Optimal premium pricing strategies for nonlife products in competitive insurance markets

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by

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Abstract

Non-life insurance pricing depends on different costs including claim and business acquisition costs, management expenses and other parameters such as margin for fluctuations in claims experience, expected profits etc. Nevertheless, in a competitive insurance market environment, company's premium should respond to changes in the level of premiums being offered by competitors. In this thesis, two major issues are being investigated. Primarily, it is explored how a company's optimal strategy can be determined in a competitive market and secondly a connection between this strategy and market's competition is established. More specifically, two functional equations for the volume of business are proposed. In the first place, the volume of business function is related to the past year's experience, the average premium of the market, the company's premium and a stochastic disturbance. Thus, an optimal premium strategy which maximizes the total expected linear discounted utility of company's wealth over a finite time horizon is defined analytically and endogenously.

In the second place, the volume of business function is enriched with company's reputation, for the first time according to the author's knowledge. Moreover, the premium elasticity and reputation elasticity of the volume of business are taking into consideration. Thus, an optimal premium strategy which maximizes the total expected linear discounted utility of company's wealth over a finite time horizon is calculated and for some special cases analytical solutions are presented. Furthermore, an upper bound or a minimum premium excess strategy is found for a company with positive reputation and positive premium elasticity of the volume of business.

Thirdly, the calculation of a fair premium in a competitive market is discussed. A nonlinear premium-reserve (P-R) model is presented and the premium is derived by minimizing a quadratic performance criterion concerns the present value of the reserve. The reserve is a stochastic equation, which includes an additive random nonlinear function of the state, premium and not necessarily Gaussian noise which is independently distributed in time, provided only that the mean value and the covariance of the random function is zero and a quadratic function of the state, premium and other parameters, respectively. In this quadratic representation of the covariance function, new parameters are implemented and enriched further the previous linear models, such

as the income insurance elasticity of demand, the number of insured and the inflation in addition to the company's reputation. Interestingly, for the very first time, the derived optimal premium in a competitive market environment is also depended on the company's reserve among the other parameters.

In each chapter numerical applications show the applicability of the proposed models and their results are further explained and analyzed.

Finally, suggestions for further research and summary of the conclusions complete the thesis.

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List of publications and presentations

List of publications

1. Pantelous A.A. and Passalidou E. (2013), Optimal premium pricing policy in a competitive insurance market environment, *Annals of Actuarial Science*, Vol 7, No. 2, pp. 175-191.

2. Pantelous A.A and Passalidou E. (2015), Optimal Premium Pricing Strategies for Competitive General Insurance Markets, *Applied Mathematics and Computation*, 259, pp. 858-874.

3. Pantelous A.A and Passalidou E. (2015), Optimal Quadratic Control of a Non-linear Premium-Reserve Model in a Competitive Insurance Market (Submitted, July 2015).

List of Presentations

1. Poster Day online, University of Liverpool, 7 - 29 April 2013.

2. Winter School on Perspectives in Actuarial Risks in Talks of Young Researches, Ascona, Swizerland, 27 Jan - 1 Feb 2013.

3. The 18^{th} International Congress on Insurance: Mathematics and Economics, Shanghai, 10 - 12 July 2014.

4. The 16^{th} Conference of the Applied Stochastic Models and Data Analysis (AS-MDA) International Society, Piraeus, 30 June - 4 July 2015.

Glossary

$\{V_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *volume of business* (or exposure) underwritten by the insurer in year $[k, k+1)$. This volume may be measured in any meaningful unit, e.g. number of claims incurred, total man-hours at risk (for workers' compensation insurance). In our thesis, we consider the number of claims incurred as the volume of exposure.

$\{\pi_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *break-even premium* in year $[k, k+1)$, i.e. risk premium plus expenses per unit exposure.

$\{p_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *premium* charged by the insurer in year $[k, k+1)$. This is our control (decision-making) parameter.

$\{\bar{p}_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *"average" premium charged by the market* in year $[k, k+1)$. We further assume that this process is stochastic, see also [18]. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the probability space and $\{\bar{p}_k | k = 1, 2, ...\}$ be the sequence of random variables defined on this probability space.

$\{w_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *company's wealth* in year $[k, k+1)$.

r: denotes *the rate of return* on equity required by shareholders of the insurer whose strategy is under consideration. We further assume that this rate is deterministic.

v: denotes the *corresponding discount factor*, $v = (1+r)^{-1}$.

$\{\gamma_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *reputation's impact* to the volume of business in year $[k, k+1)$, and $sign(\gamma_k)$ is the sign of this parameter which represents the kind of impact that reputation has on the company's volume of business in year $[k, k+1)$.

$\{\theta_k\}_{k\in\mathbb{N}}$: denotes the sequence of *the set of all other stochastic variables* (which are assumed to be independently distributed in time and not only Gaussian) and it is considered to be relevant to the demand function in year $[k, k+1)$.

$\{\alpha_k\}_{k\in\mathbb{N}}$: denotes the sequence of *the excess return of capital* in year $[k, k+1)$.

$\{R_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *company's reserve* in year $[k, k+1)$.

$\{q_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *present day value factor* of a reserve asset in year $[k, k+1)$.

$\{B_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *income elasticity of demand* concerning insurance contracts in year $[k, k+1)$.

$\{C_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *inflation rate* in year $[k, k+1)$.

$\{M_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *number of insured* in year $[k, k+1)$.

$\{N_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *financial risk of the market* in year $[k, k+1)$.

$\{h_k\}_{k\in\mathbb{N}}$: denotes the sequence of the *consumers' expectations* concerning the premium in year $[k, k+1)$.

Chapter 1

Introduction

1.1 Motivation

A non-life insurance policy is an agreement between an insurance company and a customer - the policyholder - in which the insurance company undertakes to compensate the customer for certain unpredictable losses during a time period usually six months or a year, against a fee, the premium. A non-life insurance policy may cover a damage on a car, house or other property or losses due to bodily injury to the policyholder or another person (third party liability); for a company, the insurance may cover property damages, cost for business interruption or health problems for the employees, and more.

By the insurance contract, economic risk is transferred from the policyholder to the insurer. Due to the law of large numbers, the loss of the insurance company, being the sum of a large number of comparatively small independent losses, is much more predictable than that of an individual (in relative terms): the loss should not be too far from its expected value. This leads us to the generally applied principle that the premium should be based on the expected (average) loss that is transferred from the policyholder to the insurer. There must also be a loading for administration costs, cost of capital, etc.

The need for statistical methods comes from the fact that the expected losses vary between policies: the accident rate is not the same for all policyholders and once a claim has occurred, the expected damages vary between policyholders. Most people would agree that the fire insurance premium for a large villa should be greater than for a small cottage; that a driver who is more accident-prone should pay more for a car insurance; or that installing a fire alarm which reduces the claim frequency should give a discount on the premium. Another crucial factor that a company has to take into account is market's competition.

In most countries in the western world, anti-competitive practices are prevented from competition laws which in turn are ensured by government regulators. In those markets, increasing competition does not permit monopoly profits to be earned and consequently, the gross premium prices for different insurance products are lower, and if anything there is a wider range of products supplied. In contrast to other jurisdictions where competition policy is aimed only at maximizing economic efficiency, the competition policy in the European Union (EU) has another important goal: to facilitate a common integrated market, which is a primary objective of the EU. In this context, competition policy gained a quasi-constitutional status, which affects the relationship between competition and regulation (OECD, 2005)[1]. For the sake of coherence, in this part of the introduction, it should be mentioned that the competition laws in the EU have some similarities with the laws in the United States antitrust; though there are some key differences. Insurance regulation in the EU also has a community goal aside from the usual regulation justifications: the creation of a single European insurance market. One of the main prima facie practice that the European Commission faces in insurance market, which opposes competition, are agreements between insurers concerning the premiums (OECD, 1998)[2]. Therefore, the most relevant Article of the EC Treaty is 81(1)[3].

Until the mid 1980s, it was unclear whether the insurance market is subject to the European competition policy: analogous to the special treatment granted to agriculture and transportation, and relying on some national regulations that partially exempted the insurance industry from the application of the competition policy, insurers argued that the insurance market, due to its special characteristics, should not be subjected to the competition rules [64]. In 1985 Commission clearly stated that insurance industry is subject to the competition law, according to its decision of 5 Dec 1984 [O.J. 1985 L35/20][4]. Under this framework, the insurer's union recommends to its members that they calculate the premiums at certain levels in order to stabilize the market segment. Of course, there are some further exemptions to these competition laws such as the premium calculation exemption according to the Regulation of 1992 [Reg.3932/92][5], which acknowledges the difficulty of an individual insurer to properly assess average risks and the need to have broad statistical databases. The regulation also exempts agreements for joint studies regarding claims frequency and scale.

Today, the market has been deregulated in many countries: the legislation has been

[1] OECD: Relationship between Regulators and Competition Authorities, $http://www.oecd.org/newsroom/34711139.pdf$

[2] OECD: Relationship between Regulators and Competition Authorities, http://www.oecd.org/daf/competition/sectors/1920556.pdf

[3] European Commission, Competition, Financial Services, Insurance $http://ec.europa.eu/competition/legislation/treaties/ec/art81_en.html$

[4] Official Journal of the EC, Commission Decision http://eur-lex.europa.eu/LexUriServ/LexUriServ.do?uri=OJ:L:1985:035:FULL:EN:PDF

[5] Commission Regulation (EEC) No 3932/92 http://eur-lex.europa.eu/LexUriServ/LexUriServ.do?uri=OJ:L:1992:398:0007:0014:EN:PDF

modified to ensure free competition rather than uniform pricing. The idea is that if an insurance company charges too high a premium for some policies, these will be lost to a competitor with a more fair premium. Suppose that a company charges too little for young drivers and too much for old drivers; then they will tend to loose old drivers to competitors while attracting young drivers; this adverse selection will result in economic loss both ways: by loosing profitable and gaining underpriced policies. Therefore, on a competitive market it is advantageous to charge a fair premium, if by fairness we mean that each policyholder, as far as possible, pays a premium that corresponds to the expected losses transferred to the insurance company.

Subsequently, the premium pricing process for non-life products is always a very challenging task in the insurance industry as the actuarial team needs to consider the various characteristics of the insured object, the potential demand from the policy holders, the available information about the competition of the targeted market, company's wealth and the reserve that must be kept. Thus, the main data source on which premium strategy is formulated is not only based on the insurance company's own historical data on policies and claims, but also supplementary information from external sources. At last but not least, it should be emphasized that every company's objective is either to maximize its wealth or to minimize the level of the required reserve.

An usual approach concerning non-life insurance pricing is the use of Generalized Linear Models (GLM). A number of key ratios are dependent on a set of rating factors; see [60]. For personal lines insurance which are designed to be sold in large quantities, the key ratios are often claim frequency and severity (cost per claim), while for commercial lines insurance which are designed for relatively small legal entities, the loss ratio may be also considered (claim costs per earned premium). Rating factors are grouped into classes (i.e. factor variables) and may include information about policyholder, the insured risk as well as geographic and demographic information.

In real world actuarial applications, a premium principle connects the cost of a general insurance policy to the moments of the corresponding claim arrival and severity distributions. Insurers add a loading to this cost price in order to make profit and cover their expenses. After this consideration, two main questions are raised; *"how an optimal premium can be calculated in order to maximize company's wealth or minimize the level of the required reserve?"* and *"how it is possible to find a premium strategy that takes into consideration market's competition and all the different economic parameters that affect company's wealth or reserve except for the cost of a general insurance policy?"*.

1.2 Developments in the Competitive Insurance Markets

It is generally admitted that many lines of insurance are highly competitive and as a result in the real world insurance applications, the loading depends critically on the price that other insurers charge for comparable policies. Clapp [11] demonstrates that insurance firms are able to use the quantity of insurance to compete for customers. By changing the level of indemnity while holding the premium rate constant (quantity competition), it is possible to induce customers to reveal their risk class. In [67], a competitive equilibrium may not exist and when this exists it may have strange properties.

The daily change in the exposure of a non-life insurer increases as policies are sold and decreases as policies are not renewed or canceled. In a highly competitive price-conscious market the insurer's premium relative to the rest of the insurance market is an important factor in policy sales. The size of the insurer as measured by its current exposure is also important, since larger insurers tend to attract greater volume of business than small insurers with comparable premium rates. However, there are many other factors which influence demand: the marketing of the policies, the need for insurance, the reputation of the insurer and the capacity of the insurer to underwrite policies. These factors are too numerous to incorporate into a simple non-life insurance model and they are hard to quantify, yet they all contribute to the uncertainty in how much exposure a given pricing strategy will generate.

There is little insurance literature on modeling how insurance premiums should be determined in a competitive market and how they respond to changes in the levels of premiums being offered by competitor companies.

Taylor [71] mentions that in the Australian insurance market and particularly the liability section of it, has been characterized by violent changes in premium rate. During these fluctuations in premium rates the various operators in the market appeared to act in a similar manner; generally, these individual operators followed the market as its average premium rates declined and then increased.

From the viewpoint of rational product pricing, this cyclical behavior of premium rates seems peculiar and raises questions. For instance *"what the market was attempting to achieve by such pricing"* and *"what individual insurers were attempting to achieve in following the market"*.

Analytically, he explores successfully the relation between the market's behaviour and the optimal response of an individual insurer, whose objective is to maximize the expected present value of the wealth arising over a predefined finite time horizon. He also assumes that the insurance products display a positive price-elasticity of demand. Thus, if the market as a whole begins underwriting at a loss, any attempt by a particular insurer to maintain profitability will result in a reduction of its volume of business.

Following his ideas, for a given sequence of average market prices over fixed years to the planning horizon, the demand function $f_k(.)$ is given by a relation of the following type

$$q_j = f_j(p_j, \overline{p}_j, q_{j-1}, \theta), \tag{1.1}$$

and the objective function requiring maximization is equal to

$$\mathrm{E} = \sum_{j=1}^{J} \upsilon^{j-\frac{1}{2}} q_j \left(p_j - \pi_j\right), \tag{1.2}$$

where q_j denotes the volume of exposure underwritten by the insurer in year j, π_j denotes the break-even premium rate (per unit exposure), p_j denotes the premium rate (per unit exposure) charged by the insurer in year j, $\overline{p}_j$ denotes market's average premium, θ denotes the set of all other variables considered to be relevant to the demand function and υ denotes the corresponding discount factor, $(1+r)^{-1}$. Clearly the previous demand function is far too general for any useful results to be derived. Thus, Taylor restrict it to the following

$$f_j(p_j, \overline{p}_j, q_{j-1}, \theta) = q_{j-1} f(p_j, \overline{p}_j). \tag{1.3}$$

In addition implicit in this restriction of the demand function are several assumptions:

- the demand function is stationary over time, and hence the subscript in f_j has been dropped;
- demand in year j is assumed to be proportional to demand in the preceding year;
- the discarding of the unspecified set of variables amounts effectively to treating the sequence of market rates $\overline{p}_j$, as given, exogenous to the strategy of the insurer under consideration.

The objective function which is maximized after assumptions and calculations is equal to

$$E = \sum_{j=1}^{J} \upsilon^j (p_j - \pi_j) \left[\prod_{k=1}^{j} f\left(p_k, \bar{p}_k\right) \right]. \tag{1.4}$$

In order to find an optimal solution Taylor [71] defines

$$g(p_k) = \log f\left(p_k, \bar{p}_k\right), \tag{1.5}$$

where the function $g^{'}(p)$ maybe related to the price-elasticity of demand.

He comes up to some very interesting results. Firstly, he shows that as the rate of discount of future profits increases, the projected future premium rates involved in the optimal underwriting strategy also increases. Moreover if this rate of discount is sufficiently large, then the optimal strategy will never involve any loss leaders, i.e., cases in which the average premium at which underwriting is conducted is less than the break-even premium. As price elasticity is reduced, the penalties (in loss of volume of exposure) due to high premium rates, or the gains due to low premium rates, are reduced. It is intuitive then that the strategy of optimal profitability will involve higher premium rates.

Moreover some special demand functions are studied. One of them is the negative exponential demand function in which

$$f(p_k, \bar{p}_k) = \exp -a(p_k - \bar{p}_k)/\bar{p}_k, \tag{1.6}$$

for some constant $a > 0$ which corresponds to a price elasticity which increases linearly with price. Secondly, the constant price elasticity is studied where

$$f(p_k, \bar{p}_k) = (p_k/\bar{p}_k)^{-a}, \tag{1.7}$$

for some constant $a > 0$. Finally, a demand function which prohibits loss leaders is investigated. This demand function satisfies the conditions $\left|\left[g'(p)\right]^{-1} \exp g(p)\right| < K$ and $\left|g'(p)\right| < 1/\upsilon K$ for $K > 0$ at least over a limited range of p.

According to his results [71] the optimal strategies do not follow what someone might expect. For instance, it is not the case that profitability is best served by following the market during a period of premium rate depression. In particular, the optimal strategy may well involve underwriting for important profit margins at times when the average market premium rate is well short of breaking even. Therefore, he states that the optimal response depends upon various factors including:

- the predicted time which will elapse before a return of market rates into profitability,
- the price elasticity of demand for the insurance product under consideration and
- the rate of return required on the capital supporting the insurance operation.

In particular, it is seen that the optimal strategy may well involve underwritting for significant profit margins at times when the average market premium rate is well short of breaking even. On the other hand, conditions can arise in which the optimal premium rating strategy will indicate loss leading in the near future. As a very broad generalization, this may be the case when the current average market rate lies below the break-even rate, but is expected to return to substantial profitability in the very near future. Optimal strategies will not involve loss leaders, irrespective of the degree of competition from the market, if the demand function for the insurance product assumes certain forms. These particular forms are not unrealistic. It follows that some market research on the shape of the demand function may assist an insurer seeking to determine a suitable underwriting strategy. In summary, it appears that the longer term profitability of an insurer will only comparatively rarely be best served by underwriting for deliberate losses.

Taylor in his next paper [72] notes that optimum underwriting strategies might be substantially affected by the proper marginal expense rates which must be taken into account. It is first noted that the optimal strategy is not affected by the introduction of a component of fixed expenses, irrespective of the size of that component. However, the strategy will be affected if the concomitant of the introduction of fixed expenses is the recognition of lower marginal expenses. It is possible to set limits on the effect of expenses on optimal underwriting strategy. The sharpness of these limits depends on:

- the extent of variation in marginal expense rates as demand varies;
- the price-elasticity of demand.

The case in which the marginal expense rate is constant and price-elasticity is directly proportional with price is a simple one. In this case, the optimal premium rates taking expenses into account are precisely equal to the optimal premium rates ignoring expenses, increased by the marginal expense rate.

As these two factors depart from this particular case, the behaviour of optimal premium rates with expenses taken properly into account becomes less predictable relative to the optimal rates ignoring expenses. Some empirical results have been examined and it is found that the general shape of the optimal strategy, in terms of the optimal premium rate as a function of time measured between the present and the planning time horizon, is to a large extent unaffected by whether they incorporate proper allowance for (possibly varying) marginal expenses or approximate these by assumed constant unit expenses. The general level of optimal premiums may, however, be shifted to a material extent by the proper recognition of expenses. It is also found that in examples in which the assumption of constant unit expense rates leads to optimal premium rates of substantial negative profitability. the adjustment to reflect marginal expenses properly can cause very significant changes to these low premium rates.

Emms and Haberman [17] extend significantly Taylor's ideas [71, 72] considering the continuous form of his model and more specific

$$dq = qg(p/\bar{p})dt \tag{1.8}$$

and

$$dw = -\alpha w dt + q\left(p - \pi\right)dt, \tag{1.9}$$

where q is the insurer's exposure at time t, the insurer's premium (per unit exposure) is p, the market's average premium (per unit exposure) is $\bar{p}$, the wealth process is w and the mean claim size (per unit exposure) is π. The demand function is $exp(g)$ and α represents the loss of wealth due to returns paid to shareholders. Emms et al. [17] assume that $\bar{p}$ is a positive random process with finite mean at time t and leave the distribution for the mean claim size process π unspecified. With this formulation w is an accurate reflection of the wealth of the company at time t since each policyholder pays a premium pdt per unit exposure for each dt of cover. Consequently there are no outstanding liabilities at the end of the planning horizon T.

The principal assumption of this model is that all new and existing policyholders are required to pay the current premium rate p. The change in wealth at time t due to premium income is denoted by the term $pqdt$ in the wealth equation. Such an assumption is attractive since it means that all the random processes are Markov. The objective function is

$$V = \max_{p} \left\{ \mathbb{E} \left[w\left(T\right) / S\left(0\right) \right] \right\}, \tag{1.10}$$

that is maximising the expected wealth at the end of the planning horizon T given information on the state S at time $t = 0$. The control variable is k and the state variable is the exposure q which is governed by

$$\dot{q} = qg\left(k\right). \tag{1.11}$$

For both the constant elasticity and exponential demand functions of g is a decreasing function of k. Their model is modified in a number of ways. Firstly, they suppose that the loss ratio $\gamma = \pi/\bar{p}$ is constant. If the break-even premium rate is constant this complicates the behavior of the optimal control. Specifically, if the market average premium drifts above breakeven the optimal control is necessarily a loss leader. However, one would expect the main reason for greater premiums is that claims are higher so that there is a direct correlation between the market average premium and the expected mean claim size (or breakeven premium rate). Secondly, they generalise the deterministic premium strategy to be of the form $p/\bar{p} = k\left(t\right)$. In an unconstrained model they find that the optimal control $k(t)$ is bang-bang. This is a direct consequence of the assumption that the insurer can force existing customers to pay the current premium rate. The optimal control strongly depends on how much the insurer can raise the premium rate during the course of a policy. They are led to a modification of the model which fixes the premium rate at the start of a policyholder's contract. For two choices of the demand function a smooth optimal control is calculated. They find that withdrawal from the market, setting a premium above break-even or loss-leading can be optimal and that the qualitative form of optimal premium strategy is sensitive to the form of the demand function. A loss-leading premium strategy is optimal for a linear demand function when the loss ratio is sufficiently small or the mean contract length is sufficiently large. If one adopts a parameterisation which increases the demand for insurance with a high relative premium then this leads to an unsmooth optimal control with a high terminal premium rate.

The premium strategy of loss-leading followed by profit-taking is one possible cause of the observed actuarial cycle. Many insurance companies prohibit loss-leading which imposes a restriction on the premium charged to policyholders. However, using optimal control theory the requirement becomes a constraint on the relative premium and may lead to a non-smooth control. Deterministic premium strategies can be investigated numerically for a variety of constrains including those which involve the state of the insurer. Moreover the authors compare the optimal deterministic strategies for linear demand function with the dynamic premium strategy predicted by Bellman equation. If the market average premium rate is modelled as a log-normal process they find that the deterministic premium strategy and dynamic premium are of the same form.

Emms et al. [18] model market's average premium as a geometric Brownian motion

$$\frac{d\bar{p}}{\bar{p}} = \mu dt + \sigma dZ, \tag{1.12}$$

where Z is a Wiener process and the drift μ and the volatility σ are assumed to be constant. The future market average premium is lognormally distributed (and hence positive) ie.

$$\log \bar{p}\,(t) \sim N\left(\log \bar{p}_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t, \sigma^2 t\right). \tag{1.13}$$

They define the demand process by

$$q_{k+1} = f\left(p_{k+1}, \bar{p}_{k+1}\right) q_k. \tag{1.14}$$

Therefore, the demand process is described by

$$\frac{dq}{q} = \log f\left(p, \bar{p}\right) dt, \tag{1.15}$$

where $p := p\left(\bar{p}, \pi, t\right)$ is the premium at time t. The wealth follows the stochastic process given by

$$dw = -\alpha w dt + q(p - \pi)dt, \tag{1.16}$$

where α is the excess return on capital (i.e. return on capital risk free rate) required by the shareholders of the insurer whose strategy is under consideration. Thus, is the cost of holding w in a small time interval dt. The aim is, for a given utility function of wealth $U(w,t)$ to define the value function

$$V := \max\left\{J = \mathbb{E}\int_0^T U\left(w(s),s\right)ds\right\}, \tag{1.17}$$

as the maximisation of the objective function J over a choice of strategies p. This is similar to the objective function used by Taylor [71] with profit replaced by total wealth at time T. Moreover, the authors consider two different choices of demand functions; the exponential demand function where

$$f(p,\bar{p}) = \exp\left[\frac{-a(p-\bar{p})}{\bar{p}}\right], \tag{1.18}$$

for some constant $\alpha > 0$ and the constant elasticity demand function where

$$f(p,\bar{p}) = \left(\frac{p}{\bar{p}}\right)^{-a}, \tag{1.19}$$

and the utility function is equal to

$$U(w,t) = e^{-\beta t}w, \tag{1.20}$$

where β is the intertemporal discount rate.

They study two premium strategies. The first one is to set a premium of the form

$$p(t) = k\bar{p}(t), \tag{1.21}$$

where k is constant. Thus, they assume that the premium set by the insurer is a linear function of the market average premium and calculate the objective function for a range of values of k. In this premium strategy and under the condition that $\alpha = \mu = 0$ the approximate optimal strategy is $k^* = \gamma = \frac{\pi}{\bar{p}(0)}$. Consequently the optimal premium strategy has two modes depending on the model parameters: either set an infinite premium and accumulate wealth from the existing customer base or set the premium at just above breakeven in order to maximize market exposure whilst at the same time making a profit. The existence of a finite optimal premium strategy for $\gamma < 1$ arises from two competing forces: the desire to set as low a premium as possible in order to gain new business, and the requirement to generate a profit by setting a high premium. In reality, an infinite premium rate will correspond to not selling insurance at all since no-one will buy insurance at such a price. In this premium strategy and under the condition that $\alpha = 0$ and $\mu \neq 0$ the optimal strategy is to keep the expected premium value near the break-even premium value at $t = T$.

The second premium strategy is setting a premium policy p as a function of the break-even premium π and the difference of the market average premium $\bar{p}$ and the break-even premium :

$$p(t) = \pi + r(\bar{p}(t) - \pi), \tag{1.22}$$

and hense they optimise over the single parameter r.

For $\mu = \sigma = 0$ there are two models for the optimal strategy $p = \infty$ or $r^* \approx 0$. For $\alpha = \beta = 0$ the optimal strategy is $p^* = \pi + \varepsilon\bar{p}\,(0)$. For $\alpha = \beta = \sigma = 0$ as the drift of the market average premium increases the optimal value r^* decreases. This behaviour is consistent with the idea that the optimal strategy is to aim for a large terminal exposure rather than a large profit per policyholder. By keeping the terminal premium relatively small, then $\frac{p}{\bar{p}}$ is small and so the exposure is large. As ε becomes very small then the optimal premium strategy is to set a premium just above the break-even premium. Moreover the variation of r^* is no longer linear with γ but increases only gradually.

Consequently, they investigate optimal strategies for two particular approaches to fixing the premium. The first approach is based on a linear function of the market average premium, while the second involves a linear combination of the break-even premium and the market average premium.

The qualitative behaviour of the optimal strategy in the first case is determined analytically. Thus, if the market average premium is drift-less, they demonstrate that there are two optimal strategy modes: setting an infinite premium rate when the initial market average premium rate is below the break-even premium or setting the premium rate a fraction above the break-even premium when the market is underwriting at a profit. If the market average premium has upward drift then there are again two optimal strategies: an infinite premium rate or a loss-leading strategy which makes an initial loss but gains market exposure. If the market average premium has negative drift then a non-infinite optimal strategy can exist whereby the insurer sets a premium just above the market mean. This can generate enough initial wealth to offset the loss as the market average premium drifts below break-even. The important parameters which determine the optimal strategy is:

- γ the ratio of initial market average premium to break-even premium,
- ϵ a measure of the inverse elasticity of the demand function, and
- ν the nondimensional drift of the market average premium.

The optimal form of the strategy in the second case is similar except that the drift of the market average premium does not have such a pronounced effect on the optimal strategy. Loss leading is much less likely with this form of strategy. The second strategy is also affected by the volatility of the market average premium. However, the qualitative form of the optimal strategy remains the same. As the volatility of the market average premium increases so does the wealth generated by choosing an optimal strategy.

Emms [19] determines the optimal strategy for an insurer which maximizes a particular objective over a fixed planning horizon and the premium by using a competitive demand model as well as the expected main claim size. Consequently, he supposes that the premium is partially determined by the price relative to the rest of the insurance market. According to this paper, it is not enough that an insurer set a price to cover claims if the rest of the market undercuts that price. Additionally, the demand law specifies how the insurer's income and exposure change with the relative to market premium; a low relative premium generates exposure but leads to reduced premium income.

A two factor model of the general insurance market is proposed; one factor models the randomness of the claim size and intensity, whilst the other models the market average premium. This model is used to to determine the premium which maximizes a number of possible objective functions of the insurer. In his paper the derivation of the Bellman equation is described, which gives the optimal dynamic premium strategy to maximize the expected terminal wealth of the insurer. The problem reduces to the solution of a reaction-diffusion equation, which is straightforward to solve numerically. In addition, he considers the objective of maximizing the expected total discounted utility of wealth with a utility function linear in wealth which is assumed to be a more realistic objective for the insurer given the regulatory constrains imposed over the course of the planning horizon. The resulting reaction-diffusion equation is more complex but can be solved straightforward.

In [19] the author introduces the relative loss ratio which is defined as the ratio between the breakeven premium of the insurer and the markets average premium $\gamma_t = \frac{\pi_t}{\bar{p}_t}$. The breakeven premium is the random amount a policy of length τ costs the insurance company. This is the actuarial premium without any profit margin and can be deduced form the insurer's previous claims data and a loading factor to account for expenses and interest rates [14]. In this paper, he supposes that the breakeven premium is a stochastic process. The advantage of this formulation is that there are no outstanding liabilities at the end of the planning horizon T because as soon as policies of total exposure δq_t are bought, the insurer sets aside $\pi_t \delta q_t$ to cover the resulting claims.

In a few words, the optimal premium strategy for an insurer in a competitive market using optimal control theory is found. In general, the Bellman equation arising from control theory contains a degenerate diffusion operator. For that model the author shows how this degeneracy can be removed by a change of variables, which makes the resulting problems easy to solve numerically. The choice of a linear demand function (in the relative premium) leads to a single non linear term in the Bellman equation which considerably simplifies the analysis. As a result, the ability to find the optimal premium strategy is limited by the values given to the model parameters. In general, according to this study, if the optimal control is smooth then the optimal premium gradually increases over the term of the planning horizon for the parameters sets considered there. In addition, as risk aversion is increased so does the optimal relative premium strategy. This generates lower exposure and so ultimately lower overall wealth.

One significant assumption is that the market is treated distinctly from the insurer so that whatever the insurer's premium, the market does not react with a competitive price. Another modeling specification is that of the stochastic process for the breakeven premium which represents the cost of insurance for the insurer and assessment of this quantity requires a good model for the claim process and an accurate definition of the loading factor. The benefit of using this process to define a loss-ratio is that the wealth process directly reflects the current wealth of the insurer including liabilities.

Emms again [20] studies optimal premium pricing into a competitive market with constrains. Analytically, he calculates the premium strategy which maximises the objective of the insurer subject to a constrain on the control or constrains on the reserve that the insurer must hold. Since the model is very simple an analytical solution can be found if the relative premium is bounded. Depending on the parameter values of the model this can lead to a non-smooth control. Specifically, a *"Type 1"* control represents a loss-leading strategy and the greater the loss-leading, the more likely the insurer exceeds its lower bound on the relative premium. It is shown that the premium strategy $k_t = k(t)$ is the optimal relative premium if the mean claim rate process is lognormal. For other distributions of the mean claim rate process the feedback control depends on the current value of the state variables and so it is a stochastic process. If there are no constrains then the theory in Fleming and Rishel [26] for stochastic optimisation problems can be employed.

When the insurer constrains the premium strategy, the optimal control can be non-smooth. This makes it much more difficult to obtain stochastic optimal premium strategies from the HJB equation because that equations are expected to have non-smooth solutions. Consequently, he restricts the feasible controls to be deterministic, which turns the problem into a deterministic optimisation problem even though the actual premium charged is stochastic. The resulting optimisation problem is been demostrated to be readily solved using control parameterisation. This is a general technique and allows the insurer to calculate optimal strategies for any reasonable objective or demands functions. It also permits the imposition of an arbitary number of constrains without substantially increasing the computational time.

Premium restrictions lead to control constrains, while solvency requirements lead to state constrains. A control constrain can be used to prevent negative optimal premium values. The numerical problems show that the state constraints limit the ammount of loss-leading that the insurer may experience with an optimal premium strategy. Further studies are concentrated on relaxing some of the assumptions of the model. Specifically, author parameterises the delay in the exposure equation. If one forgoes this assumption then the state equations become a system of stochastic delay differential equations. By assuming a deterministic control the optimisation problem can again be solved by control parameterisation [73]. However, in this case is needed to specify the initial curves for the state variables in order to accommodate the delay in state.

The author calculates the unconstrained optimal premium strategy numerically form the HJB equation. It is found that the optimal strategy is only weekly dependent on the volatility of the market average premium. This is because the control does not directly scale the Brownian motion in the state equations. One can view the stochastic pricing problem as a perturbed version of the deterministic model. Consequently, it is expected that the constrained stochastic model to have a similar optimal premium strategy to the constrained deterministic model.

Emms [23] introduces a simple parameterisation which represents the insurance market's response to an insurer adopting a pricing strategy determined via optimal control theory and claims are modeled using a lognormally distributed mean claim size rate. Analytically, a generalisation of the demand function which is mentioned in [20] is taken place which impacts significantly on the optimal premium strategy for an insurer. If there is no reaction in the market, then they find an analytical expression for the optimal relative premium, and if there is no insurance claims, then the optimal relative premium is zero, since there is no need for insurance. Even though the optimal premium strategy is given explicitly, it is not immediately apparent from the analytical solution how the demand function affects the optimal strategy. Consequently, he introduces a set of parameters and considered the deviation of the optimal strategy corresponding to changes in the parameter set. As the sensitivity of the market to the value of insurance is decreased, demand for insurance increases, and the optimal strategy can lead to negative premium values if the markets overprice insurance.

If the market reacts to an insurer who uses optimal control theory in order to calculate premium values, then only a numerical solution can be found for the optimal control problem, with an exit set determined as part of the optimisation problem. In addition, the numerical solution is not entirely straightforward, because the state space is separated into two regions: one where it is optimal for the insurer to leave the market, and the other where the Bellman equation yields the optimal premium strategy. The author fixes the boundary of these two regions by introducing a front-fixing coordinate transformation, which makes the Bellman equation more complicated.

Three numerical schemes are implemented, and they agree on the computed value function as the mesh is refined. This according to the author is indicative that they have a robust solution to the Bellman equation, and that the feedback law does yield an optimal control under parametric restrictions. The more implicit the numerical scheme, the more computational effort is required to solve the numerical problem. However, the fully implicit scheme allows a larger time step without introducing numerical stability. For these problems, the reduction in time step required by the explicit scheme is more than compensated for by its faster overall execution time.

If one changes the parameters far from the base set, all the numerical schemes show numerical instability. It is clear from the analytical solution that the value function is singular as the sensitivity of the demand function to the market loading is decreased, which means that infinite wealth can be generated. With market reaction, the demand function needs to be changed in order to prevent collusion. According to the author, the numerical instability indicates that the value function is singular over part of the domain.

If the insurer sets its premium sufficiently below the market average, then its exposure grows exponentially, and this growth continues indefinitely. In reality, there is a lower-bound on the insurer's reserve and a finite market for insurance policies. The constrained stochastic optimisation problem is formidable, since it is likely that there are no smooth solutions to the Bellman equation. Calculation of the maximum in the Bellman equation at each time step might lead to a more robust numerical scheme for the constrained problem. The exponential growth in exposure can be removed by introducing a saturation exposure and this may also decrease the numerical sensitivity of the optimisation problem.

Emms and Haberman [22] describe a general determinist model for pricing general insurance using optimal control theory. The theory encompasses different parametrization of the demand for policies and different objectives for the insurer. Any model tackled via control theory becomes more difficult to analyze as one increases the number of state variables to accurately model the underlying process. They focus on how the optimization problem is simplified as the assumptions of the model are changed.

The simplest problem, that of an insurer in an infinite market with a terminal wealth objective requires only backwards integration of the adjoin variable of the exposure. This has an explicit analytical solution if the price function is linear. They also find an implicit analytical solution for a non linear price function, although for this parametrization there is no cutoff in relative premium beyond which there is no demand for insurance. Thus, it becomes difficult to classify the optimal strategy because it is always optimal to sell insurance policies.

When the market is finite, the simplest optimization problem becomes a boundary value problem, where the exposure is integrated forwards in time, and simultaneously the adjoint of the exposure is integrated backwards. No analytical solutions have been found in this case. However, by analyzing the phase diagram of the state/adjoint system, they explore the optimal strategies for the insurer and find that premium strategies vary according to the equilibrium point(s) in the phase diagram, and that these points are always unstable saddle points over the parameter set of interest. The type of the optimal strategy can be classified according to in which quadrant of the saddle the insurer lies as given by its initial exposure and the position of the equilibrium point. For example, one quadrant corresponds to a loss-leading strategy where it is optimal to set an increasing premium and build up exposure if the insurer is particularly small. For the terminal wealth problem, there is an explicit expression for the position of the equilibrium point.

The demand function is the parameterization that most affects the optimal premium strategy. They are certain restrictions on the form of the demand function: most notably we require $gg^{''} < 2g^{'2}$, where g is the price function, in order that the first-order condition of the Hamiltonian gives a maximum. This is analogous result to that given by Taylor [71] and Kalish [42].

In an infinite market, the optimal premium strategy for the total wealth objective depends on the current size of the insurer if the utility function is nonlinear. The nonlinearity of the concave utility function means that low wealth is relative more favorable over high wealth, and this affects the premium strategy of a relative large insurer where the insurer is close to its saturation exposure. If the demand function is concave indicating lower demand for a given relative premium ratio, then that favors market withdrawal over a loss-leading strategy. Similarly, convex demand functions push the equilibrium point in the phase diagram toward the region of withdrawal so that loss-leading is favored.

1.3 Extending the existing literature - The new approaches

Taylor [71, 72] and Emms et al. [17, 18] study fixed premium strategies and the sensitivity of the model to its parameters involved. In their approach, the important parameters which determined the optimal strategies are the ratio of initial market average premium to break-even premium, the measure of the inverse elasticity of the demand function and the non-dimensional drift of the market average premium.

In chapter 2 we introduce a stochastic demand function for the volume of business of an insurance company into a discrete-time extending further Taylor's ideas [71, 72]. Additionally, using a linear discounted function for the wealth process of the company, as [18] have considered, we provide an analytical (endogenous) formula for the optimal premium strategy of the insurance company when it is expected to lose part of the market. Mathematically speaking, we create a maximization problem for the wealth process of a company, which has been solved using stochastic dynamic programming. Thus, the optimal controller (i.e. the premium) is defined endogenously by the market as the company struggles to increase its volume of business into a competitive environment with the same characteristics as Taylor [71, 72], Emms and Haberman [17], and Emms et al. [17, 18] have used. Finally, we consider three different strategies for the average premium of the market, and the optimal premium policy is derived and fully investigated. The results of this chapter are further evaluated by using data from the Greek Automobile Insurance Industry. Analytically, in sections 2.3 and 2.4 a discrete-time model for the insurance market is constructed. We discuss appropriate values for the model parameters and adopt suitable parameterizations. The next section 2.5 considers each strategy in turn and presents numerical applications: we find analytical forms for the optimal strategies. In Premium Strategy I, the average premium of the market is calculated considering all the competitors of the market, and their proportions regarding the volume of business. In Premium Strategy II, the average premium of market is calculated considering the top 5 competitors of the market. Finally, in Premium Strategy III the average premium of the market is calculated considering company's direct competitors.

In chapter 3, the volume of business is formed to be a general stochastic demand function extending further chapter's 2 suggestions making the model more pragmatic and realistic. Thus, here for the very first time according to the author's knowledge, for the formulation of the volume of business, the company's reputation is also considered. According to [12], company's reputation (or corporate reputation) has a strong influence on buying decisions or in other words, on the demand of the company's product. So, in our case the function for the volume of business emphasizes the ratio of the markets average to the company's premium, the past year experience, the company's reputation and a stochastic disturbance. Additionally, following the existing literature, the same linear discounted function for the wealth process of the company is used, see also [18] and chapter 2. Moreover, the optimal premium can be calculated for either negative or positive affection of the company's reputation; something that was not possible with the previous model, see chapter 2.

As in [18], [71, 72] the volume of business is directly connected to the company's product demand function. Market's demand for an insurance product is the relationship between the product's price and the product demanded by all customers. In our case, the volume of business function is derived from the equilibrium points of the competitive market (perfect competition) which are defined by the intersection of the demand and supply curves. These points determine the contracts of general insurance that purchased, and its' prices, which are equal to the marginal cost of services. Generally, an approach to competitive behavior examines the revenue and cost structure of companies, using the framework of perfect competition as the reference position. Insurance firms operating under conditions of perfect competition are unable to absorb any of the cost increase. They are forced to pass on the entire rise of input costs on output prices and revenue, leaving output unaffected. By contrast, under monopolistic conditions in equilibrium, a rise in input prices, such as wages or administrative costs, results in a reduction in output and a rise in price by a smaller amount than increase in costs, leading to shrinking of total revenue. Thus, marginally profitable firms may have to leave the industry. Moreover the insurance contracts are homogenous, customers face no quality differences, the transaction costs are zero and there are perfect information of customers and insurance companies.

In addition in chapter 3 we present the analytical solutions for some common special cases and a premium strategy concerning market's average premium. Moreover we present the stability conditions for the wealth function and for the optimal strategy and an application.

In detail, the third chapter is organized as follows: In sections 3.3 and 3.4 a discrete-time model for the insurance market is constructed. We discuss appropriate values for the model's parameters and adopt suitable parameterizations. Moreover, in 3.4 section the calculation of the optimal premium and the two main theorems are presented, both when the expected utility is being maximized and minimized. Therefore, for some special cases analytical forms for the optimal strategy are appeared. Section 3.5 considers an application data's presentation and analysis and a premium strategy regarding market's average premium.

In chapter 4 we focus on finding an optimal premium which minimizes company's reserve. The premium-reserve (P-R) process for non-life products is always a very challenging task in the insurance industry as the actuarial team needs to consider the various characteristics of the insured object, the potential demand from the policy holders, the available information about the competition of the targeted market and the reserve that must be kept. Thus, the main data source on which premium strategy is formulated is not only based on the insurance company's own historical data on policies and claims, but also supplementary information from external sources. At last but not least, it should be emphasised that every company's objective is to minimize the level of the required reserve. Consequently, the main challenge that a company faces is to set a fair premium that comes up from a reserve minimization procedure which takes into account different actuarial and financial parameters as well as the market's competition.

In this thesis the disturbance of the volume of business function denotes the set of all other stochastic variables that are considered to be relevant to the demand function (moreover, they are assumed to be independently distributed in time and Gaussian). However, this significant function should also be consisted by many other micro-macro economic factors that affect the company's volume of business and consequently, the optimal premium strategy. Thus, a more thoughtful analysis of this real world insurance problem demands that the volume of business to be modelled as a nonlinear function with respect to reserve, the premium, the noise and a quadratic performance criterion concerning the utility function to be implemented. Indeed, there are quite a few examples that nonlinear analysis to model different insurance's applications is required, see for instance [47, 48] and [25].

In this part of the introduction, let us continue with some arguments about the choice of a quadratic minimization problem. Indeed, quadratic forms are the next simplest functions after linear ones. Like linear functions, they have a matrix representation, so that studying quadratic forms reduces to studying symmetric matrices. Additionally, the second order condition that distinguish maxima from minima in economic optimization problems are stated in terms of a quadratic form. It should be mentioned that several well known economic problems are modelled using quadratic objective functions, such as the risk minimization problems in finance, where riskiness is measured by the (quadratic) variance of the returns from investments etc.; see [70, 69]. Concerning insurance's application, Lai [52] uses a quadratic utility function to find the sufficient conditions on the insurance premium and deductible to increase the production for a risk-averse firm.

Giving another dimension to the models presented in chapters 2 and 3, in chapter 4 the volume of business in year k is not only proportional to the ratio of the market's average and company's premium, but it is also related to a function of the form $f_k(R_k, \tilde{p}_k, \theta_k)$, where $F_k(R_k, \tilde{p}_k) \triangleq \mathbb{E}[f_k^2(R_k, \tilde{p}_k, \theta_k)]$. As it will be clearer in Chapter 4, the function $F_k(R_k, \tilde{p}_k)$ consists of micro-macro economic parameters which, are implemented in a competitive P-R model. These are the income insurance elasticity of demand, the numbers of insured and the inflation in addition to the fame of company. Since, it is not straightforward to define completely the function $f_k(R_k, \tilde{p}_k, \theta_k)$ because of its stochastic property, a rational approach is given by the function $F_k(R_k, \tilde{p}_k)$.

Thus, the main contribution of this chapter can be highlighted on the following key points. First, an optimal quadratic control model for the determination of the P-R strategy is developed as a minimization problem in a nonlinear framework for the very first time according to the authors' knowledge. In this approach, the present value of the company's reserve is required to be close to zero. Second, the stochastic function $f_k(R_k, \tilde{p}_k, \theta_k)$ that affects the company's reserve is analysed considering different micro-macro economic parameters, which directly or indirectly affect the optimal premium. Finally, as in [61, 62], the insurance premium is given dynamically and includes a good number of interesting and very informative parameters about the competition of the market.

Chapter 4 is organized as follows: In section 4.3 , a nonlinear model in discrete-time for the P-R strategy of an insurance market is constructed. The utility and the reserve functions are discussed and the main model's assumptions as well as their necessary economic interpretation are provided. In Section 4.4, the calculation of the optimal premium is derived which is presented using two Theorems. Additionally, in this section, some special cases of the function $f_k(R_{k,}\tilde{p}_k, \theta_k)$ are presented. The discussion of the main results is given in Section 4.5. Finally, Section 4.6 presents a numerical application to illustrate further the theoretical findings of the chapter.

In chapter 5 some recommendations concerning further research are presented. Analytically, section 5.1 presents a discussion concerning models' assumptions and the volume of business function and section 5.2 refers to the utility function. Finally, section 5.3 refers to modeling company's wealth function either introducing a wealth function which included risk investment or construct a wealth function which connects directly company's wealth and claims.

In chapter 6 the main conclusions of each chapter are presented and complete the thesis.

Chapter 2

Optimal Premium Strategy in a Competitive Market

2.1 Motivation

Nowadays, the number of products from different insurance companies has been significantly increased because of several micro and macro economical challenges, of the strong market competition and of the boosting securitization needs of the new era after the last (global) financial crisis. However, there is still little literature available in actuarial science on modelling how insurance premiums should be determined in competitive market environments, and how the competition actually affects the determination of the companys premiums; see for further discussion Daykin et al. [15] and Emms et al. [18].

It is well-known in the insurance industry that the fair pricing process for non-life products is a crucial issue for every General Insurance company, especially within the unfolding of the timebound detariffing road map by Insurance Regulatory and Development Authority (IRDA) which is once again under a great concern and publicity; see the recent article in Insurance Chronicle, Ramana [63]. Consequently, the failure of a uniform and global price in any Insurance Market, which can be based only on the premium rates, the policy terms and the conditions applicable to a particular portfolio of risks, force the insurance companies to provide more competitive prices. Especially, nowadays because of the global financial crisis, the premium strategy must be determined more accurately and competitively in order to ensure the viability of each company and to increase the volume of business in a long-term.

Inevitably, several questions can arise. For instance, in this chapter, we would like to mention just a few of them: *"What is the optimal premium strategy for an individual* *insurance company and for a specific portfolio of homogeneous or/and heterogeneous risks?"*; *"how is this related to the competitive market?"; "how does the volume of business affect the premium strategy?"* are only some of the questions that can be stated, and with non-trivial or straightforward answers.

2.2 New approach

In chapter 2, we introduce a stochastic demand function for the volume of business of an insurance company into a discrete-time extending further Taylor's ideas [71, 72]. Moreover, using a linear discounted function for the wealth process of the company, as Emms et al. [18] have considered, we provide an analytical (endogenous) formula for the optimal premium strategy of the insurance company when it is expected to lose part of the market. Mathematically speaking, we create a maximization problem for the wealth process of a company, which has been solved using stochastic dynamic programming. Thus, the optimal controller (i.e. the premium) is defined endogenously by the market as the company struggles to increase its volume of business into a competitive environment with the same characteristics as Taylor [71, 72], Emms and Haberman[17] and Emms et al.[18] have used. Therefore, we consider three different strategies for the average premium of the market, and the optimal premium policy is derived and fully investigated. The results of this chapter are further evaluated by using data from the Greek Automobile Insurance Industry. In section 2.3 a discrete-time model for the insurance market is formulated. We discuss appropriate values for the model parameters and adopt suitable parameterizations. In section 2.4 the calculation of the optimal prmium is presented. The next section 2.5 considers a numerical calculation and each strategy in turn: we find analytical forms for the optimal strategies. In Premium Strategy I, the average premium of the market is calculated considering all the competitors of the market, and their proportions regarding the volume of business. In Premium Strategy II, the average premium of market is calculated considering the top 5 competitors of the market and in Premium Strategy III considering company's direct competitors.

2.3 Model Formulation

In this paper, as in Taylor [71], and Emms et al. [18], we make the following assumptions.

- **Assumption 1:**There is positive price-elasticity of demand, i.e. if the market as a whole begins underwriting at a loss, any attempt by a particular insurer to maintain profitability will result in a reduction of his volume of business.

- **Assumption 2:** There is a finite time horizon.
- **Assumption 3:** Demand in year $k+1$ is assumed to be proportional to demand in the preceding year k.
- **Assumption 4:** θ_k affects the volume of business in a linear way (i.e. additive noise).

Additionally, extending Taylor's assumptions [71, 72], we assume that the demand function is stochastic (because of θ_k and $\bar{p}_k$).

Here, we denote the wealth process w_k as the insurer's capital at time $[k, k+1)$, following Emms et al. [18] ideas, so we obtain

$$w_{k+1} = -\alpha_k w_k + (p_k - \pi_k)V_k, \tag{2.1}$$

where α_k denotes the excess return on capital (i.e. return on capital required by the shareholders of the insurer whose strategy is under consideration). Thus, $-\alpha_k w_k$ is the cost of holding w_k in the time interval $[k, k+1)$.

Following Taylor [71, 72], the volume of business of an insurance company for a given sequence of average market prices over the k^{th} year is given by a relation of the following type

$$V_k = f_k\left(V_{k-1}, p_k, \bar{p}_k, \theta_k\right), \tag{2.2}$$

where p_k is the controller and θ_k denotes the set of all other random variables (disturbances) which are considered to be relevant to the demand function. Under this assumption V_k is stochastic and depends on k.

Our aim is to determine the strategy which maximizes the expected total utility of the wealth at time k over a finite time horizon T. As it has been also considered by Emms et al. [18] , we use a linear discounted function (of wealth).

Analytically, we want to maximize

$$\max_{p_k} \mathbb{E}\left[\sum_{k=0}^{T} U(w_k, k)\right], \tag{2.3}$$

where $U(w_k, k) = \upsilon^k w_k$ is the present value of the wealth w_k. Consequently, substituting (2.2) into (2.1), the wealth process w_k is given by (2.4)

$$w_{k+1} = -a_k w_k + (p_k - \pi_k) f_k(V_{k-1}, p_k, \bar{p}_k, \theta_k), \tag{2.4}$$

and $w_0,\ V_0, V_{-1}$ (the volume of business now and for the previous year) $a_0,\ p_0,\ \pi_0,\ \bar{p}_0$ and θ_0 are the initial conditions.

Extending Taylor's ideas [71, 72], who assumed that the volume of business in year $k+1$ is proportional to the demand of the preceding year, in this chapter we propose that the volume of business is proportional to the average premium charged by the market (see Assumption 3), but reverse proportional to the premium rate charged by the insurer in year k. Empirically speaking, this new approach might be considered a little more realistic, since it is true that whenever the average premium stays unchanged and the premium charged by the insurer increases, unavoidably the company's volume of business might decrease. On the other hand, whenever the premium calculated by the insurer stays unchanged and the average premium decreases, the volume of business might decrease as well. These thoughts lead to the assumption that the volume of business should be proportional to the rate $\frac{\bar{p}_k}{p_k}$.

Additionally, it is more realistic to assume that there might be an unexpected set of parameters, which can modify (i.e. decrease or increase) the volume of business. Consequently, we can assume that this set of parameters can be modelled using the stochastic variable θ_k, which can take either *positive* or *negative* values. In this chapter, since we are more interested in investigating the premium strategy of an insurance company when it is expected to lose part of the market, we assume that the expected values of θ_k is positive (i.e. $\mathbb{E}(\theta_k) > \mu$, where $\mu > 0$ is a deductible parameter which can be predefined by the managerial team)and then the volume of business is strictly decreasing, i.e. loosing part of the competitive market. Obviously, within the next lines, the case $\mathbb{E}(\theta_k) < \mu$ is also discussed, however this case is not very interested since it implies that the insurance company is increasing gradually its volume, and any change in its premium policy might affect it negatively.

Consequently, we can assume that the volume of business is given by

$$V_k = V_{k-1}\frac{\bar{p}_k}{p_k} - \theta_k. \tag{2.5}$$

2.4 Calculation of the Optimal Premium

After the basic notations, and the mathematical formulation of the problem, we need to calculate the optimal premium, which maximize the expected total utility of the wealth (2.3).

Following the general ideas about stochastic dynamic programming and control theory into a discrete-time framework, see for instance the classical book by Kushner [49] and Bertzekas [4], we determine the strategy which maximises the expected total utility of wealth (2.3) over a finite time horizon, and over a choice of strategies. This is similar to the objective function used by Taylor [71, 72], and Emms et al. [18].

The next Theorem provides us with the optimal premium strategy for the finite time horizon maximization problem (2.3)-(2.5), see also Jacobson [40] and Kushner

[49]

Theorem 1. *For the wealth process* $\{w_k\}_{k\in 0,1,\ldots\ldots,T-1}$ *given by*

$$w_{k+1} = -a_k w_k + (p_k - \pi_k)\left(V_{k-1}\frac{\bar{p}_k}{p_k} - \theta_k\right), \tag{2.6}$$

where $\mathbb{E}(\theta_k) > \mu > 0$*, and for the maximization problem defined by*

$$\max_{p_k} \mathbb{E}_{/w_k}\left[\sum_{i=k}^{T-1} \upsilon^i w_i\right], \tag{2.7}$$

with initial conditions w_0, V_0, a_0, p_0, π_0, $\bar{p}_0$ *and* θ_0*, the optimal strategy process* p_k^* *is given by*

$$p_k^* = \left(\frac{1}{\mathbb{E}(\theta_k)}\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)\right)^{1/2} for k \in \mathbb{N}, \tag{2.8}$$

where $\bar{p}_k$, π_k *is the "average" and the break-even premium respectively, in year* k*;* V_{k-1} *is the volume of exposure underwritten by the insurer in year* $k-1$*, and* $\mathbb{E}(\theta_k)$ *is the expectation of the (stochastic) disturbance* θ_k *in year* k *and the maximum value of (2.7) is given by*

$$w_0 d_0 + e_0, \tag{2.9}$$

$$d_k = a_k d_{k+1} - \upsilon^k \text{, and } d_T = 0, \tag{2.10}$$

$$\begin{aligned} e_k = &- d_{k+1}\left(\left(\frac{1}{\mathbb{E}(\theta_k)}\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)\right)^{1/2}\mathbb{E}(\theta_k) - V_{k-1}\mathbb{E}(\bar{p}_k)\right) + \\ &+ d_{k+1}\pi_k\left(\mathbb{E}(\theta_k) - V_{k-1}\mathbb{E}(\bar{p}_k)\left(\frac{1}{\mathbb{E}(\theta_k)}\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)\right)^{-1/2}\right) + e_{k+1,}, \text{and } e_T = 0. \end{aligned} \tag{2.11}$$

Proof. Define

$$J_k(w_k) \triangleq \max_{p_k, p_{k+1}, \ldots\ldots, p_{T-1}} \mathbb{E}_{/w_k}\left[\sum_{i=k}^{T-1} \upsilon^i w_i\right]. \tag{2.12}$$

Then, as it is known [40] the optimal performance criterion satisfied the Bellman equation

$$J_k(w_k) = \triangleq \max_{p_k} \mathbb{E}_{/w_k}\left\{\upsilon^k w_k + J_{k+1}(w_k+1)\right\}$$
$$= \max_{p_k}\left\{\upsilon^k w_k + \mathbb{E}_{/w_k} J_{k+1}(w_k+1)\right\}, \tag{2.13}$$

where $\mathbb{E}_{/w_k}(\bar{p}_k) = \mathbb{E}(\bar{p}_k)$, $\mathbb{E}_{/w_k}(\theta_k) = \mathbb{E}(\theta_k) > \mu > 0$ and $J_T(w_T) = w_T d_T + e_T = 0$; see (2.10) and ((2.11). We now show by induction that

$$J_T(w_k) = w_k d_k + e_k, \tag{2.14}$$

solves (2.13) by noting that (2.14) is true for $k = T$ by assuming that (2.14) is true for $k+1$ and by proving is true for k. Substituting the assumed expression for $J_{k+1}(w_k+1)$ into the right hand side (2.13) we obtain

$$J_k(w_k) = \max_{p_k}\left\{\upsilon^k w_k + \mathbb{E}_{/w_k} J_{k+1}(w_k+1)\right\} = \max_{p_k}\left\{\upsilon^k w_k + \mathbb{E}_{/w_k}(w_{k+1}) d_{k+1} + e_{k+1}\right\},$$

and from (2.6) we have

$$\begin{aligned}
&\max_{p_k}\left\{\upsilon^k w_k + \mathbb{E}_{/w_k}\left[-a_k w_k + (p_k - \pi_k) V_k\right] d_{k+1} + e_{k+1}\right\} \\
&= \max_{p_k}\left\{\upsilon^k w_k - a_k w_k d_{k+1} + d_{k+1}(p_k - \pi_k)\left(V_{k-1}\tfrac{\mathbb{E}(\bar{p}_k)}{p_k} - \mathbb{E}(\theta_k)\right) + e_{k+1}\right\} \\
&= \max_{p_k}\left\{\upsilon^k w_k - a_k w_k d_{k+1} - d_{k+1}(p_k \mathbb{E}(\theta_k) - V_{k-1}\mathbb{E}(\bar{p}_k))\right. \\
&\left. + d_{k+1}\pi_k\left(\mathbb{E}(\theta_k) - V_{k-1}\tfrac{\mathbb{E}(\bar{p}_k)}{p_k}\right) + e_{k+1}\right. \\
&= \max_{p_k}\left\{-w_k\left(a_k d_{k+1} - \upsilon^k\right) - d_{k+1}(p_k \mathbb{E}(\theta_k) - V_{k-1}\mathbb{E}(\bar{p}_k))\right. \\
&\left. + d_{k+1}\pi_k\left(\mathbb{E}(\theta_k) - V_{k-1}\tfrac{\mathbb{E}(\bar{p}_k)}{p_k}\right) + e_{k+1}.\right.
\end{aligned} \tag{2.15}$$

The controller that maximizes the above expression (2.15), is given by (2.8), since

$$A = w_k\left(\upsilon^k - a_k d_{k+1}\right) + (V_{k-1}\mathbb{E}(\bar{p}_k) - p_k \mathbb{E}(\theta_k)) d_{k+1} - d_{k+1}\pi_k\left(V_{k-1}\frac{\mathbb{E}(\bar{p}_k)}{p_k} - \mathbb{E}(\theta_k)\right) + e_{k+1}.$$

The first derivative of A with respect to p_k is given

$$\frac{\partial A}{\partial p_k} = d_{k+1}\pi_k V_{k-1}\mathbb{E}\left(\bar{p}_k\right)p_k^2 - d_{k+1}\mathbb{E}\left(\theta_k\right) = d_{k+1}\left(\pi_k V_{k-1}\frac{\mathbb{E}\left(\bar{p}_k\right)}{p_k^2} - \mathbb{E}\left(\theta_k\right)\right).$$

If we equalize the first derivative with zero, i.e. $\frac{\partial A}{\partial p_k} = 0$, we obtain

$$d_{k+1}\left(\pi_k V_{k-1}\frac{\mathbb{E}\left(\bar{p}_k\right)}{p_k^2} - \mathbb{E}\left(\theta_k\right)\right) = 0 \overset{d_{k+1}\neq 0, E(\theta_k)>\mu>0}{\Leftrightarrow} \pi_k V_{k-1}\frac{\mathbb{E}\left(\bar{p}_k\right)}{p_k^2} - \mathbb{E}\left(\theta_k\right) = 0.$$

The above expression gives the optimal strategy (2.8) as

$$\frac{\partial A}{\partial^2 p_k} = -2\pi_k V_{k-1}\mathbb{E}\left(\bar{p}_k\right) d_{k+1}\frac{1}{p_k^3} < 0,$$

where π_k, V_{k-1}, $\mathbb{E}\left(\bar{p}_k\right), \frac{1}{p_k^3}$ and $d_{k+1} > 0$. Now, let's substitute the above into (2.15), we obtain

$$\begin{aligned} &- w_k\left(a_k d_{k+1} - \upsilon^k\right) - d_{k+1}\left(\left(\frac{1}{\mathbb{E}\left(\theta_k\right)}\pi_k V_{k-1}\mathbb{E}\left(\bar{p}_k\right)\right)^{1/2}\mathbb{E}\left(\theta_k\right) - V_{k-1}\mathbb{E}\left(\bar{p}_k\right)\right) \\ &\qquad + d_{k+1}\pi_k\left(\mathbb{E}\left(\theta_k\right) - V_{k-1}\left(\frac{1}{\mathbb{E}\left(\theta_k\right)}\pi_k V_{k-1}\mathbb{E}\left(\bar{p}_k\right)\right)^{-1/2}\right) + e_{k+1}. \end{aligned}$$

Substituting (2.10) and (2.11) in the above expression yields the fact that (2.14) is true. Thus, the proof of the Theorem 1 by induction is complete. □

Remark 1. As it is quite likely in practice, the optimal premium strategy given by (2.6) expression depends *endogenously* on the volume of business of the previous year, the break-even premium rate, the expected value of the *average* premium rate of the market and the (stochastic) variable θ_k.

Remark 2. In order to calculate the optimal premium strategy, initially we have to calculate the expectation of θ_k which models the set of all other parameters considered to be relevant to the demand function of each company, and the insurance market (i.e. financial environment, managerial policy etc); see also Assumption 4. In particular, as it has been clearly stated in the introduction; see also Remark 3, and Proposition 1, we are interested to modify the premium strategy when our volume of business is *strictly* decreasing because of the positive $\mathbb{E}\left(\theta_k\right) > \mu$. Note that as it came clear from the relation (2.5) θ_k is equal to $V_{k-1}\frac{\bar{p}_k}{p_k} - V_k$ for each previous year.

Remark 3. In a competitive market environment, we have considered that the volume of business in each company is strictly decreasing when the expectation of the stochastic variable (disturbance) θ_k in year $k = 0, 1, ..., T-1$ takes positive values. Thus, the company should change the premium policy in order to enlarge its volume. On contrary, for negative or below the deductible point $\mu > 0$ values for the expectation of θ_k, i.e. $\mathbb{E}(\theta_k) < \mu$, the previous premium strategy might stay unchanged (see next corollary), since the company does not lose (significant) part of the market (i.e. by decreasing its volume).

The following proposition considers the case where the volume of business changes either above or below $\mu > 0$ (i.e., for decreasing or increasing the volume of business above or below the required level, respectively).

Remark 4. Moreover, we can show that the optimal expected wealth of the company at the year $k+1$ is given by (2.16).

$$\mathbb{E}\left(w^*_{k+1}\right) = V_{k-1}\mathbb{E}\left(\bar{p}_k\right) + \pi_k\mathbb{E}\left(\theta_k\right) - \left\{a_k w_k + 2(\mathbb{E}\left(\theta_k\right)\pi_k V_{k-1}\mathbb{E}\left(\bar{p}_k\right))^{1/2}\right\} \; for \; \mathbb{E}\left(\theta_k\right) > \mu. \tag{2.16}$$

As Taylor [71, 72], and Emms et al.[18] propose, and in order to take benefit of the analytical formula derived by Theorem 1 for the determination of the premium strategies into a competitive environment, in the next section we use data from the Greek automobile insurance industry, see also the tables of the Hellenic Association of Insurance Companies (2010). Moreover, we assume that the premium strategies concern the price of a contract which refers to a six-month insurance for a car that is 1400cc, 10 years old and its value estimated at 5.000 euros.

2.5 Numerical Application

2.5.1 Data

In the next section we use data from the Greek automobile insurance industry, see also the tables of the Hellenic Association of Insurance Companies (2010). Moreover, we assume that the premium strategies concern the price of a contract which refers to a six-month insurance for a car that is 1400cc, 10 years old and its value estimated at 5.000 euros. The number of the available data is limited but for the purpose of our application, this drawback is not crucial.

2.5.2 Premium strategy I: Considering the Entire Market

In the first premium strategy, the expected average premium is calculated considering all the competitors of the market, and their proportions regarding the volume of business. In mathematical terms the expected average premium of the market can be estimated by

$$\mathbb{E}\left(\bar{p}\right) = \frac{1}{m}\sum_{i=1}^{K} b_{i,n}p_{i,n}, \tag{2.17}$$

where $b_{i,n} = V_{i,n}\left(\sum\limits_{i=1}^{K} V_{i,n}\right)^{-1}$ and $\sum\limits_{i=1}^{K} b_{i,n} = 1$ for every year n, $p_{i,n}$ is the premium of the company i^{th} for the year n; K is the number of the competitors (including also our company's premium) in the insurance market and m is the number of years for the available data (i.e. we assume that we have the uniform distribution for the weight of every year). Moreover, for the calculation of the expected values of the premium of each company and the average market premium respectively, we use the available Greek data, see next paragraphs.

Proposition 1. *Considering (2.8) and (2.17), the optimal controller (i.e. premium) for the premium strategy I is equal to*

$$p_k^* = \sqrt{\frac{1}{m\mathbb{E}\left(\theta_k\right)}\pi_k V_{k-1}\sum_{i=1}^{K} b_{i,n}p_{i,n}}, \tag{2.18}$$

for $\mathbb{E}\left(\theta_k\right) > \mu > 0,\ k = 0, 1, ..., T-1.$

Proof. The proof derives straightforwardly, and it is omitted. □

2.5.3 Premium strategy II: Following the Leaders of the Market

$$\mathbb{E}\left(\bar{p}\right) = \frac{1}{m}\sum_{i=1}^{K^{top}} b_{i,n}^{top}p_{i,n}^{top}, \tag{2.19}$$

where $b_{i,n}^{top} = V_{i,n}\left(\sum\limits_{i=1}^{K^{top}} V_{i,n}\right)^{-1}$ and $\sum\limits_{i=1}^{K^{top}} b_{i,n}^{top} = 1$ for every year n, $p_{i,n}^{top}$ is the premium of the i^{th} top company for the year n; K^{top} is the number of the top competitors (including also our company's premium) in the insurance market and m is the number of years for the available data (i.e. we assume that we have the uniform distribution for the weight of every year). Next, similar to the Proposition 2, we obtain the following Proposition.

Proposition 2. *Considering (2.8) and (2.19), the optimal controller (i.e. premium) for the premium strategy II is equal to*

$$p_k^* = \sqrt{\frac{1}{m\mathbb{E}\left(\theta_k\right)} \pi_k V_{k-1} \sum_{i=1}^{K^{top}} b_{i,n}^{top} p_{i,n}^{top}}, \tag{2.20}$$

for $\mathbb{E}\left(\theta_k\right) > \mu > 0, , \; k = 0, 1, ..., T-1.$

Proof. The proof derives straightforwardly, and it is omitted. □

2.5.4 Premium strategy III: Following the Direct Competitors

$$\mathbb{E}\left(\bar{p}\right) = \frac{1}{m} \sum_{i=1}^{K^{dir}} b_{i,n}^{dir} p_{i,n}^{dir} * d_{i,n}, \tag{2.21}$$

where $b_{i,n}^{dir} = V_{i,n} \left(\sum_{i=1}^{K^{dir}} V_{i,n} \right)^{-1}$ and $\sum_{i=1}^{K^{dir}} b_{i,n}^{dir} = 1$ for every year n, $p_{i,n}^{dir}$ is the premium of the i^{th} direct competitor (company) for the year n; K^{dir} is the number of direct competitors (without including our company's premium) in the insurance market, $d_{i,n}$ is the direct competitive factor which shows in what extend the direct competitor is similar to our company and m is the number of years for the available data (i.e. we assume that we have the uniform distribution for the weight of every year).

The direct competitor factor is indicative to how the company that though as direct competitor is similar to our company's and affect our volume of business. This factor depends mainly on three other parameters:

- company's operational efficiency,
- product leadership,
- customer intimacy.

Additionally, the factors that a company will examine to identify their direct competitors in the market are the following:

- competitor's main focus and propositions,
- competitor's geographic target,
- competitor's target sector,
- type of organization the competitor choose,
- competitor's paying targets,

- service that competitor's provides,
- competitor's efficiency,
- competitor's target group.

Next, we obtain the following Proposition.

Proposition 3. *Considering (2.8) and (2.21), the optimal controller (i.e. premium) for the premium strategy III is equal to*

$$p_k^* = \sqrt{\frac{1}{m\mathbb{E}\left(\theta_k\right)}\pi_k V_{k-1}\sum_{i=1}^{K^{dir}} b_{i,n}^{dir} p_{i,n}^{dir} * d_{i,n}}\,, \tag{2.22}$$

for $\mathbb{E}\left(\theta_k\right) > \mu > 0,,\;\; k = 0, 1, ..., T-1.$

Proof. The proof derives straightforwardly, and it is omitted. □

2.5.5 Numerical Algorithm

Summarizing the discussion in the previous Section, in this sub-section, the algorithmic steps for the calculation of the optimal premium are described.

Step 1: *Collect the necessary (historical) data from the insurance market.*
The first step requires the collection of data concerning the number of companies which are in the market, their volume of business and the premium charged from each company for the previous years, respectively. Obviously if it is possible to collect data for a significant number of years the results will be more reliable.

Step 2: *Estimate market's average premium.*
Choose one of the three recommended premium strategies and estimate market's average premium for each one of the previous years and the expectation of market's average premium for the next year, $\bar{p}_k$. As it has been assumed in the previous sub-section, the average premium can be calculated either considering the entire market or considering the leaders of the market or the direct competitors, see eq. (2.17), (2.19) and (2.21). Then, step 3 follows.

Step 3: *Estimate parameter expectation of* θ_k.
Moreover, the impact of the other stochastic parameters, θ_k, needs to be estimated. Based on historical data and market's average premium one can estimate parameter θ_k which is equal to $\theta_k = V_{k-1}\frac{\bar{p}_k}{p_k} - V_k$ for each previous year, as it came clear from the

relation 2.5. After the calculation of the stochastic parameter θ_k for each previous year the calculation of the expected value of θ_k for the next year must be taken place and this parameter can be given by

$$\mathbb{E}(\theta_k) = \frac{1}{m}\sum_{i=1}^{m}\hat{\theta}_i. \tag{2.23}$$

If $\mathbb{E}(\theta_k) < \mu$, the previous premium strategy might stay unchanged (see corollary), since the company does not lose (significant) part of the market. If $\mathbb{E}(\theta_k) > \mu > 0$, step 4 follows.

Step 4: *Calculate the optimal premium.*
Using the estimated parameters $\mathbb{E}(\theta_k)$, $\bar{p}_k$ and the information collected of step 1 the next step is to calculate the optimal premium according to (2.8) for different values of break-even premium (π_k). Then, step 5 follows.

Step 5: *Design the optimal premium strategy for the insurance company.*
Now, for different values of the break-even premium π the actuary can generate different values for the optimal premium, see previous step. Then after taking into consideration the competition in the current (or targeting) insurance market and expectation of the different macroeconomic parameters (i.e. based on the random variable, θ_k), the optimal premium is calculated and agreed by the senior management of the companies.

2.5.6 Numerical Calculation and Discussion

As we have mentioned earlier premium strategy I considers the premium and the volume of business of the entire market. The expected average premium of the market is estimated using the (2.17) expression, i.e. as an expected weighted average of each competitor that gets involved in the market. Moreover, it is clear that the premium of the company with the largest volume of business affects most of the market (see also Premium Strategy II). In Table 2.1, the premium prices and the number of contracts for the 12 major non-life Greek insurance companies for a standard six-month cover of a 10-year old, 1400cc car (with 5.000 Euros covered amount) are presented for the years 2006, 2007, 2008 and 2009.

As we can observe in Table 2.1, and according to the oligopoly theory which began in 1838 with Cournot's oligopoly model, see for more details Friedman [27] and the references therein, the Greek non-life insurance industry has an oligopoly market characteristic, since there are only a few main competitors, the insurance products are almost identical (with non-significant differences) and the ownership of the key inputs and barriers imposed by the government. Thus, in the case of oligopolistic market, the revenues of the firms depend on the actions of other competitors as we have considered in our premium strategies; see also Emms et al. [18] and Taylor [71, 72].

Insurance	2006		2007		2008		2009	
Companies	premium	contracts	premium	contracts	premium	contracts	premium	contracts
A	269.09	298,269	280.30	280,991	301.00	261,196	307.35	240,698
B	282.07	303,673	293.82	308,766	306.06	278,362	315.53	250,614
C	377.06	282,224	392.77	252,630	413.44	258,683	430.67	266,414
D	371.52	304,609	404.96	255,250	437.35	263,510	451.35	278,321
E	281.56	295,769	292.96	258,181	304.71	274,382	323.68	243,294
F	377.83	796,139	397.71	687,485	432.30	726,317	469.89	779,376
G	257.88	298,304	268.62	325,836	291.98	273,470	307.35	271,487
H	366.99	200,135	386.30	182,989	402.40	258,534	423.58	267,341
I	347.58	211,314	373.74	278,174	397.59	283,295	418.52	284,889
J	351.18	299,690	377.02	318,876	392.73	316,556	426.88	338,434
K	364.11	299,995	378.67	340,898	401.39	344,771	429.09	396,112
L	291.22	319,453	302.87	287,524	314.98	246,976	331.77	241,609

Table 2.1: Premium prices in Euros and number of contracts for the 12 major non-life Greek insurance companies, see Hellenic Association of Insurance Companies (2010).

Volume of Business b (%)	2006	2007	2008	2009
A	7.63%	7.44%	6.90%	6.24%
B	7.77%	8.17%	7.35%	6.49%
C	7.22%	6.69%	6.83%	6.90%
D	7.79%	6.76%	6.96%	7.21%
E	7.57%	6.83%	7.25%	6.31%
F	20.36%	18.20%	19.18%	20.20%
G	7.63%	8.63%	7.22%	7.04%
H	5.12%	4.84%	6.83%	6.93%
I	5.41%	7.36%	7.48%	7.38%
J	7.67%	8.44%	8.36%	8.77%
K	7.67%	9.02%	9.11%	10.27%
L	8.17%	7.61%	6.52%	6.26%

Table 2.2: The volume of business, b, in % for the 12 major non-life Greek insurance companies.

According to the premium strategy I, the average premium of the market is equal to the weighted average of the premiums of all the companies involved in the market for every year. Moreover, the volume of business of each company for the years 2006-2009 is presented in Table 2.2. Finally, Table 2.3 summarizes the results of the (2.17) expression.

As it has already been mentioned above in order to calculate the optimal premium for each company first, we have to estimate the expectation of θ_k which models the set of all other parameters considered to be relevant to the demand function of each company, and the insurance market (i.e. financial environment, managerial policy etc). As it is

Average Premium (P.S.I)	Amount in Euros
$\mathbb{E}(\bar{p}_k)$	364.69

Table 2.3: The expected average premium in Euros of the market for the year 2010 is given by $\mathbb{E}(\bar{p}) = \frac{1}{m}\sum_{i=1}^{K} b_{i,n}p_{i,n}$.

clear from the relation (2.5) $\hat{\theta}_k$ (estimation of θ_k) can be calculated by $\hat{\theta}_k = V_{k-1}\frac{\bar{p}_k}{p_k} - V_k$.

Thus, considering the above expression, and for the available Greek data we are able to calculate $\hat{\theta}_k$ for the years 2007, 2008 and 2009 as it is shown at Table 2.4. So, the expected value of θ_k for the year 2010 can be given by $\mathbb{E}(\theta_k) = \frac{1}{m}\sum_{i=1}^{m}\hat{\theta}_i$.

Then, in Table 2.4, we present the expected values of θ_k (using the estimations of θ_k). As has been already mentioned before, θ_k denotes the number of contracts that the company loses or gains because of the parameters that affect the volume of business and they have not been included in the model. In our application, the large fluctuations in the expected values of θ_k occur due to

a) the limited number of the available data, and

b) the impact on each company's volume of business into the market.

(Note that since we have available data for only 4 years, it is difficult to provide a good estimation for the expected values of θ_k. However, for the purpose of our application, this drawback is not crucial.)

Companies	2006-2007		2007-2008		2008-2009		$\mathbb{E}(\theta_k)$
A	90,752	-0.19%	89,088	-0.54%	100,437	-0.66%	93,426
B	52,299	0.41%	100,178	-0.82%	103,515	-0.86%	85,331
C	-1,605	-0.53%	-29,404	0.14%	-25,304	0.07%	-18,771
D	7,533	-1.03%	-44,518	0.20%	-43,965	0.25%	-26,984
E	94,520	-0.73%	43,548	0.41%	96,983	-0.94%	78,350
F	11,841	-2.16%	-129,593	0.99%	-158,903	1.01%	-92,218
G	62,115	1.00%	145,262	-1.40%	85,678	-0.19%	97,685
H	-1,998	-0.28%	-87,902	1.98%	-22,336	0.10%	-37,412
I	-80,649	1.96%	-20,770	0.12%	-13,172	-0.10%	-38,197
J	-41,180	0.78%	-11,891	-0.08%	-40,763	0.41%	-31,278
K	-64,134	1.35%	-26,097	0.08%	-73,578	1.16%	-54,603
L	80,957	-0.56%	95,540	-1.09%	57,216	-0.26%	77,904

Table 2.4: The values of $\hat{\theta}_k$, the change in percentage for the volume of business for the years 2007-2009, and the expected values of θ_k for the year 2010.

The values of the stochastic variable θ_k can be either above or below $\mu > 0$. As we have extensively discussed in section 2, we will determine the optimal premium strategy for the year 2010 only for those companies which have positive $\mathbb{E}(\theta_k) > 0$.

These companies are A, B, E, G and L, see Figure 2.1.

In Table 2.5, we present the premium for each company for the different values of

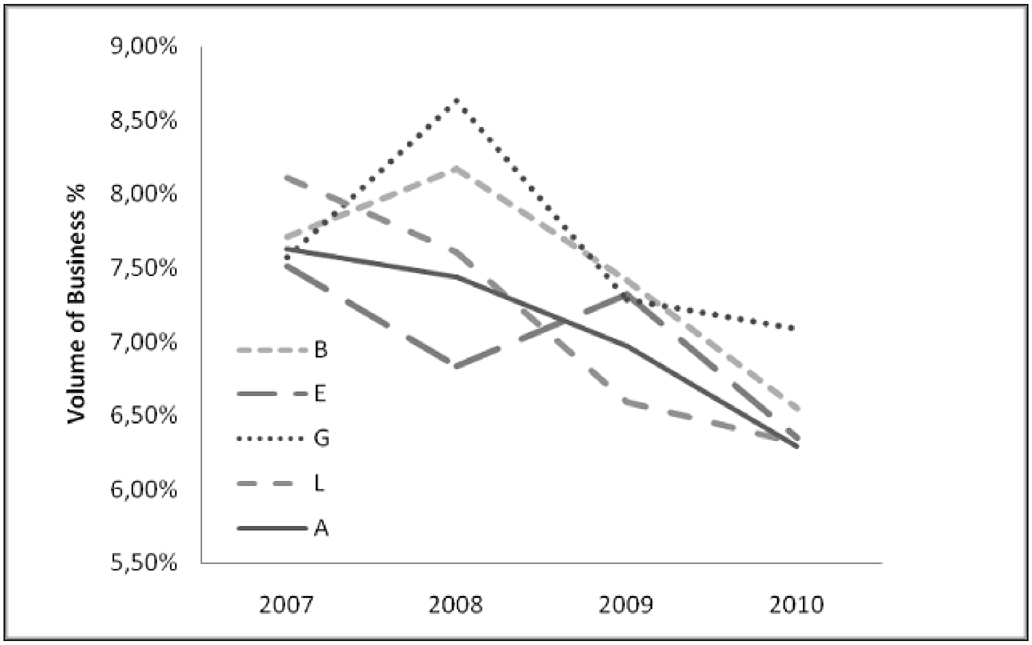

Figure 2.1: The real and the expected volume of business for the 5 Greek insurance companies that have positive $\mathbb{E}(\theta_k) > \mu$.

the break-even premium rate.

Companies	π_κ / $\mathbb{E}(\theta_k)$	20%	30%	40%	50%	60%
A	93,426	240.32	294.33	339.87	379.98	416.25
B	85,331	259.98	318.41	367.67	411.07	450.30
E	78,350	270.76	331.61	382.91	428.10	468.96
G	97,685	249.60	305.70	352.99	394.66	432.33
L	77,744	273.95	335.51	387.42	433.15	474.49

Table 2.5: The optimal premium strategy in Euros for the 5 Greek insurance companies that have positive $\mathbb{E}(\theta_k)$ for the different values of the break-even premium rate (Premium Strategy I).

As it is expected, for greater values of the π_k, greater the optimal premium values become. Consequently, since the optimal premium depends on the break-even premium rate, the company should choose its competitive strategy considering the market's construction and its marginal costs; see also Emms et al. [18]. Thus, each company should predetermine its break-even premium rate, in order to calculate the optimal premium strategy which will enlarge its volume of business. The results of Table 2.5 are shown also at Figure 2.1.

The results of Table 2.5 (see also Figure 2.2) are seemingly interesting. For the five insurance companies (A, B, E, G, and L) which expected to experience losses on their volume of business, for a break-even premium rate of 20-30 % calculated by the formula

(2.18), premiums are below the market average premium of 364.69€. Additionally, it is true that the insurance companies E and L which face similar losses (see Tables 2.1, 2.2, and 2.4) should provide similar premiums, which appear to be the most expensive premiums compared with the premiums of the other 3 companies.

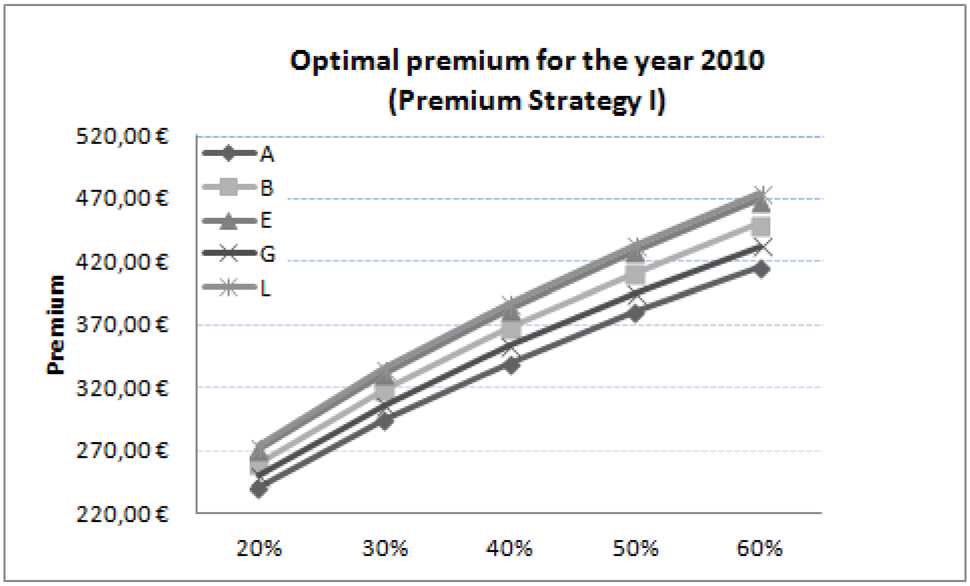

Figure 2.2: The optimal premium strategy in Euros for the 5 Greek insurance companies that have positive $\mathbb{E}(\theta_k)$ for the different values of the break-even premium rate (Premium Strategy I).

At this point, it should be mentioned that in this paper, we are not able to analyze further the results of Table 2.1, and consequently of Table 2.5 (and Figure 2.2), since the analysis of the Greek insurance market, and the micro/macro conditions that get involved for the determination of the premium strategy is far beyond the scopes of the present version of the present chapter. Additionally, the macro-micro economic analysis of the parameters that affect θ_k are partially investigated at chapter 4.

Following the second premium strategy, the average premium is calculated considering the premiums of the top K^{top} competitors of the market (including the leading company of the market). In mathematical terms the expected average premium of the market is estimated by (2.19).

For the purpose of this application, we consider the premium and the volume of business of the top 5 Greek insurance companies. Consequently, the expected average premium of the market is calculated using the (2.19) expression.

In Table 2.6, the premiums and the number of contracts for the 5 leading non-life Greek insurance companies are presenting for the years 2006, 2007, 2008 and 2009.

Thus, for the years 2006, we calculate the average premium considering the premium and the volume of business for the companies B, D, F, K, and L; for the year 2007 : B, F, G, J and K; for the year 2008 : B, F, I, J and K and for the year 2009: D, F, I, J and K. In Table 7, the volume of business is presented.

2006			2007			2008			2009		
IC	premium	contracts	IC	premium	contracts	IC	premium	contracts	IC	premium	contracts
B	282.07	303,673	B	293.82	308,766	B	306.06	278,362	D	451.35	278,321
D	371.52	304,609	F	397.71	687,485	F	432.30	726,317	F	469.89	779,376
F	377.83	304,609	G	268.62	325,836	I	397.59	283,295	I	418.52	284,889
K	364.11	796,139	J	377.02	318,876	J	392.73	316,556	J	426.88	338,434
L	291.22	319,453	K	378.67	340,898	K	401.39	344,771	K	429.09	396,112

Table 2.6: Premiums prices in Euros and number of contracts for the top 5 non-life Greek insurance companies (IC); see Friedman [27](Premium Strategy II).

According to the premium strategy II, the average premium of the market is equal to the weighted average of the premiums of the top 5 companies in the market for every year. Finally, Table 2.8 summarizes the results of the (2.19) expression.

2006	2007	2008	2009
15.00% (B)	15.58% (B)	14.28% (B)	13.40% (D)
15.05% (D)	34.69% (F)	37.26% (F)	37.52% (F)
39.34% (F)	16.44% (G)	14.53% (I)	13.72% (I)
14.82% (K)	16.09% (J)	16.24% (J)	16.29% (J)
15.78% (L)	17.20% (K)	17.69% (K)	19.07% (K)

Table 2.7: The weights in % for the calculation of the average premium for the top 5 non-life Greek insurance companies.

Average Premium (P.S.I)	Amount in Euros
$\mathbb{E}(\bar{p}_k)$	385.85

Table 2.8: The expected average premium in Euros of the market for the year 2010 is given by $\mathbb{E}(\bar{p}) = \frac{1}{m}\sum_{i=1}^{K^{top}} b_{i,n}^{top} p_{i,n}^{top}$.

Now, we calculate again the estimation of θ_k, since the average premium of the market for the years 2006, 2007, 2008 and 2009 has changed. Additionally, the average premium in the Premium Strategy I is higher than in the Premium Strategy II. So, in Table 2.9, we present the expected values of θ_k.

Next, we will determine the optimal premium strategy for the year 2010 only for those companies which have positive $\mathbb{E}(\theta_k)$ with $\mu > 10,000$ (the managerial team is not interested in modifying the premium when it expects to lose only a few thousand contracts), i.e. A, B, E, G and L.

The results of Table 2.10 (see also Figure 2.3) are also similar with those of the Premium Strategy I, since the five insurance companies have premiums significantly below the market average premium of 385.85€for a break-even premium rate of 20-

Companies	2006-2007		2007-2008		2008-2009		$\mathbb{E}(\theta_k)$
A	95,379	-0.19%	109,731	-0.54%	137,962	-0.66%	114,357
B	56,793	0.41%	122,456	-0.82%	142,470	-0.86%	107,250
C	1,520	-0.53%	-15,893	0.14%	1,219	0.07%	-4,385
D	10,804	-1.03%	-31,613	0.20%	-18,186	0.25%	-12,998
E	98,910	-0.73%	62,284	0.41%	134,414	-0.94%	98,536
F	20,546	-2.16%	-94,427	0.99%	-90,651	1.01%	-54,844
G	66,944	1.00%	169,939	-1.40%	124,967	-0.19%	120,617
H	255	-0.28%	-77,846	1.98%	4,615	0.10%	-24,326
I	-78,190	1.96%	-5,299	0.12%	16,716	-0.10%	-22,258
J	-37,724	0.78%	-6,063	-0.08%	-8,018	0.41%	-13,226
K	-60,689	1.35%	-7,317	0.08%	-38,099	1.16%	-35,368
L	85,544	-0.56%	115,725	-1.09%	90,087	-0.26%	97,118

Table 2.9: The values of $\hat{\theta}_k$, the change in percentage for the volume of business for the years 2007-2009, and the expected values of θ_k for the year 2010.

40%.

Now, if we would like to compare the findings of the two Premium Strategies, we can easily see that the Premium Strategy II is cheaper (i.e. it provides lower premiums) than the Premium Strategy I for all the A, B, E, G and L insurance companies. This result was expected, as in the Greek insurance market, the leader (dominator) companies have expensive premiums, above the average premium of the market.

	π_κ	20%	30%	40%	50%	60%
Companies	$\mathbb{E}(\theta_k)$					
A	114,357	223.43	273.65	315.98	353.28	387.00
B	107,250	238.53	292.14	337.34	377.16	413.15
E	98,536	248.34	304.16	351.21	392.67	430.14
G	120,617	231.05	282.98	326.76	365.33	400.20
L	97,118	252.38	309.10	356.91	399.04	437.13

Table 2.10: The optimal premium strategy in Euros for the 5 Greek insurance companies that have positive $\mathbb{E}(\theta_k)$ for the different values of the break-even premium rate(Premium Strategy II).

The last premium strategy calculates the average premium taking into account company's direct competitors in the market. The direct competitors are companies with similar operational efficiency, product leadership and customer intimacy. It is rational for an "average" (not leading) company not to target increasing it's volume of business by "stealing" customers from the leading company or companies but instead trying to augment its volume by targeting new customers from companies which have the same economic conditions and key marketing factors.

If for example company E choose to follow the third premium strategy then market's average premium will be calculated with companies which have the most close volume of business value and with premium lower than company's E. If we calculate the absolute

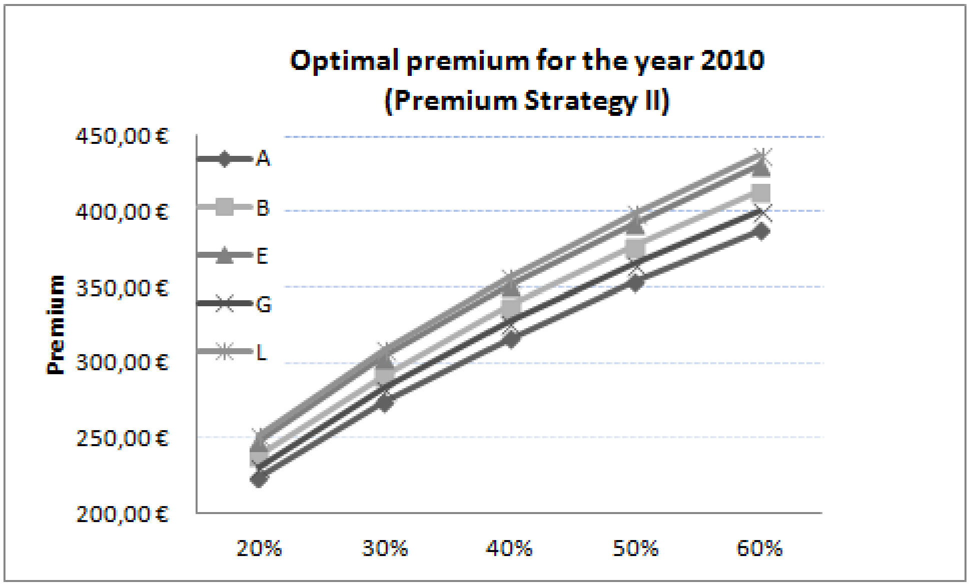

Figure 2.3: Optimal premium for the year 2010 (Premium Strategy II).

value of the difference between company's E volume of business and all the other companies in the market and the difference between company's E premium and all the other premiums for the computation of market's average premium we will use the two (because of the small number of companies in the market) companies with the minimum absolute value and positive premium difference.

Thus, for the years 2006, 2007 and 2008 we calculate the average premium considering the premium and the volume of business for the companies A and G and for the year 2009 the companies A and B. In Table 2.11, the volume of business is presented.Another crucial factor that must be taken into account is how similar and competitive are these companies. These can be taken into account by the competitive factor which affects each company's premium's affection to the average premium. The competitive factor between company's E and its direct competitors are shown in Table 2.12.

2006	2007	2008	2009
50.00% (A)	46.30% (A)	48.85% (A)	48.99% (A)
50.05% (G)	53.70% (G)	51.15% (G)	51.01% (B)

Table 2.11: The weights in % for the calculation of the average premium for Company's E direct competitors.

Company E	2006		2007		2008		2009	
	A	1.22	A	1.29	A	1.28	A	1.30
	G	1.24	G	1.41	G	1.20	B	1.24

Table 2.12: Competitive factor for Company's E direct competitors.

According to the premium strategy III, the average premium of the market is equal to the weighted average of the premiums of the direct competitors times the competitive factor between Company E and each direct competitors. Finally, Table 2.12 summarizes the results of the (2.21) expression.

Average Premium (P.S.I)	Amount in Euros
$\mathbb{E}(\bar{p}_k)$	364.40

Now, we calculate again the estimation of θ_k, since the average premium of the market for the years 2006, 2007, 2008 and 2009 has changed. Additionally, the average premium in the Premium Strategy III is different than in the Premium Strategy I and II. So, in Table 2.14, we present the expected values of θ_k.

Companies	2006-2007		2007-2008		2008-2009		$\mathbb{E}(\theta_k)$
A	113,586	-0.19%	81,809	-0.54%	95,258	-0.66%	96,884
B	74,477	0.41%	92,312	-0.82%	98,139	-0.86%	88,309
C	13,814	-0.53%	-34,169	0.14%	-28,964	0.07%	-16,440
D	23,673	-1.03%	-49,069	0.20%	-47,523	0.25%	-24,306
E	116,184	-0.73%	36,941	0.41%	91,817	-0.94%	81,648
F	54,796	-2.16%	-141,993	0.99%	-168,323	1.01%	-85,173
G	85,945	1.00%	136,561	-1.40%	80,256	-0.19%	100,921
H	9,119	-0.28%	-91,448	1.98%	-26,055	0.10%	-36,128
I	-68,516	1.96%	-26,225	0.12%	-17,298	-0.10%	-37,346
J	-24,123	0.78%	-18,222	-0.08%	-45,282	0.41%	-29,209
K	-47,134	1.35%	-32,719	0.08%	-78,475	1.16%	-52,776
L	103,590	-0.56%	88,422	-1.09%	52,679	-0.26%	81,564

Table 2.14: The values of $\hat{\theta}_k$, the change in percentage for the volume of business for the years 2007-2009, and the expected values of θ_k for the year 2010.

Next, we will determine the optimal premium strategy for the year 2010 only for company E since we are interesting only on this comapny. Company E also has positive $\mathbb{E}(\theta_k)$ with $\mu > 10,000$ (the managerial team is not interested in modifying the premium when it expects to lose only a few thousand contracts).

The results of Table 2.15 are also similar with those of the Premium Strategy I and II for a break-even premium rate of 20-30%.

	π_κ	20%	30%	40%	50%	60%
Companies	$\mathbb{E}(\theta_k)$					
E	81,648	265.13	324.71	374.95	419.20	459.21

Table 2.15: The optimal premium strategy in Euros for the Company E that has positive $\mathbb{E}(\theta_k)$ for the different values of the break-even premium rate(Premium Strategy III).

Now, if we would like to compare the findings of the three Premium Strategies, we can easily see that the Premium Strategy II is cheaper than three and one (i.e. it provides lower premiums) for company E. This result was expected, as in the Greek insurance market, the leader (dominator) companies have expensive premiums, above the average premium of the market. Moreover the premium strategy III provides lower premium than premium strategy I and higher than premium strategy II. This quite rational since the strategy for competiting the leaders of the market is to low premium in order to augement its volume of business. On the contrary when the company focus on its direct competitors it can maximize its wealth by charging higher premiums than the one that come up from premium strategies II. See also table 2.15 and figure 2.4.

Company E Premium Strategy	π_κ $\mathbb{E}(\theta_k)$	20%	30%	40%	50%	60%
I	78,350	270.76	331.61	382.91	428.10	468.96
II	98,536	248.34	304.16	351.21	392.67	430.14
III	81,648	265.13	324.71	374.95	419.20	459.21

Table 2.16: The optimal premium strategy in Euros for the Company E that has positive $\mathbb{E}(\theta_k)$ for the different values of the break-even premium rate and different premium strategies.

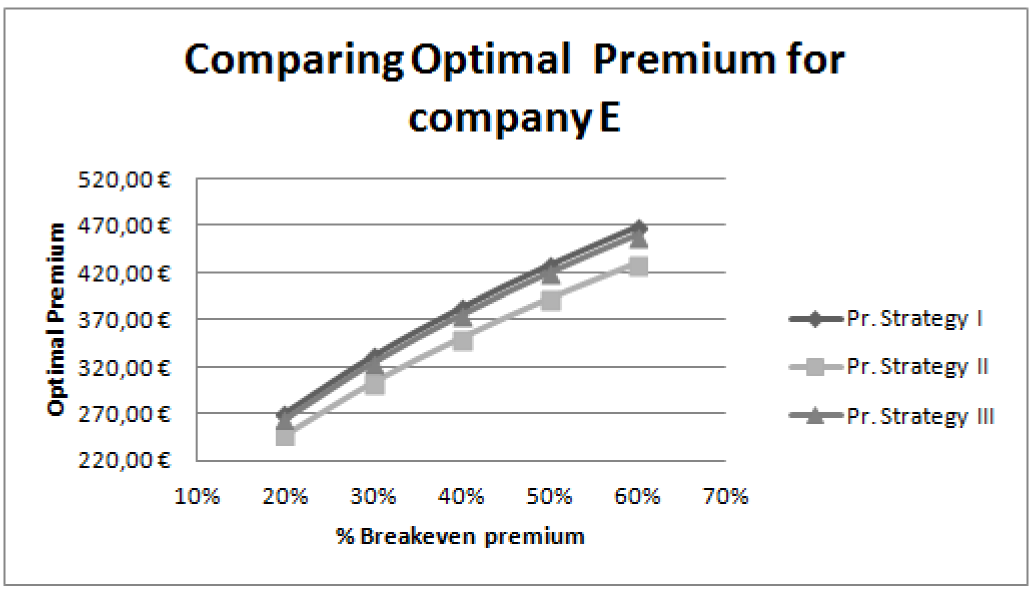

Figure 2.4: Optimal premium for the year 2010 for company E for different premium strategies.

Chapter 3

Model's enrichment with company's reputation and elasticities

3.1 Motivation

Practically speaking, the general (non-life) insurance premium pricing strategy is mainly based on the claim and business acquisition costs, the management expenses, the margin for fluctuation in claims experience and expected profits. According to Gulumser et al. [31], companies offering products and services in the general insurance markets are believed to trade under very competitive conditions. As a simple example, the case of Australia has been studied, and the outcome suggests that in the general Australian insurance industry, the firms operate in a somewhat perfect competitive environment which depicts their demand and cost structure as well. Thus, competition affects the equilibrium of the industry changing demand conditions. Consequently, it is clear that the company's optimal premium strategy, depends on the company's demand,which is also affected by competition.

This part of chapter 3 deals with two interesting questions. Actually, these will be our motivation as it will become clearer in the next sections. Firstly, *"how can the optimal premium strategy for an individual insurance company be calculated in a particular insurance market?"* and secondly *"how is this strategy related to the competitive insurance market?"*.

3.2 New Approach

With the present chapter, the volume of business is formed to be a general stochastic demand function further extending chapter's 2 suggestions making the model more pragmatic and realistic. Thus, here for the very first time according to the author's knowledge, for the formulation of the volume of business, the company's reputation is also considered. According to [12], company's reputation (or corporate reputation) has a strong influence on buying decisions or in other words, on the demand of the company's product. In our case, the function for the volume of business, emphasizes the ratio of the markets average to the company's premium, the past year's experience, the company's reputation and a stochastic disturbance. Additionally, following the existing literature, the same linear discounted function for the wealth process of the company is used, see also [18] and [61]. Moreover, the optimal premium can be calculated for either negative or positive effect of the company's reputation; which was not possible with the previous model, see [61] and chapter 2.

As in [18], [71], [72], [61], the volume of business is directly connected to the company's product demand function. Market's demand for an insurance product is the relationship between the product's price and the product demanded by all customers. In the model studied in this chapter, the volume of business function is derived from the equilibrium points of the competitive market (perfect competition) which are defined by the intersection of the demand and supply curves. These points determine the contracts of general insurance that have been purchased, and its prices, which are equal to the marginal cost of services. Generally, an approach of competitive behavior, examines the revenue and cost structure of companies, using the framework of perfect competition as a point of reference. Insurance firms operating under conditions of perfect competition are unable to absorb any of the cost increase. They are forced to pass on the entire rise of input costs on output prices and revenue, leaving output unaffected. In contrast, a rise in input prices, such as wages or administrative costs,under monopolistic conditions in equilibrium, results in a reduction of output and a rise in price in a smaller scale than the increase in costs, leading to shrinked total revenue. Thus, marginally profitable firms may be forced to leave the industry. Moreover the insurance contracts are homogenous, customers face no quality differences, the transaction costs are zero and both customers and insurance companies are fully informed.

In addition we present the analytical solutions for some common special cases, a premium strategy concerning market's average premium and an application.

This chapter is organized as follows: In section 3.3 a discrete-time model for the insurance market is constructed. We discuss appropriate values for the model's parameters and adopt suitable parameterizations. Moreover, in section 3.4 the calculation of the optimal premium and the two main theorems are presented, both when the expected utility is being maximized and minimized, respectively. Therefore, for some special cases, analytical forms for the optimal strategy are presented. Section 3.5 considers an application with data based on the Greek insurance market and deals with the applicability of the theoretical findings, the data's presentation and analysis and a premium strategy regarding market's average premium.

3.3 Model Formulation

Following again Taylor's [71], Emms et al. [18] and Pantelous and Passalidou [61], we should make the following assumptions.

Assumption 1: There is positive price-elasticity of demand, i.e. if the market as a whole begins underwriting at a loss, any attempt by a particular insurer to maintain profitability will result in a reduction of his volume of business.

Assumption 2: There is a finite time horizon.

Assumption 3: Demand in year $k+1$ is assumed to be proportional to demand in the preceding year k.

Additionally, we assume that the demand function is stochastic (because of θ_k and $\bar{p}_k$) and we denote the wealth process w_k as the insurer's capital at time $[k, k+1)$, so we obtain

$$w_{k+1} = -a_k w_k + (p_k - \pi_k)V_k, \tag{3.1}$$

where the sequence $\{a_k\}_{k\in\mathbb{N}} \in [0,1]$ denotes the excess return on capital (i.e. return on capital required by the shareholders of the insurer whose strategy is under consideration). Thus, $-a_k w_k$ is the cost of holding w_k in the time interval $[k, k+1)$.

Our aim is to determine the strategy that maximizes the expected total utility of the wealth at time k over a finite time horizon T. As it has been also considered by Emms et al. [18], we use a linear discounted function (of wealth) eq. (3.2).

Analytically, we want to maximize

$$\max_{p_k} \mathbb{E}\left[\sum_{k=0}^{T} U(w_k, k)\right], \tag{3.2}$$

where $U(w_k, k) = v^k w_k$ is the present value of the wealth w_k.

Extending the previous literature, we assume that the function for the volume of business is divided into two parts. Consistent with Taylor [71], the first part affects the volume of business in year $k+1$ more significantly, since the demand is assumed to be proportional to the demand of the preceding year k multiplied by

$$f(p_k, \bar{p}_k) = (p_k/\bar{p}_k)^\alpha \quad with \quad \alpha > 0.$$

The parameter α (which has been assumed to be equal to 1 in [61] and chapter 2) models the responsiveness (or elasticity), of the company's volume of business to a change in premium in the preceding year k. Thus, we continue to propose that the volume of business in year $k+1$ is proportional to the ratio of the market's average premium to the company's premium in year $k+1$ changed by a particular sensitivity parameter α. This important sensitivity factor is indicative of the degree to which a change in this ratio and completion affects the volume of business, see also Lemma 1. According to Moody's investor service report (May 2013)[1], all-times high levels of competition in general insurance industry challenge underwriting profitability, which should lead to upward pressure on premiums across most lines of business.

Now, the second part has a smaller influence on the volume of business, since it is related to the company's reputation and the stochastic variable θ_k. Thus, it incorporates different parameters related to the fame, reputation and to what extend this reputation influences its volume of business. As a result of the severe financial crisis in 2007-2008, nowadays buyers are more skeptical before purchasing any insurance contract even if the premium is suspiciously very low, as there is a fear of losing money due to a company's potential bankruptcy. The Reputation Review 2012 by Oxford Metrica[2], which monitors the reputation performance in the world, and *the role of reputation in the professional service firms* (University of Oxford, Novak Druce Centre for Professional Service Firms vol.6) are two characteristic examples that show the importance of the influental factor of reputation on a company's demand.

Therefore, the company's reliability is also considered very thoughtfully. The effects of reputation on the volume of business are modeled by the parameter γ_k, which considers the company's gains or losses in the market due to the reputation, and by the sensitivity parameter β. It is an important factor for the volume's of business elasticity regarding a change in fame in the year, see also Lemma 2. The aforementioned sign of γ_k is respectively positive and equal to $+1$ when the company has a good fame and reputation or -1 for the opposite.

Additionally, the stochastic parameter θ_k is considered which comprises of all other variables that are relevant to the demand function in year $[k, k+1)$. This stochastic variable can take either positive or negative values. However, the volume of business will be exponentially affected using the natural exponential function of θ_k, i.e. $e^{\theta_k} \in [0, \infty)$.

[1] UK General Insurance Outlook
$http://www.actuarialpost.co.uk/downloads/cat_1/UKGeneralInsuranceOutlook.pdf$

[2] Oxford Metrica, Reputation Review 2012
http://www.aon.com/attachments/risk-services/Aon-OM-Reputation-Review-2012.pdf

[2] University of Oxford, Novak Druce Centre for Professional Service Firms vol.6
$http://www.sbs.ox.ac.uk/sites/default/files/Novak_Druce/Doc/TheroleofreputationinPSFs.pdf$

To summarize the discussion above, we can now assume that the volume of business is given by

$$V_k = V_{k-1}\left(\frac{\bar{p}_k}{p_k}\right)^{\alpha} + sign(\gamma_k)|\gamma_k|^{\beta} e^{\theta_k}, \tag{3.3}$$

where e^{θ_k} is being involved as an additive disturbance.

With the following lemmas, the premium and reputation elasticity of the volume of business are provided. As we can easily observe, the sensitivity factors', α and β, play a key-role. The proofs of those Lemmas derive straightforwardly, and they are omitted.

Lemma 1. *The premium elasticity of the volume of business is equal to*

$$\frac{dV_k}{V_k} / \frac{dp_k}{p_k} = -\alpha \frac{V_{k-1}}{V_k}\left(\frac{\bar{p}_k}{p_k}\right)^{\alpha}.$$

□

Lemma 2. *Moreover the reputation elasticity to the volume of business is equal to*

$$\frac{dV_k}{V_k} / \frac{d\,|\gamma_k|}{|\gamma_k|} = sign\,(\gamma_k)\,\beta \frac{|\gamma_k|^{\beta} e^{\theta_k}}{V_k}.$$

□

The derived formulas can test the sensitivity of the volume of business when particular changes of the premium and reputation occur as well as how and to what extent the sensitivity factors α and β affect the volume of business. The calculation of the optimal premium is given in the next sub-section.

3.4 Calculation of the Optimal Premium

In this sub-section, the premium $p^*_{\max,k}$, that the insurance company intends to charge is calculated by maximizing the expected total utility of the wealth eq. (3.2) and (3.3), both for $sign(\gamma_k) = 1$ and $sign(\gamma_k) = -1$, over a finite time horizon T, and over a choice of strategies p. The next Theorem provides the optimal premium strategy for the finite time horizon maximization problem eqs. (3.1) - (3.3), see also [40] and [49].

Theorem 2. *For the wealth process $\{w_k\}_{k\in 0,1,\ldots,T-1}$ given by*

$$w_{k+1} = -a_k w_k + (p_k - \pi_k)\left(V_{k-1}\left(\frac{\bar{p}_k}{p_k}\right)^{\alpha} + sign\,(\gamma_k)\,|\gamma_k|^{\beta} e^{\theta_k}\right), \tag{3.4}$$

and for the maximization problem defined by

$$\max_{p_k} \mathbb{E}\left[\sum_{i=k}^{T-1} \upsilon^i w_i\right], \tag{3.5}$$

with initial conditions w_0, V_0, V_{-1}, a_0, and γ_0 the optimal strategy process $p^*_{\max,k}$ is given as a solution to the polynomial when

(a) $\alpha > 1$,

$$p_k^{\alpha+1} + b_1 p_k + b_2 = 0 \quad and \quad 0 < p_k < \left(1 + \frac{2}{\alpha - 1}\right)\pi_k, \quad for \quad k = 0, 1, ..., T-1 \tag{3.6}$$

(b) $0 < \alpha \leq 1$,

$$p_k^{\alpha+1} + b_1 p_k + b_2 = 0, \quad for \quad k = 0, 1, ..., T-1, \tag{3.7}$$

where $b_1 = \frac{(1-\alpha)V_{k-1}\mathbb{E}\left(\bar{p}_k^\alpha\right)}{sign(\gamma_k)|\gamma_k|^\beta \mathbb{E}\left(e^{\theta_\kappa}\right)}$, $b_2 = \frac{\alpha\pi_k V_{k-1}\mathbb{E}\left(\bar{p}_k^\alpha\right)}{sign(\gamma_k)|\gamma_k|^\beta \mathbb{E}\left(e^{\theta_\kappa}\right)}$ *(or* $b_2 = \frac{\alpha\pi_k}{1-\alpha} b_1, for\ \alpha \neq 1$*), and* $\bar{p}_k$, π_k *is the average and the break-even premium respectively, in year k; V_{k-1} is the volume of exposure underwritten by the insurer in year $k-1$, and $\mathbb{E}\left(e^{\theta_k}\right)$ is the expectation of the natural exponential function of the (stochastic) disturbance θ_k in year k, γ_k denotes the reputation effect in year k and the maximum value of eq. (3.5) is given by*

$$w_0 d_0 + e_0. \tag{3.8}$$

Moreover, we define

$$d_k = \upsilon^k - a_k d_{k+1} > 0 \quad and \quad d_T = 0, \tag{3.9}$$

$$e_k = \left(p^*_{\max,k} - \pi_k\right)\left[V_{k-1}\frac{\mathbb{E}\left(\bar{p}_k^\alpha\right)}{\left(p^*_{\max,k}\right)^\alpha} + sign\left(\gamma_k\right)|\gamma_k|^\beta \mathbb{E}\left(e^{\vartheta_k}\right)\right] d_{k+1} + e_{k+1},$$

and

$$e_T = 0. \tag{3.10}$$

Proof. First, let's define

$$J_k(w_k) \triangleq \max_{p_k, p_{k+1}, ..., p_{T-1}} \mathbb{E}_{|w_k}\left[\sum_{i=k}^{T-1} \upsilon^i w_i\right]. \tag{3.11}$$

Then, as it is known from [40], the optimal performance criterion satisfies the Bellman equation

$$J_k(w_k) = \max_{p_k} \mathbb{E}_{|w_k} \left\{ \upsilon^k w_k + J_{k+1}(w_{k+1}) \right\} = \max_{p_k} \left\{ \upsilon^k w_k + \mathbb{E}_{|w_k} J_{k+1}(w_{k+1}) \right\}, \quad (3.12)$$

where $\mathbb{E}_{|w_k}\left(\bar{p}_k^{\alpha}\right) = \mathbb{E}\left(\bar{p}_k^{\alpha}\right)$ and $\mathbb{E}_{|w_k}\left(e^{\theta_k}\right) = \mathbb{E}\left(e^{\theta_k}\right)$, and $J_T(w_T) = w_T d_T + e_T = 0$; see eqs. (3.9), (3.10) and (3.12).

We now show by induction that

$$J_k(w_k) = w_k d_k + e_k, \quad (3.13)$$

solves (3.12) by noting that (3.13) is true for $k = T$ and by assuming that (3.12) is true for $k+1$ and by proving is true for k. Substituting the assumed expression for $J_{k+1}(w_{k+1})$ into the right hand side (3.12) we obtain

$$J_k(w_k) = \max_{p_k} \left\{ \upsilon^k w_k + \mathbb{E}_{|w_k} J_{k+1}(w_{k+1}) \right\} = \max_{p_k} \left\{ \upsilon^k w_k + \mathbb{E}_{|w_k}(w_{k+1}) d_{k+1} + e_{k+1} \right\},$$

and from (3.4) we have

$$\max_{p_k} \left\{ \upsilon^k w_k + \left[-a_k w_k + (p_k - \pi_k)\, \mathbb{E}_{|w_k} V_k \right] d_{k+1} + e_{k+1} \right\} =$$

$$\max_{p_k} \{ \upsilon^k w_k - a_k w_k d_{k+1} + d_{k+1}\,(p_k - \pi_k) \left(V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{p_k^{\alpha}} + sign\left(\gamma_k\right) |\gamma_k|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) \right) + e_{k+1} \}. \quad (3.14)$$

The controller that maximizes the eq. (3.14), is given by eq. (3.6) or (3.7), since

$$\begin{aligned} A &= w_k \left(\upsilon^k - a_k d_{k+1} \right) + d_{k+1}\,(p_k - \pi_k) \left(V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{p_k^{\alpha}} + sign\left(\gamma_k\right) |\gamma_k|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) \right) + e_{k+1} \\ &= w_k \left(\upsilon^k - a_k d_{k+1} \right) + p_k V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{p_k^{\alpha}} d_{k+1} + sign\left(\gamma_k\right) p_k |\gamma_k|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) d_{k+1} \\ &- \pi_k V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{p_k^{\alpha}} d_{k+1} - \pi_k sign\left(\gamma_k\right) |\gamma_k|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) d_{k+1} + e_{k+1}. \end{aligned}$$

The first derivative of A with respect to p_k is given by

$$\frac{\partial A}{\partial p_k} = (1-\alpha)\, V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{p_k^{\alpha}} d_{k+1} + sign\left(\gamma_k\right) |\gamma_k|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) d_{k+1} + \alpha \pi_k V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{a}\right)}{p_k^{\alpha+1}} d_{k+1}.$$

If we equalize the first derivative with zero, i.e. $\frac{\partial A}{\partial p_k} = 0$, we obtain

$$(1-\alpha)\, V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{p_k^{\alpha}} d_{k+1} + sign\left(\gamma_k\right) \left|\gamma_k\right|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) d_{k+1} + \alpha \pi_k V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{p_k^{\alpha+1}} d_{k+1} = 0,$$

$$d_{k+1}\left[(1-\alpha)\, V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{p_k^{\alpha}} + sign\left(\gamma_k\right) \left|\gamma_k\right|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) + \alpha \pi_k V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{p_k^{\alpha+1}}\right] = 0,$$

$$(1-\alpha)\, V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{p_k^{\alpha}} + sign\left(\gamma_k\right) \left|\gamma_k\right|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) + \alpha \pi_k V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{p_k^{\alpha+1}} = 0,$$

$$sign\left(\gamma_k\right) \left|\gamma_k\right|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) {p_k}^{\alpha+1} + (1-\alpha)\, V_{k-1} \mathbb{E}\left(\bar{p}_k^{\alpha}\right) p_k + \alpha \pi_k V_{k-1} \mathbb{E}\left(\bar{p}_k^{\alpha}\right) = 0.$$

Finding the solution of the polynomial, see eq. (3.6) (or eq. (3.7)) and substituting it into eq. (3.14) and then if we substitute eq. (3.9) and (3.10), we conclude that eq. (3.14) is true.

The above expression gives the optimal strategy eq. (3.6) (or eq. (3.7)) when

$$\frac{\partial A}{\partial^2 p_k} = -\alpha V_{k-1} \mathbb{E}\left(\bar{p}_k^{\alpha}\right) d_{k+1} {p_k}^{-(\alpha+1)} \left[(1-\alpha) + (\alpha+1)\frac{\pi_k}{p_k}\right] < 0.$$

It is known that $\alpha,\ V_{k-1},\ \mathbb{E}(\bar{p}_k)^{\alpha},\ d_{k+1},\ {p_k}^{-(\alpha+1)}$ are positive so two cases are derived:

(a) for $\alpha > 1$: the optimal premium pricing strategy is derived when the inequality $0 < p_k < \left(1 + \frac{2}{\alpha-1}\right) \pi_k$ holds. In other words, we have an upper bound for the optimal premium that the company has to charge in year k. It is interesting to note that the upper bound is related to the break-even premium and the elasticity parameter α.

(b) for $0 < \alpha \leq 1$: we always have an optimal strategy. □

Remark 5. Since the root of the polynomial given by eq. (3.6) (or eq. (3.7)) is the optimal premium, it must be a real and positive number. Thus, we have to ensure that the polynomial function (3.6) (or (3.7)) will have at least one real and positive root.

Thus, considering the Descartes' rule of signs and elements of the polynomial theory, in order to have at least one positive real root for the polynomial (3.6) (or (3.7)), the sign variations in the sequence of coefficients of the polynomial must be two or one. Then, if the polynomial has one sign variation, then it has exactly one positive real root and if the polynomial has two sign variations, then it has either two or zero real positive roots.

More precisely, in the case that $sign(\gamma_k) = 1$, there are two possible sub-cases for the polynomial's coefficients:

- For $\alpha > 1$, $b_1 < 0$ and $b_2 > 0$, the polynomial has *two* or *zero positive* roots.
- For $0 < \alpha \leq 1$, $b_1 > 0$ and $b_2 > 0$, the polynomial has *no positive* root.

Moreover, in the case that $sign(\gamma_k) = -1$, there are also two possible sub-cases for the polynomial's coefficients (see also Remark 7):

- For $\alpha > 1$, $b_1 > 0$ and $b_2 < 0$, the polynomial has exactly *one positive* root.
- For $0 < \alpha \leq 1$, $b_1 < 0$ and $b_2 < 0$, the polynomial has exactly *one positive* root.

Combing the Theorem's 2 results and requesting at least one *real positive* root lead to the following corollary which gives us three possible directions that guarantee the existence of an optimal strategy, i.e. the calculation of the insurance premium:

Corollary 1. *For the parameters of the Theorem 2, we have positive optimal solution,* $p^*_{\max,k}$*:*

I *For* $sign(\gamma_k) = 1$*, with* $\alpha > 1$ *and* $0 < p_k < \left(1 + \frac{2}{\alpha-1}\right)\pi_k$*, when the polynomial (3.6) has two positive roots.*

II *For* $sign(\gamma_k) = -1$ *with* $0 < \alpha \leq 1$*, when the polynomial (3.7) has exactly one positive root.*

III *For* $sign(\gamma_k) = -1$ *with* $\alpha > 1$ *and* $0 < p_k < \left(1 + \frac{2}{\alpha-1}\right)\pi_k$*, when the polynomial (3.6) has exactly one positive root.*

Remark 6. Let's further investigate the case that $sign(\gamma_k) = 1$, and eventually the polynomial (3.6) has *zero* positive roots (either for $\alpha > 1$ or $0 < \alpha \leq 1$). Since V_{k-1}, $\left(\frac{\bar{p}_k}{p_k}\right)^\alpha$, $|\gamma_k|^\beta$ and e^{θ_k} are positive when $sign(\gamma_\kappa) = 1$, the company's volume of business increases as a result of the *good* fame as long as all the other parameters remain unchanged, i.e. positive. Practically speaking, if the company's reputation and the parameter θ_k tend to be particularly high, i.e. go to positive infinity, the volume of business will tend to be positive infinite as well. Thus, in this case, the wealth function (3.1) will increase as well, and tend to infinity too, as the excess return on capital is small, corresponding to the product of the difference between p_k and the break-even premium π_k multiplied by the volume of business.

Consequently, as it derives from Remark 6, when the company has a very *good* reputation, and the factor $0 < \alpha \leq 1$ or $\alpha > 1$ (without a positive root), the optimal premium *cannot* be defined by solving another maximization problem. In other words, the company is very flexible to choose any premium it wishes, since the very good reputation guarantee the increase of its profits. Obviously, as it has been also discussed by Pantelous and Passalidou [61], the previous year premium strategy, i.e. p_{k-1} is characterized as a very successful choice, so the company can preserve it for one more year. However, it might also be useful to calculate the minimum excess premium strategy, $\tilde{p}^*_{\min,k}$, that the company could charge in order to have a positive expected wealth (3.1) and to stay competitive in the insurance market.

Thus, the next theorem provides an upper bound for the optimal premium strategy, the minimum premium excess strategy process $\tilde{p}^*_{\min,k}$, when the company is targeting a particular wealth $\{W_k\}_{k\in 0,1,\ldots,T-1}$ for the finite time horizon minimization problem, eqs. (3.15) - (3.16), see also [40] and [49] when $sign(\gamma_\kappa) = 1$ and $\alpha > 0$.

Theorem 3. *For the wealth process $\{w_k\}_{k\in 0,1,\ldots,T-1}$ given by*

$$w_{k+1} = -a_k w_k + (\pi_k + \tilde{p}_k)\left(V_{k-1}\left(\frac{\bar{p}_k}{\tilde{p}_k}\right)^\alpha + |\gamma_k|^\beta e^{\theta_k}\right), \tag{3.15}$$

where $\alpha > 0$ and for the minimization problem defined by

$$\min_{p_k} \mathbb{E}\left[\sum_{i=k}^{T-1} \upsilon^i (W_i - w_i)\right], \tag{3.16}$$

with initial conditions w_0, V_0, V_{-1}, a_0, γ_0 and targeted wealth $\{W_k\}_{k\in 0,1,\ldots,T-1}$, the minimum premium excess strategy process $\tilde{p}^*_{\min,k}$ is given by

$$\tilde{p}_k^{\alpha+1} + \tilde{b}_1 \tilde{p}_k + \tilde{b}_2 = 0 \quad \text{and} \quad \left(1 + \frac{2}{\alpha - 1}\right)\pi_k < \tilde{p}_k, \quad \text{for} \quad k = 0, 1, ..., T-1 \tag{3.17}$$

where $\tilde{b}_1 = -\frac{(1-\alpha)V_{k-1}\mathbb{E}\left(\bar{p}_k^a\right)}{|\gamma_k|^\beta \mathbb{E}\left(e^{\theta_\kappa}\right)}$, $\tilde{b}_2 = -\frac{a\pi_k V_{k-1}\mathbb{E}\left(\bar{p}_k^a\right)}{|\gamma_k|^\beta \mathbb{E}\left(e^{\theta_\kappa}\right)}$ or $\frac{a\pi_k}{1-a}\tilde{b}_1$ when $a \neq 1$. The parameters $\bar{p}_k$, π_k, V_{k-1} and $\mathbb{E}\left(e^{\theta_k}\right)$ have been defined in Theorem 2. Then, the minimum value of (3.15) is given by

$$(W_0 - w_0)\, d_0 + \varepsilon_0. \tag{3.18}$$

Moreover, we define

$$d_k = \upsilon^k - a_k d_{k+1} > 0 \quad \text{and} \quad d_T = 0, \tag{3.19}$$

$$\varepsilon_k = a_k d_{k+1} W_k + W_{k+1} d_{k+1} - \tilde{p}^*_{\min,k} V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^a\right)}{\left(\tilde{p}^*_{\min,k}\right)^a} d_{k+1}$$
$$-\tilde{p}^*_{\min,k} |\gamma_k|^\beta \mathbb{E}\left(e^{\vartheta_k}\right) d_{k+1} \;\; - \pi_k V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^a\right)}{\left(\tilde{p}^*_{\min,k}\right)^a} d_{k+1} - \pi_k |\gamma_k|^\beta \mathbb{E}\left(e^{\vartheta_k}\right) d_{k+1} + \varepsilon_{k+1},$$

and

$$e_T = 0. \tag{3.20}$$

Proof. First, let's define

$$J_k(W_k - w_k) \triangleq \min_{\tilde{p}_k, \tilde{p}_{k+1}, \ldots, \tilde{p}_{T-1}} \mathbb{E}_{|w_k} \left[\sum_{i=k}^{T-1} \upsilon^i (W_i - w_i) \right]. \tag{3.21}$$

Then, the Bellman equation is given by

$$J_k(w_k) = \min_{\tilde{p}_k} \left\{ \upsilon^k (W_k - w_k) + \mathbb{E}_{|w_k} J_{k+1}(W_{k+1} - w_{k+1}) \right\}, \tag{3.22}$$

where $\mathbb{E}_{|w_k}\left(\bar{p}_k^\alpha\right) = \mathbb{E}\left(\bar{p}_k^\alpha\right)$ and $\mathbb{E}_{|w_k}\left(e^{\theta_k}\right) = \mathbb{E}\left(e^{\theta_k}\right)$, and $J_T(W_T - w_T) = (W_T - w_T)d_T + e_T = 0$; see eqs. (3.19), (3.20) and (3.22).

We now show by induction that

$$J_k(W_k - w_k) = (W_k - w_k)d_k + e_k, \tag{3.23}$$

solves (3.22) by noting that (3.23) is true for $k = T$ and by assuming that (3.22) is true for $k+1$ and by proving is true for k. Substituting the assumed expression for $J_{k+1}(W_{k+1} - w_{k+1})$ into the right hand side (3.22) we obtain

$$J_k(w_k) = \min_{\tilde{p}_k} \left\{ \upsilon^k (W_k - w_k) + \mathbb{E}_{|w_k} (W_{k+1} - w_{k+1}) d_{k+1} + e_{k+1} \right\},$$

and from (3.4) we have

$$\begin{aligned} &\min_{\tilde{p}_k} \{ (W_k - w_k) \left(\upsilon^k - a_k d_{k+1} \right) + W_k a_k d_{k+1} + W_{k+1} d_{k+1} \\ &-d_{k+1} \left(\pi_k + \tilde{p}_k \right) \left(V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^\alpha\right)}{\tilde{p}_k^\alpha} + |\gamma_k|^\beta \mathbb{E}\left(e^{\vartheta_k}\right) \right) + \varepsilon_{k+1} \} \end{aligned} \tag{3.24}$$

The controller that minimizes the above expression, eq. (3.24), is given by eq. (3.6) or

(3.7), since

$$\begin{aligned}A = &(W_k - w_k)\left(\upsilon^k - a_k d_{k+1}\right) + W_k a_k d_{k+1} + W_{k+1} d_{k+1} \\ &-\tilde{p}_k V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{\tilde{p}_k^{\alpha}} d_{k+1} - \tilde{p}_k |\gamma_k|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) d_{k+1} \\ &- \pi_k V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{\tilde{p}_k^{\alpha}} d_{k+1} - \pi_k |\gamma_k|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) d_{k+1} + \varepsilon_{k+1}.\end{aligned}$$

The first derivative of A with respect to $\tilde{p}_k$ is given by

$$\frac{\partial A}{\partial \tilde{p}_k} = (1-a)\, V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{a}\right)}{\tilde{p}_k^{a}} d_{k+1} - |\gamma_k|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) d_{k+1} + \alpha \pi_k V_{k-1} \frac{\mathbb{E}\left(\bar{p}_k^{\alpha}\right)}{\tilde{p}_k^{\alpha+1}} d_{k+1}.$$

If we equalize the first derivative with zero, i.e. $\frac{\partial A}{\partial \tilde{p}_k} = 0$, we obtain

$$|\gamma_k|^{\beta} \mathbb{E}\left(e^{\vartheta_k}\right) \tilde{p}_k^{\alpha+1} - (1-\alpha)\, V_{k-1} \mathbb{E}\left(\bar{p}_k^{\alpha}\right) \tilde{p}_k - \alpha \pi_k V_{k-1} \mathbb{E}\left(\bar{p}_k^{\alpha}\right) = 0$$

Finding the solution of the polynomial, see eq. (3.6) (or eq. (3.7)) and substituting it into eq. (3.24) and then if we substitute eq. (3.19) and eq. (3.20), we conclude to the fact that eq. (3.23) is true. The above expression gives the optimal strategy, see eq. (3.7) as

$$\frac{\partial A}{\partial^2 \tilde{p}_k} = -\alpha V_{k-1} \mathbb{E}\left(\bar{p}_k^{\alpha}\right) d_{k+1} \tilde{p}_k^{-(\alpha+1)} \left[(1-\alpha) + (\alpha+1)\frac{\pi_k}{\tilde{p}_k}\right] > 0.$$

□

Remark 7. It appears that, the optimal premium strategy given by eq. (3.18) derives quite naturally using elements of dynamic programming. Thus, as in [61], the optimal strategy depends endogenously on the previous year's volume of business, the break-even premium rate, the expected value of the average premium rate, the company's fame and reputation, and the expected value of the natural exponential function of the variable θ_k.

Remark 8. Since the root of the polynomial given by eq. (3.17) is the optimal premium, it must be again a real and positive number. Thus, considering the fundamental Descartes' rule of signs and elements of the polynomial theory, the sign variations in the sequence of coefficients of the polynomial (3.17) appears to be one (either + + - or + - -), and thus, it has always exactly *one* positive real root.

In the remaining part of section 3.4, some special cases for the parameter α are considered, when analytical formulae for the root of polynomial (3.6) (or (3.7) or (3.17)) is derived.

First case: When $\alpha = 1, 2, 3..... \in \mathbb{N}$

Proposition 4. *For $\alpha = 1$, then the optimal premium is given by*

$$p^*_{\max,k} = \sqrt{\frac{V_{k-1}\pi_k \mathbb{E}\left(\bar{p}_k\right)}{|\gamma_k|^{\beta}\mathbb{E}\left(e^{\theta_k}\right)}} \in \mathbb{R}_+ \quad when \quad sign(\gamma_k) = -1.$$

Proof. For $\alpha = 1$ then eq. (3.6) is equal to

$$p^2_{\max,k} + \frac{\pi_k V_{k-1}\mathbb{E}\left(\bar{p}_k\right)}{sign(\gamma_k)|\gamma_k|^{\beta}\mathbb{E}\left(e^{\theta_k}\right)} = 0 \Leftrightarrow p^*_{\max,k} = \sqrt{\frac{\pi_k V_{k-1}\mathbb{E}\left(\bar{p}_k\right)}{|\gamma_k|^{\beta}\mathbb{E}\left(e^{\theta_k}\right)}}.$$

when $sign(\gamma_k) = -1$.

The above expression gives always the optimal strategy since

$$\frac{\partial A}{\partial^2 p_k} = -V_{k-1}\mathbb{E}\left(\bar{p}_k\right) d_{k+1} p_k^{-2} 2\frac{\pi_k}{p_k} < 0.$$

$\square$

Remark 9. This is the optimal premium that has derived in Theorem 1; see [61] and chapter 2.

Proposition 5. *For $\alpha = 2$, then the optimal premium is given by*

$$p^*_{\max,k} = \nu_1 + \nu_2, \quad when \quad \Delta > 0 \quad and \quad p_k < 3\pi_k,$$

$$where\ \nu_1 = sign(w_1)\sqrt[3]{|w_1|}\ ,\ \nu_2 = sign(w_2)\sqrt[3]{|w_2|},\ sign(w_i) = \left\{\begin{array}{ccc} 1 & if & w_i > 0 \\ -1 & if & w_i < 0 \\ 0 & if & w_i = 0 \end{array}\right\},$$

and w_1, w_2 *are the roots of the equation* $w^2 - 2\pi_k b_1 w - \frac{b_1^3}{27} = 0$, *and*

$$\Delta = \pi_k^2 b_1^2 + \frac{4}{27} {b_1}^3, \quad b_1 = -\frac{V_{k-1}\mathbb{E}\left(\bar{p}_k^2\right)}{sign(\gamma_k)|\gamma_k|^\beta \mathbb{E}\left(e^{\theta_\kappa}\right)}, \quad b_2 = \frac{2\pi_k V_{k-1}\mathbb{E}\left(\bar{p}_k^2\right)}{sign(\gamma_k)|\gamma_k|^\beta \mathbb{E}\left(e^{\theta_\kappa}\right)} = -2\pi_k b_1.$$

Proof. For $\alpha = 2$ then eq. (3.6) is equal to

$$p_k^3 + b_1 p_k - 2\pi_k b_1 = 0.$$

Now, following Cardano's method[3] we substitute $p_k = u + z$, then we take

$$p_k^3 = u^3 + z^3 + 3uz\,(u+z)\,,$$

and the equation above is equal to

$$u^3+z^3+3uz\,(u+z)+b_1\,(u+z)-2\pi_k b_1 = 0 \Leftrightarrow \left(u^3+z^3-2\pi_k b_1\right)+(u+z)\,(3uz+b_1) = 0.$$

Thus, all the pairs (u, z) that verify the system $\left.\begin{array}{rr} u^3+z^3-2\pi_k b_1 = & 0 \\ 3uz+b_1 = & 0 \end{array}\right\}$ are solutions. Moreover,

$$\left.\begin{array}{rr} u^3+z^3-2\pi_k b_1 = & 0 \\ 3uz+b_1 = & 0 \end{array}\right\} \Rightarrow \left.\begin{array}{rr} u^3+z^3 = & 2\pi_k b_1 \\ uz = & -\frac{b_1}{3} \end{array}\right\} \Rightarrow \left\{\begin{array}{rr} u^3+z^3 = & 2\pi_k b_1 \\ u^3 z^3 = & -\frac{b_1^3}{27}. \end{array}\right.$$

Obviously, u^3, z^3 are roots of the equation $w^2-2\pi_k b_1 w-\frac{b_1^3}{27} = 0$, where $\Delta = \pi_k^2 b_1^2+\frac{4}{27}{b_1}^3$. When $\Delta = \pi_k^2 b_1^2+\frac{4}{27}{b_1}^3 > 0$, eq. (3.6) has *two* real solutions $w_1, w_2 \in \mathbb{R}$ and the optimal strategy is given by $p_k^* = \nu_1 + \nu_2$, where

$$v_1 = sign(w_1)\sqrt[3]{|w_1|}, \quad v_2 = sign(w_2)\sqrt[3]{|w_2|} \text{ and } sign(w_i) = \left\{\begin{array}{ccc} 1 & if & w_i > 0 \\ -1 & if & w_i < 0 \\ 0 & if & w_i = 0 \end{array}\right. .$$

In order to have an optimal strategy $\frac{\partial A}{\partial^2 p_k} = -2V_{k-1}\mathbb{E}(\bar{p}_k)^2 \frac{d_{k+1}}{p_k^3}\left[-1+3\frac{\pi_k}{p_k}\right]$ must be below zero from where the condition $p_k < 3\pi_k$ is derived. □

[3] Jacobson, Nathan (2009), Basic algebra 1 (2nd ed.), Dover, p.210

Remark 10. When $sign(\gamma_k) = 1$, and $b_1 < 0$, then $\Delta = \pi_k^2 b_1^2 + \frac{4}{27}{b_1}^3$ can take any value, and the polynomial given by eq. (3.6) has two sign changes and *two* or *zero* positive real roots.

However, when $sign(\gamma_k) = -1$, $b_1 > 0$, then $\Delta = \pi_k^2 b_1^2 + \frac{4}{27}{b_1}^3 > 0$, the polynomial given by eq. (3.6) has one sign change and exactly *one* positive real root.

Proposition 6. *For $\alpha = 3$, then the optimal premium is given by*

- $p^*_{\max,k_1} = \frac{1}{2}\left(k + \sqrt{\Delta_2}\right)$,
- $p^*_{\max,k_2} = \frac{1}{2}\left(k - \sqrt{\Delta_2}\right)$,
- $p^*_{\max,k_3} = \frac{1}{2}\left(k + \sqrt{\Delta_3}\right)$,
- $p^*_{\max,k_4} = \frac{1}{2}\left(k - \sqrt{\Delta_3}\right)$,

when $p^*_{\max,k_1}$, $p^*_{\max,k_2}$, $p^*_{\max,k_3}$, $p^*_{\max,k_4} \in \mathbb{R}_+$ and $p^*_{\max,k} < 2\pi_k$, where $\Delta_2 = k^2 - 4g$, $\Delta_3 = k^2 - 4h$, $k = \sqrt{v_1 + v_2}$, $g = v_1 + v_2 - \frac{b_1}{k}$, $h = v_1 + v_2 + \frac{b_1}{k}$, and $v_1 = sign(w_1)\sqrt[3]{|w_1|}$ and $v_2 = sign(w_2)\sqrt[3]{|w_2|}$, where $sign(w_i) = \begin{cases} 1 & if \quad w_i > 0 \\ -1 & if \quad w_i < 0 \\ 0 & if \quad w_i = 0 \end{cases}$, and w_1, w_2 are the roots of the equation $w^2 - b_1^2 w - 8(\pi_k b_1)^3 = 0$, and $b_1 = -2\frac{V_{k-1}\mathbb{E}(\bar{p}_k)^3}{sign(\gamma_k)|\gamma_k|^\beta \mathbb{E}(\bar{p}_k)^\alpha}$, $b_2 = \frac{3\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)^3}{sign(\gamma_k)|\gamma_k|^\beta \mathbb{E}\left(e^{\theta_k}\right)} = -\frac{3}{2}b_1$.

Proof. For $\alpha = 3$, then the eq. (3.6) is equal to

$$p_k^4 + b_1 p_k - \frac{3}{2}\pi_k b_1 = 0 \text{ where } b_1 = -2\frac{V_{k-1}\mathbb{E}(\bar{p}_k)^3}{sign(\gamma_k)|\gamma_k|^\beta \mathbb{E}(\bar{p}_k)^\alpha}, \; b_2 = \frac{3\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)^3}{sign(\gamma_k)|\gamma_k|^\beta \mathbb{E}\left(e^{\theta_k}\right)}.$$

According to Descrates-Euler-Cardano[4] algorithm, in order to solve the quartic equation, we first need to solve a particular cubic equation, the coefficients of which are derived from those of quartic. A root of the cubic is then used to factorize the quartic into quadratics, which is solved. Following the above algorithm, we use a subsidiary cubic with the coefficients: $y^3 - 4\left(-\frac{3}{2}\pi_k b_1\right)y - b_1^2 = 0 \Leftrightarrow y^3 + 6\pi_k b_1 y - b_1^2 = 0$.

[4]Solving quartics and cubics for graphics, Dr Don Herbison-Evans, The university of Sydney (1994) http://sydney.edu.au/engineering/it/research/tr/tr487.pdf

Following again Cardano's method (see proof of proposition 2) this cubic polynomial

$$w^2 - b_1^2 w - \frac{(6\pi_k b_1)^3}{27} = 0 \Leftrightarrow w^2 - b_1^2 w - 8(\pi_k b_1)^3 = 0.$$

The discriminant of this quadratic equation is equal to

$$\Delta = (b_1)^4 + 4 * 8(\pi_k b_1)^3 = 0 \Leftrightarrow \Delta = (b_1)^4 + 32(\pi_k b_1)^3 = 0.$$

When $\Delta = (b_1)^4 + 32(\pi_k b_1)^3 > 0$, eq. (3.6) has *two* real solutions $w_1, w_2 \in \mathbb{R}$ and the solution of the cubic polynomial is given by $p_k^* = \nu_1 + \nu_2$, where $v_1 = sign(w_1)\sqrt[3]{|w_1|}$ and $v_2 = sign(w_2)\sqrt[3]{|w_2|}$ where

$$sign(w_i) = \begin{cases} 1 & if \quad w_i > 0 \\ -1 & if \quad w_i < 0 \\ 0 & if \quad w_i = 0. \end{cases}$$

In this algorithm the solution of the quartic is obtained by the quadratics:

$$p_k^2 + kp_k + g = 0 \; and \; p_k^2 - kp_k + h = 0.$$

where $k = (y)^{1/2}$, $g = y - \frac{b_1}{k}$, $h = y + \frac{b_1}{k}$. □

Remark 11. When $sign(\gamma_k) = 1$, and $b_1 < 0$, the polynomial which is given by eq. (3.6) has two sign changes and *two* or *zero* positive real roots. However, when $sign(\gamma_k) = -1$, $b_1 > 0$, the polynomial which is given by eq. (3.6) has one sign change and exactly *one* positive real root.

Remark 12. For $\alpha > 3$, no general formula exists (or more precisely, no formula in terms of addition, subtraction, multiplication, division, arbitrary constants and n-th roots). This result is proved in Galois Theory and is known as the Abel-Ruffini theorem. Nevertheless, finding solutions to higher order polynomial formulas is affordable using numerical methods, e.g., Newton's method.

Second case: When $sign(\gamma_k) = -1$ and $\alpha \in (0, 1)$.

If we write α as a ratio of numbers i.e. $\alpha = {}^{\kappa}\!/_{\lambda}$, $\kappa < \lambda$, $\lambda \neq 0$ and $\kappa, \lambda = 1, 2.... \in \mathbb{N}$

and we substitute this ratio into eq. (3.6), then we will get the following function

$$p_k^{\kappa/\lambda+1} + b_1 p_k + b_2 = 0 \Leftrightarrow p_k^{\frac{\kappa+\lambda}{\lambda}} + b_1 p_k + b_2 = 0 \Leftrightarrow \left(p_k^{1/\lambda}\right)^{\kappa+\lambda} + b_1 p_k + b_2.$$

If we pose $p_k^{1/\lambda} = y, y > 0$ *then* $p_k = y^\lambda, p_k, y > 0$. Then the above polynomial is equal to $y^{\kappa+\lambda} + b_1 y^\lambda + b_2 = 0$ and for $\kappa + \lambda = 1, 2, 3, 4$ there is a general formula for the polynomial's solution. For $\kappa + \lambda > 5$ see Remark 11.

In the next section, the numerical algorithm is presented and an interesting application is considered.

3.5 Numerical Application

3.5.1 Data

In order to illustrate the main theoretical findings of this paper, a numerical example based on data from the Greek Automobile Insurance market is presented; see also [61]. Unfortunately, the real data publicly available is not analytic and several assumptions have to be made, leading us to subjective numerical results which illustrate only the applicability of our methodology.

The Greek insurance market, see [61], is an oligopoly comprising of less than 10 key-companies. However, for the purpose of this application, we will focus our attention on the main three companies. So, the approximate number of contracts for the 3 major non-life Greek insurance companies for a standard six-month cover of a 10-year old, 1300cc car (with 2.000 Euros covered amount) is presented for the year 2013 in table 1.

Insurance Companies	Volume of business
A	1,290,320
B	736,621
C	548.861

Table 3.1: The volume of business (i.e. number of contracts) for the automobile insurance market for the three main Greek companies in 2013.

Empirically speaking, see Table 3.1[5] in [61], the elasticity parameter, α, for A, B and C insurance companies has been estimated and assumed to be equal to 5, 2 and 2 respectively.

[5]The financial crisis in 2008 has revealed structural weaknesses in the Greek insurance and financial environment, and thus several companies from those sectors have decided lastly to be emerged. However, as this is just a numerical application for illustrating further the applicability of our theoretical findings, we believe that more justification is not necessary, and it is omitted.

In the next sub-section, the methodology for calculating the average premium is considered and then the algorithm for the optimum pricing stategy is provided.

3.5.2 Premium Strategy: A Generic Multiplicative process for the Market's Average Premium

As it has been considered in [17], [18] and [61], the optimal premium strategy can be calculated only if the average premium process is defined appropriately. Indeed, in the existing main literature, Emms et al. [18] have assumed a simple geometric stochastic differential equation for the evolution of the average premium into a continuous-time framework. However, in our approach the expectation (first moment) of the market's average premium needs to be calculated, and in order to make it more realistic, we will further assume that the market's average premium process has some kind of "memory" or is related to the past year experience. In this way, we have the flexibility to model other kind of macroeconimic events, such as emerges in the market, increment in taxes etc. Consequently, using a point process is an effective way to model the average premium. Point processes with noise $1/f$ were introduced in 1998 by Kaulakys et al. [44] and later on, they were generalized for $1/f^{\beta}$ for $0.5 < \beta < 2$, see Kaulakys et al. [45]. Influenced by this model, we can adopt the main idea and adjust it to our model. Thus, the market's average premium values can be represented as a sequence of correlated pulses or series of events, i.e.

$$x\left(\bar{p}\right) = \bar{a}\sum_{k} \delta\left(\bar{p} - \bar{p}_k\right), \tag{3.25}$$

where $\delta\left(\bar{p}\right)$ is the Dirac δ-function and $\bar{a}$ is an average contribution to the signal $x(\bar{p})$ of one event. Additionally, there is a generic multiplicative process for the market's average premium

$$\bar{p}_k = \bar{p}_{k-1} + \gamma \bar{p}_{k-1}^{2\mu-1} + \sigma \bar{p}_{k-1}^{\mu}\varepsilon_k, \tag{3.26}$$

generating the power law distributed $P_k\left(\bar{p}_k\right) \sim \bar{p}_k^{\alpha}$, $\alpha = \frac{2\gamma}{\sigma^2} - 2\mu$ sequence of the average premium values $\bar{p}_k$ in k-space and $1/f^{\beta}$ power spectral density of the signal (3.25), $S\left(f\right) \sim \frac{1}{f^{\beta}}$, $\beta = 1 + \frac{\alpha}{3-2\mu}$. In this approach the average market's premium fluctuates due to the random perturbations by a sequence of uncorrelated normally distributed random variable $\{\varepsilon_k\}$ with zero expectation and unit variance; σ is the standard deviation of the white noise and γ is the coefficient of the non-linear damping.

In the Figure 3.1, a realistic presentation of the market's average premium is provided. Thus, as it will be assumed in the numerical part of this section, for $\mu = 0.5$, $\sigma = 0.2$ and $\gamma = 2$ into eq. (3.26), the market's average premium is 281.21 Euros in 2013 for a standard six-month cover of a 10-year old, 1300cc car (with 2.000 Euros

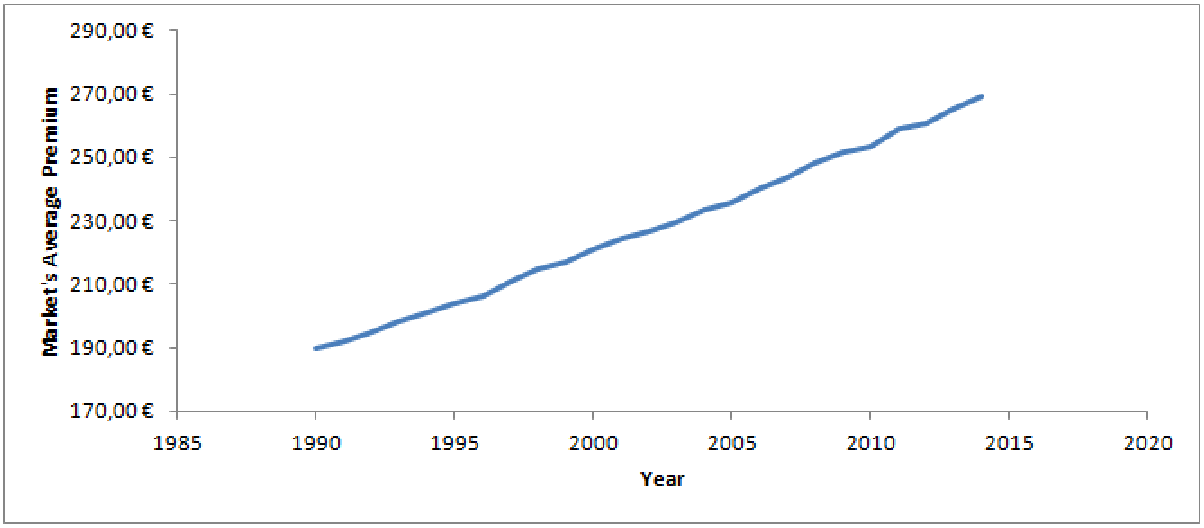

Figure 3.1: A realization of the market's average premium assuming that it was 190 Euros in 1990 for a standard six-month cover of a 10-year old, 1300cc car (with 2.000 Euros covered amount).

covered amount). The following calculations are based on this assumption, see also table 3.2.

Values for α	$\mathbb{E}\left(\bar{p}_k^a\right)$
1	283.21
2	80,489.11
5	1,885,856,128,571.30

Table 3.2: The values of $\mathbb{E}\left(\bar{p}_k^a\right)$ for different α and market's average premium 281.21 Euros for 2013.

3.5.3 Numerical Algorithm

Summarizing the discussion in Section 3.3, in this sub-section, the algorithmic steps for the calculation of the optimal premium are described.

Step 1: *Collect the necessary (historical) data from the insurance market.*
The first step requires the collection of data concerning the number of companies which are in the market, the volume of business, V_{k-1}, and the break-even premium, π_k, from each company for the previous and current financial year, respectively. Moreover, the impact of the reputation, γ_k, and the other stochastic parameters, θ_k, need to be estimated. Obviously, as it will also be the case later, the last two parameters might be unknown or not easily estimated, so different scenarios need to be considered from the insurance company's point of view and strategy design, see also next subsection. Then, step 2 follows.

Step 2: *Estimate parameter α.*

Based on historical data one can estimate parameter α of the eq. (3.3) for each company using data fitting tools. The parameter is a good indication about the elasticity of the market's average premium over the company's premium. Then, step 3 follows.

Step 3: *Estimate the parameters of market's average premium process, $\bar{p}_k$.*

As it has been assumed in the previous sub-section, the average premium can be modelled as a point process, see eq. (3.2). Again with the appropriate data fitting, parameters γ, μ and σ for $\alpha > 1$ are calculated. Thus, the moments for $\bar{p}_k$, i.e. $\mathbb{E}\left(\bar{p}_k^a\right)$ are derived. Then, step 4 follows.

Step 4: *Calculate the coefficients b_1 and b_2 of the polynomial (3.6) (or (3.7)).*

Using the estimated parameter α and the information collected in step 1, the coefficients b_1 and b_2 can be estimated, then step 5 follows.

Step 5: *Calculate the roots of the polynomial (3.6) (or (3.7)).*

I For $sign(\gamma_k) = 1$, with $\alpha > 1$ and $0 < p_k < \left(1 + \frac{2}{\alpha-1}\right)\pi_k$, then the polynomial (2.6) has zero or two positive roots.

II For $sign(\gamma_k) = -1$ with $0 < \alpha \leq 1$, then the polynomial (3.7) has exactly one positive root.

III For $sign(\gamma_k) = -1$ with $\alpha > 1$ and $0 < p_k < \left(1 + \frac{2}{\alpha-1}\right)\pi_k$, then the polynomial (2.6) has exactly one positive root.

The main possible directions for this step are related to the negative or positive effect of the reputation on the volume of business. There are many different ways in order to "measure" the reputation effects, such as sentiment analysis, purchase intent, opinion polls etc. Consequently, when we are able to determine the impact of the reputation, one of the above three possible cases is considered further (see also Corollary 1). Then, step 6 follows.

Step 6: *Design and agreement on the optimal premium strategy for the insurance company.*

Now, for different values of the elasticity parameter β (or/and break-even premium, π_k or/and the effect of reputation, γ_k; see also Step 1) the actuary can generate different values for the optimal premium after solving the polynomial mentioned in step 4; see also Proposition 2, 3, 4 and the second case in section 3.3. Then after taking into consideration the competition in the current (or targeting) insurance market, the reputation of the company, the break-even premium, expectation of the different macroeconomic parameters (i.e. based on the random variable, θ_k). Finally, the optimal premium is calculated and agreed by the senior management of the companies.

In the next sub-section, the numerical calculations are provided for the insurance companies A, B and C.

3.5.4 Numerical Calculations and Discussion

For Steps 1, 2 and 3, it is assumed that the market's average premium is 281.21 Euros in 2013 for a standard six-month cover of a 10-year old, 1300cc car (with 2.000 Euros covered amount), the elasticity parameter, α, for A, B and C insurance companies has been estimated and assumed to be equal to 5, 2 and 2 respectively, and the break-even premium takes a range of values from 200 to 240 Euros, i.e. around $60 - 70\%$ of the average premium.

3.5.5 Optimal premium for different values of β

Then, the following scenarios are considered for companies A, B and C for negative/positive reputation, and $\beta = 2$, 1.5, 1 and 0.5. In the first scenario (see Figure 3.2, 3.4 and 3.5), the companies' reputation has *negative* effect on the volume of their business, since $sign(\gamma_k) = -1$, $|\gamma_k| = 2$, and $\mathbb{E}\left(e^{\theta_\kappa}\right) = 59,874$. Thus, for $\beta = 2$, 1.5, 1 and 0.5, the company is approximately losing 239,496, 169,349, 119,748 and 84,674 contracts respectively. In the second scenario (see Figure 3.3), the companies' reputation has *positive* effect on the volume of business, since $sign(\gamma_k) = 1$, also $|\gamma_k| = 2$, and $\mathbb{E}\left(e^{\theta_\kappa}\right) = 59,874$. Thus, for $\beta = 2$, 1.5, 1 and 0.5, the company is approximately gaining 239,496, 169,349, 119,748 and 84,674 contracts respectively.

I) Starting with *company A*, the volume of business (i.e. 1,290,320 contracts, see table 3.1 is very elastic to one potential change of the ratio of the company's average premium to the company's premium since the factor α is equal to 5. In the first scenario (see Figure 3.2), the company's reputation has *negative* effect on the volume of business, then, $b_1 > 0$ and $b_2 < 0$, so one positive solution for the polynomial (3.6) is obtained, see also Corollary 1. In Figure 3.2 and table 3.3, the premium takes different values depending on the different level of the break even premium. As expected the premium of the leading company of the Greek insurance market, can be discounted significantly when the break even premium is lower, and our approach recommends lower values when $\beta = 2$, i.e. when it loses bigger part of the market.

In the second scenario (see Figure 3.3 and table 3.4), the company's reputation has *positive* effect on the volume of business, then, $b_1 < 0$ and $b_2 > 0$, so two or zero positive solutions for the polynomial (3.6) are expected, see also Corollary 1. In our case, there is positive root, and as also expected, the premium takes higher values compared to the first scenario and can be discounted significantly when the break even

π_k	200€	205€	210€	215€	220€	225€	230€	235€	240€
$\beta = 0.5$	247,98	253,92	259,82	265,69	271,51	277,30	283,03	288,71	294,34
$\beta = 1$	247,19	253,02	258,80	264,53	270,21	275,83	281,39	286,89	292,32
$\beta = 1.5$	246,13	251,81	257,44	262,99	268,48	273,90	279,25	284,52	289,71
$\beta = 2$	244,72	250,21	255,63	260,98	266,24	271,41	276,50	281,51	286,42

Table 3.3: The premium of the insurance company A for $sign(\gamma_k) = -1$, different values of β and for the break-even premium π_k.

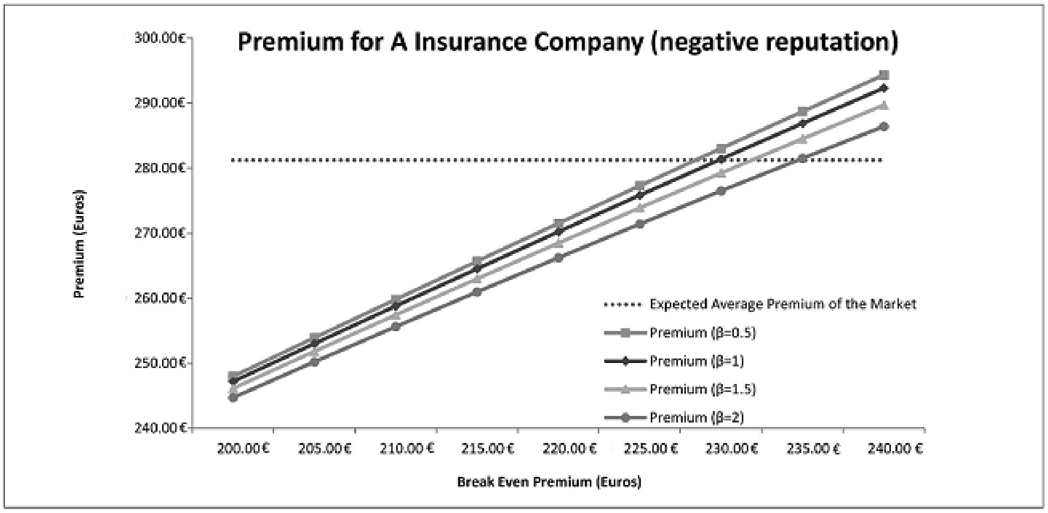

Figure 3.2: The premium of the insurance company A for $sign(\gamma_k) = -1$, different values of β and for the break-even premium.

premium is lower, and our approach recommends higher values when $\beta = 2$, i.e. when it gains bigger part of the market. In both cases, for some break even premiums the recommended premium is significantly lower than the average premium, giving a good indication of the leading position of this company.

π_k	200€	205€	210€	215€	220€	225€	230€	235€	240€
$\beta = 0.5$	252,24	258,87	265,55	272,30	279,11	286,01	293,00	300,11	307,33
$\beta = 1$	253,25	260,06	266,95	273,95	281,07	288,32	295,73	303,33	311,17
$\beta = 1.5$	254,76	261,86	269,11	276,53	284,16	292,04	300,25	308,85	317,99
$\beta = 2$	257,11	264,72	272,60	280,82	289,48	298,74	308,86	320,34	334,42

Table 3.4: The premium of insurance company A for $sign(\gamma_k) = 1$, different values of β and for the break-even premium π_k.

II) Continuing with *company B*, the volume of business (i.e. 736,621 contracts, see table 3.1) is less elastic to the previous one, since the factor α is equal to 2. In the first scenario (see Figure 3.4 and table 3.5), the company's reputation has *negative* effect

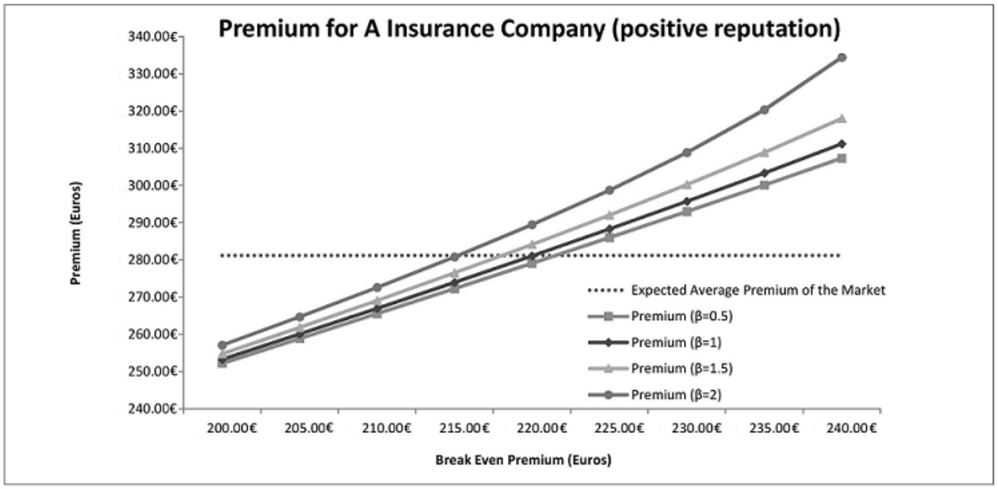

Figure 3.3: The premium of insurance company A for $sign(\gamma_k) = 1$, different values of β and for the break-even premium π_k.

on the volume of business, then $b_1 > 0$ and $b_2 < 0$, so one positive solution for the polynomial (3.6) is obtained, see also Corollary 1. In Figure 3.4, the premium depends on the different level of the break even premium and changes accordingly. Again for the 2nd biggest company of the Greek insurance market, the premium can be discounted significantly when the break even premium is lower, and our approach recommends lower values when $\beta = 2$, i.e. when it loses bigger part of the market. For all the different values of break even premiums, the recommended premium is significantly higher than the average premium, giving a good indication of the struggling position of this company when suffering big losses to the volume of its business. For the *positive* effect in the company's reputation to the volume of business $b_1 < 0$ and $b_2 > 0$, and it can be shown that zero positive solutions for the polynomial (3.6) are derived. Thus, the senior management of the company can keep the same premium strategy with the previous year.

π_k	200€	205€	210€	215€	220€	225€	230€	235€	240€
$\beta = 0.5$	342,58	349,19	355,72	362,16	368,52	374,80	381,01	387,14	393,19
$\beta = 1$	328,44	334,45	340,36	346,20	351,95	357,62	363,22	374,19	379,56
$\beta = 1.5$	312,68	318,08	323,39	328,63	333,78	338,86	343,86	348,80	353,66
$\beta = 2$	295,63	300,45	305,18	309,84	314,43	318,94	323,39	332,08	336,33

Table 3.5: The premium of insurance company B for $sign(\gamma_k) = -1$, different values of β and for the break-even premium π_k.

III) Finally for *company C*, the volume of business (i.e. 548.861 contracts, see table 3.1) is equally elastic to the previous one, since the factor α is also 2. In the first

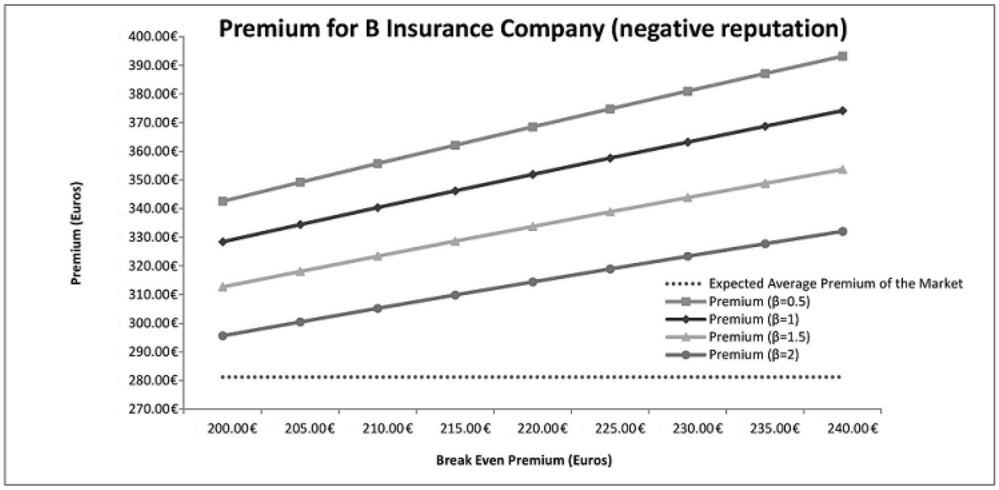

Figure 3.4: The premium of insurance company B for $sign(\gamma_k) = -1$, different values of β and for the break-even premium π_k.

scenario (see Figure 3.5), the company's reputation has *negative* effect on the volume of business, then $b_1 > 0$ and $b_2 < 0$, so one positive solution for the polynomial (3.6) is obtained, see also Corollary 1. In Figure 3.5 and 3.6, the premium's values changes according to the different level of the break even premium. Again for the 3rd biggest company of the Greek insurance market, the premium can be discounted significantly when the break even premium is lower, and our approach recommends lower values when $\beta = 2$, i.e. when it loses bigger part of the market. For all the different values of break even premiums, the recommended premium is higher than the average premium, but it is lower when compared to that of Company B. This is a very interesting result as company B and C have common elasticity factors and their volumes are comparable. Thus, as company C has less contracts compared to company B, it reduces a little more the recommended premium in order to retake the lost volume. Again for the *positive* effect in the company's reputation to the volume of business $b_1 < 0$ and $b_2 > 0$, and it can be also shown that zero positive solutions for the polynomial (3.6) is derived. Thus, the senior management of the company can keep the same premium strategy with the previous year.

3.5.6 Optimal premium for diferrent values of $|\gamma_k|$

Then, the following scenarios are considered for companies A, B and C for negative/positive reputation, and $|\gamma_k| = 2$, 1.5, 1 and 0.5. Again in the first scenario (see Figure 3.6, 3.8 and 3.9), the companies' reputation has *negative* effect on the volume of their business, since $sign(\gamma_k) = -1$, $\beta = 2$, and $\mathbb{E}\left(e^{\theta_\kappa}\right) = 59,874$. Thus, for $|\gamma_k| =$

π_k	200€	205€	210€	215€	220€	225€	230€	235€	240€
$\beta = 0.5$	330,69	336,78	342,79	348,72	354,56	360,33	366,02	371,63	377,16
$\beta = 1$	315,15	320,64	326,05	331,37	336,61	341,78	346,87	351,89	356,84
$\beta = 1.5$	298,27	303,18	308,00	312,74	317,41	322,01	326,53	330,99	335,38
$\beta = 2$	280,44	284,79	289,06	293,26	297,40	301,47	305,47	309,41	313,29

Table 3.6: The premium of insurance company C for $sign(\gamma_k) = -1$, different values of β and for the break-even premium π_k.

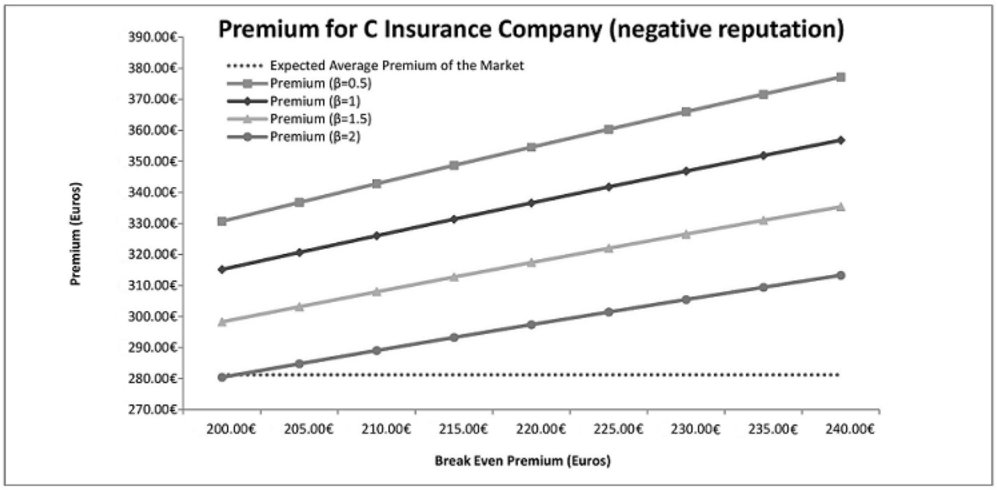

Figure 3.5: The premium of insurance company C for $sign(\gamma_k) = -1$, different values of β and for the break-even premium π_k.

2, 1.5, 1 and 0.5, the company is approximately losing 239,496, 134,717, 59,874 and 14,969 contracts respectively. In the second scenario (see Figure 3.7), the companies' reputation has *positive* effect on the volume of business, since $sign(\gamma_k) = 1$, also $\beta = 2$, and $\mathbb{E}\left(e^{\theta_\kappa}\right) = 59,874$. Thus, for $|\gamma_k| = 2$, 1.5, 1 and 0.5, the company is approximately gaining 239,496, 134,717, 59,874 and 14,969 contracts respectively.

I) Starting with *company A*, as we have said earlier the volume of business (i.e. 1,290,320 contracts, see table 3.1) is very elastic to one potential change of the ratio of the company's average premium to the company's premium since the factor α is equal to 5. In the first scenario (see Figure 3.6 and 3.7), the company's reputation has *negative* effect on the volume of business, then, $b_1 > 0$ and $b_2 < 0$, so one positive solution for the polynomial (2.6) is obtained, see also Corollary 1. In Figure 3.6 and table 3.7, the premium takes different values depending on the different level of the break even premium. As expected the premium of the leading company of the Greek insurance market, can be discounted significantly when the break even premium is lower, and our approach recommends lower values when $|\gamma_k| = 2$, i.e. when it loses bigger part of the market.

π_k	200€	205€	210€	215€	220€	225€	230€	235€	240€
$\lvert\gamma_k\rvert = 0.5$	249,63	255,82	262,00	268,18	274,34	280,50	286,65	292,78	298,90
$\lvert\gamma_k\rvert = 1$	248,55	254,58	260,57	266,54	272,48	278,39	284,25	290,08	295,87
$\lvert\gamma_k\rvert = 1.5$	246,87	252,65	258,38	264,06	269,68	275,23	280,73	286,15	291,51
$\lvert\gamma_k\rvert = 2$	244,72	250,21	255,63	260,98	266,24	271,41	276,50	281,51	286,42

Table 3.7: The premium of the insurance company A for $sign(\gamma_k) = -1$, different values of γ and for the break-even premium π_k.

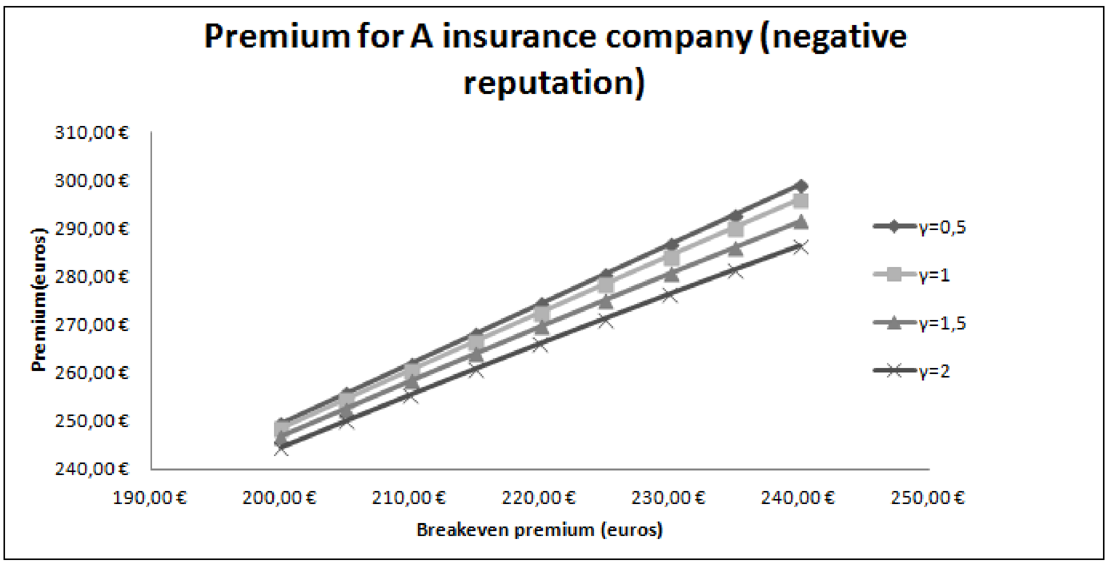

Figure 3.6: The premium of the insurance company A for $sign(\gamma_k) = -1$, different values of $|\gamma_k|$ and for the break-even premium π_k.

In the second scenario (see Figure 3.7), the company's reputation has *positive* effect on the volume of business, then, $b_1 < 0$ and $b_2 > 0$, so two or zero positive solutions for the polynomial (3.6) are expected, see also Corollary 1. In our case, there is positive root, and as also expected, the premium takes higher values compared to the first scenario and can be discounted significantly when the break even premium is lower, and our approach recommends higher values when $|\gamma_k| = 2$ see also table 3.8, i.e. when it gains bigger part of the market. In both cases, for some break even premiums the recommended premium is significantly lower than the average premium, giving a good indication of the leading position of this company.

II) Continuing with *company B*, the volume of business (i.e. 736,621 contracts, see table 3.1) is less elastic to the previous one, since the factor α is equal to 2. In the first

π_k	200€	205€	210€	215€	220€	225€	230€	235€	240€
$\lvert\gamma_k\rvert = 0.5$	250,38	256,69	263,01	269,34	275,67	282,02	288,38	294,76	301,15
$\lvert\gamma_k\rvert = 1$	251,56	258,07	264,61	271,20	277,83	284,51	291,26	298,06	304,95
$\lvert\gamma_k\rvert = 1.5$	253,69	260,58	267,58	274,70	281,95	289,38	297,00	304,86	313,02
$\lvert\gamma_k\rvert = 2$	257,11	264,72	272,60	280,82	289,48	298,74	308,86	320,34	334,42

Table 3.8: (The premium of the insurance company A for $sign(\gamma_k) = 1$, different values of $|\gamma_k|$ and for the break-even premium π_k).

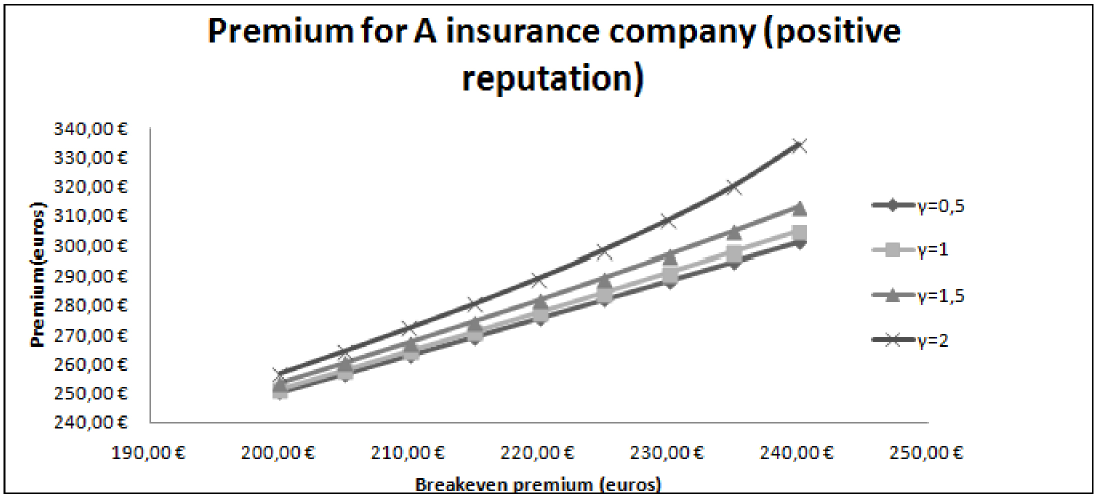

Figure 3.7: The premium of the insurance company A for $sign(\gamma_k) = 1$, different values of $|\gamma_k|$ and for the break-even premium π_k.

scenario (see Figure 3.8 and table 3.9), the company's reputation has *negative* effect on the volume of business, then $b_1 > 0$ and $b_2 < 0$, so one positive solution for the polynomial (3.6) is obtained, see also Corollary 1. In Figure 3.8, the premium depends on the different level of the break even premium and changes accordingly. Again for the 2nd biggest company of the Greek insurance market, the premium can be discounted significantly when the break even premium is lower, and our approach recommends lower values when $|\gamma_k| = 2$, i.e. when it loses bigger part of the market. For all the different values of break even premiums, the recommended premium is significantly higher than the average premium, giving a good indication of the struggling position of this company when suffering big losses to the volume of its business. For the *positive* effect in the company's reputation to the volume of business $b_1 < 0$ and $b_2 > 0$, and it can be shown that zero positive solutions for the polynomial (3.6) are derived. Thus, the senior management of the company can keep the same premium strategy with the previous year.

III) Finally for *company C*, the volume of business (i.e. 548.861 contracts, see table

π_k	200€	205€	210€	215€	220€	225€	230€	235€	240€
$\lvert\gamma_k\rvert$=0.5	385,53	394,50	403,42	412,30	421,14	429,94	438,69	447,39	456,05
$\lvert\gamma_k\rvert$=1	354,87	362,07	369,19	376,22	383,18	390,07	396,87	410,26	416,85
$\lvert\gamma_k\rvert$=1.5	323,25	329,05	334,76	340,39	345,94	351,40	356,80	362,11	367,36
$\lvert\gamma_k\rvert$=2	295,63	300,45	305,18	309,84	314,43	318,94	323,39	332,08	336,33

Table 3.9: The premium of insurance company B for $sign(\gamma_k) = -1$, different values of $|\gamma_k|$ and for the break-even premium π_k.

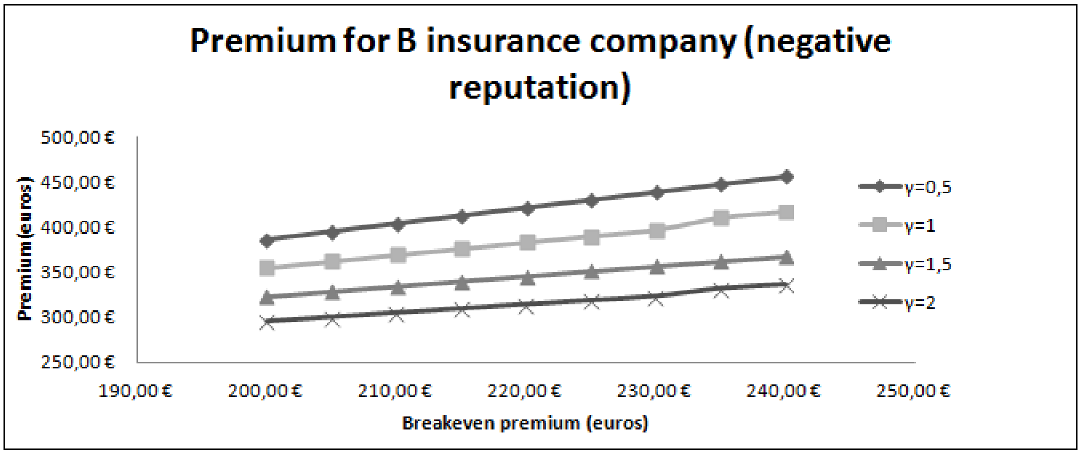

Figure 3.8: The premium of insurance company B for $sign(\gamma_k) = -1$, different values of $|\gamma_k|$ and for the break-even premium π_k.

3.1) is equally elastic to the previous one, since the factor α is also 2. In the first scenario (see Figure 3.8 and table 3.10), the company's reputation has *negative* effect on the volume of business, then $b_1 > 0$ and $b_2 < 0$, so one positive solution for the polynomial (3.6) is obtained, see also Corollary 1. In Figure 3.8, the premium's values changes according to the different level of the break even premium. Again for the 3rd biggest company of the Greek insurance market, the premium can be discounted significantly when the break even premium is lower, and our approach recommends lower values when $\gamma = 2$, i.e. when it loses bigger part of the market. For all the different values of break even premiums, the recommended premium is higher than the average premium, but it is lower when compared to that of Company B. This is a very interesting result as company B and C have common elasticity factors and their volumes are comparable. Thus, as company C has less contracts compared to company B, it reduces a little more the recommended premium in order to retake the lost volume. Again for the *positive* effect in the company's reputation to the volume of business $b_1 < 0$ and $b_2 > 0$, and it can be also shown that zero positive solutions for the polynomial (3.6) is derived. Thus, the senior management of the company can keep the same premium strategy with the previous year.

π_k	200€	205€	210€	215€	220€	225€	230€	235€	240€
$\lvert\gamma_k\rvert = 0.5$	381,23	389,91	398,55	407,13	415,67	424,15	432,57	440,95	449,27
$\lvert\gamma_k\rvert = 1$	315,15	320,64	326,05	331,37	336,61	341,78	346,87	351,89	356,84
$\lvert\gamma_k\rvert = 1.5$	309,55	314,84	320,04	325,16	330,21	335,17	340,07	344,89	349,65
$\lvert\gamma_k\rvert = 2$	280,44	284,79	289,06	293,26	297,40	301,47	305,47	309,41	313,29

Table 3.10: The premium of insurance company C for $sign(\gamma_k) = -1$, different values of $|\gamma_k|$ and for the break-even premium π_k.

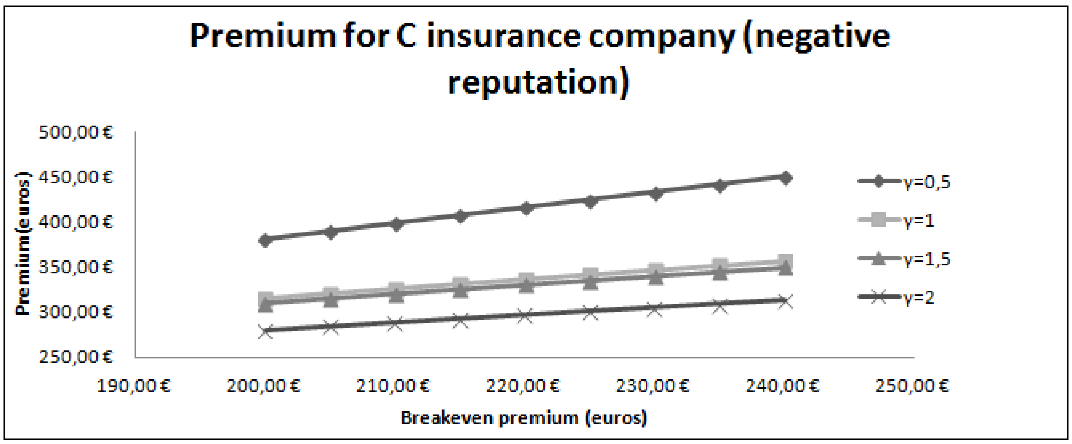

Figure 3.9: The premium of insurance company C for $sign(\gamma_k) = -1$, different values of $|\gamma_k|$ and for the break-even premium π_k.

Chapter 4

Calculation of fair premium

4.1 Motivation

The premium-reserve (P-R) process for non-life products is always a very challenging task in the insurance industry as the actuarial team needs to consider the various characteristics of the insured object, the potential demand from the policy holders, the available information about the competition of the targeted market and the reserve that must be kept. Thus, the main data source on which premium strategy is formulated is not only based on the insurance company's own historical data on policies and claims, but also supplementary information from external sources. At last but not least, it should be emphasised that every company's objective is to minimize the level of the required reserve. Consequently, the main challenge that a company faces is to set a fair premium that comes up from a reserve minimization procedure which takes into account different actuarial and financial parameters as well as the market's competition.

In real world actuarial applications, a premium principle connects the cost of a general insurance policy to the moments of the corresponding claim arrival and severity distributions. Insurers add a loading to this cost price in order to make profit and cover their expenses. After this consideration, two main questions are raised; *"How an optimal premium can be calculated in order to minimize the level of the required reserve?"* and *"how it is possible to find a premium strategy that takes into consideration market's competition and all the different economic parameters that affect company's reserve except for the cost of a general insurance policy?"*.

A usual approach concerning non-life insurance pricing is the use of Generalized Linear Models (GLM). A number of key ratios are dependent on a set of rating factors; see [60]. For personal lines insurance which are designed to be sold in large quantities, the key ratios are often claim frequency and severity (cost per claim), while for commercial lines insurance which are designed for relatively small legal entities, the loss ratio may be also considered (claim costs per earned premium). Rating factors are grouped into classes (i.e. factor variables) and may include information about policyholder, the insured risk as well as geographic and demographic information.

4.2 New approach: Demand for a Nonlinear Optimal Control Framework

In chapter 2 and 3, the disturbance of the volume of business function denotes the set of all other stochastic variables that are considered to be relevant to the demand function (moreover, they are assumed to be independently distributed in time and Gaussian). However, this significant function should also be consisted by many other micro-macro economic factors that affect the company's volume of business and consequently, the optimal premium strategy. Thus, a more thoughtful analysis of this real world insurance problem demands that the volume of business to be modelled as a nonlinear function with respect to reserve, the premium, the noise and a quadratic performance criterion concerning the utility function to be implemented. Indeed, there are quite a few examples that nonlinear analysis to model different insurance's applications is required, see for instance [47, 48] and [25].

First, let us continue with some arguments about the choice of a quadratic minimization problem. Indeed, quadratic forms are the next simplest functions after linear ones. Like linear functions, they have a matrix representation, so that studying quadratic forms reduces to studying symmetric matrices. Additionally, the second order condition that distinguish maxima from minima in economic optimization problems are stated in terms of a quadratic form. It should be mentioned that several well known economic problems are modelled using quadratic objective functions, such as the risk minimization problems in finance, where riskiness is measured by the (quadratic) variance of the returns from investments etc.; see [70, 69]. Concerning insurance's application, Lai [52] uses a quadratic utility function to find the sufficient conditions on the insurance premium and deductible to increase the production for a risk-averse firm.

Giving another dimension to chapters' 2 and 3 models, in the present paper the volume of business in year k is not only proportional to the ratio of the market's average and company's premium, but it is also related to a function of the form $f_k(R_k, \tilde{p}_k, \theta_k)$, where $F_k(R_k, \tilde{p}_k) \triangleq \mathbb{E}[f_k^2(R_k, \tilde{p}_k, \theta_k)]$. As it will be clearer in the next section, the function $F_k(R_k, \tilde{p}_k)$ consists of micro-macro economic parameters which, are implemented in a competitive P-R model. These are the income insurance elasticity of demand, the numbers of insured and the inflation in addition to the fame of company. Since, it is not straightforward to define completely the function $f_k(R_k, \tilde{p}_k, \theta_k)$ because of its stochastic property, a rational approach is given by the function $F_k(R_k, \tilde{p}_k)$.

Thus, the main contribution of this chapter can be highlighted on the following key points. First, an optimal quadratic control model for the determination of the P-R strategy is developed as a minimization problem in a nonlinear framework for the very first time according to the author's knowledge. In this approach, the present value of the company's reserve is required to be close to zero. Second, the stochastic function $f_k(R_k, \tilde{p}_k, \theta_k)$ that affects the company's reserve is analysed considering different micro-macro economic parameters, which directly or indirectly affect the optimal premium. Finally, as in chapter 2 and 3, the insurance premium is given dynamically and includes a good number of interesting and very informative parameters about the competition of the market.

The chapter is organized as follows: In Section 4.3, a nonlinear model in discrete-time for the P-R strategy of an insurance market is constructed. The utility and the reserve functions are discussed and the main model's assumptions as well as their necessary economic interpretation are provided. In Section 4.4, the calculation of the optimal premium is derived which is presented using two Theorems. Additionally, in this section, some special cases of the function $f_k(R_{k,}\tilde{p}_k, \theta_k)$ are presented. The discussion of the main results is given in Section 4.5. Then, Section 4.6 presents a numerical application to illustrate further the theoretical findings of the chapter.

4.3 Model Formulation

4.3.1 Utility and Reserve Function

Borch [6, 7] and Gerber and Pafumi [28] show the importance of the utility theory to formulate and model some real world problems that were relevant to insurance industry. Following Von Neuman and Morgenstern [75], who argue that the existence of a utility function can derive from a set of axioms governing a preference ordering, thus our suggested reserve utility function has the following two basic properties:
(a) $U(R_k)$ is a decreasing function of reserve R_k.
(b) $U(R_k)$ is a convex function of R_k.
The first property deals with the required evidence that less reserve is better, which is a reasonable target for every insurance company. One way to justify the second property is to require the marginal utility $U(R_k)$ to be an increasing function of reserve R_k or equivalently, that the gain of utility resulting from a premium gain of g, $U(R_k + g) - U(R_k)$ to be an increasing function of the reserve. The utility function which is proposed is equal to the sum of the present value of company's reserve from the starting year which is equal to zero till year $N-1$ times $\frac{1}{2}$ plus the present value of the company's reserve in year N times $\frac{1}{2}$.

In a linear framework, we can denote the process R_k as the insurer's reserve at time

$[k, k+1)$ which is given by:

$$R_{k+1} = -a_k R_k + (p_k - \pi_k) V_k, \tag{4.1}$$

where $a_k \in [0,1]$ denotes the *excess return* on capital (i.e. return on capital required by the shareholders of the insurer whose strategy is under consideration). Thus, $-a_k R_k$ is the cost of holding R_k in the time interval $[k, k+1)$.

As in [71, 72, 61, 62], the volume of business in year k is assumed to be proportional to the ratio of the market's average premium to the company's premium in year k times the company's volume of business in the preceding year. Furthermore, the volume of business is stochastic due to the stochastic parameter θ_k which is assumed to be independently distributed in time and not always Gaussian, and indirectly affects the premium and finally the company's reserve; see also Assumption 2 (see next sub-section).

Consequently, in this paper, a minimization problem with respect to $\tilde{p}_k = \frac{1}{p_k}$ is considered, where the following quadratic performance criterion is valid,

$$\mathbb{E}_{/} R_0 \left\{ \sum_{k=0}^{N-1} \frac{1}{2} q_k R_k^2 + \frac{1}{2} q_N R_N^2 \right\}, \tag{4.2}$$

subject to the stochastic dynamic system of the P-R process

$$R_{k+1} = -a_k R_k + m_k + Z_k \tilde{p}_k + sign(f_k) f_k(R_k, \tilde{p}_k, \theta_k), \tag{4.3}$$

where R_0 is known, $m_k = V_{k-1}\overline{p}_k$, $Z_k = -V_{k-1}\overline{p}_k \pi_k$, $R_k, \tilde{p}_k, \theta_k \in \mathbb{R}$, $f_k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, q_k, q_N, α_k, Z_k and $m_k \in \mathbb{R}$ and $sign(f_k) = \pm 1$ denotes the way that the function $f_k(R_k, \tilde{p}_k, \theta_k)$ affects company's reserve. Since market's average premium is stochastic, Z_k and m_k are also stochastic. In other words $\mathbb{E}(m_k) = V_{k-1}\mathbb{E}(\bar{p}_k)$, $\mathbb{E}(Z_k) = -V_{k-1}\mathbb{E}(\bar{p}_k)\pi_k = -\mathbb{E}(m_k)\pi_k$.

Remark 13. The Eq. (4.3) is an interesting and significant extension of the relevant equations in [71, 72, 61] and [62] in discrete-time framework. Actually, if $f_k(R_k, \tilde{p}_k, \theta_k)$ is eliminated completely, then Taylor's [71], and Pantelous and Passalidou's [61] simplified approach is deriving. It should be also emphasised that Taylor [71] is proposing a nonlinear framework, but instead a linear approach of the wealth (equivalently here, of the reserve) process is discussed eventually.

4.3.2 Assumptions

The following assumptions are made:
Assumption 1: $f_k(R_k, \tilde{p}_k) \triangleq \mathbb{E}\left[f_k(R_k, \tilde{p}_k, \theta_k)\right]$ is zero for all $R_k, \tilde{p}_k \in \mathbb{R}, k = 0, ..., N-1$. There is no loss of generality to assume that $f_k(R_k, \tilde{p}_k) \triangleq 0$, because appropriate choices of $-a_k$, Z_k, m_k will model any mean value of f_k which is linear in $R_k, \tilde{p}_k$.
Assumption 2: $F_k(R_k, \tilde{p}_k) \triangleq \mathbb{E}\left[f_k^2(R_k, \tilde{p}_k, \theta_k)\right]$ exists and is a general quadratic function of R_k, $\tilde{p}_k$ for $k = 0, ..., N-1$. $F_k(R_k, \tilde{p}_k)$ has the following representation

$$F_k(R_k, \tilde{p}_k) \triangleq B_k\left(\frac{1}{2}R_k^2 C_k + \tilde{p}_k \gamma_k R_k + \frac{1}{2}\tilde{p}_k^2 M_k\right),$$

where $B_k, C_k, M_k, \gamma_k \in \mathbb{R}$.

Remark 14. The income elasticity, B_k, of non-life insurance measures the responsiveness of the demand for general insurance contracts to a change in the income of the people demanding them ceteris paribus (all other factors held constant). Lee et al. [53] conclude that insurance, like other developed financial services, has grown in quantitative importance as part of the general advancement of financial sectors and that there is a relationship between non-life insurance premiums and real income. According to their study, income elasticity of non-life insurance premiums are larger than one.

Remark 15. The inflation, C_k, can change dramatically company's reserve since inflation reflects a reduction in the purchasing power per unit of money or loss of real value in the medium of exchange and unit of account within the economy. D' Arcy [13] finds that both the underwriting profit margin and insurance investment returns are negatively correlated with the inflation rate during the period 1951-1976. Krivo [51] determines that although inflation and the underwriting profit margin are not significantly correlated over the subsequent period 1977-2006, investment returns and the year-to-year change in underwriting profit margin are both significantly negatively correlated with inflation over that period. Lowe and Warren [57] describe the negative impact of inflation on property-liability insurers' claim costs, loss reserves and asset portfolios. They express concern that most current actuaries, underwriters and claim staff have never experienced severe inflation, so could be slow to adapt to any change in the economic environment. In other words, property-liability insurers are impacted by inflation in several ways. The clearest impact is the cost of future claims on current policies according to Ahlgrim and D' Arcy [2].

Remark 16. A critical factor that affects demand for non-life insurance and indirectly premium and reserve is the number of insureds, M_k, in the market. As new insureds enter the market this has a direct effect on insurance contracts that insureds are willing and able to buy. An increase in the number of buyers means that there are more individual demand curves to add up to get the general insurance demand curve, so market's demand increases. An increase in demand shifts the demand curve to the right so at each premium, the quantity of contracts demand increases. The excess demand causes the premium to rise and equilibrium is restored at a different point.

Remark 17. As it has been discussed in details in [62], the reputation of the company affects the product's demand and consequently the optimal premium as well as the company's reserve. The reputation of a business, γ_k is essential to its survival. The trust and confidence of the client can have a direct and profound effect on a company's bottom line.

Assumption 3: $F_k(R_k, \tilde{p}_k) \geq 0, \forall R_k \in \mathbb{R}$, $\tilde{p}_k \in \mathbb{R}$.

Assumption 3 is necessary (hence not at all restrictive) in order that $F_k(R_k, \tilde{p}_k)$ be a covariance function for each $R_k, \tilde{p}_k \in \mathbb{R}$. The optimal control sequence $\{\tilde{p}_k\}$ is to be drawn from sequences of closed loop controllers i.e. of the form $\tilde{p}_k = D_k(\mathcal{R}_k); \mathcal{R}_k \triangleq \{R_0, R_1, ..., R_k\}$ where $D_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}; k = 0, ..., N-1$. Note that because θ_k is a sequence of random variables independently disturbed in time, knowledge of is equivalent to knowledge of $\mathcal{R}_k$ so that the sequences of closed loop controllers can be written $\tilde{p}_k = D_k(R_k); k = 0, ..., N-1$.

Assumption 4: There is positive price-elasticity of demand, i.e. if the market as a whole begins underwriting as a whole begins underwriting at a loss, any attempt by a particular insurer to maintain profitability will result in a reduction of his volume of business.

Assumption 5: There is a finite time horizon.

Assumption 6: Demand in year $k+1$ is assumed to be proportional to demand in the preceding year k.

4.4 Calculation of the Optimal Premium in the P-R Process

4.4.1 The Main Result

In this section, the optimal premium, $\tilde{p}_k^*$, that the insurance company intends to charge is calculated by minimizing the expected total utility of the reserve Eq. (4.2) and (4.3) over a finite time horizon T, and over a choice of strategies p. The next theorem provides the optimal premium strategy for the finite time horizon minimization problem Eqs. (4.1) - (4.3), see also [40] and [49]. Let us define first

$$\tilde{u}_k = 2V_{k-1}^2 \pi_k^2 \mathbb{E}\left(\bar{p}_k^2\right) S_{k+1} + B_k M_k S_{k+1}, \tag{4.4}$$

$$\tilde{a}_k = 2a_k V_{k-1} \pi_k \mathbb{E}\left(\bar{p}_k\right) S_{k+1} + B_k \gamma_k S_{k+1}, \tag{4.5}$$

$$\tilde{m}_k = -2V_{k-1}^2 \pi_k \mathbb{E}\left(\bar{p}_k^2\right) S_{k+1} - V_{k-1} \pi_k \mathbb{E}\left(\bar{p}_k\right) d_{k+1}, \tag{4.6}$$

where

$$S_k = q_k + 2a_k^2 S_{k+1} + B_k C_k S_{k+1} - \tilde{u}_k^{-1} \tilde{a}_k^2, \; S_N = q_N, \tag{4.7}$$

$$d_k = -2a_k V_{k-1} \mathbb{E}\left(\bar{p}_k\right) S_{k+1} - d_{k+1} a_k - \tilde{a}_k \tilde{u}_k^{-1} \tilde{m}_k, \; d_N = 0, \tag{4.8}$$

$$e_k = V_{k-1}^2 \mathbb{E}\left(\bar{p}_k^2\right) S_{k+1} + V_{k-1} \mathbb{E}\left(\bar{p}_k\right) d_{k+1} + e_{k+1} - \frac{1}{2} \tilde{u}_k^{-1} \tilde{m}_k^2, \; e_N = 0. \tag{4.9}$$

Theorem 4. *Using (4.4)-(4.9) and*

$$\tilde{u}_k \geqslant 0 \; for \; all \; k \in \{0, ..., N-1\}, \tag{4.10}$$

the optimal premium strategy is given by

$$\tilde{p}_k^* = -\left[\frac{\tilde{a}_k R_k + \tilde{m}_k}{\tilde{u}_k}\right]; k = 0, ..., N-1, \tag{4.11}$$

and the minimum value is given by

$$\frac{1}{2} R_0^2 S_0 + d_0 R_0 + e_0. \tag{4.12}$$

Proof. Let us define

$$J_k\left(R_k\right) \triangleq \min_{p_k,\cdots,p_{N-1}} \mathbb{E}_{/R_k}\left\{\sum_{i=\kappa}^{N-1}\frac{1}{2}R_i^2 q_i + \frac{1}{2}R_N^2 q_N\right\}. \tag{4.13}$$

The optimal performance criterion satisfies Bellman equation

$$J_k\left(R_k\right) \triangleq \min_{p_k,\cdots,p_{N-1}} \mathbb{E}_{/R_k}\left\{\frac{1}{2}R_k^2 q_k + J_{k+1}\left(R_{k+1}\right)\right\},$$

$$J_k\left(R_k\right) \triangleq \min_{p_k,\cdots,p_{N-1}} \left\{\frac{1}{2}R_k^2 q_k + \mathbb{E}_{/R_k}\left(J_{k+1}\left(R_{k+1}\right)\right)\right\}, \tag{4.14}$$

and $J_N(R_N) = \frac{1}{2}R_N^2 u_N$. We now show by induction that

$$J_k(R_k) \triangleq \frac{1}{2}S_k R_k^2 + d_k R_k + e_k. \tag{4.15}$$

solves (4.14), by noting that (4.15) is true for $k = N$, assuming that (4.15) is true for $k+1$ and proving that is true for k. Substituting the assumed expression for into the right hand side of (4.14) yields

$$\begin{aligned}
&\min_{p_k}\{\tfrac{1}{2}R_k^2 q_k + \mathbb{E}_{/R_k}[-a_k R_k - V_{k-1}\pi_k \bar{p}_k \tilde{p}_k + V_{k-1}\bar{p}_k + sign(f_k) f_k(R_{k,}\tilde{p}_k,\theta_k)]^2 S_{k+1}\\
&+\mathbb{E}_{/R_k}\left[-a_k R_k - V_{k-1}\pi_k \bar{p}_k \tilde{p}_k + V_{k-1}\bar{p}_k + sign(f_k) f_k(R_{k,}\tilde{p}_k,\theta_k)\right] d_{k+1} + e_{k+1}\}\\
&= \min_{p_k}\{\tfrac{1}{2}R_k^2 q_k + \mathbb{E}_{/R_k}[a_k^2 R_k^2 + V_{k-1}^2\pi_k^2\bar{p}_k^2\tilde{p}_k^2 + V_{k-1}^2\bar{p}_k^2 + 2a_k R_k V_{k-1}\pi_k\bar{p}_k\tilde{p}_k\\
&-2a_k R_k V_{k-1}\bar{p}_k - 2a_k R_k sign(f_k) f_k(R_{k,}\tilde{p}_k,\theta_k)\\
&-2V_{k-1}^2\pi_k\bar{p}_k^2\tilde{p}_k - 2V_{k-1}\pi_k\bar{p}_k\tilde{p}_k sign(f_k) f_k(R_{k,}\tilde{p}_k,\theta_k)\\
&+2V_{k-1}\bar{p}_k sign(f_k) f_k(R_{k,}\tilde{p}_k,\theta_k) + [sign(f_k) f_k(R_{k,}\tilde{p}_k,\theta_k)]^2] S_{k+1}\\
&+\mathbb{E}_{/R_k}\left[-a_k R_k - V_{k-1}\pi_k \bar{p}_k \tilde{p}_k + V_{k-1}\bar{p}_k + sign(f_k) f_k(R_{k,}\tilde{p}_k,\theta_k)\right] d_{k+1} + e_{k+1}\}\\
&= \min_{p_k}\{\tfrac{1}{2}R_k^2 q_k + [a_k^2 R_k^2 + V_{k-1}^2\pi_k^2\mathbb{E}\left(\bar{p}_k^2\right)\tilde{p}_k^2 + V_{k-1}^2\mathbb{E}\left(\bar{p}_k^2\right)\\
&+2a_k R_k V_{k-1}\pi_k\mathbb{E}\left(\bar{p}_k\right)\tilde{p}_k - 2a_k R_k V_{k-1}\mathbb{E}\left(\bar{p}_k\right) - 2V_{k-1}^2\pi_k\mathbb{E}\left(\bar{p}_k^2\right)\tilde{p}_k + \mathbb{E}[f_k(R_{k,}\tilde{p}_k,\theta_k)]^2] S_{k+1}\\
&+\left[-a_k R_k - V_{k-1}\pi_k\mathbb{E}\left(\bar{p}_k\right)\tilde{p}_k + V_{k-1}\mathbb{E}\left(\bar{p}_k\right)\right] d_{k+1} + e_{k+1}\}
\end{aligned}$$

$$
\begin{aligned}
&= \min_{p_k}\{\tfrac{1}{2}R_k^2 q_k + a_k^2 R_k^2 S_{k+1} + V_{k-1}^2 \pi_k^2 \mathbb{E}\left(\bar{p}_k^2\right)\tilde{p}_k^2 S_{k+1} + V_{k-1}^2 \mathbb{E}\left(\bar{p}_k^2\right) S_{k+1} \\
&+ 2a_k R_k V_{k-1}\pi_k \mathbb{E}\left(\bar{p}_k\right)\tilde{p}_k S_{k+1} - 2a_k R_k V_{k-1}\mathbb{E}\left(\bar{p}_k\right) S_{k+1} - 2V_{k-1}^2 \pi_k \mathbb{E}\left(\bar{p}_k^2\right)\tilde{p}_k S_{k+1} \\
&+ B_k\left(\tfrac{1}{2}R_k^2 C_k + \tilde{p}_k \gamma_k R_k + \tfrac{1}{2}\tilde{p}_k^2 M_k\right) S_{k+1} \\
&- a_k R_k d_{k+1} - V_{k-1}\pi_k \mathbb{E}\left(\bar{p}_k\right)\tilde{p}_k d_{k+1} + V_{k-1}\mathbb{E}\left(\bar{p}_k\right) d_{k+1} + e_{k+1}\}.
\end{aligned}
\tag{4.16}
$$

Because of (4.10), the control that minimizes (4.16) is given by (4.11). When this is substituted into (4.16), we obtain

$$
\begin{aligned}
&\frac{1}{2}R_k^2\left[q_k + 2{a_k}^2 S_{k+1} + B_k C_k S_{k+1} - \tilde{u}_k^{-1}\tilde{a}_k^2\right] \\
&\qquad + R_k\left[-2a_k V_{k-1}\mathbb{E}\left(\bar{p}_k\right) S_{k+1} - d_{k+1}a_k - \tilde{u}_k^{-1}\tilde{a}_k^2\right] \\
&\qquad + \left[V_{k-1}^2 \mathbb{E}\left(\bar{p}_k^2\right) S_{k+1} + V_{k-1}\mathbb{E}\left(\bar{p}_k\right) d_{k+1} + e_{k+1} - \frac{1}{2}\tilde{u}_k^{-1}\tilde{m}_k^2\right].
\end{aligned}
\tag{4.17}
$$

Using (4.4), (4.5), (4.6) in (4.17) yields the fact that (4.15) is true. Thus, the proof by induction is complete. □

Remark 18. In practice, the optimal premium given by Eq. (4.8), makes really sense when $\tilde{a}_k R_k + \tilde{m}_k < 0$. Indeed, the inequality is satisfied if the accumulated reserve retains small, which is actually the main output of the control process. Otherwise, if the inequality is not satisfied, then the previous year premium is considered as the desirable one, and no further changes are suggested, see [61].

Remark 19. In [40], Theorem 5 shows that under certain conditions the inequality (4.11) is satisfied. Similar result could be implied here and is presented in theorem 5.

The next theorem shows that under certain reasonable conditions inequality (4.11) is satisfied.

Theorem 5. *Suppose that $q_N \geq 0$ and that $q_k 0$ for all $k = 0, ..., N-1$, then $\tilde{u}_k > 0$ and*

$$
S_{k+1} \geqslant 0; k = 0, ..., N-1. \tag{4.18}
$$

Proof. We prove this by induction. First we note that because of Assumption 3 and the equation $F_k(R_k, \tilde{p}_k)$ which is mentioned in Assumption 2 we have $B_k M_k S_{k+1} \geq 0$, if

$$S_{k+1} \geq 0. \tag{4.19}$$

Now we proceed by induction. Clearly we have $\tilde{u}_{N-1} > 0$ and $S_N \geq 0$. We now assume

$$S_i \geq 0, i = k+1, ..., N. \tag{4.20}$$

which implies

$$\widetilde{u}_{i-1} > 0, i = k+1, ..., N. \tag{4.21}$$

Because $S_{k+1} \geqslant 0$ it follows from (4.19) and (4.4) that $\tilde{u}_k > 0$.

This, by virtue of (4.7), permits the computation of S_k, d_k, e_k and (4.15) yields

$$J_k(R_k) \;=\; \frac{1}{2}R_k^2 S_k \,+\, d_k R_k \,+\, e_k \;=\; \max_{p_k,\dots,p_{N-1}} \mathbb{E}_{/R_k}\left\{\sum_{i=\kappa}^{N-1}\frac{1}{2}R_i^2 q_i + \frac{1}{2}R_N^2 q_N\right\}. \tag{4.22}$$

Now because of our assumption on $q_N, q_k; k = 0, ..., N-1$ the right-hand side of (4.22) is non-negative for all w_k which implies $S_k \geqslant 0$ and hence it follows from (4.19) and (4.4) that $\tilde{u}_{k-1} > 0$ so that (4.20), (4.21) are true for $i = k, ..., N$ and the proof by induction is complete. □

4.4.2 Three Special Cases

In the previous subsection, a minimization problem using a quadratic performance criterion (4.2) is considered for the nonlinear wealth function (4.3), where $f_k(R_k, \tilde{p}_k, \theta_k)$ is not defined explicitly; instead the function $F_k(R_k, \tilde{p}_k)$ is formulated. In this subsection, three cases of the function $f_k(R_k, \tilde{p}_k, \theta_k)$ are presented.

-Standard Linear-Quadratic Case

Assume that

$$f_k(R_k, \tilde{p}_k, \theta_k) = \Delta_k \theta_k,\ \Delta_k \in \mathbb{R}, \tag{4.23}$$

with $\mathbb{E}\left[\theta_k\right] = 0,\ \mathbb{E}\left[\theta_k^2\right] = \Lambda_k$. In this case $F_k(R_k, \tilde{p}_k) = \Lambda_k \Delta_k^2,\ \forall R_k, \tilde{p}_k, k$ and the optimal premium is independent of $\Lambda_k \Delta_k^2$. This is the simplest case that can be considered explicitly, where the company's reserve function (4.3) is affected by the stochastic parameter θ_k times the factor Δ_k. As we have mentioned earlier, θ_k denotes the set of all other stochastic variables and it is also considered to be relevant to the company's demand function and it is indirectly connected to the company's reserve function. Δ_k measures the impact that the parameter has to the reserve function through the demand function. Since all the other parameters of the reserve function are known, the factor $\Delta_k\theta_k$ can be calculated and further analysed. In this special case the optimal premium strategy is given by Eq. (4.11), where

$$\tilde{u}_k = 2V_{k-1}^2\pi_k^2\mathbb{E}\left(\bar{p}_k^2\right)S_{k+1},\ \tilde{a}_k = 2a_kV_{k-1}\pi_k\mathbb{E}\left(\bar{p}_k\right)S_{k+1},$$

$$S_k = q_k + 2a_k^2S_{k+1} - \tilde{u}_k^{-1}\tilde{a}_k^2,\ S_N = q_N,$$

and $\tilde{m}_k$ is given by Eq. 4.6 all the other parameters are the same as in Theorem 4.

-Norm Dependent Random Vector

Assume now that

$$R_{k+1} = -a_kR_k + Z_k\tilde{p}_k + m_k + \left(\frac{1}{2}C_1R_k^2 + \gamma_1\tilde{p}_kR_k + \frac{1}{2}M_1\tilde{p}_k^2\right)^{1/2}\Delta_k\theta_k, \qquad (4.24)$$

where $C_1 \geq 0$. The the number of consumers, respectively. Here, these three factors are constant for the whole duration, and they are also multiplied by the stochastic parameter θ_k ($\mathbb{E}\left[\theta_k\right] = 0,\ \mathbb{E}\left[\theta_k^2\right] = \Lambda_k$) and the factor Δ_k. Indeed, the inflation rate can change dramatically company's reserve as it reflects the reduction of the purchasing power per unit of money or loss of real value in the medium of exchange and unit of account within the economy. Furthermore, company's reputation has an impact on both company's reserve and premium, and finally the number of the consumers directly affects company's demand and indirectly premium. Again, since all the other parameters of the reserve function are known, the factor $\Delta_k\theta_k$ can be calculated and further analysed. Note that if $C_1 = 1, \gamma_1 = M_1 = 0$ we have noise $\Delta_k\theta_k$ multiplied by $\|R_k\|$. Equation (4.24) is nonlinear,

$$f_k(R_k, \tilde{p}_k) = 0 \text{ and } F_k(R_k, \tilde{p}_k) = \left(\tfrac{1}{2}C_1R_k^2 + \gamma_1\tilde{p}_kR_k + \tfrac{1}{2}M_1\tilde{p}_k^2\right)\Lambda_k\Delta_k^2.$$

In this special case the optimal premium strategy is given by Eq. (4.11), where

$$\tilde{u}_k \triangleq 2V_{k-1}^2\pi_k^2\mathbb{E}\left(\bar{p}_k^2\right)S_{k+1} + M_1\Lambda_k\Delta_k^2,$$

$$\tilde{a}_k \triangleq 2a_kV_{k-1}\pi_k\mathbb{E}\left(\bar{p}_k\right)S_{k+1} + \gamma_1\Lambda_k\Delta_k^2S_{k+1},$$

$$S_k = q_k + 2a_k^2 S_{k+1} + C_1 \Lambda_k \Delta_k^2 - \tilde{u}_k^{-1} \tilde{a}_k^2,\ S_N = q_N,$$

and $\tilde{m}_k$ is given by Eq. (4.3) all the other parameters are the same as in Theorem 4.

-Random Vector Dependent Upon Absolute Value of Linear Combination of R_k & $\tilde{p}_k$

Finally, assume that

$$R_{k+1} = -a_k R_k + Z_k \tilde{p}_k + m_k + |N_k R_k + h_k \tilde{p}_k| \Delta_k \theta_k, \tag{4.25}$$

where N_k and h_k denote the financial risk that the market confronts and the future expectations of the insured in time $[k, k+1]$, respectively; see [56], [5], [68], [39], [53, 54]. Again the statistics of θ_k are $\mathbb{E}[\theta_k] = 0,\ \mathbb{E}\left[\theta_k^2\right] = \Lambda_k$. Thus, in this case

$$f_k(R_k, \tilde{p}_k, \theta_k) = |N_k R_k + h_k \tilde{p}_k| \Delta_k \theta_k.$$

Here, the absolute value of this factor is used as $\Gamma_k \theta_k$ takes either positive or negative values. As all the other parameters of the reserve function are known, the factor $\Gamma_k \theta_k$ can be calculated and further analysed. Note that (3.23) includes, if $\tilde{p}_k \in \mathbb{R}$, $R_{k+1} = -a_k R_k + Z_k \tilde{p}_k + m_k + |\tilde{p}_k| \Delta_k \theta_k$. Again it is easy to see that (3.23) satisfies our assumptions and

$$f_k(R_k, \tilde{p}_k) = 0 \text{ and } F_k(R_k, \tilde{p}_k) = |N_k R_k + h_k \tilde{p}_k| \Lambda_k \Delta_k^2.$$

Remark 20. Another variable that affects reserve process and the premium is the financial risk, N_k, of the market. Studies have documented a correlation between financial development and the development of the insurance market. Lorent [56] indicates that insurance and banking are increasingly intricate and Billio et al. [5] also show that the insurance sector has over time become highly interrelated with other sectors in financial system such as banks, hedge funds etc. due to the involvement of insurance companies in non-core activities such as credit defaults swaps, derivatives trading and investment management. Nevertheless, it is essential to notice that individuals and corporations affront insurable and uninsurable risks in the non-life insurance market; see, for instance, [68] and [39]. Particularly, Lee et al. [53, 54] study the impact of country risks, including political, financial and economic risks, on the income elasticity of insurance demand and conclude that there is a significant effect between them. In other words, these risks affect the company's premium-reserve strategy through the income elasticity of insurance demand. Insurance exists because of risks, and therefore risks and insurance are highly correlated.

Remark 21. Finally, the future expectation of consumers, h_k, concerning the premiums can also affect how many contracts one is willing and able to buy. The expectations that buyers have, concerning the future premium, are assumed constant when a demand curve is constructed. Clients' expectations are one of the five demand determinants that shift the demand curve when they change. It is important to realize that buyers make decisions based on a comparison of current and future premiums. They are motivated to purchase a non-life insurance contract at the lowest possible price. If that lowest price is the one existing today, then they will buy today. If that lowest price is expected to occur in the future, then they will wait until later to buy. Thus, if potential clients think next month's premium will be higher than they had initially expected, they may buy an insurance contract today and not next month. That means that the demand for general insurance contracts today will increase.

In this special case the optimal premium strategy is given by Eq. (4.11), where

$$\tilde{u}_k = 2V_{k-1}^2 \pi_k^2 \mathbb{E}\left(\bar{p}_k^2\right) S_{k+1},\ \tilde{a}_k = 2a_k V_{k-1} \pi_k \mathbb{E}\left(\bar{p}_k\right) S_{k+1},$$

$$\tilde{m}_k = -2V_{k-1}^2 \pi_k \mathbb{E}\left(\bar{p}_k^2\right) S_{k+1} - V_{k-1} \pi_k \mathbb{E}\left(\bar{p}_k\right) d_{k+1} + \left|h_k\right| \Lambda_k \Delta_k^2,$$

$$S_k = q_k + 2a_k^2 S_{k+1} - \tilde{u}_k^{-1} \tilde{a}_k^2,\ S_N = q_N,$$

$$d_k = -2a_k V_{k-1} \mathbb{E}\left(\bar{p}_k\right) S_{k+1} - d_{k+1} a_k \left|N_k\right| \Lambda_k \Delta_k^2 - \tilde{a}_k \tilde{u}_k^{-1} \tilde{m}_k.$$

4.5 Discussion about the Optimal Premium

Initially, it is essential to mention that the optimal premium is calculated based on three main factors, i.e. $\tilde{u}_k,\ \tilde{a}_k,\ \tilde{m}_k$, which have the three following significant parameters:

- The break-even premium;
- The company's volume of business of the preceding year;
- The expectation of market's average premium.

These factors are also appeared in [61, 62]. The expectation of the market's average premium is directly related to the market's competition, which affects the company's premium. Thus, the expectation of market's average premium is directly related to the market's competition which affects company's premium. Moreover the break-even premium is directly related to the company's profitability as well as its reserve and the volume of business of the preceding year is indicative to the company's volume of business and optimal premium as well.

Apart from the parameters mentioned above, one other parameter is also appeared. This is S_{k+1}, which is calculated based on the following:

- The present day value factor;
- The market's inflation;
- The income elasticity of demand.

These economic factors play the most crucial role concerning the company's reserve on which optimal premium is depended on. The present day value factor is used to simplify the calculation for finding the present value of a series of values in the future. It is based on a discount interest rate and the number of periods. The inflation and the interest rates are linked, and frequently referenced in macroeconomics. Inflation refers to the rate at which prices for goods and services rises. In general, as interest rates are lowered, people are able to borrow more money. The result is that consumers have more money to spend, causing the economy to grow and inflation to increase. The opposite holds true for rising interest rates. As interest rates are increased, consumers tend to have less money to spend. With less spending, the economy slows and inflation decreases. The income elasticity of demand affects every financial factor of every market and every business in it since both are related directly or indirectly to the consumer's income. These parameters are related to company's reserve directly since the main difference between the optimal premium calculated in this paper and the ones mentioned in our previous papers is that the optimal premium is related directly to the company's reserve.

Now, the factor $\tilde{a}_k$ depends also on:

- The excess return of capital i.e. return on capital required by the shareholders of the insurer whose strategy is under consideration;
- Company's reputation.

Excess return on capital refers to principal payments back to "capital owners" (shareholders, partners, unit holders) that exceeds the growth (net income/taxable income) of an insurance business or investment. As the financial risk in a market becomes higher the shareholders probably will ask for a higher excess return of capital. Moreover, firms with strong positive reputations attract better people. They are perceived as providing more value, which often allows them to charge a premium. Their customers are more loyal and buy broader ranges of products and services. Because the market believes that such companies will deliver sustained earnings and future growth, they have higher price-earnings multiples and market values and lower costs of capital.

Another crucial factor is the future expectations of the insured. Buyers make decisions based on a comparison of current and future prices. They are motivated to purchase an insurance contract at the lowest possible price. If that lowest price is the one existing today, then they will buy today. If that lowest price is expected to occur in the future, then they will wait until later to buy. Finally, the optimal premium depends on the parameter $\tilde{u}_k$ which calculates on the number of the insured.

Consequently, the main parameters that were appeared to affect the optimal premium pricing policy in Pantelous and Passalidou [61, 62] continue to be present in the new optimal premium Eq. (4.11) (i.e. break-even premium, previous year's volume and the expectation of the market's average premium). This equation is also enriched further with the level of the company's reserve which affection on the optimal premium depends on three main parameters mentioned above. Now, the optimal premium depends on many more parameters. Thus, the proposed new optimal premium is getting closer to reality since it takes into consideration different market's financial factors, which affect indirectly the company's optimal premium strategy.

4.6 Numerical Application

4.6.1 Data

In order to illustrate the main theoretical finding of this paper, a simple numerical example is presented. Unfortunately, since the real data are not available in public, we cannot be analytic and, thus, several assumptions for the data have to be implemented. Consequently, the derived numerical results are subjective and they just illustrate the applicability of our theoretical findings. The following lines present the main parameters which are needed for the calculation of the company's optimal premium according to the proposed model.

$$\mathbb{E}(\bar{p}_k) = 200 \text{ €},\ g_k = 0.2,\ V_{k-1} = 5,000,\ a_k = 0.8,\ \pi_k = 80 \text{ €},\ d_{k+1} = 2.1,$$
$$Var(\bar{p}_k) = 41,\ S_{k+1} = 0.5,\ B_k = 1.2,\ R_k = 720,000 \text{ €and } M_k = 10^6.$$

4.6.2 Premium Strategy

In this application the expectation of the market's average premium will be found following one of the suggested average premium strategies. In this strategy, the average premium is calculated considering all the competitors of the market, and their proportions regarding to the volume of business. In mathematical terms the expected average premium of the market is equal to

$$\mathbb{E}\left(\bar{p}\right)=\frac{1}{m}\sum_{i=1}^{K}b_{i,n}p_{i,n}\text{ where }b_{i,n}=V_{i,n}\left(\sum_{i=1}^{K}V_{i,n}\right)^{-1}\text{ and }\sum_{i=1}^{K}b_{i,n}=1$$

for every year n, $p_{i,n}$ is the premium of the company i^{th} for the year n; K is the number of the competitors (including also our company's premium) in the insurance market and is the number of years for the available data (i.e. we assume that we have the uniform distribution for the weight of every year).

4.6.3 Numerical Algorithm

Summarizing the discussion in the previous Section, in this sub-section, the algorithmic steps for the calculation of the optimal premium are described.

Step 1: *Collect the necessary (historical) data from the company and the insurance market.*
The first step requires the collection of data concerning company's volume of business of the previous year, the excess return of capital, the income elasticity of demand, the inflation rate, the number of insured, the reputation's impact to the volume of business and the variance of market's average premium.

Step 2: *Estimate market's average premium.*
Choose one of the three recommended premium strategies (see also chapter 2) and estimate market's average premium for each one of the previous years and the expectation of market's average premium for the next year, $\bar{p}_k$. As it has been assumed in previous chapter, the average premium can be calculated either considering the entire market or considering the leaders of the market or the direct competitors, see eq. (2.17), (2.19) and (2.21). Then, step 3 follows.

Step 3: *Estimate parameters $\tilde{\alpha}_k, \tilde{m}_k$ and $\tilde{u}_k$.*
After collecting the necessary data and estimating the expectation of market's average premium the next step is to calculate the parameters $\tilde{\alpha}_k, \tilde{m}_k$ and $\tilde{u}_k$. If $\tilde{\alpha}_k R_k + \tilde{m}_k > 0$, the previous premium strategy might stay unchanged, since the company already charges a fair premium. If $\tilde{\alpha}_k R_k + \tilde{m}_k < 0$, step 4 follows.

Step 4: *Calculate the optimal premium.*
Using the estimated parameters$\tilde{\alpha}_k, \tilde{m}_k$ and $\tilde{u}_k$ and the information collected of step 1 the next step is to calculate the optimal premium according to (4.11) for different

Company's Reserve	Premium
700,000€	181.58€
710,000€	184.94€
720,000€	188.42€
730,000€	192.03€
740,000€	195.79€

Table 4.1: Company's premium for different levels of reserve in Euros.

values of reserve (R_k) or breakeven premium (π_k). Then, step 5 follows.

Step 5: *Design the optimal premium strategy for the insurance company.*
Now, for different values of the break-even premium π_k or different values of reserve (R_k) the actuary can generate different values for the optimal premium. Then after taking into consideration the competition in the current (or targeting) insurance market and expectation of the different macroeconomic parameters the optimal premium is calculated and agreed by the senior management of the company.

4.6.4 Numerical Calculation and discussion

After the calculations the optimal premium that comes up is calculated $\tilde{p}_k{}^* = 0.00531$ and since $\tilde{p}_k = p_k^{-1}$ the optimal premium strategy is equal to $p_k{}^* = 188.42$ €. As it is mentioned above, an important parameter that affects company's optimal premium is company's reserve. Now, table 4.1 and figure 4.1 present the optimal premium for different levels of reserve ceteris paribus.

As company's reserves turns out to be bigger the optimal premium is also getting bigger. This is quite rational, since companies with bigger reserve tend to charge their clients a higher premium. On the other hand, companies with a smaller reserve charge a smaller premium in order to attract more new customers.

Another important parameter that affects the company's optimal premium is market's competition or in other words market's average premium. Here, table 4.2 and figure 4.2 present the optimal premium for different levels of market's average premium ceteris paribus. As the market's average premium turns out to be higher the optimal premium does getting lower. In other words, the optimal premium is not always follow market's swing. In fact, the optimal premium is lower than market's average premium till company's competitive equilibrium point which is 195 €and after this point is lower than market's average premium.

Moreover as we have mentioned in the previous chapters, a parameter that affects significantly company's optimal premium is the break-even premium. Here, table 4.3 and figure 4.3 present the optimal premium for different levels of break-even premium ceteris paribus. As the break-even premium turns out to be higher the optimal premium

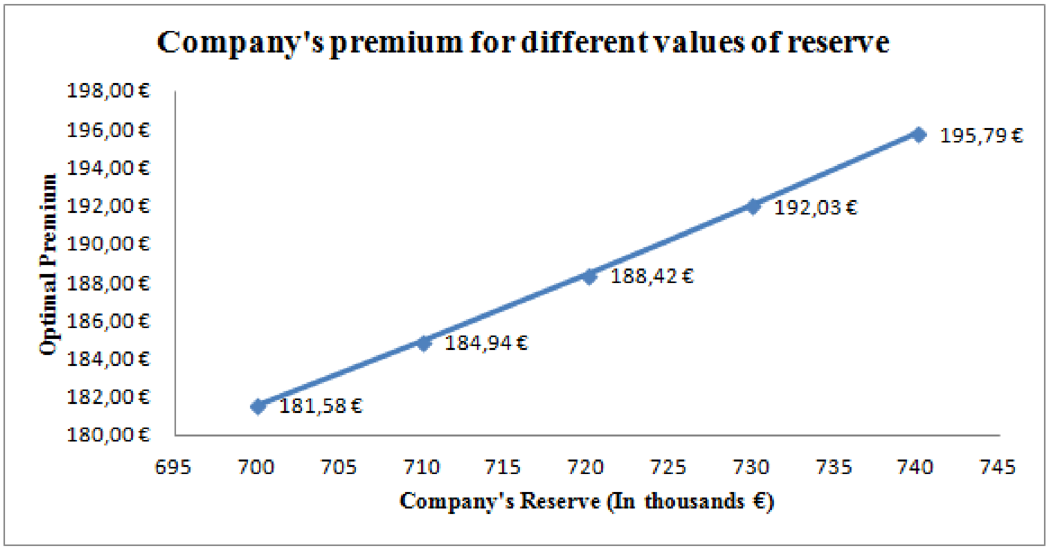

Figure 4.1: Company's premium for different levels of reserve in Euros.

Market's Average Premium	Premium
180.00 €	211.32 €
190.00 €	202.85 €
200.00 €	188.42 €
210.00 €	177.01 €
220.00 €	167.78 €

Table 4.2: Company's premium for different values of market's average premium in Euros.

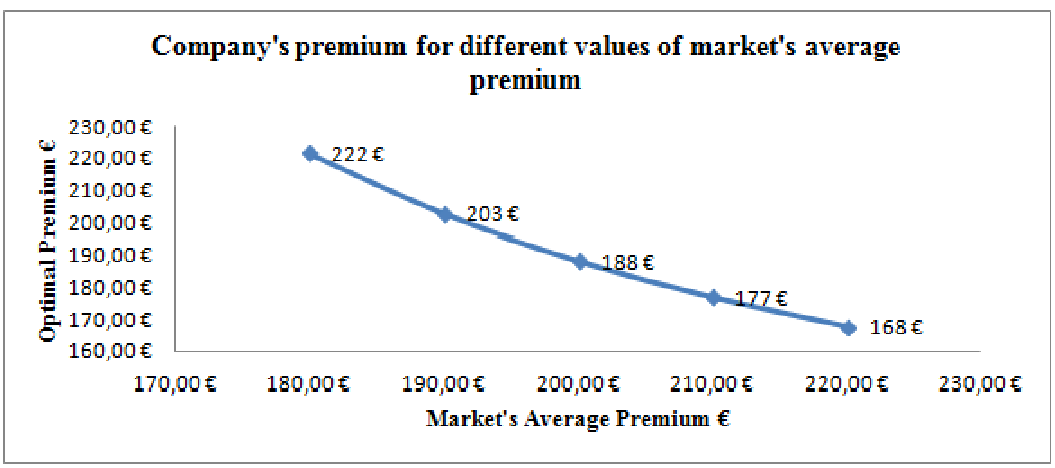

Figure 4.2: Company's premium for different values of market's average premium in Euros.

is getting higher. From this table it is obvious that the optimal premium is very elastic to a change in the breakeven premium.

Break-even Premium	Premium
80.00 €	167.78 €
90.00 €	188,75 €
100.00 €	209,73 €
110.00 €	230,70 €
120.00 €	251,67 €

Table 4.3: Company's premium for different values of break-even premium in Euros.

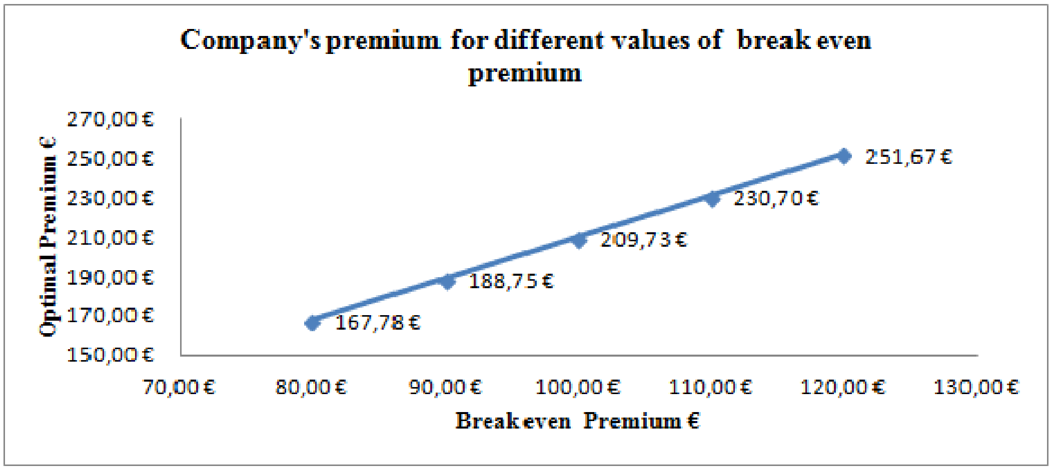

Figure 4.3: Company's premium for different values of break-even premium in Euros.

Furthermore, company's volume of business affects company's optimal premium. Here, table 4.4 and figure 4.4 present the optimal premium for different levels of company's volume of business ceteris paribus. As volume of business turns out to be higher the optimal premium is getting lower. In other words, when company has a big volume of business due to the law of large numbers then it has lower potential claims and can charge a lower premium.

Moreover, table 4.5 and figure 4.5 present the optimal premium for different levels of the value α_k which denotes the excess return of capital. As α_k turns out to be higher the optimal premium is getting higher. From this table it is obvious that the optimal premium is not very elastic to a change in α_k.

Volume of business	Premium
3.500	316,73 €
3.600	292,67 €
3.700	273,05 €
3.800	256,75 €
3.900	242,98 €

Table 4.4: Company's premium in Euros for different values of volume of business

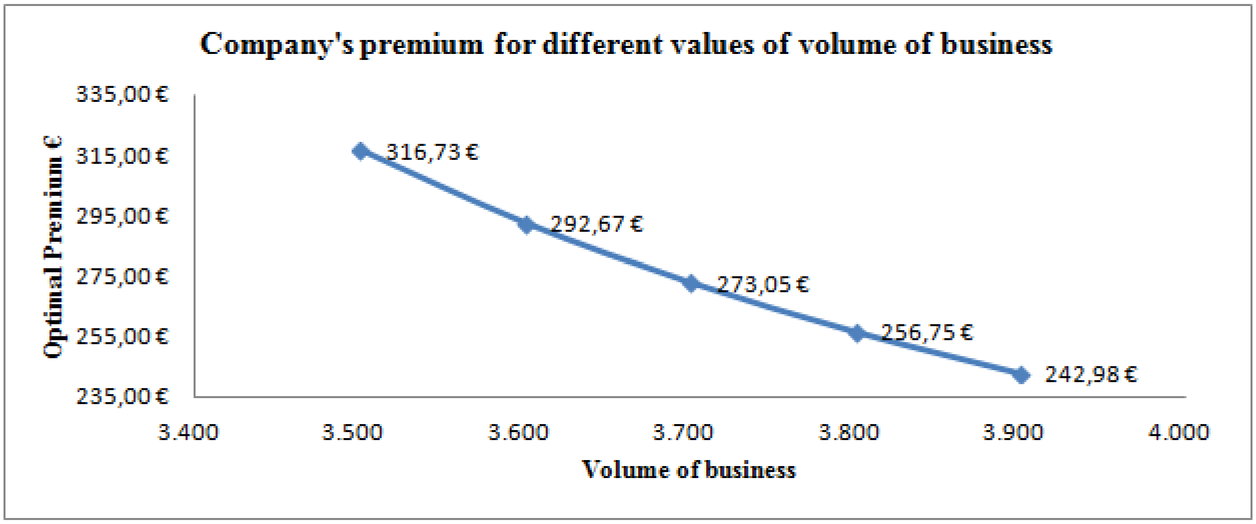

Figure 4.4: Company's premium in Euros for different values of volume of business

Excess return on capital	Premium
0.80	167,78 €
0.70	165,99 €
0.60	164,58 €
0.50	163,45 €
0.40	162,51 €

Table 4.5: Company's premium for different values of excess return of capital in Euros.

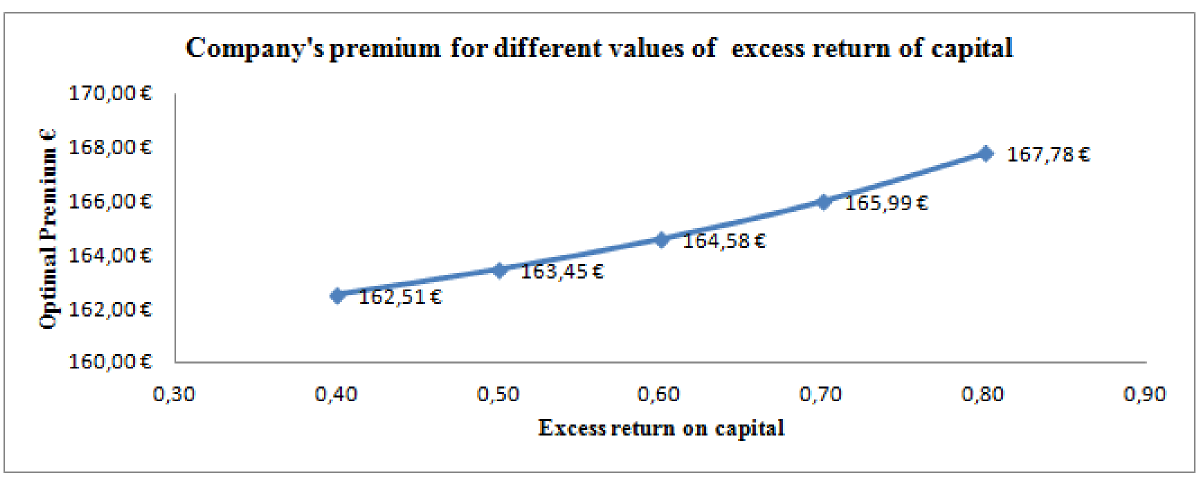

Figure 4.5: Company's premium for different values of excess return of capital in Euros.

Chapter 5

Further research

As it has been already mentioned, there is not a lot of literature concerning pricing of general insurance in a competitive market. In the previous chapters three main models are formulated and solved and analytical solutions have been presented. According to them, simple but useful numerical algorithms are suggested concerning optimal premium pricing strategy under different economic parameters and company's targets.

5.1 Discussion on models' assumptions and volume of business function

As we have already mentioned in chapter 2 and 3, concerning our first and second model, we make the following assumptions

- There is positive price-elasticity of demand.
- There is a finite time horizon.
- Demand in year $k+1$ is assumed to be proportional to demand in the preceding year k.
- θ_k affects the volume of business in a linear way (i.e. additive noise).

Considering the previous assumptions several questions are raised. For example if there is any assumption which can be relaxed or changed in order to have a more realistic approach of the problem. The first assumption is that there is a positive price-elasticity of demand ie. if the market as a whole begins underwriting at a loss, any attempt by a particular insurer to maintain profitability will result in a reduction of his volume of business. Price elasticity of demand is a measure used in economics to show the responsiveness, or elasticity, of the quantity demanded of a good or service to a change in its price, ceteris paribus (i.e. holding constant all the other determinants of demand). More precisely, it gives the percentage change in quantity demanded in response to a one percent change in price.

The first assumption can be changed into *there is either negative or positive price-elasticity of demand* i.e. if the market as a whole begins underwriting at a loss, any attempt by a particular insurer to maintain profitability will not result necessarily in a reduction of his volume of business.

The third assumption is that the demand in year $k+1$ is assumed to be proportional to demand in the preceding year k. Obviously there must be a direct connection between the demand of two successive years but this connection must not be necessary proportional times the ratio of the market's average premium to company's premium. One suggestion will be that demand in year $k+1$ is assumed to affect the demand of the preceding year in a linear way. Moreover, the fourth assumption can be changed. More specific, θ_k can affect the volume of business in a proportional way.

Under these new assumptions the volume of business function can take the following formulation

$$V_k = \frac{\bar{p}_k}{p_k}\theta_k \pm V_{k-1}, \tag{5.1}$$

where θ_k is again a stochastic parameter and measures the affection of the ratio average premium to company's premium, to company's volume of business of the preceding year plus or minus company's volume of business of the previous year.

Since the assumptions in chapter 3 and 2 are the same and under the considerations that have been just presented the recommended volume of business in chapter 3 can be changed into the following

$$V_k = \left(\frac{\bar{p}_k}{p_k}\right)^a e^{\theta_k} + sign(\gamma_k)|\gamma_k|^\beta V_{k-1}. \tag{5.2}$$

Here the volume of business of the preceding year is directly connecting to company's reputation which can either be positive or negative and the stochastic parameter θ_k can take either positive or negative values and weights the competition's affection (ratio of market's average premium to company's premium) to the volume of business.

In chapter 4 new assumptions enrich our model. According to the first assumption of chapter 4 $f_k(R_k, \tilde{p}_k) \triangleq \mathbb{E}\left[f_k(R_k, \tilde{p}_k, \theta_k)\right]$ is zero for all $R_k, \tilde{p}_k \in \mathbb{R}, k = 0, ..., N-1$. There is no loss of generality assuming this but in a wider approach the $\mathbb{E}\left[f_k(R_k, \tilde{p}_k, \theta_k)\right]$ could be a linear combination of the function's factor i.e.

$$\mathbb{E}\left[f_k(R_k, \tilde{p}_k, \theta_k)\right] = \mathrm{A_k} R_k \tilde{p}_k + \mathrm{B_k} \tilde{p}_k \theta_k + \Gamma_k R_k \theta_k. \tag{5.3}$$

It would be very interesting to explore the meaning and amount of the factors $\mathrm{A_k}, \mathrm{B_k}, \Gamma_k$ in order to have a more complete understanding of the model.

Additionally, following the second assumption $F_k(R_k, \tilde{p}_k) \triangleq \mathbb{E}\left[f_k^2(R_k, \tilde{p}_k, \theta_k)\right]$ exists, is a general quadratic function of R_k, $\tilde{p}_k$ for $k = 0, ..., N-1$. $F_k(R_k, \tilde{p}_k)$ and has the following representation $F_k(R_k, \tilde{p}_k) \triangleq B_k\left(\frac{1}{2}R_k^2 C_k + \tilde{p}_k g_k R_k + \frac{1}{2}\tilde{p}_k^2 M_k\right)$, where $B_k, C_k, M_k, g_k \in \mathbb{R}$. More elements can enrich the equation $F_k(R_k, \tilde{p}_k)$ i.e.

$$F_k(R_k, \tilde{p}_k) \triangleq B_k\left(\frac{1}{2}R_k^2 C_k + \tilde{p}_k g_k R_k + \frac{1}{2}\tilde{p}_k^2 M_k + \Delta_k R_k + \mathrm{E_k}\tilde{p}_k\right). \tag{5.4}$$

The third assumption must be maintained $F_k(R_k, \tilde{p}_k) \geq 0, \forall R_k \in \mathbb{R}$, $\tilde{p}_k \in \mathbb{R}$ in order to $F_k(R_k, \tilde{p}_k)$ be a covariance matrix.

Concerning the volume of business function, following Emms [19] the change in exposure is split up into the lost due to policy termination and that gained due to new business (or renewals) and entered the parameter n which denotes the rate of generation of new business and is equal to

$$n = qG\left(p/\bar{p}\right),$$

where G is a non-negative demand function. This parameterisation reflects the idea that the reputation of a company is proportional to its exposure in the market and that it is the reputation of an insurer which partially increases its likelihood to generate new business. New business generation is also determined by the premium that the insurer sets relative to the market, which is represented by the demand function G.

Adjusting this main idea to our volume of business function in discrete time, we can assume that the rate of generation of new business is equal to

$$n_k = V_k G\left(p_k/\bar{p}_k\right), \tag{5.5}$$

which relates the rate of generation of new business with the volume of business and ratio of company's premium to market's average premium.

In addition the volume of business following Emms [19] may assumed to be equal to

$$V_k = n_k - \xi V_{k-1}, \tag{5.6}$$

where ξV_{k-1} is equal to the loss of exposure due to policy termination (and not renewal).

5.2 Discussion on utility function

In chapter 2 and 3 the utility function is the present value of wealth $U(w_k, k) = \upsilon^k w_k$.

Many different alternatives are for the utility functions.

One simple case is the logarithmic form of u favored by Bernoulli, which can be written as

$$u(w) = \log(1 + cw), \tag{5.7}$$

for positive constant c.Bernoulli argued for (5.7) with an early expression of the law of diminishing marginal utility which says that the increment of utility for the next bit of wealth ought to be inversely proportional to the amount of wealth prior to the incremental increase.

Another utility function that could be used to extend our model is von Neumann-Morgenstern utility function with the following representation

$$u(w) = aw - be^{-cw}, \tag{5.8}$$

for constants $a \geq 0, b, c > 0$. This utility function is not only increasing $(u^{'} > 0)$ but also risk averse $(u^{''} < 0)$.

Moreover, a utility function that can be used is a quadratic one

$$u(w) = aw^2 + bw, \tag{5.9}$$

with $a \geq 0, b \geq 0$ and $a + b > 0$ to satisfy the assumption that more wealth is preferred to less (for constant a and b) which can be also expressed as $x \succ y$ when $x > y \geq 0$.If wealth were unbounded below, with $x \succ y \Leftrightarrow x > y$, then the quadratic case would reduce to the linear case of $u(w) = w$ because any nonzero a would violate monotonicity.

An alternative proposition for the utility function is an exponential utility

$$u(w) = 1 - e^{-aw}, \tag{5.10}$$

with coefficient of absolute risk aversion $A(w) = a$ or

$$U(w) = \frac{1-\gamma}{\gamma}\left(\frac{aW}{1-\gamma} + b\right)^{\gamma}, \tag{5.11}$$

with $a > 0$ and $b + \frac{aW}{1-\gamma} > 0$.

Concerning the utility function of company's reserve which is presented in chapter 4 the utility function could be equal to

$$\mathrm{E}_{/w_0}\left\{\sum_{k=0}^{N-1}\frac{1}{2}\left(w_k^{\mathrm{T}}Q_k w_k + \tilde{p}_k^{\mathrm{T}}\Gamma_k\tilde{p}_k\right) + \frac{1}{2}w_N^{\mathrm{T}}Q_N w_N\right\}, \tag{5.12}$$

where Γ_k denotes the marginal utility of insurance contracts in year $[k, k{+}1)$. In order to calculate a fair premium company's reserve must be minimized but the marginal utility of insurance contracts must be maximized. In other words a min-max optimization problem must be solved.

5.3 Discussion on models' wealth function

5.3.1 Wealth's risk investement

In chapter 2 and 3 the company's wealth function is equal to

$$w_{k+1} = -\alpha_k w_k + (p_k - \pi_k) V_k,$$

where α_k is the excess return of capital required by shareholder etc.

If the company chooses to invest the rest of its wealth to a portofolio with different investing products with a positive or zero return income then the wealth function will have the following representation

$$w_{k+1} = -a_k w_{k1} + (1 - a_k) w_{k2} I_k + (p_k - \pi_k) V_k \tag{5.13}$$

where $w_{k1} + w_{k2} = w_k$ and I_k is the return of the investment of the w_{k2} which is not risk free and could be either zero or positive and is a stochastic variable.

5.3.2 Direct connection between company's wealth and claims

According to Emms [20] the breakeven premium π_t (per unit exposure) is related to the mean claim size rate u_s

$$\pi_t = \mathrm{E}\left[\int_t^{t+\tau} u_s d_s / F_t\right],$$

and the wealth process is equal to

$$dw_t = -a w_t dt + q_t \left(G\left(k_t\right) p_t - u_t\right) dt,$$

where the rate of increase in exposure caused by new business and renewals is $q_t G(k_t, t)$ and $G(k_t, t)$ is the demand for insurance of relative price k_t at time t.

One suggesting wealth function concerning our model can be calculated

$$w_{k+1} = -a_k w_k + V_k \left(G_k\left(\frac{p_k}{\bar{p}_k}\right) p_k - C_k\right), \tag{5.14}$$

where $G_k\left(\frac{p_k}{\bar{p}_k}\right)$ is the demand function of insurance and C_k is the claims for year k.

Chapter 6

Conclusions

Taylor [71, 72] and Emms et al. [17, 18] study fixed premium strategies and the sensitivity of the model to its parameters involved. In their approach, the important parameters which determined the optimal strategies are the ratio of initial market average premium to break-even premium, the measure of the inverse elasticity of the demand function and the non-dimensional drift of the market average premium. However, the main purpose of our thesis is formulating different volume of business functions incorporating new economic parameters and calculate a premium which *derives straightforward from company's and market's historical data.*

In the second chapter, we articulate and answer three main questions. The first one is *"What is the optimal premium strategy for an individual insurance company and for a specific portfolio of homogeneous or/and heterogeneous risks?"*. The second is *"how is this related to the competitive market?";* and finally *"how does the volume of business affect the premium strategy?"*.

In order to answer these questions, extending further the ideas proposed by Taylor [71, 72], Emms Haberman [17] and Emms et al.[18], we develop a model for the optimal premium pricing policy of a non-life insurance company into a competitive market environment using elements of dynamic programming into a stochastic, discrete-time framework when the insurance company is expected *to lose part of the market* competition. For that reason, a stochastic demand function for the volume of business of an insurance company into a discrete-time has been applied according to which the volume of business is proportional to the volume of business of the presenting year (past year experience) times the rate market's average premium to company's premium (which is a control function) minus a stochastic parameter θ_k. Thus, by maximizing the total expected linear discounted utility of the wealth over a finite time horizon, the optimal premium strategy is defined analytically and endogenously for $\mathbb{E}(\theta_k) > \mu > 0$.

Thus, the optimal controller (i.e. the premium) is defined endogenously by the market as the company struggles to increase its volume of business into a competitive environment with the same characteristics as Taylor [71, 72], Emms and Haberman[17] , and Emms et al.[18] have used.

Finally, we consider three different strategies for the average premium of the market. In the Premium Strategy I, the average premium is calculated considering all the competitors of the market, and their proportions regarding the volume of business (i.e. we assume that we have the uniform distribution for the weight of every year). Moreover, in the Premium Strategy II, the average premium is calculated considering the premiums of the top K^{top} competitors of the market (including the leading company of the market) and finally in Premium Strategy III, the average premium is calculated considering the premium of company's direct competitors. The direct competitor factor is indicative to how the company that though as direct competitor is similar to our company's and affect our volume of business. This factor depends mainly on three other factors which are company's operational efficiency, product leadership and customer intimacy.

In chapter 3, we articulate and answer two main questions. The first one is *"how a company's optimal strategy can be determined into a general competitive market environment"* and secondly *"how this strategy is connected to the market's competition"*. A functional equation for the volume of business is proposed, which relates the company's premium with the past year experience, the average premium of the market, company's reputation and a stochastic disturbance, and it can be seen as a nice extension to the ideas proposed in chapter 2. Specifically market's volume of business is equal to the volume of business of the preceding year multiplied by the ratio market's average premium to company's premium raised to a factor α plus company's reputation raised to a factor β times the natural exponential function of the stochastic parameter θ_k. Company's reputation can either a have positive or negative impact on company's volume of business and the sign changes respectively.

Using again a linear discounted function for the company's wealth an optimal premium strategy can be investigated which maximizes its present value or minimizes the present value of the difference between a targeted wealth and the company's wealth. Thus, the main results are presented in two interesting theorems. The first theorem calculates the premium that the insurance company intends to charge by maximizing the expected total utility of wealth both for negative or positive reputation over a finite time horizon T and over a choice of strategies p. According to the results of the second theorem we may not have always find an optimal positive solution. Then the previous year's strategy is charectirized as a very successful choice. However we can also calculate an upper bound for the optimal premium strategy which is presented in theorem three of the third chapter when the company is targeting a particular wealth for the finite time horizon and the company tries to minimize the present value of the difference between a targeted wealth and the company's wealth when the company has positive reputation and $\alpha > 0$.

Analytical solutions of some special and common cases, for $\alpha = 1, 2, 3$, are presented where the optimal premium depends endogenously on the dynamics of the insurance market. then the optimal premium is given by Finally, an optimal premium strategy is proposed for the calculation of the markets' average premium and an application based on data from the Greek insurance market is presented for a complete understanding of the model. Market's average premium values can be represented as a sequence of correlated pulses or series of events and there is a generic multiplicative process for market's average premium which fluctuated due to random perturbations by a sequence of uncorrelated normally distributed rendom variable with zero expectation and unit variance.

Indeed, as far as computational techniques are concerned, the proposed process in this chapter to calculate the optimal premium is not challenging. Firstly, we define the volume of business function that fits better to our data. Afterwards, with the premium strategy mentioned above we calculate the expected market's average premium for the preceding year using an advanced point process with memory. Then, the polynomial can be solved and the optimal premium strategy which should be a positive number is finally derived.

In this chapter, two main questions are trying to be answered *"How an optimal premium can be calculated in order to minimize the level of the required reserve?"* and *"how it is possible to find a premium strategy that takes into consideration market's competition and all the different economic parameters that affect company's reserve except for the cost of a general insurance policy?"*.

For this purpose an optimisation process for the calculation of a *fair premium* is described. This topic is of a greatest interest for the practitioners as well as the whole insurance industry. Analytically, the reserve is considered to be a stochastic equation which has an additive random nonlinear function of the state, premium and not necessarily Gaussian noise (θ_k) which is, however, independently distributed in time, provided only that the mean value and the covariance of the random function is zero and a quadratic function of the state, premium and other parameters, respectively.

The new premium does not only capture the break-even premium, the company's volume of business of the preceding year, the expectation of market's average premium as it did in the linear models, but also the income insurance elasticity of demand, the number of consumers, the inflation in addition to the company's reputation. Finally, the derived optimal premium depends on the company's reserve as well as the other already mentioned above factors.

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