure 5.1.

4. The system of curves cuts the surface P into $l_h + l_p + l_e + 1$ connected components of which $l_p + l_e$ are homeomorphic to an annulus, $l_h + 1$ are discs. The last disc has boundary

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c_1 \cdots c_n.$$

If $(a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n)$ is a canonical system of curves, then we call the corresponding set of elements in the orbifold fundamental group $\pi(P, p)$ a *standard basis* or a *standard set of generators* of $\pi(P, p)$.

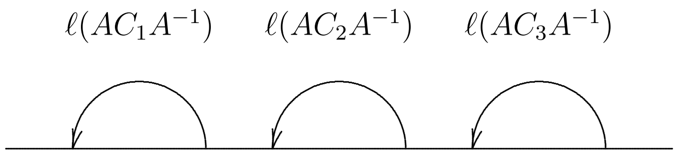

Figure 5.2: Axes of a sequential set of signature $(0; 3, 0, 0)$

Definition 5.3.8 For two elements C_1 and C_2 in G with finite fixed points in $\mathbb{R}$ we say that $C_1 < C_2$ if all fixed points of C_1 are smaller than any fixed point of C_2.

Definition 5.3.9 A *sequential set of signature* $(0; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ with $l_h+l_p+l_e = 3$ is a triple of elements (C_1, C_2, C_3) in G^+ such that the element C_i is hyperbolic for $i = 1, \dots, l_h$, parabolic for $i = l_h+1, \dots, l_h+l_p$ and elliptic of order $p_{i-l_h-l_p}$ for $i = l_h+l_p+1, \dots, l_h+l_p+l_e = 3$, their product $C_1 \cdot C_2 \cdot C_3 = 1$, and for some element $A \in G$ the elements $\{C_i' = AC_iA^{-1}\}_{i=1,2,3}$ are positive, have finite fixed points and satisfy $C_1' < C_2' < C_3'$. (Figure 5.2 illustrates the position of the axes of the elements C_i' for a sequential set of signature $(0; 3, 0, 0)$, i.e. when all elements are hyperbolic.)

Definition 5.3.10 A *sequential set of signature* $(0; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ is a tuple of elements $(C_1, \dots, C_{l_h+l_p+l_e})$ in G^+ such that the element C_i is hyperbolic for $i = 1, \dots, l_h$, is parabolic for $i = l_h+1, \dots, l_h+l_p$ and elliptic of order $p_{i-l_h-l_p}$ for $i = l_h+l_p+1, \dots, l_h+l_p+l_e$, and for any $i \in \{2, \dots, l_h + l_p + l_e - 1\}$ the triple $(C_1 \cdots C_{i-1}, C_i, C_{i+1} \cdots C_{l_h+l_p+l_e})$ is a sequential set.

Definition 5.3.11 A *sequential set of signature* $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ is a tuple of elements

$$(A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_{l_h+l_p+l_e})$$

in G^+ such that the elements $A_1, \dots, A_g, B_1, \dots, B_g$ are hyperbolic, the element C_i is hyperbolic for $i = 1, \dots, l_h$, parabolic for $i = l_h+1, \dots, l_h+l_p$ and elliptic of order $p_{i-l_h-l_p}$ for $i = l_h+l_p+1, \dots, l_h+l_p+l_e$, and the tuple

$$(A_1, B_1A_1^{-1}B_1^{-1}, \dots, A_g, B_gA_g^{-1}B_g^{-1}, C_1, \dots, C_{l_h+l_p+l_e})$$

is a sequential set of signature $(0; 2g + l_h, l_p, l_e : p_1, \dots, p_{l_e})$.

Definition 5.3.12 We say that a Fuchsian group Γ is of signature

$$(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$$

if the corresponding orbifold $\mathbb{H}/\Gamma$ is of this signature.

Theorem 5.3.13 *Let V be a sequential set of signature*

$$(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e}).$$

For $i = 1, \ldots, l_e$ let $y_i \in \mathbb{H}$ be the fixed point of the corresponding elliptic element of order p_i in V. Let $P = \mathbb{H}/\Gamma$ and let $\Psi : \mathbb{H} \to P$ be the natural projection. Let $Q = \{(\Psi(y_1), p_1), \ldots, (\Psi(y_{l_e}), p_{l_e})\}$. Then the sequential set V generates a Fuchsian group Γ such that the Riemann factor orbifold $(P = \mathbb{H}/\Gamma, Q)$ is of signature $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$. The natural projection $\Psi : \mathbb{H} \to P$ maps the sequential set V to a canonical system of curves on the factor surface (P, Q).

Theorem 5.3.14 *Let Γ be a Fuchsian group such that the factor orbifold $P = \mathbb{H}/\Gamma$ is of signature $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ with $l_h + l_p + l_e = n$. Let p be a point in P which does not belong to the marking. Let $\Psi : \mathbb{H} \to P$ be the natural projection. Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \to \pi(P, p)$ be the induced isomorphism. Let*

$$v = \{\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_g, \tilde{b}_g, \tilde{c}_1, \ldots, \tilde{c}_n\}$$

be a canonical system of curves on P, then

$$V = \Phi^{-1}(v) = \{\Phi^{-1}(a_1), \Phi^{-1}(b_1), \ldots, \Phi^{-1}(a_g), \Phi^{-1}(b_g), \Phi^{-1}(c_1), \ldots, \Phi^{-1}(c_n)\}$$

is a sequential set of signature $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$.

5.3.3 Lifting Fuchsian groups of genus 0

In this section we recall some results from [16] needed for the generalisation in section 5.5.

Lemma 5.3.15 *(Lemma 4.5 in [16]) Let $(0; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ with $l_h + l_p + l_e = n$ be the signature of the sequential set $(\bar{C}_1, \ldots, \bar{C}_n)$. For $i = 1, \ldots, n$, let $\tilde{C}_i$ be the canonical lift of $\bar{C}_i$ into G_m^+. Let U be the generator of the centre $Z(G_m^+)$ given by the element $R_x(2\pi)$. Then the elements $\tilde{C}_1, \ldots, \tilde{C}_n$ satisfy the following relations: $\tilde{C}_{l_h+l_p+i}^{p_i} = U$ for $i = 1, \ldots, l_e$ and $\tilde{C}_1 \cdots \tilde{C}_n = U^{n-2}$.*

Proof (This proof is inspired by an idea from [10]). A canonical lift $\tilde{C}_{l_h+l_p+i}$ of an elliptic element $\rho_x(2\pi/p_i) = \bar{C}_{l_h+l_p+i}$ is of the form $\tilde{C}_{l_h+l_p+i} = R_x(2\pi/p_i)$ for some x. Hence

$$\tilde{C}^{p_i}_{l_h+l_p+i} = (R_x(2\pi/p_i))^{p_i} = R_x(2\pi) = U.$$

Let Π be the canonical fundamental polygon for the group generated by the elements $\bar{C}_1, \ldots, \bar{C}_n$ such that the generators $\bar{C}_i$ can be described by products $\bar{C}_i = \sigma_i\sigma_{i+1}$ of reflections $\sigma_1, \ldots, \sigma_n$ in the edges of the polygon Π (suitably numbered). Then $\sigma_i^2 = \text{id}$ therefore

$$\bar{C}_1 \cdots \bar{C}_n = (\sigma_1\sigma_2)(\sigma_2\sigma_3)\cdots(\sigma_{n-1}\sigma_n)(\sigma_n\sigma_1) = \text{id}.$$

The product $\bar{C}_1 \cdots \bar{C}_n = \text{id} \in G$ lifts to $\tilde{C}_1 \cdots \tilde{C}_n \in G^+_m$. So $\tilde{C}_1 \cdots \tilde{C}_n$ is in the preimage of the identity, therefore belongs to the centre of G^+_m.

As we vary Π continuously, this central element must also vary continuously. But $Z(G_m)$ is a discrete group, so $\tilde{C}_1 \cdots \tilde{C}_n$ must remain constant. We can shrink the polygon Π down towards a point x. In the course of this continuous deformation of the fundamental polygon Π the hyperbolic and parabolic elements of the sequential set will become elliptic. As we continue shrinking the polygon towards the point x, the angles $\pi/p_1, \ldots, \pi/p_n$ of the polygon tend to the angles $\beta_1, \ldots, \beta_n$ of some Euclidian n-sided polygon. Thus the element $\tilde{C}_i = R_x(2\pi/p_i) \in G^+_m$ tends towards the limit $R_x(2\beta_i)$, while the product $\tilde{C}_1 \cdots \tilde{C}_n$ tends toward the product

$$R_x(2\beta_1)\cdots R_x(2\beta_n) = R_x(2\beta_1 + \cdots + 2\beta_n).$$

Therefore, using the formula

$$\beta_1 + \cdots + \beta_n = (n-2)\pi$$

for the sum of the angles in a Euclidian n-sided polygon, we see that the product $\tilde{C}_1 \cdots \tilde{C}_n$ must be equal to

$$R_x(2(n-2)\pi) = U^{n-2}.$$

Thus $\tilde{C}^{p_i}_{l_h+l_p+i} = U$ and $\tilde{C}_1 \cdots \tilde{C}_n = U^{n-2}$. □

Recall from the proof of Lemma 5.3.1 that if $(p, m) = 1$, $\bar{C} = \rho_x(2\pi/p) \in G^+$ and $\tilde{C} = R_x(2\pi/p)$ is the canonical lift of $\bar{C}$ into G^+_m, then $\tilde{C}^p = R_x(2\pi) =$

$U \neq \tilde{e}$. This implies that the canonical lift $\tilde{C}$ is not a good candidate for lifting $\bar{C}$. Try $C = \tilde{C} \cdot U^t$, where $t \cdot p + 1 \equiv 0 \mod m$. Then

$$C^p = (\tilde{C} \cdot U^t)^p = \tilde{C}^p \cdot U^{tp} = U \cdot U^{tp} = U^{1+tp} = \tilde{e}.$$

This implies that $C = \tilde{C} \cdot U^t$ is a good candidate for lifting $\bar{C}$.

Lemma 5.3.16 *(Lemma 4.6 in [16]). Let $(C_1, \ldots, C_n)$ be an n-tuple of elements in G_m^+ such that their images $(\bar{C}_1, \ldots, \bar{C}_n)$ in G form a sequential set of signature*

$$(0; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$$

with $l_h + l_p + l_e = n$. Then $C_1 \cdots C_n = \tilde{e}$ if and only if

$$s_m(C_1) + \cdots + s_m(C_n) \equiv -(n-2) \mod m.$$

Proof For $i = 1, \ldots, n$, let $\tilde{C}_i$ be the canonical lift of $\bar{C}_i$ into G_m^+. The elements C_i can be written in the form $C_i = \tilde{C}_i \cdot U^{s_m(C_i)}$, therefore

$$C_1 \cdots C_n = (\tilde{C}_1 \cdots \tilde{C}_n) \cdot U^{s_m(C_1)+\cdots+s_m(C_n).}$$

Using Lemma 5.3.15 we obtain $C_1 \cdots C_n = U^{n-2+s_m(C_1)+\cdots s_m(C_n)}$. The product $C_1 \cdots C_n$ is equal to $\tilde{e}$ if and only if the exponent of U in the last equation is divisible by m, i.e. if

$$n - 2 + s_m(C_1) + \cdots + s_m(C_n) \equiv 0 \mod m$$

that is

$$s_m(C_1) + \cdots + s_m(C_n) \equiv -(n-2) \mod m.$$

□

Corollary 5.3.17 *(Lemma 4.7 in [16]). Let (C_1, C_2, C_3) be a triple of elements in G_m^+ with*

$$C_1 \cdot C_2 \cdot C_3 = \tilde{e}.$$

Let $\bar{C}_i$ be the image of the element C_i in G. Let $(\bar{C}_1, \bar{C}_2, \bar{C}_3)$ be a sequential set of signature $(0; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ with $l_h + l_p + l_e = 3$. Then

$$s_m(C_1 \cdot C_2) = s_m(C_1) + s_m(C_2) + 1 \text{ if the element } C_3 \text{ is not of order } 2,$$

$$s_m(C_1 \cdot C_2) = -s_m(C_1) - s_m(C_2) - 1 \text{ if the element } C_3 \text{ is of order } 2.$$

Proof According to Lemma 5.3.16 the elements C_i satisfy

$$s_m(C_1) + s_m(C_2) + s_m(C_3) \equiv -1 \mod m.$$

On the other hand $C_1C_2C_3 = \tilde{e}$ implies $C_1C_2 = C_3^{-1}$, hence

$$s_m(C_1C_2) = s_m(C_3^{-1}) = -s_m(C_3) = s_m(C_1) + s_m(C_2) + 1$$

if the element C_3 is not of order 2 and

$$s_m(C_1C_2) = s_m(C_3^{-1}) = s_m(C_3) = -s_m(C_1) - s_m(C_2) - 1$$

if the element C_3 is of order 2. □

5.3.4 Lifting sets of generators of Fuchsian groups

In this section we collect some results from [16] that will be needed in section 5.5. We have added some details to the proof of Proposition 5.3.20.

Lemma 5.3.18 *(Lemma 4.8 in [16]). Let Γ be a Fuchsian group of signature*

$$(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$$

generated by the sequential set $\bar{V} = \{\bar{A}_1, \bar{B}_1, \dots, \bar{A}_g, \bar{B}_g, \bar{C}_1, \dots, \bar{C}_n\}$, where $n = l_h + l_p + l_e$. Let $V = \{A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_n\}$ be a set of lifts of the elements of the sequential set $\bar{V}$ into G_m^+, i.e. the image of A_i, B_i resp, C_j in G is $\bar{A}_i$, $\bar{B}_i$ resp. $\bar{C}_j$. Then the subgroup Γ^* of G_m generated by V is a lift of Γ into G_m^+ if and only if

$$[A_1, B_1] \cdots [A_g, B_g] \cdot C_1 \cdots C_n = e, \quad C_{l_h+l_p+i}^{p_i} = e \quad for\ i = 1, \dots, l_e.$$

Proof For any choice of the set of lifts V the restriction of the covering map $G_m \to G$ to the group Γ^* generated by V is a homomorphism with image Γ. If the conditions of the lemma hold true, then the group Γ^* satisfies the same relations as the group Γ, hence this homomorphism is injective. □

Lemma 5.3.19 *(Lemma 4.9 in [16]). Let $\{A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_n\}$ be a tuple of elements in G_m^+ such that the images $\{\bar{A}_1, \bar{B}_1, \dots, \bar{A}_g, \bar{B}_g, \bar{C}_1, \dots, \bar{C}_n\}$ in G form a sequential set of signature $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ with $l_h + l_p + l_e = n$. Then*

$$\prod_{i=1}^{g} [A_i, B_i] \cdot \prod_{j=1}^{n} C_j = \tilde{e} \Longleftrightarrow \sum_{j=1}^{n} s_m(C_j) \equiv (2 - 2g) - n \mod m$$

(in the case $n = 0$ this means $2 - 2g \equiv 0 \mod m$) and for any $i = 1, \ldots, l_e$

$$C_{l_h+l_p+i}^{p_i} = \tilde{e} \Longleftrightarrow p_i \cdot s_m(C_{l_h+l_p+i}) + 1 \equiv 0 \mod m.$$

Proof The case $g = 0$ was discussed in Lemma 5.3.16. We shall now reduce the general case to the case $g = 0$. By definition of sequential sets the set

$$(\bar{A}_1, \bar{B}_1\bar{A}_1^{-1}\bar{B}_1^{-1}, \ldots, \bar{A}_g, \bar{B}_g\bar{A}_g^{-1}\bar{B}_g^{-1}, \bar{C}_1, \ldots, \bar{C}_n)$$

is a sequential set of signature $(0; 2g + l_h, l_p, l_e)$, hence

$$\prod_{i=1}^{g}[A_i, B_i] \cdot \prod_{i=1}^{n} C_i = \prod_{i=1}^{g}(A_i \cdot B_iA_i^{-1}B_i^{-1}) \cdot \prod_{i=1}^{n} C_i = \tilde{e}$$

if and only if

$$\sum_{i=1}^{g}(s_m(A_i) + s_m(B_iA_i^{-1}B_i^{-1})) + \sum_{i=1}^{n} s_m(C_i)$$
$$\equiv -((2g + n) - 2) \equiv (2 - 2g) - n \mod m.$$

Invariance of the level function s_m under conjugation implies that

$$s_m(B_iA_i^{-1}B_i^{-1}) = s_m(A_i^{-1}).$$

Since A_i is not an element of order 2, $s_m(A_i^{-1}) = -s_m(A_i)$, and hence

$$s_m(A_i) + s_m(B_iA_i^{-1}B_i^{-1}) = s_m(A_i) - s_m(A_i) = 0.$$

The last statement follows from Lemma 5.3.1. □

Proposition 5.3.20 *(Lemma 4.10 in [16]). Let Γ be a Fuchsian group of signature*

$$(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e}).$$

Let $\bar{V} = \{\bar{A}_1, \bar{B}_1, \ldots, \bar{A}_g, \bar{B}_g, \bar{C}_1, \ldots, \bar{C}_n\}$, where $n = l_h+l_p+l_e$, be a sequential set that generates Γ. That is $\bar{A}_1, \bar{B}_1, \ldots, \bar{A}_g, \bar{B}_g$ are hyperbolic and correspond to handles. The element $\bar{C}_i$ is hyperbolic for $i = 1, \ldots, l_h$ and corresponds to a hole, is parabolic for $i = l_h+1, \ldots, l_h+l_p$ and corresponds to a puncture and is elliptic of order $p_{i-l_h-l_p}$ for $i = l_h + l_p + 1, \ldots, l_h + l_p + l_e$ and corresponds to a marking (orbifold) point. Then there exists a lift of Γ into G_m^+ with

$\bar{C}_i$ being lifted to level n_i for $i = 1, \ldots, l_h + l_p$, if and only if the signature $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ satisfies the following liftability conditions: $(p_i, m) = 1$ for $i = 1, \ldots, l_e$ and

$$\sum_{i=1}^{l_e} \frac{1}{p_i} - \sum_{i=1}^{l_h+l_p} n_i - (2g-2) - n \equiv 0 \mod m.$$

Moreover, if the liftability conditions are satisfied then any set of levels can be realised for lifting $A_1, B_1, \ldots, A_g, B_g$.

Proof First assume there exists a lift of Γ into G_m^+. Let$\{A_i, B_i, C_j\}$ be a set of lifts of $\bar{V}$ as in Lemmas 5.3.18 and 5.3.19. Let $n_i = s_m(C_i)$. Then according to Lemma 5.3.19 we have $p_i \cdot n_{i+l_h+l_p} + 1 \equiv 0 \mod m$ for $i = 1, \ldots, l_e$ and

$$(2g-2) + n + \sum_{i=1}^{l_h+l_p+l_e} n_i \equiv 0 \mod m.$$

The congruence $p_i \cdot n_{i+l_h+l_p} + 1 \equiv 0 \mod m$ implies that p_i is prime with m for $i = 1, \ldots, l_e$. Furthermore, since

$$(p_1 \cdots p_{l_e}) \cdot n_{i+l_h+l_p} \equiv \frac{p_1 \cdots p_{l_e}}{p_i} \cdot (p_i \cdot n_{i+l_h+l_p}) \equiv \frac{p_1 \cdots p_{l_e}}{p_i} \cdot (-1) \equiv -(p_1 \cdots p_{l_e}) \cdot \frac{1}{p_i},$$

we obtain that

$$(p_1 \cdots p_{l_e}) \cdot \sum_{i=1}^{l_e} n_{i+l_h+l_p} \equiv (p_1 \cdots p_{l_e}) \left(- \sum_{i=1}^{l_e} \frac{1}{p_i} \right).$$

Hence

$$\begin{aligned}
&(p_1 \cdots p_{l_e}) \left(\sum_{i=1}^{l_e} \frac{1}{p_i} - \sum_{i=1}^{l_h+l_p} n_i - (2g-2) - n \right) \\
&\equiv (p_1 \cdots p_{l_e}) \left(- \sum_{i=l_h+l_p+1}^{l_h+l_p+l_e} n_i - \sum_{i=1}^{l_h+l_p} n_i - (2g-2) - n \right) \\
&\equiv -(p_1 \cdots p_{l_e}) \left(\sum_{i=1}^{l_h+l_p+l_e} n_i + (2g-2) + n \right) \\
&\equiv 0 \mod m.
\end{aligned}$$

Now assume the liftability conditions are satisfied. We want to construct a lift of Γ into G_m^+. Since p_i is prime with m, we can choose $n_{i+l_h+l_p} \in \mathbb{Z}/m\mathbb{Z}$ such that $p_i \cdot n_{i+l_h+l_p} + 1 \equiv 0 \mod m$ for $i = 1, \dots, l_e$. Then

$$(p_1 \cdots p_{l_e}) \cdot n_{i+l_h+l_p} \equiv -(p_1 \cdots p_{l_e}) \cdot \frac{1}{p_i}$$

and hence

$$(p_1 \cdots p_{l_e}) \cdot \left((2g-2) + n + \sum_{i=1}^{l_h+l_p+l_e} n_i \right) \equiv$$

$$(p_1 \cdots p_{l_e}) \cdot \left((2g-2) + n + \sum_{i=1}^{l_h+l_p} n_i - \sum_{i=1}^{l_e} \frac{1}{p_i} \right) \equiv 0.$$

Since $p_1 \cdots p_{l_e}$ is prime with m, we conclude that

$$(2g-2) + n + \sum_{i=1}^{n} n_i \equiv 0,$$

i.e.

$$\sum_{i=1}^{n} n_i \equiv (2-2g) - n.$$

Let $V = \{A_i, B_i, C_j\}$ be any set of lifts of $\bar{V}$ such that $s_m(C_i) = n_i$ for $i = 1, \dots, l_h + l_p + l_e$. We have $p_i \cdot n_{i+l_h+l_p} + 1 \equiv 0 \mod m$ for $i = 1, \dots, l_e$ and

$$\sum_{i=1}^{n} n_i \equiv (2-2g) - n.$$

Hence according to Lemma 5.3.19 the set V generates a lift of Γ into G_m. Since Lemma 5.3.19 does not impose any conditions on the values $s_m(A_i)$ and $s_m(B_i)$ for $i = 1, \dots, g$, any of m^{2g} choices of these $2g$ values leads to a different lift of Γ into G_m. Finally note that we can divide by $p_1 \cdots p_{l_e}$ in $\mathbb{Z}/m\mathbb{Z}$ since $(p_i, m) = 1$. $\square$

Lemma 5.3.21 *If m is even and $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ is as in Proposition 5.3.20 then*

$$\sum_{i=1}^{l_e} \frac{1}{p_i} - l_e \equiv 0 \mod 2.$$

Proof If $\gcd(p_i, m) = 1$ and m is even then p_i is odd, hence $\frac{1}{p_i}$ is odd in $\mathbb{Z}/m\mathbb{Z}$ and therefore $\frac{1}{p_i} - 1$ is even in $\mathbb{Z}/m\mathbb{Z}$. Thus

$$\sum_{i=1}^{l_e} \frac{1}{p_i} - l_e = \sum_{i=1}^{l_e} (\frac{1}{p_i} - 1) \equiv 0 \mod 2.$$

$\square$

5.4 Lifts of Fuchsian groups, Level Function and Arf Functions

The following result was proved in [15] and [16]:

Theorem 5.4.1 *For a Fuchsian group Γ, there is a 1-1-correspondence between the lifts of Γ into G_m^+ and m-Arf functions on $P = \mathbb{H}/\Gamma$.*

When we lift Γ into G_m^+, for each $g \in \Gamma$ we choose $g^* \in G_m^+$ above g in such a way that the set of all g^* forms a subgroup. For each element that we lift we can choose a number in $\mathbb{Z}/m\mathbb{Z}$ to specify the level that we lift to. This gives a function $s_m : \Gamma \to \mathbb{Z}/m\mathbb{Z}$.

So we can check the conditions of Arf functions to see if there is a lift, rather than checking if the lifted elements form a subgroup.

By Theorems 5.1.2 and 5.4.1 there is a 1-1-correspondence between m-spin bundles on Riemann orbifolds and m-Arf functions.

5.5 Higher Arf functions

In sections 5.5.1 and 5.5.2 we recall the results about higher Arf functions from [16] and [15]. In section 5.5.3 we generalise the results of [16] and [15] to the case of orbifolds with holes and punctures. In [15] only the case of surfaces with holes and punctures but without marking points was considered, while in [16] surfaces with markings but without holes or punctures were dealt with.

5.5.1 Definition of higher Arf functions on orbifolds

Higher Arf functions were introduced in [16] and [15] to describe higher spin structures. Here we follow section 5.1 in [16].

Let Γ be a Fuchsian group of signature $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ and $P = \mathbb{H}/\Gamma$ the corresponding orbifold. Let $p \in P$. Let $\pi(P, p)$ be the orbifold fundamental group of P (see Definition 5.3.4). Let $\Psi : \mathbb{H} \to P$ be the natural projection. Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \to \pi(P, p)$ be the induced isomorphism. (See Proposition 5.3.6).

Definition 5.5.1 We denote by $\pi^0(P, p)$ the set of all non trivial elements of $\pi(P, p)$ that can be represented by simple contours. An *m-Arf function* is a function

$$\sigma : \pi^0(P, p) \to \mathbb{Z}/m\mathbb{Z}$$

satisfying the following conditions

1. $\sigma(bab^{-1}) = \sigma(a)$ for any elements a, $b \in \pi^0(P, p)$,
2. $\sigma(a^{-1}) = -\sigma(a)$ for any element $a \in \pi^0(P, p)$ that is not of order 2,
3. $\sigma(ab) = \sigma(a) + \sigma(b)$ for any elements a and b which can be represented by a pair of simple contours in P intersecting at exactly one point p with intersection number $\langle a, b \rangle \neq 0$,
4. $\sigma(ab) = \sigma(a) + \sigma(b) - 1$ for any elements a, $b \in \pi^0(P, p)$ such that the element ab is in $\pi^0(P, p)$ and the elements a and b can be represented by a pair of simple contours in P intersecting at exactly one point p with intersection number $\langle a, b \rangle = 0$ and placed in a neighbourhood of the point p as shown in Figure 5.3,

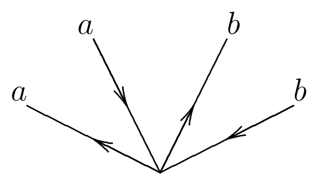

Figure 5.3: $\sigma(ab) = \sigma(a) + \sigma(b) - 1$

5. for any elliptic element c of order q we have $q \cdot \sigma(c) + 1 \equiv 0 \mod m$.

Definition 5.5.2 Let Γ^* be a lift of Γ into G_m^+. Let us consider a function $\hat{\sigma}_{\Gamma^*} : \pi(P,p) \to \mathbb{Z}/m\mathbb{Z}$ such that the following diagram commutes

$$\begin{tikzcd} \Gamma \arrow[r, "\cong"] \arrow[d, "\Phi"'] & \Gamma^* \arrow[d, "s_m|_{\Gamma^*}"] \\ \pi(P,p) \arrow[r, "\hat{\sigma}_{\Gamma^*}"] & \mathbb{Z}/m\mathbb{Z} \end{tikzcd}$$

It was shown in [16] and [15] that the function $\sigma_{\Gamma^*} = \hat{\sigma}_{\Gamma^*}|\pi^0(P,p)$ is an m-Arf function. We call this function the m-Arf function associated to the lift Γ^*.

The following result was proved in [16] and [15].

Theorem 5.5.3 *For a Fuchsian group Γ, there is a 1-1-correspondence between*

1. *lifts of Γ into G_m^+,*
2. *m-Arf functions on $P = \mathbb{H}/\Gamma$,*
3. *m-spin bundles on $P = \mathbb{H}/\Gamma$.*

Hence the results of section 5.3 on lifts of Fuchsian groups can be rewritten in terms of the corresponding Arf functions.

Proposition 5.5.4 *Let Γ be a Fuchsian group of signature*

$$(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$$

and let $P = \mathbb{H}/\Gamma$. Let $(a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n)$ be a standard system of generators of Γ with $c_1, \dots, c_{l_h}$ hyperbolic, $c_{l_h+1}, \dots, c_{l_h+l_p}$ parabolic and $c_{l_h+l_p+1}, \dots, c_n$ elliptic of orders $p_1, \dots, p_{l_e}$. Then there exists an m-Arf function $\sigma : \pi^0(P,p) \to \mathbb{Z}/m\mathbb{Z}$ with $\sigma(c_i) = n_i$ for $i = 1, \dots, l_h + l_p$ if and only if the following liftability conditions are satisfied: $(p_i, m) = 1$ for $i = 1, \dots, l_e$ and

$$\sum_{i=1}^{l_e} \frac{1}{p_i} - \sum_{i=1}^{l_h+l_p} n_i - (2g-2) - n \equiv 0 \mod m.$$

Moreover, if the liftability conditions are satisfied, any tuple of $2g$ values of σ on $a_1, b_1, \dots, a_g, b_g$ leads to a unique m-Arf function.

5.5.2 Higher Arf functions and autohomeomorphisms of orbifolds

We follow section 5.2 in [16].

Let Γ be a Fuchsian group of signature $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ and $P = \mathbb{H}/\Gamma$ the corresponding orbifold. Let $p \in P$. Let $\Psi : \mathbb{H} \to P$ be the natural projection. Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \to \pi^0(P, p)$ be the induced isomorphism. Let Γ^* be a lift of Γ in G_m^+. Consider the following transformations of a standard basis

$$v = \{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n\}$$

of $\pi^0(P, p)$ with $n = l_h + l_p + l_e$ to another standard basis

$$v' = \{a'_1, b'_1, \dots, a'_g, b'_g, c'_1, \dots, c'_n\} :$$

1. $a'_1 = a_1 b_1$.

2. $a'_1 = (a_1 a_2) a_1 (a_1 a_2)^{-1}$,
   $b'_1 = (a_1 a_2) a_1^{-1} a_2^{-1} b_1 (a_1 a_2)^{-1}$,
   $a'_2 = a_1 a_2 a_1^{-1}$,
   $b'_2 = b_2 a_2^{-1} a_1^{-1}$.

3. $a'_g = (b_g^{-1} c_1) b_g^{-1} (b_g^{-1} c_1)^{-1}$,
   $b'_g = (b_g^{-1} c_1 b_g) c_1^{-1} b_g a_g b_g^{-1} (b_g^{-1} c_1 b_g)^{-1}$,
   $c'_1 = b_g^{-1} c_1 b_g$.

4. $a'_k = a_{k+1}$, $b'_k = b_{k+1}$,
   $a'_{k+1} = (d_{k+1}^{-1} d_k) a_k (d_{k+1}^{-1} d_k)^{-1}$,
   $b'_{k+1} = (d_{k+1}^{-1} d_k) b_k (d_{k+1} d_k)^{-1}$.

5. $c'_k = c_{k+1}$, $c_{k+1} = c_{k+1}^{-1} c_k c_{k+1}$.

Here $d_i = [a_i, b_i]$ for $i = 1, \dots, g$, in 4 we consider $k \in \{1, \dots, g-1\}$, in 5 we consider $k \in \{1, \dots, n-1\}$ such that either c_k, c_{k+1} are both hyperbolic, both parabolic or both elliptic of the same order. If a'_i, b'_i resp. c'_i is not described explicitly, this means $a'_i = a_i$, $b'_i = b_i$ resp. $c'_i = c_i$.

We will call these transformations *generalised Dehn twists*. Each generalised Dehn twist induces a homotopy class of autohomeomorphisms of the orbifold P, which maps holes to holes, punctures to punctures and marking points to marking points of the same order. The group of all homotopy classes of autohomeomorphisms of the orbifold P is generated by the homotopy classes of generalised Dehn twists as described above.

In Lemma 5.5 in [16] it was shown how to compute the values of an m-Arf function σ on the standard basis v' from the values of σ on the standard basis v for each of the generalised Dehn twists described above. The generalisation to the case of orbifolds is straightforward.

Lemma 5.5.5 *Let $\sigma : \pi^0(P, p) \to \mathbb{Z}/m\mathbb{Z}$ be an m-Arf function. Let D be a generalised Dehn twist of the type described above. Suppose that D maps the standard basis*

$$v = \{a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n\}$$

into the standard basis

$$v' = D(v) = \{a'_1, b'_1, \ldots, a'_g, b'_g, c'_1, \ldots, c'_n\}.$$

Let α_i, β_i, γ_i resp. α'_i, β'_i, γ'_i be the values of σ on the elements of v resp. v'. Then for the Dehn twists of types $1-5$ we obtain

1. $\alpha'_1 = \alpha_1 + \beta_1$.
2. $\beta'_1 = \beta_1 - \alpha_1 - \alpha_2 - 1$, $\beta'_2 = \beta_2 - \alpha_2 - \alpha_1 - 1$.
3. $\alpha'_g = -\beta_g$, $\beta'_g = \alpha_g - \gamma_1 - 1$.
4. $\alpha'_k = \alpha_{k+1}$, $\beta'_k = \beta_{k+1}$, $\alpha'_{k+1} = \alpha_k$, $\beta'_{k+1} = \beta_k$.
5. $\gamma'_k = \gamma_{k+1}$, $\gamma'_{k+1} = \gamma_k$.

5.5.3 Topological classification of higher Arf functions

In this section we combine the ideas from [15] and [16] to understand the case of orbifolds with holes and punctures.

Let P be a Riemann surface of type $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ with $l_h + l_p + l_e = n$. Let $p \in P$.

Definition 5.5.6 Let $\sigma : \pi^0(P,p) \to \mathbb{Z}/m\mathbb{Z}$ be an m-Arf function. We define the *Arf invariant* $\delta = \delta(P,\sigma)$ of σ as follows: If $g > 1$ and m is even then we set $\delta = 0$ if there is a standard basis $\{a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n\}$ of the orbifold fundamental group $\pi(P,p)$ such that

$$\sum_{i=1}^{g}(1-\sigma(a_i))(1-\sigma(b_i)) \equiv 0 \mod 2$$

and we set $\delta = 1$ otherwise. If $g > 1$ and m is odd then we set $\delta = 0$. If $g = 0$ then we set $\delta = 0$. If $g = 1$ and there is a standard basis $\{a_1, b_1, c_1, \ldots, c_n\}$ of the fundamental group $\pi(P,p)$ with $c_1, \ldots, c_{l_h}$ corresponding to holes, $c_{l_h+1}, \ldots, c_{l_h+l_p}$ corresponding to punctures and $c_{l_h+l_p+1}, \ldots, c_{l_h+l_p+l_e}$ corresponding to marking points then we set

$$\begin{aligned}\delta &= \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1)+1, \ldots, \sigma(c_n)+1) && (5.1)\\ &= \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1)+1, \ldots, \sigma(c_{l_h+l_p})+1, p_1-1, \ldots, p_{l_e}-1). && (5.2)\end{aligned}$$

Lemma 5.5.7 *Formulas (5.1) and (5.2) are equivalent.*

Proof Let d be a common divisor of m, $\sigma(c_{i+l_h+l_p})+1$ for $i = 1, \ldots, l_e$. Then $m \equiv 0 \mod d$ and $\sigma(c_{i+l_h+l_p}) \equiv -1 \mod d$. We know that $p_i\sigma(c_{i+l_h+l_p}) + 1 \equiv 0 \mod m$ but $m \equiv 0 \mod d$ hence $p_i\sigma(c_{i+l_h+l_p}) + 1 \equiv 0 \mod d$. Since $\sigma(c_{i+l_h+l_p}) \equiv -1 \mod d$ we have $-p_i + 1 \equiv 0 \mod d$. Hence d is a divisor of $p_i - 1$.

Now let d be a common divisor of m, $p_i - 1$. Then $m \equiv 0 \mod d$ and $p_i \equiv 1 \mod d$. We know that $p_i\sigma(c_{i+l_h+l_p}) + 1 \equiv 0 \mod m$ implying $p_i\sigma(c_{i+l_h+l_p}) + 1 \equiv 0 \mod d$. Since $p_i \equiv 1 \mod d$ we have $\sigma(c_{i+l_h+l_p}) + 1 \equiv 0 \mod d$. Hence d is a divisor of $\sigma(c_{i+l_h+l_p}) + 1$. □

Expanding on a remark after Definition 6.1 in [16], we will now show that the Arf invariant δ does not change under Dehn twists (the transformations described in Lemma 5.5.5), i.e. it is indeed an invariant of an Arf function.

Lemma 5.5.8 *If $g > 1$, m is even and $\sigma(c_i)$ are odd for $i = 1, \ldots, l_h + l_p$ then*

$$\sum_{i=1}^{g}(1-\sigma(a_i))(1-\sigma(b_i))$$

does not change parity under Dehn twists. If $g = 1$ then

$$\gcd(\sigma(a_1), \sigma(b_1), \sigma(c_1) + 1, \ldots, \sigma(c_n) + 1)$$

does not change under Dehn twists.

Proof Let $D, v, v', \alpha_i, \beta_i, \gamma_i, \alpha_i', \beta_i', \gamma_i'$ be as in Lemma 5.5.5. Let us first consider the case $g > 1$ and m even:

For a Dehn twist of type 1 we have

$$\begin{aligned}(1-\alpha_1')(1-\beta_1') &= (1-(\alpha_1+\beta_1))(1-\beta_1)\\ &= (1-\alpha_1)(1-\beta_1) - \beta_1(1-\beta_1) \equiv (1-\alpha_1)(1-\beta_1) \mod 2\end{aligned}$$

since $\beta_1(1-\beta_1)$ is always even.

For a Dehn twist of type 2 we have

$$\begin{aligned}&(1-\alpha_1')(1-\beta_1') + (1-\alpha_2')(1-\beta_2')\\ &= (1-\alpha_1)(1-\beta_1+\alpha_1+\alpha_2+1) + (1-\alpha_2)(1-\beta_2+\alpha_1+\alpha_2+1)\\ &= (1-\alpha_1)(1-\beta_1) + (1-\alpha_2)(1-\beta_2) + (2-(\alpha_1+\alpha_2))((\alpha_1+\alpha_2)+1)\\ &= (1-\alpha_1)(1-\beta_1) + (1-\alpha_2)(1-\beta_2) \mod 2\end{aligned}$$

since $(2-(\alpha_1+\alpha_2))((\alpha_1+\alpha_2)-1)$ is always even.

For a Dehn twist of type 3, if $l_h + l_p > 0$ then $\gamma_1 = \sigma(c_1)$ is odd by assumption. If $l_h + l_p = 0$ then since m is even and $p_1 \cdot \gamma_1 + 1 \equiv 0 \mod m$, we have that γ_1 is odd. Then $\gamma_1 + 1 \equiv 0 \mod 2$ and $1 + \beta_g \equiv 1 - \beta_g \mod 2$ imply

$$\begin{aligned}(1-\alpha_g')(1-\beta_g') &= (1+\beta_g)(1-\alpha_g+(\gamma_1+1))\\ &\equiv (1+\beta_g)(1-\alpha_g) \equiv (1-\beta_g)(1-\alpha_g) \mod 2.\end{aligned}$$

Dehn twists of type 4 do not change δ since they only permute (α_k, β_k) with $(\alpha_{k+1}, \beta_{k+1})$. Dehn twists of type 5 do not change δ since they do not change α_i and β_i.

Let us now consider the case $g = 1$: Dehn twists of types 2 and 4 involve pairs a_i, b_i and a_j, b_j, i.e. they are not applicable in the case $g = 1$. Dehn twists of type 5 do not change δ since they do not change α_i, β_i.

For a Dehn twist of type 1 we obtain $\alpha_1' = \alpha_1 + \beta_1$, $\beta_1' = \beta_1$ and $\gamma_i' = \gamma_i$. Thus

$$\gcd(\alpha_1', \beta_1') = \gcd(\alpha_1 + \beta_1, \beta_1) = \gcd(\alpha_1, \beta_1)$$

and therefore

$$\gcd(m, \alpha_1', \beta_1', \gamma_1' + 1, \dots, \gamma_{l_h+l_p}' + 1, p_1 - 1, \dots, p_{l_e} - 1)$$

$$= \gcd(m, \alpha_1, \beta_1, \gamma_1 + 1, \dots, \gamma_{l_h+l_p} + 1, p_1 - 1, \dots, p_{l_e} - 1).$$

For a Dehn twist of type 3 we have $\alpha_1' = -\beta_1$, $\beta_1' = \alpha_1 - \gamma_1 - 1$ and $\gamma_i' = \gamma_i$. Let d be a common divisor of $m, \alpha_1, \beta_1, \gamma_1+1, \dots, \gamma_{l_h+l_p}+1, p_1-1, \dots, p_{l_e}-1$, i.e.

$$m \equiv \alpha_1 \equiv \beta_1 \equiv 0 \mod d,$$

$$\gamma_1 \equiv \dots \equiv \gamma_{l_h+l_p} \equiv -1 \mod d,$$

$$p_1 \equiv \dots \equiv p_{l_e} \equiv 1 \mod d.$$

Since d divides α_1, β_1 and $\gamma_1 + 1$ then d is a divisor of $\alpha_1' = -\beta_1$ and $\beta_1' = \alpha_1 - (\gamma_1 + 1)$. So every common divisor of $m, \alpha_1, \beta_1, \gamma_1 + 1, \dots, \gamma_{l_h+l_p} + 1, p_1 - 1, \dots, p_{l_e} - 1$ is a common divisor of $m, \alpha_1', \beta_1', \gamma_1' + 1, \dots, \gamma_{l_h+l_p}' + 1, p_1 - 1, \dots, p_{l_e} - 1$. Similarly, every common divisor of $m, \alpha_1', \beta_1', \gamma_1' + 1, \dots, \gamma_{l_h+l_p}' + 1, p_1 - 1, \dots, p_{l_e} - 1$ is a common divisor of $m, \alpha_1, \beta_1, \gamma_1 + 1, \dots, \gamma_{l_h+l_p} + 1, p_1 - 1, \dots, p_{l_e} - 1$. Thus

$$\gcd(m, \alpha_1', \beta_1', \gamma_1' + 1, \dots, \gamma_{l_h+l_p}' + 1, p_1 - 1, \dots, p_{l_e} - 1)$$

is equal to

$$\gcd(m, \alpha_1, \beta_1, \gamma_1 + 1, \dots, \gamma_{l_h+l_p} + 1, p_1 - 1, \dots, p_{l_e} - 1).$$

□

Definition 5.5.9 Let n_j^h be the number of holes c_i with $\sigma(c_i) = j$, n_j^p the number of punctures c_i with $\sigma(c_i) = j$ and n_j^e the number of elliptics $c_{l_h+l_p+i}$ with $\sigma(c_i) = j$ (that is, the number of elliptics with $p_i j + 1 \equiv 0 \mod m$.) The *type of the m-Arf function* (P, σ) is the tuple

$$(g, \delta, n_0^h, \dots, n_{m-1}^h, n_0^p, \dots, n_{m-1}^p, p_1, \dots, p_{l_e}),$$

where δ is the Arf invariant of σ defined above.

Lemma 5.5.10 *Let $\sigma : \pi^0(P,p) \to \mathbb{Z}/m\mathbb{Z}$ be an m-Arf function.*

(a) If $g > 1$ then there is a standard basis $v = \{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n\}$ of $\pi(P,p)$ such that

$$(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, \xi, 1, \dots, 1)$$

with $\xi \in \{0,1\}$ and

$$\sigma(c_1) \leq \cdots \leq \sigma(c_{l_h}),\ \sigma(c_{l_h+1}) \leq \cdots \leq \sigma(c_{l_h+l_p+l_e}).$$

If m is odd or there is a contour around a hole or puncture such that the value of σ on this contour is even, then the basis can be chosen in such a way that $\xi = 1$, i.e. so that

$$(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, 1, 1, \dots, 1).$$

(b) If $g = 1$ then there is a standard basis $v = \{a_1, b_1, c_1, \dots, c_n\}$ of $\pi(P,p)$ such that $(\sigma(a_1), \sigma(b_1)) = (\delta, 0)$, where δ is the Arf invariant of σ and

$$\sigma(c_1) \leq \ldots \leq \sigma(c_{l_h}), \sigma(c_{l_h+1}) \leq \ldots \leq \sigma(c_{l_h+l_p}).$$

(The inequalities between $\sigma(c_i) \in \mathbb{Z}/m\mathbb{Z}$ are to be understood as inequalities between elements of $\{0, \dots, m-1\}$).

Proof a) Let us fix some standard basis

$$v_0 = \{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n\}$$

and consider the sequence of values of the Arf function σ on the basis v_0

$$(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_n)$$

$$= (\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g), \sigma(c_1), \dots, \sigma(c_n)).$$

Any other standard basis v is an image of this basis v_0 under an auto-homeomorphism of the surface, i.e. under a sequence of Dehn twists. Hence according to Lemma 5.5.5 the sequence of values of σ on the basis v is the image of the corresponding sequence with respect to the basis v_0 under the group generated by the transformations that change the first $2g$ components $(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ as follows

1. $(\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \ldots, \alpha_g, \beta_g) \mapsto (\alpha_1, \beta_1, \ldots, \alpha_i \pm \beta_i, \beta_i, \ldots, \alpha_g, \beta_g)$
2. $(\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \ldots, \alpha_g, \beta_g) \mapsto (\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i \pm \alpha_i, \ldots, \alpha_g, \beta_g)$
3. $(\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \ldots, \alpha_j, \beta_j, \ldots, \alpha_g, \beta_g)$
   $\mapsto (\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i - \alpha_j - 1, \ldots, \alpha_j, \beta_j - \alpha_i - 1, \ldots, \alpha_g, \beta_g)$
4. $(\alpha_1, \beta_1, \ldots, \alpha_{g-1}, \beta_{g-1}, \alpha_g, \beta_g)$
   $\mapsto (\alpha_1, \beta_1, \ldots, \alpha_{g-1}, \beta_{g-1}, -\beta_g, \alpha_g - \gamma_1 - 1)$
5. $(\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \ldots, \alpha_j, \beta_j, \ldots, \alpha_g, \beta_g)$
   $\mapsto (\alpha_1, \beta_1, \ldots, \alpha_j, \beta_j, \ldots, \alpha_i, \beta_i, \ldots, \alpha_g, \beta_g)$
6. $(\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \ldots, \alpha_g, \beta_g) \mapsto (\alpha_1, \beta_1, \ldots, -\alpha_i, -\beta_i, \ldots, \alpha_g, \beta_g)$
7. $(\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \ldots, \alpha_g, \beta_g) \mapsto (\alpha_1, \beta_1, \ldots, -\beta_i, \alpha_i, \ldots, \alpha_g, \beta_g)$

and change $(\gamma_1, \ldots, \gamma_n)$ by all possible permutations of $(\gamma_1, \ldots, \gamma_{l_h})$ and $(\gamma_{l_h+1}, \ldots, \gamma_{l_h+l_p})$.

The inequalities between the values of σ on the elements c_i are easy to satisfy, because the transformation group contains all possible permutations of $(\gamma_1, \ldots, \gamma_{l_h})$ and $(\gamma_{l_h+1}, \ldots, \gamma_{l_h+l_p})$. Our aim is to show that if m is odd or one of the numbers $\gamma_1, \ldots, \gamma_{l_h+l_p}$ is even, any tuple $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ can be transformed into the $2g$-tuple

$$(0, 1, 1, \ldots, 1)$$

while otherwise any such tuple can be transformed into one of the tuples

$$(0, 0, 1, \ldots, 1) \quad \text{or} \quad (0, 1, 1, \ldots, 1).$$

b) We claim that the group of transformations described in (a) contains the transformation of the form

$$(\ldots, \alpha_i, \beta_i, \ldots, \alpha_j, \beta_j, \ldots,) \mapsto (\ldots, \alpha_i, \beta_i - 2, \ldots, \alpha_j, \beta_j, \ldots).$$

Assume $(i, j) = (1, 2)$ to simplify notation. We apply transformations 3, 6, again 3 and again 6 and obtain

$$\begin{aligned}
&(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots) \mapsto (\alpha_1, \beta_1 - \alpha_2 - 1, \alpha_2, \beta_2 - \alpha_1 - 1, \ldots) \\
&\mapsto (\alpha_1, \beta_1 - \alpha_2 - 1, -\alpha_2, -(\beta_2 - \alpha_1 - 1), \ldots) \\
&\mapsto (\alpha_1, \beta_1 - \alpha_2 - 1 - (-\alpha_2) - 1, -\alpha_2, -\beta_2 + \alpha_1 + 1 - \alpha_1 - 1, \ldots) \\
&= (\alpha_1, \beta_1 - 2, -\alpha_2, -\beta_2, \ldots) \\
&\mapsto (\alpha_1, \beta_1 - 2, \alpha_2, \beta_2, \ldots).
\end{aligned}$$

c) Furthermore, we claim that the group of transformations contains a transformation of the form

$$(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_g, \beta_g) \mapsto (0, \xi, 1, 1, \dots, 1, 1),$$

where $\xi \in \{0, 1\}$.

With the help of the transformation described in (b) we transform

$$(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_g, \beta_g) \mapsto (\alpha'_1, \beta'_1, \alpha'_2, \beta'_2, \dots, \alpha'_g, \beta'_g),$$

where $\alpha'_i, \beta'_i \in \{0, 1\}$.

If we have some $i, j \in \{1, \dots, g\}$ such that $\alpha'_i = \beta'_i = \alpha'_j = \beta'_j = 0$, then applying the inverse of transformation 3 we obtain $\alpha''_i = \alpha'_i = 0$, $\beta''_i = \beta'_i + \alpha'_j + 1 = 1$, $\alpha''_j = \alpha'_j = 0$, $\beta''_j = \beta'_j + \alpha'_i + 1 = 1$. By successive use of this transformation we can achieve the situation where every pair (α''_i, β''_i) except at most one contains at least one 1. Using transformation 5 we can assume that $(\alpha'_i, \beta'_i) \in \{(0, 1), (1, 0), (1, 1)\}$ for $i = 2, \dots, g$. Applying transformations 1 and 2 respectively we can change $(\alpha''_i, \beta''_i) = (1, 0)$ or $(0, 1)$ to $(1, 1)$. Hence we obtain $\alpha''_2 = \beta''_2 = \dots = \alpha''_g = \beta''_g = 1$. It remains to consider (α''_1, β''_1). Applying transformations 1 and 2 respectively we can change $(\alpha''_1, \beta''_1) = (1, 1)$ and $(1, 0)$ to $(0, 1)$.

d) If $m = 2r + 1$ is odd, then we use the transformation described in (b) to map

$$\begin{aligned} (0, 1, 1, \dots, 1) &\mapsto (1, 1 - 2 \cdot (r + 1), 1, \dots, 1) \\ &= (0, -m, 1, \dots, 1) = (0, 0, 1, \dots, 1). \end{aligned}$$

e) If m is even and $\gamma_1 = 2r$ is even, then we use the transformation 5, the transformation 4, successive application of the transformation described in (b) and the transformation 5 again to map

$$\begin{aligned} (0, 0, 1, \dots, 1) &\mapsto (1, \dots, 1, 0, 0) \\ &\mapsto (1, \dots, 1, 0, 0 - 2r - 1) \\ &\mapsto (1, \dots, 1, 0, 1) \\ &\mapsto (0, 1, 1, \dots, 1). \end{aligned}$$

hence also in this case the $2g$-tuple $(0, 0, 1, \ldots, 1)$ can be transformed into the tuple $(0, 1, 1, \ldots, 1)$.

f) If m is even and γ_i is even for some $i \in \{2, \ldots, l_h\}$, then we can apply the transformation that permutes γ_1 and γ_i to obtain the situation as in (e). The situation is more complicated if m is even and γ_i is even for some $i \in \{l_h + 1, \ldots, l_h + l_p\}$ as we cannot permute elements of different type (c_1 is hyperbolic, c_i is parabolic). In such a case we need to introduce a different kind of standard basis, one where $c_1, \ldots, c_{l_p}$ are parabolic and $c_{l_p+1}, \ldots, c_{l_p+l_h}$ are hyperbolic. □

The following theorem gives conditions under which an m-Arf function and hence an m-spin structure exists on an orbifold with holes, punctures and marking points.

Theorem 5.5.11 *A tuple $t = (g, \delta, n_0^h, \ldots, n_{m-1}^h, n_0^p, \ldots, n_{m-1}^p, p_1, \ldots, p_{l_e})$ is the topological type of an m-Arf function on a hyperbolic Riemann orbifold of type $(g, l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ if and only if it has the following properties:*

(a) If $g > 1$ and m is odd, then $\delta = 0$.

(b) If $g > 1$, m is even and $n_j^h + n_j^p \neq 0$ for some even j, then $\delta = 0$.

(c) If $g = 1$ then δ is a divisor of m, $\gcd\{j + 1 | n_j^h + n_j^p \neq 0\}$ and $\gcd(p_1 - 1, \ldots, p_{l_e} - 1)$.

(d) The following degree conditions are satisfied: $(p_i, m) = 1$ for $i = 1, \ldots, l_e$ and

$$\sum_{j=0}^{m-1} j(n_j^h + n_j^p) - \sum_{i=1}^{l_e} \frac{1}{p_i} = (2 - 2g) - (l_h + l_p + l_e) \mod m.$$

Proof Let us first assume that the tuple t is the type of an m-Arf function on a hyperbolic Riemann orbifold of type $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$. Then according to Proposition 5.5.4 the signature $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ satisfies the liftability condition (d) (here $\sum_{i=1}^{l_h+l_p} n_i = \sum_{i=0}^{m-1} j(n_j^h+n_j^p)$ and $n = l_h+l_p+l_e$).

If $g > 1$ and m is odd, $\delta(P, \sigma) = 0$ by definition.

If m is even and $n_j^h + n_j^p \neq 0$ for some even $j \in \mathbb{Z}/m\mathbb{Z}$ then according to Lemma 5.5.10 there is a standard basis $v = \{a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n\}$ of $\pi^0(P, p)$ such that

$$(\sigma(a_1), \sigma(b_1), \ldots, \sigma(a_g), \sigma(b_g)) = (0, 1, 1, \ldots, 1),$$

hence $\delta(P, \sigma) = 0$ by definition.

If $g = 1$ then δ is a divisor of m, $\sigma(a_1)$, $\sigma(b_1)$, $\sigma(c_1) + 1, \ldots, \sigma(c_{l_h + l_p}) + 1, p_1 - 1, \ldots, p_{l_e} - 1$ by definition.

Let us assume that $t = (g, \delta, n_j^h, n_j^p, p_1, \ldots, p_{l_e})$ is a tuple satisfying the conditions (a) to (d). Let P be a Riemann orbifold of signature $(g, l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ and let

$$\{a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n\}$$

be a standard basis of $\pi^0(P, p)$. According to Proposition 5.5.4, any tuple of $2g$ values in $\mathbb{Z}/m\mathbb{Z}$ can be realised as a set of values on a_i, b_i of an m-Arf function on $\pi^0(P, p)$. In particular, if $g > 1$ then for any $\delta \in \{0, 1\}$ there exists an m-Arf function σ such that $(\sigma(a_1), \sigma(b_1), \ldots, \sigma(a_g), \sigma(b_g)) = (0, 1 - \delta, 1, \ldots, 1)$ and if $g = 1$ then for any divisor δ of m, $\sigma(a_1)$, $\sigma(b_1)$, $\sigma(c_1) + 1, \ldots, \sigma(c_{l_h + l_p}) + 1, p_1 - 1, \ldots, p_{l_e} - 1$ there exists an m-Arf function σ such that $(\sigma(a_1), \sigma(b_1)) = (\delta, 0)$.

Let $g > 1$. For the chosen basis $v = \{a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_{l_h + l_p + l_e}\}$ we have $\sigma(a_1) = 0$, $\sigma(b_1) = 1 - \delta$ and $\sigma(a_i) = \sigma(b_i) = 1$ for all $i > 1$, hence $(1 - \sigma(a_1))(1 - \sigma(b_1)) = \delta$ and $(1 - \sigma(a_i))(1 - \sigma(b_i)) = 0$ for all $i > 1$. Therefore

$$\sum_{i=1}^{g} (1 - \sigma(a_i))(1 - \sigma(b_i)) = \delta.$$

If $\delta = 0$ then we have just shown that there exists a standard set of generators with

$$\sum_{i=1}^{g} (1 - \sigma(a_i))(1 - \sigma(b_i)) = 0,$$

hence $\delta(\sigma) = 0 = \delta$.

If $\delta = 1$ then we have

$$\sum_{i=1}^{g}(1-\sigma(a_i))(1-\sigma(b_i)) = 1$$

for one set of generators. We assume that conditions (a) and (b) are satisfied, hence we can only have $\delta = 1$ if m is even and $\sigma(c_i)$ are odd for all $i = 1, \ldots, l_h + l_p$. Then Lemma 5.5.8 shows that

$$\sum_{i=1}^{g}(1-\sigma(a_i))(1-\sigma(b_i))$$

does not change under Dehn twists, hence:

$$\sum_{i=1}^{g}(1-\sigma(a_i))(1-\sigma(b_i)) = 1$$

for any standard generating set, i.e. $\delta(\sigma) = 1 = \delta$.

Now let $g = 1$. Then

$$\begin{aligned}
\delta(\sigma) &= \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1) + 1, \ldots, \sigma(c_{l_h+l_p}), p_1 - 1, \ldots, p_{l_e} - 1) \\
&= \gcd(m, \delta, 0, \sigma(c_1) + 1, \ldots, \sigma(c_{l_h+l_p}), p_1 - 1, \ldots, p_{l_e} - 1) \\
&= \gcd(\delta, \gcd(m, \sigma(c_1) + 1, \ldots, \sigma(c_{l_h+l_p}), p_1 - 1, \ldots, p_{l_e} - 1)).
\end{aligned}$$

We assume that condition (c) is satisfied, hence δ is a divisor of $m, \sigma(c_1) + 1, \ldots, \sigma(c_{l_h+l_p}) + 1, p_1 - 1, \ldots, p_{l_e} - 1$. Hence

$$\delta(\sigma) = \gcd(\delta, \gcd(m, \sigma(c_1) + 1, \ldots, \sigma(c_{l_h+l_p}), p_1 - 1, \ldots, p_{l_e} - 1)) = \delta.$$

$\square$

Chapter 6

Higher Spin Bundles on Riemann Orbifolds and Gorenstein Quasi-Homogeneous Surface Singularities

6.1 Higher Spin Bundles on Riemann Orbifolds and Gorenstein Quasi-Homogeneous Surface Singularities

Definition 6.1.1 We consider the space $\mathbb{C}^n$ with fixed coordinates

$$x_1, \ldots, x_n.$$

A holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is said to be *quasi-homogeneous of degree d with indices $\alpha_1, \ldots, \alpha_n$*, if for any $\lambda > 0$ we have

$$f(\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n) = \lambda^d f(x_1, \ldots, x_n).$$

The index α_s is also called the *weight* of the variable x_s. In terms of the Taylor series $f = \sum f_k x^k$ the condition of quasihomogeneity *of degree d means that all the indicies $(k_1, \ldots, k_n)$ of the non-zero terms $f_{(k_1,\ldots,k_n)} x_1^{k_1} \cdots x_n^{k_n}$ of the series lie on the hyperplane*

$$\gamma = \{(k_1, \ldots, k_n) : \alpha_1 k_1 + \cdots + \alpha_n k_n = d\}.$$

Definition 6.1.2 A normal isolated singularity of dimension n is Gorenstein if and only if there exists a nowhere vanishing n-form on a punctured neighbourhood of the singular point. For example all isolated singularities of complete intersections are Gorenstein.

Definition 6.1.3 A normal isolated singularity of dimension at least 2 is $\mathbb{Q}$-Gorenstein if there is a natural number r such that the divisor $r \cdot \mathcal{K}_X$ is defined on a punctured neighbourhood of the singular point by a function. Here $\mathcal{K}_X$ is the canonical divisor of X.

Automorphy factors were introduced by Dolgachev [5] , Milnor [10], Neumann [19] and Pinkham [20] to describe quasi-homogeneous surface singularities.

Definition 6.1.4 A *negative unramified automorphy factor* (U, Γ, L) is a complex line bundle L over a simply connected Riemann surface U together with a discrete co-compact subgroup $\Gamma \subset Aut(U)$ acting compatibly on U and on the line bundle L, such that the following two conditions are satisfied:

1. The action of Γ is free on L^*, the complement of the zero-section in L.

2. Let $\tilde{\Gamma} \triangleleft \Gamma$ be a normal subgroup of finite index, which acts freely on U, and let $E \to P$ be the complex line bundle $E = L/\tilde{\Gamma}$ over the compact Riemann surface $P = U/\tilde{\Gamma}$. Then E is a negative line bundle, i.e. the self-intersection number $P \cdot P$ is negative.

There are three possibilities for a simply-connected Riemann surface U, it is either the sphere $\mathbb{C}P^1$, the complex plane $\mathbb{C}$ or the real hyperbolic plane $\mathbb{H}$. We call the corresponding automorphy factors spherical, euclidian and hyperbolic respectively.

Dolgachev in [5] gave the following characterisation of those automorphy factors that correspond to the Gorenstein singularities.

Theorem 6.1.5 *A quasi-homogeneous surface singularity is Gorenstein if and only if for the corresponding automorphy factor (U, Γ, L) there is an integer m (called the level or the exponent of the automorphy factor) such that the m-th tensor power L^m is Γ-equvariantly isomorphic to the tangent bundle T_U of the surface U.*

This result puts Gorenstein quasi-homogeneous surface singularities in relation with higher spin bundles on Riemann orbifolds.

The following correspondence was shown in [16]:

Theorem 6.1.6 *Let Γ be a Fuchsian group of signature*

$$(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$$

and $P = \mathbb{H}/\Gamma$ the corresponding orbifold. Let $p \in P$. There is a 1-1-*correspondence between*

1. *hyperbolic Gorenstein automorphy factors of level m associated to the Fuchsian group Γ,*

2. *m-spin structures on P,*

3. *lifts of Γ into G_m^+,*

4. *m-Arf functions $\sigma : \pi^0(P, p) \to \mathbb{Z}/m\mathbb{Z}$.*

Chapter 7

Introduction to Klein Surfaces

In this section we follow [17] with slight modifications from Klein surfaces to a certain type of Klein orbifolds.

7.1 Introduction to Klein Surfaces

Definition 7.1.1 A *Klein surface* (or *non-singular real algebraic curve*) is a topological surface with a maximal atlas whose transition maps are *dianalytic*, i.e. either holomorphic or anti-holomorphic.

Definition 7.1.2 A *real form* of a complex algebraic curve P is a pair (P, τ), where P is a compact Riemann surface and $\tau : P \to P$ is an anti-holomorphic involution on P. The set P^τ of fixed points of τ is called the set of real points of (P, τ).

Example 7.1.3 *If a complex algebraic curve is given by an equation*

$$F(x, y) = 0,$$

then the complex conjugation of both components x and y induces an anti-holomorphic involution. The fixed points of this involution are the points of the curve with both components real.

The category of Klein surfaces is equivalent to the category of such pairs (P, τ), see [1]. The involution acts on all structures related to the Riemann surface, for example on vector bundles.

Definition 7.1.4 A *Klein orbifold* is a pair (P, τ), where P is a compact Riemann orbifold and $\tau : P \to P$ is an anti-holomorphic involution on P.

Definition 7.1.5 We say (P, τ) is a *nice Klein orbifold* if the fixed point set P^τ does not contain any marking points of P. A nice Klein orbifold has an even number of marking points, paired up by the involution τ.

There are two kinds of contours which are invariant under the involution τ, the *ovals* and the *twists*. Ovals are simple closed smooth contours consisting of fixed points of τ. A twist is a simple contour in P which is invariant under the involution τ but does not contain any fixed points of τ.

Definition 7.1.6 We say (P, τ) is *separating* if the set $P \backslash P^\tau$ is not connected. Otherwise we say it is *non-separating.*

We can decompose P into two orbifolds P_1 and P_2 by removing a number of invariant contours. The decomposition is unique in the case of a separating Klein orbifold but not unique in the case of a non-separating Klein orbifold since the twists can be chosen in different ways.

Definition 7.1.7 Let (P, τ) be a nice Klein orbifold. The *topological type* of (P, τ) is the tuple $(g; 2r, k, \epsilon : p_1, \ldots, p_r)$, where g is the genus of the Riemann orbifold P, $2r$ the number of marked points on P, $p_1, p_1, p_2, p_2, \ldots, p_r, p_r$ their orders, k is the number of connected components of the set of real points P^τ, $\epsilon = 0$ if (P, τ) is non-separating and $\epsilon = 1$ otherwise.

Definition 7.1.8 Given two Klein orbifolds (P_1, τ_1) and (P_2, τ_2), we say that they are *topologically equivalent* if there exists a homeomorphism $\phi : P_1 \to P_2$ such that $\phi \circ \tau_1 = \tau_2 \circ \phi$.

The following generalisation of the result of Weichold [23] gives a classification of nice Klein surfaces up to topological equivalence, see also [4]:

Theorem 7.1.9 *Two nice Klein orbifolds are topologically equivalent if and only if they are of the same topological type. A tuple* $(g, 2r, k, \epsilon, p_1, \ldots, p_r)$ *is a topological type of some nice Klein orbifold if and only if either* $\epsilon = 1$, $1 \leq k \leq g+1$, $k \equiv g+1 \mod 2$ *or* $\epsilon = 0$, $0 \leq k \leq g$.

Any separating nice Klein orbifold can be obtained by gluing together a Riemann orbifold with boundary without marking points on the boundary with its copy via the identity map along the boundary components. If we replace the identity map with a half-turn on some of the boundary components, we obtain a non-separating nice Klein orbifold. Moreover, all non-separating nice Klein orbifolds are obtained in this way. This is illustrated in figures 7.1 and 7.2.

All Klein surfaces can be constructed from real Fuchsian groups:

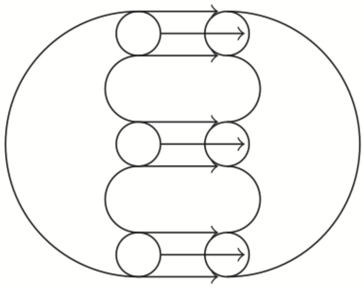

Figure 7.1: Separating Klein surface.

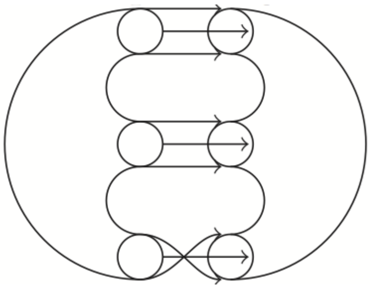

Figure 7.2: Non-Separating Klein surface.

Definition 7.1.10 A *non-Euclidian crystallographic group* or *NEC group* is a discrete subgroup of $Aut(\mathbb{H})$. We will call a NEC group $\hat{\Gamma}$ a *real Fuchsian group* if the intersection $\hat{\Gamma}^+ = \hat{\Gamma} \cap Aut_+(\mathbb{H})$ is a Fuchsian group, $\hat{\Gamma} \neq \hat{\Gamma}^+$ and the quotient $P = \mathbb{H}/\hat{\Gamma}^+$ is a compact orbifold.

Let $\hat{\Gamma}$ be a real Fuchsian group. Let $\hat{\Gamma}^{\pm} = \hat{\Gamma} \cap Aut_{\pm}(\mathbb{H})$, $P_{\hat{\Gamma}} = \mathbb{H}/\hat{\Gamma}^+$ and let $\Phi : \mathbb{H} \to P_{\hat{\Gamma}}$ be the natural projection. Then for any automorphism $g \in \hat{\Gamma}^-$, the map $\tau_{\hat{\Gamma}} = \Phi \circ g \circ \Phi^{-1}$ is an anti-holomorphic involution on $P_{\hat{\Gamma}}$. Thus a real Fuchsian group $\hat{\Gamma}$ defines the Klein orbifold $[\hat{\Gamma}] = (P_{\hat{\Gamma}}, \tau_{\hat{\Gamma}})$.

Definition 7.1.11 The topological type of a real Fuchsian group $\hat{\Gamma}$ is the topological type of the corresponding Klein orbifold $(P_{\hat{\Gamma}}, \tau_{\hat{\Gamma}})$.

The following is a generalisation of the results in [14] to the case of nice Klein orbifolds, see also [4]:

Theorem 7.1.12 *For a hyperbolic automorphism $c \in Aut_+(\mathbb{H})$, let $\bar{c}$ be the reflection whose mirror coincides with the axis of c, let $\sqrt{c}$ be the hyperbolic automorphism such that $(\sqrt{c})^2 = c$ and let $\tilde{c} = \bar{c}\sqrt{c}$.*

1. *Let* $(g, 2r, k, 1, p_1, \dots, p_r)$ *be a topological type of a nice Klein orbifold, i.e.* $1 \le k \le g+1$ *and* $k \equiv g+1 \mod 2$. *Let* $n = k$. *Let* $\tilde{g} = (g+1-n)/2$. *Let*

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, e_1, \dots, e_r, c_1, \dots, c_n)$$

with $e_1, \dots, e_r$ *elliptic and* $c_1 \dots c_n$ *hyperbolic be a standard set of generators of a Fuchsian group of signature* $(\tilde{g}; k, 0, r : p_1, \dots, p_r)$, *then*

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, e_1, \dots, e_r, c_1, \dots, c_n, \bar{c}_1, \dots, \bar{c}_n)$$

is a set of generators of a real Fuchsian group $\hat{\Gamma}$ *of topological type*

$$(g; 2r, k, 1 : p_1, \dots, p_r).$$

Any real Fuchsian group of topological type

$$(g; 2r, k, 1 : p_1, \dots, p_r)$$

is obtained in this way.

2. *Let* $(g, 2r, k, 0, p_1, \dots, p_r)$ *be a topological type of a nice Klein orbifold, i.e.* $0 \le k \le g$. *Let us choose* $n \in \{k+1, \dots, g+1\}$ *such that* $n \equiv g+1 \mod 2$. *Let* $\tilde{g} = (g+1-n)/2$. *Let* $(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, e_1, \dots, e_r, c_1, \dots, c_n)$ *be a standard set of generators of a Fuchsian group of signature* $(\tilde{g}; n, 0, r : p_1, \dots, p_r)$, *then*

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, e_1, \dots, e_r, c_1, \dots, c_n, \bar{c}_1, \dots, \bar{c}_k, \tilde{c}_{k+1}, \dots, \tilde{c}_n)$$

is a set of generators of a real Fuchsian group of topological type

$$(g; 2r, k, 0, p_1, \dots, p_r).$$

Any real Fuchsian group of topological type $(g; 2r, k, 0, p_1, \dots, p_r)$ *is obtained in this way.*

3. *Let* $\hat{\Gamma}$ *be a real Fuchsian group as in part* 1 *or* 2 *and let* (P, τ) *be the corresponding nice Klein orbifold. We now think of elements*

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n)$$

as loops in P *without a base point (up to homotopy of free loops) rather than generators of* $\hat{\Gamma}$. *We have* $P^\tau = c_1 \cup \dots \cup c_k$. *The contours*

$c_1, \ldots, c_k$ correspond to ovals, the contours $c_{k+1}, \ldots, c_n$ correspond to twists. Let P_1 and P_2 be the connected components of the complement of the contours $c_1, \ldots, c_n$ in P. Each of these components is an orbifold of genus $\tilde{g} = (g+1-n)/2$ with n holes and r marking points. We have $\tau(P_1) = P_2$. We will refer to P_1 and P_2 as a decomposition of (P, τ) in two halves. (Note that such a decomposition is unique if (P, τ) is separating, but is not unique if (P, τ) is non - separating since the twists $c_{k+1}, \ldots, c_n$ can be chosen in different ways). Let $a'_i = (\tau a_i)^{-1}$ and $b'_i = (\tau b_i)^{-1}$ for $i = 1, \ldots, \tilde{g}$, let $e'_i = (\tau e_i)^{-1}$ for $i = 1, \ldots, r$. Then

$$(a_1, b_1, \ldots, a_{\tilde{g}}, b_{\tilde{g}}, e_1, \ldots, e_r, c_1, \ldots, c_n)$$

and

$$(a'_1, b'_1, \ldots, a'_{\tilde{g}}, b'_{\tilde{g}}, e'_1, \ldots, e'_r, c_1, \ldots, c_n))$$

are sets of generators of $\pi(P_1)$ and $\pi(P_2)$.

4. *Let P_1 and P_2 be a decomposition of (P, τ) in two halves. A bridge between invariant contours c_i and c_j is a contour of the form*

$$r_i \cup (\tau l)^{-1} \cup r_j \cup l,$$

where l is a simple path in P_1 starting on c_j and ending on c_i, r_i is the path along c_i from the end point of l to the end point of τl and r_j is the path along c_j from the starting point of τl to the starting point of l. (If c_i or c_j is an oval, the path r_i or r_j respectively consists of just one point). Let $d_1, \ldots, d_{n-1}$ be contours which only intersect at the base point, such that d_i is a deformation of a bridge between c_i and c_n. Then

$$(a_1, b_1, \ldots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \ldots, a'_{\tilde{g}}, b'_{\tilde{g}}, e_1, \ldots, e_r, e'_1 \cdots, e'_r,$$

$$c_1, \ldots, c_{n-1}, d_1, \ldots, d_{n-1})$$

is a set of generators of $\pi(P)$. Note that $\tau c_i = c_i$. We will refer to such sets of generators as symmetric.

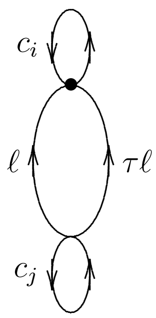

Figure 7.3: A bridge between ovals c_i and c_j

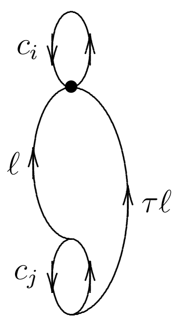

Figure 7.4: A bridge between an oval c_i and a twist c_j

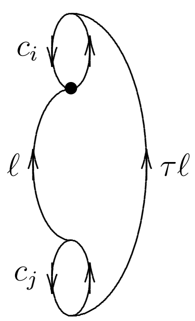

Figure 7.5: A bridge between two twists c_i and c_j

Chapter 8

Higher Spin Bundles on Klein Orbifolds

This chapter contains the main results of the thesis. I follow the scheme of the proof of a similar result in the Klein surface case in [17] and [18] and show that the generalisation of the claim to the Klein orbifold setting holds.

Let (P, τ) be a Klein orbifold. The involution τ acts on all structures related to the Riemann orbifold P, for example on the spin structures. The question is, given an m-spin bundle on P invariant under the involution, what properties does the corresponding Arf function have? In this chapter we study these properties. We get that for odd m the Arf function σ vanishes on all ovals, while for even m we have $\sigma = 0$ or $m/2$ on all ovals and σ vanishes on all twists. We call Arf functions that satisfy these conditions real m-Arf functions.

In [17] Natanzon and Pratoussevitch describe topological invariants of m-spin bundles on Klein surfaces and give conditions under which such bundles exist. I have extended their results to the case of nice Riemann orbifolds. I will outline the main results in section **??** Using these results and Theorem 5.5.11 I next describe all connected components of the space of higher spin bundles on Klein orbifolds and prove that any connected component is homeomorphic to a quotient of $\mathbb{R}^d$ by a discrete group.

8.1 From Lifts of Real Fuchsian Groups to Higher Spin Bundles on Klein Orbifolds

Definition 8.1.1 An *m-spin bundle on a Klein orbifold* (P, τ) is a pair $(e : L \to P, \beta)$, where $e : L \to P$ is an m-spin bundle on P and $\beta : L \to L$ is an anti-holomorphic involution on L such that $e \circ \beta = \tau \circ e$.

Definition 8.1.2 Two m-spin bundles $(e_1 : L_1 \to P_1, \beta_1)$ and $(e_2 : L_2 \to P_2, \beta_2)$ on Klein orbifolds (P_1, τ_1) and (P_2, τ_2) are *isomorphic* if there exist biholomorphic maps $\phi_L : L_1 \to L_2$ and $\phi_P : P_1 \to P_2$ such that the obvious diagrams commute: $e_2 \circ \phi_L = \phi_P \circ e_1$, $\beta_2 \circ \phi_L = \phi_L \circ \beta_1$ and $\tau_2 \circ \phi_P = \phi_P \circ \tau_1$.

Definition 8.1.3 A *lift of a real Fuchsian group* $\hat{\Gamma}$ into G_m is a subgroup $\hat{\Gamma}^*$ of G_m such that the projection $\pi|_{\hat{\Gamma}^*} : \hat{\Gamma}^* \to \hat{\Gamma}$ is an isomorphism.

Recall the following correspondence (Theorem 5.1.2):

Theorem 8.1.4 *There is a* 1-1*-correspondence between m-spin bundles on Riemann orbifolds and lifts of Fuchsian groups into the m-fold cover of $Aut_+(\mathbb{H})$.*

We will sketch the proof here as it will be useful for the study of lifts of real Fuchsian groups. Let Γ be a Fuchsian group. A lift of Γ is of the form

$$\Gamma^* = \{(g, \delta_g) | g \in \Gamma, \delta_g \in \mathrm{Hol}(\mathbb{H}, \mathbb{C}^*), \delta_g^m = \frac{d}{dz} g\}.$$

The corresponding m-spin bundle $e_{\Gamma^*} : L_{\Gamma^*} \to P = \mathbb{H}/\Gamma$ is of the form

$$L_{\Gamma^*} = (\mathbb{H} \times \mathbb{C})/\Gamma^* \to \mathbb{H}/\Gamma = P,$$

where the action of Γ^* on $\mathbb{H} \times \mathbb{C}$ is given by

$$(g, \delta_g) \cdot (z, x) = (g(z), \delta_g(z) \cdot x).$$

Every m-spin bundle on $P = \mathbb{H}/\Gamma$ is obtained as e_{Γ^*} for some lift Γ^* of Γ. Here we generalize Proposition 3.4 in [17] and give a more detailed proof.

Proposition 8.1.5 *To any lift of a real Fuchsian group into the m-fold cover G_m of* $\mathrm{Aut}(\mathbb{H})$ *we can associate an m-spin bundle on the corresponding Klein orbifold.*

Proof Let $\hat{\Gamma}^*$ be a lift of a real Fuchsian group $\hat{\Gamma}$ into the m-fold cover G_m of $\mathrm{Aut}(\mathbb{H})$. Let $\Gamma = \hat{\Gamma}^+ = \hat{\Gamma} \cap \mathrm{Aut}_+(\mathbb{H})$ be the corresponding Fuchsian group and $\Gamma^* = \hat{\Gamma}^* \cap G_m^+$ be the corresponding lift of Γ. Take $P = \mathbb{H}/\Gamma$ and $L_{\Gamma^*} = (\mathbb{H} \times \mathbb{C})/\Gamma^*$. Let $e_{\Gamma_*} : L_{\Gamma^*} \to P$ be the corresponding m-spin bundle as in Theorem 8.1.4. For any $(g, \delta_g) \in \hat{\Gamma}^* \cap G_m^-$ consider the mapping $(z, x) \mapsto (g(z), \delta_g(z) \cdot \bar{x})$. I will show that this mapping maps points in the same orbit of Γ^* on $\mathbb{H} \times \mathbb{C}$ to points in the same orbit, i.e. it induces a map $\beta_{\Gamma^*} : L_{\Gamma^*} \to L_{\Gamma^*}$. Moreover, I will show that the map β_{Γ^*} does not depend on the choice of the element $(g, \delta_g) \in \hat{\Gamma}^* \cap G_m^-$. Finally I will show that β_{Γ^*} is an anti-holomorphic involution. Let us discuss the details:

If (z', x') and (z, x) correspond to the same point in $L_{\Gamma^*} = (\mathbb{H} \times \mathbb{C})/\Gamma^*$,

$$(z', x') = (h, \delta_h) \cdot (z, x) = (h(z), \delta_h(z) \cdot x)$$

for some $(h, \delta_h) \in \Gamma^*$. Then

$$(g(z'), \delta_g(z') \cdot \bar{x}') = (g(h(z)), \delta_g(h(z)) \cdot \overline{\delta_h(z) \cdot x}) = ((g \circ h)(z), ((\delta_g \circ h) \cdot \bar{\delta_h})(z) \cdot \bar{x}).$$

On the other hand

$$(g \circ h \circ g^{-1}, \delta_{g \circ h \circ g^{-1}}) \cdot (g(z), \delta_g(z) \cdot \bar{x}) = ((g \circ h \circ g^{-1})(g(z)), \delta_{g \circ h \circ g^{-1}}(g(z)) \cdot \delta_g(z) \cdot \bar{x}).$$

Note that

$$\begin{aligned} \delta_{g \circ h} &= (\delta_g \circ h) \cdot \bar{\delta_h}, \\ \delta_{g \circ h \circ g^{-1}} &= (\delta_{g \circ h} \circ g^{-1}) \cdot \overline{\delta_{g^{-1}}} = (((\delta_g \circ h) \cdot \bar{\delta_h}) \circ g^{-1}) \cdot \overline{\delta_{g^{-1}}}, \end{aligned}$$

hence

$$\begin{aligned} &(g \circ h \circ g^{-1}, \delta_{g \circ h \circ g^{-1}}) \cdot (g(z), \delta_g(z) \cdot \bar{x}) \\ &= ((g \circ h)(z), (((\delta_g \circ h) \cdot \bar{\delta_h}) \circ g^{-1})(g(z)) \cdot \overline{\delta_{g^{-1}}}(g(z)) \cdot \delta_g(z) \cdot \bar{x}). \end{aligned}$$

Note that $(e_G, 1) = (g^{-1}, \delta_{g^{-1}}) \cdot (g, \delta_g)$ implies $\overline{\delta_{g^{-1}}}(g(z)) \cdot \delta_g(z) = 1$, hence

$$(g \circ h \circ g^{-1}, \delta_{g \circ h \circ g^{-1}}) \cdot (g(z), \delta_g(z) \cdot \bar{x}) = ((g \circ h)(z), ((\delta_g \circ h) \cdot \bar{\delta_h})(z) \cdot \bar{x}).$$

Thus we obtain

$$\begin{aligned} (g(z'), \delta_g(z') \cdot \bar{x}') &= ((g \circ h)(z), ((\delta_g \circ h) \cdot \bar{\delta_h})(z) \cdot \bar{x}) \\ &= (g \circ h \circ g^{-1}, \delta_{g \circ h \circ g^{-1}}) \cdot (g(z), \delta_g(z) \cdot \bar{x}), \end{aligned}$$

i.e. $(g(z'), \delta_g(z') \cdot \bar{x}')$ and $(g(z), \delta_g(z) \cdot \bar{x})$ correspond to the same point in L_{Γ^*}.

Thus the mapping $(z, x) \mapsto (g(z), \delta_g(z) \cdot \bar{x})$ induces a map $\beta_{\hat{\Gamma}^*} : L_{\Gamma^*} \to L_{\Gamma^*}$.

If we choose different $(g_1, \delta_{g_1}), (g_2, \delta_{g_2}) \in \hat{\Gamma}^* \cap G_m^-$, then

$$\begin{aligned}
&(g_2, \delta_{g_2}) \cdot (g_1^{-1}, \delta_{g_1}^{-1}) \\
&= (g_2 \circ g_1^{-1}, (\delta_{g_2} \circ g_1^{-1}) \cdot \overline{\delta_{g_1^{-1}}}), \\
&(g_2 \circ g_1^{-1}, \delta_{g_2 \circ g_1^{-1}}) \cdot (g_1(z), \delta_{g_1}(z) \cdot \bar{x}) \\
&= ((g_2 \circ g_1^{-1})(g_1(z)), (\delta_{g_2} \circ g_1^{-1})(g_1(z)) \cdot \overline{\delta_{g_1^{-1}}}(g_1(z)) \cdot \delta_{g_1}(z) \cdot \bar{x})).
\end{aligned}$$

Note that $(e_G, 1) = ({g_1}^{-1}, \delta_{{g_1}^{-1}}) \cdot (g_1, \delta_{g_1})$ implies $\overline{\delta_{{g_1}^{-1}}}(g_1(z)) \cdot \delta_{g_1}(z) = 1$, hence

$$(g_2 \circ g_1^{-1}, \delta_{g_2 \circ g_1^{-1}}) \cdot (g_1(z), \delta_{g_1}(z) \cdot \bar{x}) = (g_2(z), \delta_{g_2}(z) \cdot \bar{x}),$$

i.e. $(g_1(z), \delta_{g_1}(z) \cdot \bar{x})$ and $(g_2(z), \delta_{g_2}(z) \cdot \bar{x})$ correspond to the same point in L_{Γ^*}. Thus the map $\beta_{\hat{\Gamma}^*}$ does not depend on the choice of the element $g \in \hat{\Gamma}^* \cap G_m^-$.

If we apply $\beta_{\hat{\Gamma}^*}$ twice we get

$$\begin{aligned}
(z, x) &\mapsto (g(z), \delta_g(z) \cdot \bar{x}) \\
&\mapsto (g(g(z)), \delta_g(g(z)) \cdot \overline{\delta_g(z) \cdot \bar{x}}) \\
&= ((g \circ g)(z), ((\delta_g \circ g) \cdot \bar{\delta}_g)(z) \cdot x) \\
&= ((g \circ g)(z), \delta_{g \circ g}(z) \cdot x) \\
&= (g \circ g) \cdot (z, x).
\end{aligned}$$

We have $g \circ g \in \hat{\Gamma}^*$ since $g \in \hat{\Gamma}^*$ and we have $g \circ g \in G_m^+$ for any $g \in G_m$, hence $g \circ g \in \hat{\Gamma}^* \cap G_m^+ = \Gamma^*$. Thus (z, x) and $(g \circ g) \cdot (z, x)$ are equal modulo the action of Γ^*. We have therefore shown that $\beta_{\hat{\Gamma}^*}$ is indeed an involution. We can now associate with the lift $\hat{\Gamma}^*$ of the real Fuchsian group $\hat{\Gamma}$ the m-spin bundle $e_{\hat{\Gamma}^*} := (e_{\Gamma^*}, \beta_{\hat{\Gamma}^*})$. □

Proposition 8.1.6 *To any m-spin bundle on the Klein orbifold (P, τ) we can associate a lift of a real Fuchsian group into the m-fold cover G_m of* $\mathrm{Aut}(\mathbb{H})$.

Proof Any m-spin bundle on (P, τ) is obtained as e_{Γ^*} for some lift Γ^* of Γ into G_m. Let $(e : L \to P, \beta : L \to L)$ be an m-spin bundle on (P, τ). We have $e \circ \beta = \tau \circ e$. Consider a lift $\tilde{\beta}$ of $\beta : L \to L$ to the universal cover $\tilde{L} = \mathbb{H} \times \mathbb{C}$ of L. Let $\tilde{e}$ be the projection $\mathbb{H} \times \mathbb{C} \to P$. The map $\tilde{\beta}$ is bi-anti-holomorphic, invariant under Γ^* and with the property $\tilde{e} \circ \tilde{\beta} = \tau \circ \tilde{e}$, hence $\tilde{\beta}$ is of the form

$$\tilde{\beta}(z, x) = (g(z), f(z, x)),$$

where g is some element of $\hat{\Gamma}^-$ and f is some anti-holomorphic map. For a fixed z the map $x \mapsto f(z, x)$ is a bi-anti-holomorphic map $\mathbb{C} \to \mathbb{C}$, hence $f(z, x) = a(z) \cdot \bar{x} + b(z)$, where $a : \mathbb{H} \to \mathbb{C}^*$ and $b : \mathbb{H} \to \mathbb{C}$ are holomorphic functions. Since β is a bundle map, it preserves the zero section of L, hence $b(z) = 0$ for all z. Thus $\tilde{\beta}$ is of the form

$$\tilde{\beta}(z, x) = (g(z), a(z) \cdot \bar{x}),$$

where $a : \mathbb{H} \to \mathbb{C}^*$ is a holomorphic function. Considering the m-fold tensor products, we obtain an anti-holomorphic involution given by

$$\tilde{\beta}^{\otimes m}(z, x) = (g(z), a^m(z) \cdot \bar{x})$$

on the cotangent bundle of P, hence

$$a^m = \frac{d}{d\bar{z}} g.$$

Therefore $\tilde{g} = (g, \delta_g)$ with $\delta_g = a$ defines a lift of the element g into G_m^-.

The map $\tilde{\beta}$ is invariant under the action of Γ^* on $\mathbb{H} \times \mathbb{C}$:

For an element $\tilde{h} = (h, \delta_h) \in \Gamma^*$, the elements (z, x) and $\tilde{h} \cdot (z, x)$ represent the same elements in L, hence the elements

$$\tilde{\beta}(z, x) = \tilde{g} \cdot (z, x)$$

and

$$\tilde{\beta}(\tilde{h} \cdot (z, x)) = \tilde{g} \cdot \tilde{h} \cdot (z, x)$$

must represent the same elements in L. Note that if $\widetilde{ghg^{-1}}$ is the lift of ghg^{-1} into Γ^* then the element

$$\widetilde{ghg^{-1}} \cdot \tilde{g} \cdot (z, x) = (g(h(z)), \dots)$$

represents the same element in L as $\tilde{g} \cdot (z, x)$. Thus the elements

$$\widetilde{ghg^{-1}} \cdot \tilde{g} \cdot (z, x) \quad \text{and} \quad \tilde{g} \cdot \tilde{h} \cdot (z, x)$$

are equivalent under the action of Γ^* on $\mathbb{H} \times \mathbb{C}$. The action of Γ^* on $\mathbb{H} \times \mathbb{C}$ is free, hence these elements can only be equivalent if

$$\widetilde{ghg^{-1}} \cdot \tilde{g} = \tilde{g} \cdot \tilde{h}.$$

Thus

$$\tilde{g} \cdot \tilde{h} \cdot \tilde{g}^{-1} = \widetilde{ghg^{-1}} \in \Gamma^*.$$

Hence the element $\tilde{g} = (g, \delta_g)$ normalises the lift Γ^*, i.e. $\tilde{g}\Gamma^*\tilde{g}^{-1} = \Gamma^*$.

β is an involution, hence the element

$$\tilde{\beta}(\tilde{\beta}(z, x)) = \tilde{g}^2 \cdot (z, x)$$

represents the same element in L as (z, x). The action of Γ^* on $\mathbb{H} \times \mathbb{C}$ is free, hence these elements can only be equivalent if $\tilde{g}^2 = \tilde{e}$. The fact that $\tilde{g} \cdot \Gamma^* \cdot \tilde{g}^{-1} = \Gamma^*$ and $\tilde{g}^2 = \tilde{e}$ implies that the subgroup of G_m generated by Γ^* and $\tilde{g}$ is a lift of $\hat{\Gamma}$ into G_m. $\square$

8.2 Lifts of Real Fuchsian Groups and Real Arf Functions.

According to Theorem 5.5.3 there is a 1-1 correspondence between lifts of a Fuchsian group Γ into the m-fold cover G_m^+ of $G^+ = PSL(2, \mathbb{R})$, m-spin bundles on the Riemann orbifold $P = \mathbb{H}/\Gamma$ and m-Arf functions on P. A lift $\hat{\Gamma}^*$ of a real Fuchsian group $\hat{\Gamma}$ into the m-fold cover G_m of G induces a lift $\Gamma^* = \hat{\Gamma}^* \cap G_m^+$ of the Fuchsian group $\Gamma = \hat{\Gamma} \cap G^+$ into G_m^+, whence an m-Arf function $\sigma_{\hat{\Gamma}^*}$ on $\mathbb{H}/\Gamma$. We study the properties of such m-Arf functions.

Lemma 8.2.1 *Let $\hat{\Gamma}$ be a real Fuchsian group, $\Gamma = \hat{\Gamma}^+ \cap G^+$ the corresponding Fuchsian group, $[\hat{\Gamma}] = (P = \mathbb{H}/\Gamma, \tau)$ the corresponding Klein orbifold and $\hat{\Gamma}^*$ a lift of $\hat{\Gamma}$ into G_m. Assume that Γ has no elements of order 2. Then the induced m-Arf function $\sigma = \sigma_\Gamma^*$ on P has the following property: $\sigma(\tau c) = -\sigma(c)$ for any $c \in \pi_1^0(P, p)$.*

Proof The anti-holomorphic involution on P is given by $\tau = \Phi \circ f \circ \Phi^{-1}$, where f is any automorphism in $\hat{\Gamma}^- = \hat{\Gamma} \cap G^-$ and Φ is the natural projection $\mathbb{H} \to P$.

The induced involution on $\pi_1(P, p) \cong \Gamma \cong \Gamma^*$ is given by conjugation by an element of $(\hat{\Gamma}^*)^- = \hat{\Gamma}^* \cap G_m^-$, which by proposition 5.2.6 changes the sign of s_m, hence $\sigma(\tau c) = -\sigma(c)$ for any $c \in \pi_1^0(P, p)$. $\square$

Definition 8.2.2 We call an m-Arf function on a Klein surface *compatible* (with the involution τ) if $\sigma(\tau c) = -\sigma(c)$ for any $c \in \pi_1^0(P, p)$.

Lemma 8.2.3 *Let σ be a compatible m-Arf function on a Klein surface (P, τ). If m is odd, σ vanishes on all ovals and all twists. If m is even, then $\sigma(c)$ is either equal to 0 or to $m/2$ for any oval and any twist c.*

Proof If an element $c \in \pi_1^0(P, p) \cong \Gamma$ corresponds to an oval, then $\tau(c) = c$ and $\sigma(\tau c) = \sigma(c)$. If c corresponds to a twist, then $\tau(c)$ is conjugate to c, that is $\tau(c) = gcg^{-1}$ and $\sigma(\tau c) = \sigma(gcg^{-1}) = \sigma(c)$. So in both cases we have $\sigma(\tau c) = \sigma(c)$. On the other hand σ is compatible, so $\sigma(\tau c) = -\sigma(c)$ for any c. Hence we must have $2\sigma(c) = 0$ modulo m. For odd m this implies $\sigma(c) = 0$, for even m either $\sigma(c) = 0$ or $\sigma(c) = m/2$. $\square$

Recall from Theorem 7.1.12 that for a hyperbolic automorphism $c \in Aut_+(\mathbb{H})$, $\bar{c}$ is the reflection whose mirror coincides with the axis of c, $\sqrt{c}$ is the hyperbolic automorphism such that $(\sqrt{c})^2 = c$ ($\sqrt{c}$ shifts half the distance of c) and $\tilde{c} = \bar{c}\sqrt{c}$.

The results of [14] imply:

Lemma 8.2.4 *If $c \in \hat{\Gamma}$ is a hyperbolic element that corresponds to an oval on $P = \mathbb{H}/\Gamma$, then $\hat{\Gamma}$ contains the reflection $\bar{c}$. If $c \in \hat{\Gamma}$ is a hyperbolic element that corresponds to a twist on $P = \mathbb{H}/\Gamma$, then $\hat{\Gamma}$ contains the element $\tilde{c} = \bar{c}\sqrt{c}$.*

Lemma 8.2.5 *Let $\hat{\Gamma}$ be a real Fuchsian group, $\Gamma = \hat{\Gamma}^+ = \hat{\Gamma} \cap G^+$ the corresponding Fuchsian group, $[\hat{\Gamma}] = (P = \mathbb{H}/\Gamma, \tau)$ the corresponding Klein surface and $\hat{\Gamma}^*$ a lift of $\hat{\Gamma}$ into G_m. Then the induced m-Arf function $\sigma = \sigma_{\Gamma^*}$ on P vanishes on all twists.*

Proof By Lemma 8.2.3 we know that if m is odd then σ vanishes on any oval and any twist. For even m we have $\sigma(c) = 0$ or $\sigma(c) = m/2$ for any oval and any twist c. We need to show that the case m even, c a twist and $\sigma(c) = m/2$ is not possible.

Let c be a hyperbolic element in $\Gamma \cong \pi_1^0(P, p)$ which corresponds to a twist. According to Lemma 8.2.4 the group $\hat{\Gamma}$ contains the element $\tilde{c} = \bar{c}\sqrt{c}$. Let $C \in (\hat{\Gamma}^*)^+$ and $\tilde{C} \in (\hat{\Gamma}^*)^-$ be the lifts of c and $\tilde{c}$. Without loss of generality we can assume that $c = \tau_{0,\infty}(\lambda)$ so that $\tilde{c} = j \cdot \tau_{0,\infty}(\lambda/2)$, where j is the reflection in the imajinary axis and $\tau_{0,\infty}(\lambda/2)$ shifts half the distance of c. The lift of $\tilde{c}$ in $\hat{\Gamma}^*$ is of the form $\tilde{C} = JT_{0,\infty}(\lambda/2) \cdot U^q$ for some integer q.

Using identities $JT_{0,\infty}(\lambda) = T_{0,\infty}(\lambda)J$ and $JU^q = U^{-q}J$ from Proposition 2.3.1 we obtain:

$$\begin{aligned}
\tilde{C}^2 = (JT_{0,\infty}(\lambda/2)U^q)^2 &= JT_{0,\infty}(\lambda/2)U^q JT_{0,\infty}(\lambda/2)U^q \\
&= T_{0,\infty}(\lambda/2)JU^q JT_{0,\infty}(\lambda/2)U^q \\
&= T_{0,\infty}(\lambda/2)U^{-q}JJT_{0,\infty}(\lambda/2)U^q \\
&= T_{0,\infty}(\lambda/2)U^{-q}U^q T_{0,\infty}(\lambda/2) \\
&= T_{0,\infty}(\lambda/2)T_{0,\infty}(\lambda/2) \\
&= T_{0,\infty}(\lambda/2 + \lambda/2) \\
&= T_{0,\infty}(\lambda).
\end{aligned}$$

The element $\tilde{C} \in \hat{\Gamma}^*$, so $(\tilde{C})^2 \in \hat{\Gamma}^*$ since lifted elements form a subgroup. The element $(\tilde{C})^2$ is a preimage of $(\tilde{c})^2 = \tilde{c}\tilde{c} = \bar{c}\sqrt{c}\bar{c}\sqrt{c} = \bar{c}\bar{c}\sqrt{c}\sqrt{c} = c$. Thus $(\tilde{C})^2 = T_{0,\infty}(\lambda)$ is the lift of c in $\hat{\Gamma}^*$. By definition of an induced m-Arf function, we know that $\sigma(c)$ is equal to the value of s_m on the lift of c, hence $\sigma(c) = s_m(T_{0,\infty}(\lambda))$. On the other hand we have $s_m(T_{0,\infty}(\lambda)) = 0$, compare with section 5.2.1. Therefore $\sigma(c) = s_m(T_{0,\infty}(\lambda)) = 0$. $\square$

Definition 8.2.6 A *real m-Arf function* on a Klein orbifold (P, τ) is an m-Arf function σ on P such that

1. σ is compatible with τ, i.e. $\sigma(\tau c) = -\sigma(c)$ for any $c \in \pi_1^0(P, p)$.
2. σ vanishes on all twists.

Corollary 8.2.7 *Let $\hat{\Gamma}^*$ be a lift of a real Fuchsian group $\hat{\Gamma}$. Assume that $\Gamma = \hat{\Gamma} \cap Aut_+(\mathbb{H})$ has no elements of order 2. Then the induced m-Arf function $\sigma = \sigma_{\hat{\Gamma}^*}$ is a real m-Arf function on the Klein orbifold $[\hat{\Gamma}]$.*

Definition 8.2.8 Two lifts $(\hat{\Gamma}^*)_1$ and $(\hat{\Gamma}^*)_2$ of a real Fuchsian group $\hat{\Gamma}$ are *similar* if $(\hat{\Gamma}^*)_1 = (\hat{\Gamma}^*)_2 \cdot U^q$ for some $q \in \mathbb{Z}$.

Following the same reasoning as in [17] we obtain

Theorem 8.2.9 *Let $\hat{\Gamma}$ be a real Fuchsian group. Assume that $\Gamma = \hat{\Gamma} \cap Aut_+(\mathbb{H})$ contains no elements of order 2. Let $(P = \mathbb{H}/\Gamma, \tau)$ be the corresponding Klein orbifold. Let $p \in P$. Then there is a 1-1-correspondence between:*

1. *m-spin structures on the Klein orbifold (P, τ).*
2. *Similarity classes of lifts of $\hat{\Gamma}$ into the m-fold cover G_m of $Aut(\mathbb{H})$.*
3. *Real m-Arf functions $\sigma : \pi_1^0(P, p) \to \mathbb{Z}/m\mathbb{Z}$.*

8.3 Topological Invariants of Higher Arf Functions

We want to know whether the space of all m-spin structures is connected, and if not, how many connected components it has. It turns out that we have either one component (connected) or two components, and that there is an invariant of an m-Arf structure, namely the *Arf invariant*, introduced in Definition 5.5.6 which tells us which component the m-Arf structure is in.

For convenience we will recall the results of section 5.5.3, specialising to the case that P is a Riemann orbifold of type $(g; l_h, l_p, l_e : p_1, p_1, \dots, p_r, p_r)$ with $l_p = 0$, $l_e = 2r$ and $l_h + 2r = n$. Let $p \in P$. Let $\sigma : \pi^0(P, p) \to \mathbb{Z}/m\mathbb{Z}$ be an m-Arf function. The *Arf invariant* $\delta = \delta(P, \sigma)$ of σ is defined as follows: If $g > 1$ and m is even then we set $\delta = 0$ if there is a standard basis $\{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n\}$ of the orbifold fundamental group $\pi(P, p)$ such that

$$\sum_{i=1}^{g} (1 - \sigma(a_i))(1 - \sigma(b_i)) \equiv 0 \mod 2$$

and we set $\delta = 1$ otherwise. If $g > 1$ and m is odd then we set $\delta = 0$. If $g = 0$ then we set $\delta = 0$. If $g = 1$ and there is a standard basis $\{a_1, b_1, c_1, \ldots, c_n\}$ of the fundamental group $\pi(P, p)$ with $c_1, \ldots, c_{l_h}$ holes and $c_{l_h+1}, \ldots, c_{l_h+2r}$ elliptics then we set

$$\begin{aligned}\delta &= \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1) + 1, \ldots, \sigma(c_n) + 1) \\ &= \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1) + 1, \ldots, \sigma(c_{l_h}) + 1, p_1 - 1, \ldots, p_r - 1).\end{aligned}$$

Let n_j^h be the number of holes c_i with $\sigma(c_i) = j$. Then the *topological type* of σ is a tuple

$$t = (g, \delta, n_0^h, \ldots, n_{m-1}^h, p_1, p_1, \ldots, p_r, p_r),$$

where g is the genus, δ the Arf invariant of σ, $\sum_{j=0}^{m-1} n_j^h = l_h$ and $p_1, p_1, \ldots, p_r, p_r$ are the orders of elliptic elements.
The following theorem is a special case of Theorem 5.5.11 and Proposition 5.5.4.

Theorem 8.3.1 *1) Let n_j^h be the number of holes c_i with $\sigma(c_i) = j$. A tuple*

$$t = (g, \delta, n_0^h, \ldots, n_{m-1}^h, p_1, p_1, \ldots, p_r, p_r)$$

is the topological type of a hyperbolic m-Arf function on a Riemann orbifold of type $(g; l_h, l_p = 0, l_e = 2r : p_1, p_1, \ldots, p_r, p_r)$ if and only if it has the following properties:

(a) If $g > 1$ and m is odd, then $\delta = 0$.

(b) If $g > 1$, m is even and $n_j^h \neq 0$ for some even j, then $\delta = 0$.

(c) If $g = 1$ then δ is a divisor of m, $\gcd\{j + 1 | n_j^h \neq 0\}$ and $\gcd(p_1 - 1, \ldots, p_r - 1)$.

(d) The following degree condition is satisfied in $\mathbb{Z}/m\mathbb{Z}$:

$$\sum_{j=0}^{m-1} j \cdot n_j^h - 2\sum_{i=1}^{r} \frac{1}{p_i} = (2 - 2g) - (l_h + 2r).$$

2) If $(g, n_0^h, n_1^h, \ldots, n_{m-1}^h, p_1, p_1, \ldots, p_r, p_r)$ with $\sum_{j=0}^{m-1} n_j^h = l_h$ satisfy the condition (d), then for any choice of a standard (or symmetric) set of generators

$$(a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n)$$

of $\pi(P)$ *and any choice of the values* $\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)$ *there exists an* m*-Arf function on* P *of type* $(g, \delta, n_0^h, n_1^h, \dots, n_{m-1}^h, p_1, p_1, \dots, p_r, p_r)$ *for some* δ *with the given values on* $a_1, b_1, \dots, a_g, b_g$.

8.4 Topological Invariants of Real Arf Functions - Non-Separating Case

Let (P, τ) be a non-separating nice Klein orbifold of type $(g, 2r, k, 0, p_1, \dots, p_r)$.

Definition 8.4.1 The *topological type* of a real m-Arf function σ on P is:

(i) If m is odd, then the topological type is the tuple $(g, k, p_1, \dots, p_r)$, where k is the number of ovals of (P, τ).

(ii) If m is even, then the topological type is the tuple $(g, \delta, k_0, k_1, p_1, \dots, p_r)$, where δ is the m-Arf invariant of σ and k_j is the number of ovals of (P, τ) with value of σ equal to $m \cdot j/2$.

8.5 Topological Invariants of Real Arf Functions - Separating Case

Let (P, τ) be a separating nice Klein orbifold of type $(g, 2r, k, 1, p_1, \dots, p_r)$. Let P_1 and P_2 be the connected components of $P \backslash P^\tau$.

Definition 8.5.1 If m is odd, then the *topological type* of a real m-Arf function σ on P is the tuple $(g, \tilde{\delta}, k, p_1, \dots, p_r)$, where $\tilde{\delta}$ is the m-Arf invariant of $\sigma|_{P_1}$ and k is the number of ovals of (P, τ).

Real m-Arf functions with even m on separating Klein orbifolds have additional topological invariants.

Definition 8.5.2 We say that two ovals are *similar*, $c_1 \sim c_2$, if $\sigma(l \cup (\tau l)^{-1})$ is odd, where l is a simple path in P_1 connecting c_1 and c_2.

Note that two ovals are similar with respect to σ if and only if they are similar with respect to $\sigma \mod 2$, hence we obtain using Theorem 3.3 in [14]:

Proposition 8.5.3 *Similarity of ovals is well-defined. Similarity is an equivalence relation on the set of all ovals with at most two equivalence classes.*

Definition 8.5.4 Let c be an oval in P^τ. If m is even, then the *topological type* of a real m-Arf function σ on P for even m is the tuple

$$(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \ldots, p_r),$$

where $\tilde{\delta}$ is the m-Arf invariant of $\sigma|_{P_1}$, k_j^0 is the number of ovals similar to c with value of σ equal to $j \cdot m/2$ and k_j^1 is the number of ovals not similar to c with value of σ equal to $j \cdot m/2$ on (P, τ). (The invariants k_j^i are defined up to the swap of k_j^0 and k_j^1.)

8.6 Real Arf Functions on Bridges

Lemma 8.6.1 *Let (P, τ) be a nice Klein orbifold and let σ be a (not necessarily real) m-Arf function on (P, τ). Assume that the orbifold fundamental group $\pi(P, p)$ has no elements of order 2. Let d be a bridge as defined in Theorem 7.1.12.*
• *Let (P, τ) be separating. Then $\sigma(\tau d) = -\sigma(d)$.*
• *Let (P, τ) be non-separating. Let $c_1, \ldots, c_n$ be invariant contours as in Theorem 7.1.12, with $c_1, \ldots, c_k$ corresponding to ovals and $c_{k+1}, \ldots, c_n$ corresponding to twists. If σ vanishes on all twists $c_{k+1}, \ldots, c_n$ then $\sigma(\tau d) = -\sigma(d)$.*

Proof Let $d = r_i \cup (\tau l)^{-1} \cup r_j \cup l$ be a bridge between c_i and c_j as in Theorem 7.1.12.

• If c_i and c_j are both ovals, as shown in Figure 8.1 we have:

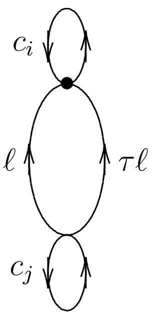

Figure 8.1: A bridge between ovals c_i and c_j

$$d = (\tau l)^{-1} \cup l,$$
$$\tau d = \tau((\tau l)^{-1} \cup l) = l^{-1} \cup \tau l = d^{-1}.$$

Now we see that $\sigma(\tau d) = \sigma(d^{-1}) = -\sigma(d)$.

• If c_i is an oval and c_j is a twist, as in Figure 8.2, we have:

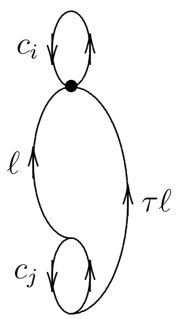

Figure 8.2: A bridge between an oval c_i and a twist c_j

$$\begin{aligned} d &= (\tau l)^{-1} \cup r_j \cup l, \\ \tau d &= \tau((\tau l)^{-1} \cup r_j \cup l) = l^{-1} \cup \tau r_j \cup \tau l, \\ (\tau d)^{-1} &= (\tau l)^{-1} \cup (\tau r_j)^{-1} \cup l \\ &= ((\tau l)^{-1} \cup r_j \cup l) \cup (l^{-1} \cup r_j^{-1} \cup (\tau r_j)^{-1} \cup l) \\ &= d \cup c_j^{-1}. \end{aligned}$$

Using Properties 3 and 2 of Arf functions we get

$$\sigma((\tau d)^{-1}) = \sigma(d) + \sigma(c_j^{-1}) = \sigma(d) - \sigma(c_j).$$

Using Property 2 we obtain $\sigma(\tau d)^{-1} = -\sigma(\tau d)$ and therefore

$$\sigma(\tau d) = -\sigma(d) + \sigma(c_j).$$

Now we see that $\sigma(c_j) = 0$ implies that $\sigma(\tau d) = -\sigma(d)$.

• If c_i and c_j are both twists , as in Figure 8.3 we have:

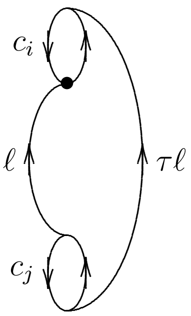

Figure 8.3: A bridge between two twists c_i and c_j

$$\begin{aligned}
d &= r_i \cup (\tau l)^{-1} \cup r_j \cup l, \\
\tau d &= \tau(r_i \cup (\tau l)^{-1} \cup r_j \cup l) = \tau r_i \cup l^{-1} \cup \tau r_j \cup \tau l = l^{-1} \cup \tau r_j \cup \tau l \cup \tau r_i, \\
(\tau d)^{-1} &= (\tau r_i)^{-1} \cup (\tau l)^{-1} \cup (\tau r_j)^{-1} \cup l \\
&= ((\tau r_i)^{-1} \cup r_i^{-1}) \cup (r_i \cup (\tau l)^{-1} \cup r_j \cup l) \cup (l^{-1} \cup r_j^{-1} \cup (\tau r_j)^{-1} \cup l) \\
&= c_i^{-1} \cup d \cup c_j^{-1}.
\end{aligned}$$

Using Properties 3 and 2 of Arf functions we obtain

$$\sigma((\tau d)^{-1}) = \sigma(c_i^{-1}) + \sigma(d) + \sigma(c_j^{-1}) = \sigma(d) - \sigma(c_i) - \sigma(c_j).$$

Using Property 2 of Arf functions we obtain $\sigma((\tau d)^{-1}) = -\sigma(\tau d)$ and therefore

$$\sigma(\tau d) = \sigma(c_i) - \sigma(d) + \sigma(c_j).$$

Now we see that $\sigma(c_i) = \sigma(c_j) = 0$ implies $\sigma(\tau d) = -\sigma(d)$. □

We will need the following generalisation of Lemma 4.5 in [17]:

Lemma 8.6.2 *Let (P, τ) be a Klein orbifold of type $(g, 2r, k, \epsilon, p_1, \dots, p_r)$, $p_i \geq 3$, and σ an m-Arf function on P. Let $c_1, \dots, c_n$ be invariant contours as in Theorem 7.1.12, with $c_1, \dots, c_k$ corresponding to ovals and $c_{k+1}, \dots, c_n$ corresponding to twists. Let*

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}},$$

$$e_1, \dots, e_r, e'_1, \dots, e'_r, c_1, \dots, c_{n-1}, d_1, \dots, d_{n-1})$$

be a symmetric generating set of $\pi_1^0(P)$. Assume that *(i)* $\sigma(a_i) = \sigma(a'_i)$, $\sigma(b_i) = \sigma(b'_i)$ *for* $i = 1, \ldots, \tilde{g}$,

(ii) $2\sigma(c_i) = 0$ *for* $i = 1, \ldots, n-1$,

(iii) $\sigma(c_i) = 0$ *for* $i = k+1, \ldots, n$.

Then σ *is a real* m*-Arf function on* (P, τ).

Remark. Condition (iii) means that σ vanishes on all twists.

Proof To be real, the m-Arf function σ must vanish on all twists and be compatible with τ, i.e. satisfy the equation $\sigma(\tau x) = -\sigma(x)$ for all $x \in \pi_1^0(P)$. Condition (iii) implies that σ vanishes on all twists. We will first check the equation $\sigma(\tau x) = -\sigma(x)$ for all x in $\mathcal{B}$.

• $x = a_i, b_i$, $i = 1, \ldots, \tilde{g}$: Recall that $a'_i = (\tau a_i)^{-1}$, hence $\tau a_i = (a'_i)^{-1}$ and $\sigma(\tau a_i) = \sigma((a'_i)^{-1}) = -\sigma(a'_i)$. Condition (i) implies $\sigma(a'_i) = \sigma(a_i)$, hence $\sigma(\tau a_i) = -\sigma(a'_i) = -\sigma(a_i)$. Similarly $\sigma(\tau b_i) = -\sigma(b_i)$.

• $x = a'_i, b'_i$, $i = 1, \ldots, \tilde{g}$: Recall that $a'_i = (\tau a_i)^{-1}$, hence $\tau a'_i = (a_i)^{-1}$ and $\sigma(\tau a'_i) = \sigma((a_i)^{-1}) = -\sigma(a_i)$. Condition (i) implies $\sigma(a_i) = \sigma(a'_i)$, hence $\sigma(\tau a'_i) = -\sigma(a_i) = -\sigma(a'_i)$. Similarly $\sigma(\tau b'_i) = -\sigma(b'_i)$.

• $x = e_i, e'_i$, $i = 1, \ldots, r$. $\sigma(\tau e_i) = \sigma(e_i^{-1}) = -\sigma(e_i)$ since τe_i and e_i^{-1} are elliptic elements of the same order (see Lemma 5.3.1).

• $x = c_i$, $i = 1, \ldots, n-1$: Recall that $\tau c_i = c_i$ for an oval c_i, $i = 1, \ldots, k$, while τc_i is conjugate to c_i for a twist c_i, $i = k+1, \ldots, n-1$. In both cases $\sigma(\tau c_i) = \sigma(c_i)$. Condition (ii) implies $2\sigma(c_i) = 0$ for $i = 1, \ldots, n-1$, hence $\sigma(\tau c_i) = \sigma(c_i) = -\sigma(c_i)$.

• $x = d_i$, $i = 1, \ldots, n-1$: Condition (iii) implies that σ vanishes on all twists. According to Lemma 8.6.1 it follows that $\sigma(\tau d_i) = -\sigma(d_i)$ for $i = 1, \ldots, n-1$. Consider $\hat{\sigma} : \pi_1(P) \to \mathbb{Z}/m\mathbb{Z}$ given by $\hat{\sigma}(x) = -\sigma(\tau x)$. The involution τ is an orientation-reversing homeomorphism, so it is easy to check using Definition 5.5.1 that if σ is an m-Arf function then so is $\hat{\sigma}$. We have checked that for any x in $\mathcal{B}$ we have $\sigma(\tau x) = -\sigma(x)$, hence $\hat{\sigma}(x) = -\sigma(\tau x) = \sigma(x)$. Thus we have two m-Arf functions, σ and $\hat{\sigma}$, which coincide on a generating set. We can conclude that σ an d$\hat{\sigma}$ coincide everywhere on $\pi_1(P)$, i.e. for any $x \in \pi_1(P)$ we have $\sigma(\tau x) = -\hat{\sigma}(x) = -\sigma(x)$. This shows that the Arf function σ is real.
□

8.7 Topological Equivalence of Arf Functions

Definition 8.7.1 Two m-Arf functions σ_1 and σ_2 on a Klein orbifold (P, τ) are *topologically equivalent* if there exists a homeomorphism $\phi : P \to P$ such that $\phi \circ \tau = \tau \circ \phi$ and $\sigma_1 = \sigma_2 \circ \phi_*$ for the induced automorphism ϕ_* of $\pi(P, p)$.

Theorem 8.7.2 *Let (P, τ) be a Klein orbifold of type $(g, 2r, k, \epsilon, p_1, \ldots, p_r)$, $p_i \geq 3$. Let σ be a real m-Arf function on (P, τ). For $\epsilon = 1$ assume that $k \leq g - 1$ and let $n = k$. For $\epsilon = 0$ assume that $k \leq g - 2$ and take $n \in \{k+1 \ldots, g-1\}$ such that $n = g - 1 \mod 2$. (The assumption $k \leq g-2$ implies $k + 1 \leq g - 1$, hence $\{k+1, \ldots, g-1\} \neq \emptyset$). Let*

$$(a_1, b_1, \ldots, a_{\tilde{g}}, b_{\tilde{g}}, e_1, \ldots, e_r, c_1, \ldots, c_n)$$

be a generating set as in Theorem 7.1.12. Then we can choose bridges $d_1, \ldots, d_{n-1}$ in such a way that

$$(a_1, b_1, \ldots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \ldots, a'_{\tilde{g}}, b'_{\tilde{g}},$$

$$e_1, \ldots, e_r, e'_1, \ldots, e'_r, c_1, \ldots, c_{n-1}, d_1, \ldots, d_{n-1})$$

with $a'_i = (\tau a_i)^{-1}$, $b'_i = (\tau b_i)^{-1}$, $e'_i = (\tau e_i)^{-1}$ is a symmetric generating set of $\pi_1(P, p)$ and

(i) If m is odd, then $\sigma(d_i) = 0$ for $i = 1, \ldots, n - 1$.

(ii) If m is even and (P, τ) is separating, then $\sigma(d_i) \in \{0, 1\}$ for $i = 1, \ldots, n - 1$.

(iii) If m is even and (P, τ) is non - separating, then $\sigma(d_1) = \cdots = \sigma(d_{n-1}) \in \{0, 1\}$.

Proof In the case $\epsilon = 1$ we have $\tilde{g} = (g + 1 - k)/2$ and the assumption $k \leq g - 1$ implies $\tilde{g} \geq 1$. In the case $\epsilon = 0$ we have $\tilde{g} = (g + 1 - n)/2$ and the assumption $n \leq g - 1$ implies $\tilde{g} \geq 1$. Our aim is to choose disjoint bridges $d_1, \ldots, d_{n-1}$ which satisfy the conditions above.

• If m is even we can consider the 2-Arf function $\sigma \mod 2$ on P. If m is even and (P, τ) is non-separating, then according to Lemma 11.2 in [14], we can choose the bridges $d_1, \ldots, d_{n-1}$ in such a way that

$$(\sigma \mod 2)(d_1) = \cdots = (\sigma \mod 2)(d_{n-1}).$$

This means for the original m-Arf function σ that

$$\sigma(d_1) \equiv \cdots \equiv \sigma(d_{n-1}) \mod 2.$$

• Let Q_1 be the compact orbifold of genus $\tilde{g}$ with one hole obtained from P_1 after removing all bridges $d_1, \ldots, d_{n-1}$. Let $\tilde{\delta}$ be the Arf invariant of $\sigma|_{Q_1}$.

(i) Consider the case $\tilde{g} \geq 2$. According to Lemma 5.5.10 we can choose a standard set of generators $(a_1, b_1, \ldots, a_{\tilde{g}}, b_{\tilde{g}}, e_1, \ldots, e_r, \tilde{c})$ of $\pi(Q_1, p)$ in such a way that

$$(\sigma(a_1), \sigma(b_1), \ldots, \sigma(a_{\tilde{g}}), \sigma(b_{\tilde{g}})) = (0, 1 - \tilde{\delta}, 1, \ldots, 1),$$

hence $\sigma(a_1) = 0$.

(ii) Consider the case $\tilde{g} = 1$. According to Lemma 5.5.10 we can choose a standard set of generators $(a_1, b_1, e_1, \ldots, e_r, \tilde{c})$ of $\pi(Q_1, p)$ in such a way that

$$(\sigma(a_1), \sigma(b_1)) = (\tilde{\delta}, 0),$$

hence $\sigma(b_1) = 0$.

Thus if $\tilde{g} \geq 1$ then there exists a non-trivial contour a in P_1 with $\sigma(a) = 0$, which does not intersect any of the bridges $d_1, \ldots, d_{n-1}$. If we replace d_i by $(\tau a)^{-1} d_i a$, then

$$\sigma((\tau a)^{-1} d_i a) = \sigma((\tau a)^{-1}) + \sigma(d_i) + \sigma(a) - 2.$$

Taking into account the fact that $\sigma(a) = 0$ we obtain

$$\sigma((\tau a)^{-1} d_i a) = \sigma(d_i) - 2.$$

Repeating this operation we can obtain $\sigma(d_i) = 0$ for odd m and $\sigma(d_i) \in \{0, 1\}$ for even m.

• Note that the property $\sigma(d_1) \equiv \cdots \equiv \sigma(d_{n-1}) \mod 2$ (if m is even and (P, τ) is non-separating) is preserved during this process, hence $\sigma(d_1) = \cdots = \sigma(d_{n-1})$ at the end of the process. □

8.8 Moduli of Klein Orbifolds

For the following results about moduli spaces of Klein orbifolds we follow [15] and [16] with minor modifications.

Let $\mathcal{M}_{g,2r,k,0,p_1,\ldots,p_r}$ be the moduli space of Klein orbifolds of topological type $(g, 2r, k, 0, p_1, \ldots, p_r)$. Let $\Gamma_{g,2r,k,p_1,\ldots,p_r}$ be the group generated by the elements

$$v = \{a_1, b_1, \ldots, a_g, b_g, e_1, \ldots, e_r, c_1, \ldots, c_k\}$$

with the defining relations

$$\prod_{i=1}^{g}[a_i, b_i] \cdot \prod_{i=1}^{r} e_i \cdot \prod_{i=1}^{k} c_i = 1$$

and $e_i^{p_i} = 1$ for $i = 1, \dots, r$.

The *Fricke space* $\tilde{T}_{g,2r,k,p_1,\dots,p_r}$ is the set of all monomorphisms

$$\psi : \Gamma_{g,2r,k,p_1,\dots,p_r} \to \mathrm{Aut}_+(\mathbb{H})$$

such that

$$\{\psi(a_i), \psi(b_i)(i = 1, \dots, g), \psi(e_i)(i = 1, \dots, r), \psi(c_i)(i = 1, \dots, k)\}$$

is a set of generators of a Fuchsian group of signature

$$(g; l_h = k, l_p = 0, l_e = 2r : p_1, p_1, \dots, p_r, p_r).$$

For $(g; l_h = k, l_p = 0, l_e = 2r : p_1, p_1, \dots, p_r, p_r)$ to be a signature of a group of hyperbolic isometries, we have to assume that $\sum_{i=1}^{r} \frac{1}{p_i} < r+g-1$. The Fricke space $\tilde{T}_{g,2r,k,p_1,\dots,p_r}$ is homeomorphic to $\mathbb{R}^{6g-6+3k+2r}$. The group $\mathrm{Aut}_+(\mathbb{H})$ acts on $\tilde{T}_{g,2r,k,p_1,\dots,p_r}$ by conjugation. The *Teichmuller space* is $T_{g,2r,k,p_1,\dots,p_r} = \tilde{T}_{g,2r,k,p_1,\dots,p_r}/\mathrm{Aut}_+(\mathbb{H})$.

Theorem 8.8.1 *Let $(g, 2r, k, 1, p_1, \dots, p_r)$ be a topological type of a Klein orbifold, i.e. $1 \le k \le g+1$ and $k \equiv g+1 \mod 2$. Let $\tilde{g} = (g+1-k)/2$. The moduli space $\mathcal{M}_{g,2r,k,1,p_1,\dots,p_r}$ of Klein orbifolds of topological type $(g, 2r, k, 1, p_1, \dots, p_r)$ is the quotient of the Teichmuller space $T_{\tilde{g},2r,k,p_1,\dots,p_r}$ by a discrete group action of automorphisms. $T_{\tilde{g},2r,k,p_1,\dots,p_r}$ is homeomorphic to a quotient of $\mathbb{R}^{6\tilde{g}-6+3k+2r} = \mathbb{R}^{3g-3+2r}$.*

Theorem 8.8.2 *Let $(g, 2r, k, 0, p_1, \dots, p_r)$ be a topological type of a Klein orbifold, i.e. $0 \le k \le g$. Let us choose $n > k$ such that $n \equiv g+1 \mod 2$. Let $\tilde{g} = (g+1-n)/2$. The moduli space $\mathcal{M}_{g,2r,k,0,p_1,\dots,p_r}$ of Klein orbifolds of topological type $(g, 2r, k, 0, p_1, \dots, p_r)$ is the quotient of the Teichmuller space $T_{\tilde{g},2r,n,p_1,\dots,p_r}$ by a discrete group action of automorphisms. $T_{\tilde{g},2r,n,p_1,\dots,p_r}$ is homeomorphic to a quotient of $\mathbb{R}^{6\tilde{g}-6+3n+2r} = \mathbb{R}^{3g-3+2r}$.*

Theorem 8.8.3 *The moduli space of Klein orbifolds of genus g decomposes into connected components $\mathcal{M}_{g,2r,k,\epsilon,p_1,\ldots,p_r}$. Each connected component is homeomorphic to a quotient of $\mathbb{R}^{3g-3+2r}$ by a discrete group action.*

We can use Theorem 5.5.11 to give conditions for Arf functions on each component and we have conditions on the ovals and the twists. We now combine these to determine Arf functions which are invariant under the involution. We consider four cases, covering separating and non-separating Klein orbifolds for even and odd m.

8.9 Non-Separating Case, Even m

In this section we describe all topological types and the moduli space in the non-separating case for even m.

In the separating case, the invariant contours are all ovals. When we decompose the surface into two halves each component has genus $\tilde{g} = (g+1-k)/2$, where k is the number of ovals. In the non-separating case we have at least one twist. Natanzon showed in [13] that we can cut the surface in such a way that the resulting components have genus 0.

Proposition 8.9.1 *Let m be even. Let (P, τ) be a nice Klein surface of type $(g, 2r, k, 0, p_1, \ldots, p_r)$, $g \geq 2$, $p_i \geq 3$. A tuple $(g, \delta, k_0, k_1, p_1, \ldots, p_r)$ with $k_0 + k_1 = k$ is a topological type of some real m-Arf function on (P, τ) if and only if the following condition is satisfied:*

$$\frac{m}{2} \cdot k_1 - \sum_{i=1}^{r} \frac{1}{p_i} = (1 - g - r) \mod m.$$

If this condition is satisfied, then for any choice of a set of generators and values on the set of generators which give us the right topological invariants we get a real Arf function.

Proof Let $\mathcal{B} = (e_1, \ldots, e_r, e'_1, \ldots, e'_r, c_1, \ldots, c_g, d_1, \ldots, d_g)$ be a symmetric set of generators of $\pi(P, p)$ as in Theorem 7.1.12.

1) Let σ be a real m-Arf function on (P, τ) of topological type

$$(g, \delta, k_0, k_1, p_1, \ldots, p_r).$$

The Arf function σ is real, hence vanishes on all twists, $\sigma(c_i) = 0$ for $i = k+1, \ldots, g+1$. As for the ovals $c_1, \ldots, c_k$, the Arf function σ is of the topological type $(g, \delta, k_0, k_1, p_1, \ldots, p_r)$, hence σ takes the value 0 on k_0 ovals and the value $m/2$ on k_1 ovals. Thus the topological type of the m-Arf function $\sigma|_{P_1}$ is:

$$(\tilde{g}, \tilde{\delta}, n_0^h, \ldots, n_{m-1}^h, p_1, \ldots, p_r)$$

with $\tilde{g} = 0$, $n_0^h = k_0 + (g+1-k)$, $n_{\frac{m}{2}}^h = k_1$ and $n_j^h = 0$ for $j \neq 0, \frac{m}{2}$.

According to Theorem 5.5.11, the tuple $(\tilde{g}, \tilde{\delta}, n_0^h, \ldots, n_{m-1}^h, p_1, \ldots, p_r)$ must satisfy the following:

$$\sum_{j=0}^{m-1} j \cdot n_j^h - \sum_{i=1}^{r} \frac{1}{p_i} = (2 - 2\tilde{g}) - (g + 1 + r) \mod m.$$

For $\tilde{g}$ and n_j^h as above, we obtain:

$$\frac{m}{2} \cdot k_1 - \sum_{i=1}^{r} \frac{1}{p_i} = (1 - g - r) \mod m.$$

2) Now let (k_0, k_1) be any tuple with $k_0 + k_1 = k$ and $\frac{m}{2} \cdot k_1 - \sum_{i=1}^{r} \frac{1}{p_i} = (1 - g - r) \mod m$. We will define an m-Arf function σ on P as follows: Choose a subset A in $\{1, \ldots, k\}$ with $|A| = k_1$. Set $\sigma(c_i) = m/2$ for $i \in A$, $\sigma(c_i) = 0$ for $i \in \{1, \ldots, g\}, i \notin A$. Theorem 5.5.11 implies that on the compact surface P the Arf invariant δ of σ satisfies

$$\delta = \sum_{i=1}^{g} (1 - \sigma(c_i))(1 - \sigma(d_i)) \mod 2.$$

By varying the values $\sigma(d_i)$ we can always achieve both $\delta = 0 \mod 2$ and $\delta = 1 \mod 2$ unless all $\sigma(c_i)$ are odd. Recall that $\sigma(c_1), \ldots, \sigma(c_k) \in \{0, m/2\}$ and $\sigma(c_{k+1}) = \ldots = \sigma(c_g) = 0$, hence the case that all $\sigma(c_i)$ are odd is only possible if $m = 2 \mod 4$, $k_0 = 0$, $k_1 = k = g$. On the other hand $\frac{m}{2} k_1 - \sum_{i=1}^{r} \frac{1}{p_i} = 1 - g - r \mod m$. Substituting $k_1 = g$ we obtain

$$\frac{m}{2} g - \sum_{i=1}^{r} \frac{1}{p_i} = 1 - g - r \mod m.$$

Let us consider this equation modulo 2. We have $\frac{m}{2} = 1 \mod 2$ since $m = 2 \mod 4$. By Lemma 5.3.21 have $\sum_{i=1}^{r} \frac{1}{p_i} = r \mod 2$. Therefore $g = 1 - g \mod 2$, i.e. $0 = 1 \mod 2$. This contradiction shows that with an appropriate choice of $\sigma(d_i)$ we can achieve any value of δ in $\{0, 1\}$.

The values $\sigma(e_i)$ and $\sigma(e_i')$ are determined by p_i. According to Theorem 8.3.1, these values on $\mathcal{B}$ determine a unique m-Arf function σ on P. We have $\sigma(c_1)+\dots+\sigma(c_{g+1}) = k_1 \cdot m/2 + \sigma(c_{g+1})$. On the other hand Theorem 8.3.1 applied to P_1 implies that

$$\begin{aligned}\sigma(c_1) + \cdots + \sigma(c_{g+1}) &= \sum_{i=1}^{r} \frac{1}{p_i} + (2 - 2\tilde{g}) - (g + 1 + r) \\ &= \sum_{i=1}^{r} \frac{1}{p_i} + 1 - g - r \mod m.\end{aligned}$$

Hence $\sigma(c_{g+1}) = \sum_{i=1}^{r} \frac{1}{p_i} + 1 - g - r - k_1 \cdot m/2 \mod m$. We know that $\sum_{i=1}^{r} \frac{1}{p_i} + 1 - g - r - k_1 \cdot m/2 = 0 \mod m$, hence $\sigma(c_{g+1}) = 0 \mod m$. Lemma 8.6.2 implies that σ is a real Arf function. This construction gives all real Arf functions of type (g, δ, k_0, k_1) as it realizes all choices of values on a symmetric set of generators that conforms to the given topological type. □

Example 8.9.2 *For example, look at an orbifold P of genus $g = 4$ in the case $m = 6$. Theorem 7.1.9 implies $0 \leq k \leq 4$. We will look at the case when $k = 2$. For $k = 2$ we can take $(k_0, k_1) = (0, 2), (1, 1)$ or $(2, 0)$.*

Let $p_1, p_1, \dots, p_r, p_r$ be the orders of the marking points. By Theorem 5.5.11 for an m-spin bundle to exist on the compact orbifold P we require

$$-2\sum_{i=1}^{r} \frac{1}{p_i} = (2 - 2g) - 2r = -6 - 2r = -2r \mod 6. \tag{8.1}$$

By Proposition 8.9.1 for there to be an involution on the m-spin bundle we require

$$3 \cdot k_1 - \sum_{i=1}^{r} \frac{1}{p_i} = -3 - r \mod 6. \tag{8.2}$$

Note that the condition (8.1) is implied by (8.2).

Recall that m and p are relatively prime. So $(6, p_i) = 1$ implying $p_i = 1 \mod 6$ or $p_i = 5 \mod 6$. $\frac{1}{p_i}$ is the inverse of p_i in $\mathbb{Z}/6\mathbb{Z}$. For $p_i = 1 \mod 6$ we have $\frac{1}{p_i} = 1 \mod 6$ and for $p_i = 5 \mod 6$, $\frac{1}{p_i} = 5 \mod 6$.

Out of r orders of marking points, let $x \leq r$ be $5 \mod 6$ and $r - x$ be $1 \mod 6$. For (8.1) to be satisfied we need $-2(5x + (r - x)) = -2r \mod 6$, that is $-8x = 0 \mod 6$ and $x = 0, 3, 6, 9, \ldots$. So if $r = 1$ we have a 6-spin bundle for $p_1 = 1 \mod 6$, if $r = 2$ we have a bundle if $p_1 = p_2 = 1 \mod 6$ and if $r = 3$ we have a bundle if $p_1 = p_2 = p_3 = 1 \mod 6$ or if $p_1 = p_2 = p_3 = 5 \mod 6$.

To have an involution on the bundle we require (8.2) to be satisfied. That is $3 \cdot k_1 - (5x + (r - x)) = -3 - r \mod 6 \Rightarrow 4x = 3k_1 + 3 \mod 6$. For $k_1 = 0$ that is $4x = 3 \mod 6$ which is never satisfied. For $k_1 = 1$ we have $4x = 6 \mod 6$ that is $4x = 0 \mod 6$ so $x = 0, 3, 6, 9, \ldots$. For $k = 2$ we have $4x = 9 \mod 6$ which is never satisfied.

While we can have 6-spin bundles on Klein orbifolds of types

$$(g, 2r, 2, 0, p_1, \ldots, p_r)$$

with $(k_0, k_1) = (2, 0)$ and $(k_0, k_1) = (0, 2)$, we can only extend the involution from the orbifold to the bundle if $(k_0, k_1) = (1, 1)$. Then $x = 0, 3, 6, 9, \ldots$ i.e if $r = 1$ then $p_1 = 1 \mod 6$, if $r = 2$ then $p_1 = p_2 = 1 \mod 6$ and if $r = 3$ then $p_1 = p_2 = p_3 = 1 \mod 6$ or $p_1 = p_2 = p_3 = 5 \mod 6$.

Proposition 8.9.3 *Let m be even and let (P, τ) be a nice Klein orbifold of type $(g, 2r, k, 0, p_1, \ldots, p_r)$, $g \geq k + 2$, $p_i \geq 3$. Two m-Arf functions on (P, τ) are topologically equivalent if and only if they have the same topological type $(g, \delta, k_0, k_1, p_1, \ldots, p_r)$.*

Proof Let σ be a real m-Arf function on (P, τ) of topological type

$$(g, \delta, k_0, k_1, p_1, \ldots, p_r).$$

Let $n \in \{k + 1, \ldots g - 1\}$ such that $n = g - 1 \mod 2$. (The assumption $k \leq g - 2$ implies $k + 1 \leq g - 1$, hence $\{k + 1, \ldots, g - 1\} \neq \emptyset$).

Let $c_1, \ldots, c_n$ be invariant contours as in Theorem 7.1.12, with $c_1, \ldots, c_k$ corresponding to ovals and $c_{k+1}, \ldots, c_n$ corresponding to twists. The Arf function σ is real, hence it vanishes on all twists,

$$\sigma(c_{k+1}) = \cdots = \sigma(c_n) = 0.$$

The Arf function σ is of topological type (g, δ, k_0, k_1), hence $(\sigma(c_1), \ldots, \sigma(c_k))$ is a permutation of k_0 times zero and k_1 times $m/2$. Let P_1 and P_2 be connected components of the complement of the contours $c_1, \ldots, c_n$ in P. Each of these components is a surface or genus $\tilde{g} = (g + 1 - n)/2$ with n holes. The condition $n \leq g - 1$ implies $\tilde{g} \geq 1$.

(i) Consider the case $\tilde{g} \geq 2$, i.e. $n \leq g - 3$. Note that there are contours around holes in P_1 such that the values of σ on these contours are even, namely the value of σ on $c_{k+1}, \ldots, c_n$ is zero. Thus, according to Lemma 5.5.10, we can choose a standard generating set

$$(a_1, b_1, \ldots, a_{\tilde{g}}, b_{\tilde{g}}, e_1, \ldots, e_r, c_1, \ldots, c_n)$$

of $\pi(P_1)$ in such a way that

$$(\sigma(a_1), \sigma(b_1), \ldots, \sigma(a_{\tilde{g}}), \sigma(b_{\tilde{g}})) = (0, 1, 1, \ldots, 1)$$

and

$$\sigma(c_1) = \cdots = \sigma(c_{k_0}) = 0, \quad \sigma(c_{k_0+1}) = \cdots = \sigma(c_k) = m/2.$$

(ii) Consider the case $\tilde{g} = 1$, i.e. $n = g - 1$. According to Lemma 5.5.10 we can choose a standard generating set $(a_1, b_1, e_1, \ldots, e_r, c_1, \ldots, c_n)$ of $\pi(P_1)$ in such a way that

$$(\sigma(a_1), \sigma(b_1)) = (\tilde{\delta}, 0),$$

where

$$\tilde{\delta} = \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1) + 1, \ldots, \sigma(c_n) + 1, p_1 - 1, \ldots, p_r - 1),$$

and

$$\sigma(c_1) = \cdots = \sigma(c_{k_0}) = 0, \quad \sigma(c_{k_0+1}) = \cdots = \sigma(c_k) = m/2.$$

Note that there exist i such that $\sigma(c_i) + 1 = 1$, for example all i in $\{k + 1, \ldots, n\}$, therefore the greatest common divisor is $\tilde{\delta} = 1$. Hence we can choose a standard generating set $(a_1, b_1, e_1, \ldots, e_r, c_1, \ldots, c_n)$ of $\pi(P_1)$ in such a way that

$$(\sigma(a_1), \sigma(b_1)) = (1, 0),$$

$$\sigma(c_1) = \cdots = \sigma(c_{k_0}) = 0,$$

$$\sigma(c_{k_0+1}) = \cdots = \sigma(c_k) = m/2.$$

Let

$$\mathcal{B} = (a_1, b_1, \ldots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \ldots, a'_{\tilde{g}}, b'_{\tilde{g}},$$
$$e_1, \ldots, e_r, e'_1, \ldots, e'_r, c_1, \ldots, c_{n-1}, d_1, \ldots, d_{n-1})$$

be a symmetric generating set of $\pi(P)$ as in Theorem 8.7.2, i.e. there exists $\xi \in \{0, 1\}$ such that

$$\sigma(d_1) = \cdots = \sigma(d_{n-1}) = \xi.$$

The Arf invariant of σ is:

$$\begin{aligned}
\delta &\equiv \sum_{i=1}^{n-1}(1-\sigma(c_i))(1-\sigma(d_i)) + \sum_{i=1}^{\tilde{g}}(1-\sigma(a_i))(1-\sigma(b_i)) \\
&+ \sum_{i=1}^{\tilde{g}}(1-\sigma(a'_i))(1-\sigma(b'_i)) \\
&\equiv \sum_{i=1}^{n-1}(1-\sigma(c_i))(1-\sigma(d_i)) \\
&\equiv (1-\xi)[\underbrace{(1-0)+\cdots+(1-0)}_{k_0} + \underbrace{(1-m/2)+\cdots+(1-m/2)}_{k_1} + \\
&\underbrace{(1-0)+\cdots+(1-0)}_{n-1-k}] \\
&\equiv (1-\xi)(k_0 + k_1(1-m/2) + n - 1 - k) \\
&\equiv (1-\xi)(n-1-k_1\cdot m/2) \\
&\equiv (1-\xi) \mod 2.
\end{aligned}$$

To see that $n-1-k_1\cdot\frac{m}{2} = 1 \mod 2$, note that by Proposition 8.9.1 we have

$$\frac{m}{2}\cdot k_1 = \sum_{i=1}^{r}\frac{1}{p_i} + 1 - r - g \mod m,$$

hence $(n-1)-k_1\cdot\frac{m}{2} = (n-1)-\sum_{i=1}^{r}\frac{1}{p_i} - 1 + r + g = (n-2) + g + r - \sum_{i=1}^{r}\frac{1}{p_i}$ mod m. By the choice of n we have $n = g-1 \mod 2$, hence $(n-2)+g = (g-3)+g = 2g-3 = 1 \mod 2$. So we need to show that $r - \sum_{i=1}^{r}\frac{1}{p_i} = 0$ mod 2. This holds due to Lemma 5.3.21.

Hence $\sigma(d_i) = 1-\delta$ for $i = 1, \ldots, n-1$. The values of σ on e_i and e'_i are determined by their order p_i. Thus for any real m-Arf function σ of topological type $(g, \delta, k_0, k_1, p_1, \dots, p_r)$ we can choose a symmetric set of generators $\mathcal{B}$ with the values of σ on $\mathcal{B}$ determined by $(g, \delta, k_0, k_1, p_1, \dots, p_r)$ only. Hence any two real m-Arf functions of topological type $(g, \delta, k_0, k_1, p_1, \dots, p_r)$ are topologically equivalent. $\square$

Theorem 8.9.4 *The space of m-spin bundles on non-separating nice Klein orbifolds with $g \geq k+2$, marking points of orders ≥ 3 and even m decomposes into the connected components $S(g, \delta, k_0, k_1, p_1, \dots, p_r)$, where $t = (g, \delta, k_0, k_1, , p_1, \dots, p_r)$ satisfies the conditions of Proposition 8.9.1 and $S(t)$ is the set of all real m-spin bundles such that the associated m-Arf function is of type t. Each of the components $S(g, \delta, k_0, k_1, p_1, \dots, p_r)$ is a branched covering of $\mathcal{M}_{(g,2r,k_0+k_1,0,p_1,\dots,p_r)}$ and is diffeomorphic to*

$$\mathbb{R}^{3g-3+2r}/\mathrm{Mod}_{g,\delta,k_0,k_1,p_1,\dots,p_r},$$

where $\mathrm{Mod}_{g,\delta,k,0,k_1,p_1,\dots,p_r}$ is a discrete group of diffeomorphisms.

Proof Let $k = k_0 + k_1$. By definition, to any $\psi \in \tilde{T}_{0,2r,g+1,p_1,\dots,p_r}$ corresponds a sequential set

$$V = (E_1, \dots, E_r, C_1, \dots, C_{g+1})$$

of type $(0, 2r, g+1, p_1, \dots, p_r)$ which, together with

$$(\bar{C}_1, \dots, \bar{C}_k, \tilde{C}_{k+1}, \dots, \tilde{C}_{g+1}),$$

generates a real Fuchsian group Γ_ψ. On the Klein orbifold $(P, \tau) = [\Gamma_\psi]$, we consider the symmetric set of generators $(e_1, \dots, e_r, e'_1, \dots, e'_r, c_1, d_1, \dots, c_g, d_g)$, where $c_1, d_1, \dots, c_g, d_g$ correspond to the shifts

$$C_1, \tilde{C}_{g+1}\bar{C}_1, \dots, C_k, \tilde{C}_{g+1}\bar{C}_k, C_{k+1}, \tilde{C}_{g+1}\tilde{C}_{k+1}, \dots, C_g, \tilde{C}_{g+1}\tilde{C}_g.$$

We introduce an Arf function $\sigma = \sigma_\psi$ given by

$$\sigma(c_i) = 0 \quad \text{for} \quad i = 1, \dots, k_0,$$

$$\sigma(c_i) = m/2 \quad \text{for} \quad i = k_0+1, \dots, k_0+k_1 = k,$$

$$\sigma(c_i) = 0 \quad \text{for} \quad i = k+1, \dots, g,$$

$$\sigma(d_i) = 1 - \delta \quad \text{for} \quad i = 1, \dots, g.$$

$$\text{and } \sigma(e_i) = \sigma(e'_i) = n_i \text{ such that } n_i p_i + 1 = 0 \mod m.$$

Similar to the construction of an Arf function in the proof of Proposition 8.9.1 we show that $\sigma(c_{g+1}) = 0$ and hence Lemma 8.6.2 implies that σ is a real Arf function.

The Arf invariant of σ is equal modulo 2 to

$$\begin{aligned}\sum_{i=1}^{g}(1-\sigma(c_i))(1-\sigma(d_i)) &= \left(\sum_{i=1}^{g}(1-\sigma(c_i))\right)\cdot\delta \\ &= \left(g-\sum_{i=1}^{g}\sigma(c_i)\right)\cdot\delta \\ &= (g-k_1\cdot\frac{m}{2})\cdot\delta \\ &= \delta.\end{aligned}$$

$t = (g, \delta, k_0, k_1, p_1, \dots, p_r)$ satisfies the conditions of Proposition 8.9.1, hence $g-k_1\frac{m}{2} = 2g+r-1-\sum_{i=1}^{r}\frac{1}{p_i} \mod m$ and therefore $g-k_1\frac{m}{2} = r-\sum_{i=1}^{r}\frac{1}{p_i}-1 \mod 2$. Lemma 5.3.21 implies $r-\sum_{i=1}^{r}\frac{1}{p_i} = 0 \mod 2$ hence $g-k_1\frac{m}{2} = 1 \mod 2$. We conclude that the Arf invariant of σ is equal to $\delta \mod 2$.

Hence σ is a real Arf function of type $(g, \delta, k_0, k_1, p_1, \dots, p_r)$. By Theorem 8.2.9, an m-spin bundle $\Omega(\psi) \in S(g, \delta, k_0, k_1, p_1, \dots, p_r)$ is associated with this Arf function. The correspondence $\psi \mapsto \Omega(\psi)$ induces a map $\Omega : T_{0,2r,g+1,p_1,\dots,p_r} \to S(g, \delta, k_0, k_1, p_1, \dots, p_r)$. Let us prove that

$$\Omega(T_{0,2r,g+1,p_1,\dots,p_r}) = S(g, \delta, k_0, k_1, p_1, \dots, p_r).$$

Indeed, by Theorem 8.8.2, the map

$$\Psi = \Phi\circ\Omega : T_{0,2r,g+1,p_1,\dots,p_r} \to S(g, \delta, k_0, k_1, p_1, \dots, p_r) \to \mathcal{M}_{g,2r,k,0,p_1,\dots,p_r},$$

where Φ is the natural projection, satisfies the condition

$$\Psi(T_{0,2r,g+1,p_1,\dots,p_r}) = \mathcal{M}_{g,2r,k,0,p_1,\dots,p_r}.$$

The fibre of the map Ψ is represented by the group $\mathrm{Mod}_{g,2r,k,0,p_1,\dots,p_r}$ of all self-homeomorphisms of the orbifold (P,τ). By Proposition 8.9.3, this group acts transitively on the set of all real Arf functions of type $(g, \delta, k_0, k_1, p_1, \dots, p_r)$ and hence, by Theorem 8.2.9, transitively on the fibres $\Phi^{-1}((P,\tau))$. Thus

$$\Omega(T_{0,2r,g+1,p_1,\dots,p_r}) = S(g, \delta, k_0, k_1, p_1, \dots, p_r)$$

$$S(g,\delta,k_0,k_1,p_1,\dots,p_r)=T_{0,2r,g+1,p_1,\dots,p_r}/\mathrm{Mod}_{g,\delta,k_0,k_1,p_1,\dots,p_r},$$

where

$$\mathrm{Mod}_{g,\delta,k_0,k_1,,p_1,\dots,p_r}\subset\mathrm{Mod}_{g,2r,k,0,p_1,\dots,p_r}.$$

According to Theorem 8.8.2, the space $T_{0,2r,g+1,p_1,\dots,p_r}$ is diffeomorphic to $\mathbb{R}^{3g-3+2r}$. □

8.10 Non-Separating Case, Odd m

In this section we describe all topological types and the moduli space in the non-separating case for odd m.

Proposition 8.10.1 *Let m be odd. Let (P,τ) be a nice Klein orbifold of type $(g,2r,k,0,p_1,\dots,p_r)$, $g\geq 2$, $p_i\geq 3$. A tuple $(g,k,p_1,\dots,p_r)$ is a topological type of some real m-Arf function on (P,τ) if and only if*

$$-\sum_{i=1}^{r}\frac{1}{p_i}=(1-g-r)\mod m.$$

If this condition is satisfied, then for any choice of a set of generators and values on the set of generators which give us the right topological invariants we get a real Arf function.

Proof Let

$$\mathcal{B}=(e_1,\dots,e_r,e'_1,\dots,e'_r,c_1,\dots,c_g,d_1,\dots,d_g)$$

be a symmetric set of generators as in Theorem 7.1.12.

1) Let σ be a real m-Arf function on (P,τ) of topological type $(g,k,p_1,\dots,p_r)$. Since m is odd, Lemma 8.2.3 implies that σ vanishes on all ovals and twists, $\sigma(c_i)=0$ for $i=1,\dots,g+1$. Thus the topological type of the m-Arf function $\sigma|_{P_1}$ is

   $$(\tilde{g};\tilde{\delta},n_0^h,\dots,n_{m-1}^h:p_1,\dots,p_r)$$

   with $\tilde{g}=0$, $n_0^h=g+1$, $n_j^h=0$ for $j\neq 0$.

   According to Theorem 5.5.11, the tuple $(\tilde{g},\tilde{\delta},n_0^h,\dots,n_{m-1}^h,p_1,\dots,p_r)$ must satisfy the following:

   $$\sum_{j=0}^{m-1}j\cdot n_j^h-\sum_{i=1}^{r}\frac{1}{p_i}=(2-2\tilde{g})-(g+1+r)\mod m.$$

For $\tilde{g}$ and n_j^h as above, we obtain:

$$-\sum_{i=1}^{r} \frac{1}{p_i} = (1 - g - r) \mod m.$$

2) Now let $(g, k, p_1, \ldots, p_r)$ be any tuple such that $-\sum_{i=1}^{r} \frac{1}{p_i} = 1 - g - r \mod m$. We will define an m-Arf function σ on P as follows: Set $\sigma(c_i) = 0$ for $i = 1, \ldots, g$ and choose arbitrary values for $\sigma(d_i)$ for $i = 1, \ldots g$. The values $\sigma(e_i)$ and $\sigma(e'_i)$ are determined by p_i. According to Theorem 8.3.1, these values on $\mathcal{B}$ determine a unique m-Arf function σ on P. We have $\sigma(c_1) + \cdots + \sigma(c_{g+1}) = \sigma(c_{g+1})$. On the other hand Theorem 8.3.1, applied to P_1, implies that

$$\begin{aligned} \sigma(c_1) + \cdots + \sigma(c_{g+1}) &= \sum_{i=1}^{r} \frac{1}{p_i} + (2 - 2\tilde{g}) - (g + 1 + r) \\ &= \sum_{i=1}^{r} \frac{1}{p_i} + 1 - g - r \mod m. \end{aligned}$$

Hence $\sigma(c_{g+1}) = \sum_{i=1}^{r} \frac{1}{p_i} + 1 - g - r \mod m$. We know that $0 = \sum_{i=1}^{r} \frac{1}{p_i} + 1 - g - r \mod m$ hence $\sigma(c_{g+1}) = 0 \mod m$. Lemma 8.6.2 implies that σ is a real Arf function. This construction gives all real Arf functions of type $(g, k, p_1, \ldots, p_r)$ as it realizes all choices of values on a symmetric set of generators that conforms to the given topological type. $\square$

Proposition 8.10.2 *Let m be odd and let (P, τ) be a nice Klein orbifold of type $(g, 2r, k, 0, p_1, \ldots, p_r)$, $p_i \geq 3$, $g \geq k + 2$. Two m-Arf functions on (P, τ) are topologically equivalent if and only if they have the same topological type $(g, k, p_1, \ldots, p_r)$.*

Proof Let σ be a real m-Arf function on (P, τ) of topological type

$$(g, k, p_1, \ldots, p_r).$$

Let $n \in \{k + 1, \ldots g - 1\}$ such that $n = g - 1 \mod 2$. (The assumption $k \leq g - 2$ implies $k + 1 \leq g - 1$, hence $\{k + 1, \ldots, g - 1\} \neq \emptyset$).
Let $c_1, \ldots, c_n$ be invariant contours as in Theorem 7.1.12 with $c_1, \ldots, , c_k$ corresponding to ovals and $c_{k+1}, \ldots, c_n$ corresponding to twists. Since m is odd, Lemma 8.2.3 implies that σ vanishes on all ovals and twists,

$$\sigma(c_1) = \cdots = \sigma(c_n) = 0.$$

Let P_1 and P_2 be the connected components of the complement of the contours $c_1, \dots, c_n$ in P. Each of these components is a surface of genus $\tilde{g} = (g+1-n)/2$ with n holes. The condition $n \leq g-1$ implies $\tilde{g} \geq 1$.

(i) Consider the case $\tilde{g} \geq 2$, i.e. $n \leq g-3$. According to Lemma 5.5.10 we can now choose a standard set of generators

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, e_1, \dots, e_{l_r}, c_1, \dots, c_n)$$

of $\pi(P_1)$ in such a way that

$$(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_{\tilde{g}}), \sigma(b_{\tilde{g}})) = (0, 1, 1, \dots, 1).$$

(ii) Consider the case $\tilde{g} = 1$, i.e. $n = g-1$. According to Lemma 5.5.10 we can choose a standard generating set $(a_1, b_1, e_1, \dots, e_r, c_1, \dots, c_n)$ of $\pi(P_1)$ in such a way that

$$(\sigma(a_1), \sigma(b_1)) = (\tilde{\delta}, 0),$$

where

$$\tilde{\delta} = \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1)+1, \dots, \sigma(c_n)+1, p_1-1, \dots, p_r-1).$$

Note that $\sigma(c_i)+1 = 1$ for all i in $\{1, \dots, n\}$, therefore the greatest common divisor is $\tilde{\delta} = 1$. Hence we can choose a standard generating set $(a_1, b_1, e_1, \dots, e_r, c_1, \dots, c_n)$ of $\pi_1(P_1)$ in such a way that

$$(\sigma(a_1), \sigma(b_1)) = (1, 0)$$

Let

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}},$$
$$e_1, \dots, e_r, e'_1, \dots, e'_r, c_1, \dots, c_{n-1}, d_1, \dots, d_{n-1})$$

be a symmetric generating set of $\pi(P)$ as in Theorem 8.7.2, i.e.

$$\sigma(d_1) = \cdots = \sigma(d_{n-1}) = 0.$$

The values of σ on e_i and e'_i are determined by their order p_i. Thus for any real m-Arf function σ of topological type $(g, k, p_1, \dots, p_r)$ we can choose a symmetric set of generators $\mathcal{B}$ with the values of σ on $\mathcal{B}$ determines by $(g, k, p_1, \dots, p_r)$ only. Hence any two real m-Arf functions of topological type $(g, k, p_1, \dots, p_r)$ are topologically equivalent. □

Theorem 8.10.3 *The space of m-spin bundles on non-separating nice Klein orbifolds with $g \geq 2$, marking points of orders ≥ 3 and with odd m decomposes into the connected components $S(g, k, p_1, \ldots, p_r)$, where $t = (g, k, p_1, \ldots, p_r)$ satisfies the conditions of Proposition 8.10.1 and $S(t)$ is the set of all real m-spin bundles such that the associated m-Arf function is of type t. Each of the components $S(g, k, p_1, \ldots, p_r)$ is a branched covering of $\mathcal{M}_{(g,2r,k,0,p_1,\ldots,p_r)}$ and is diffeomorphic to*

$$\mathbb{R}^{3g-3+2r}/\mathrm{Mod}_{g,k,p_1,\ldots,p_r},$$

where $\mathrm{Mod}_{g,k,p_1,\ldots,p_r}$ is a discrete group of diffeomorphisms.

Proof By definition, to any $\psi \in \tilde{T}_{0,2r,g+1,p_1,\ldots,p_r}$ corresponds a sequential set

$$V = (E_1, \ldots, E_r, C_1, \ldots, C_{g+1})$$

of type $(0, 2r, g+1, p_1, \ldots, p_r)$ which, together with

$$(\bar{C}_1, \ldots, \bar{C}_k, \tilde{C}_{k+1}, \ldots, \tilde{C}_{g+1}),$$

generates a real Fuchsian group Γ_ψ. On the Klein orbifold $(P, \tau) = [\Gamma_\psi]$, we consider the symmetric set of generators $(e_1, \ldots, e_r, e'_1, \ldots, e'_r, c_1, d_1, \ldots, c_g, d_g)$, where $c_1, d_1, \ldots, c_g, d_g$ correspond to the shifts

$$C_1, \tilde{C}_{g+1}\bar{C}_1, \ldots, C_k, \tilde{C}_{g+1}\bar{C}_k, C_{k+1}, \tilde{C}_{g+1}\tilde{C}_{k+1}, \ldots, C_g, \tilde{C}_{g+1}\tilde{C}_g.$$

We introduce a real Arf function $\sigma = \sigma_\psi$ given by

$$\sigma(c_i) = \sigma(d_i) = 0 \quad \text{for} \quad i = 1, \ldots, g$$

and $\sigma(e_i) = \sigma(e'_i) = n_i$ such that $n_i p_i + 1 = 0 \mod m$.

Similar to the construction of an Arf function in the proof of proposition 8.10.1 we show that $\sigma(c_{g+1}) = 0$ and hence Lemma 8.6.2 implies that σ is a real Arf function of type $(g, k, p_1, \ldots, p_r)$.

By Theorem 8.2.9, an m-spin bundle $\Omega(\psi) \in S(g, k, p_1, \ldots, p_r)$ is associated with this Arf function. The correspondence $\psi \mapsto \Omega(\psi)$ induces a map $\Omega : T_{0,2r,g+1,p_1,\ldots,p_r} \to S(g, k, p_1, \ldots, p_r)$. Let us prove that $\Omega(T_{0,2r,g+1,p_1,\ldots,p_r}) = S(g, k, p_1, \ldots, p_r)$. Indeed, by Theorem 8.8.2, the map

$$\Psi = \Phi \circ \Omega : T_{0,2r,g+1,p_1,\ldots,p_r} \to S(g, k, p_1, \ldots, p_r) \to \mathcal{M}_{g,2r,k,0,p_1,\ldots,p_r},$$

where Φ is the natural projection, satisfies the condition

$$\Psi(T_{0,2r,g+1,p_1,\ldots,p_r}) = \mathcal{M}_{g,2r,k,0,p_1,\ldots,p_r}.$$

The fibre of the map Ψ is represented by the group $\mathrm{Mod}_{g,2r,k,0,p_1,\ldots,p_r}$ of all self-homeomorphisms of the orbifold (P,τ). By Proposition 8.10.2, this group acts transitively on the set of all real Arf functions of type $(g,k,p_1,\ldots,p_r)$ and hence, by Theorem 8.2.9, transitively on the fibres $\Phi^{-1}((P,\tau))$. Thus

$$\Omega(T_{0,2r,g+1,p_1,\ldots,p_r}) = S(g,k,p_1,\ldots,p_r)$$

and

$$S(g,k,p_1,\ldots,p_r) = T_{0,2r,g+1,p_1,\ldots,p_r}/\mathrm{Mod}_{g,\delta,k,p_1,\ldots,p_r},$$

where

$$\mathrm{Mod}_{g,k,p_1,\ldots,p_r} \subset \mathrm{Mod}_{g,2r,k,0,p_1,\ldots,p_r}.$$

According to Theorem 8.8.2, the space $T_{0,2r,g+1,p_1,\ldots,p_r}$ is diffeomorphic to $\mathbb{R}^{3g-3+2r}$. $\square$

8.11 Separating Case, Even m

In this section we describe all topological types and the moduli spaces in the separating case for even m.

In the separating case, i.e. if $P\backslash P^\tau$ has two connected components, we consider the restriction of the Arf function σ to these components and use the classification result of Theorem 5.5.11. On the other hand, we can start with an appropriate Arf function on one of the components and extend it to the other component and to the whole surface.

Proposition 8.11.1 *Let m be even. Let (P,τ) be a nice Klein orbifold of type $(g,2r,k,1,p_1,\ldots,p_r)$, $g \geq 2$, $p_i \geq 3$. A tuple*

$$(g,\tilde{\delta},k_0^0,k_1^0,k_0^1,k_1^1,p_1,\ldots,p_r)$$

is a topological type of some real m-Arf function on (P,τ) if and only if

$$(\tilde{g},\tilde{\delta},n_0^h = k_0^0+k_0^1,0,\ldots,0,n_{m/2}^h = k_1^0+k_1^1,0,\ldots,0,p_1,\ldots,p_r)$$

is a topological type of some m-Arf function on an orbifold of genus $\tilde{g} = (g+1-k)/2$ with k holes and r marking points of orders $p_1,\ldots,p_r$, i.e. if and only if the following conditions are satisfied:

(a) If $g > k+1$ and $k_0^0 + k_0^1 \neq 0$ then $\tilde{\delta} = 0$.

(b) If $g > k+1$ and $m \equiv 0 \mod 4$ then $\tilde{\delta} = 0$.

(c) If $g = k+1$ and $k_0^0 + k_0^1 \neq 0$ then $\tilde{\delta} = 1$.

(d) If $g = k+1$ and $k_0^0 + k_0^1 = 0$ and $m \equiv 0 \mod 4$ then $\tilde{\delta} = 1$.

(e) If $g = k+1$ and $k_0^0 + k_0^1 = 0$ and $m \equiv 2 \mod 4$ then $\tilde{\delta} \in \{1, 2\}$.

(f) The following degree condition is satisfied

$$\frac{m}{2} \cdot (k_1^0 + k_1^1) - \sum_{i=1}^{r} \frac{1}{p_i} = 1 - g - r \mod m.$$

If these conditions are satisfied, then for any choice of a set of generators and values on the set of generators which give us the right topological invariants we get a real Arf function.

If $m \equiv 2 \mod 4$, then the Arf invariant $\delta \in \{0, 1\}$ of such an Arf function satisfies

$$\delta = k_0^0 = k_0^1 \mod 2.$$

If $m \equiv 0 \mod 4$, them the Arf invariant $\delta \in \{0, 1\}$ of such an Arf function satisfies

$$\delta = k_0^0 + k_1^0 = k_0^1 + k_1^1 \mod 2.$$

Proof Let P_1 and P_2 be the connected components of $P \backslash P^\tau$. Each of these components is a surface of genus $\tilde{g} = (g+1-k)/2$ with k holes. Let

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, e_1, \dots, e_r,$$
$$e'_1, \dots, e'_r, c_1, \dots, c_{n-1}, d_1, \dots, d_{n-1})$$

be a symmetric set of generators as in Theorem 7.1.12

1) Let $(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r)$ be the topological type of a real m-Arf function σ on (P, τ). The boundary components of P_1 correspond to the ovals of P. The values on the ovals are 0 repeated $k_0^0 + k_0^1$ times and $m/2$ repeated $k_1^0 + k_1^1$ times. So

$$(\tilde{g}, \tilde{\delta}, n_0^h = k_0^0 + k_0^1, 0, \dots, 0, n_{m/2}^h = k_1^0 + k_1^1, 0, \dots, 0, p_1, \dots, p_r)$$

is the topological type of the m-Arf function $\sigma|_{P_1}$ on P_1.

2) Now let $(\tilde{g}, \tilde{\delta}, n_0^h = k_0^0 + k_0^1, 0, \dots, 0, n_{m/2}^h = k_1^0 + k_1^1, 0, \dots, 0, p_1, \dots, p_r)$ be the topological type of an m-Arf function $\tilde{\sigma}$ on P_1. We will define an m-Arf function σ on P as follows:

$$\sigma(a_i) = \tilde{\sigma}(a_i), \quad \sigma(b_i) = \tilde{\sigma}(b_i) \quad \text{for} \quad i = 1, \dots, \tilde{g},$$

$$\sigma(a_i') = \tilde{\sigma}(a_i), \quad \sigma(b_i') = \tilde{\sigma}(b_i) \quad \text{for} \quad i = 1, \dots, \tilde{g},$$

$$\sigma(c_i) = \tilde{\sigma}(c_i) \quad \text{for} \quad i = 1, \dots, k-1.$$

The values $\sigma(e_i)$ and $\sigma(e_i')$ are determined by p_i.

Let A_0 and A_1 be the following subsets of $\{1, \dots k\}$:

$$A_0 = \{i \mid \sigma(c_i) = 0\}, \quad A_1 = \{i \mid \sigma(c_i) = m/2\}.$$

Note that $|A_j| = k_j^0 + k_j^1$. Let us partition each set A_j into subsets A_j^0 and A_j^1 such that $|A_j^\alpha| = k_j^\alpha$. Let $A^\alpha = A_0^\alpha \cup A_1^\alpha$ for $\alpha = 0, 1$. Let $\beta \in \{0, 1\}$ be such that $k \in A^\beta$. For $i = 1, \dots, k-1$, choose an arbitrary odd value for $\sigma(d_i)$ if $i \in A^\beta$ and an arbitrary even value for $\sigma(d_i)$ if $i \in A^{1-\beta}$. According to Theorem 8.3.1, these values on $\mathcal{B}$ determine a unique m-Arf function σ on P. Lemma 8.6.2 implies that σ is a real m-Arf function. This construction gives all real Arf functions of type $(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r)$ as it realizes all choices of values on a symmetric set of generators that conforms to the given topological type.

3) According to Theorem 5.5.11, for even m, the tuple

$$(\tilde{g}, \tilde{\delta}, n_0^h = k_0^0 + k_0^1, 0, \dots, 0, n_{m/2}^h = k_1^0 + k_1^1, 0, \dots, 0, p_1, \dots, p_r)$$

is a topological type of some m-Arf function on an orbifold of genus $\tilde{g} = (g - 1 + k)/2$ with k holes and r marking points if and only if

(i) If $\tilde{g} > 1$ and $n_j^h \neq 0$ for some even j, then $\tilde{\delta} = 0$.

(ii) If $\tilde{g} = 1$ then $\tilde{\delta}$ is a divisor of m, $\gcd(\{j + 1 | n_j^h \neq 0\})$ and $\gcd(p_1 - 1, \dots, p_r - 1)$.

(iii) $\sum_{j=0}^{m-1} j \cdot n_j^h - \sum_{i=1}^{r} \frac{1}{p_i} = (2 - 2\tilde{g}) - (k + r)$.

Note that $\tilde{g} = 0 \Leftrightarrow g = k+1$, $\tilde{g} > 1 \Leftrightarrow g > k+1$, $(2-2\tilde{g}) - (k+r) = 1 - g - r$ and $\sum_{j=0}^{m-1} j \cdot n_j^h = 0 \cdot n_0^h + \frac{m}{2} \cdot n_{\frac{m}{2}}^h = \frac{m}{2}(k_1^0 + k_1^1)$.

Let $\tilde{g} > 1$, that is $g > k+1$. By Theorem 5.5.11 we have $\tilde{\delta} = 0$ if $n_j^h \neq 0$ for some even j. Only $n_0^h = k_0^0 + k_0^1$ and $n_{m/2}^h = k_1^0 + k_1^1$ are non-zero. Taking $n_0^h = k_0^0 + k_0^1 \neq 0$ we have condition (a). The value $m/2$ is even when $m \equiv 0 \mod 4$, giving condition (b).

Let $\tilde{g} = 1$, that is $g = k+1$. If $n_0^h \neq 0$ and $n_{m/2}^h = 0$ then we have by Theorem 5.5.11 that $\tilde{\delta}$ is a divisor of m, $\gcd(1) = 1$ and $\gcd(p_1 - 1, \ldots, p_r - 1)$, so $\tilde{\delta} = 1$. If $n_0^h \neq 0$ and $n_{m/2}^h \neq 0$ then $\tilde{\delta}$ is a divisor of m, $\gcd\{1, \frac{m}{2}+1\} = 1$ and $\gcd(p_1 - 1, \ldots, p_r - 1)$, so $\tilde{\delta} = 1$, giving condition (c).

If $n_0^h = 0$ and $n_{m/2}^h \neq 0$ then $\tilde{\delta}$ is a divisor of m, $\gcd\{\frac{m}{2}+1\} = \frac{m}{2}+1$ and $\gcd(p_1 - 1, \ldots, p_r - 1)$.

(i) For $m \equiv 0 \mod 4$, $\tilde{\delta}$ must be a divisor of $\gcd(m, \frac{m}{2}+1) = 1$ so $\tilde{\delta} = 1$. This gives condition (d).

(ii) For $m \equiv 2 \mod 4$, $\tilde{\delta}$ is a divisor of $\gcd(m, \frac{m}{2}+1) = 2$ so $\tilde{\delta} \in \{1, 2\}$, giving condition$(e)$.

Finally the degree condition

$$\sum_{j=0}^{m-1} j \cdot n_j^h - \sum_{i=1}^{r} \frac{1}{p_i} = (2 - 2\tilde{g}) - (k + r)$$

can be rewritten as

$$\frac{m}{2}(k_1^0 + k_1^1) - \sum_{i=1}^{r} \frac{1}{p_i} = 1 - g - r \mod m.$$

4) The Arf invariant δ of the Arf function σ on the compact orbifold P is equal to

$$\sum_{i=1}^{\tilde{g}}(1-\sigma(a_i))(1-\sigma(b_i)) + \sum_{i=1}^{\tilde{g}}(1-\sigma(a_i')(1-\sigma(b_i')) + \sum_{i=1}^{k-1}(1-\sigma(c_i))(1-\sigma(d_i))$$

modulo 2. Note that

$$\sum_{i=1}^{\tilde{g}}(1 - \sigma(a_i))(1 - \sigma(b_i)) \equiv \sum_{i=1}^{\tilde{g}}(1 - \sigma(a_i'))(1 - \sigma(b_i')) \mod 2,$$

hence

$$\delta \equiv \sum_{i=1}^{k-1}(1-\sigma(c_i))(1-\sigma(d_i)) \mod 2.$$

5) If $m \equiv 2 \mod 4$, then

$$\sum_{i=1}^{k-1}(1-\sigma(c_i))(1-\sigma(d_i)) \equiv |A_0^{1-\beta}| = k_0^{1-\beta} \mod 2.$$

In this case $m/2$ is odd, hence condition (f) can be reduced modulo 2 to

$$k_1^0 + k_1^1 - \sum_{i=1}^{r} \frac{1}{p_i} = 1 - g - r \mod 2.$$

By Lemma 5.3.21 we have $\sum_{i=1}^{r} \frac{1}{p_i} - r = 0 \mod 2$. Hence we have

$$k_1^0 + k_1^1 = 1 - g \mod 2.$$

On the other hand, Theorem 7.1.9 implies that $k = g + 1 \mod 2$. Hence

$$k_0^0 + k_0^1 = k - (k_1^0 + k_1^1) = (g+1) - (1-g) = 0 \mod 2,$$

i.e.

$$k_0^0 \equiv k_0^1 \mod 2.$$

6) If $m \equiv 0 \mod 4$, then

$$\sum_{i=1}^{k-1}(1-\sigma(c_i))(1-\sigma(d_i)) \equiv |A^{1-\beta}| = k_0^{1-\beta} + k_1^{1-\beta} \mod 2.$$

In this case $m/2$ is even, hence condition (f) can be reduced modulo 2 to

$$-\sum_{i=1}^{r} \frac{1}{p_i} = 1 - g - r \mod 2.$$

By Lemma 5.3.21 we have $\sum_{i=1}^{r} \frac{1}{p_i} - r = 0 \mod 2$, hence

$$0 = 1 - g \mod 2.$$

On the other hand, Theorem 7.1.9 implies that $k = g + 1 \mod 2$. Hence $k = (k_0^0 + k_0^1) + (k_1^0 + k_1^1)$ is even, i.e.

$$k_0^0 + k_1^0 \equiv k_0^1 + k_1^1 \mod 2.$$

$\square$

Proposition 8.11.2 *Let m be even and let (P,τ) be a nice Klein orbifold of type $(g, 2r, k, 1, p_1, \dots, p_r)$, $g \geq k+1$, $p_i \geq 3$. Two m-Arf functions on (P,τ) are topologically equivalent if and only if they have the same topological type*

$$(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r).$$

Proof Let P_1 and P_2 be the connected components of $P \backslash P^\tau$. Each of these components is an orbifold of genus $\tilde{g} = (g+1-k)/2$ with k holes and r marking points. Let σ be a real m-Arf function on (P,τ) of topological type

$$(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r).$$

The condition $k \leq g-1$ implies $\tilde{g} \geq 1$.

(i) Consider the case $\tilde{g} \geq 2$, i.e. $k \leq g-3$. According to Lemma 5.5.10 we can choose a standard set of generators

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, e_1, \dots, e_r, c_1, \dots, c_k)$$

of $\pi(P_1, p)$ in such a way that

$$(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_{\tilde{g}}), \sigma(b_{\tilde{g}})) = (0, 1-\tilde{\delta}, 1, \dots, 1).$$

(ii) Consider the case $\tilde{g} = 1$, i.e. $k = g-1$. According to Lemma 5.5.10 we can choose a standard set of generators $(a_1, b_1, e_1, \dots, e_r, c_1, \dots, c_k)$ of $\pi(P_1, p)$ in such a way that

$$(\sigma(a_1), \sigma(b_1)) = (\tilde{\delta}, 0).$$

Let

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}},$$
$$e_1, \dots, e_r, e'_1, \dots, e'_r, c_1, \dots, c_{k-1}, d_1, \dots, d_{k-1})$$

be a symmetric set of generators of $\pi(P)$ as in Theorem 8.7.2, i.e. all $\sigma(d_i) \in \{0,1\}$ for $i = 1, \dots, k-1$. The values on e_i and e'_i are determined by their orders $p_1, \dots, p_r$. Let A_0^0 be the first k_0^0 elements of $\{1, \dots k\}$, let A_1^0 be the next k_1^0 elements, let A_0^1 be the next k_0^1 elements and finally let A_1^1 be the last k_1^1 elements. We can assume that $c_1, \dots, c_k$ are arranged in order such that

$$\sigma(c_i) = jm/2 \quad \text{for } i \in A_j^\alpha,\ j = 0, 1\ \alpha = 0, 1,$$

$\sigma(d_i)$ is odd if $i, k \in A_0^\alpha \cup A_1^\alpha$ for some $\alpha \in \{0, 1\}$ and $\sigma(d_i)$ is even otherwise. Since $\sigma(d_i) \in \{0, 1\}$, we have

$$\sigma(d_i) = 1 \quad \text{if } i, k \in A_0^\alpha \cup A_1^\alpha \quad \text{for some } \alpha \in \{0, 1\} \text{ and } \sigma(d_i) = 0 \text{ otherwise.}$$

Thus for any real m-Arf function σ of topological type

$$(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r)$$

we can choose a symmetric set of generators $\mathcal{B}$ with the values of σ on $\mathcal{B}$ determined by $(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r)$ only. Hence any two real m-Arf functions of topological type $(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r)$ are topologically equivalent. $\square$

Theorem 8.11.3 *The space of m-spin bundles on separating nice Klein orbifolds with $g \geq k + 1$, marking points of orders ≥ 3 and with even m decomposes into the connected components*

$$S(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r),$$

where

$$t = (g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r)$$

satisfies the conditions of Proposition 8.11.1 and $S(t)$ is the set of all real m-spin bundles such that the associated m-Arf function is of type t. Each of the components $S(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r)$ is a branched covering of

$$\mathcal{M}_{g, 2r, k_0^0 + k_1^0 + k_0^1 + k_1^1, 1, p_1, \dots, p_r}$$

and is diffeomorphic to

$$\mathbb{R}^{3g-3+2r} / \mathrm{Mod}_{g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r},$$

where $\mathrm{Mod}_{g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1, p_1, \dots, p_r}$ *is a discrete group of diffeomorphisms.*

Proof Let $k = k_0^0 + k_1^0 + k_0^1 + k_1^1$. By definition, to any $\psi \in \tilde{T}_{\tilde{g}, 2r, k, p_1, \dots, p_r}$ corresponds a sequential set

$$V = (A_1, B_1, \dots, A_{\tilde{g}}, B_{\tilde{g}}, E_1, \dots, E_r, C_1, \dots, C_k)$$

of type $(\tilde{g}, 2r, k, p_1, \ldots, p_r)$ which, together with

$$(\bar{C}_1, \ldots, \bar{C}_k),$$

generates a real Fuchsian group Γ_ψ. On the Klein orbifold $(P, \tau) = [\Gamma_\psi]$, we consider the symmetric set of generators

$$(a_1, b_1, \ldots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \ldots, a'_{\tilde{g}}, b'_{\tilde{g}}, e_1, \ldots, e_r, e'_1, \ldots, e'_r, c_1, d_1, \ldots, c_{k-1}, d_{k-1}),$$

where $a_1, b_1, \ldots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \ldots, a'_{\tilde{g}}, b'_{\tilde{g}}, c_1, d_1, \ldots, c_{k-1}, d_{k-1}$ correspond to the shifts

$$A_1, B_1, \ldots, A_{\tilde{g}}, B_{\tilde{g}},$$

$$\bar{C}_k A_1 \bar{C}_k, \bar{C}_k B_1 \bar{C}_k, \ldots, \bar{C}_k A_{\tilde{g}} \bar{C}_k, \bar{C}_k B_{\tilde{g}} \bar{C}_k,$$

$$C_1, \bar{C}_k \bar{C}_1, \ldots, C_{k-1}, \bar{C}_k \bar{C}_{k-1}.$$

Let us consider the case $k_1^1 > 0$, other cases are similar. We introduce a real Arf function $\sigma = \sigma_\psi$ determined by the following conditions.

If $\tilde{g} \geq 2$, i.e. $k \leq g - 3$, then

$$\sigma(a_1) = \sigma(a'_1) = 0, \quad \sigma(b_1) = \sigma(b'_1) = 1 - \tilde{\delta},$$

$$\sigma(a_i) = \sigma(a'_i) = \sigma(b_i) = \sigma(b'_i) = 1 \text{ for } i = 2, \ldots, \tilde{g}.$$

If $\tilde{g} = 1$, i.e. $k = g - 1$, then

$$\sigma(a_1) = \sigma(a'_1) = \tilde{\delta}, \quad \sigma(b_1) = \sigma(b'_1) = 0$$

and $\sigma(e_i) = \sigma(e'_i) = n_i$ such that $n_i p_i + 1 = 0 \mod m$.

Furthermore let

$$A_0^0 = \{1, \ldots, k_0^0\}, \quad A_0^1 = \{k_0^0 + 1, \ldots, k_0\},$$

$$A_1^0 = \{k_0^1, \ldots, k_0 + k_1^0\}, \quad A_1^1 = \{k_0 + k_1^0 + 1, \ldots, k - 1\}$$

and

$$\sigma(c_i) = 0 \text{ for } i \in A_0^0 \cup A_0^1,$$

$$\sigma(c_i) = 1 \text{ for } i \in A_1^0 \cup A_1^1,$$

$$\sigma(d_i) = 0 \text{ for } i \in A_0^0 \cup A_1^0,$$

$$\sigma(d_i) = 1 \text{ for } i \in A_0^1 \cup A_1^1.$$

Similar to the proof of Proposition 8.11.1, Lemma 8.6.2 implies that σ is a real Arf function and calculation shows that the Arf invariant of $\sigma|_{P_1}$ is equal to $\tilde{\delta}$. Hence σ is a real m-Arf function of type $t = (g, \tilde{\delta}, k_i^j, p_1, \dots, p_r)$.

By Theorem 8.2.9, an m-spin bundle $\Omega(\psi) \in S(g, \tilde{\delta}, k_i^j, p_1, \dots, p_r)$ is associated with this Arf function. The correspondence $\psi \mapsto \Omega(\psi)$ induces a map $\Omega : T_{\tilde{g},2r,k,p_1,\dots,p_r} \to S(g, \tilde{\delta}, k_i^j, p_1, \dots, p_r)$. Let us prove that $\Omega(T_{\tilde{g},2r,k,p_1,\dots,p_r}) = S(g, \tilde{\delta}, k_i^j, p_1, \dots, p_r)$. Indeed, by Theorem 8.8.1, the map

$$\Psi = \Phi \circ \Omega : T_{\tilde{g},2r,k,p_1,\dots,p_r} \to S(g, \tilde{\delta}, k_i^j, p_1, \dots, p_r) \to \mathcal{M}_{g,2r,k,1,p_1,\dots,p_r},$$

where Φ is the natural projection, satisfies the condition

$$\Psi(T_{\tilde{g},2r,k,p_1,\dots,p_r}) = \mathcal{M}_{g,2r,k,1,p_1,\dots,p_r}.$$

The fibre of the map Ψ is represented by the group $\mathrm{Mod}_{g,k,1,p_1,\dots,p_r}$ of all self-homeomorphisms of the orbifold (P, τ). By Proposition 8.11.2, this group acts transitively on the set of all real Arf functions of type $(g, \tilde{\delta}, k_i^j, p_1, \dots, p_r)$ and hence, by Theorem 8.2.9, transitively on the fibres $\Phi^{-1}((P, \tau))$. Thus

$$\Omega(T_{\tilde{g},2r,k,p_1,\dots,p_r}) = S(g, \tilde{\delta}, k_i^j, p_1, \dots, p_r),$$

$$S(g, \tilde{\delta}, k_i^j, p_1, \dots, p_r) = T_{\tilde{g},2r,k,p_1,\dots,p_r} / \mathrm{Mod}_{g,\tilde{\delta},k_i^j,p_1,\dots,p_r},$$

where

$$\mathrm{Mod}_{g,\tilde{\delta},k_i^j,p_1,\dots,p_r} \subset \mathrm{Mod}_{g,k,1,p_1,\dots,p_r}.$$

According to Theorem 8.8.1, the space $T_{\tilde{g},2r,k,p_1,\dots,p_r}$ is diffeomorphic to $\mathbb{R}^{3g-3+2r}$. $\square$

8.12 Separating Case, Odd m

In this section we describe all topological types and the moduli spaces in the separating case for odd m.

Proposition 8.12.1 *Let m be odd. Let (P, τ) be a nice Klein orbifold of type $(g, 2r, k, 1, p_1, \dots, p_r)$, $g \geq 2$, $p_i \geq 3$. A tuple $(g, \tilde{\delta}, k, p_1, \dots, p_r)$ is a topological type of some real m-Arf function on (P, τ) if and only if*

$$(\tilde{g}, \tilde{\delta}, k, 0, \dots, 0, p_1, \dots, p_r)$$

is a topological type of some m-Arf function on an orbifold of genus

$$\tilde{g} = (g+1-k)/2$$

with k holes and r marking points of orders $p_1, \dots, p_r$, i.e. if and only if the following conditions are satisfied:

(a) If $g > k+1$ then $\tilde{\delta} = 0$.

(b) If $g = k+1$ then $\tilde{\delta} = 1$.

(c) $-\sum_{i=1}^{r} \frac{1}{p_i} = (1-g-r) \mod m$.

If these conditions are satisfied, then for any choice of a set of generators and values on the set of generators which give us the right topological invariants we get a real Arf function.

Proof Let P_1 and P_2 be the connected components of $P \backslash P^{\tau}$. Each of these components is a surface of genus $\tilde{g} = (g+1-k)/2$ with k holes and r marked points. Let

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, e_1, \dots, e_r,$$

$$e'_1, \dots, e'_r, c_1, \dots, c_{n-1}, d_1, \dots, d_{n-1})$$

be a symmetric set of generators as in Theorem 7.1.12.

1) Let $(g, \tilde{\delta}, k, p_1, \dots, p_r)$ be the topological type of a real m-Arf function σ on (P, τ). Since m is odd, Lemma 8.2.3 implies that σ vanishes on all ovals and twists, therefore

$$(\tilde{g} = (g+1-k)/2, \tilde{\delta}, n_0^h = k, n_1^h = 0, \dots, n_{m-1}^h = 0, p_1, \dots, p_r)$$

is the topological type of the m-Arf function $\sigma|_{P_1}$ on P_1.

2) Now let $(\tilde{g} = (g+1-k)/2, \tilde{\delta}, k, 0, \dots, 0, p_1, \dots, p_r)$ be the topological type of an m-Arf function $\tilde{\sigma}$ on P_1. We will define an m-Arf function σ on P_1 as follows:

$$\sigma(a_i) = \tilde{\sigma}(a_i), \quad \sigma(b_i) = \tilde{\sigma}(b_i) \quad \text{for} \quad i = 1, \dots, \tilde{g},$$

$$\sigma(a'_i) = \tilde{\sigma}(a_i), \quad \sigma(b'_i) = \tilde{\sigma}(b_i) \quad \text{for} \quad i = 1, \dots, \tilde{g},$$

$$\sigma(c_i) = \tilde{\sigma}(c_i) \quad \text{for} \quad i = 1, \dots, k.$$

Choose arbitrary values for $\sigma(d_i)$ for $i = 1, \dots k-1$. The values $\sigma(e_i)$ and $\sigma(e'_i)$ are determined by p_i. According to Theorem 8.3.1, these values on $\mathcal{B}$ determine a unique m-Arf function σ on P. Lemma 8.6.2 implies that σ is a real Arf function.

This construction gives all real Arf functions of type $(g, \tilde{\delta}, k, p_1, \dots, p_r)$ as it realizes all choices of values on a symmetric set of generators that conforms to the given topological type.

3) According to Theorem 5.5.11, for odd m, the tuple

$$(\tilde{g}, \tilde{\delta}, k, 0, \dots, 0, p_1, \dots, p_r)$$

is a topological type of some m-Arf function on an orbifold of genus $\tilde{g} = (g + 1 - k)/2$ with k holes and r marking points if and only if

(a) If $\tilde{g} > 1$ then $\tilde{\delta} = 0$.

(b) If $\tilde{g} = 1$ then $\tilde{\delta}$ is a divisor of m, $\gcd\{j + 1 | n^h_j \neq 0\}$ and $\gcd(p_1 - 1, \dots, p_r - 1)$. Note that $n^h_0 \neq 0$, hence $\tilde{\delta}$ is a divisor of 1, i.e. $\tilde{\delta} = 1$.

(c) The degree condition

$$\sum_{j=0}^{m-1} j \cdot n^h_j - \sum_{i=1}^{r} \frac{1}{p_i} = (2 - 2\tilde{g}) - (k + r) \mod m,$$

where $n^h_0 = k, n^h_1 = 0, \dots, n^h_{m-1} = 0$, $\tilde{g} = (g + 1 - k)/2$ and $l_h = \sum_{j=0}^{m-1} n^h_j = k$. We obtain:

$$0 \cdot k + 1 \cdot 0 + \dots (m-1) \cdot 0 - \sum_{i=1}^{r} \frac{1}{p_i} = 2 - 2(\frac{g + 1 - k}{2}) - (k + r)$$

$$-\sum_{i=1}^{r} \frac{1}{p_i} = (1 - g - r) \mod m.$$

□

Proposition 8.12.2 *Let m be odd and let (P, τ) be a nice Klein orbifold of type $(g, 2r, k, 1, p_1, \dots, p_r)$, $g \geq k+1$, $p_i \geq 3$. Two m-Arf functions on (P, τ) are topologically equivalent if and only if they have the same topological type $(g, \tilde{\delta}, k, p_1, \dots, p_r)$.*

Proof Let P_1 and P_2 be the connected components of $P \backslash P^\tau$. Each of these components is an orbifold of genus $\tilde{g} = (g+1-k)/2$ with k holes and r marking points. Let σ be a real m-Arf function on (P, τ) of topological type $(g, \tilde{\delta}, k, p_1, \dots, p_r)$. The condition $k \leq g-1$ implies $\tilde{g} \geq 1$.

(i) Consider the case $\tilde{g} \geq 2$, i.e. $k \leq g-3$. Then $\tilde{\delta} = 0$. According to Lemma 5.5.10 we can choose a standard set of generators

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, e_1, \dots, e_r, c_1, \dots, c_k)$$

of $\pi(P_1, p)$ in such a way that

$$(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_{\tilde{g}}), \sigma(b_{\tilde{g}})) = (0, 1, 1, \dots, 1).$$

(ii) Consider the case $\tilde{g} = 1$, i.e. $k = g-1$. Then $\tilde{\delta} = 1$. According to Lemma 5.5.10 we can choose a standard set of generators

$$(a_1, b_1, e_1, \dots, e_r, c_1, \dots, c_k)$$

in such a way that

$$(\sigma(a_1), \sigma(b_1)) = (\tilde{\delta}, 0).$$

Let

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, e_1, \dots, e_r, e'_1, \dots, e'_r,$$

$$c_1, \dots, c_{k-1}, d_1, \dots, d_{k-1})$$

be a symmetric set of generators of $\pi(P)$ as in Theorem 8.7.2, i.e. $\sigma(d_i) = 0$ for $i = 1, \dots, k-1$. The values on e_i and e'_i are determined by their order p_i. Since m is odd, Lemma 8.2.3 implies that σ vanishes on all ovals,

$$\sigma(c_i) = 0 \quad \text{for} \quad i = 1, \dots, k.$$

Thus for any real m-Arf function σ of topological type $(g, \tilde{\delta}, k, p_1, \dots, p_r)$ we can choose a symmetric set of generators $\mathcal{B}$ with the values of σ on $\mathcal{B}$ determined by $(g, \tilde{\delta}, k, p_1, \dots, p_r)$ only. Hence any two real m-Arf functions of topological type $(g, \tilde{\delta}, k, p_1, \dots, p_r)$ are topologically equivalent. $\square$

Theorem 8.12.3 *The space of m-spin bundles on separating nice Klein orbifolds with $g \geq k+1$, marking points of orders ≥ 3 and with odd m decomposes into the connected components $S(g, \tilde{\delta}, k, p_1, \dots, p_r)$, where $t = (g, \tilde{\delta}, k, p_1, \dots, p_r)$ satisfies the conditions of Proposition 8.12.1 and $S(t)$ is the set of all real m-spin bundles such that the associated m-Arf function is of type t. Each of the components $S(g, \tilde{\delta}, k, p_1, \dots, p_r)$ is a branched covering of $\mathcal{M}_{(g,2r,k,1,p_1,\dots,p_r)}$ and is diffeomorphic to*

$$\mathbb{R}^{3g-3+2r}/\mathrm{Mod}_{g,\tilde{\delta},k,p_1,\dots,p_r},$$

where $\mathrm{Mod}_{g,\tilde{\delta},k,p_1,\dots,p_r}$ *is a discrete group of diffeomorphisms.*

Proof By definition, to any $\psi \in \tilde{T}_{\tilde{g},k}$ corresponds a sequential set

$$V = (A_1, B_1, \dots, A_{\tilde{g}}, B_{\tilde{g}}, E_1, \dots, E_r, C_1, \dots, C_k)$$

of type $(\tilde{g}, 2r, k, p_1, \dots, p_r)$ which, together with

$$(\bar{C}_1, \dots, \bar{C}_k),$$

generates a real Fuchsian group Γ_ψ. On the Klein orbifold $(P, \tau) = [\Gamma_\psi]$, we consider the symmetric set of generators

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, e_1, \dots, e_r, e'_1, \dots, e'_r, c_1, d_1, \dots, c_{k-1}, d_{k-1})$$

where

$$a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, c_1, d_1, \dots, c_{k-1}, d_{k-1}$$

correspond to the shifts

$$A_1, B_1, \dots, A_{\tilde{g}}, B_{\tilde{g}},$$

$$\bar{C}_k A_1 \bar{C}_k, \bar{C}_k B_1 \bar{C}_k, \dots, \bar{C}_k A_{\tilde{g}} \bar{C}_k, \bar{C}_k B_{\tilde{g}} \bar{C}_k,$$

$$C_1, \bar{C}_k \bar{C}_1, \dots, C_{k-1} \bar{C}_k \bar{C}_{k-1}.$$

We introduce a real Arf function $\sigma = \sigma_\psi$ determined by the following conditions. If $\tilde{g} \geq 2$, i.e. $k \leq g-3$, then

$$\sigma(a_1) = \sigma(a'_1) = 0,$$

$$\sigma(b_1) = \sigma(b'_1) = 1 - \tilde{\delta},$$

$$\sigma(a_i) = \sigma(a'_i) = \sigma(b_i) = \sigma(b'_i) = 1 \quad \text{for} \quad i = 2, \dots, \tilde{g}.$$

If $\tilde{g} = 1$, i.e. $k = g - 1$, then

$$\sigma(a_1) = \sigma(a_1') = \tilde{\delta}, \quad \sigma(b_1) = \sigma(b_1') = 0.$$

Furthermore let

$$\sigma(c_i) = \sigma(d_i) = 0 \quad \text{for} \quad i = 1, \dots k - 1$$

and $\sigma(e_i) = \sigma(e_i') = n_i$ such that $n_i p_i + 1 = 0 \mod m$.

Similar to the proof of Proposition 8.12.1, Lemma 8.6.2 implies that σ is a real Arf function and calculation shows that the Arf invariant of $\sigma|_{P_1}$ is equal to $\tilde{\delta}$. Hence σ is a real m-Arf function of type $t = (g, \tilde{\delta}, k, p_1, \dots, p_r)$.

By Theorem 8.2.9, an m-spin bundle $\Omega(\psi) \in S(g, \tilde{\delta}, k, p_1, \dots, p_r)$ is associated with this Arf function. The correspondence $\psi \mapsto \Omega(\psi)$ induces a map $\Omega : T_{\tilde{g},2r,k,p_1,\dots,p_r} \to S(g, \tilde{\delta}, k, p_1, \dots, p_r)$. Let us prove that $\Omega(T_{\tilde{g},2r,k,p_1,\dots,p_r}) = S(g, \tilde{\delta}, k, p_1, \dots, p_r)$. Indeed, by Theorem 8.8.1, the map

$$\Psi = \Phi \circ \Omega : T_{\tilde{g},2r,k,p_1,\dots,p_r} \to S(g, \tilde{\delta}, k, p_1, \dots, p_r) \to \mathcal{M}_{g,2r,k,1,p_1,\dots,p_r},$$

where Φ is the natural projection, satisfies the condition

$$\Psi(T_{\tilde{g},2r,k.p_1,\dots,p_r}) = \mathcal{M}_{g,2r,k,1,p_1,\dots,p_r}.$$

The fibre of the map Ψ is represented by the group $\mathrm{Mod}_{g,k,1,p_1,\dots,p_r}$ of all self-homeomorphisms of the orbifold (P, τ). By Proposition 8.12.2, this group acts transitively on the set of all real Arf functions of type $(g, \tilde{\delta}, k, p_1, \dots, p_r)$ and hence, by Theorem 8.2.9, transitively on the fibres $\Phi^{-1}((P, \tau))$. Thus

$$\Omega(T_{\tilde{g},2r,k,p_1,\dots,p_r}) = S(g, \tilde{\delta}, k, p_1, \dots, p_r)$$

and

$$S(g, \tilde{\delta}, k, p_1, \dots, p_r) = T_{\tilde{g},2r,k,p_1,\dots,p_r} / \mathrm{Mod}_{g,\tilde{\delta},k,p_1,\dots,p_r},$$

where

$$\mathrm{Mod}_{g,\tilde{\delta},k,p_1,\dots,p_r} \subset \mathrm{Mod}_{g,k,1,p_1,\dots,p_r}.$$

According to Theorem 8.8.1, the space $T_{\tilde{g},2r,k,p_1,\dots,p_r}$ is diffeomorphic to $\mathbb{R}^{3g-3+2r}$.
$\square$

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