Modelling Catastrophe Risk Bonds

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by

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Abstract

Insurance companies are seeking more adequate liquidity funds to cover the insured property losses related to nature and man-made disasters. Past experience shows that the losses caused by catastrophic events, such as earthquakes, tsunamis, floods or hurricanes, are extremely large. One of the alternative methods of covering these extreme losses is to transfer part of the risk to the financial markets, by issuing catastrophe-linked bonds.

This thesis focuses on model and value Catastrophe (CAT) risk bonds. The findings of this thesis is twofold. First, we study the pricing process for CAT bonds with different model setups. Second, based on different framework, we structured three catastrophe based (earthquake, general and nuclear risk) bonds, estimated the parameters of the model by employing real world data and obtained numerical results using Monte Carlo simulation. Comparison between different models is also conducted.

The first model employed the structure of n financial and m catastrophe-independent risks, and obtain the valuation framework. This generalized extension allows an easier application in the industry. As an illustration, a structured earthquake is considered with parametric trigger type – annual maximum magnitude of the earthquake – and the pricing formulas are derived. The second model presents a contingent claim model with the aggregate claims following compound forms where the claim inter-arrival times are dependent on the claim sizes by employing a two-dimensional semi-Markov process. The final model derives nuclear catastrophe (N-CAT) risk bond prices by extending the previous model. A two-coverage type trigger CAT bond is analysed by adding a perturbed state into the claims system, i.e. the system stops (N-CAT bond contract terminated) immediately after a major catastrophe.

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Appendices

Publications

- Shao, J., Pantelous, A., and Papaioannou, A. **(2015)**. Catastrophe Risk Bonds With Applications To Earthquakes. *European Actuarial Journal*, **5(1)**, pages **113 –138**.

- Ayyub, B. M., Pantelous, A., and Shao, J. **(2015)**. Towards Resilience to Catastrophic Events: Financing Liabilities via Catastrophe Risk Bonds. *Economics of Community Disaster Resilience (NIST-ASCE) Workshop: Background Information and Technical Papers*, pages **94 – 105**.

- Ayyub, B. M., Pantelous, A., and Shao, J. **(2015)**. Towards Resilience to Nuclear Accidents: Financing Nuclear Liabilities via Catastrophe Risk Bonds. Under review since May 2015, *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part B: Mechanical Engineering*.

- Shao, J., Papaioannou, A., and Pantelous, A. **(2015)**. Pricing and Simulating CAT Bonds in a Markov-dependent Environment. Under review since December 2014, *Insurance: Mathematics and Economics*.

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Chapter 1

Introduction

Due to the potentially enormous financial demands on insurance (reinsurance) businesses and the increasing difficulty of covering catastrophic losses by reinsurance, it is considerable to introduce a securitization method to protect vulnerable individuals. Insurance companies alleviate part of their risks by introducing securitization mechanics to achieve a more adequate liquidity fund. An alternative method is to issue catastrophe (CAT) bonds, which transfer the financial consequences of catastrophic events from issuers to investors in a contract to cover huge liabilities through traditional reinsurance providers or governmental budgets.

CAT bonds spread the risks to another level – global financial markets. Investors take on a specific set of risks (generally catastrophe and natural disaster risks) of a specified catastrophe or event occurring in return for attractive rates of investment. If a predetermined catastrophe or event occurs, the investors will lose the principal they invested and the issuer (often insurance or reinsurance companies) will receive that money to cover their losses.

The aim of this thesis is to model and value the price of the catastrophe risk bonds. Our structure of catastrophe risk bonds involve different catastrophic perils (earthquake and nuclear power risk) with different payoff functions and interest rate models. This thesis gives a dynamic view of modelling CAT bonds, and finally numerically computes and then compares between the prices under the different scenarios.

In this chapter, we will explain the motivation behind this work. Section 1.1

presents the definition of catastrophe and catastrophic risks. It also answers the question of what is the size of loss of a catastrophic event and what is the probability of having a catastrophic event occur in the given period. Section 1.2 illustrates the current nuclear liability conventions and also the liability limitation regimes. Section 1.3 provides the definition and structure of CAT bonds and then introduces the CAT bonds history. And finally in Section 1.4 is the literature review of pricing CAT bonds.

1.1 Catastrophic Events and Catastrophic Risks

A catastrophic event is defined to be a sudden event that causes one person or a group of people to suffer, or that makes difficulties. Catastrophic accidents include earthquakes, nuclear and chemical accidents, extreme storms, super-volcanoes, outer space related events, pandemics, etc. Such events occur infrequently, but cause massive losses over a short period. The Insurance Service Office's (ISO's) Property Claim Service (PCS)[1] declared 254 catastrophes (in United States) that incurred damages of approximately US$112 billion between 1990 and 1996, while the losses due to Hurricane Andrew in 1992 reached US$ 26 billion[2]. Thus, even a single event can led to the insolvency of insurance companies.

Some arguments state that due to catastrophic accidents rarely occurring, an insurance company may not face such an event during its life time. Take nuclear accident risks as an example, a report to the United States Congress from the Presidential Commission on Catastrophic Nuclear Accidents in 1990, see Griffith et al. (1990), provides an estimate of a catastrophic nuclear accident probability in the United States of about 1 in a billion year per nuclear power plant (NPP) unit, i.e. a reactor. Expressing this

[1] ISO's Property Claim Service unit is the internationally recognized authority on insured property losses due to catastrophes in the USA, Puerto Rico, and the US Virgin Islands. It contains information on all the historical catastrophes since 1949, including the states affected, perils, and associated loss estimates. http://www.verisk.com/property-claim-services/.

[2] An illustration of the PCS catastrophe loss data converted to 2014 dollars using the Consumer Prices Index (CPI) in US is given in Figure 1.1, e.g. the Northridge earthquake (1994) with losses of US$20 billion, 9/11 Terrorist Attacks (2001) with losses of US$25 billion, Hurricane Katrina (2005) with losses of US$50 billion and Hurricane Sandy with losses of US$20 billion. Data from PCS.

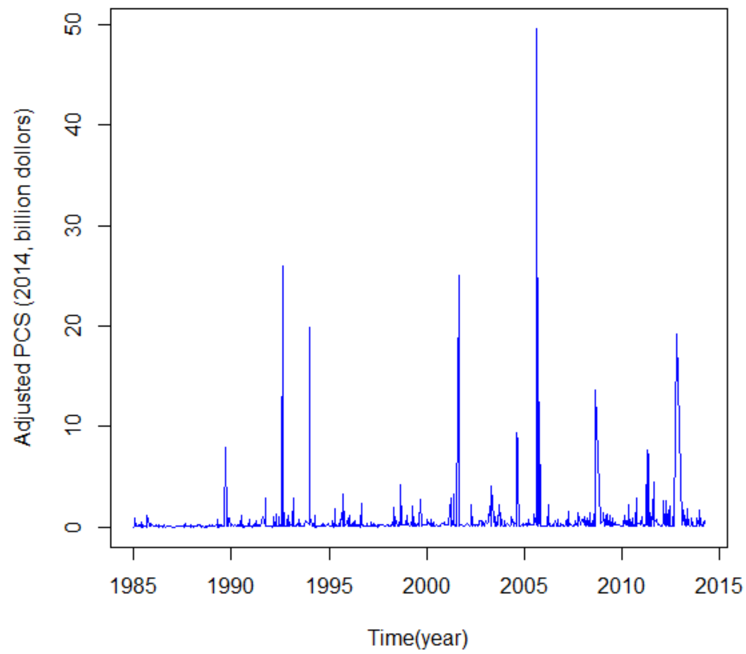


Figure 1.1: Annual catastrophe loss in the USA in 1985–2013, data from PCS.

best estimate in this manner implicitly assumes no enhancements to the safety of nuclear plants in the 1 billion years which is unrealistic.

Societies and decision makers desire to know "what is the probability of a catastrophic event in the remaining lifetime of today's plants?" The underlying computations are approximate to allow for using simple multiplication to account for the increase in unit-years. These computations show catastrophic nuclear accidents would approach inevitability based on current practices, and growth in number of units and their ages as demonstrated in Figure 1.2. The curves in Figure 1.2 are based on Griffith et al. (1990) updated by the increase in NPP units and are not updated to account for accidents which occurred afterwards, such as the Fukushima and Chernobyl disasters. With an assumed remaining lifetime of about 30 years per plant and about 100 NPP units operating in 1990, we obtain about 1 in 0.33 million years. These estimates can be refined based on the scenarios postulated in Nuclear Regulatory Commission (NRC) reports, such as NUREG–1150 U.S.NRC (2012a) and WASH–1400 U.S.NRC

(2012b) and recent Probabilistic Risk Analysis (PRA) of NPPs. Considering the uncertainties associated with underlying random variables, parameters and assumptions, the best estimate of 1 in 0.33 million can be expressed as a range of 1 in 1.66 million to 1 in 0.066 million.

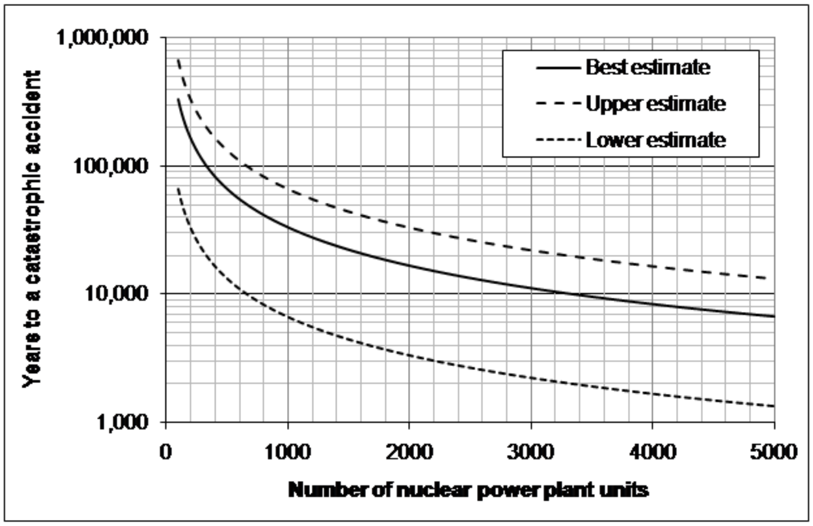
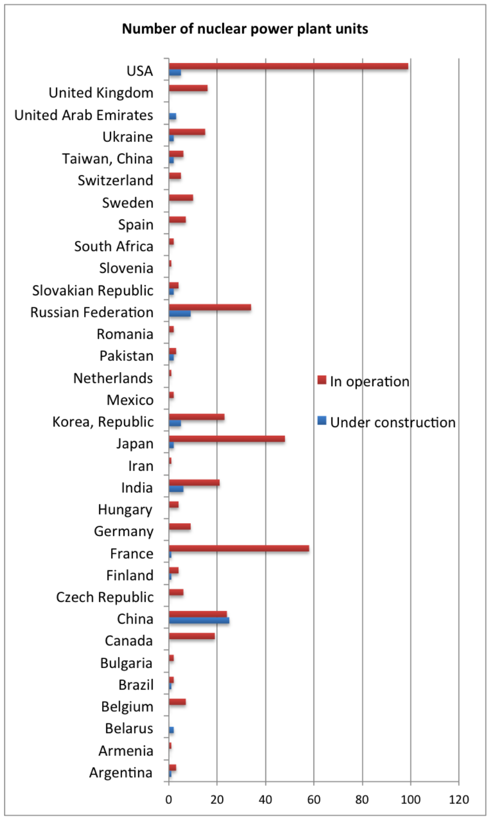


Figure 1.2: Time to a catastrophic nuclear accident as a function of the number of nuclear power plant units worldwide, Griffith et al. (1990).

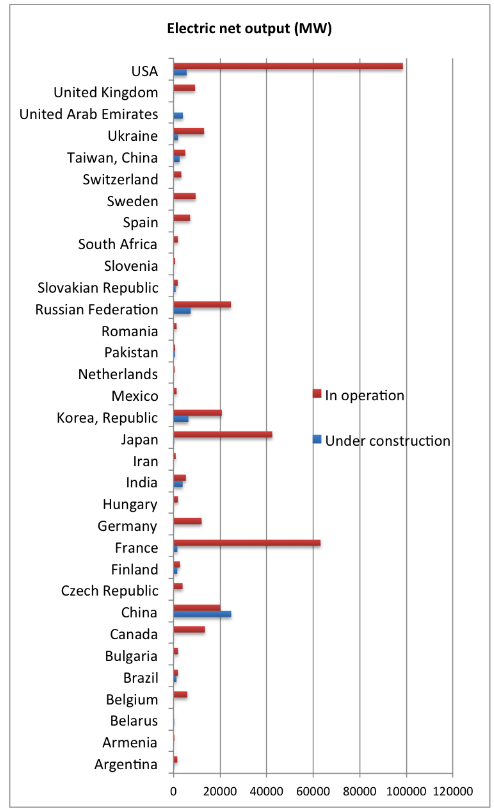
In light of the 2011 Fukushima disaster, recent discussion has focused on maximizing the oversight power of global institutions and strengthening safety measures. Without accounting for the variation in nuclear technology, regulatory regimes, operators' experience and NPP units' ages, the worldwide probability of a catastrophic nuclear accident can be estimated as significantly greater than, by orders of magnitude, the levels provided by Griffith et al. (1990). The Fukushima and Chernobyl disasters of 2011 and 1986, respectively, provide empirical evidence for such levels. With a nuclear renaissance underway, the worldwide inventory of NPP units is expected to increase from 439 to 508, with corresponding increases in net electric outputs as shown in Figure 1.3, European Nuclear Society (2015).

Assessing the adequacy of liability coverage requires examining the consequences of historic and postulated nuclear accidents. Most notable nuclear accidents[3] in the civil power sector include: the 1979 Three Mile Island in which the containment re-

[3]More detail check Appendix A.



(a) Number of plant units



(b) Electric net output

Figure 1.3: Nuclear power plant units worldwide, in operation and under construction, as of March 10, 2015, European Nuclear Society (2015).

mained intact and resulted in 1993 US$1 billion dollar cleaning-up cost performed over 14 years; and the 1986 Chernobyl disaster in the former Soviet Union resulting in 56 lives lost, over 4000 people with long-term effects, and US$15 billion of direct loss. It is estimated that the damages could accumulate to US$305 billion for Ukraine and US$261 billion for Belarus in the thirty years following the Chernobyl accident. Various estimates of the total damage which could be caused by accidents at nuclear power plants range from US$110 billion to as much as US$7 trillion, Friends of the Earth Europe (2007); Raju and Ramana (2010).

1.2 Nuclear Liability Conventions and Liability Limitation Regimes

Most countries with commercial nuclear programs adhere to one of the international conventions and concurrently have their own legislative regimes for nuclear liability, see Balachandran (2010); American Nuclear Insurers (2013). The national regimes implement the conventions' principles and impose the financial security requirements that vary from country to country. The thirty-four countries that possess NPPs can be grouped as follows:

1. The first group includes those countries that are parties to one or more of the conventions, and which have their own legislative regimes. Prominent examples are France, Germany, Spain and the United Kingdom, all of which are parties to the Paris Convention (PC) and Revised Paris Protocol (RPC, not yet in force). Since 1988, parties to the Joint Protocol (JP) are treated as if they are parties to both the Vienna Convention (VC) and the PC. Seventeen countries have signed the Convention on Supplementary Compensation for Nuclear Damage (CSC), including Czech Republic, Canada, Ukraine and India, but most have not yet ratified it. In 2014 Japan and UAE passed legislation to ratify the CSC.

2. The second group includes those countries that are not parties to the conventions, but which have their own legislative regimes. Prominent examples are USA, Canada, Japan and Republic of Korea (South Korea). These countries impose strict liability on their nuclear installation operators. So they conform with the channeling requirements of the Paris and Vienna Conventions, despite not being parties to those conventions.

3. The final group of countries neither being parties to the conventions nor having their own legislative regimes. Although these countries are relatively few, China is the most prominent example that has issued directives stating its position on nuclear liability, but has not yet developed a specific regime. China's nuclear liability directives were issued in 1986 as an interim measure in connection with the French-designed Daya Bay nuclear power plant. It contains most of the

elements of the international nuclear liability conventions, e.g. channeling of absolute nuclear liability to the plant operator and exclusive court jurisdiction. Other countries in this group include Pakistan, with 3 NPPs. Pakistan is neither members of any international convention nor have any national legislation.

Table 1.1 provides a summary of the convention and membership by country, World Nuclear Association (2015).

In addition, the US enacted a nuclear liability regime – the Price Anderson Act – to manage the risk of a nuclear accident in 1957. It has created a favorable climate for the nuclear American industry and provides US$13.6 billion in cover without cost to the public or government and without fault needing to be proven. The Act was amended over the years. Someone could arguably demonstrate that the US government is providing subsidies since the coverage is far less than the potential loss, see Balachandran (2010); GAO (2004); World Nuclear Association (2015).

So far in this section, we have presented exposures from the perspectives of the public, operators and government; however, what does all this mean for a designer, builder or supplier? If the products or services are provided to a nuclear installation in a country subject to the PC or VC, the supplier likely does not need nuclear liability insurance. The supplier should not be held liable for damages resulting from a nuclear incident. Liability should be channeled to the facility operator.

The two exposures for suppliers, according to World Nuclear Association (2015) are: to have nuclear legislation which legally channels liability to the facility operator and to have cross-border liability. These exposures might necessitate suppliers to purchase their own insurance. The decision whether to purchase insurance often reflects how risk averse a supplier and its risk-management philosophy. American Nuclear Insurers (2013) makes available a foreign Supplier's and Transporter's policy (called S&T policy) for this purpose, that indemnifies the insured for third-party bodily injury or property damage resulting from the nuclear energy hazard, which is defined as the radioactive, toxic, explosive or other hazardous properties of nuclear material. The policy's current maximum limit of liability is US$50 million available in all insured countries except Japan, Mexico, South Africa, Spain and Sweden, where the available limit is US$15 million because of reinsurance commitments. The policy excludes

Table 1.1: Nuclear power countries and liability conventions to which they are party, World Nuclear Association (2015).

Countries	**Conventions party to**	**Countries**	**Conventions party to**
Argentina	VC; RVC; CSC	Lithuania	VC; JP; (CSC signed)
Armenia	VC;	Mexico	VC
Belgium	PC; BSC; RPC; RBSC	Netherlands	PC; BSC; JP; RPC; RBSC
Brazil	VC	Pakistan	
Bulgaria	VC; JP	Romania	VC; JP; RVC; CSC
Canada	(CSC signed)	Russia	VC
China		Slovakia	VC; JP
Czech Rep.	VC; JP; (CSC signed)	Slovenia	PC; BSC; JP; RPC; RBSC
Finland	PC; BSC; JP; RPC; RBSC	South Africa	
France	PC; BSC; RPC; RBSC	Spain	PC; BSC; RPC; RBSC
Germany	PC; BSC; JP; RPC; RBSC	Sweden	PC; BSC; JP; RPC; RBSC
Hungary	VC; JP	Switzerland	PC; RPC; BSC; RBSC
India	(CSC signed)*	Taiwan, China	
Iran		Ukraine	VC; JP; (CSC signed)
Japan	CSC	UAE	RVC; CSC
Kazakhstan	RVC	United Kingdom	PC; BSC; RPC; RBSC
Korea		United States	CSC

PC = Paris Convention (PC).

RPC = 2004 Revised Paris Protocol. Not yet in force.

BSC = Brussels Supplementary Convention.

RBSC = 2004 Revised Brussels Supplementary Convention. Not yet in force.

VC = Vienna Convention.

RVC = Revised Vienna Convention.

JP = 1988 Joint Protocol.

CSC = Convention on Supplementary Compensation for Nuclear Damage (CSC), in force from 15 April 2015.

*India has not ratified CSC; domestic liability law may not conforms with the requirements of the convention.

several countries for various reasons, most notably the United States, Canada, China, India and Russia.

1.3 Catastrophe Risk Bonds

Losses and recovery costs from catastrophic accidents are typically covered by a combination of utility companies, special insurance programs and/or governments. For example losses from the 2011 Fukushima disaster were covered primarily by the government of Japan. Resources for this purpose are often inadequate and require a cash reserve that could be challenging to maintain. Low penetration rates for insurance leaves individuals, companies and governments to shoulder the financial losses arising from catastrophic events. In emerging markets with nonexistent or immature legal regimes, liability could lead to international tensions and potentially wars, particularly in cases of cross-border exposures.

According to the information in Section 1.1, using a nuclear accident rate of 10^{-6} per year, assuming 500 policies, loss per accident of US5 trillion, and price of a policy for the break-even point can be computed to be US$10,000$ per year. Obviously, an insurance model of this type would not sustain itself and would bankrupt upon the occurrence of the first catastrophic accident within the life of the present NNPs population. Insurers covering other catastrophic perils – earthquake risk – may also face problems. According to historical information from the National Earthquake Information Center (NEIC), 12,000–14,000 earthquakes are recorded annually throughout the world[4]. In California, two or three earthquakes of magnitude 5.5 and higher occur annually, and these are large enough to cause moderate damage[5]. Although infrequent, earthquakes and their side effects, including landslides, surface fault ruptures, liquefaction, aftershock fires, and tsunamis, have huge potential to cause injury, loss of life, and property damage. The California Geological Survey[6] has reported that more than

[4] Accessed on 01/07/2015, `http://earthquake.usgs.gov/earthquakes/?source=sitenav`.

[5] `http://www.conservation.ca.gov/index/earthquakes/Pages/qh_earthquakes.aspx`, accessed on 01/07/2015.

[6] Accessed on 01/07/2015, `http://www.consrv.ca.gov/CGS/Pages/Index.aspx`.

70% of California residents live within an area where significant earthquakes could occur in the next 50 years, according to slip rates in geological time. Therefore, the potential financial demands on insurance and reinsurance businesses make it realistic to introduce a mechanism for individuals against nature and man-made disasters.

The requirement to achieve adequate liability coverage is to have a system that has adequate financial depth to fulfill claims. To succeed, financing is essential using special purpose instruments from the global market. Figure 1.4 provides an estimate of the 2012 global outstanding bonds and loans to be US$175 Trillion out of the total US$225 Trillion of capital stock (outstanding bonds, loans and equity) with stocks at US$50 Trillion, Lund et al. (2013). Despite the 2008 financial crisis, global bonds and loan markets have increased consistently over the past twenty years from US$45 Trillion in 1990.

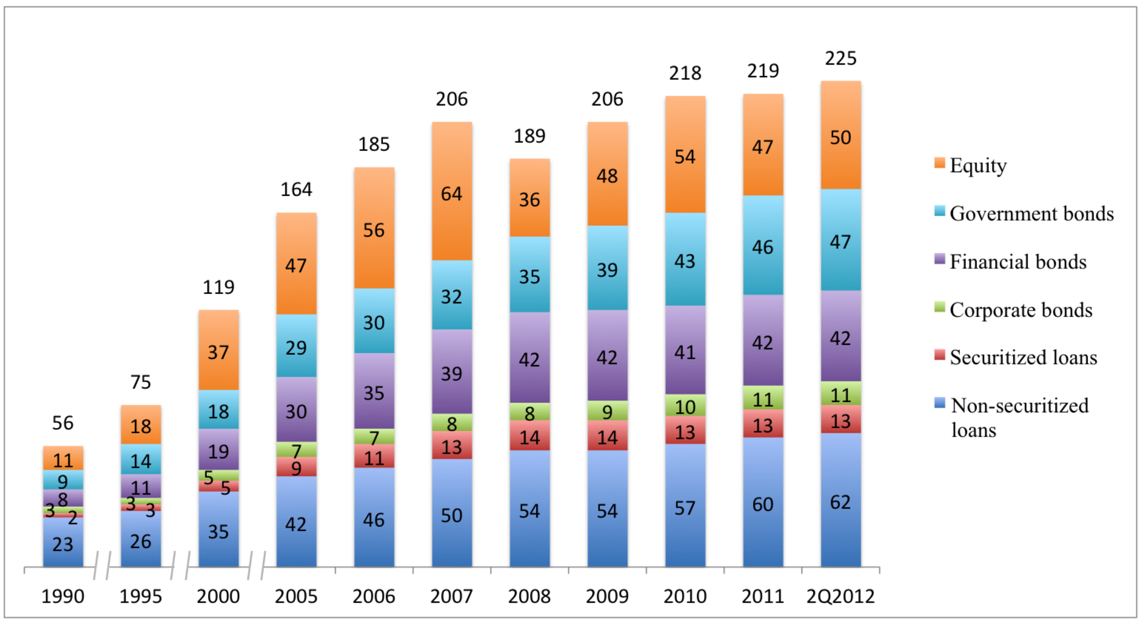


Figure 1.4: Global stock of debt and equity outstanding, US$ Trillion, end of period, constant 2011 exchange rates, Lund et al. (2013).

CAT risk bonds (or Act-of God bonds) are born for these extreme events and sharing the risk to another level – global financial markets as the only pool of cash large enough to underwrite such losses lies in capital markets and the collection of big investors like pension funds, hedge funds and sovereign wealth funds that normally invest in stocks and bonds. CAT risk bonds are the most popular insurance-linked financial securities and their use has been accelerating in the last decade.

The first experimental transaction was completed in the mid-1990s after Hurricane Andrew and the Northridge earthquake, which incurred insurance losses of US$\$15.5$ billion and US$\$12.5$ billion, respectively, by a number of specialized catastrophe-oriented insurance and reinsurance companies in the USA, including AIG, Hannover Re, St Paul Re, and USAA, GAO (2002). The CAT bonds market has boomed over the years. The issued capital has increased tenfold within ten years, from less than US$\$0.8$ billion in 1997 to over US$\$8$ billion in 2007. The issuers raised more than US$\$9$ billion of new CAT bonds in 2014[7]. CAT bonds are inherently risky, non-indemnity-based multi-period deals, which pay a regular coupon to investors at end of each period and a final principal payment at the maturity date, if no predetermined catastrophic events occur. A major catastrophe in the secured region before the CAT bond maturity date leads to full or partial loss of the capital.

CAT bonds structure including where the capital flows from one party to another is presented in Figure 1.5. The issuer does not directly issue the CAT bond, but uses Special Purpose Vehicle (SPV) for the transaction. SPV can be interpreted as a focused insurer whose only purpose is to write one insurance contract. The existence of SPV, which is equal to a focused one-policy insurer, minimises the frictional cost of capital. Furthermore, sufficient high endowment of the SPV eliminates the counterparty risk. SPV enters into a reinsurance agreement with a sponsor or counterparty (e.g. insurer, reinsurer, or government) by issuing CAT bonds to investors and receives premiums from the sponsor in exchange for providing a pre-specified coverage. Therefore, sponsors can transfer part of the risks to investors who bear the risk in return for higher expected returns. The SPV collects the capital (principal and premium) and invests the proceeds into a collateral account (trust account, which is typically highly related to short-term securities, e.g. Treasure bonds). The returns generated from collateral accounts are swapped for floating returns based on London Interbank Offered Rate

[7]Accessed on 17/08/2015, `http://www.artemis.bm/deal_directory/cat_bonds_ils_issued_outstanding.html`. ARTEMIS is an online website since 1999, Artemis provides news, analysis and data on catastrophe bonds, insurance-linked securities and alternative reinsurance capital.

(LIBOR) in order to immunize the sponsor and the investors from interest rate risk and default risk, Cummins (2008).

The investors coupon payments are made up of SPV investment returns, plus the premiums from the sponsor. If no trigger event occurs during the term time of the CAT bond, then the collateral is liquidated at maturity date of CAT bond and investors are repaid principal plus a compensation for bearing the catastrophe risks (solid line in Figure 1.5). However, if a trigger event occurs before the maturity, the SPV will liquidate collateral required to make the payment and reimburse the counterparty according to the terms of the catastrophe bond transaction, and CAT bond investors will only receive part of the capital (dashed line in Figure 1.5).

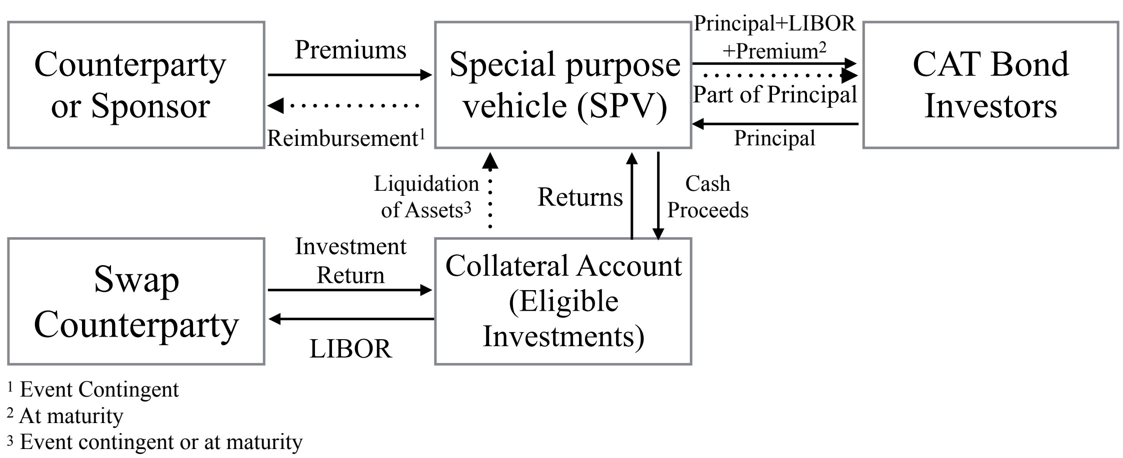


Figure 1.5: Structure of CAT bonds.

Finally, the feature of correlation of the traditional stock market allows CAT bond investors to still gain in a bad economic circumstance. CAT bonds reduce barriers to entry and increase the contestability of the reinsurance market, Froot (2001).

To bear the catastrophe risks, CAT bonds carry a 3 to 5 year maturity and compensate for a floating London Interbank Offered Rate (LIBOR) coupon plus a premium at a rate between 2% and 20%, see Cummins (2008); GAO (2002). Detailed information of CAT bonds premium level is given in Figure 1.6[8]. One of the key elements of any CAT bond is the terms under which the securities begin to experience a loss. Catas-

[8]Accessed on 01/07/2015, `http://www.artemis.bm/deal_directory/cat_bonds_ils_by_coupon_pricing.html`.

trophe bonds utilise triggers with defined parameters which have to be met to start accumulating losses. Only when these specific conditions are met do investors begin to lose their investment. Triggers can be structured in many ways from a sliding scale of actual losses experienced by the issuer (indemnity) to a trigger which is activated when industry wide losses from an event hit a certain point (industry loss trigger) to an index of weather or disaster conditions, which means actual catastrophe conditions above a certain severity will trigger a loss (parametric index trigger) etc, Hagedorn et al. (2009); Burnecki et al. (2011). Figure 1.7 presents the amount and percentage of CAT bonds issued by trigger type[9]. Indemnity trigger type is subject to the highest degree of moral hazard due to the fact that loss is controlled by sponsor. To tackle this problem, a better choice would be using industry loss trigger or parametric index trigger, although these might bear a relatively higher basis risk.

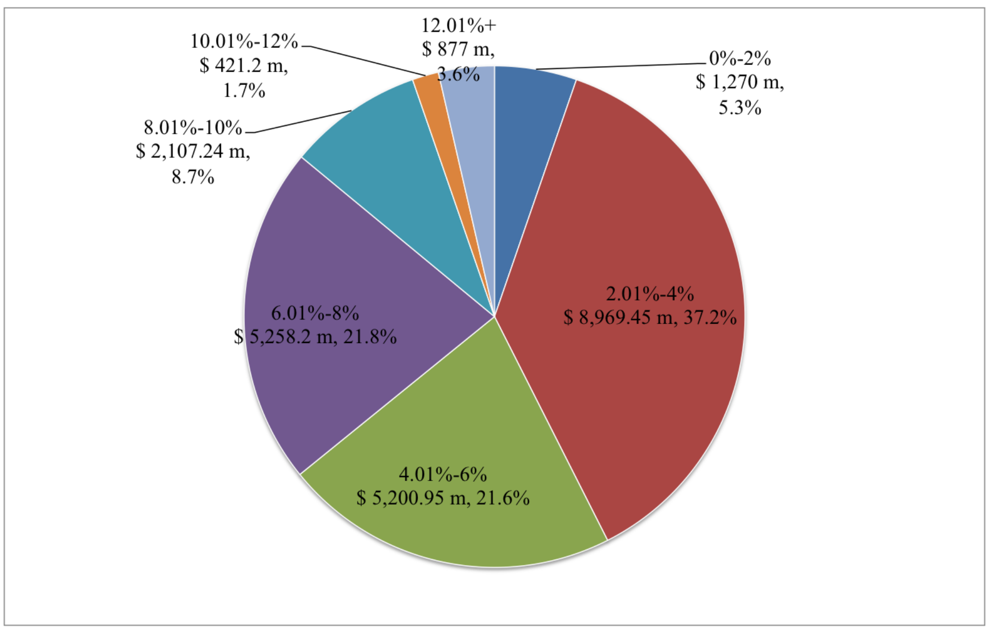


Figure 1.6: Catastrophe bonds & ILS outstanding by coupon pricing, data from Artemis, accessed on 01/07/2015.

CAT bonds can be structured to provide per-occurrence cover, so exposure to a single major loss event (currently US$ 14,850.33 million which account for 64.2%), or to provide aggregate cover, exposure to multiple events over the course of each

[9]Accessed on 01/07/2015, http://www.artemis.bm/deal_directory/cat_bonds_ils_by_trigger.html.

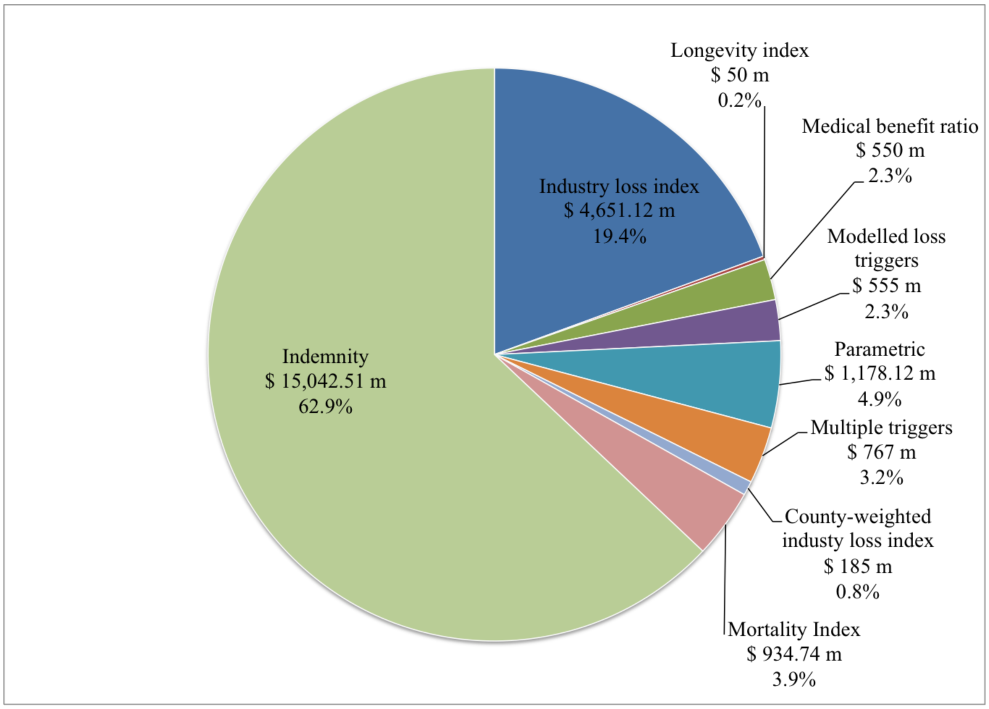


Figure 1.7: Catastrophe bonds & ILS outstanding by trigger type, data from Artemis, accessed on 01/07/2015.

annual risk-period (US$ 8,269.14 million which account for 35.8%)[10]. Some CAT bonds transactions work on a multiple loss approach and so are only triggered (or portions of the deals are) by second and subsequent events. This means that sponsors can issue a deal that will only be triggered by a second landfalling hurricane to hit a certain geographical location, for example.

1.4 Literature Review

Despite the raising popularity, the number of previous studies devoted to CAT bonds pricing is relatively limited. Among the current pricing literature, authors mainly devoted to modelling CAT bonds by different approaches, and a few have attempted to model and price from the real world perspective, in order to provide a tradeable CAT bond for a given catastrophe.

[10]Accessed on 01/07/2015, http://www.artemis.bm/deal_directory/cat_bonds_ils_by_aggregate_occurrence.html.

The prediction of catastrophe risks requires an incomplete markets framework to evaluate the CAT bonds price, because the catastrophe risk cannot be replicated by a portfolio of primitive securities, see Harrison and Kreps (1979); Cox et al. (2000); Cox and Pedersen (2000); Vaugirard (2003). In the case of an incomplete market, there is no universal pricing theory that successfully addresses issues such as specification of hedging strategies and price robustness, see Young (2004). For example, Wang (2004) addressed market incompleteness using the Wang transform, an approach adopted by Lin and Cox (2005, 2008); Pelsser (2008); Galeotti et al. (2011). Froot and Posner (2000, 2002) derived an equilibrium pricing model for the uncertain parameters of multi-events risks. Follmer and Schweizer (1991) introduced a minimal martingale measure for option pricing, whereas Schweizer (1995) used a variance optimal martingale measure.

Another common technique used in an incomplete market setting is the principle of equivalent utility for obtaining indifferent pricing. Young (2004) calculated the price of a contingent claim under a stochastic interest rate for an exponential utility function. An extension was proposed by Egami and Young (2008), who introduced a more complex payment structure based on the assumption of utility indifference. Dieckmann (2011) applied a CAT bond model based on consumption, while Zhu (2011) detailed the premium spread using an intertemporal equilibrium framework. Braun (2012) analysed the premium using OLS regression with robust standard errors. Cox and Pedersen (2000) used a time-repeatable representative agent utility. Their approach was based on a model of the term structure of interest rates and a probability structure for catastrophe risks, which assumed that the agent uses a utility function to make choices about consumption streams. They applied their theoretical results to Morgan Stanley, Winterthur, USAA, and Winterthur-style bonds. Reshetar (2008) used a similar setting for multiple-event CAT bonds for the first time. Zimbidis et al. (2007) adopted the Cox and Pedersen (2000) framework to price a Greek bond using equilibrium pricing theory with dynamic interest rates.

Several studies have used stochastic processes to price CAT bonds. Under the assumption of continuous time, one of the approaches is to model the probability of credit default which follows the methodology of pricing credit derivatives in finance. Baryshnikov et al. (2001) presented a continuous time no-arbitrage price of zero coupon and non-zero coupon CAT bonds that incorporated a compound doubly stochastic Poisson process. The main weakness of this paper is the authors assumed that the arbitrage measure and real world measures coincide. Burnecki and Kukla (2003) corrected and then applied their results with PCS data to calculate the arbitrage-free price of zero-coupon and coupon CAT bonds. Burnecki et al. (2011) illustrated the value of CAT bonds with loss data provided by PCS when the flow of events was an inhomogeneous Poisson process. These approaches were utilized by Härdle and Cabrera (2010) for calibrating CAT bonds prices for Mexican earthquakes. Jarrow (2010) obtained a simple closed form CAT bond solution with a LIBOR term structure of interest rate.

Another approach in continuous time is to model the trigger involving aggregate loss process. It is important to note that Vaugirard (2003) was the first to develop a simple arbitrage approach for evaluating catastrophe risk insurance-linked securities, although they employed a non-traded underlying framework. In this paper, CAT bondholders have a short position on an option. Lin et al. (2008) applied a Markov-modulated Poisson process for catastrophe occurrences using a similar approach to that of Vaugirard (2003). Lee and Yu (2002, 2007) also introduced the default risk, moral hazard, and basis risk with stochastic interest rate. Pérez-Fructuoso (2008) developed a CAT bond with index triggers. Ma and Ma (2013) proposed a mixed approximation method to simplify the distribution of aggregate loss and to find the numerical solutions of CAT bonds with general pricing formulae. In addition, Nowak and Romaniuk (2013) expanded Vaugirard's model and obtained CAT bond prices using Monte Carlo simulations with different payoff functions and spot interest rates.

This thesis is organized as follows. Chapter 2 introduces the preliminary presentation for CAT bond pricing under an assumption that the occurrence of a localized catastrophe is independent of the global financial market behaviour. In this chapter, we first answer the question of why we need to model in an incomplete framework. Then, we list three universal model assumptions made within this thesis. Classical probabilistic structure and valuation formula are also given, which will be used in Chapters 4 and 5. And finally, we analysis two interest rate models (ARIMA and CIR) and the extreme value theory.

Chapter 3 develops a model with multiple catastrophes and financial risks framework in a discrete-time period as an extension of the approach of Cox and Pedersen (2000). It applies an incomplete and no-arbitrage framework and assumes that all risks are mutually independent, while aggregate consumptions depend only on financial risk variables. Then, we apply theoretical results to construct a structured parametric index earthquake multi-variable CAT bond for one-period and multi-period. As a numerical example, a CAT bond with historical data from California is proposed in which the magnitude, latitude, longitude, and depth are included in the model. In addition, appropriate models are constructed for the term structure of interest and inflation rate dynamics, and a stochastic process for the coupon rate. Finally, on the basis of analysis for the aforementioned catastrophe and financial market risks, we use equilibrium pricing theory to find a certain value price for the CAT California earthquake bond.

Chapter 4 derives CAT bond pricing formulae under the special case of the previous chapter, one financial and one catastrophic risk. We make three main contributions to the area of CAT bond pricing. First, we construct our model in a Markov-dependent environment as an extension of the approach proposed by Ma and Ma (2013). For the first time in the CAT bonds area, we model the dependency between the claims sizes and the claim inter-arrival times for the aggregate claims as a semi-Markov process. The main benefit of this extension is the development of a more realistic model, where the occurrence time before the next claim is partially dependent on the previous claim size, which indicates that a major catastrophic event triggers many other catastrophic events in a short period. Second, in order to obtain a more complete example, we structure four different payoff functions (classical zero-coupon and coupon, multi-threshold zero-coupon, and defaultable) and we give analytical formulae for CAT bonds. Third, we apply our theoretical results to construct a CAT bond and we then use PCS data to estimate relevant parameters to obtain analytical solutions, thereby providing clear guidance for practitioners.

As a further extension and application, Chapter 5 conducts, for the first time, a two-coverage type trigger nuclear catastrophe risk bond (N-CAT) for potentially supplementing the covering of US commercial nuclear power plants beyond the coverage per the Price Anderson Act as amended, and potentially other plants worldwide are proposed and designed. The N-CAT peril is categorized by three risk layers: incident, accident and major accident. Assume that if a major accident (e.g. nuclear reactor core failure) occurs, the N-CAT expire immediately, which means there is an absorbing state in the system. The pricing formula is derived by using a semi-Markovian dependence structure in continuous time with a perturbed state, which is also an extension of the previous chapter. A numerical application illustrates the main findings of the paper.

Chapter 2

Preliminaries

2.1 Incomplete Market

Brigo and Mercurio (2007) defined in Definition 2.1.3.: '*A financial market is complete if and only if every contingent claim is attainable.*' Harrison and Kreps (1979); Harrison and Pliska (1981, 1983) stated the following two fundamental arbitrage-free theorem: Firstly, if a market exists an equivalent martingale measure (risk-neutral probability measure), then the market is arbitrage free. Secondly, if this risk-neutral probability measure is unique, then the financial market is complete. However, an arbitrage-free market does not necessarily need to be complete. In a complete market, the derivation of a unique price equals the discounted expected value of the future payoff under the risk-neutral measure. In an incomplete market, the derivative price is not unique due to the fact that one can construct several different hedging portfolios. Therefore, in order to evaluate derivatives under an incomplete framework, one can choice a suitable risk-neutral probability measure and then take the conditional expectation under this measure.

In this study, the introduction of catastrophe risk requires an incomplete market framework to evaluate the CAT bond price, because the catastrophe risk cannot be replicated by a portfolio of primitive securities, see Harrison and Kreps (1979); Cox et al. (2000); Cox and Pedersen (2000); Vaugirard (2003). In this section, we are going to briefly discuss the nature of incomplete markets with catastrophe risks. Assuming the same setting as in Section 4 of Cox and Pedersen (2000): a single period model with two tradeable zero-coupon bonds (a one-period bond and a two-period bond), when interest rate (6%) will go 'up' (7%) or 'down'(5%) during this period with equal probabilities. Denote n_1 as the number of one-period bonds and n_2 as two -period bonds in this portfolio, with cost

$$\frac{1}{1.06}n_1 + \frac{1}{1.06}0.5(\frac{1}{1.07} + \frac{1}{1.05})n_2. \tag{2.1}$$

Then, the value of the portfolio at time 1 is equal to the cash flow at time 1:

$$\begin{bmatrix} c^u \\ c^d \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{1.07} \\ 1 & \frac{1}{1.05} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}. \tag{2.2}$$

Then, solve this equation to obtain:

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{1.07} \\ 1 & \frac{1}{1.05} \end{bmatrix}^{-1} \begin{bmatrix} c^u \\ c^d \end{bmatrix} = \begin{bmatrix} 53.5c^u - 52.5c^d \\ -56.175c^u + 56.175c^d \end{bmatrix}.$$

Substituting into Eq. (2.1) and price of cash flow $[c^u, c^d]^T$ at time 1 is equal to

$$\frac{1}{1.06}(0.5c^u + 0.5c^d) = 0.4717c^u + 0.4717c^d,$$

and this means that the model is complete. Assuming that there is additional catastrophic risks with the condition that occurrence of catastrophic event is independent of the financial variables. Therefore, the cash flows at time 1 is:

$$\begin{bmatrix} c^{u,+} \\ c^{u,-} \\ c^{d,+} \\ c^{d,-} \end{bmatrix}.$$

Comparing with Eq. (2.2), there are cash flows at time 1 which cannot be obtained by the portfolio, which means that the model is incomplete. According to Cox and Pedersen (2000), in order to obtain such a bond with a unique price, we need to give the probability distribution of the catastrophic risks and assume that the prices are discounted expected values. However, because of the catastrophic risks, CAT bonds cannot be perfectly hedged in an incomplete market and the high yields received may not be sufficient to balance investor risk. In the next section, a formal classical assumption is given in order to model and obtain explicit prices for catastrophe risk bonds.

2.2 Classical Probabilisitic Structure and Valuation Theory

We price CAT bonds under the following assumptions: (i) an arbitrage-free investment market exists with equivalent martingale measure, (ii) the financial market behaves independently of the occurrence of catastrophes, and (iii) the interest rate changes can be replicated using existing financial instruments.

In this section, the probabilistic structure and valuation theory for the classical model is given. We will use this structure in Chapters 4 and 5 and extend to multi-dimension in Chapter 3. Let $0 < T < \infty$ be the maturity date of the continuous time trading interval $[0, T]$. The market uncertainty is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where $\mathcal{F}_t$ is an increasing family of σ-algebras, which is given by $\mathcal{F}_t = \mathcal{F}_t^{(1)} \times \mathcal{F}_t^{(2)} \subset \mathcal{F}$, for $t \in [0, T]$, where $\mathcal{F}_t^{(1)}$ represents the investment information (e.g. past security prices and interest rates) available to the market at time t and $\mathcal{F}_t^{(2)}$ represents the catastrophic risk information (e.g. insured property losses). The financial risk variables and the catastrophic risk variables can be modelled on $(\Omega^{(1)}, \mathcal{F}^{(1)}, (\mathcal{F}_t^{(1)})_{t\in[0,T]}, \mathbb{P}^{(1)})$ and $(\Omega^{(2)}, \mathcal{F}^{(2)}, (\mathcal{F}_t^{(2)})_{t\in[0,T]}, \mathbb{P}^{(2)})$, respectively. Moreover, define two filtrations $\mathcal{A}^{(1)}$ ($\mathcal{A}_t^{(1)} = \mathcal{F}_t^{(1)} \times \{\emptyset, \Omega_2\}$ for $t \in [0, T]$) and $\mathcal{A}^{(2)}$ ($\mathcal{A}_t^{(2)} = \{\emptyset, \Omega_1\} \times \mathcal{F}_t^{(2)}$ for $t \in [0, T]$). It is proved by Lemma 5.1 of Cox and Pedersen (2000) that σ-algebras $\mathcal{A}_t^{(1)}$ and $\mathcal{A}_t^{(2)}$ are independent under the probability measure $\mathbb{P}$. Thus, an $\mathcal{A}_T^{(\kappa)}$ measurable random variable X on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ (or an $\mathcal{A}^{(\kappa)}$ adapted stochastic process Y) is said to depend only on the financial risk variables ($\kappa = 1$) or catastrophic risk variables ($\kappa = 2$).

The presence of catastrophic risks that are uncorrelated with the underlying financial risks leads us to consider an incomplete market, and there is no universal theory addresses all aspects of pricing. The benchmark to price uncertain cash flow under an incomplete framework is the representative agent. For valuation purposes, similar to Merton (1976), we assume that under the risk-neutral pricing measure $\mathbb{Q}$, the overall economy depends only on financial risk variables. This is a fairly natural approximation because the global economic circumstances in terms of exchange and production are only marginally influenced by localized catastrophes. For more information, see

Merton (1976); Doherty (1997); Cox and Pedersen (2000); Lee and Yu (2002); Ma and Ma (2013). According to Lemma 5.2 in Cox and Pedersen (2000), under an assumption that the aggregate consumption is $\mathcal{A}^{(1)}$ adapted (assumption (ii)), for any random variable X that is $\mathcal{A}_T^{(2)}$ measurable, that

$$E^{\mathbb{Q}}[X] = E^{\mathbb{P}}[X]. \tag{2.3}$$

Thus, a $\mathcal{A}^{(2)}$ adapted aggregate loss process $\{L(t) : t \in [0, T]\}$ retains its original distributional characteristics after changing from the historical estimated actual probability measure $\mathbb{P}$ to the risk-neutral probability measure $\mathbb{Q}$. And the σ-algebras $\mathcal{A}_T^{(1)}$ and $\mathcal{A}_T^{(2)}$ are independent under the risk-neutral probability measure $\mathbb{Q}$. In an arbitrage-free market (assumption (i)) at any time t, the price of an attainable contingent claim with payoff $\{P(T) : T > t\}$ can be expressed by the fundamental theorem of asset pricing in the following form,

$$V(t) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r(s)ds} P(T)|\mathcal{F}_t), \tag{2.4}$$

see Delbaen and Schachermayer (1994).

2.3 Interest Rate Process

There are different types of interest rates, such as government and interbank rates. Zero-coupon rates can be either from government rates which are usually deduced by bonds issued by governments or from interbank rates which are exchanged deposits between banks. The most important interbank rate usually considered as a reference for contracts is the LIBOR (London InterBank Offered Rate) rate, fixed daily in London. For the purpose of bond prices, all kinds of rates are available. The first stochastic interest rate model was proposed by Merton (1973), followed by the pioneering approach of Vasicek (1977) and some other classical models, such as Dothan (1978); Cox et al. (1985); Ho and Lee (1986); Hull and White (1990); Black et al. (1990). In this section, we provide analysis for two interest rate models[11] (ARIMA and CIR), which will be used in this thesis.

[11]Nowak and Romaniuk (2013) compared the CAT bond prices under the assumption of the spot interest rate described by the Vasicek, Hull-White, and CIR models. However, we are not interested

2.3.1 ARIMA

The Auto-Regressive Integrated Moving Average (ARIMA) is a typical form to analyse time series data in statistics and econometrics and can be denoted as ARIMA(p, d, q), where p the number of lags of the stationarized series in the prediction equation, or formally called 'autoregressive terms'; d is the number of nonseasonal differences a time series needs for stationarity, called 'integrated'; and q is the number of lagged forecast errors in the equation, called 'moving average terms'.

Special cases of ARIMA models are as follows:

- random-walk (ARIMA$(0, 1, 0)$ without constant),

$$X(t) = X(t-1) + \epsilon(t);$$

- exponential smoothing models (ARIMA$(0, 1, 1)$ without constant),

$$X(t) = X(t-1) - (1-\alpha)\epsilon(t-1);$$

- first-order autoregressive models (ARIMA$(1, 0, 0)$),

$$X(t) = C + \theta X(t-1) + \epsilon(t);$$

- first-order moving average models (ARIMA$(0, 0, 1)$),

$$X(t) = C + \epsilon(t) + \alpha\epsilon(t-1);$$

- damped-trend linear exponential smoothing (ARIMA$(1, 1, 1)$),

$$X(t) - X(t-1) = C + \theta(X(t-1) - X(t-2)) + \epsilon(t) + \alpha\epsilon(t-1);$$

where C is a constant, θ, α are parameters and $\epsilon(t)$ is a white noise process. In particular, if slope coefficient θ is close to 0, then the process looks like white noise; as θ approaches 1, the model describes mean-reverting behaviour.

For the purposes of estimating the parameters and predicting by the ARIMA model, we use **arima** and **predict** functions in R.

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in the pricing process which is affected by the interest rate dynamics. Readers can refer to Brigo and Mercurio (2007) for more information on interest rate dynamics.

2.3.2 CIR

A typical instantaneous interest rate dynamics proposed by Cox, Ingersol, and Ross (CIR model, Cox et al. (1985)) assumed a 'square-root' term in the diffusion coefficient. This model is a benchmark because it provides analytical bonds and bond options pricing. The short-rate dynamics $\{r(t) : t \in [0, T]\}$ under the risk-neutral measure $\mathbb{Q}$ can be expressed as follows,

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), \tag{2.5}$$

with the condition

$$2k\theta > \sigma^2, \tag{2.6}$$

where $r(0), k, \theta$, and σ are positive constants. The condition Eq. (2.6) guarantees that the process $r(t)$ remains in the positive domain and the origin is inaccessible. Assuming the spot interest rate under the real world measure $\mathbb{P}$ with the form:

$$dr(t) = [k\theta - (k + \lambda_r)r(t)]dt + \sigma\sqrt{r(t)}dW^*(t), \tag{2.7}$$

where $W^*(t) = W(t) + \int_0^t \frac{\lambda_r\sqrt{r(s)}}{\sigma}ds$ is a Brownian motion under the risk measure $\mathbb{P}$ and λ_r is a positive constant[12] contributing to the market price of risk. Assuming $\mathbb{Q}$ and $\mathbb{P}$ are equivalent measures, then compare Eq. (2.5) and Eq. (2.7) and we obtain Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\int_0^t \frac{\lambda_r^2 r(s)}{\sigma^2}ds + \int_0^t \frac{\lambda_r\sqrt{r(s)}}{\sigma}dW^*(s)\right).$$

The market price of risk process $\lambda_r^*(t)$ is a stochastic process with the functional form

$$\lambda_r^*(t) = \frac{\lambda_r}{\sigma}\sqrt{r(t)},\ t \in [0, T].$$

For detailed information about this transformation, please refer to Ma and Ma (2013); Nowak and Romaniuk (2013); Shirakawa (2002); Lee and Yu (2002); Remillard (2013).

According to Brigo and Mercurio (2007), we can price a pure-discount T-bond at time t by the following equalities:

$$B_{CIR}(t, T) = A(t, T)e^{-B(t,T)r(t)}, \tag{2.8}$$

[12]For the case $\lambda_r = 0$, dynamics Eq. (2.5) and Eq. (2.7) coincide, where risk neutral world and objective world are identical.

where

$$A(t,T) = \left[\frac{2he^{(k+\lambda_r+h)(T-t)/2}}{2h+(k+\lambda_r+h)(e^{(T-t)h}-1)}\right]^{\frac{2k\theta}{\sigma^2}}, \tag{2.9}$$

$$B(t,T) = \left[\frac{2(e^{(T-t)h}-1)}{2h+(k+\lambda_r+h)(e^{(T-t)h}-1)}\right], \tag{2.10}$$

$$h = \sqrt{(k+\lambda_r)^2+2\sigma^2}. \tag{2.11}$$

We complete this subsection by giving maximum likelihood estimation of the CIR model. Glasserman (2003) stated that in the CIR model, the increments of the short-rate follows a non-central chi-square distribution and the transition density of Eq. (2.5) can be written as:

$$r(t) = \frac{\sigma^2(1-e^{-k(t-u)})}{4k}\chi^2_d\left(\frac{4ke^{-k(t-u)}}{\sigma^2(1-e^{-k(t-u)})}r(u)\right),\ t>u,$$

where,

$$d = \frac{4\theta k}{\sigma^2} \text{ and } \lambda = \frac{4ke^{-k(t-u)}}{\sigma^2(1-e^{-k(t-u)})}r(u).$$

The cumulative distribution function is

$$\mathbb{P}(r(t)\leq y|r(u)) = F_{\chi^2_{(d,\lambda)}}\left(\frac{4ky}{\sigma^2(1-e^{-k(t-u)})}\right),$$

and the probability density function is given as

$$\mathbb{P}_{r(t)}(y|r(u)) = cp_{\chi^2_{(d,\lambda)}}(cy),$$

where $p_{\chi^2_{(d,\lambda)}}(\cdot)$ is the density of the non-central χ^2 distribution, where

$$c = \frac{4k}{\sigma^2(1-e^{-k(t-u)})}.$$

Finally, one can have the log-likelihood function:

$$l(\theta,k,\sigma;y) = \sum_{i=2}^{n} log(c) + \sum_{i=2}^{n} log(p_{\chi^2_{(d,\lambda)}}(cy_i|y_{i-1})),$$

where $y = r_1,\ldots,r_n$ is given according to the data. We use numerical optimization to find the maximum likelihood estimation of the parameters and the R-function of the model is given in Appendix C.2. Alternatively, one can use the R Package **SMFI5** with **LogLikCIR** function.

2.4 Extreme Value Theory

Extreme value theory deals with the stochastic of the minimum or the maximum of a very large collection of random observations from the same arbitrary distribution. The first statement of extremal limit theorem was by Fisher and Tippett (1928), and they suggested that the behaviour of the maxima can be described by only a few forms. Thereafter, Gnedenko (1943) gave convergence to a unified version type theorem – the Generalized Extreme Value distribution (GEV). Gumbel (1958) showed statistical application of theory to estimate extremes.

Suppose $X_1, X_2, \ldots$ are independent and identically distributed random variables with common cumulative distribution function F. Let $M_\delta = \max\{X_1,\ X_2,\ \ldots,\ X_\delta\}$ denote the maximum of the first δ random variables. In theory, the exact distribution of M_δ can be derived by

$$\begin{aligned}\mathbb{P}(M_\delta \leq z) &= \mathbb{P}(X_1 \leq z, \ldots, X_\delta \leq z)\\ &= \mathbb{P}(X_1 \leq z) \cdots \mathbb{P}(X_\delta \leq z) = (F(z))^\delta.\end{aligned}$$

However, this is not immediately helpful in practice, since the distribution function F is not always available. There are two possible methods to solve this problem, first is the Central Limit Theorem (CLT) and second is Fisher-Tippett-Gnedenko theorem which is discussed in this section, see Fisher and Tippett (1928); Embrechts et al. (1997); Coles et al. (2001).

Theorem 2.4.1. *(Fisher–Tippett–Gnedenko)*
If there exist sequences of constants $\{\sigma_\delta : \sigma_\delta > 0, \forall\delta \in \mathbb{N}\}$ *and* $\{\beta_\delta : \delta \in \mathbb{N}\}$ *such that*

$$\mathbb{P}\left\{\frac{M_\delta - \beta_\delta}{\sigma_\delta} \leq z\right\} \to G(z) \quad \textit{as } \delta \to \infty, z \in \mathbb{R},$$

then

$$G(z) \propto \exp\left\{-(1 + \alpha z)^{-1/\alpha}\right\},$$

where α depends on the tail shape of the distribution. When normalized, G is a non-degenerate distribution function and belongs to one of the following forms ($\gamma > 0$):

I. (Gumbel) $G(z) = \exp\left\{-\exp\left(-\left(\frac{z-\beta}{\sigma}\right)\right)\right\}$ *when the distribution of M_δ has an exponential tail.*

II. (Fréchet) $G(z) = \begin{cases} 0 & z \leq \beta \\ \exp\left\{-\left(\frac{z-\beta}{\sigma}\right)^{-\gamma}\right\} & z > \beta. \end{cases}$ *when the distribution of* M_δ *has a heavy tail (including polynomial decay).*

III. (Weibull) $G(z) = \begin{cases} \exp\left\{-\left(-\left(\frac{z-\beta}{\sigma}\right)\right)^{\gamma}\right\} & z < \beta \\ 1 & z \geq \beta \end{cases}$ *when the distribution of* M_δ *has a light tail with finite upper bound.*

These can be grouped into the the single distribution called ***Generalized Extreme Value (GEV)*** *distribution, with c.d.f.*

$$G(z) = \exp\left\{-\left[1 + \alpha\left(\frac{z-\beta}{\sigma}\right)\right]^{-1/\alpha}\right\}, \tag{2.12}$$

defined on $\{z : 1 + \alpha(z-\beta)/\sigma > 0\}$*, where* $\beta \in \mathbb{R}$*,* $\sigma > 0$ *and* $\alpha \in \mathbb{R}$*.*

The model has three parameters: location parameter β*, scale parameter* σ*, and shape parameter* α*. The case* $\alpha = 0$ *is interpreted as the limit* $\alpha \to 0$ *and Eq.* (2.12) *corresponds to the Gumbel family. For the cases* $\alpha > 0$ *(*$\alpha = \frac{1}{\gamma}$*) and* $\alpha < 0$ *(*$\alpha = -\frac{1}{\gamma}$*), Eq.* (2.12) *leads to Frechét and Weibull family distributions, respectively.*

We complete this section by giving maximum likelihood estimation for GEV distribution parameters (α, σ, β). Assuming $M_1, \ \ldots, \ M_\delta$ are independent variables with GEV distribution, then the log-likelihood for parameters (α, σ, β) $(\alpha \neq 0)$ is given by

$$l(\alpha, \sigma, \beta) = n \log \sigma - \left(1 + \frac{1}{\alpha}\right) \sum_{i=1}^{\delta} \log\left[1 + \alpha\left(\frac{M_i - \beta}{\sigma}\right)\right] - \sum_{i=1}^{\delta}\left[1 + \alpha\left(\frac{M_i - \beta}{\sigma}\right)\right]^{\frac{-1}{\alpha}}, \tag{2.13}$$

provided that

$$1 + \alpha\left(\frac{M_i - \beta}{\sigma}\right) > 0, \ \text{for}\ i = 1, \ 2, \ \ldots, \delta.$$

There is no analytic solution for maximize Eq. (2.13), but for any given dataset the maximization is obtained straightforwardly by using standard numerical algorithms, Coles et al. (2001). In the following chapters, we use R Package **fExtremes** with **gevFit** function (or R library **ismev** with **gev.fit** function) to estimate GEV parameters.

Chapter 3

Multi Variables CAT Bond Model

In this chapter, a model with m catastrophe risks and n financial risks in a discrete-time period is developed as an extension of the approach of Cox and Pedersen (2000). Theoretical results are applied to construct a multi-variable CAT bond, and then use California earthquakes data to derive the price density function for a 5-year structured parametric earthquake CAT bond. This chapter works under an incomplete and no-arbitrage framework, assuming that all risks (both financial and catastrophic risks) are pairwise independent.

The reminder of this chapter is organized as follows. Section 3.1 describes the probability structure for the model and a valuation framework of CAT bonds. The fact that catastrophic risks are uncorrelated with financial risk movements makes the problem much simpler. In Section 3.2, one-period and multi-period payoff formulas for earthquake CAT bonds are specified, and the term structures or the distributions of the risk variables relative to the bond are analysed. The distribution of the annual maximum earthquake magnitude in California is estimated using extreme value theory. It is assumed that the dynamics of the LIBOR rate is a CIR model and that the interest and inflation rates follow ARIMA processes. Section 3.3 presents numerical examples for 1-year and 5- year CAT bonds. The density plot for the price is derived to illustrate the applicability of our results. Finally, Section 3.4 discusses the results.

3.1 Modeling CAT bonds

3.1.1 Model Description and Preliminaries

In this subsection, a preliminary presentation for the CAT bond structure is given. Generalizing the ideas of Cox and Pedersen (2000), a CAT bond that combines n financial market variables and m catastrophic risk variables is designed. The model set-up requires a probabilistic structure which is given as follows.

Assume that issuers are trading CAT bonds in an investment market that is arbitrage-free. The time of the catastrophe(s) is independent of the term structure(s) under the relevant probability measure. We assume that there are n financial risk variables, each modelled on a filtered probability space $(\Omega_{1,i}, \mathcal{F}^{(1,i)}, (\mathcal{F}_t^{(1,i)})_{t=0,\ 1,\dots,\ T}, \mathbb{P}_{1,i})$ for $i = 1,\ 2,\dots,\ n$. Let $T < \infty$ be the maturity time of the trading interval. Let $\mathcal{F}_t^{(1,i)}$ be the σ-algebras of $\Omega_{1,i}$ representing the investment information available to the market at time t ($t = 0,\ 1,\dots,\ T$), where $\mathcal{F}^{(1,i)}$ ($i = 1,\ 2,\dots,\ n$) are corresponding filtrations. Thus, each probability measure $\mathbb{P}_{1,i}$ is defined for all events belonging to the $\mathcal{F}_t^{(1,i)}$ σ-algebra, $t \leq T$. Note that the measures $\mathbb{P}_{1,i}$ do not necessarily have the same distributions.

Then consider m catastrophic risk variables, which are modelled on a filtered probability space $(\Omega_{2,j}, \mathcal{F}^{(2,j)}, (\mathcal{F}_t^{(2,i)})_{t=0,\ 1,\dots,\ T}, \mathbb{P}_{2,j})$, where $\mathcal{F}_t^{(2,j)}$ are the σ-algebras of $\Omega_{2,j}$ representing the catastrophic risk information available at time t ($t = 0,\ 1,\dots,\ T$) and $\mathbb{P}_{2,j}$ ($j = 1,\ 2,\dots,\ m$) are the probability measures governing the catastrophe structure (not necessarily with the same distribution). The filtrations $\mathcal{F}^{(2,j)}$ are indexed by the same times $t = 0,\ 1,\dots,\ T$ as previously. The sample space of the full model can be constructed, such that

$$\Omega = \Big(\Omega_{1,1} \times \Omega_{1,2} \times \cdots \times \Omega_{1,n}\Big) \times \Big(\Omega_{2,1} \times \Omega_{2,2} \times \cdots \times \Omega_{2,m}\Big).$$

A typical event of the full model sample space is of the form $\omega = (\widetilde{\omega}_{1,n},\ \widetilde{\omega}_{2,m})$, where $\widetilde{\omega}_{\kappa,\ell} = (w_{\kappa,1},\ w_{\kappa,2},\dots,\ w_{\kappa,\ell})$, $\kappa = 1,\ 2$, $\ell = n,\ m$, such that $w_{1,i} \in \Omega_{1,i}$ ($i = 1,\ 2,\dots,\ n$) and $w_{2,j} \in \Omega_{2,j}$ ($j = 1,\ 2,\dots,\ m$).

Assuming that the events $w_{\kappa,1},\ w_{\kappa,2},\dots,\ w_{\kappa,\ell}$ ($\kappa = 1,\ 2$, $\ell = n,\ m$) are pairwise independent, then the probability measure on the sample space Ω is given by

$$\mathbb{P}(\omega) = \prod_{i=1}^{n} \mathbb{P}_{1,i}(\omega_{1,i}) \cdot \prod_{j=1}^{m} \mathbb{P}_{2,j}(\omega_{2,j}), \quad i = 1,\ 2, \dots,\ n,\ j = 1,\ 2, \dots,\ m.$$

In addition, the natural filtration produced by the σ-algebras of Ω is denoted by $\mathcal{F}$ and given by

$$\mathcal{F}_t = \Big(\mathcal{F}_t^{(1,1)} \times \mathcal{F}_t^{(1,2)} \times \cdots \times \mathcal{F}_t^{(1,n)}\Big) \times \Big(\mathcal{F}_t^{(2,1)} \times \mathcal{F}_t^{(2,2)} \times \cdots \times \mathcal{F}_t^{(2,m)}\Big),$$

for $t = 0,\ 1, \dots,\ T$. Thus, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ 1,\dots,\ T}, \mathbb{P})$ constitutes a probability space for the full model with all the elements defined as above. In order to define random variables in the full model that depends only on either financial variables or catastrophic variables, let us introduce the increasing filtrations $\mathcal{A}_t^{(1)} \subset \mathcal{A}^{(1)}$ and $\mathcal{A}_t^{(1,i)} \subset \mathcal{A}^{(1,i)}$ ($i = 1, \dots,\ n$), and similarly $\mathcal{A}_t^{(2)} \subset \mathcal{A}^{(2)}$ and $\mathcal{A}_t^{(2,j)} \subset \mathcal{A}^{(2,j)}$ ($j = 1, \dots,\ m$) generated from the following σ-algebras:

$$\begin{aligned}
\mathcal{A}_t^{(1)} &= \mathcal{F}_t^{(1,1)} \times \cdots \times \mathcal{F}_t^{(1,n)} \times \{\emptyset, \Omega_{2,1}, \dots, \Omega_{2,m}\}, \\
\mathcal{A}_t^{(1,i)} &= \mathcal{F}_t^{(1,i)} \times \{\emptyset, \Omega_{2,1}, \dots, \Omega_{2,m}\}, \quad i = 1, \dots, n, \\
\mathcal{A}_t^{(2)} &= \{\emptyset, \Omega_{1,1}, \dots, \Omega_{1,n}\} \times \mathcal{F}_t^{(2,1)} \times \cdots \times \mathcal{F}_t^{(2,m)}, \\
\mathcal{A}_t^{(2,j)} &= \{\emptyset, \Omega_{1,1}, \dots, \Omega_{1,n}\} \times \mathcal{F}_t^{(2,j)}, \quad j = 1, \dots, m,
\end{aligned}$$

for $t = 1, \dots,\ T$. An $\mathcal{A}_T^{(\kappa)}$ measurable random variable X on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ 1,\dots,\ T}, \mathbb{P})$ (or an $\mathcal{A}^{(\kappa)}$ adapted stochastic process Y) depends on financial risk variables ($\kappa = 1$) or catastrophic risk variables ($\kappa = 2$). Let financial events be $\alpha_{1,i} \in \mathcal{A}_T^{(1,i)}$ and catastrophic events be $\alpha_{2,j} \in \mathcal{A}_T^{(2,j)}$. We need the independent notion of $\mathcal{A}_T^{(\kappa,\ell)}$ because we cannot refer to $\mathcal{F}_T^{(\kappa,\ell)}$ as being independent under $\mathbb{P}$, since each of $\mathcal{F}_T^{(\kappa,\ell)}$ does not contain events defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ 1,\dots,\ T}, \mathbb{P})$.

Lemma 3.1.1. *For $i = 1, \dots, n$ and $j = 1, \dots, m$, the σ-algebras $\mathcal{A}_T^{(1,i)}$ and $\mathcal{A}_T^{(2,j)}$ are independent under the probability measure $\mathbb{P}$.*

Proof. For $i = 1, \dots, n$ and $j = 1, \dots, m$, we have $\alpha_{1,i} \in \mathcal{A}_T^{(1,i)}$ and $\alpha_{2,j} \in \mathcal{A}_T^{(2,j)}$. Therefore, $\alpha_{1,i} = A_{1,i} \times \Omega_{2,1} \times \cdots \times \Omega_{2,m}$ for some $A_{1,i} \in \mathcal{F}_t^{(1,i)}$, and $\alpha_{2,j} = \Omega_{1,1} \times \cdots \times \Omega_{1,n} \times A_{2,j}$ for some $A_{2,j} \in \mathcal{F}_t^{(2,j)}$, and we have that

$$\mathbb{P}\left[\Big(\bigcap_{i=1}^{n} \alpha_{1,i}\Big) \bigcap \Big(\bigcap_{j=1}^{m} \alpha_{2,j}\Big) \right] = \mathbb{P}\Big(A_{1,1} \times \cdots \times A_{1,n} \times A_{2,1} \times \cdots \times A_{2,m}\Big)$$

$$
\begin{aligned}
&= \prod_{i=1}^{n} \mathbb{P}_{1,i}(A_{1,i}) \cdot \prod_{j=1}^{m} \mathbb{P}_{2,j}(A_{2,j}) \\
&= \prod_{i=1}^{n} \mathbb{P}_{1,i}(A_{1,i}) \prod_{j=1}^{m} \mathbb{P}_{2,j}(\Omega_{2,j}) \prod_{i=1}^{n} \mathbb{P}_{1,i}(\Omega_{1,i}) \prod_{j=1}^{m} \mathbb{P}_{2,j}(A_{2,j}) \\
&= \prod_{i=1}^{n} \mathbb{P}(\alpha_{1,i}) \cdot \prod_{j=1}^{m} \mathbb{P}(\alpha_{2,j}).
\end{aligned}
$$

And the result follows. □

3.1.2 The Valuation Framework

In this subsection, we show how to implement valuation under the full model by choosing the equivalent measure. Similar to Cox and Pedersen (2000) and Magill and Quinzii (2002), the setting of a representative agent is adopted to price uncertain cash flow streams, as which is the benchmark financial economics technique. By this technique, we need to assume a representative utility function and an aggregate consumption process.

Assume a T-period economy, in which agents can make choices and consume during each period. An agent makes choices about his future consumption, represented by the stochastic process $\{c(t); t = 0,\ 1, \ldots,\ T\}$. The aggregate consumption stochastic process is denoted by $\{C^*(t); k = 0,\ 1, \ldots,\ T\}$. Both these processes are adapted to filtration of the full model. Only the first choice is known with certainty at time $t = 0$. For $i = 1, 2, \ldots, n$, let $\{r_i(t); t = 0,\ 1, \ldots,\ T-1\}$ be the one-period financial market rates. Then these one-period financial market rates can be defined through the conditional expectation

$$
\prod_{i=1}^{n} \frac{1}{1 + r_i(t)} = \frac{1}{u'_k(C^*(t))} \mathbb{E}^{\mathbb{P}}\left[u'_{t+1}(C^*(t+1)) | \mathcal{F}_t\right], \quad t = 0,\ 1, \ldots,\ T-1, \tag{3.1}
$$

where $u_0,\ u_1, \ldots,\ u_T$ represent the utility functions, and also assume representative agent's utility is additively separable and differentiable. The Randon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined in the same vein as Cox and Pedersen (2000)

$$
\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{i=1}^{n} \prod_{s=0}^{T-1} [1 + r_i(s)] \left[\frac{u'_T(C^*(T))}{u'_0(C^*(0))}\right]. \tag{3.2}
$$

Note that this new random variable is measurable with respect to $\mathcal{F}_T$. In addition, we clearly need to ensure that $\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$ (Lemma 3.1.2). First, for notation simplicity, denote the one-period financial market discount rates by

$$B(k) = \begin{cases} \prod_{i=1}^{n}\prod_{s=0}^{t-1}[1 + r_i(s)], & \text{for } t = 1,\ 2, \dots,\ T, \\ 1, & \text{for } t = 0. \end{cases}$$

Then, define the stochastic processes $\{\xi(t); t = 0,\ 1, \dots,\ T\}$ and $\{\zeta(t); t = 0,\ 1 \dots,\ T\}$ as

$$\xi(t) = \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right] = \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} \text{ and } \zeta(t) = B(t)\cdot\frac{u_t'(C^*(t))}{u_0'(C^*(0))},$$

with $t = 1, \dots, T$ and $B(0) = 1$, which leads to $\zeta(0) = 1$. By Eq. (3.2) it holds that $\zeta(T) = \frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{F}_T$. Similar to Lemma B.1 and Theorem B.1 of Cox and Pedersen (2000), we have the following lemma and theorem.

Lemma 3.1.2. *The process $\{\zeta(t); t = 0,\ 1, \dots,\ T\}$ is a $\mathbb{P}$-martingale on the filtration $\mathcal{F}$, and $\zeta(t) = \xi(t)$ for $t = 0,\ 1, \dots,\ T$.*

Proof. First, note that the process $\{\zeta(t); t = 0,\ 1, \dots,\ T\}$ is $\mathcal{F}$ adapted, since the processes $r_i(t)$ and $C^*(t)$ are $\mathcal{F}$ adapted processes, as well. Furthermore,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[\zeta(t+1)|\mathcal{F}_t] \\
=&\mathbb{E}^{\mathbb{P}}\left[B(t)\prod_{i=1}^{n}[1 + r_i(t)]\frac{u_k'(C^*(t+1))}{u_0'(C^*(0))}\middle|\mathcal{F}_t\right] \\
=&\mathbb{E}^{\mathbb{P}}\left[\zeta(t)\prod_{i=1}^{n}[1 + r_i(t)]\frac{u_{t+1}'(C^*(t+1))}{u_t'(C^*(t))}\middle|\mathcal{F}_t\right] \\
=&\zeta(t)\prod_{i=1}^{n}[1 + r_i(t)]\frac{1}{u_t'(C^*(t))}\mathbb{E}^{\mathbb{P}}\left[u_{t+1}'(C^*(t+1))\middle|\mathcal{F}_t\right] = \zeta(t),
\end{aligned}$$

where the last equality is obtained by using Eq. (3.1). Finally, by using the fact that the process $\{\zeta(t); k = 0,\ 1, \dots,\ T\}$ forms a martingale, we conclude that

$$\zeta(t) = \mathbb{E}^{\mathbb{P}}\left[\zeta(T)|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right] = \xi(t).$$

$\square$

Remark 3.1.1. *An immediate consequence of Lemma 3.1.2 is that*

$$1 = \mathbb{E}^{\mathbb{P}}\Big[\zeta(0)\Big] = \mathbb{E}^{\mathbb{P}}\Big[\zeta(T)\Big] = \mathbb{E}^{\mathbb{P}}\Big[\xi(T)\Big] = \mathbb{E}^{\mathbb{P}}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big],$$

which ensures that the Radon-Nikodym derivative in Eq. (3.2) *indeed defines a new probability measure.*

Intuitively, the probability measure $\mathbb{Q}(\cdot)$ is equivalent to knowledge of the representative investor's utility function and the aggregate consumption process.

Theorem 3.1.1. *Under the assumptions of the representative agent pricing model, the value of a generic future cash flow process* $\{P_{CAT}(t); t = 1,\ 2, \ldots,\ T\}$ *at time* 0 *is given by*

$$V(P_{CAT}) = \mathbb{E}^{\mathbb{Q}}\Big[\sum_{t=1}^{T} \frac{1}{\prod_{i=1}^{n}\prod_{s=0}^{t-1}[1 + r_i(s)]} P_{CAT}(t)\Big] = \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \frac{1}{B(t)} P_{CAT}(t)\right]. \tag{3.3}$$

Remark 3.1.2. *When in incomplete markets, there is no unique interpretation for the prices that we assign to CAT bonds unless we introduce the probability distribution of the catastrophe risk, see Section 2.1.*

Using similar arguments to those in Theorem B.2 of Cox and Pedersen (2000), the general intertemporal valuation of a future cash flow can be expressed in terms of the equivalent measure $\mathbb{Q}(\cdot)$.

Theorem 3.1.2. *Under the assumptions of the representative agent pricing model, the value of a generic future cash flow process* $\{d(t); t = k+1,\ p+2, \ldots,\ T\}$ *at time* k *is given by*

$$\mathbb{E}^{\mathbb{P}}\Big[\sum_{t=k+1}^{T} \frac{u_t'(C^*(t))}{u_k'(C^*(k))} P_{CAT}(t)\Big|\mathcal{F}_k\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\sum_{t=k+1}^{T} \frac{B(k)}{B(t)} P_{CAT}(t)\Big|\mathcal{F}_k\Big],$$

where $k = 0,\ 1, \ldots, T$, *with the convection* $\sum_{b}^{a} = 0$ *for* $a < b, a, b \in \mathbb{N}$.

For analysis of CAT bonds, hereafter we assume that the aggregate consumption depends only on financial risks, given as $C^*(\tilde{\omega}_{1,n}, \tilde{\omega}_{2,m};\ t) = C^*(\tilde{\omega}_{1,n}; t)$ for $\omega \equiv (\tilde{\omega}_{1,n}, \tilde{\omega}_{2,m}) \in \Omega$. Then C^* is $\mathcal{A}^{(1)}$ adapted. This is quite a natural approximation since global economic conditions in terms of exchange and production are not strongly related to localized catastrophes, see Cox and Pedersen (2000). Assuming that the aggregate consumption process depends only on financial risk information available at time t, and that the structure at time 0 is known.

Lemma 3.1.3. *Under the assumption that C^* is $\mathcal{A}^{(1)}$ adapted, for any random variable X that is $\mathcal{A}_T^{(2)}$ measurable we have*

$$E^{\mathbb{Q}}[X] = E^{\mathbb{P}}[X].$$

In particular, for any catastrophic events $\alpha_{2,j}$ ($j = 1,\ 2, \ldots,\ m$) that are $\mathcal{A}_T^{(2,j)}$ measurable, it holds that

$$\mathbb{Q}(\bigcap_{j=1}^{m}(\alpha_{2,j})) = \mathbb{P}(\bigcap_{j=1}^{m}(\alpha_{2,j})) = \prod_{j=1}^{m} \mathbb{P}_{2,j}(A_{2,j}), \tag{3.4}$$

where $A_{2,j} \in \mathcal{F}_T^{(2,j)}$.

Proof. Note that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in Eq. (3.2) is $\mathcal{A}_T^{(1)}$ measurable, because of the fact that C^* and $B(T)$ are $\mathcal{A}^{(1)}$ adapted. Therefore, for any random variable X that is $\mathcal{A}_T^{(2)}$ measurable we have that X and $\frac{d\mathbb{Q}}{d\mathbb{P}}$ are independent under $\mathbb{P}$. Together with Lemma 3.2.5 of Shreve (2004), one can prove that

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}\left[X\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathbb{E}^{\mathbb{P}}[X]\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathbb{E}^{\mathbb{P}}[X] \cdot 1 = \mathbb{E}^{\mathbb{P}}[X].$$

Moreover, define

$$X = \prod_{j=1}^{m} \mathbb{1}_{\alpha_{2,j}} = \mathbb{1}_{\bigcap_{j=1}^{m} \alpha_{2,j}},$$

where $\alpha_{2,j} \in \mathcal{A}_T^{(2,j)}$, $j = 1, 2, \ldots, m$. Substituting into Eq. (3.4), and obtain

$$\begin{aligned}
\mathbb{Q}(\bigcap_{j=1}^{m}(\alpha_{2,j})) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\bigcap_{j=1}^{m}\alpha_{2,j}}\right] = \mathbb{E}^{\mathbb{Q}}[X] = E^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\bigcap_{j=1}^{m}\alpha_{2,j}}\right] \\
&= \mathbb{P}(\bigcap_{j=1}^{m}(\alpha_{2,j})) = \mathbb{P}\left[\bigcap_{j=1}^{m}\{\Omega_{1,1} \times \cdots \times \Omega_{1,m} \times A_{2,j}\}\right] \\
&= \prod_{j=1}^{m}\left[(\prod_{i=1}^{n}\mathbb{P}(\Omega_{1,j}))\mathbb{P}(A_{2,j})\right] = \prod_{j=1}^{m}\mathbb{P}_{2,j}(A_{2,j}).
\end{aligned}$$

□

Remark 3.1.3. *Under the measure $\mathbb{P}(\cdot)$ and the assumption that C^* depends only on financial risk variables, we can conclude that the catastrophic events $\alpha_{2,j}$ that depend on the jth catastrophic risk ($j = 1, \ldots, m$) are independent.*

To implement Theorems 3.1.1 and 3.1.2, it is crucial to assume that the events are mutually independent, that is, they depend only on financial risks and only on catastrophic risks, under the measure $\mathbb{Q}$.

Lemma 3.1.4. *Under the assumption that C^* is $\mathcal{A}^{(1)}$ adapted, the σ-algebras $\mathcal{A}_T^{(1)}$ and $\mathcal{A}_T^{(2)}$ are independent under $\mathbb{Q}$.*

Proof. Let $\alpha_{1,i} \in \mathcal{A}_T^{(1,i)}$ and $\alpha_{2,j} \in \mathcal{A}_T^{(2,j)}$. Applying Lemma 3.2.5 of Shreve (2004), then have

$$\begin{aligned}\mathbb{Q}\Big((\bigcap_{i=1}^{n} \alpha_{1,i}) \bigcap (\bigcap_{j=1}^{m} \alpha_{2,j}) \Big) &= \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\bigcap_{i=1}^{n} \alpha_{1,i}} \mathbb{1}_{\bigcap_{j=1}^{m} \alpha_{2,j}} \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[\mathbb{1}_{\bigcap_{i=1}^{n} \alpha_{1,i}} \mathbb{1}_{\bigcap_{j=1}^{m} \alpha_{2,j}} \frac{d\mathbb{Q}}{d\mathbb{P}} \right].\end{aligned}$$

Since $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in Eq. (3.2) is $\mathcal{A}_T^{(1)}$ measurable,

$$\mathbb{1}_{\bigcap_{i=1}^{m} \alpha_{1,i}} \frac{d\mathbb{Q}}{d\mathbb{P}} \quad \text{and} \quad \mathbb{1}_{\bigcap_{j=1}^{m} \alpha_{2,j}}$$

are independent under $\mathbb{P}$. Consequently,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}} \left[\mathbb{1}_{\bigcap_{i=1}^{n} \alpha_{1,i}} \mathbb{1}_{\bigcap_{j=1}^{m} \alpha_{2,j}} \frac{d\mathbb{Q}}{d\mathbb{P}} \right] &= \mathbb{E}^{\mathbb{P}} \left[\mathbb{1}_{\bigcap_{i=1}^{n} \alpha_{1,i}} \frac{d\mathbb{Q}}{d\mathbb{P}} \right] \mathbb{E}^{\mathbb{P}} \left[\mathbb{1}_{\bigcap_{j=1}^{m} \alpha_{2,j}} \right] \\ &= \mathbb{E}^{\mathbb{Q}} [\mathbb{1}_{\bigcap_{i=1}^{n} \alpha_{1,i}}] \mathbb{P}[\bigcap_{j=1}^{m} \alpha_{2,j}] \\ &= \mathbb{Q}[\bigcap_{i=1}^{n} \alpha_{1,i}] \prod_{j=1}^{m} \mathbb{P}_{2,j}[\alpha_{2,j}].\end{aligned}$$

Referring back to Lemma 3.1.3, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}} \left[\mathbb{1}_{\bigcap_{i=1}^{n} \alpha_{1,i}} \mathbb{1}_{\bigcap_{j=1}^{m} \alpha_{2,j}} \frac{d\mathbb{Q}}{d\mathbb{P}} \right] &= \mathbb{Q}[\bigcap_{i=1}^{n} \alpha_{1,i}] \prod_{j=1}^{m} \mathbb{P}_{2,j}[\alpha_{2,j}] \\ &= \mathbb{Q}[\bigcap_{i=1}^{n} \alpha_{1,i}] \mathbb{Q}[\bigcap_{j=1}^{m} \alpha_{2,j}].\end{aligned}$$

Therefore, we conclude that under $\mathbb{Q}$ the σ-algebras $\mathcal{A}_T^{(1)}$ and $\mathcal{A}_T^{(2)}$ are independent. $\square$

As a direct implication of Lemmas 3.1.3 and 3.1.4, the current value of cash flows X depending on catastrophic risks has the simple form as below. For notation simplicity, we denote the current value of non-defaultable zero-coupon bond maturing at time t with face amount 1 as $P(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(t)}\right]$.

Corollary 3.1.1. *The current value of an $\mathcal{A}_t^{(2)}$ measurable cash flow X paid at time t is given by*

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(t)}X\right] = P(t)\mathbb{E}^{\mathbb{P}}[X].$$

Under the discrete time framework, we can express the valuation measure as a product measure of the probability measures $\mathbb{Q}_1$ and $\mathbb{P}_{2,j}$,

$$\begin{aligned}
\mathbb{Q}(\omega) &= \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega)\mathbb{P}(\omega) \\
&= B(\omega; T)\frac{u'_T(C^*(\omega; T))}{u'_0(C^*(\omega; 0))}\mathbb{P}(\omega) \\
&= \prod_{s=0}^{T-1}\left[\prod_{i=1}^{n}[1 + r_i(\omega_{1,i}; s)]\right]\frac{u'_T(C^*(\tilde{\omega}_{1,n}; T))}{u'_0(C^*(\tilde{\omega}_{1,n}; 0))}\prod_{i=1}^{n}\mathbb{P}_{1,i}(\omega_{1,i})\prod_{j=1}^{m}\mathbb{P}_{2,j}(\omega_{2,j}) \\
&= \mathbb{Q}_1(\tilde{\omega}_{1,n})\prod_{j=1}^{m}\mathbb{P}_{2,j}(\omega_{2,j}),
\end{aligned} \tag{3.5}$$

where

$$\mathbb{Q}_1(\tilde{\omega}_{1,n}) = \prod_{s=0}^{T-1}\left[\prod_{i=1}^{n}[1 + r_i(\omega_{1,i}; s)]\right]\frac{u'_T(C^*(\tilde{\omega}_{1,n}; T))}{u'_0(C^*(\tilde{\omega}_{1,n}; 0))}\prod_{i=1}^{n}\mathbb{P}_{1,i}(\omega_{1,i}). \tag{3.6}$$

The probability measure in Eq. (3.6) is generated in terms of the term structure of financial risks, see Pedersen (1994). It is practical to have Eq. (3.5) since the empirical probabilities of catastrophic events can be used for the probability measures $\mathbb{P}_{2,j}$, where $j = 1, \ldots, m$.

3.1.3 Implication for Valuation

In this subsection, a concrete form for pricing certain CAT bonds is presented under the discrete time framework. The valuation structure of CAT bonds can be further simplified because the discount factors $B(t)$ are $\mathcal{A}_t^{(1)}$ measurable and depend only on financial risks. Consider a generic future cash flow process $P_{CAT}(\omega; t) =$

$P_{CAT}(\tilde{\omega}_{1,n}, \tilde{\omega}_{2,m}; t)$ depending on financial and catastrophic risks. In addition, define an associated process of future cash flow as

$$\overline{P_{CAT}}(t) = \mathbb{E}^{\mathbb{Q}}[P_{CAT}(t)|\mathcal{A}_t^{(1)}],$$

which is the conditional expectation over the loss distribution of catastrophic risks given fixed financial risk variables. The value of $\overline{P_{CAT}}$ reflects the financial events by filtration $\mathcal{A}^{(1)}$; thus, $\overline{P_{CAT}}(t)$ is $\mathcal{A}_t^{(1)}$ measurable. We now reformulate Eq. (3.3) using the process $\overline{P_{CAT}}$, with $B(t)$ and $\overline{P_{CAT}}(t)$ are both $\mathcal{A}_t^{(1)}$ measurable. We have

$$V(\overline{P_{CAT}}) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \frac{1}{B(t)} \overline{P_{CAT}}(t)\right] = \mathbb{E}^{\mathbb{Q}_1}\left[\sum_{t=1}^{T} \frac{1}{B(t)} \overline{P_{CAT}}(t)\right], \tag{3.7}$$

where $\mathbb{Q}_1$ is the valuation measure in terms of n financial risk variables given in Eq. (3.6). This is practical since one can use Eq. (3.7) to value the CAT bond by choosing a term structure for arbitrage-free financial risks and calculating the expected cash flow conditionally on the financial risk process.

However, to complete the valuation, we also need to verify the structure of the cash flow process. A direct deduction from Corollary 3.1.1 is the case in which the CAT bond cash flows depend only on the catastrophic risk variables.

Theorem 3.1.3. *For CAT bond cash flows that are $\mathcal{A}^{(2)}$ adapted,*

$$\overline{P_{CAT}}(t) = \mathbb{E}^{\mathbb{Q}}\left[d(t)|\mathcal{A}_t^{(1)}\right] = \mathbb{E}^{\mathbb{P}}[P_{CAT}(t)]$$

and the value of the CAT bond can be given as

$$V(t) = \sum_{t=1}^{T} P(t)\mathbb{E}^{\mathbb{P}}[P_{CAT}(t)]. \tag{3.8}$$

The pricing formulas for CAT bonds given in Eqs. (3.7) and (3.8), which are an extension of work by Cox and Pedersen (2000), are the core results of this chapter.

3.2 Application of The Results for Earthquakes

In this section, a model with three financial risks ($n = 3$ with LIBOR, real interest, and inflation rates) and two catastrophe risks ($m = 2$ with earthquake magnitude and depth) is introduced. The maximum earthquake magnitude in one region is selected as the parametric index trigger for this CAT bond. It becomes clearer to the reader later in this section that the region in which the earthquake occurs is included in the CAT bond payoff function.

We model one financial market risk, the real interest rate, via a discrete process $\{r_1(t); t = 1, 2, \ldots, T\}$ within the triple $(\Omega_{1,1}, \mathcal{F}^{(1,1)}, \mathbb{P}_{1,1})$, that is equipped with the filtration $\mathcal{F}^{(1,1)}$. Similarly, the inflation rate process $\{r_2(t); t = 1, 2, \ldots, T\}$ is modelled on another complete probability triple $(\Omega_{1,2}, \mathcal{F}^{(1,2)}, \mathbb{P}_{1,2})$ equipped with the filtration $\mathcal{F}^{(1,2)}$. The final financial risk US LIBOR rate $\{R(t); t = 1, 2, \ldots, T\}$ is modelled within $(\Omega_{1,3}, \mathcal{F}^{(1,3)}, \mathbb{P}_{1,3})$, which is equipped with the filtration $\mathcal{F}^{(1,3)}$.

Catastrophic risks are modelled via two random variables. We model the annual maximum-magnitude earthquake using the random variable $\{M(t); t = 1, 2, \ldots, T\}$ within the probability space $(\Omega_{2,1}, \mathcal{F}^{(2,1)}, \mathbb{P}_{2,1})$, which is equipped with the filtration $\mathcal{F}^{(2,1)}$, and the depth $\{D(t); t = 1, 2, \ldots, T\}$ within $(\Omega_{2,2}, \mathcal{F}^{(2,2)}, \mathbb{P}_{2,2})$, which is equipped with the filtration $\mathcal{F}^{(2,2)}$.

One-period and multi-period models are developed and the CAT bond valuation is performed in three stages. In the first stage we specify cash flows to the bondholder, which are dependent on the above risk variables. In this application, parametric trigger type (annual maximum magnitude earthquake in the region as the triggering event) is used for this structured CAT bond, and investors can benefit from no moral hazard risk while sponsors can enjoy a quicker reimbursement. In the next stage we analyse the dynamics of financial risks and catastrophic risks by assuming a suitable distribution function and estimating parameters from historical data. In the final stage we generate sequences of a discrete-time process for future risks and obtain the price of CAT bonds in an arbitrage-free framework.

3.2.1 One-period (basic) Model

In this subsection, a simple one-period model is formulated. Under the discrete-time framework of the analysis, we first define the following symbols and notations:

T : maturity date for the CAT bond.

Z : face amount of the CAT bond.

$r_1(t)$: risk-free real interest rate at time t (e.g. 1-year US Treasury securities rate).

$r_2(t)$: inflation rate at time t (e.g. represented by the consumer price index (CPI)).

$R(t)$: deterministic coupon payment rate at time t (e.g. 12-month US LIBOR rate on the bond issuance date).

e : extra premium loading for the earthquake risk (normally positive considering risk-averse investors).

$M(t)$: maximum earthquake magnitude at t^{th} year within all selected regions. If we have two regions, $M(t) = \max\{M^1(t), M^2(t)\}$, where $M^1(t)$ and $M^2(t)$ represent the maximum-magnitude earthquake in each of the two regions, which have the common distributions described in sub-Section 3.2.3.

$D(t)$: depth (km) of the earthquake at time t.

$V(P_{CAT})$: value of the CAT bond at time of issuance.

Due to the fact that we are working in a one period model, for the remainder of this subsection we from this point in this subsection assume that $T = t = 1$. Thus one can simplify the notations as $r_1,\ r_2,\ R,\ M$, and D and assume the dynamics of financial risks (real interest rate, inflation rate, and LIBOR rate) are constant.

Denote $P_{CAT}(R; M, D)$ as payoff function of the CAT bond with piecewise cash flow on maturity. Zimbidis et al. (2007) gave a similar expression for CAT bond cash flows that depend on M and D. As an illustration, the structure of the cash value is given by

$$P_{CAT}(R;M,D)=\begin{cases} Z\cdot(1+f(R)), & M\in[0,\mu_1], \text{with } \{D\leq\delta_1\} \text{ or } \{D>\delta_1\}\\ Z\cdot(1+g(R)), & M\in(\mu_1,\mu_2], \text{with } \{D\leq\delta_2\} \text{ or } \{D>\delta_2\}\\ Z\cdot(1+h(R)), & M\in(\mu_2,\mu_3], \text{with } \{D\leq\delta_3\} \text{ or } \{D>\delta_3\}\\ Z, & M\in(\mu_3,\mu_4]\\ \phi(Z), & M\in(\mu_4,\mu_5], \text{with } \{D\leq\delta_4\} \text{ or } \{D>\delta_4\}\\ \gamma(Z), & M\in(\mu_5,\mu_6], \text{with } \{D\leq\delta_5\} \text{ or } \{D>\delta_5\}\\ \eta(Z), & M\in(\mu_6,\infty), \end{cases}$$

where the trigger points $\mu_1,\mu_2,\ldots,\mu_6$ and $\delta_1,\delta_2,\ldots,\delta_5\in\mathbb{R}^+$ are the pre-determined levels in the CAT bonds contract for magnitude and depth, respectively, and $0<\mu_1<\mu_2<\ldots<\mu_6$, $0<\delta_1<\delta_2<\ldots<\delta_5$. Selection of $\mu_1,\mu_2,\ldots,\mu_6$ affects the securitization level of the bond, which an individual company should balance between profit and marketability by analysing historical earthquake loss data. Finally, coupon payment functions $f(R)$, $g(R)$, $h(R)$, $\phi(Z)$, $\gamma(Z)$, and $\eta(Z)$ are normally designed according to company policy. Here we illustrate a possible example:

$$\begin{aligned}
f(R) &= \begin{cases} 2.6R\cdot\mathbb{1}_{\{D\leq\delta_1\}}+2.8R\cdot\mathbb{1}_{\{D>\delta_1\}}, \text{for } M=M^1\\ 2.9R\cdot\mathbb{1}_{\{D\leq\delta_1\}}+3R\cdot\mathbb{1}_{\{D>\delta_1\}}, \text{for } M=M^2 \end{cases}\\
g(R) &= \begin{cases} 1.6R\cdot\mathbb{1}_{\{D\leq\delta_2\}}+1.9R\cdot\mathbb{1}_{\{D>\delta_2\}}, \text{for } M=M^1\\ 1.8R\cdot\mathbb{1}_{\{D\leq\delta_2\}}+2R\cdot\mathbb{1}_{\{D>\delta_2\}}, \text{for } M=M^2 \end{cases}\\
h(R) &= \begin{cases} 0.5R\cdot\mathbb{1}_{\{D\leq\delta_3\}}+0.6R\cdot\mathbb{1}_{\{D>\delta_3\}}, \text{for } M=M^1\\ R\cdot\mathbb{1}_{\{D\leq\delta_3\}}+1.1R\cdot\mathbb{1}_{\{D>\delta_3\}}, \text{for } M=M^2 \end{cases}\\
\phi(Z) &= \begin{cases} 0.8Z\cdot\mathbb{1}_{\{D\leq\delta_4\}}+0.85Z\cdot\mathbb{1}_{\{D>\delta_4\}}, \text{for } M=M^1\\ 0.95Z\cdot\mathbb{1}_{\{D\leq\delta_4\}}+0.98Z\cdot\mathbb{1}_{\{D>\delta_4\}}, \text{for } M=M^2 \end{cases}\\
\gamma(Z) &= \begin{cases} 0.55Z\cdot\mathbb{1}_{\{D\leq\delta_5\}}+0.6Z\cdot\mathbb{1}_{\{D>\delta_5\}}, \text{for } M=M^1\\ 0.7Z\cdot\mathbb{1}_{\{D\leq\delta_5\}}+0.75Z\cdot\mathbb{1}_{\{D>\delta_5\}}, \text{for } M=M^2 \end{cases}
\end{aligned}$$

$$\text{and} \quad \eta(Z) = \begin{cases} 0.2Z, \text{for } M = M^1 \\ 0.5Z, \text{for } M = M^2. \end{cases}$$

In the one-period case, we assume that Z, r_1, r_2, R, and e are constant. Therefore, cash flow is independent of financial risks, and we can apply Eq. (3.8) and obtain the price of the CAT bond:

$$V(P_{CAT}) = \frac{1}{1+(r_1+e)} \cdot \frac{1}{1+r_2} \mathbb{E}^{\mathbb{P}} \left[P_{CAT}(R; M, D) \right], \tag{3.9}$$

where $\mathbb{P}$ is the probability measure corresponding to the distribution of M^1, M^2 (obtained in Table 3.2), and D. It is important to note that one of our financial market rates $(r_1 + e)$ is a shift of the interest rate, which makes CAT bonds more attractive than normal return bonds.

Assuming that expectation exists in Eq. (3.9), CAT bond prices can be approximated by using the same logic as Zimbidis et al. (2007) according to equilibrium pricing theory:

$$V(P_{CAT}) = \lim_{h \to \infty} V^{(h)}(P_{CAT}),$$

where

$$V^{(h)}(P_{CAT}) = \frac{1}{1+(r_1+e)} \frac{1}{1+r_2} \frac{1}{h} \sum_{l=1}^{h} P_{CAT}(R; M^{(l)}, D^{(l)}). \tag{3.10}$$

Here $M^{(l)}, D^{(l)}$ represents the l^{th} simulated value in h simulations. Therefore, we approximate the value of $V(P_{CAT})$ based on Eq. (3.10) by generating h events, see Boyle et al. (1997); Romaniuk (2003).

3.2.2 Multi-period (advanced) Model

Under the discrete-time framework, we now introduce the notation for multi-period models. T, Z, e, $M(t)$, $D(t)$ and the coupon payment functions $f(R)$, $g(R)$, $h(R)$, $\phi(Z)$, $\gamma(Z)$, and $\eta(Z)$ have the same form as in the one-period model.

$r_1(t)$: market yield at the 1-year US Treasury securities rate at time t. More precisely, $r_1(t)$ gives the annual compounded interest discount rate of a typical cash flow for the period $t+1$. We assume that $r_1(t)$ is ARIMA $(1,1,1)$ model with parameters θ_1 and α_1 for any $t = 1,\ 2, \dots,\ T$, which simulates the interest rate well, see Box et al. (2011); Dhaene (1989)[13]. For $r_1(t) > 0$,

$$\Delta r_1(t) = C_1 + \theta_1 \Delta r_1(t-1) + \varepsilon_1(t) + \alpha_1 \varepsilon_1(t-1),$$

where $\Delta r_1(t) = r_1(t) - r_1(t-1)$, C_1 is constant, and the error terms $\varepsilon_1(t)$ are assumed to be independent, identically distributed variables sampled from a normal distribution with zero mean.

$r_2(t)$: 1-year inflation rate at time t. In a similar setting as for the treasury rate, we assume that $r_2(t)(r_2(t) > 0)$ follows an ARIMA $(0,0,1)$ model with parameter α_2 for any $t = 1,\ 2, \dots,\ T$. For $r_2(t) > 0$,

$$r_2(t) = C_2 + \varepsilon_2(t) + \alpha_2 \varepsilon_2(t-1),$$

where $\varepsilon_2(t)$ are i.i.d. normal distributed random variables representing the white noise of the model.

$R(t)$: the 12-month LIBOR rate at time t. Here, we assume that the fundamental process for the instantaneous LIBOR rate $\{R(t); t = 1,\ 2, \dots,\ T\}$ is the CIR process Cox et al. (1985) given by the following stochastic differential equation[14]:

$$dR(t) = \alpha_3(\beta_3 - R(t))dt + \sigma_3 \sqrt{R(t)} dW(t), \tag{3.11}$$

where $\theta_3 = (\alpha_3, \beta_3, \sigma_3)$ are the model parameters and $W(t)$ is standard Brownian motion.

$P_{CAT}(R(t); M(t), D(t))$: coupon payment value received by the CAT bondholder at time $t = 1,\ 2, \dots,\ T$, constructed in the following form:

$$P_{CAT}(R(t); M(t), D(t)) =$$

[13]Detailed information is shown in sub-Section 2.3.1.

[14]Detailed information is shown in sub-Section 2.3.2.

$$
\begin{cases}
Zf(R(t))\mathbb{1}_{\{0\le M(t)\le \mu_1\}} + Zg(R(t))\mathbb{1}_{\{\mu_1< M(t)\le \mu_2\}} + Zh(R(t))\mathbb{1}_{\{\mu_2< M(t)\le \mu_3\}}, \\
\text{for } t=1,\ 2,\dots,\ T-1, \\
Z(1+f(R(t)))\mathbb{1}_{\{0\le M(t)\le \mu_1\}} + Z(1+g(R(t)))\mathbb{1}_{\{\mu_1< M(t)\le \mu_2\}} + \\
+Z(1+h(R(t)))\mathbb{1}_{\{\mu_2< M(t)\le \mu_3\}} + Z\mathbb{1}_{\{\mu_3< M(t)\le \mu_4\}} + \phi(Z)\mathbb{1}_{\{\mu_4< M(t)\le \mu_5\}} + \\
+\gamma(Z)\mathbb{1}_{\{\mu_5< M(t)\le \mu_6\}} + \eta(Z)\mathbb{1}_{\{M(t)> \mu_6\}}, \text{ for } t=T.
\end{cases}
\tag{3.12}
$$

Therefore,

$$
\overline{P_{CAT}}(R(t);M(t),D(t)) = \mathbb{E}^{\mathbb{Q}}[P_{CAT}(k)|\mathcal{A}(t)^{(1)}] =
$$

$$
\begin{cases}
\mathbb{E}^{\mathbb{Q}}\left[Zf(R(t))\mathbb{1}_{\{0\le M(t)\le \mu_1\}} + Zg(R(t))\mathbb{1}_{\{\mu_1< M(t)\le \mu_2\}} + Zh(R(t))\mathbb{1}_{\{\mu_2< M(t)\le \mu_3\}}\right], \\
\text{for } t=1,\ 2,\dots,\ T-1, \\
\mathbb{E}^{\mathbb{Q}}[Z(1+f(R(t)))\mathbb{1}_{\{0\le M(t)\le \mu_1\}} + Z(1+g(R(t)))\mathbb{1}_{\{\mu_1< M(t)\le \mu_2\}} + \\
+Z(1+h(R(t)))\mathbb{1}_{\{\mu_2< M(t)\le \mu_3\}} + Z\mathbb{1}_{\{\mu_3< M(t)\le \mu_4\}} + \phi(Z)\mathbb{1}_{\{\mu_4< M(t)\le \mu_5\}} + \\
+\gamma(Z)\mathbb{1}_{\{\mu_5< M(t)\le \mu_6\}} + \eta(Z)\mathbb{1}_{\{M(t)> \mu_6\}}], \text{ for } t=T.
\end{cases}
$$

Assume that random variables $\{M(t); t=1,\ 2,\dots,\ T\}$, $\{D(t); t=1,\ 2,\dots,\ T\}$, $\{\varepsilon_1(t); t=1,\ 2,\dots,\ T\}$, and $\{\varepsilon_2(t); t=1,\ 2,\dots,\ T\}$ and stochastic processes $\{r_1(t); t=1,\ 2,\dots,\ T\}$, $\{r_2(t); t=1,\ 2,\dots,\ T\}$, $\{R(t); t=1,\ 2,\dots,\ T\}$, and $\{W(t); t=1,\ 2,\dots,\ T\}$ are pairwise independent.

According to Eq. (3.7), the value of a T-period CAT bond is

$$
V(P_{CAT}) = \mathbb{E}^{\mathbb{Q}_1}\left[\sum_{t=1}^{T}\frac{1}{\prod_{s=0}^{t-1}[1+r_1(s)+e][1+r_2(s)]}\overline{P_{CAT}}(R(t);M(t),D(t))\right], \tag{3.13}
$$

which can be calculated using the same method as for Eq. (3.10). Assuming that expectation exists in Eq. (3.13), similar to the one-period model, the CAT bond price can be approximated by the strong law of large numbers:

$$
V(P_{CAT}) = \lim_{h\to\infty} V^{(h)}(P_{CAT}),
$$

where

$$
V^{(h)}(P_{CAT}) = \frac{1}{h}\sum_{l=1}^{h}\sum_{t=1}^{T}\frac{\overline{P_{CAT}}(R^{(l)}(t);M^{(l)}(t),D^{(l)}(t))}{\prod_{s=0}^{t-1}[1+r_1^{(l)}(s)+e][1+r_2^{(l)}(s)]}. \tag{3.14}
$$

Here $r_1^{(l)}(\cdot),\ r_2^{(l)}(\cdot),\ R^{(l)}(\cdot),\ M^{(l)}(\cdot),\ D^{(l)}(\cdot)$ represents the l^{th} simulated value in h simulations. For future convenience, we use the magnitude and depth trigger points $\mu_1 = 5.4, \mu_2 = 5.8, \mu_3 = 6.2, \mu_4 = 6.6, \mu_5 = 7.0, \mu_6 = 7.4$, and $\delta_1 = 20, \delta_2 = 15.\delta_3 = 10, \delta_4 = 10, \delta_5 = 10$ in the numerical example. A catastrophe might or might not occur before the maturity date T. According to the cash flow stream given in Eq. (3.12), a CAT bond with face amount US$\$K$ will pay coupons $f(R)$, $g(R)$, and $h(R)$ to bondholders at the end of each period if an earthquake of maximum magnitude in the intervals $(0, 5.4]$, $(5.4, 5.8]$, and $(5.8, 6.2]$, respectively, occurred in this period, or no coupon payment if the magnitude is greater than 6.2. On the maturity date, the CAT bond is scheduled to repay the full principal payment plus its normal coupon when maximum magnitude earthquake level is $(0, 6.6]$; or only partial of the principle, which is $\phi(Z)$, $\gamma(Z)$, or $\eta(Z)$, if the magnitude is in the interval $(6.6, 7.0]$, $(7.0, 7.4]$, or $(7.4, \infty)$, respectively.

3.2.3 California Earthquake Data for Catastrophic Risk Variables

Earthquake data from California is used in order to carry out this example in the following section. Figure 3.1[15] shows recent significant earthquakes in California, with a darker colour representing more severe earthquakes. The two circles denote locations where the most significant earthquakes occurred. In this example, we analyse the earthquakes that hit these circled areas, San Francisco (region 1) and Los Angeles (region 2), over the period 1968 – 2011. Table 3.1[16] lists the annual maximum-magnitude (M) earthquakes in each region and the corresponding latitude (La), longitude (Lo), and depth (D). These two regions include the biggest cities in California which claim the majority of the economic losses. The elements of the data set is presented according to Coles et al. (2001).

[15]Source by NOAA National Geophysical Data Center, on 30/12/2011, `http://maps.ngdc.noaa.gov/viewers/hazards/`.

[16]Data from Southern California Earthquake Data Center (SCEC), `http://www.data.scec.org/`.

Table 3.1: Annual maximum-magnitude earthquakes in two regions in California, data from SCEC.

	Region 1				Region 2					Region 1				Region 2			
Y	La	Lo	D	M	La	Lo	D	M	Y	La	Lo	D	M	La	Lo	D	M
1968	37.0	-121.7	12	4.3	33.2	-116.1	10	6.6	1990	36.9	-121.7	6	5.4	34.1	-117.7	4	5.5
1969	38.5	-122.8	5	5.7	33.3	-116.4	6	5.5	1991	35.8	-121.3	10	5.2	34.3	-118.0	9	5.8
1970	36.8	-121.4	10	4.7	34.3	-117.5	11	5.2	1992	38.9	-122.8	4	4.5	34.2	-116.4	1	7.3
1971	36.6	-121.2	4	4.7	34.4	-118.4	9	5.8	1993	37.0	-121.5	8	4.8	35.1	-119.1	21	5.2
1972	36.6	-121.2	4	5.1	34.8	-118.9	4	4.3	1994	38.8	-122.4	12	4.2	34.2	-118.5	18	6.7
1973	39.3	-123.4	18	4.8	34.1	-119.0	8	5.3	1995	36.6	-121.2	7	4.9	34.4	-118.7	13	5.0
1974	36.9	-121.5	5	5.2	34.4	-118.5	24	4.7	1996	36.9	-121.6	5	4.9	34.0	-116.3	6	4.5
1975	39.4	-121.5	5	5.7	34.5	-116.5	0	5.3	1997	39.9	-120.8	5	4.3	35.0	-116.8	2	5.3
1976	36.5	-120.5	12	4.6	33.1	-115.6	5	5.1	1998	36.8	-121.5	9	5.1	34.3	-116.8	6	4.8
1977	39.4	-123.3	15	4.8	34.4	-118.5	11	4.4	1999	37.9	-122.7	8	4.6	34.6	-116.3	0	7.1
1978	39.2	-123.2	12	4.5	34.3	-119.7	18	5.1	2000	38.4	-122.4	10	4.9	32.9	-115.5	10	4.5
1979	37.1	-121.5	9	5.8	32.7	-115.4	15	5.8	2001	39.8	-120.6	18	5.3	33.5	-116.5	15	5.0
1980	37.8	-121.8	15	5.8	33.5	-116.5	19	5.3	2002	37.0	-121.6	7	4.9	34.8	-116.3	5	4.8
1981	37.4	-121.7	9	4.8	33.1	-115.6	19	5.8	2003	35.7	-121.1	9	6.5	34.3	-116.8	1	5.0
1982	36.6	-121.3	11	4.6	33.5	-116.7	13	4.8	2004	36.0	-120.5	11	4.9	35.4	-118.6	4	5.0
1983	35.8	-121.4	10	5.2	32.5	-117.4	10	4.7	2005	38.8	-122.8	0	4.3	33.5	-116.6	14	5.2
1984	37.3	-121.7	9	6.2	34.0	-119.0	13	4.6	2006	38.8	-122.8	5	4.7	32.7	-116.0	14	4.6
1985	39.5	-120.6	7	4.6	35.5	-118.9	6	4.6	2007	37.4	-121.8	10	5.5	33.7	-117.5	13	4.7
1986	37.5	-121.7	9	5.7	34.0	-116.6	10	5.7	2008	40.0	-120.9	0	4.5	34.0	-117.8	15	5.4
1987	36.2	-120.7	28	4.8	33.0	-115.9	11	6.6	2009	35.6	-121.0	6	4.4	33.3	-115.7	6	4.8
1988	37.4	-121.7	10	5.3	34.9	-118.7	7	5.4	2010	37.5	-121.8	11	4.1	32.7	-115.9	5	5.7
1989	37.0	-121.9	17	7.0	33.9	-118.6	12	5.0	2011	36.6	-121.2	8	4.6	32.7	-115.7	15	4.4

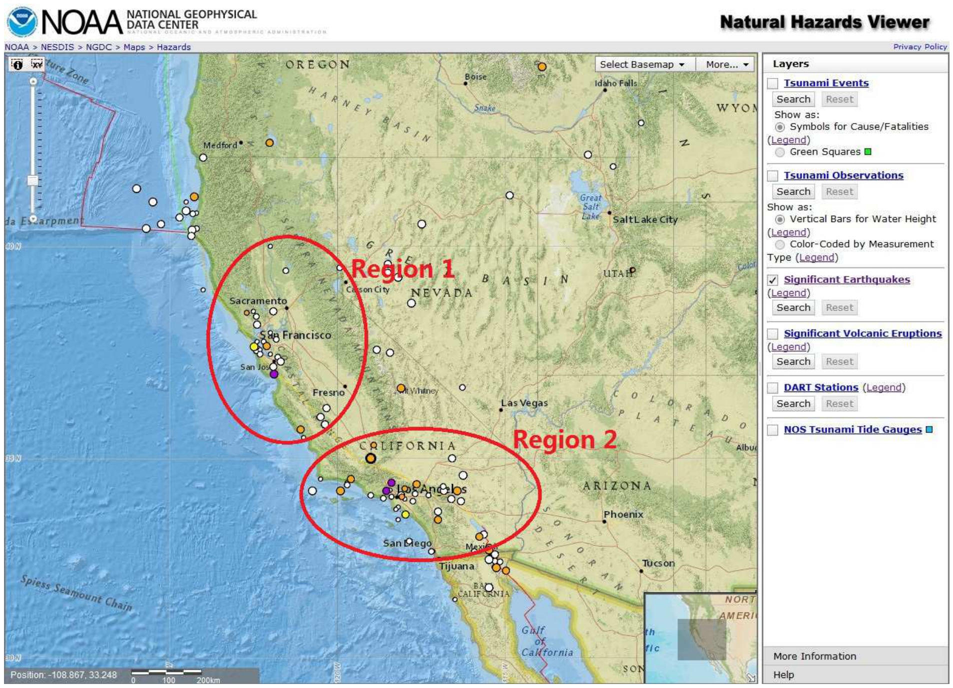


Figure 3.1: Recent significant earthquakes in California with highlighted regions, source by NOAA.

Magnitude

The traditional approach for defining extremes is to focus on the statistical behavior of

$$M_\delta^q(t) = \max\{X_1^q(t), X_2^q(t), \ldots, X_\delta^q(t)\},$$

where $q = 1, 2$ and $X_1^q(t), X_2^q(t), \ldots, X_\delta^q(t)$ is a sequence of $\delta = 365$ independent random variables with a universal distribution function F that measures the daily maximum-magnitude earthquake in each region for the period $[t, t{+}1)$. $X_\delta^q(t) = 0$ if no earthquake occurs in region q on the day. The sequence $M_\delta(t) = \max\{M_\delta^1(t),\ M_\delta^2(t)\}$ corresponds to the t^{th} year maximum-magnitude earthquake. The distribution of $M_\delta^q(t)$ can be derived using the GEV distribution. The rescaled sample maxima $(M_\delta^q(t))^* = (M_\delta^q(t) - b_\delta(t))/a_\delta(t)$ is a heavy-tailed distribution and the possible distribution is provided by GEV family[17], as $\delta \to \infty$

$$G(z) = \exp\left\{-\left[1 + \alpha_4\left(\frac{z - \beta_4}{\sigma_4}\right)\right]^{-1/\alpha_4}\right\} \tag{3.15}$$

[17] For more information, please check Section 2.4 Theorem 2.4.1.

defined on $\{z : 1 + \alpha_4(z - \beta_4)/\sigma_4 > 0\}$, where $-\infty < \beta_4 < \infty$, $\sigma_4 > 0$, $-\infty < \alpha_4 < \infty$, and $\beta_4 = \mathbb{E}(M_\delta^q(t))$, $\sigma_4 = \sqrt{\mathrm{Var}(M_\delta^q(t))}$.

The model has three parameters: location parameter β_4, scale parameter σ_4, and shape parameter α_4. When $\alpha_4 = 0$ is the limit of Eq. (3.15) as $\alpha_4 \to 0$, the model corresponds to the Gumbel family. For the cases $\alpha_4 > 0$ and $\alpha_4 < 0$, Eq. (3.15) leads to Frechét and Weibull family distributions, respectively.

According to time series plots of the maxima for both regions (Figure 3.2), it is reasonable to assume that the patterns of variation have stayed constant over the observed period, which suggests that the data are independent observations from the GEV distribution, see Coles et al. (2001); Zimbidis et al. (2007).

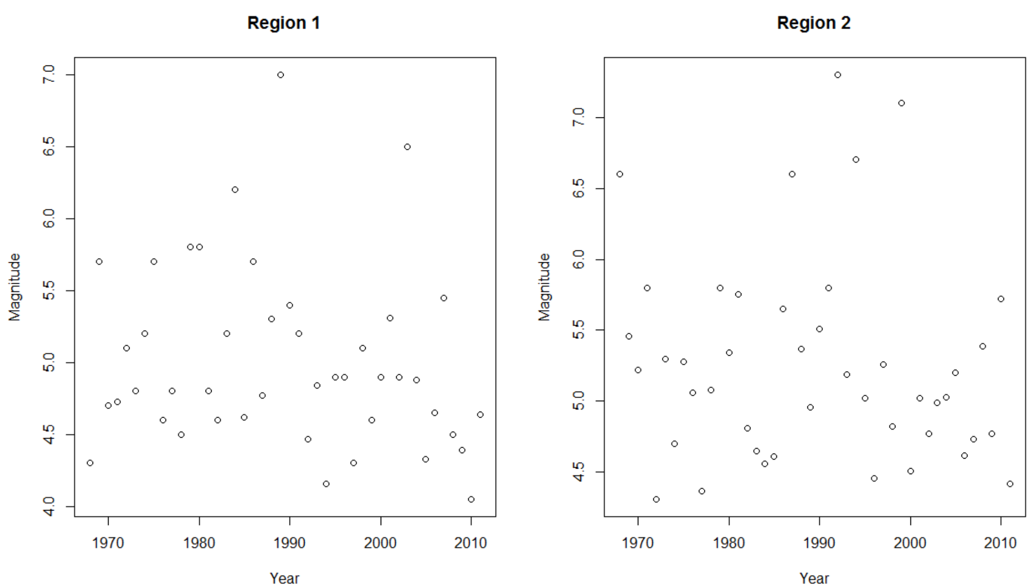


Figure 3.2: Scatter plot of the annual maximum-magnitude earthquakes $M^1(t)$ in region 1 and $M^2(t)$ in region 2 in California, 1968 – 2011.

In this subsection, we take region 1 as an example for analysis. Maximize the GEV log-likelihood for these data and achieve the estimate

$$(\hat{\beta_4}, \hat{\sigma_4}, \hat{\alpha_4}) = (4.71946946, 0.44861472, 0.05866229),$$

for which the log-likelihood is 35.72926. The approximate variance-covariance matrix of the parameter estimates is

$$V = \begin{bmatrix} 0.005854675 & 0.001935385 & -0.003127097 \\ 0.001935385 & 0.003228341 & -0.001542433 \\ -0.003127097 & -0.001542433 & 0.013764031 \end{bmatrix}.$$

Therefore, one can easily obtain standard errors 0.0765, 0.0568, and 0.1173 for β_4, σ_4, and α_4, respectively, with approximate 95% confidence intervals of $\beta_4 \in [4.64, 4.80]$, $\sigma_4 \in [0.39, 0.51]$, and $\alpha_4 \in [-0.06, 0.18]$.

To assess the accuracy of the GEV model fitted to the California earthquake data, various diagnostic plots of $M^1(t)$ are shown in Figure 3.3. The probability and quantile plots are close to linear, which confirms the validity of the fitted model. The estimate of α_4 is close to zero, and the estimated curve in the return level plot is nearly linear. According to the histogram density plot of the data, the density estimate is consistent. Consequently, the analysis provides strong evidence that the GEV model provides a good fit.

Furthermore, the tail behaviour of the distribution displayed in Figure 3.4 reflects the sample mean excess, and the downward trend suggests a very short tail behaviour for the annual maximum-magnitude earthquakes in region 1 in California, see Beirlant et al. (1996); Embrechts et al. (1997).

Similar analysis can be conducted in region 2 and estimate the GEV distribution with parameters $(\hat{\mu}, \hat{\sigma}, \hat{\alpha_4}) = (4.9275121,\ 0.4833782,\ 0.1181457)$, and the exceeding probabilities intervals $M^1(t)$ and $M^2(t)$ for the GEV distributions are listed in Table 3.2. The possibility of an earthquake of magnitude greater than 6.6 occurring in the target regions is less than 8%, so we can introduce a bond with 92% capital guarantee.

Depth

The next stage is to analyse the earthquake depth distribution. According to the density plot in Figure 3.5, earthquake depth follows a right-skewed heavy-tailed distribution and we fit it as a gamma distribution:

$$f(x; \alpha_5, \beta_5) = \beta_5^{\alpha_5} \frac{1}{\Gamma(\alpha_5)} x^{\alpha_5 - 1} e^{-\beta_5 x}$$

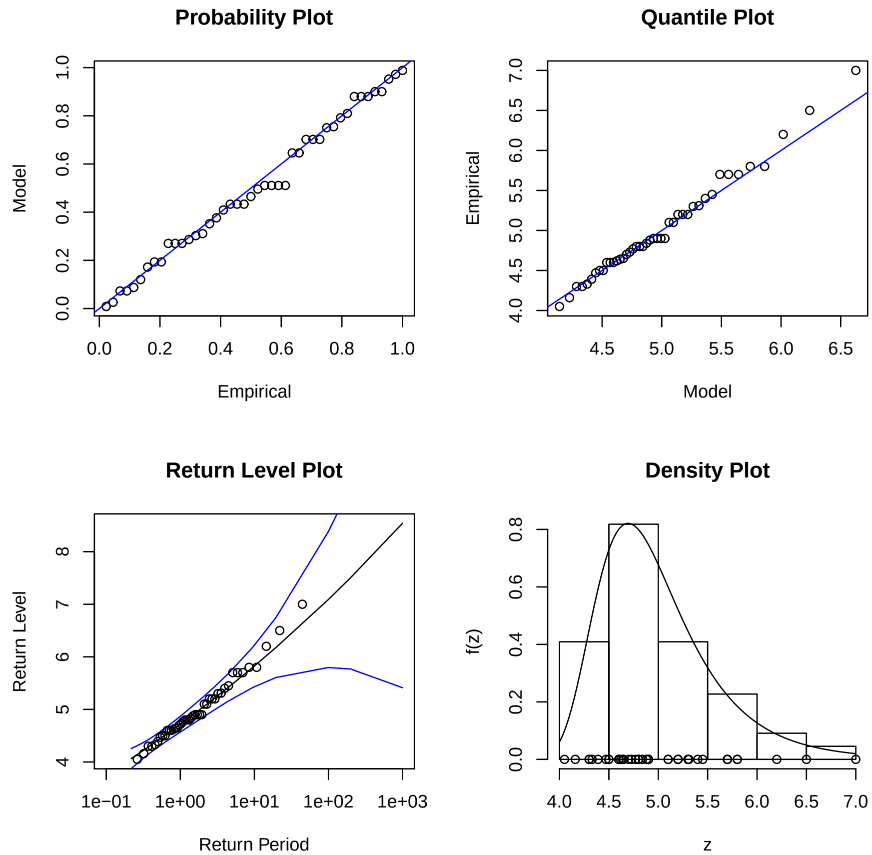


Figure 3.3: Diagnostic plots for GEV fitting to the annual maximum-magnitude earthquakes $M^1(t)$ in region 1 in California.

for which the estimated parameters are $(\hat{\alpha}_5, \hat{\beta}_5) = (2.35378504, 0.25460951)$ and $(\hat{\alpha}_5, \hat{\beta}_5) = (1.44878306, 0.14585340)$ for regions 1 and 2, respectively. This model is realistic since earthquakes that occur near the surface tend to be of higher magnitude compared with deeper earthquakes, see Fujikura et al. (1999).

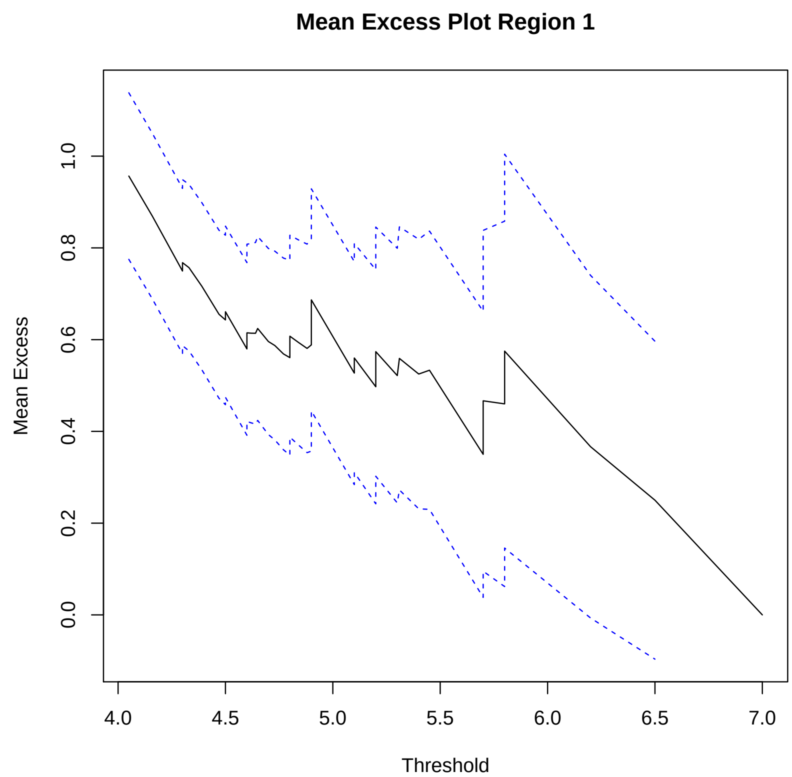


Figure 3.4: Sample mean excess for annual maximum-magnitude earthquakes $M^1(t)$ in region 1 in California, with 95% confidence interval.

3.3 Numerical Examples

3.3.1 Numerical Example For The One-period Model

Consider a one-period model with face value Z =US\$1000, interest rate $r_1 = 0.12\%$, and inflation rate[18] $r_2 = 3.16\%$. Given risk premium $e = 3\%$ and LIBOR rate[19] $R = 1.13\%$, Eq. (3.10) yields[20] the value of a one-period CAT bond as US\$940.

[18] Board of Governors of the Federal Reserve System, accessed on 30/12/2011, `http://www.federalreserve.gov/`.

[19] Accessed on 30/12/2011, `http://www.bba.org.uk/`.

[20] Code for one-period model, please check the Appendix B.1.

Table 3.2: Annual maximum-magnitude earthquakes exceeding probabilities for the GEV model in regions 1 and 2 in California.

	Region 1	Region 2
$\mathbb{P}(5.0 < M \leq 5.4)$	0.205599827	0.250286661
$\mathbb{P}(5.4 < M \leq 5.8)$	0.105005438	0.150399888
$\mathbb{P}(5.8 < M \leq 6.2)$	0.049947001	0.080828401
$\mathbb{P}(6.2 < M \leq 6.6)$	0.023619037	0.042623696
$\mathbb{P}(6.6 < M \leq 7.0)$	0.011371076	0.022816466
$\mathbb{P}(7.0 < M \leq 7.4)$	0.005618113	0.012543028
$\mathbb{P}(M > 7.4)$	0.006178647	0.01813135

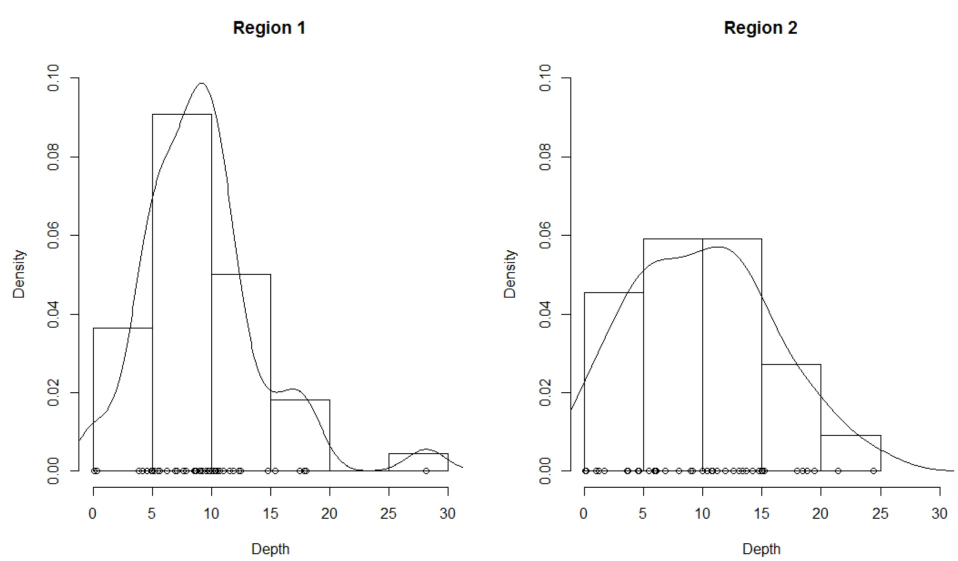


Figure 3.5: Density depth plot for the annual maximum-magnitude earthquakes $D^1(t)$ in region 1 and $D^2(t)$ in region 2 in California.

3.3.2 Pricing For The Multi-period Model

Consider a 5-year period CAT bond with payments depending on earthquake magnitude in selected areas. Because the probability of large-magnitude earthquakes is low, large number of events need to be considered to estimate the price of CAT bonds with a relatively small error, see Romaniuk (2003). One can build the simulation using the following five steps[21].

Step 1:

First, generate $100,000$ 5-year period sequence values via GEV distributions representing the maximum-magnitude earthquakes in each region. Similarly, we can generate $100,000$ depth sequences for both regions by gamma distributions. Then we select depth in the larger-magnitude branch for future simulation.

Step 2:

Second, obtain $100,000$ paths for the LIBOR rate $R(t)$ for $t \in [0,5]$ using Monte Carlo simulations. Following Romaniuk (2003), we use an iterative stochastic equation with the concept of local characterizations for the Levy process.

In this simulation, let $[0,T]$ be the lifetime interval for the CAT bond and discretize this into δ different steps. The time moments are $\tau = \{\tau(0) = 0, \tau(1), \ldots, \tau(\delta) = T\}$, where δ is the number of steps. The steps are constant at 1 day (250 business days a year), with $\Delta\tau = \tau_{k+1} - \tau_k$, where $k = 1, 2, \ldots, \delta - 1$. The discrete version of Eq. (3.11) given by Kladıvko (2007) takes the form

$$R(\tau + \Delta\tau) - R(\tau) = \alpha_3(\beta_3 - R(\tau))\Delta\tau + \sigma_3\sqrt{R(\tau)}\sqrt{\Delta\tau}\varepsilon_3(\tau), \qquad (3.16)$$

where $\varepsilon_3(\tau)$ follow $N(0, \Delta\tau)$ as a white noise process for $\tau = 1, 2, \ldots$.

The MATLAB implementation of the estimation processes provided by Kladıvko (2007) suggests use of the ordinary least square of Eq. (3.16) to find the starting point for the parameters. Then the log-likelihood function of the CIR process is maximized. Statistical analysis of 12-month LIBOR historical data for 2000 – 2011 yields the parameter estimates $\hat{\theta} = (\hat{\alpha_3}, \hat{\beta}_3, \hat{\sigma_3}) = (0.212421, 1.084655, 0.420791)$. For the initial value in Eq. (3.16) we set $R(0) = 1.13\%$, which was the actual LIBOR rate in December 2011.

[21] Code for multi-period model, please check the Appendix B.2.

Step 3:

The next step is the generation of sequences for the annual interest and inflation rates[22]. Recall from Section 3.2.1 that $r_1(t)$ follows an ARIMA $(1, 1, 1)$ model with parameters $(\hat{C}_1, \hat{\theta}_1, \hat{\alpha_1}) = (-0.0976, -0.2833, 1)$, and $r_2(t)$ follows an ARIMA $(1, 0, 0)$ model with parameters $(\hat{C}_2, \hat{\alpha_2}) = (0.7867, 0.7867)$, for any $t = 1,\ 2, \ldots$ according to the maximum log-likelihood estimate of the 1–year US Treasury securities rate and inflation rate for 1968 – 2011.

Step 4:

The next step is to calculate the coupon payments (cash flows $P_{CAT}(R(t); M(t), D(t))$) of the CAT bond for the 5-year period. It should be mentioned that this procedure is quite complex and involves logical functions and many subroutines. According to the cash flow stream in Eq. (3.12), the capital of our CAT bond may decrease if and only if an earthquake of magnitude greater than 6.6 hits California before the maturity date. Moreover, we assume a face amount of US$$1000$ and a risk premium of $e = 3\%$.

Step 5:

The final step is to calculate the present value of cash flows for every year, and then average over all the discounted values based on $r_1(t), r_2(t)$ for each period. According to Eq. (3.14), the price of the $T = 5$ CAT bond is approximately US$$779.73$.

To test the validity of the results, we ran the algorithm 100 times to generate 100 possible value of the CAT bond, for which the variance equals 0.91. It can easily be derived that the price variance dramatically decreases as h increases, and is asymptotically equal to zero after $10,000$. Figure 3.6 is a density plot of price values in which the density reaches the mode at US$$778.62$ at a density of 0.43. This is quite a promising result since the low volatility level suggests that our pricing model is both consistent and computationally efficient. Compared to a zero-coupon bond with price US$935, which depends only on financial risks, this CAT bond with a 92% capital guarantee is

[22]Data from Board of Governors of Federal Reserve System, `http://www.federalreserve.gov/`. for the period 1968–2011

very attractive to investors.

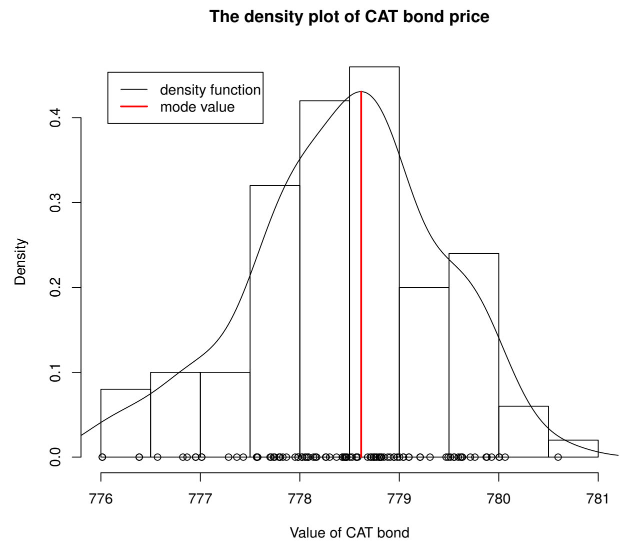


Figure 3.6: Density plot and cumulative density plot of the CAT bond price after running the algorithm 100 times with $h = 100,000$.

3.4 Summary

This chapter built a valuation framework for earthquake CAT bonds with n financial and m catastrophic independent risks. These securitization products can play a vital role in the financial sustainability of insurers and re-insurers, as well as for governmental authorities. The high return of the CAT bond identified here can generate sufficient funds to pay claims and post-disaster reconstruction costs if a significant catastrophic event occurs in an area. Furthermore, the assumptions made are quite standard and realistic, so the valuation model is easy to modify further and apply in industry. To simplify the model, all the risks are mutually independent. It is quite natural that earthquakes occur only in certain regions, and such events generally do not affect exchange and production levels and the economic environment on a global scale.

We also demonstrated how to construct a practical pricing model for earthquakes in California from 1968 to 2011 by adapting parametric triggers. Extreme value theory for the maximum-magnitude earthquakes in each year was used and we concluded that they follow a Frechét distribution in this case. In addition, earthquake depth fitted a gamma distribution. For financial risks, a classical ARIMA model for interest and inflation rates was chosen, and a CIR model for the stochastic process of coupon payment as a predetermined function of the annual LIBOR rate. Consequently, we were able to identify an equilibrium price for an earthquake CAT bond that depends on the risk variables above. The model, as an extension of the Cox and Pedersen (2000) approach, provides a more accurate approximation of price by considering multiple variables cross financial and catastrophic risks.

The dependence between the different risk variables cannot be used within our methodology and framework for bond pricing. Consequently, it should be characterized as a separate problem. This issue will be considered in the next chapter.

Chapter 4

Semi-Markov CAT Bond Model

In this chapter, as a special case of the previous chapter, CAT bonds are modelled with one financial risk (interest rate) and one catastrophe risk (PCS loss). We present a contingent claim model similar to Ma and Ma (2013), for pricing catastrophe risk bonds. Firstly, analytical bond pricing formulas are derived into a stochastic interest rate environment with the aggregate claims following compound forms where the claim inter-arrival times are dependent on the claim sizes, by employing a two-dimensional semi-Markov process. Secondly, explicit CAT bond price formulas are obtained in terms of four different payoff functions. Thirdly, estimates and calibrates of the parameters of the pricing models are made, using catastrophe loss data provided by Property Claim Services (PCS) from 1985–2013. According to Lin and Wang (2009) and Ma and Ma (2013), the estimates provided by the PCS are widely accepted as the reference industry index triggers in financial-market derivatives, including exchange-traded futures and options, CAT bonds, catastrophe swaps, industry loss warranties (ILWs), and other catastrophe-linked instruments. Thus, it is reasonable to use the PCS index losses from the entire property and casualty industry in the USA to estimate the parameters related to aggregate losses for pricing CAT bonds in the present study. Furthermore, we assume that the CAT loss industry indices are instantaneously measurable and updatable. Industrial trigger helps to tackle moral hazard and prevents insurers from disclosing detailed information to their competitors, Ma and Ma (2013). Finally, Monte Carlo simulations are used to analyze the numerical results for the aforementioned

CAT bonds pricing formulas.

The reminder of this chapter is organized as follows. Section 4.1 presents the pricing model of CAT bonds including: assumptions, aggregate claims processes, and payoff functions. Section 4.2 presents a numerical analysis of the PCS data and comparison between different models. In Section 4.3, we provide a discussion of the results.

4.1 Modelling CAT bonds

4.1.1 Modelling Assumptions

In this subsection, we provide preliminary details of the CAT bond structure. We follow the classical modelling assumption (Section 2.2) and then define stochastic processes and random variables with respect to the probability measure $\mathbb{P}$. Denote the CAT bond price process by $\{V_\ell^{(\varrho)}(t) : t \in [0,T]\}$, which is characterized by the aggregate loss process $\{L(t) : t \in [0,T]\}$, and the payoff functions $P_{CAT}^{(\varrho)}$, where $\ell = 0, 1, 2$ and $\varrho = 1, 2, 3, 4$. For each $t \in [0,T]$, the process $\{N(t) : t \in [0,T]\}$ describes the number of claims that occur until the time t. Also define $\{X_k : k \in \mathbb{N}^+\}$ as a sequence of i.i.d. random variables that represent the size of individual claims and $\{T_k : k \in \mathbb{N}^+\}$ represents a sequence of epoch times for the claims. In addition, define the spot interest rate process by $\{r(t) : t \in [0,T]\}$ and $\{W(t) : t \in [0,T]\}$ is a standard Brownian motion.

According to the CAT bonds payment structure, CAT bond investors receive premiums if trigger has not been pulled. In this study, an insurance industry index trigger is utilized to price CAT bonds. This means that investors might lose their capital if the estimated aggregate losses from the whole industry exceeds a predetermined level. Furthermore, in this chapter, we use valuation theory which is given in Section 2.2 and price the CAT bond formulae with interest rate $r(t)$ following a CIR interest rate process, as discussed in Section 2.3.

4.1.2 Aggregate Claims Process

In the classical actuarial literature, Bowers Jr. et al. (1986) stated that risk models are characterized by the following two stochastic processes: the claim number process, which counts the claims; and the claim amounts process, which determines the losses when a claim occurs. All previous studies of CAT bonds assumed that these two processes are mutually independent. However, because the independence assumption is restrictive in many applications, a more appropriate option, especially for CAT bonds, is to add dependence between the claim sizes and the inter-arrival times in the claims process when modelling the aggregate losses. In this chapter, for the first time in this area, a CAT bond's aggregate claims process, where the dependency among the characterized processes is described by a semi-Markov risk model. This model was first introduced by Miller (1962) and fully developed by Janssen (1969); Janssen and Manca (2007). In addition, a special case of this model is introduced where the claim arrival process is a continuous time Markov process with an exponential inter-arrival time.

Define the claim number process $\{N(t) : t \in [0, T]\}$, which follows a Poisson process with parameter $\lambda > 0$, to describe the number of future catastrophes in the insured region. The claim sizes $\{X_k : k \in \mathbb{N}^+\}$, which are independent of the process $\{N(t) : t \in [0, T]\}$, comprise a sequence of positive i.i.d. random variables with a common distribution function $F(x) = \mathbb{P}\{X_k < x\}$, which describes the amount of losses incurred by the k^{th} event. Then, the aggregate loss process $\{L(t) : t \in [0, T]\}$ is modelled by a compound Poisson process, as follows:

$$L(t) = \sum_{k=1}^{N(t)} X_k, \tag{4.1}$$

with the convention that $L(t) = 0$ when $N(t) = 0$.

Consider a semi-Markovian dependence structure in continuous time, where the process $\{J_n, n \geq 0\}$ represents the successive type of claims or environment states, which take their values in $J = \{1, ..., m\}$ $(m \in \mathbb{N}^+)$. Define $\{X_n, n \geq 1\}$ as a sequence of successive claim sizes, $X_0 = 0$ a.s. and $X_n > 0, \forall n$, and $\{T_n, n \in \mathbb{N}^+\}$ is the epoch time of the n^{th} claim. Suppose that $0 < T_1 < T_2 < \ldots < T_n < T_{n+1} < \ldots$,

$T_0 = U_0 = 0$ a.s., and $U_n = T_n - T_{n-1}$ $(n \in \mathbb{N}^+)$ denotes the sojourn time in state J_{n-1}. Suppose that the trivariate process $\{(J_n, U_n, X_n); n \geq 0\}$ is a semi-Markovian dependence process defined by the following matrix $\mathbf{Q}(= (Q_{ij}), i, j \in J)$,

$$Q_{ij}(t, x) = \mathbb{P}(J_n = j, U_n \leq t, X_n \leq x | (J_k, U_k, X_k), k = 1, 2, ..., n-1, J_{n-1} = i), \tag{4.2}$$

where the process of successive claims $\{J_n\}$ is an irreducible homogeneous continuous time Markov chain with state space J and transition matrix $\mathbf{P}(= (p_{ij}), i, j \in J)$, where $\lim_{t\to\infty, x\to\infty} Q_{ij}(t, x) = p_{ij}, i, j \in J$. The process changes its state at every claim instance based on the transition matrix $\mathbf{P}$, and an interpretation of this model in terms of CAT bonds is that the arrival time before the next catastrophic event U_{k+1} depends partially on the severity of the previous catastrophic event X_k, for all $k = 0, 1, 2, \ldots$.

Assuming that the random variable $J_n, n \geq 0$ and the two-dimensional random variable $(U_n, X_n), n \geq 1$ are conditionally independent, then

$$\begin{aligned} G_{ij}(t, x) =& \mathbb{P}(U_n \leq t, X_n \leq x | J_0, ..., J_{n-1} = i, J_n = j) \\ =& \begin{cases} Q_{ij}(t, x)/p_{ij}, \text{ for } p_{ij} > 0, \\ \mathbb{1}\{t \geq 0\}\mathbb{1}\{x \geq 0\}, \text{ for } p_{ij} = 0, \end{cases} \end{aligned} \tag{4.3}$$

where $\mathbb{1}\{\cdot\}$ represents an indicator function. The random variable $J_n, n \geq 0$ is conditionally dependent on the random variable $U_n, n \geq 1$ and also dependent on the random variable $X_n, n \geq 1$. Denote

$$G_{ij}(t, \infty) = \mathbb{P}(U_n \leq t | J_0, ..., J_{n-1} = i, J_n = j), \tag{4.4}$$

$$G_{ij}(\infty, x) = \mathbb{P}(X_n \leq x | J_0, ..., J_{n-1} = i, J_n = j), \tag{4.5}$$

and obtain the following equations by suppressing the condition J_n,

$$H_i(t, x) = \mathbb{P}(U_n \leq t, X_n \leq x | J_0, ..., J_{n-1} = i) = \sum_{j=1}^{m} p_{ij} G_{ij}(t, x),$$

$$H_i(t, \infty) = \mathbb{P}(U_n \leq t | J_0, ..., J_{n-1} = i), \tag{4.6}$$

$$H_i(\infty, x) = \mathbb{P}(X_n \leq x | J_0, ..., J_{n-1} = i). \tag{4.7}$$

Assuming that the sequences $\{U_n, n \geq 1\}$, $\{X_n, n \geq 1\}$ are conditionally independent and given the sequence $\{J_n, n \geq 0\}$, then

$$G_{ij}(t, x) = G_{ij}(t, \infty) G_{ij}(\infty, x), \forall t, x \in \mathbb{R}, \forall i, j \in J.$$

Thus, the semi-Markov kernel $\mathbf{Q}$ can be expressed as

$$Q_{ij}(t,x) = p_{ij}G_{ij}(t,\infty)G_{ij}(\infty,x), \forall t,x \in \mathbb{R}, \forall i,j \in J.$$

Define ${}^A\mathbf{Q}(= ({}^AQ_{ij}), i,j \in J)$ as the kernel of the process $\{(J_n, U_n); n \geq 0\}$ and ${}^B\mathbf{Q}(= ({}^BQ_{ij}), i,j \in J)$ as the kernel of the process $\{(J_n, X_n); n \geq 0\}$, then

$$\begin{aligned} {}^AQ_{ij}(t) &= Q_{ij}(t,\infty) = p_{ij}G_{ij}(t,\infty), \forall t \in \mathbb{R}, \forall i,j \in J, \\ {}^BQ_{ij}(x) &= Q_{ij}(\infty,x) = p_{ij}G_{ij}(\infty,x), \forall x \in \mathbb{R}, \forall i,j \in J. \end{aligned}$$

In order to calculate the distribution function of the accumulated claims amount, consider the following random walk process, as presented in Janssen and Manca (2007). Let L_n be the successive total claims amount after the arrival of the n^{th} claim, which is defined as:

$$L_n = \sum_{k=1}^{n} X_k, \forall n \geq 1, \forall i,j \in J. \tag{4.8}$$

Then, the joint probability of the process $\{(J_n, T_n, L_n); n \geq 0\}$ is denoted as

$$\mathbb{P}[J_n = j, T_n \leq t, L_n \leq x | J_0 = i] = Q_{ij}^{*n}(t,x).$$

This n-fold convolution matrix $\mathbf{Q}^{(n)}(= (Q_{ij}^{(n)}), i,j \in J)$ can be valued recursively by:

$$\begin{aligned} Q_{ij}^{*0}(t,x) &= \begin{cases} (1 - G_{ij}(0,\infty))(1 - G_{ij}(\infty,0)), & \text{if } i = j \\ 0, & \text{elsewhere,} \end{cases} \\ Q_{ij}^{*1}(t,x) &= Q_{ij}(t,x), \quad \ldots \\ Q_{ij}^{*n}(t,x) &= \sum_{l=1}^{m} \int_0^t \int_0^x Q_{lj}^{*(n-1)}(t-t', x-x') dQ_{il}(t',x'). \end{aligned}$$

Similarly to the processes $\{(J_n, T_n); n \geq 0\}$ and the process $\{(J_n, X_n); n \geq 0\}$ have

$$\mathbb{P}[J_n = j, T_n \leq t | J_0 = i] = {}^AQ_{ij}^{*n}(t) = \sum_{l=1}^{m} \int_0^t {}^AQ_{lj}^{*(n-1)}(t-t') d^AQ_{il}(t'), \tag{4.9}$$

$$\mathbb{P}[J_n = j, L_n \leq x | J_0 = i] = {}^BQ_{ij}^{*n}(x) = \sum_{l=1}^{m} \int_0^x {}^BQ_{lj}^{*(n-1)}(x-x') d^BQ_{il}(x').$$

Then, one can obtain the following equations:

$$\mathbb{P}[J_n = j | J_0 = i] = p_{ij}^{*n} = \sum_{l=1}^{m} p_{lj}^{*(n-1)} p_{il},$$

$$\mathbb{P}[L_n \le x | J_0 = i, J_n = j] = G_{ij}^{*n}(\infty, x) = \begin{cases} \frac{{}^{B}Q_{ij}^{*n}(x)}{p_{ij}^{*n}}, \text{ for } p_{ij}^{*n} > 0, \\ \mathbb{1}\{x \ge 0\}, \text{ for } p_{ij}^{*n} = 0, \end{cases} , \quad (4.10)$$

$$Q_{ij}^{*n}(t,x) = {}^{A}Q_{ij}^{*n}(t) G_{ij}^{*n}(\infty, x).$$

Let the counting process $\{N_i(t), t \ge 0\}$ denote the total number of type i claims that occur in $(0,t]$, for all $i \in J$. Thus, the total number of claims $\{N(t), t \ge 0\}$ that occur in $(0,t]$ is

$$N(t) = \sum_{i=1}^{m} N_i(t),$$

with the convention that $N(0) = 0, N_i(0) = 0$. Moreover, define $J_{N(t)}$ as the type of the last claim that occurred before or on t, and thus the aggregate claims process can be expressed as

$$L(t) = L_{N(t)} = \sum_{k=1}^{N(t)} X_k,$$

which is the same form as the classical aggregate claims process Eq. (4.1). Moreover, suppose that the embedded Markov Chain $\{J_n; n \ge 0\}$ is ergodic and that a sequence of unique probabilities $(\Pi_1, ..., \Pi_M)$ exists, which represents the stationary probability distribution, $\Pi_1 + ... + \Pi_M = 1$ and $\Pi_1, ..., \Pi_M \in [0,1]$.

Proposition 4.1.1 produces the density function of the aggregate loss, which is very useful in the CAT bonds pricing procedure in subsection 4.1.3.

Proposition 4.1.1. *Let $F_1(t,D)$ denotes the probability function that aggregate claims $L(t)$ which are less than or equal to the threshold D, at time t. Then,*

$$F_1(t,D) = \sum_{i=1}^{m} \sum_{j=1}^{m} \Pi_i \sum_{n=0}^{\infty} \int_0^t (1 - H_j(t - t', \infty)) d[{}^{A}Q_{ij}^{*n}(t') G_{ij}^{*n}(\infty, D)].$$

Proof. Starting with the stationary probability for J_0, Eq. (4.2) in Janssen (1980) gives

$$F_1(t,D) = \mathbb{P}(\sum_{k=1}^{N(t)} X_k \le D) = \sum_{i=1}^{m} \sum_{j=1}^{m} \Pi_i \mathbb{P}(\sum_{k=1}^{N(t)} X_k \le D, J_{N(t)} = j | J_0 = i).$$

Furthermore, according to Chapter 7, Eq. (3.32) in Janssen and Manca (2007), the following equality holds:

$$\mathbb{P}(\sum_{k=1}^{N(t)} X_k \le D, J_{N(t)} = j | J_0 = i) = \sum_{n=0}^{\infty} \int_0^t (1 - H_j(t - t', \infty)) dQ_{ij}^{*n}(t', D),$$

and the result follows by simple substitution. $\square$

Introduce the SM'/SM model as a particular case of the previous model, and the matrix $\mathbf{G}(t,\infty)(= G_{ij}(t,\infty), i,j \in J)$ is defined as:

$$G_{ij}(t,\infty) = \begin{cases} 0, \ t < 0 \\ 1 - e^{-\lambda_i t}, \ t \geq 0. \end{cases}$$

Thus, the distribution function of the sojourn time depends uniquely on the current state i, which is exponentially distributed with the parameter λ_i. Furthermore, assume that the Markov chain jumps to state j at each claim instance with a claim size distribution of $F_j(D) = \mathbb{P}_j(X_k \leq D)$. This has a practical meaning because a bigger catastrophic event can trigger many other events as side effects. Formally, we have the following assumptions:

$$G_{ij}(t,\infty) = G_i(t,\infty), G_{ij}(\infty, D) = G_j(\infty, D) = F_j(D), i,j \in J, t, x > 0.$$

More precisely, the process $\{J_n, U_n, X_n; n \geq 0\}$ has the following probabilistic structure:

$$\begin{aligned} Q_{ij}(t,D) &= \mathbb{P}[J_n = j, U_n \leq t, X_n \leq D | (J_k, U_k, X_k), k = 1,2,...,n-1, J_{n-1} = i] \\ &= \mathbb{P}[J_1 = j, U_1 \leq t, X_1 \leq D | J_0 = i] \\ &= p_{ij} F_j(D)(1 - e^{-\lambda_i t}), \end{aligned}$$

$\forall t, D \in \mathbb{R}, \forall i, j \in J$. Thus, J_n, W_n, and X_n are independent of the past given J_{n-1}, and the sequences $\{U_n, n \geq 1\}$, $\{X_n, n \geq 1\}$ are conditionally independent given the sequence $\{J_n, n \geq 0\}$. Rewrite the Eq. (4.9), Eq. (4.10), and Eq. (4.6) as:

$$\begin{aligned} {}^{A}Q_{ij}^{*n}(t) &= (p_{ij}(1 - e^{-\lambda_i t}))^{*n}, \\ G_{ij}^{*n}(\infty, D) &= \frac{(p_{ij} F_j(D))^{*n}}{p_{ij}^{*n}}, \\ H_j(t,\infty) &= \sum_{i=1}^{m} p_{ji}(1 - e^{-\lambda_j t}) = 1 - e^{-\lambda_j t}. \end{aligned}$$

Substituting in Proposition 4.1.1, following corollary can be easily obtained.

Corollary 4.1.1. *At time t, the probability that the total loss amount $L(t)$ is less than or equal to the predefined level D can be computed as:*

$$F_2(t,D) = \sum_{i=1}^{m} \sum_{j=1}^{m} \Pi_i \sum_{n=0}^{\infty} \int_0^t e^{-\lambda_j (t-t')} d \left[\frac{(p_{ij}(1 - e^{-\lambda_i t'}))^{*n} (p_{ij} F_j(D))^{*n}}{p_{ij}^{*n}} \right].$$

Remark 4.1.1. *For $m = 1$, this model is the classical Poisson process model with parameter λ. It is also possible to have the matrix $\boldsymbol{G}(t, \infty)$ as:*

$$G_{ij}(t, \infty) = \begin{cases} 0, \; t < 0 \\ 1 - e^{-\lambda_i(t)t}, \; t \geq 0, \end{cases}$$

where $\lambda_i(t)$ represents the intensity of the Poisson point process in state i, $i \in J$. Therefore, if we assume that $m = 1$ in this example, the model will reduce to a model that employs the number-of-claims process $\{N(t) : t \in [0, T]\}$ using a nonhomogeneous Poisson process (NHPP) with parameters $\lambda(t) > 0$, as utilized by Ma and Ma (2013). One can easily show that the probability of aggregate claims $L(t)$ less than or equal to the threshold D, at time t is equal to:

$$F_0(t, D) = \sum_{n=0}^{\infty} e^{-\lambda(t)t} \frac{(\lambda(t)t)^n}{n!} F^{*n}(D), \tag{4.11}$$

where $F^{*n}(x) = \mathbb{P}(X_1 + X_2 + \cdots + X_n \leq x)$ denotes the n-fold convolution of F, which is the same as Eq. (21) in Ma and Ma (2013).

4.1.3 Pricing Model For The CAT Bonds

In this subsection, we show how to price CAT bonds using the standard tool of a risk-neutral valuation measure with the following payoff functions for T time maturity one-period CAT bonds[23]. Their valuation is a consequence of Eq. (2.4).

Defining a hypothetical zero coupon CAT bond at the maturity date, as follows:

$$P_{CAT}^{(1)} = \begin{cases} Z, \; \text{for } L(T) \leq D, \\ rpZ, \; \text{for } L(T) > D, \end{cases} \tag{4.12}$$

where $L(T)$ is the total insured loss value at the expiry date T, D denotes the threshold value agreed in the bond contract, and rp ($rp \in [0, 1)$) is the fraction of the principle Z, which the bondholders must pay when a trigger event occurs.

The next payoff function with a multi-threshold value is given by the equation

$$P_{CAT}^{(2)} = rp_k Z \quad \forall D_{k-1} < L(T) \leq D_k, \tag{4.13}$$

[23]We only discuss one-period bonds in this study because multi-period coupon bonds can be treated as a portfolio of zero-coupon bonds with different maturities.

where $k = 1, 2, \ldots, h$ with $rp_1 = 1 > rp_2 > \cdots > rp_h \geq 0$ and $D_0 = 0 < D_1 < \cdots < D_h = D$. In general, an investor's rate of return is inversely proportional to the total catastrophe claims.

Another payoff function with a coupon payment at the maturity date, if the trigger has not occurred, is of the form

$$P_{CAT}^{(3)} = \begin{cases} Z + C, \text{ for } L(T) \leq D, \\ Z, \text{ for } L(T) > D, \end{cases} \tag{4.14}$$

where $C > 0$ is the coupon payment level.

In order to introduce the final payoff function, consider a CAT bond issuer with the asset value A_{issue} and debt value B_{issue} at the bond maturity time. Define the default risk as the risk when the sponsor is unable to pay their obligations (i.e. the premium of the CAT bond). Thus, a CAT bondholder would not receive the full amount of capital even if the aggregate loss is less than the predetermined level. Let $\{N_{issue} : N_{issue} \geq 0\}$ be the number of this issued CAT bond. Furthermore, assume that the issuer's financial situation is independent of the aggregate industry-estimated catastrophic loss process. If the issuing company obtains sufficient funds to pay the bondholders at the maturity date T, the face value Z will be paid on the condition that a trigger event has not occurred, or a proportion will be paid according to the principle employed. If the issuing company fails to meet its obligation, the bondholders will lose all of their capital. More precisely, the structure of the defaultable payoff function is equal to

$$P_{CAT}^{(4)} = \begin{cases} Z, \text{ if } L(T) \leq D \text{ and } A_{issue} > B_{issue} + ZN_{issue} \\ rpZ, \text{ if } L(T) > D \text{ and } A_{issue} > B_{issue} + rpZN_{issue} \\ 0, \text{ otherwise.} \end{cases} \tag{4.15}$$

In this case, the payoff of the CAT bond depends not only on the listed catastrophic events, but also on the issuer's financial position. However, at this stage, we are not interested in the performance of the issuing company throughout the trading period.

According to the payoff structures of the CAT bonds Eqs (4.12)–(4.15), the interest rate dynamics Eq. (2.5) and the aggregate loss process Eq. (4.1), we present the prices of the CAT bonds in Theorem 4.1.1–4.1.4. These are the main results of this study.

Zero-coupon CAT bond prices at time t when paying principal Z at the time of maturity T with payoff function Eq. (4.12) is shown in Theorem 4.1.1.

Theorem 4.1.1. *Let $V_\ell^{(1)}(t)$ ($\ell = 0, 1, 2$) be the prices of the T-maturity zero-coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with payoff function $P_{CAT}^{(1)}$, as defined in Eq.* (4.12)*. Then,*

$$V_\ell^{(1)}(t) = B_{CIR}(t,T)Z(rp + (1-rp)F_\ell(T-t,D)),\ \ell = 0,1,2,$$

where $F_\ell(T-t,D)$ represents the accumulated function of the aggregate loss in the alternative models given in Proposition 4.1.1, Corollary 4.1.1, and Remark 4.1.1, respectively, and the pure discounted bond price $B_{CIR}(t,T)$ with the CIR interest rate model is given by Eqs (2.8)–(2.11).

Proof. Cox and Pedersen (2000) suggested that the payoff function is independent of the financial risks variable (interest rate) under the risk-neutral measure $\mathbb{Q}$. Then, according to Eq. (2.4), we have

$$V_\ell^{(1)}(t) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s ds} P_{CAT}^{(1)}(T)|\mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s ds}|\mathcal{F}_t)\mathbb{E}^{\mathbb{Q}}(P_{CAT}^{(1)}(T)|\mathcal{F}_t).$$

Using the result of the zero-coupon bond price with the CIR interest rate model, as discussed in Section 4.2 we have $\mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s ds}) = B_{CIR}(t,T)$. With Eq. (2.3), the above equation can be written as

$$B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}(P_{CAT}^{(1)}(T)|\mathcal{F}_t).$$

By simply applying the payoff function Eq. (4.12) and rearranging the formula, the CAT bond price can be formulated as

$$\begin{aligned} V_\ell^{(1)}(t) =& B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}(Z\mathbb{1}\{L(T) \le D\} + rpZ\mathbb{1}\{L(T) > D\}|\mathcal{F}_t) \\ =& B_{CIR}(t,T)(Z\mathbb{P}(L(T) \le D) + rpZ\mathbb{P}(L(T) \ge D)) \\ =& B_{CIR}(t,T)Z(F_\ell(T,D) + rp(1 - F_\ell(T,D))), \end{aligned}$$

where $\ell = 0, 1, 2$ and the result follows. □

Similarly, in the next theorem, we compute the value of the zero-coupon CAT bond at time t when paying principal Z at the time of maturity T, with payoff function Eq. (4.13) determining by the amount of the aggregate claims.

Theorem 4.1.2. *Let $V_\ell^{(2)}(t)$ ($\ell = 0, 1, 2$) be the price of the T-maturity zero-coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with the payoff function $P_{CAT}^{(2)}$, as defined in Eq.* (4.13)*. Then,*

$$V_\ell^{(2)}(t) = B_{CIR}(t,T)Z\sum_{k=1}^{h} rp_k(F_\ell(T-t, D_k) - F_\ell(T-t, D_{k-1})),\ \ell = 0, 1, 2,$$

where $F_\ell(T-t, x)$ represents the accumulated function of the aggregate loss in the alternative models given in Proposition 4.1.1, Corollary 4.1.1, and Remark 4.1.1, respectively, and the pure discounted bond price $B_{CIR}(t,T)$ with the CIR interest rate model is given by Eqs (2.8)–(2.11)*.*

Proof. Similar to the proof in Theorem 4.1.1, let the payoff function follow Eq. (4.13), and can easily obtain that

$$\begin{aligned} V_\ell^{(2)}(t) =& B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}(\sum_{k=1}^{h} Zrp_k\mathbb{1}\{D_{k-1} < L(T) \le D_k\}|\mathcal{F}_t) \\ =& B_{CIR}(t,T)(Z\sum_{k=1}^{h} rp_k\mathbb{P}(D_{k-1} < L(T) \le D_k)) \\ =& B_{CIR}(t,T)Z\sum_{k=1}^{h} rp_k(F_\ell(T, D_k) - F_\ell(T, D_{k-1})), \end{aligned}$$

where $\ell = 0, 1, 2$ and the result follows. □

In the next theorem, we show that the value of the coupon CAT bond at time t when paying principal Z and a coupon C at the time to maturity T depends on the payoff function Eq. (4.14).

Theorem 4.1.3. *Let $V_\ell^{(3)}(t)$ ($\ell = 0, 1, 2$) be the price of the T-maturity coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with the payoff function $P_{CAT}^{(3)}$, as defined in Eq.* (4.14)*. Then,*

$$V_\ell^{(3)}(t) = B_{CIR}(t,T)(Z + CF_\ell(T-t, D)),\ \ell = 0, 1, 2,$$

where $F_\ell(T-t, x)$ represents the accumulated function of the aggregate loss in the alternative models given in Proposition 4.1.1, Corollary 4.1.1, and Remark 4.1.1, respectively, and the pure discounted bond price $B_{CIR}(t,T)$ with the CIR interest rate model is given by Eqs (2.8)–(2.11)*.*

Proof. Similar to the proof in Theorem 4.1.1 and if we let the payoff function follow Eq. (4.14), we can easily obtain

$$\begin{aligned} V_\ell^{(3)}(t) =& B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}((Z+C)\mathbb{1}\{L(T)\le D\} + Z\mathbb{1}\{L(T)>D\}|\mathcal{F}_t) \\ =& B_{CIR}(t,T)((Z+C)\mathbb{P}(L(T)\le D) + Z\mathbb{P}(L(T)\ge D)) \\ =& B_{CIR}(t,T)((Z+C)F_\ell(T,D) + Z(1-F_\ell(T,D))) \end{aligned}$$

where $\ell = 0,1,2$ and the result follows. □

In the next theorem, we show that the price of the zero-coupon CAT bond at time t when paying principal Z at time to maturity T depends on the amount of the aggregate claims, which is also associated with the probability of the issuing company defaulting at time T.

Theorem 4.1.4. *Let $V_\ell^{(4)}(t)$ ($\ell = 0,1,2$) be the price of the T-maturity zero-coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with the payoff function $P_{CAT}^{(4)}$, as defined in Eq.* (4.15). *Then*

$$V_\ell^{(4)}(t) = B_{CIR}(t,T)Z[rp + (1-rp-\tilde{F}(Z)-rp\tilde{F}(rpZ))F_\ell(T-t,D)) + p\tilde{F}(rpZ)],$$

where $\ell = 0,1,2$ and $F_\ell(T-t,D)$ represents the accumulated function of the aggregate loss in the alternative models given in Proposition 4.1.1, Corollary 4.1.1, and Remark 4.1.1, respectively, and the pure discounted bond price $B_{CIR}(t,T)$ with the CIR interest rate model is given by Eqs (2.8)–(2.11). *$\tilde{F}(x)$ denotes the issuing company's default probability at time T and*

$$\tilde{F}(x) = \mathbb{P}(\frac{A_{issue} - B_{issue}}{N_{issue}} \le x).$$

Proof. Similar to the proof in Theorem 4.1.1, we have

$$V_\ell^{(4)}(t) = B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}(P_{CAT}^{(3)}(T)|\mathcal{F}_t).$$

Let the payoff function follow Eq. (4.15) and denote $M = \frac{A_{issue}-B_{issue}}{N_{issue}}$. According to the assumption that the default risk and catastrophe risk are independent, i.e. $L(T)$ and M are independent under the measure $\mathbb{P}$, the following equalities hold:

$$\mathbb{E}^{\mathbb{P}}(P_{CAT}^{(4)}(T)|\mathcal{F}_t) = \mathbb{E}^{\mathbb{P}}[Z\mathbb{1}\{L(T)\le D, A_{issue} > B_{issue} + ZN_{issue}\}$$

$$
\begin{aligned}
&+ rpZ\mathbb{1}\{L(T) > D, A_{issue} > B_{issue} + rpZN_{issue} + 0\} \\
=&Z\mathbb{P}(L(T) \leq D, M > Z) + rpZ\mathbb{P}(L(T) > D, M > rpZ) \\
=&Z\mathbb{P}(L(T) \leq D)\mathbb{P}(M > Z) + rpZ\mathbb{P}(L(T) > D)\mathbb{P}(M > rpZ),
\end{aligned}
$$

where $\ell = 0, 1, 2$ and the result follows. □

4.2 Numerical Analysis

In this section, we compute the value of the CAT bonds modelled in Section 4.1 with face value Z =US\$1 at time $t = 0$. In order to apply pricing formulas to the real world and to obtain the CAT bond prices, we need to compute the exact distribution of the aggregate loss $F_\ell(T, D)$ ($\ell = 0, 1, 2$). However, as in Ma and Ma (2013), this is extremely difficult to calculate because the closed form solutions of these high-order convolutions are not available. Therefore, we employ Monte Carlo simulations for the analysis and we approximate the CAT bonds prices via numerical computation.

We calculate the CAT bond price where the spot interest rate process followed the CIR model. In this experiment, we employ 3-month maturity US monthly Treasury bill data (1994 – 2013)[24] to estimate the parameters of the CIR model[25]. Based on the MLE method, we conclude that both the initial short-term interest rate r_0 and the long-term mean interest rate θ were 2.04% annually, the mean-reverting force $k = 0.0984$, and the volatility parameter $\sigma = 4.77\%$. Furthermore, we assume that the market price of risk λ_r was a constant -0.01.

In actuarial research, an event is referred to as catastrophic if it occurs with a low probability and it causes severe damage. Empirical studies are conducted for the data provided by ISO's PCS unit, which describe insured property losses in the USA caused by catastrophic events over a predetermined threshold that occurred between 1985 and 2013. And then inflation is adjusted for a set of 870 original loss data using the CPI.

[24]It is not necessary to use a dataset with the same time period as the PCS data because the financial risks and catastrophe risks are independent, Cox and Pedersen (2000).

[25]Detailed information in sub-Section 2.3.2.

Figure 4.1 shows the annual adjusted PCS loss and the total annual number of qualified catastrophes between 1985 and 2013. The 20 most expensive insured CAT losses are listed in Table 4.1. (An illustration of the individual CAT loss is shown in Figure 1.1, where the peaks in the figure represent the most costly events.) Thus, we can conclude that the PCS loss data are heavy-tailed, see Ma and Ma (2013).

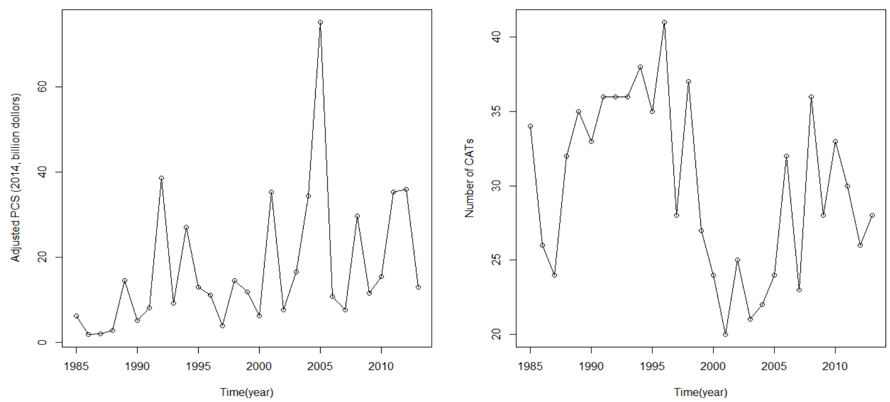


Figure 4.1: PCS annual catastrophe losses (left) and the number of catastrophes (right) in the USA during 1985 – 2013.

In this study, details of the processes used for parameter estimation and the non-parametric tests is omitted[26]. We fit the distribution of PCS losses by the general extreme value (GEV, as discussed in Section 2.4) distribution with the following parameters: shape parameter = 0.9273133, location parameter = 10.2718058, and scale parameter = 10.6295782, which we compare with the next best fit lognormal distribution with the parameters: $\mu_2 = 2.858557$ and $\sigma_2 = 1.26377$. In the classical non-homogenous model, by applying the nonlinear least squares procedure, we conclude that the quantity of loss process could be modelled as an inhomogeneous Poisson process with intensity $\lambda(s) = 31.067647 - 1.122352 \sin^2(s - 0.473033) + 1.167737 \exp\{\cos(\frac{2\pi s}{7.704062})\}$. This allows us to model the catastrophic data in changing economic or natural environments. Figure 4.2 shows a real catastrophe loss tra-

[26]The choice of the distribution is very important because it varies the bond price. Readers can refer to Ma and Ma (2013) for details of the use of MLE to estimate parameters and selecting the best fit model with nonparametric tests.

Table 4.1: The 20 most costly insured CAT losses in the USA during 1985 – 2013.

Event	Date	PCS loss (US$ billion)	2014 dollars (US$ billion)
Hurricane Katrina	25/08/2005	41.10	49.56
Hurricane Andrew	24/08/1992	15.50	26.02
Terrorist attacks	11/09/2001	18.78	24.97
Northridge Earthquake	17/01/1994	12.50	19.86
Hurricane Sandy	28/10/2012	18.75	19.23
Hurricane Ike	12/09/2008	12.50	13.67
Hurricane Wilma	24/10/2005	10.30	12.42
Hurricane Charley	13/08/2004	7.47	9.32
Hurricane Ivan	15/09/2004	7.11	8.86
Hurricane Hugo	17/09/1989	4.20	7.97
Wind and Thunderstorm Event	22/04/2011	7.30	7.64
Wind and Thunderstorm Event	20/05/2011	6.90	7.22
Hurricane Rita	20/09/2005	5.63	6.79
Hurricane Frances	03/09/2004	4.59	5.73
Hurricane Jeanne	15/09/2004	3.65	4.56
Hurricane Irene	26/08/2011	4.30	4.50
Hurricane Georges	21/09/1998	2.96	4.27
Wind and Thunderstorm Event	02/05/2003	3.21	4.10
Tropical Storm Allison	05/06/2001	2.50	3.32
Hurricane Opal	04/10/1995	2.10	3.25

jectory (in green) and sample trajectories of the aggregate claims process generated under the assumptions of a GEV distribution (red) and a lognormal distribution (blue) with non-homogeneous Poisson intensity, respectively, between 1985 and 2013. This suggests that the GEV distribution fitted better to the long-term real-world aggregate loss process.

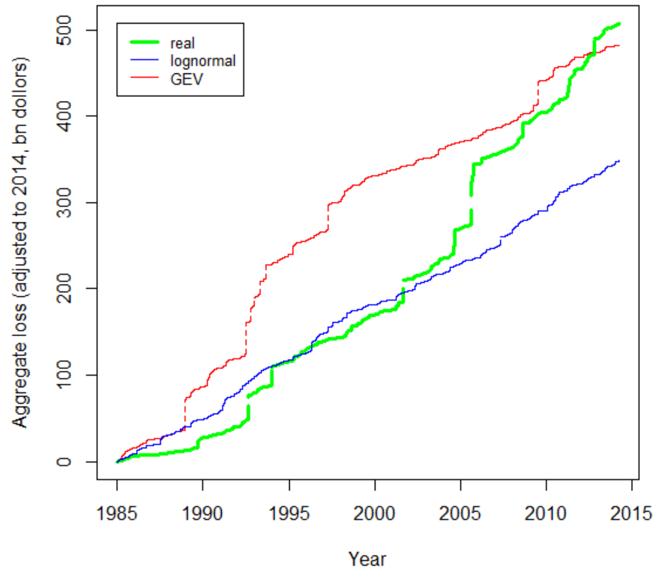


Figure 4.2: Sample trajectories of the aggregate loss process in the USA during 1985–2013.

In order to analyse the semi-Markov process model [27], assuming that we are working in a two-state (m=2) environment, i.e. a many claims period (state 1, a stormy season with claim frequency λ_1, Siegl and Tichy (1999)) and a few claims period (state 2 with claim frequency λ_2). Define a period as a stormy season (or many claims period) based on the following conditions:

1. more than one claim per month during each month of the stormy season;
2. the next claim after the stormy season occurred at least 10 days after the last claim in the stormy season;
3. the first claim in the stormy season occurred at least 10 days after the previous claim;
4. the gap between two stormy seasons (i.e. a non-stormy season or a few claims period) lasted at least 3 months;

[27] The SM'/SM model is a special case of a general Markov model, so in this part of the application, we considered the SM'/SM model as an example.

5. less than one claim per month during non-stormy seasons.

By analysing the dates of occurrence for the PCS loss data, we can observe that there were 19 stormy seasons and the parameters of the model are given in Table 4.2.

Table 4.2: Parameters of the semi-Markov process model.

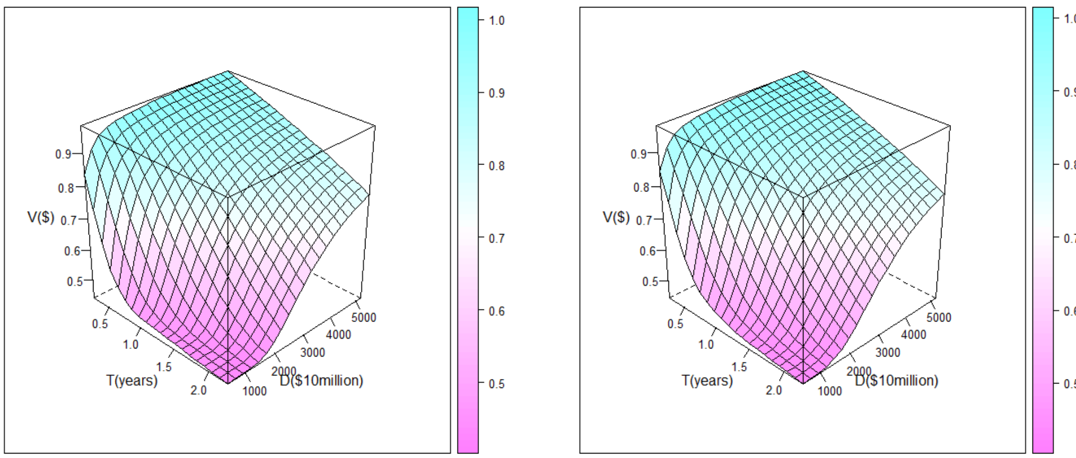
Parameters		State 1	State 2
GEV distribution	k	0.9417813	0.7368964
	σ_3	10.5249838	12.8791786
	μ	10.0954148	11.6702931
Lognormal distribution	μ	2.8487725	2.9874937
	σ_3	1.2730741	1.1261473
Intensity of Poisson process	λ	34.967753	10.520688
Transition probabilities	p_{1j} for $j \in 1, 2$	0.9767442	0.0232558
	p_{2j} for $j \in 1, 2$	0.3064516	0.6935484

For each model, we obtain the $T \in [0.25,\, 2.25]$ years maturity zero-coupon CAT bond prices in Monte-Carlo simulations[28]. For the payoff functions Eq. (4.12) and Eq. (4.15), we assumed that $rp = 0.5$ when the aggregate loss $L(T)$ exceeds the threshold level $D \in [434, 5210]$ US$10 million, i.e. the threshold level in the interval of quarterly to three times the annual average loss. In addition, for the payoff function Eq. (4.13), arbitrarily set the following parameters: $h = 3$, $rp_1 = 1$, $rp_2 = 0.5$, $rp_3 = 0.25$, $D_1 = 434$ US$10 million, $D_2 \in [434, 5210]$ US$10 million, and $D_3 = \infty$. For a real-world CAT bond, the issuing company might use a multi-threshold payoff structure to reduce the risk of investment and to attract more investors. Furthermore, the probabilities of the issuing company defaulting at time T were $\tilde{F}(Z) = 0.1$ and $\tilde{F}(rpZ) = 0.05$. Finally, we assume that the coupon payment rate in Eq. (4.14) is US$ 0.1.

[28]For **R** code for valuing CAT bond, see Appendix C.4.

Figure 4.3 illustrates the CAT bond prices for the payoff functions $P_{CAT}^{(1)}$ with the threshold level D and time to maturity T under the stochastic interest rate assumptions. We show the CAT bond values for the accumulated distributed function of the classical aggregate loss process $F_0(t, D)$ given in Remark 4.1.1, where the loss distribution followed the GEV distribution and the intensity of the claims was a non-homogenous Poisson process in Figure 4.3a. In Figure 4.3b, we show the CAT bond prices where the c.d.f. of the aggregate loss process $F_2(t, D)$ follows the SM'/SM model given in Corollary 4.1.1. With similar settings, Figure 4.3 and Figures 4.4 – 4.6 illustrate the CAT bonds prices with the GEV distribution for the payoff functions $P_{CAT}^{(2)}$, $P_{CAT}^{(3)}$, and $P_{CAT}^{(4)}$, respectively. Using the payoff function $P_{CAT}^{(1)}$ as an example, the price differences between the CAT bond prices with the classical and SM'/SM models are shown in Figure 4.7a under the GEV, the NHPP, and stochastic interest rates assumptions. In Figure 4.7b, we show how the bond prices are affected by the distribution of the severity of the losses (lognormal and GEV distributions). The differences are particularly evident in the tails (higher threshold level); therefore, a heavy-tailed distribution is a more appropriate choice for modelling catastrophe loss, as demonstrated by Ma and Ma (2013).

Figures 4.3 – 4.6 show that there are few differences in shape between the different aggregate loss models because we used the same dataset. In general, the CAT bond price decreases as the maturity time and threshold level increases. By comparing the different payoff functions, it is clear that CAT bond prices decreased with increasing threshold and when the default risk is added to the payoff function, while the coupon CAT bonds has higher prices compared with the zero-coupon CAT bonds. This indicates that the choice of different payoff functions has a major impact on the CAT bond prices. According to Figure 4.7a, the differences in the bond price change significantly, by as much as 3.5%. We notice that the prices in our model are slightly higher than those in the model of Ma and Ma (2013). This might because our model has more information (longer estimate period) for both catastrophe risks and financial risks, and this might also because we make the model more realistic by considering the dependency between the claim size and intensity. We can protect the sponsors of the CAT bonds from the default risks with a higher bond price. Our results also demonstrate that the choice of the aggregate loss process model affects the bond prices. An illustration of the characteristic CAT bond prices is presented in Table 4.3. The case using GEV decrease at a faster rate than the lognormal case. This is an interesting result because the trajectory of the aggregate loss process GEV distribution was always larger than the lognormal distribution process, as shown in Figure 4.2. Our model with a GEV distribution yields a fairer price than others in the market.



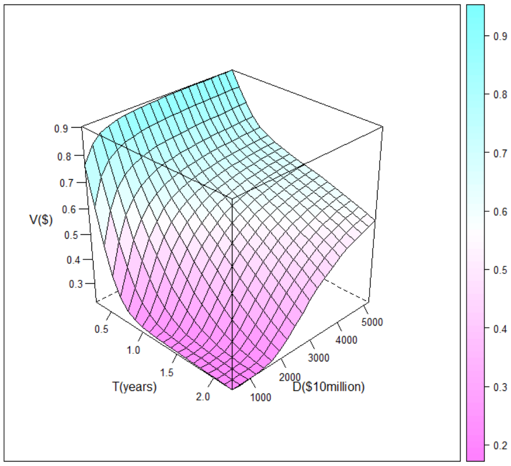
(a) $V_0^{(1)}(t)$ (based on Ma and Ma (2013) model) with GEV distribution.

(b) $V_2^{(1)}(t)$ (based on our model) with GEV distribution.

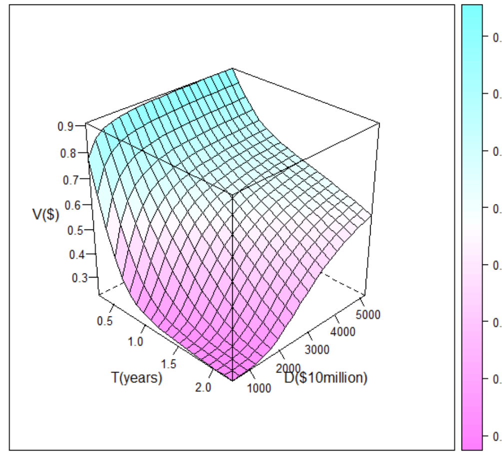
Figure 4.3: CAT bonds prices (z-coordinate axes) for the payoff function $P_{CAT}^{(1)}$ under the GEV, the NHPP, and stochastic interest rate assumptions. The time to maturity (T) decreases on the left axes and threshold level (D) increases on the right axes.

4.3 Summary

This chapter developed a contingent claim process to price CAT bonds using models with a risk-free spot interest rate under assumptions of a no-arbitrage market, independently of the financial risks and catastrophe risks, as well as the possibility of replicated interest rate changes with existing financial instruments. Under the risk-neutral pricing measure, bond price formulae is derived for four types of payoff functions (the classic zero coupon, the multi-threshold zero coupon, the defaultable zero coupon, and the

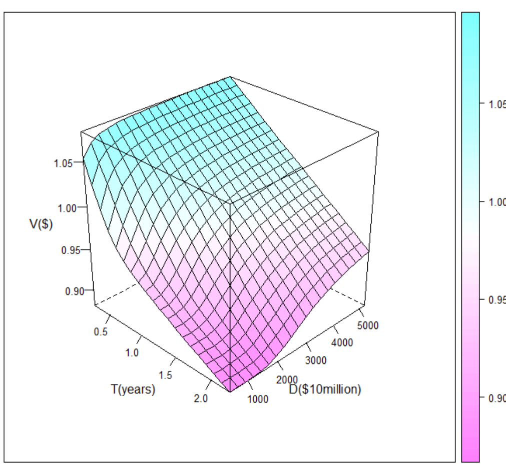


(a) $V_0^{(2)}(t)$ (based on Ma and Ma (2013) model) with GEV distribution.

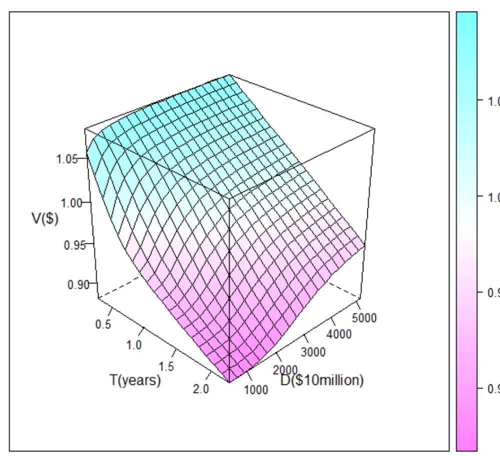


(b) $V_2^{(2)}(t)$ (based on our model) with GEV distribution.

Figure 4.4: CAT bonds prices (z-coordinate axes) for the payoff function $P_{CAT}^{(2)}$ under the GEV, the NHPP, and stochastic interest rate assumptions. The time to maturity (T) decreases on the left axes and the threshold level (D) increases on the right axes.

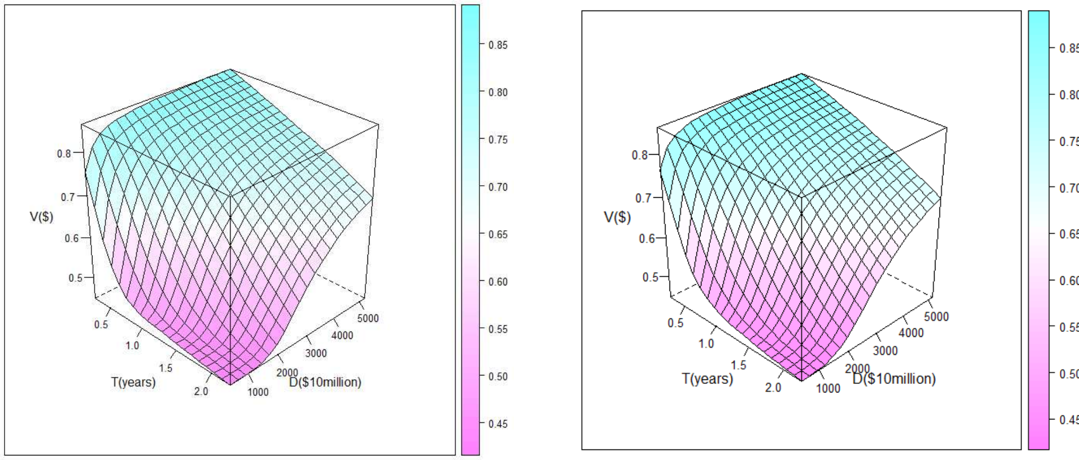


(a) $V_0^{(3)}(t)$ (based on Ma and Ma (2013) model) with GEV distribution.



(b) $V_2^{(3)}(t)$ (based on our model) with GEV distribution.

Figure 4.5: CAT bonds prices (z-coordinate axes) for the payoff function $P_{CAT}^{(3)}$ under the GEV, the NHPP, and stochastic interest rate assumptions. The time to maturity (T) decreases on the left axes and the threshold level (D) increases on the right axes.



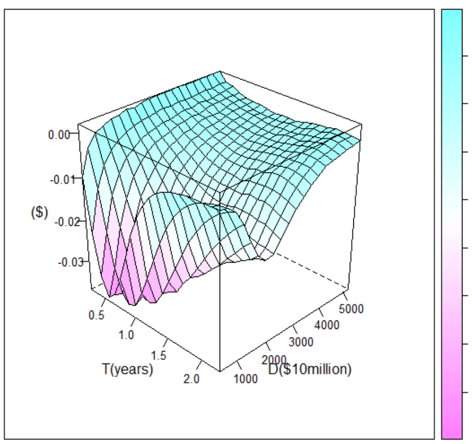
(a) $V_0^{(4)}(t)$ (based on Ma and Ma (2013) model) with GEV distribution.

(b) $V_2^{(4)}(t)$ (based on our model) with GEV distribution.

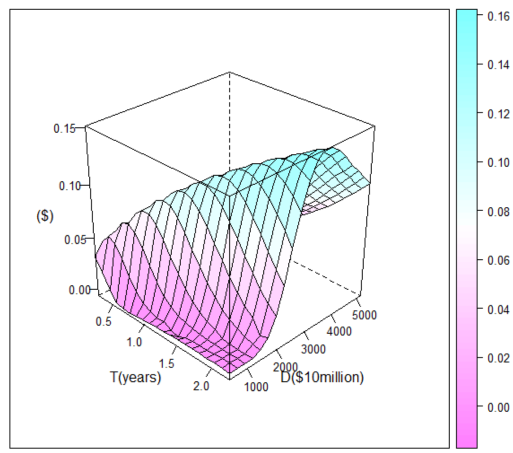
Figure 4.6: CAT bonds prices (z-coordinate axes) for the payoff function $P_{CAT}^{(4)}$ under the GEV, the NHPP, and stochastic interest rate assumptions. The time to maturity (T) decreases on the left axes and the threshold level (D) increases on the right axes.

coupon payoff functions) when trigger is determined by the aggregate loss process with a semi-Markov-dependent structure. Here the spot interest rate followed CIR model and the inter-arrival time followed an exponential distribution.

The numerical experiments utilized Monte Carlo simulations with data from the PCS loss index in the USA during 1985 – 2013. The numerical analyses showed that the CAT bond prices decreased as the threshold level decreased, as the time to maturity increased, and with the existence of a default probability. The CAT bond prices increased after the introduction of coupons. Furthermore, we showed that the choice of the fitted loss severity distribution had a great impact on the bond prices. The additional dependency between the claim sizes and the claim inter-arrival times is a significant factor when pricing CAT bonds, thereby yielding higher and fairer CAT bond prices.



(a) Differences between $V_0^{(1)}$ (based on Ma and Ma (2013) model) and $V_2^{(1)}$ with GEV.



(b) Differences between the lognormal and GEV distribution of $V_2^{(1)}$.

Figure 4.7: Differences (z-coordinate axes) in the CAT bond prices for $P_{CAT}^{(1)}$ under the GEV (or lognormal), the NHPP, and stochastic interest rate assumptions. The time to maturity (T) decreases on the left axes and the threshold level (D) increases on the right axes.

Table 4.3: Characteristic CAT bond prices for the payoff function $P_{CAT}^{(1)}$.

Maturity	Threshold	Ma and Ma (2013) model	Our model	
T	D	$V_0^{(1)}$ with GEV	$V_2^{(1)}$ with GEV	lognormal
0.25	434.2084	0.8364458	0.8462135	0.8782797
0.25	2948.0462	0.9728978	0.9710456	0.9829963
0.25	5210.5003	0.9788091	0.9767331	0.9827252
0.6710526	434.2084	0.527049	0.5616225	0.5572698
0.6710526	2948.0462	0.9184668	0.9224868	0.9589832
0.6710526	5210.5003	0.9415366	0.9434254	0.9599925
1.0921053	434.2084	0.4704685	0.4830596	0.4817231
1.0921053	2948.0462	0.8501765	0.8581766	0.9336625
1.0921053	5210.5003	0.9022568	0.9050835	0.936414
1.5131579	434.2084	0.4583024	0.4618092	0.4619588
1.5131579	2948.0462	0.7685499	0.7775374	0.8952628
1.5131579	5210.5003	0.8606659	0.8629707	0.914194
2.25	434.2084	0.4403813	0.441893	0.4420322
2.25	2948.0462	0.6046435	0.6143239	0.745787
2.25	5210.5003	0.7750711	0.7765762	0.8780913

Chapter 5

Towards Resilience to Nuclear Accidents: Financing Nuclear Liabilities via Catastrophe Risk Bonds

In light of the 2011 Fukushima disaster, recent discussion has focused on maximizing the oversight power of global institutions and strengthening safety measures. In addition to these, the development of dependable liability coverage that can be tapped in an emergency is also needed and should be considered thoughtfully. To succeed, financing is essential using special purpose instruments from the global bond market which is as big as US$175 trillion. In the global financial market nuclear CAT risk bonds are nonexistent, which specifically are designed for covering losses from nuclear disasters, like the 2011 Fukushima disaster. Expanding the use of nuclear power, particularly in emerging markets, could contribute towards addressing global climate change and sustainability concerns. This expansion can be facilitated by nuclear CAT risk bonds covering nuclear related perils. It shifts the liability to the market and helps this sector to grow through increased participation of various service and product providers.

In this chapter, for the very first time, a catastrophe risk bond for financing nuclear liability is proposed based on a concept conceived by Ayyub and Parker (2011). The model is fashioned after the CAT bond financial products, and similarly tied to the global bond market. Previous literature focused on one type of coverage (either per-occurrence or annual aggregate). In this chapter a nuclear perils focused CAT risk bond with multi coverage type is proposed. An example of two-coverage type CAT risk bond is Residential Reinsurance 2012 Ltd., on behalf of USAA, which provide per-occurrence coverage for the all perils and also provides coverage on an annual aggregate basis. This is an extension of the previous chapter.

The reminder of this chapter is organized as follows. Section 5.1 presents the pricing model of CAT bonds including: assumptions, aggregate claims processes, and the payoff function. An explicit closed form solution is given for valuing nuclear CAT bond.s Section 5.2 presents a numerical example of nuclear CAT bonds with a discussion of the results. Section 5.3 is the summary of the chapter.

5.1 Modelling N-CAT Risk Bond

According to The International Nuclear Event Scale (INES), IAEA (2013), events are classified on the Scale at 7 levels which can be categorized by three risk layers: *incident* (level 1 to 3), *accident* (level 4 to 6) and *major accident* (level 7). Figure 5.1 provides examples of risk perils for each layer. In this chapter, a nuclear CAT risk bond, termed N-CAT risk bonds, covers all nuclear power plants (104 operating reactors) in US which triggers are determined by the losses due to each peril, is modelled. An incident is defined to include, for instance, strike, failures in safety provisions and lost or stolen highly radioactive sealed sources, where the event with insignificant off-site impact and affordable in-site impact. An accident includes the release of radioactive material, cost of fitting a core machine, etc, which has severe in-site and off-site impact. A major accident is defined to include nuclear reactor core failure with widespread health and environmental effects, such as 2011 Fukushima disaster which caused total economic losses of US$$210$ billion.

5.1.1 Modelling Assumptions

Similar to the previous Chapter 4, let $0 < T < \infty$ be the maturity date of the continuous time trading interval $[0, T]$. The market uncertainty is defined on a filtered

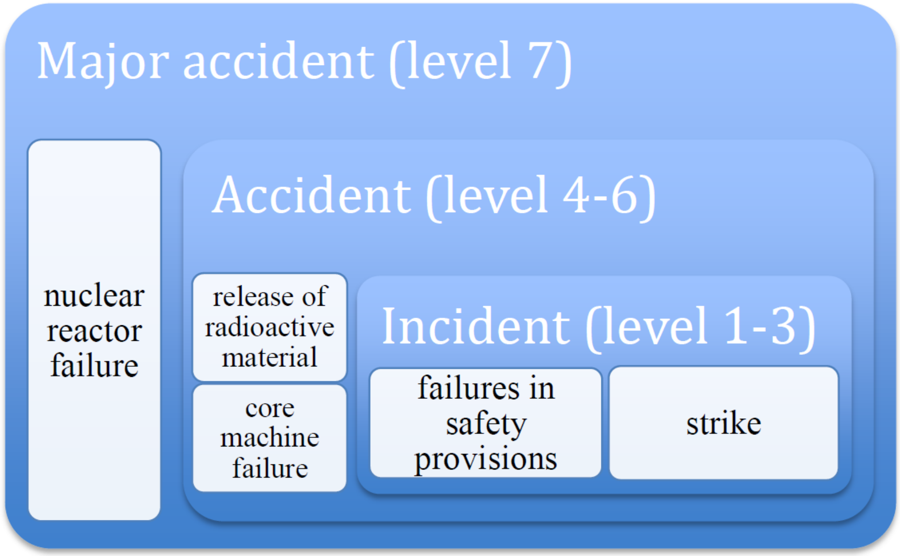


Figure 5.1: The nuclear power risks (five risks as an example) with respect to the risk layers: incident, accident and major accident.

probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where $\mathcal{F}_t$ is an increasing family of σ-algebras given by $\mathcal{F}_t \subset \mathcal{F}$, for $t \in [0, T]$. All stochastic processes and random variables are defined with respect to probability measure $\mathbb{P}$. Denote the CAT bond price process by $\{V(t) : t \in [0, T]\}$, which is characterized by the aggregate loss process $\{L(t) : t \in [0, T]\}$, and the payoff functions P_{CAT}. In addition, the aggregate loss process is determined by the following two processes: claim number process $\{N(t) : t \in [0, T]\}$, which describes the number of claims that occur until the time t, and claim size process $\{X_k : k \in \mathbb{N}^+\}$, which is a sequence of i.i.d. random variables that represent the size of individual claims. Let $\{T_k : k \in \mathbb{N}^+\}$ represent a sequence of epoch times for the claims. In addition, define the spot interest rate process by $\{r(t) : t \in [0, T]\}$ and $\{W(t) : t \in [0, T]\}$ is a standard Brownian motion.

As for the CAT bonds payment structure, CAT bond investors receive premiums (or coupons) if trigger has not been pulled. This chapter utilizes an insurance indemnity, two-coverage type trigger: per-occurrence trigger and aggregate loss trigger.

5.1.2 Aggregate Claims Process

The aggregate loss process is modelled as a compound distribution process, which is characterized by the frequency (claim number process) and the severity (claim amounts process) of catastrophic events, see Klugman et al. (2012), Tse (2009) and Ma and Ma (2013). As an extension of Chapter 4, a perturbed (absorbing) state model is introduced in order to model the per-occurrence trigger. In this model, time before the next claim occurs depends on the state where the system stays, and the system stops (N-CAT bond contract is terminated) when it has jumped to the perturbed state.

The model considers a semi-Markovian dependence structure in continuous time, where the process $\{J_n, n \geq 0\}$ represents the successive type of claims or environment states take their values in $J = \{0, 1, 2, 3, 4\}$. For notation convenience, denote $J' = \{1, 2, 3, 4\}$, therefore, $J = \{0\} + J'$. Here states J' are called work of the system, which refers to the incident and accident risks, and state 0 is the failure of the system (perturbed state) and we refer to the nuclear reactor failure risk in this case. Figure 5.2 shows the possible state changes of the system that contrasted in this paper. The transition matrix $\mathbf{P}(= p_{ij}, i, j \in J)$ can be written as

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ p_{10} & p_{11} & p_{12} & p_{13} & p_{14} \\ p_{20} & p_{21} & p_{22} & p_{23} & p_{24} \\ p_{30} & p_{31} & p_{32} & p_{33} & p_{34} \\ p_{40} & p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix},$$

where $\sum_{j=0}^{4} p_{ij} = 1, i \in J$. To interpret this N-CAT bond more precisely, if an incident or accident level loss occurs, N-CAT bond stay in the period of work of the system (state i, where $i \in J'$), the probability to have a state j ($j \in J$) type risk is p_{ij}. If a state 0 major accident loss occurs, the N-CAT bond contract will terminate immediately, i.e. the system will stay in the state 0.

Define $\{X_n, n \geq 1\}$ to be a sequence of successive claim sizes from all 104 NPPs in US, $X_0 = 0$ a.s. and $X_n > 0, \forall n$, and $\{T_n, n \in \mathbb{N}^+\}$ is the epoch time of the n^{th} claim. Suppose that $0 < T_1 < T_2 < \ldots < T_n < T_{n+1} < \ldots$, $T_0 = U_0 = 0$ a.s., and $U_n = T_n - T_{n-1}$ ($n \in \mathbb{N}^+$) denotes the sojourn time in state J_{n-1}. Suppose that

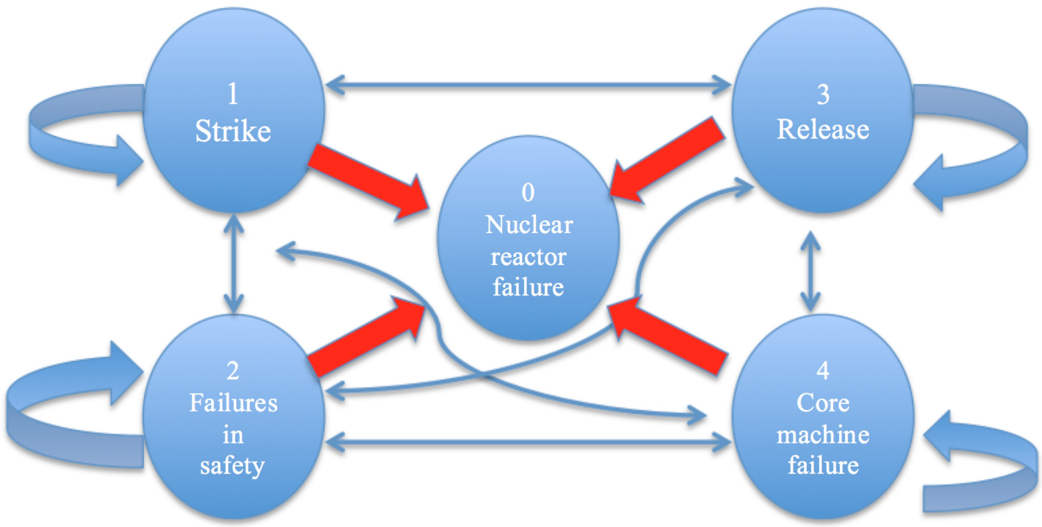


Figure 5.2: Possible states changes of the system with respect to the five listed risks.

the trivariate process $\{(J_n, U_n, X_n); n \geq 0\}$ is a semi-Markovian dependency process defined by the matrix $\mathbf{Q}(= Q_{ij}, i, j \in J)$, which in the same form as Eq. (4.2) in Chapter 4. An explanation of the special case in terms of US N-CAT bonds will be given in the end of this subsection.

In the same vein as in Chapter 4, assuming that the random variable $J_n, n \geq 0$ and the two-dimensional random variable $(U_n, X_n), n \geq 1$ are conditionally independent, then $G_{ij}(t, x) = \mathbb{P}(U_n \leq t, X_n \leq x | J_0, ..., J_{n-1} = i, J_n = j)$ is same as Eq. (4.3). Then the similar definition is given for $G_{ij}(t, \infty)$ and $G_{ij}(\infty, x)$ by Eq. (4.4) and Eq. (4.5), respectively. Suppressing the condition J_n,

$$H_i(t, x) = \mathbb{P}(U_n \leq t, X_n \leq x | J_0, ..., J_{n-1} = i) = \sum_{j=0}^{4} p_{ij} G_{ij}(t, x),$$

and $H_i(t, \infty), H_i(\infty, x)$ is defined by Eq. (4.6) and Eq. (4.7), respectively. Assuming that the sequences $\{U_n, n \geq 1\}$, $\{X_n, n \geq 1\}$ are conditionally independent and given the sequence $\{J_n, n \geq 0\}$, then

$$G_{ij}(t, x) = G_{ij}(t, \infty) G_{ij}(\infty, x), \forall t, x \in \mathbb{R}, \forall i, j \in J.$$

Thus, the semi-Markov kernel $\mathbf{Q}$ can be expressed as the following product

$$Q_{ij}(t, x) = p_{ij} G_{ij}(t, \infty) G_{ij}(\infty, x), \forall t, x \in \mathbb{R}, \forall i, j \in J.$$

Define the claim number process $\{N(t) : t \in [0,T]\}$ ($N(0) = 0$), which describes the number of claims in 104 NPPs which are insured in US. The claim sizes $\{X_k : k \in \mathbb{N}^+\}$ are independent of the process $\{N(t) : t \in [0,T]\}$. Then, the aggregate loss process $\{L(t) : t \in [0,T]\}$ is modelled by a compound Poisson process, as follows:

$$L(t) = \sum_{k=1}^{N(t)} X_k,$$

with the convention that $L(t) = 0$ when $N(t) = 0$. And $J_{N(t)}$ is the state where the last claim stays. Let L_n be the successive total claims amount after the arrival of the n^{th} claim, which is defined in the same form as given in Eq. (4.8). Then, the joint probability of the process $\{(J_n, T_n, L_n); n \geq 0\}$ can be denoted as

$$\mathbb{P}[J_n = j, T_n \leq t, L_n \leq x | J_0 = i] = Q_{ij}^{*n}(t,x),$$
$$\mathbb{P}[J_n = 0, T_n \leq t, L_{n-1} \leq x | J_0 = i] = Q_{i0}^{*n}(t,x),$$

where $i, j \in J'$. This n-fold convolution matrix $\mathbf{Q}^{(n)}(= Q_{ij}^{(n)}, i, j \in J)$ can be valued recursively by the following two parts:

$$\begin{aligned}
Q_{ij}^{*0}(t,x) &= \begin{cases} (1 - G_{ij}(0,\infty))(1 - G_{ij}(\infty,0)), \text{ if } i = j, \\ 0, \text{ elsewhere,} \end{cases} \\
Q_{ij}^{*1}(t,x) &= Q_{ij}(t,x), \quad \ldots \\
Q_{ij}^{*n}(t,x) &= \sum_{l=1}^{4} \int_0^t \int_0^x Q_{lj}^{*(n-1)}(t-t', x-x') dQ_{il}(t',x').
\end{aligned}$$

and

$$\begin{aligned}
Q_{i0}^{*0}(t,x) &= 0, \\
Q_{i0}^{*1}(t,x) &= Q_{i0}(t,x) = G_{i0}(t,\infty)p_{i0}, \quad \ldots \\
Q_{i0}^{*n}(t,x) &= \mathbb{P}[J_n = 0, J_{n-1} = J', ..., J_1 = J', L_{n-1} \leq x, T_n \leq t | J_0 = i] \\
&= \sum_{l=1}^{4} \int_0^t \mathbb{Q}_{il}^{*(n-1)}(t-t', x) d(G_{l0}(t',\infty)p_{l0}),
\end{aligned}$$

where $i, j \in J'$.

Moreover, suppose that there exist a sequence of probabilities $(\Pi_1, \Pi_2, \Pi_3, \Pi_4)$ exists (here we assume $\Pi_0 = 0$, a.s.), which represents the starting probability distribution for the embedded Markov Chain $\{J_n; n \geq 0\}$, $\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 = 1$ and $\Pi_1, \Pi_2, \Pi_3, \Pi_4 \in [0,1]$.

The following probabilities are essential for the purposes of pricing N-CAT bonds. At time t, for the predetermined threshold level D ($D \geq 0$), we derive that

$$\begin{aligned} F_1(t,D) &= \mathbb{P}(L(t) \leq D, J_{N(t)} \neq 0) \\ &= \sum_{i=1}^{4}\sum_{j=1}^{4}\Pi_i \sum_{n=0}^{\infty}\int_0^t (1 - H_j(t-t',\infty))dQ_{ij}^{*n}(t',D), \qquad (5.1) \\ F_2(t,D) &= \mathbb{P}(J_{N(t)} = 0) = \sum_{i=1}^{4}\Pi_i \sum_{n=1}^{\infty} Q_{i0}^{*n}(t,D), \qquad (5.2) \\ F_3(t,D) &= \mathbb{P}(L(t) > D, J_{N(t)} \neq 0) = 1 - F_1(t,D) - F_2(t,D). \qquad (5.3) \end{aligned}$$

In particular, a special case of this US N-CAT bond is SM'/SM model, which can be structured as $(i \in J', j \in J)$:

1. The inter-arrival time distribution only depends on the current state, and is given by matrix $G_{ij}(t,\infty) = \mathbb{P}_i(U_k \leq t)$.

2. Claim size distribution is given by $G_{ij}(\infty, x) = \mathbb{P}_j(X_k \leq x)$.

That is, the process changes its state at every claim instance based on the transition matrix $\mathbf{P}$, with the claim size distribution dependant on the future state . While the arrival time before the next catastrophic claim U_k depends on the severity of the current event X_k, for all $k = 0, 1, 2, \ldots$.

5.1.3 Pricing Model For The N-CAT Bonds

Defining a hypothetical zero coupon N-CAT bond at the maturity date T with face value Z, the price of the N-CAT risk bond is given to be the following payoff structure:

1. If at expiring time T, $L(T) \geq D$ $(D \geq 0)$ and $J_k \neq 0$ $(\forall k)$, that is, the total loss is greater than a predefined level and no major accident occurred prior to T, bond holder will lose part of their capital and receive $rp_1 Z (rp_1 > 0)$;

2. If a major accident (state 0 event) $(J_k = 0)$ occurs before the expiry date T, the N-CAT bond expires immediately and bond holder will receive a partial amount of their principle $rp_2 Z$ (normally $0 < rp_2 < rp_1$);

3. Otherwise bond holder will receive Z.

Formally, the payoff function described above is given mathematically by

$$P_{CAT} = \begin{cases} Z, \text{ for } L(T) \le D \text{ and } J_{N(t)} \neq 0, \\ rp_1 Z, \text{ for } L(T) > D \text{ and } J_{N(t)} \neq 0, \\ rp_2 Z, J_{N(t)} = 0. \end{cases} \tag{5.4}$$

Zero-coupon N-CAT bond prices at time t paying principal Z at time to maturity T is given in the following Theorem 5.1.1.

Theorem 5.1.1. *Let $V(t)$ be the value of T-maturity zero-coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with payoff function P_{CAT} Eq.* (5.4)*. Then*

$$V(t) = B_{CIR}(t,T)Z[rp_1 + (1-rp_1)F_1(T-t,D) + (rp_2 - rp_1)F_2(T-t,D)],$$

where $F_1(T-t,D)$ and $F_2(T-t,D')$ represent the probabilities given in Eq. (5.1) *and Eq.* (5.2)*, respectively, and pure discounted bond price $B_{CIR}(t,T)$ with CIR interest rate model is given by Eq.* (2.8)–(2.11)*.*

Proof. Similar to the proof in Chapter 4, Theorem 4.1.1 and apply the payoff function Eq. (5.4), the value of N-CAT bond price can be formulated as

$$\begin{aligned} V(t) =& B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}(Z\mathbb{1}\{L(T) \le D, J_{N(t)} \neq 0\} + rp_1 Z\mathbb{1}\{L(T) > D, J_{N(t)} \neq 0\} \\ & + rp_2 Z\mathbb{1}\{J_{N(t)} = 0\}|\mathcal{F}_t) \\ =& B_{CIR}(t,T)(Z\mathbb{P}(L(T) \le D, J_{N(t)} \neq 0) + rp_1 Z\mathbb{P}(L(T) > D, J_{N(t)} \neq 0) \\ & + rp_2 Z\mathbb{P}(J_{N(t)} = 0)) \\ =& B_{CIR}(t,T)Z(F_1(T,D) + rp_1 F_3(T,D) + rp_2 F_2(T,D)) \\ =& B_{CIR}(t,T)Z[rp_1 + (1-rp_1)F_1(T-t,D) + (rp_2 - rp_1)F_2(T-t,D)] \\ =& B_{CIR}(t,T)Z[rp_1 + (1-rp_1)\sum_{i=1}^{4}\sum_{j=1}^{4}\Pi_i \sum_{n=0}^{\infty}\int_0^T (1 - H_j(T-t',\infty))dQ_{ij}^{*n}(t',D) \\ & + (rp_2 - rp_1)\sum_{i=1}^{4}\Pi_i \sum_{n=1}^{\infty} Q_{i0}^{*n}(T,D)]. \end{aligned}$$

□

5.2 Numerical Example of N-CAT Risk Bond

In this section, a numerical example illustrates the applicability of the theoretical model which has been presented previously. Due to data limitations, the following assumptions need to be made.

For the US N-CAT bond SM'/SM model, the inter-arrival time distribution is assumed to be a Poisson process with parameter λ_i, and it can be given by matrix $\mathbf{G}(t,\infty)(=G_{ij}(t,\infty), i \in J', j \in J)$,

$$G_{ij}(t,\infty) = \begin{cases} 0, \ t < 0 \\ 1 - e^{-\lambda_i t}, \ t \geq 0. \end{cases}$$

Furthermore, arbitrarily assume that $\lambda_i = 10, \ 30, \ 5, \ 20$, for $i = 1, \ 2, \ 3, \ 4$, respectively. That is to say, if an event occurs (termed to be either strike, failures in safety provisions, release of radioactive or core machine failure), the time before the next event follows an exponential distribution with parameter λ_i. The claim size distribution is assumed to follow a lognormal distribution with mean μ_j and variance σ_j,

$$G_{ij}(\infty, x) = \frac{1}{x\sigma_j\sqrt{2\pi}} \exp\left(-\frac{(lnx - \mu_j)^2}{2\sigma_j^2}\right), i \in J', j \in J.$$

Furthermore, assume that $\mu_j = 2, \ 1, \ 2.5, \ 3$ and $\sigma_j = 1, \ 0.8, \ 1.5, \ 1.2$, for $j = 1, \ 2, \ 3, \ 4$, respectively. Due to the properties of the catastrophic events, the loss cost by each type of the peril is a heavy tailed distribution. In this case study, it is also assumed that core machine failure tends to cause more losses, while failure in safety provisions causes less losses. Moreover the transition matrix **P** is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.003 & 0.397 & 0.3 & 0.2 & 0.1 \\ 0.004 & 0.4 & 0.096 & 0.3 & 0.2 \\ 0.001 & 0.4 & 0.4 & 0.199 & 0.1 \\ 0.001 & 0.2 & 0.2 & 0.5 & 0.098 \end{pmatrix},$$

and the starting distribution $(\Pi_1, \ \Pi_2, \ \Pi_3, \ \Pi_4) = (0.3476325, 0.2609975, 0.2642861, 0.1264328)$. Here, p_{i0} $(i \in J')$ is very small because the probability of having a nuclear reactor failure is very rare. The first row $(p_{0j}, j \in J)$ are zeros because if a major accident (state 0 event, nuclear reactor failure) occurs, the system stops and the N-CAT bond is terminated immediately. $p_{11} = 0.397$ means that the probability of having a strike after a strike is 0.397, and $p_{12} = 0.3$ means that the probability of failures in safety provisions after a strike is 0.3, and so on.

We obtain the $T \in [0.5,\ 2]$ years maturity zero-coupon CAT bond prices with face amount of US$\$1,000$ in Monte-Carlo simulations[29]. For the payoff function Eq. (5.4), it is assumed that $rp_1 = 0.5$ when the aggregate loss $L(T)$ exceeds the threshold level $D = [100,\ 1600]$ in million US\$, and $rp_2 = 0.25$.

In this case study, the same data set as Chapter 4 is fitted to the interest rate model. Thus, both the initial short-term interest rate r_0 and the long-term mean interest rate θ were 2.04% annually, the mean-reverting force $k = 0.0984$, and the volatility parameter $\sigma = 4.77\%$. Furthermore, it is assumed that the market price of risk λ_r is a constant -0.01.

Table 5.1 and Figure 5.3 illustrate the value of N-CAT bonds for the payoff functions Eq. (5.4) with the CAT threshold level D and time to maturity T under the stochastic interest rate assumptions, where the loss distribution follows the lognormal distribution and the intensity of the claims is a Poisson distribution. For example, an N-CAT bond buyer needs to pay US$\$781.16$ now in order to buy this N-CAT bond with face value US$\$1,000$ which will mature in six months and with threshold level US$\$100$ million. With fixed threshold level US$\$100$ million, the bond value decreases from US$\$781.16$ to US$\$455.31$ for the maturity time from half year to 2 years. This is a quicker rate for threshold level US$\$1,600$ million, with the bond value decreasing from US$\$948.42$ to US$\$855.32$. For fixed time to maturity, the N-CAT bond value increases when the threshold level increases (from US$\$100$ million to US$\$1,600$ million.), and with a quicker rate for longer maturity time (from US$\$781.16$ to US$\$948.42$ for $T = 0.5$ and US$\$455.31$ to US$\$855.32$ for $T = 2$.)

Obviously, the value of the N-CAT (V) decreases in relation with the maturity time (T). Moreover, with higher pre-determined threshold levels (D), the N-CAT bond value (V) increases accordingly, although a change of D won't affect the probability of having a major accident claim.

[29]For **R** code for valuing CAT bond, see Appendix D.

Table 5.1: Vaule of N-CAT bonds with face value US$1,000 for time to the maturity ($T = 0.5,\ 1,\ 1.5,\ 2$, years) and threshold level ($D = 100,\ 600, 1000,\ 1600$ in million US$).

N-CAT value (V, US$)		Time to maturity (T, years)			
		0.5	1	1.5	2
Threshold (D, US$ millions)	100	781.16	603.19	500.64	455.31
	600	942.38	905.84	853.43	792.51
	1000	947.09	919.22	881.91	843.62
	1600	948.42	922.55	888.49	855.32

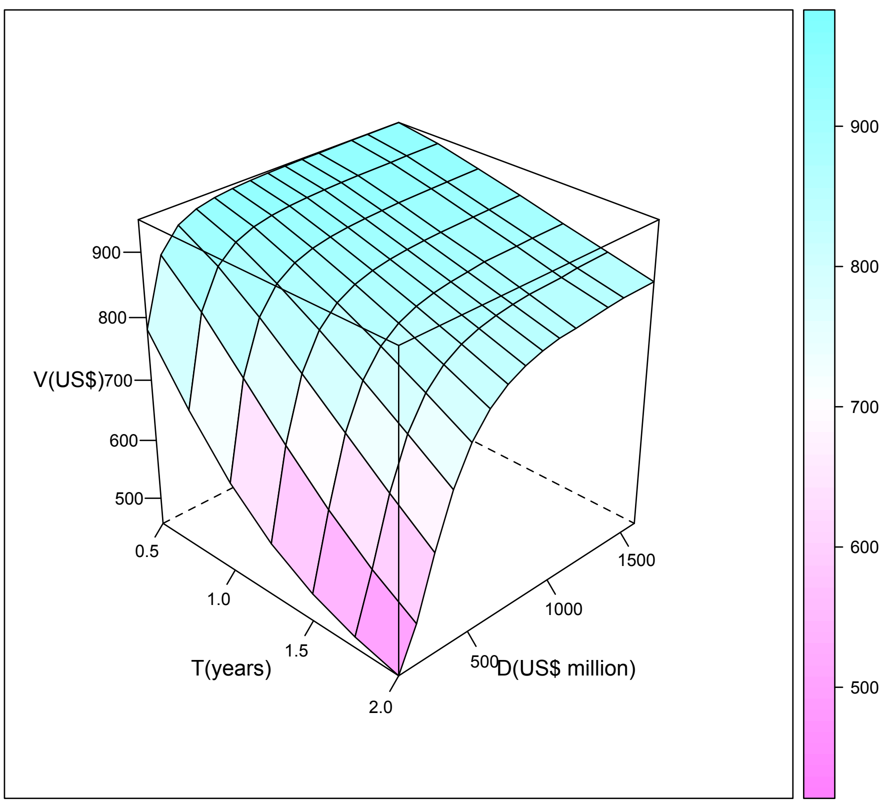


Figure 5.3: Vaule of N-CAT bonds (z-coordinate axes) under the lognormal, the NHPP and stochastic interest rates assumptions. Here, time to the maturity (T) decreases on the left axes and threshold level (D) increases on the right axes.

5.3 Summary

In this chapter, for the very first time, a two-coverage type trigger nuclear catastrophe (N-CAT) bond was proposed for financing nuclear liability, which can be categorized by three risk layers: major accident, accident and incident. In the Global CAT risk bond market, a similar trigger type of a total size of US$400 million has been proposed recently by Residential Reinsurance 2012 Ltd. (Series 2012-2)[30] for covering U.S. hurricane, U.S. earthquake, U.S. severe thunderstorm, U.S. winter storm and California wildfire. After the 2011 Fukushima disaster, the development of dependable liability coverage that can be tapped in an emergency is of significant importance to the many countries with a significant number of NPPs. For instance, with 104 operating reactors, the U.S. has a total of about US$12 billion in coverage (as of 2011), Griffith et al. (1990), before congressional authorization for additional funding, estimating the damage due to a catastrophic accident from US$110 billion to as much as US$7 trillion.

In this study, the value of N-CAT bonds was formulated under assumptions of a no-arbitrage market, independent of the financial risks and catastrophe risks, and the possibility of replicated interest rate changed with existing financial instruments. Under the risk-neutral pricing measure, the pricing formula was derived by using a semi-Markov dependent structure in continuous time where the claim inter-arrival times were dependent on the claim sizes together with CIR interest rate model and two-coverage type payoff function. Numerical experiments utilized Monte Carlo simulations by assuming the distributions and parameters. The values of the N-CAT bonds were obtained under the lognormal, the NHPP and stochastic interest rates assumptions for different threshold levels (D) and time to the maturities (T). The numerical analysis showed that the CAT bond prices decreased as the threshold level decreased, as the time to maturity increased.

[30] See Artemis 2012, Residential Reinsurance 2012 Ltd. (Series 2012-2) `http://www.artemis.bm/deal_directory/residential-reinsurance-2012-ltd-series-2012-2/`.

Chapter 6

Conclusion

The study was set out to explore the concept of modelling CAT bonds in order to help CAT bond sponsors to provide a fair price. The study had also sought to compare the differences between the models with different probability structures.

The reasons and motivation behind this work is because the economic losses caused by catastrophic events are huge, sometimes even a single event can cause insurance and reinsurance companies to face bankruptcy. As an alternative solution, CAT bonds are developed. CAT bond markets have boomed over the past twenty years and valuing CAT bonds have become a hot topic in academic literature. However, current literature of this topic is limited to building pricing models of CAT bonds by different approaches. To fill the gap, this study sought to model CAT bonds in a multi dimensional view:

1. Developing two different probabilistic structures (generalized multiple financial and catastrophic risk variables structure in Chapter 3, and back to classical single financial and catastrophic structure in Chapters 4 and 5).

2. Raising different CAT bond payoff functions (Chapter 3 studied a parametric trigger type, while Chapter 4 analysed four payoff functions with industry index trigger, and in Chapter 5 payoff function are triggered by both per-occurrence and aggregate loss).

3. Assuming different dependency between variables (variables in Chapter 3 are

pairwise independent, and Chapters 4 and 5 assumed the size of the claim depended on the inter-arrival time between the claims).

4. Applying to different catastrophic events (utilized earthquake data, PCS data and nuclear, respectively).

The pattern in Chapter 3 was consistent with that presented by Cox and Pedersen (2000). It provided a generalized multi-variables valuation formulae for easy application in the industry. Here, the n financial risk variables and m catastrophic risk variables are pairwise independent. The numerical example of a one-period and multi-period parametric CAT bonds with the California earthquake data can be the guidelines for earthquake CAT bond issuers when issuing similar bonds.

Instead of assuming total independence, Chapter 4 assumed the aggregate claims follows compound forms where the claim inter-arrival times are dependent on the claim sizes by employing a two-dimensional semi-Markov process. Softening the of dependence argument helps to develop a more realistic model, where claims sizes depend on the current state the system in. And the application of PCS data makes it possible to compare with the models in the other literature, i.e. Ma and Ma (2013).

Finally, Chapter 5 also works in a semi-Markov environment, however, the model was of a more complex design due to the addition of a perturbed state. This extra term allowed us to model a CAT bond which will redeem at the per-determined expiry date, and will expire immediately after a certain event hits the insured region. The N-CAT bond introduced in this chapter is new to the literature and shifted liability to the capital market in any nuclear liability limitation regimes.

The limitations of this thesis are mainly in the application part. The estimation of the parameters involved in the model is always a challenging aspect as the collection of historical data for losses due to catastrophic events in commercial NPPs is rather limited, which makes the accuracy of the pricing method even more challenging. In addition, one could always employ the data with longer periods to obtain a better estimation of the model. In this thesis, we used Monte-Carlo simulation to compute the CAT prices, however, this method is very computationally expensive. It is very challenging to find a more efficient alternative approximation method to reduce the computation time in order to use the programme in the industry.

Instead of having inter-arrival time depend on claims sizes, there is potential for a new model with different types of dependency for other catastrophe risk random variables using the probabilistic structure we generalized. Although we solved the problem of characterizing the dependency between catastrophe claims, the dependency between the CAT market and the financial market cannot be used within our framework. Thus, the problem of the dependency between CAT risks and the financial market risks is very interesting, and thus it will be addressed in future research. Another direction of extension can be the reliability of the CAT bonds.

As the continuity of the final chapter, an ongoing project is carried on for N-CAT bond, and we are going to extend the Chapter 5 in following five directions:

1. Model under the probabilistic structure in Chapter 3 with n financial and m catastrophic risks. Moreover, consider the dependency among the catastrophe risks, i.e. a multi-perils product.

2. Present a generalized model by extending state sizes from 5 to n. Generally speaking, the 5 states case in Chapter 5 is an example of how one can model nuclear risks. However, it is worthwhile to have more risks perils and with different combinations. For instance, two types of multi-peril CAT bonds can be modelled: one with earthquakes, tsunami and nuclear power plant failure and another one with hurricanes, tornadoes, flood and nuclear power plant failure.

3. Different interest rate models, e.g. Vasicek, Hull-White and CIR, then compare the sensitivity of the CAT prices.

4. Structure different payoff functions.

5. And apply different scenarios by employing different types of distribution for claims sizes and claims inter-arrival times processes.

Also, the analysis of the impact of N-CAT risk bonds-specific variables on premiums is also a very interesting question, in particular considering complexity in terms of the number of insured peril types or regions. Finally, it is useful to investigate how the future nuclear disasters (especially in emerging markets, like China and India) and financial crises might affect N-CAT bond premiums and demand.

The benefit of this work is to model CAT bonds under different scenarios and obtain realistic and comparable prices for different perils by numerical simulation as a benchmark for the future CAT bonds. It also highlights how the dependence argument can influence the value of a CAT bond with the same payoff function. Therefore, readers can have a complete picture of all aspects of CAT bonds pricing.

Appendices

Appendix A

Nuclear power plant accidents and incidents with multiple fatalities and/or more than US$100 million in property damage, 1961-2011[31]

[31] http://en.wikipedia.org/wiki/Nuclear_and_radiation_accidents_and_incidents.

Date	Location	Cost US$m 2006 (Fatalities)
3/1/1961	Idaho Falls, Idaho, US	22 (3)
5/10/1966	Frenchtown Charter Township, US	132 (0)
7/12/1975	Greifswald, East Germany	443 (0)
5/1/1976	Jaslovské Bohunice, Czechoslovakia	(2)
22/2/1977	Jaslovské Bohunice, Czechoslovakia	1,700 (0)
28/31979	Three Mile Island, Pennsylvania, US	2,400 (0)
15/9/1984	Athens, Alabama, US	110 (0)
9/3/1985	Athens, Alabama, US	1,830 (0)
11/4/1986	Plymouth, Massachusetts, US	1,001 (0)
26/4/1986	Chernobyl disaster, Ukrainian SSR	(56 direct)
4/5/1986	Hamm-Uentrop, Germany	267 (0)
31/3/1987	Delta, Pennsylvania, US	400 (0)
19/12/1987	Lycoming, New York, US	150 (0)
17/3/1989	Lusby, Maryland, US	120 (0)
20/2/1996	Waterford, Connecticut, US	254 (0)
2/9/1996	Crystal River, Florida, US	384 (0)
30/9/1999	Ibaraki Prefecture, Japan	54 (2)
16/2/2002	Oak Harbor, Ohio, US	143 (0)
9/8/2004	Fukui Prefecture, Japan	9 (4)
25/7/2006	Forsmark, Sweden	100 (0)
11/3/2011	Fukushima, Japan	(2+)
12/9/2011	Marcoule, France	(1)

Appendix B

R Code For Chapter 3

B.1 One-period Model

Parameters used in this section.

```
library(fExtremes)
T<-1  # time period
m<-100000  # simulation times
premium<-0.03 # the extra risk premium
Int<-0.0012 # risk-free interest rate
Infl<-0.0316 # inflation rate
r1<-rep(0.013,m) # LIBOR rate
K=1000 # face value of the CAT bond

## generate m cases for Magnitude and Depth
Mag1<-rgev(m, xi = 0.05866229, mu = 4.71946946, beta =
    0.44861472)
Mag2<-rgev(m, xi = 0.1181457, mu = 4.9275121, beta =
    0.4833782)
Depth1<-rgamma(m, shape=2.35378504, rate = 0.25460951)
Depth2<-rgamma(m, shape=1.44878306, rate = 0.14585340)

```

```
C=rep(NA,m)   #value of payoff function
f=rep(NA,m)
g=rep(NA,m)
h=rep(NA,m)
p=rep(NA,m)
q=rep(NA,m)
s=rep(NA,m)
# thresholds levels
aa<-2.6  ;ab<-2.8  ;ac<-1.6  ;ad<-1.8  ;ae<-0.5  ;af<-0.6
ba<-2.9   ;bb<-3  ;bc<-1.8  ;bd<-2  ;be<-1  ;bf<-1.1
ag<-0.8  ;ah<-0.85  ;ai<-0.55  ;aj<-0.6  ;ak<-0.2
bg<-0.95  ;bh<-0.98  ;bi<-0.7  ;bj<-0.75  ;bk<-0.5
```

Price payoff function P_{CAT}.

```
for (i in 1:m)
{
if(Mag1[i]>Mag2[i])
{
if(Mag1[i]<5.4)
{if(Depth1[i]<=20)f[i]<-aa*r1[i]
if(Depth1[i]>20)f[i]<-ab*r1[i]
C[i]=K*(1+f[i])  }

if(Mag1[i] < 5.8 && Mag1[i]>=5.4)
{if(Depth1[i]<=15)g[i]<-ac*r1[i]
if(Depth1[i]>15)g[i]<-ad*r1[i]
C[i]=K*(1+g[i])  }

if(Mag1[i] < 6.2 && Mag1[i]>=5.8)
{if(Depth1[i]<=10)h[i]<-ae*r1[i]
if(Depth1[i]>10)h[i]<-af*r1[i]
```

```
C[i]=K*(1+h[i])}

if(Mag1[i] < 6.6 && Mag1[i]>=6.2)
C[i]=K

if(Mag1[i] < 7.0 && Mag1[i]>=6.6)
{if(Depth1[i]<=10)p[i]<-ag*K
if(Depth1[i]>10)p[i]<-ah*K
C[i]=p[i]}

if(Mag1[i] < 7.4 && Mag1[i]>=7.0)
{if(Depth1[i]<=10)q[i]<-ai*K
if(Depth1[i]>10)q[i]<-aj*K
C[i]=q[i]}

if(Mag1[i] >7.4)
{s[i]<-ak*K
C[i]=s[i]}
}

else
{
if(Mag2[i]<5.4)
{if(Depth2[i]<=20)f[i]<-ba*r1[i]
if(Depth2[i]>20)f[i]<-bb*r1[i]
C[i]=K*(1+f[i])}

if(Mag2[i] < 5.8 && Mag2[i]>=5.4)
{if(Depth2[i]<=15)g[i]<-bc*r1[i]
if(Depth2[i]>15)g[i]<-bd*r1[i]
C[i]=K*(1+g[i]) }
```

```
if(Mag2[i] < 6.2 && Mag2[i]>=5.8)
{if(Depth2[i]<=10)h[i]<-be*r1[i]
if(Depth2[i]>10)h[i]<-bf*r1[i]
C[i]=K*(1+h[i])}

if(Mag2[i] < 6.6 && Mag2[i]>=6.2)
C[i]=K

if(Mag2[i] < 7.0 && Mag2[i]>=6.6)
{if(Depth2[i]<=10)p[i]<-bg*K
if(Depth2[i]>10)p[i]<-bh*K
C[i]=p[i]}

if(Mag2[i] < 7.4 && Mag2[i]>=7.0)
{if(Depth2[i]<=10)q[i]<-bi*K
if(Depth2[i]>10)q[i]<-bj*K
C[i]=q[i]}

if(Mag2[i] >7.4)
{s[i]<-bk*K
C[i]=s[i]}
}
}
```

Final value of the CAT bond according to Eq. (3.10).

```
discount<-(1+Infl)*(1+premium+Int)
P<-C/discount
mean(P) # this is the final price of the CAT bond
```

B.2 Multi-period Model

Parameters used in this section.

```
library(fExtremes)
T<-5  # time period
m<-100000  # simulation times
premium<-0.03 # the extra risk premium
K=1000 # face value of the CAT bond
C=matrix (0, T, m) # value of payoff function
f=matrix (0, T, m)
g=matrix (0, T, m)
h=matrix (0, T, m)
p=matrix (0, T, m)
q=matrix (0, T, m)
s=matrix (0, T, m)
aa<-2.6 ;ab<-2.8 ;ac<-1.6 ;ad<-1.8 ;ae<-0.5 ;af<-0.6
ba<-2.9  ;bb<-3 ;bc<-1.8 ;bd<-2 ;be<-1 ;bf<-1.1
ag<-0.8 ;ah<-0.85 ;ai<-0.55 ;aj<-0.6 ;ak<-0.2
bg<-0.95 ;bh<-0.98 ;bi<-0.7 ;bj<-0.75 ;bk<-0.5
```

Generating m cases for LIBOR rate, interest rate and inflation rate.

```
## LIBOR rate
alpha = +0.212421
mu    = +1.084655
sigma = +0.420791
delta<-1
r<-matrix (NA, T, m)
r[1,]<-1.13 # initial value
for (i in 1:(T-1))
{
r[i+1,]<-r[i,]
```

```
for (days in 1:250)
{
e<-rnorm(m, mean=0, sd=sqrt(delta))
for (j in 1:m)
{
r[i+1,j] <- r[i+1,j] + alpha*(mu-r[i+1,j])*delta + sigma*
    sqrt(r[i+1,j])*e[j]+days-days
if(r[i+1,j]<0)
r[i+1,j]<- r[1,j] + alpha*(mu-r[1,j])*delta
r[1,j]<-r[i+1,j]
}}}
r[1,]<-1.13
rt<-r/100

# interest rate
ar<--0.2833
ma<-1
const<--0.0976
Int<-matrix(NA,T+2,m)
e <-matrix(NA,T,m)
Int[1,]<-0.29
Int[2,]<-0.12
for (j in 1:T)
{e[j,]<-rnorm(m)
   for (i in 1:m)
   {
     Int[j+2,i]<- Int[j+1,i]+ar*Int[j+1,i]-ar*Int[j,i]+ma*e
         [j,i]-const
     if(Int[j+2,i]<0)
     Int[j+2,i]<- Int[j+1,i]+ar*Int[j+1,i]-ar*Int[j,i]
   }
```

```
}
Int<-Int/100
Int<-rbind(Int[2:(T+2),])

# inflation rate
e<- matrix(NA, T, m)
Infl<-matrix(NA,T+1,m)
Infl[1,]<-3.16
for(i in 1:T)
{e[i,]<-rnorm(m)
Infl[i+1,]<-0.8899+0.7867*Infl[i,]+e[i,]
for (j in 1:m)
  {
if (Infl[i+1,j]<0)
Infl[i+1,j]<-0.8899+0.7867*Infl[i,j]
}}
Infl<-Infl/100
Infl<-rbind(Infl[1:(T+1),])
```

Generating m cases for Magnitude and Depth in both regions.

```
Mag<-array (NA, c(T, m, 2))
for (i in 1:T)
{
Mag[i, ,1]<-rgev(m, xi = 0.05866229, mu = 4.71946946,
    beta = 0.44861472)
Mag[i, ,2]<-rgev(m, xi = 0.1181457, mu = 4.9275121, beta
    = 0.4833782)
}

Dep<-array (NA, c(T, m, 2))
for (i in 1:T)
```

```
{
Dep[i, ,1]<-rgamma(m, shape=2.35378504, rate =
    0.25460951)
Dep[i, ,2]<-rgamma(m, shape=1.44878306, rate =
    0.14585340)
}
mag<-array (NA, c(m))
for(i in 1:m)
{
mag[i]<-max(Mag[T,i,1],Mag[T,i,2])
}
```

Price payoff function P_{CAT}.

```
# Coupon payment for the period of 1 to (T-1).
for (j in 1:(T-1))
{
for (i in 1:m)
{
  if(Mag[j,i,1]>Mag[j,i,2]) #Magnitude of region 1 is
      larger
  {
    if(Mag[j,i,1]<5.4)
     {
      if(Dep[j,i,1]<=20)f[j,i]<-aa*rt[j,i]
      if(Dep[j,i,1]>20)f[j,i]<-ab*rt[j,i]
      C[j,i]=K*(0+f[j,i])
     }

if(Mag[j,i,1] < 5.8 && Mag[j,i,1]>=5.4)
{if(Dep[j,i,1]<=15)g[j,i]<-ac*rt[j,i]
if(Dep[j,i,1]>15)g[j,i]<-ad*rt[j,i]
```

```
C[j, i]=K*(0+g[j, i])  }

 if(Mag[j, i, 1] < 6.2 && Mag[j, i, 1]>=5.8)
{if(Dep[j, i, 1]<=10)h[j, i]<-ae*rt[j, i]
 if(Dep[j, i, 1]>10)h[j, i]<-af*rt[j, i]
C[j, i]=K*(0+h[j, i])}
  }
  if(Mag[j, i, 1]<Mag[j, i, 2]) #Magnitude of region 2 is
      larger
  {
    if(Mag[j, i, 2]<5.4)
     {
       if(Dep[j, i, 2]<=20)f[j, i]<-ba*rt[j, i]
       if(Dep[j, i, 2]>20)f[j, i]<-bb*rt[j, i]
      C[j, i]=K*(0+f[j, i])
     }

 if(Mag[j, i, 2] < 5.8 && Mag[j, i, 2]>=5.4)
{if(Dep[j, i, 2]<=15)g[j, i]<-bc*rt[j, i]
 if(Dep[j, i, 2]>15)g[j, i]<-bd*rt[j, i]
C[j, i]=K*(0+g[j, i])  }

 if(Mag[j, i, 2] < 6.2 && Mag[j, i, 2]>=5.8)
{if(Dep[j, i, 2]<=10)h[j, i]<-be*rt[j, i]
 if(Dep[j, i, 2]>10)h[j, i]<-bf*rt[j, i]
C[j, i]=K*(0+h[j, i])}
  }
}
}
# For the final payment
for (i in 1:m)
```

```
{
j=T
  if(Mag[j,i,1]>Mag[j,i,2]) #Magnitude of region 1 is
      larger
  {
    if(Mag[j,i,1]<5.4)
     {    if(Dep[j,i,1]<=20)f[j,i]<-aa*rt[j,i]
        if(Dep[j,i,1]>20)f[j,i]<-ab*rt[j,i]
        C[j,i]=K*(1+f[j,i])}

 if(Mag[j,i,1] < 5.8 && Mag[j,i,1]>=5.4)
 {if(Dep[j,i,1]<=15)g[j,i]<-ac*rt[j,i]
 if(Dep[j,i,1]>15)g[j,i]<-ad*rt[j,i]
C[j,i]=K*(1+g[j,i])  }

 if(Mag[j,i,1] < 6.2 && Mag[j,i,1]>=5.8)
 {if(Dep[j,i,1]<=10)h[j,i]<-ae*rt[j,i]
 if(Dep[j,i,1]>10)h[j,i]<-af*rt[j,i]
C[j,i]=K*(1+h[j,i])}

 if(Mag[j,i,1] < 6.6 && Mag[j,i,1]>=6.2)
C[j,i]=K

 if(Mag[j,i,1] < 7.0 && Mag[j,i,1]>=6.6)
 {if(Dep[j,i,1]<=10)p[j,i]<-ag*K
 if(Dep[j,i,1]>10)p[j,i]<-ah*K
C[j,i]=p[j,i]}

 if(Mag[j,i,1] < 7.4 && Mag[j,i,1]>=7.0)
 {if(Dep[j,i,1]<=10)q[j,i]<-ai*K
 if(Dep[j,i,1]>10)q[j,i]<-aj*K
```

```
C[j,i]=q[j,i]}

 if(Mag[j,i,1]  >7.4)
{s[j,i]<-ak*K
C[j,i]=s[j,i]}

  }
  if(Mag[j,i,1]<Mag[j,i,2]) #Magnitude of region 2 is
     larger
  {
    if(Mag[j,i,2]<5.4)
     {if(Dep[j,i,2]<=20)f[j,i]<-ba*rt[j,i]
      if(Dep[j,i,2]>20)f[j,i]<-bb*rt[j,i]
      C[j,i]=K*(1+f[j,i])}

 if(Mag[j,i,2] < 5.8 && Mag[j,i,2]>=5.4)
{if(Dep[j,i,2]<=15) g[j,i]<-bc*rt[j,i]
 if(Dep[j,i,2]>15) g[j,i]<-bd*rt[j,i]
C[j,i]=K*(1+g[j,i]) }

 if(Mag[j,i,2] < 6.2 && Mag[j,i,2]>=5.8)
{if(Dep[j,i,2]<=10)h[j,i]<-be*rt[j,i]
 if(Dep[j,i,2]>10)h[j,i]<-bf*rt[j,i]
C[j,i]=K*(1+h[j,i])}

 if(Mag[j,i,2] < 6.6 && Mag[j,i,2]>=6.2)
C[j,i]=K

 if(Mag[j,i,2] < 7.0 && Mag[j,i,2]>=6.6)
{if(Dep[j,i,2]<=10)p[j,i]<-bg*K
 if(Dep[j,i,2]>10)p[j,i]<-bh*K
```

```
C[j,i]=p[j,i]}

 if(Mag[j,i,2]< 7.4 && Mag[j,i,2]>=7.0)
{if(Dep[j,i,2]<=10)q[j,i]<-bi*K
 if(Dep[j,i,2]>10)q[j,i]<-bj*K
C[j,i]=q[j,i]}

 if(Mag[j,i,2] >7.4)
{s[j,i]<-bk*K
C[j,i]=s[j,i]}
  }
}
```

Final value of the CAT bond according to Eq. (3.14).

```
discount=matrix(NA,T,m)
discount[1,]<-(1+Int[1,]+premium)*(1+Infl[1,])

for (i in 1:(T-1))
{
for (j in 1:m)
{discount[i+1,j]<-discount[i,j]*(1+Int[i+1,j]+premium)*
    (1+Infl[i+1,j])
}
}
P<-colSums(C/discount)
mean(P) # this is the final price of the CAT bond
```

Appendix C

R Code For Chapter 4

C.1 Data

In this example, we are using PSC data set with ‘Claim number’,‘State’,‘Gross Loss’, ‘CPI index’,‘Loss’,‘Time of year’,‘Start time’ and ‘End time’. Size of the data set = 3950, total claim number = 879.

```r
df = read.table("…/PCSCATDATA.txt", head=T)
dfs<- data.frame(Number=rep(NA, 879), Loss=rep(NA, 879),
Time=rep(NA,879), CPI=rep(NA, 879), Start=rep(NA, 879),
stringsAsFactors=FALSE)
# In the data set, the original ‘Start time’ and ‘End time’ are in the form of day/month/year.
df$Start<-as.Date(df$Start, format ="%d/%m/%Y")
dfs$Start<-as.Date(dfs$Start, format ="%d/%m/%Y")
# load data the first line of original data set
dfs[1, ] <-df[1,c(1,5,6,4,7)]
# calculate the total loss in terms of each claim
len<-1
for (i in 1:3950)
{
if (df$Number[i]!=df$Number[i+1])
```

```
{len<-len+1
dfs[len,]<-df[i+1,c(1,5,6,4,7)]
}
if (df$Number[i]==df$Number[i+1])
{dfs$Loss[len]<-dfs$Loss[len]+df$Loss[i+1]}
}
for (i in 1:879)
{
if (dfs$Loss[i]>4.51e8)
{dfs$State[i]<-1}
else
{dfs$State[i]<-0}
}
# Total loss for each claim in US$ 10 millions.
dfs$Loss<-dfs$Loss/1e7
# Calculate the total loss for each year.
totalloss<-c()
v<-dfs$Loss[1]
for (i in 1:878)
{
if (dfs$CPI[i]==dfs$CPI[i+1])
{v<-v+dfs$Loss[i+1]}
else
{totalloss<-c(totalloss,v)
v<-dfs$Loss[i+1]}
}
# Make a plot for annual total loss
par(mfrow=c(1,2))
plot(1985:2013,totalloss/100,type="o",xlab="Time(year)",
    ylab="Adjusted PCS (2014, billion dollors)")
# Calculate the number of claims for each year
```

```r
Noclaim<-c()
for (i in 1:878)
{
if (dfs$CPI[i]!=dfs$CPI[i+1])
{Noclaim<-c(Noclaim,sum(dfs$CPI==dfs$CPI[i]))}
}
# Make a plot for annual total number of claims
plot(1985:2013,Noclaim,type="o",xlab="Time(year)", ylab="
    Number of CATs")
```

And this is the plot of Figure 4.1.

In this example, we value the CAT bond prices for time to maturity $T \in [0.25, 2.25]$, and threshold level $D \in$ [mean(totalloss)$/4$, $3 \times$ mean(totalloss)].

```r
t<-seq(from=0.25,to=2.25,length=20)
D<-seq(from=mean(totalloss)/4, to=mean(totalloss)*3,
    length.out=20)
myGrid <- data.frame(expand.grid(t,D))
colnames(myGrid) <- c("t","D")
tau<-myGrid$t # tau ...(T-t), time to maturity
```

We defined two different states in the model (many claims period and few claims period), and separate original data into those two states.

```r
staten<-length(dfs$State)-sum(dfs$State)
statenn<-staten+1
a<-dfs$State  # a sequence of index of states
#initial probability matrix pi0
pi<-sum(a)/length(a)
pi0<-matrix(c(pi,(1-pi)),nrow=2,ncol=2,byrow=F)

# Transition probability matrix P
# First find the total number of transition
```

```
alen<-length(a)-1
np1<-0
np2<-0
np3<-0
np4<-0
for (i in 1:alen)
{
if (a[i]==1 && a[i+1]==1) #from state 1 to 1
{np1<-np1+1}
if (a[i]==1 && a[i+1]==0) #from state 1 to 0
{np2<-np2+1}
if (a[i]==0 && a[i+1]==1) #from state 0 to 1
{np3<-np3+1}
if (a[i]==0 && a[i+1]==0) #from state 0 to 0
{np4<-np4+1}
}
p1<-np1/(np1+np2)
p2<-np2/(np1+np2)
p3<-np3/(np3+np4)
p4<-np4/(np3+np4)
P<-array(c(p1,p3,p2,p4),dim=c(2,2,length(t)))

#we need the data in two sets of states
df<-dfs[order(State),]
df1<-df[1:staten ,]
df2<-df[statenn:length(df$State),]
```

C.2 CIR Interest Rate

Estimate parameter of CIR model.

```
CIRloglike<-function(param,data,times,test=F,addsign=T){
#CIR log-likelihood Function
#param ... parameters of the CIR model
#dt ... time interval (in years) between the data points
#c... multiplying term for the chi-square distribution
#df... degree of freedom
#ncp... non-centrality parameter
theta=param[1]
k=param[2]
sigma=param[3]
N<-length(data)
if(test==T)
dt=times
else
dt<-diff(times,1)
rate=data[1:(N-1)]
lagrate=data[2:N]
ncp= rate*((4*k*exp(-k*dt))/(sigma^2*(1-exp(-k*dt))))
d=4*theta*k/sigma^2
c=4*k/(sigma^2*(1-exp(-k*dt)))
res<-sum(dchisq(c*lagrate,df=d,ncp=ncp,log=TRUE)+log(c))
if(addsign)
return(-res)
else
return(res)
}

MLE_CIR=optim(par=c(0.1,0.1,0.1),fn=CIRloglike,method=
"L-BFGS-B", lower=c(0.01,0.01,0.01),upper=c(1,1,1),data
=interestrate$interest,times=1/12,test=T)$par
# results >[1] 0.02040567 0.09839527 0.04772081
```

Value of a zero coupon bond with interest rate follow CIR model, which can be calculated by Eq. (2.8)–(2.11).

```
theta<-0.09839527*0.02040567/0.08839527 #\theta*=k\theta/
    (k+\lambda_r)
k<-0.08839527 #k*=k+\lambda_r=0.09839527-0.01
sigma2<-0.04772081^2
CirPriceYield<- function(r,tau,Param,priceyn=F){
# r ... r(t) current value of short rate
# tau ...(T-t), time to maturity
# Param ... vector holding the parameters of the CIR
    model
h=sqrt(k^2+2*sigma2)
B= 2*(exp(h*tau)-1) / (2*h+(k+h)*(exp(tau*h)-1))
A= ((2*h*exp((k+h)*(tau)/2)) / (2*h+(k+h)*(exp(tau*h)-1))
    )^(2*k*theta/sigma2)
if(priceyn){
if(tau==0) return(1) #price is par-value(1) at maturity
else return(A*exp(-B*r))
}
else return((r*B-log(A))/tau)
}
priceCIR<-c()
for (j in 1:length(tau))
{
priceCIR<-c(priceCIR,CirPriceYield(r,tau[j],c(theta,k,
    sigma2),T))
}
```

C.3 Claim Frequency Distribution

```
## Having the data set dfs, need to estimate parameters
    Poisson
attach(dfs)
xdata <- cumsum(dfs$Time)
ydata <- seq_along(xdata)/cumsum(dfs$Time)
a<-seq(from=40,to=879,by=15)
xdata<-xdata[5:879]
ydata<-ydata[5:879]
data<-data.frame(ydata,xdata)
detach(dfs)
attach(data)
# some starting values
a = 27
b = -3
c = -0.3
d = 4
o = 8
# fit the model
fit = nls (ydata~a+b*(sin(xdata+c))^2+d*exp(cos((2*pi*
xdata)/o)), start=list(a=a,b=b,c=c,d=d,o=o), data=data)
#          a          b          c          d          o
#31.067647  -1.122352  -0.473033  1.167737  7.704062
a=coef(fit)[1]
b=coef(fit)[2]
c=coef(fit)[3]
d=coef(fit)[4]
o=coef(fit)[5]
NHP<-function(time){a+b*(sin(time+c))^2+d*exp(cos((2*pi*
    time)/o))}
```

```
# Intensity of Poisson process
fitdistr(df1$Time,"exponential")
fitdistr(df2$Time,"exponential")
```

C.4 Value of CAT Bond

As a demonstrator, here we calculate the probability function given in Proposition 4.1.1, with lognormal distribution and NHPP. Other distributions, one can easily change the distribution code, i.e. to **pgev** function.

```
rate1 = 34.967753 #rate of Poisson process in state 1
rate2 = 10.520688 #rate of Poisson process in state 2
#claim size distribution
mulog1<-2.84877246
sigmalog1<- 1.27307413
mulog2<- 2.9874937
sigmalog2<-1.1261473

deltat<-0.0002 # step of dt
deltax<-0.5      # step of dx

resultinfsum<-c()

for (seno in 1:400)
{
D=myGrid[seno,2]
T=myGrid[seno,1]
runs<-floor(T/deltat)
runss<-runs*2+1
choicemiddlet<-seq(2,runss,by=2)
```

```
noindext<-seq(1,runss,by=2)
t<-seq(from=0, to=T, length.out=runss)
indext<-t[noindext]
t<-t[choicemiddlet]

runsx<-floor(D/deltax)
runssx<-runsx*2+1
choicemiddlex<-seq(2,runssx,by=2)
noindexx<-seq(1,runssx,by=2)
d<-seq(from=0, to=D, length.out=runssx)
indexx<-d[noindexx]
d<-d[choicemiddlex]

#Conditional probability of sojourn time matrix AF
ag1<-pexp(t, rate = rate1 , lower.tail = TRUE, log.p =
    FALSE)
ag2<-pexp(t, rate = rate2 , lower.tail = TRUE, log.p =
    FALSE)
dag1<- pexp(indext, rate = rate1 , lower.tail = TRUE, log
    .p = FALSE)
dag2<- pexp(indext, rate = rate2 , lower.tail = TRUE, log
    .p = FALSE)
AG <-array(NA, dim=c(2,2,runs))
AG[1,1,] <- ag1
AG[1,2,] <- ag1
AG[2,1,] <- ag2
AG[2,2,] <- ag2
dAG <-array(NA, dim=c(2,2,runs))
dAG[1,1,] <- diff(dag1)
dAG[1,2,] <- diff(dag1)
dAG[2,1,] <- diff(dag2)
```

```
dAG[2,2,] <- diff(dag2)
AQ<-array(P,dim=c(2,2,runs))*AG
dAQ<-array(P,dim=c(2,2,runs))*dAG

#Conditional probability of claim sizes matrix BF
bg1<-plnorm(d, meanlog = mulog1, sdlog = sigmalog1)
bg2<-plnorm(d, meanlog = mulog2, sdlog = sigmalog2)
BG <-array(NA, dim=c(2,2,runsx))
BG[1,1,] <- bg1
BG[2,1,] <- bg1
BG[1,2,] <- bg2
BG[2,2,] <- bg2
dbg1<-plnorm(indexx, meanlog = mulog1, sdlog = sigmalog1)
dbg2<-plnorm(indexx, meanlog = mulog2, sdlog = sigmalog2)
dBG <-array(NA, dim=c(2,2,runsx))
dBG[1,1,] <- diff(dbg1)
dBG[1,2,] <- diff(dbg2)
dBG[2,1,] <- diff(dbg1)
dBG[2,2,] <- diff(dbg2)
BQ<-array(P,dim=c(2,2,runsx))*BG
dBQ<-array(P,dim=c(2,2,runsx))*dBG

#relax the dependence on J_n=j we need the AH
AH <-array(NA, dim=c(1,2,runs))
AH[1,1,]<-1-exp(-rate1*t)
AH[1,2,]<-1-exp(-rate2*t)

######### n=0-times convolution
M0<-array(0, dim=c(2,2))
M0[1,1]<-exp(-rate1*T)*(1-plnorm(D, meanlog = mulog1,
    sdlog = sigmalog1))
```

```
M0[2,2]<-exp(-rate2*T)*(1-plnorm(D, meanlog = mulog2,
    sdlog = sigmalog2))
M<-M0
############## n=1
dQn<-array(NA, dim=c(2,2,runs-1))
dQn[1,1,] <- diff(AQ[1,1,]*BG[1,1,runsx])
dQn[1,2,] <- diff(AQ[1,2,]*BG[1,2,runsx])
dQn[2,1,] <- diff(AQ[2,1,]*BG[2,1,runsx])
dQn[2,2,] <- diff(AQ[2,2,]*BG[2,2,runsx])
dQn<-array(c(dQn,0,0,0,0),dim=c(2,2,runs))
M1<-array(0, dim=c(2,2))
M1[1,1]<-AQ[1,1,runs]*BG[1,1,runsx]-sum(rev(AH[1,1,])*dQn
    [1,1,])
M1[1,2]<-AQ[1,2,runs]*BG[1,2,runsx]-sum(rev(AH[1,2,])*dQn
    [1,2,])
M1[2,1]<-AQ[2,1,runs]*BG[2,1,runsx]-sum(rev(AH[1,1,])*dQn
    [2,1,])
M1[2,2]<-AQ[2,2,runs]*BG[2,2,runsx]-sum(rev(AH[1,2,])*dQn
    [2,2,])
M<-M+M1
############## n=n
AQnn<-AQ
BQnn<-BQ
Pnn<-P
nfoldtimes<-200
for (times in 1:nfoldtimes)
{
AQn<-array(0, dim=c(2,2,runs))
BQn<-array(0, dim=c(2,2,runsx))
Mn<-array(0, dim=c(2,2))
for(j in 1:runs)
```

```
{
AQn[1,1,j]<-sum(rev(AQnn[1,1,1:j])*dAQ[1,1,1:j]+rev(AQnn[2,1,1:j])*dAQ[1,2,1:j])
AQn[1,2,j]<-sum(rev(AQnn[1,2,1:j])*dAQ[1,1,1:j]+rev(AQnn[2,2,1:j])*dAQ[1,2,1:j])
AQn[2,1,j]<-sum(rev(AQnn[1,1,1:j])*dAQ[2,1,1:j]+rev(AQnn[2,1,1:j])*dAQ[2,2,1:j])
AQn[2,2,j]<-sum(rev(AQnn[1,2,1:j])*dAQ[2,1,1:j]+rev(AQnn[2,2,1:j])*dAQ[2,2,1:j])
}
for(j in 1:runsx)
{
BQn[1,1,j]<-sum(rev(BQnn[1,1,1:j])*dBQ[1,1,1:j]+rev(BQnn[2,1,1:j])*dBQ[1,2,1:j])
BQn[1,2,j]<-sum(rev(BQnn[1,2,1:j])*dBQ[1,1,1:j]+rev(BQnn[2,2,1:j])*dBQ[1,2,1:j])
BQn[2,1,j]<-sum(rev(BQnn[1,1,1:j])*dBQ[2,1,1:j]+rev(BQnn[2,1,1:j])*dBQ[2,2,1:j])
BQn[2,2,j]<-sum(rev(BQnn[1,2,1:j])*dBQ[2,1,1:j]+rev(BQnn[2,2,1:j])*dBQ[2,2,1:j])
}
Pn<-array(NA,dim=c(2,2))
Pn[1,1]<-Pnn[1,1]*P[1,1]+Pnn[1,2]*P[2,1]
Pn[1,2]<-Pnn[1,1]*P[1,2]+Pnn[1,2]*P[2,2]
Pn[2,1]<-Pnn[2,1]*P[1,1]+Pnn[2,2]*P[2,1]
Pn[2,2]<-Pnn[2,1]*P[1,2]+Pnn[2,2]*P[2,2]
BGn<-BQn[,,runsx]/Pn
dQn<-array(NA, dim=c(2,2,runs-1))
dQn[1,1,] <- diff(AQn[1,1,]*BGn[1,1])
dQn[1,2,] <- diff(AQn[1,2,]*BGn[1,2])
dQn[2,1,] <- diff(AQn[2,1,]*BGn[2,1])
```

```
dQn[2,2,] <- diff(AQn[2,2,]*BGn[2,2])
dQn<-array(c(dQn,0,0,0,0),dim=c(2,2,runs))
Mn[1,1]<-AQn[1,1,runs]*BGn[1,1]-sum(rev(AH[1,1,])*dQn
    [1,1,])
Mn[1,2]<-AQn[1,2,runs]*BGn[1,2]-sum(rev(AH[1,2,])*dQn
    [1,2,])
Mn[2,1]<-AQn[2,1,runs]*BGn[2,1]-sum(rev(AH[1,1,])*dQn
    [2,1,])
Mn[2,2]<-AQn[2,2,runs]*BGn[2,2]-sum(rev(AH[1,2,])*dQn
    [2,2,])

M<-M+Mn
AQnn<-AQn
BQnn<-BQn
Pnn<-Pn
if (mean(Mn)<1e-9 && times>40){break}
}
##### stationary probability
pi1<-817/879
pi2<-62/879
infsum<-(M[1,1]+M[1,2])*pi1+(M[2,1]+M[2,2])*pi2
resultinfsum<-c(resultinfsum,infsum)
print(c(seno,infsum))
}
```

And finally obtain the price of the CAT bond for each payoff function.

```
bond1<-priceCIR*(p+(1-p)*resultinfsum)
bond2<-priceCIR*(0.25+0.5*resultinfsum+0.25*resultinfsum
    [1:20])
bond3<-priceCIR*(1+0.1*resultinfsum)
bond4<-priceCIR*(p+(1-p-0.1-p*0.05)*resultinfsum+p*0.01)
```

```
# 3D plot of the price in terms of D and t, here plot
    bond1 as an example.
library(lattice)
wireframe(bond1 ~ myGrid$t * myGrid$D,
 scales = list(arrows = FALSE),
  xlab = "T(years)", ylab = "D($10million)", zlab="V($)",
  drape = TRUE, colorkey = TRUE,
 screen = list(z = -45, x = -60),
)
```

Appendix D

R Code For Chapter 5

```
rate0 = 0.5  #rate of Poisson process in state 0
rate1 = 5  #rate of Poisson process in state 1
rate2 = 20 #rate of Poisson process in state 2
rate3 = 10 #rate of Poisson process in state 3
rate4 = 30 #rate of Poisson process in state 4

#claim size distribution
mulog1<-2.5
sigmalog1<- 1.5
mulog2<- 3
sigmalog2<-1.2
mulog3<-2
sigmalog3<- 1
mulog4<-1
sigmalog4<-0.8
P<-array(c(0.199,0.5,0.2,0.3, 0.1,0.098,0.1,0.2, 0.3,0.2,0.397,0.4, 0.4,0.2,0.3,0.096),dim=c(4,4))
P0<-array(c(0.001,0.002,0.003,0.004),dim=c(4,1))

#stationary probability
```

```
pi0<-0
pi1<-0.2642861
pi2<-0.1264328
pi3<-0.3476325
pi4<-0.2609975

t<-seq(from=0.5,to=2,length=7)
D<-seq(from=100, to=1600, length.out=16)
myGrid <- data.frame(expand.grid(t,D))
colnames(myGrid) <- c("t","D")
tau<-myGrid$t # tau ...(T-t), time to maturity
```

Here we obtain $F_1(t,D)$ and $F_2(t,D)$ given in Eq. (5.1) and (5.2), respectively.

```
resultinfsum1<-c()
resultinfsum2<-c()
deltat<-0.0002
deltax<-0.5
for (seno in 1:112)
{
D=myGrid[seno,2]
T=myGrid[seno,1]
runs<-floor(T/deltat)
runss<-runs*2+1
choicemiddlet<-seq(2,runss,by=2)
noindext<-seq(1,runss,by=2)
t<-seq(from=0, to=T, length.out=runss)
indext<-t[noindext]
t<-t[choicemiddlet]
runsx<-floor(D/deltax)
runssx<-runsx*2+1
choicemiddlex<-seq(2,runssx,by=2)
```

```
noindexx<-seq(1,runssx,by=2)
d<-seq(from=0, to=D, length.out=runssx)
indexx<-d[noindexx]
d<-d[choicemiddlex]

#Conditional probability of sojourn time matrix AF
ag1<-pexp(t, rate = rate1 , lower.tail = TRUE, log.p =
   FALSE)
ag2<-pexp(t, rate = rate2 , lower.tail = TRUE, log.p =
   FALSE)
ag3<-pexp(t, rate = rate3 , lower.tail = TRUE, log.p =
   FALSE)
ag4<-pexp(t, rate = rate4 , lower.tail = TRUE, log.p =
   FALSE)
dag1<- pexp(indext, rate = rate1 , lower.tail = TRUE, log
   .p = FALSE)
dag2<- pexp(indext, rate = rate2 , lower.tail = TRUE, log
   .p = FALSE)
dag3<- pexp(indext, rate = rate3 , lower.tail = TRUE, log
   .p = FALSE)
dag4<- pexp(indext, rate = rate4 , lower.tail = TRUE, log
   .p = FALSE)
AG <-array(NA, dim=c(4,4,runs))
AG[1,1,] <- ag1
AG[2,1,] <- ag2
AG[3,1,] <- ag3
AG[4,1,] <- ag4
AG[1,2,] <- ag1
AG[2,2,] <- ag2
AG[3,2,] <- ag3
AG[4,2,] <- ag4
```

```
AG[1,3,] <- ag1
AG[2,3,] <- ag2
AG[3,3,] <- ag3
AG[4,3,] <- ag4
AG[1,4,] <- ag1
AG[2,4,] <- ag2
AG[3,4,] <- ag3
AG[4,4,] <- ag4
dAG <-array(NA, dim=c(4,4,runs))
dAG[1,1,] <- diff(dag1)
dAG[2,1,] <- diff(dag2)
dAG[3,1,] <- diff(dag3)
dAG[4,1,] <- diff(dag4)
dAG[1,2,] <- diff(dag1)
dAG[2,2,] <- diff(dag2)
dAG[3,2,] <- diff(dag3)
dAG[4,2,] <- diff(dag4)
dAG[1,3,] <- diff(dag1)
dAG[2,3,] <- diff(dag2)
dAG[3,3,] <- diff(dag3)
dAG[4,3,] <- diff(dag4)
dAG[1,4,] <- diff(dag1)
dAG[2,4,] <- diff(dag2)
dAG[3,4,] <- diff(dag3)
dAG[4,4,] <- diff(dag4)
AQ<-array(P,dim=c(4,4,runs))*AG
dAQ<-array(P,dim=c(4,4,runs))*dAG

#Conditional probability of claim sizes matrix BF
bg1<-plnorm(d, mulog1, sigmalog1)
bg2<-plnorm(d, mulog2, sigmalog2)
```

```
bg3<-plnorm(d, mulog3, sigmalog3)
bg4<-plnorm(d, mulog4, sigmalog4)
BG <-array(NA, dim=c(4,4,runsx))
BG[1,1,] <- bg1
BG[1,2,] <- bg2
BG[1,3,] <- bg3
BG[1,4,] <- bg4
BG[2,1,] <- bg1
BG[2,2,] <- bg2
BG[2,3,] <- bg3
BG[2,4,] <- bg4
BG[3,1,] <- bg1
BG[3,2,] <- bg2
BG[3,3,] <- bg3
BG[3,4,] <- bg4
BG[4,1,] <- bg1
BG[4,2,] <- bg2
BG[4,3,] <- bg3
BG[4,4,] <- bg4
dbg1<-plnorm(indexx, mulog1, sigmalog1)
dbg2<-plnorm(indexx, mulog2, sigmalog2)
dbg3<-plnorm(indexx, mulog3, sigmalog3)
dbg4<-plnorm(indexx, mulog4, sigmalog4)
dBG <-array(NA, dim=c(4,4,runsx))
dBG[1,1,] <- diff(dbg1)
dBG[1,2,] <- diff(dbg2)
dBG[1,3,] <- diff(dbg3)
dBG[1,4,] <- diff(dbg4)
dBG[2,1,] <- diff(dbg1)
dBG[2,2,] <- diff(dbg2)
dBG[2,3,] <- diff(dbg3)
```

```
dBG[2,4,] <- diff(dbg4)
dBG[3,1,] <- diff(dbg1)
dBG[3,2,] <- diff(dbg2)
dBG[3,3,] <- diff(dbg3)
dBG[3,4,] <- diff(dbg4)
dBG[4,1,] <- diff(dbg1)
dBG[4,2,] <- diff(dbg2)
dBG[4,3,] <- diff(dbg3)
dBG[4,4,] <- diff(dbg4)
BQ<-array(P,dim=c(4,4,runsx))*BG
dBQ<-array(P,dim=c(4,4,runsx))*dBG

#relax the dependence on J_n=j we need the AH
AH <-array(NA, dim=c(1,4,runs))
AH[1,1,]<-pexp(rate1*t) #P[1,1,1]*pexp(rate1*t)+P[1,2,1]*pexp(rate2*t)+P[1,3,1]*pexp(rate2*t)+P[1,4,1]*pexp(rate4*t)
AH[1,2,]<-pexp(rate2*t) #P[2,1,1]*pexp(rate1*t)+P[2,2,1]*pexp(rate2*t)+P[2,3,1]*pexp(rate2*t)+P[2,4,1]*pexp(rate4*t)
AH[1,3,]<-pexp(rate3*t) #P[3,1,1]*pexp(rate1*t)+P[3,2,1]*pexp(rate2*t)+P[3,3,1]*pexp(rate2*t)+P[3,4,1]*pexp(rate4*t)
AH[1,4,]<-pexp(rate4*t) #P[4,1,1]*pexp(rate1*t)+P[4,2,1]*pexp(rate2*t)+P[4,3,1]*pexp(rate2*t)+P[4,4,1]*pexp(rate4*t)

######### n=0-times convolution
M0<-array(0, dim=c(4,4)) #state in J'
M<-M0
F20<-array(0, dim=c(4,1)) #from i to state 0
```

```r
F2<-F20
############## n=1
dQn<-array(NA, dim=c(4,4,runs-1))
dQn[1,1,] <- diff(AQ[1,1,]*BG[1,1,runsx])
dQn[1,2,] <- diff(AQ[1,2,]*BG[1,2,runsx])
dQn[1,3,] <- diff(AQ[1,3,]*BG[1,3,runsx])
dQn[1,4,] <- diff(AQ[1,4,]*BG[1,4,runsx])
dQn[2,1,] <- diff(AQ[2,1,]*BG[2,1,runsx])
dQn[2,2,] <- diff(AQ[2,2,]*BG[2,2,runsx])
dQn[2,3,] <- diff(AQ[2,3,]*BG[2,3,runsx])
dQn[2,4,] <- diff(AQ[2,4,]*BG[2,4,runsx])
dQn[3,1,] <- diff(AQ[3,1,]*BG[3,1,runsx])
dQn[3,2,] <- diff(AQ[3,2,]*BG[3,2,runsx])
dQn[3,3,] <- diff(AQ[3,3,]*BG[3,3,runsx])
dQn[3,4,] <- diff(AQ[3,4,]*BG[3,4,runsx])
dQn[4,1,] <- diff(AQ[4,1,]*BG[4,1,runsx])
dQn[4,2,] <- diff(AQ[4,2,]*BG[4,2,runsx])
dQn[4,3,] <- diff(AQ[4,3,]*BG[4,3,runsx])
dQn[4,4,] <- diff(AQ[4,4,]*BG[4,4,runsx])
dQn<-array(c(dQn,0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0),dim=
    c(4,4,runs))
M1<-array(0, dim=c(4,4))
M1[1,1]<-AQ[1,1,runs]*BG[1,1,runsx]-sum(rev(AH[1,1,])*dQn
    [1,1,])
M1[1,2]<-AQ[1,2,runs]*BG[1,2,runsx]-sum(rev(AH[1,2,])*dQn
    [1,2,])
M1[1,3]<-AQ[1,3,runs]*BG[1,3,runsx]-sum(rev(AH[1,3,])*dQn
    [1,3,])
M1[1,4]<-AQ[1,4,runs]*BG[1,4,runsx]-sum(rev(AH[1,4,])*dQn
    [1,4,])
```

```
M1[2,1]<-AQ[2,1,runs]*BG[2,1,runsx]-sum(rev(AH[1,1,])*dQn[2,1,])
M1[2,2]<-AQ[2,2,runs]*BG[2,2,runsx]-sum(rev(AH[1,2,])*dQn[2,2,])
M1[2,3]<-AQ[2,3,runs]*BG[2,3,runsx]-sum(rev(AH[1,3,])*dQn[2,3,])
M1[2,4]<-AQ[2,4,runs]*BG[2,4,runsx]-sum(rev(AH[1,4,])*dQn[2,4,])
M1[3,1]<-AQ[3,1,runs]*BG[3,1,runsx]-sum(rev(AH[1,1,])*dQn[3,1,])
M1[3,2]<-AQ[3,2,runs]*BG[3,2,runsx]-sum(rev(AH[1,2,])*dQn[3,2,])
M1[3,3]<-AQ[3,3,runs]*BG[3,3,runsx]-sum(rev(AH[1,3,])*dQn[3,3,])
M1[3,4]<-AQ[3,4,runs]*BG[3,4,runsx]-sum(rev(AH[1,4,])*dQn[3,4,])
M1[4,1]<-AQ[4,1,runs]*BG[4,1,runsx]-sum(rev(AH[1,1,])*dQn[4,1,])
M1[4,2]<-AQ[4,2,runs]*BG[4,2,runsx]-sum(rev(AH[1,2,])*dQn[4,2,])
M1[4,3]<-AQ[4,3,runs]*BG[4,3,runsx]-sum(rev(AH[1,3,])*dQn[4,3,])
M1[4,4]<-AQ[4,4,runs]*BG[4,4,runsx]-sum(rev(AH[1,4,])*dQn[4,4,])
M<-M+M1
 F21 <-array(NA, dim=c(4,1))
 F21[1,1]<-ag1[runs]*P0[1,1]
 F21[2,1]<-ag2[runs]*P0[2,1]
 F21[3,1]<-ag3[runs]*P0[3,1]
 F21[4,1]<-ag4[runs]*P0[4,1]
 F2<-F2+F21
```

```
dF2<-array(NA, dim=c(4,1,runs))
dF2[1,1,]<-diff(dag1*P0[1,1])
dF2[2,1,]<-diff(dag2*P0[2,1])
dF2[3,1,]<-diff(dag3*P0[3,1])
dF2[4,1,]<-diff(dag4*P0[4,1])
############## n=n
AQnn<-AQ
BQnn<-BQ
Pnn<-P
nfoldtimes<-200
for (times in 1:nfoldtimes)
{
AQn<-array(0, dim=c(4,4,runs))
BQn<-array(0, dim=c(4,4,runsx))
Pn<-array(NA,dim=c(4,4))
Mn<-array(0, dim=c(4,4))
F2n<-array(0, dim=c(4,1))

#for the case of jump into state 0
F2n[1,1]<-sum(rev(AQnn[1,1,1:runs]*(BQnn[1,1,runsx]/Pnn
    [1,1]))*dF2[1,1,1: runs]+rev(AQnn[1,2,1:runs]*(BQnn
    [1,2,runsx]/Pnn[1,2]))*dF2[2,1,1: runs]
+rev(AQnn[1,3,1: runs]*(BQnn[1,3,runsx]/Pnn[1,3]))*dF2
    [3,1,1: runs]+rev(AQnn[1,4,1: runs]*(BQnn[1,4,runsx]/
    Pnn[1,4]))*dF2[4,1,1: runs])
F2n[2,1]<-sum(rev(AQnn[2,1,1:runs]*(BQnn[2,1,runsx]/Pnn
    [2,1]))*dF2[1,1,1: runs]+rev(AQnn[2,2,1:runs]*(BQnn
    [2,2,runsx]/Pnn[2,2]))*dF2[2,1,1: runs]
+rev(AQnn[2,3,1: runs]*(BQnn[2,3,runsx]/Pnn[2,3]))*dF2
    [3,1,1: runs]+rev(AQnn[2,4,1: runs]*(BQnn[2,4,runsx]/
    Pnn[2,4]))*dF2[4,1,1: runs])
```

```
F2n[3,1]<-sum(rev(AQnn[3,1,1:runs]*(BQnn[3,1,runsx]/Pnn[3,1]))*dF2[1,1,1: runs]+rev(AQnn[3,2,1:runs]*(BQnn[3,2,runsx]/Pnn[3,2]))*dF2[2,1,1: runs]
+rev(AQnn[3,3,1: runs]*(BQnn[3,3,runsx]/Pnn[3,3]))*dF2[3,1,1: runs]+rev(AQnn[3,4,1: runs]*(BQnn[3,4,runsx]/Pnn[3,4]))*dF2[4,1,1: runs])
F2n[4,1]<-sum(rev(AQnn[4,1,1:runs]*(BQnn[4,1,runsx]/Pnn[4,1]))*dF2[1,1,1: runs]+rev(AQnn[4,2,1:runs]*(BQnn[4,2,runsx]/Pnn[4,2]))*dF2[2,1,1: runs]
+rev(AQnn[4,3,1: runs]*(BQnn[4,3,runsx]/Pnn[4,3]))*dF2[3,1,1: runs]+rev(AQnn[4,4,1: runs]*(BQnn[4,4,runsx]/Pnn[4,4]))*dF2[4,1,1: runs])
for(j in 1:runs)
{
AQn[1,1,j]<-sum(rev(AQnn[1,1,1:j])*dAQ[1,1,1:j]+rev(AQnn[2,1,1:j])*dAQ[1,2,1:j]+ rev(AQnn[3,1,1:j])*dAQ[1,3,1:j]+rev(AQnn[4,1,1:j])*dAQ[1,4,1:j])
AQn[1,2,j]<-sum(rev(AQnn[1,2,1:j])*dAQ[1,1,1:j]+rev(AQnn[2,2,1:j])*dAQ[1,2,1:j]+ rev(AQnn[3,2,1:j])*dAQ[1,3,1:j]+rev(AQnn[4,2,1:j])*dAQ[1,4,1:j])
AQn[1,3,j]<-sum(rev(AQnn[1,3,1:j])*dAQ[1,1,1:j]+rev(AQnn[2,3,1:j])*dAQ[1,2,1:j]+ rev(AQnn[3,3,1:j])*dAQ[1,3,1:j]+rev(AQnn[4,3,1:j])*dAQ[1,4,1:j])
AQn[1,4,j]<-sum(rev(AQnn[1,4,1:j])*dAQ[1,1,1:j]+rev(AQnn[2,4,1:j])*dAQ[1,2,1:j]+ rev(AQnn[3,4,1:j])*dAQ[1,3,1:j]+rev(AQnn[4,4,1:j])*dAQ[1,4,1:j])

AQn[2,1,j]<-sum(rev(AQnn[1,1,1:j])*dAQ[2,1,1:j]+rev(AQnn[2,1,1:j])*dAQ[2,2,1:j]+ rev(AQnn[3,1,1:j])*dAQ[2,3,1:j]+rev(AQnn[4,1,1:j])*dAQ[2,4,1:j])
```

```
AQn[2,2,j]<-sum(rev(AQnn[1,2,1:j])*dAQ[2,1,1:j]+rev(AQnn[2,2,1:j])*dAQ[2,2,1:j]+ rev(AQnn[3,2,1:j])*dAQ[2,3,1:j]+rev(AQnn[4,2,1:j])*dAQ[2,4,1:j])
AQn[2,3,j]<-sum(rev(AQnn[1,3,1:j])*dAQ[2,1,1:j]+rev(AQnn[2,3,1:j])*dAQ[2,2,1:j]+ rev(AQnn[3,3,1:j])*dAQ[2,3,1:j]+rev(AQnn[4,3,1:j])*dAQ[2,4,1:j])
AQn[2,4,j]<-sum(rev(AQnn[1,4,1:j])*dAQ[2,1,1:j]+rev(AQnn[2,4,1:j])*dAQ[2,2,1:j]+ rev(AQnn[3,4,1:j])*dAQ[2,3,1:j]+rev(AQnn[4,4,1:j])*dAQ[2,4,1:j])

AQn[3,1,j]<-sum(rev(AQnn[1,1,1:j])*dAQ[3,1,1:j]+rev(AQnn[2,1,1:j])*dAQ[3,2,1:j]+ rev(AQnn[3,1,1:j])*dAQ[3,3,1:j]+rev(AQnn[4,1,1:j])*dAQ[3,4,1:j])
AQn[3,2,j]<-sum(rev(AQnn[1,2,1:j])*dAQ[3,1,1:j]+rev(AQnn[2,2,1:j])*dAQ[3,2,1:j]+ rev(AQnn[3,2,1:j])*dAQ[3,3,1:j]+rev(AQnn[4,2,1:j])*dAQ[3,4,1:j])
AQn[3,3,j]<-sum(rev(AQnn[1,3,1:j])*dAQ[3,1,1:j]+rev(AQnn[2,3,1:j])*dAQ[3,2,1:j]+ rev(AQnn[3,3,1:j])*dAQ[3,3,1:j]+rev(AQnn[4,3,1:j])*dAQ[3,4,1:j])
AQn[3,4,j]<-sum(rev(AQnn[1,4,1:j])*dAQ[3,1,1:j]+rev(AQnn[2,4,1:j])*dAQ[3,2,1:j]+ rev(AQnn[3,4,1:j])*dAQ[3,3,1:j]+rev(AQnn[4,4,1:j])*dAQ[3,4,1:j])

AQn[4,1,j]<-sum(rev(AQnn[1,1,1:j])*dAQ[4,1,1:j]+rev(AQnn[2,1,1:j])*dAQ[4,2,1:j]+ rev(AQnn[3,1,1:j])*dAQ[4,3,1:j]+rev(AQnn[4,1,1:j])*dAQ[4,4,1:j])
AQn[4,2,j]<-sum(rev(AQnn[1,2,1:j])*dAQ[4,1,1:j]+rev(AQnn[2,2,1:j])*dAQ[4,2,1:j]+ rev(AQnn[3,2,1:j])*dAQ[4,3,1:j]+rev(AQnn[4,2,1:j])*dAQ[4,4,1:j])
AQn[4,3,j]<-sum(rev(AQnn[1,3,1:j])*dAQ[4,1,1:j]+rev(AQnn[2,3,1:j])*dAQ[4,2,1:j]+ rev(AQnn[3,3,1:j])*dAQ[4,3,1:
```

```
j]+rev(AQnn[4,3,1:j])*dAQ[4,4,1:j])
AQn[4,4,j]<-sum(rev(AQnn[1,4,1:j])*dAQ[4,1,1:j]+rev(AQnn[2,4,1:j])*dAQ[4,2,1:j]+ rev(AQnn[3,4,1:j])*dAQ[4,3,1:j]+rev(AQnn[4,4,1:j])*dAQ[4,4,1:j])
}
for(j in 1:runsx)
{
BQn[1,1,j]<-sum(rev(BQnn[1,1,1:j])*dBQ[1,1,1:j]+rev(BQnn[2,1,1:j])*dBQ[1,2,1:j]+ rev(BQnn[3,1,1:j])*dBQ[1,3,1:j]+rev(BQnn[4,1,1:j])*dBQ[1,4,1:j])
BQn[1,2,j]<-sum(rev(BQnn[1,2,1:j])*dBQ[1,1,1:j]+rev(BQnn[2,2,1:j])*dBQ[1,2,1:j]+ rev(BQnn[3,2,1:j])*dBQ[1,3,1:j]+rev(BQnn[4,2,1:j])*dBQ[1,4,1:j])
BQn[1,3,j]<-sum(rev(BQnn[1,3,1:j])*dBQ[1,1,1:j]+rev(BQnn[2,3,1:j])*dBQ[1,2,1:j]+ rev(BQnn[3,3,1:j])*dBQ[1,3,1:j]+rev(BQnn[4,3,1:j])*dBQ[1,4,1:j])
BQn[1,4,j]<-sum(rev(BQnn[1,4,1:j])*dBQ[1,1,1:j]+rev(BQnn[2,4,1:j])*dBQ[1,2,1:j]+ rev(BQnn[3,4,1:j])*dBQ[1,3,1:j]+rev(BQnn[4,4,1:j])*dBQ[1,4,1:j])

BQn[2,1,j]<-sum(rev(BQnn[1,1,1:j])*dBQ[2,1,1:j]+rev(BQnn[2,1,1:j])*dBQ[2,2,1:j]+ rev(BQnn[3,1,1:j])*dBQ[2,3,1:j]+rev(BQnn[4,1,1:j])*dBQ[2,4,1:j])
BQn[2,2,j]<-sum(rev(BQnn[1,2,1:j])*dBQ[2,1,1:j]+rev(BQnn[2,2,1:j])*dBQ[2,2,1:j]+ rev(BQnn[3,2,1:j])*dBQ[2,3,1:j]+rev(BQnn[4,2,1:j])*dBQ[2,4,1:j])
BQn[2,3,j]<-sum(rev(BQnn[1,3,1:j])*dBQ[2,1,1:j]+rev(BQnn[2,3,1:j])*dBQ[2,2,1:j]+ rev(BQnn[3,3,1:j])*dBQ[2,3,1:j]+rev(BQnn[4,3,1:j])*dBQ[2,4,1:j])
BQn[2,4,j]<-sum(rev(BQnn[1,4,1:j])*dBQ[2,1,1:j]+rev(BQnn[2,4,1:j])*dBQ[2,2,1:j]+ rev(BQnn[3,4,1:j])*dBQ[2,3,1:
```

```
j]+rev(BQnn[4,4,1:j])*dBQ[2,4,1:j])

BQn[3,1,j]<-sum(rev(BQnn[1,1,1:j])*dBQ[3,1,1:j]+rev(BQnn[2,1,1:j])*dBQ[3,2,1:j]+ rev(BQnn[3,1,1:j])*dBQ[3,3,1:j]+rev(BQnn[4,1,1:j])*dBQ[3,4,1:j])
BQn[3,2,j]<-sum(rev(BQnn[1,2,1:j])*dBQ[3,1,1:j]+rev(BQnn[2,2,1:j])*dBQ[3,2,1:j]+ rev(BQnn[3,2,1:j])*dBQ[3,3,1:j]+rev(BQnn[4,2,1:j])*dBQ[3,4,1:j])
BQn[3,3,j]<-sum(rev(BQnn[1,3,1:j])*dBQ[3,1,1:j]+rev(BQnn[2,3,1:j])*dBQ[3,2,1:j]+ rev(BQnn[3,3,1:j])*dBQ[3,3,1:j]+rev(BQnn[4,3,1:j])*dBQ[3,4,1:j])
BQn[3,4,j]<-sum(rev(BQnn[1,4,1:j])*dBQ[3,1,1:j]+rev(BQnn[2,4,1:j])*dBQ[3,2,1:j]+ rev(BQnn[3,4,1:j])*dBQ[3,3,1:j]+rev(BQnn[4,4,1:j])*dBQ[3,4,1:j])

BQn[4,1,j]<-sum(rev(BQnn[1,1,1:j])*dBQ[4,1,1:j]+rev(BQnn[2,1,1:j])*dBQ[4,2,1:j]+ rev(BQnn[3,1,1:j])*dBQ[4,3,1:j]+rev(BQnn[4,1,1:j])*dBQ[4,4,1:j])
BQn[4,2,j]<-sum(rev(BQnn[1,2,1:j])*dBQ[4,1,1:j]+rev(BQnn[2,2,1:j])*dBQ[4,2,1:j]+ rev(BQnn[3,2,1:j])*dBQ[4,3,1:j]+rev(BQnn[4,2,1:j])*dBQ[4,4,1:j])
BQn[4,3,j]<-sum(rev(BQnn[1,3,1:j])*dBQ[4,1,1:j]+rev(BQnn[2,3,1:j])*dBQ[4,2,1:j]+ rev(BQnn[3,3,1:j])*dBQ[4,3,1:j]+rev(BQnn[4,3,1:j])*dBQ[4,4,1:j])
BQn[4,4,j]<-sum(rev(BQnn[1,4,1:j])*dBQ[4,1,1:j]+rev(BQnn[2,4,1:j])*dBQ[4,2,1:j]+ rev(BQnn[3,4,1:j])*dBQ[4,3,1:j]+rev(BQnn[4,4,1:j])*dBQ[4,4,1:j])
}
Pn[1,1]<-Pnn[1,1]*P[1,1]+Pnn[1,2]*P[2,1]+Pnn[1,3]*P[3,1]+Pnn[1,4]*P[4,1]
```

```
Pn[1,2]<-Pnn[1,1]*P[1,2]+Pnn[1,2]*P[2,2]+Pnn[1,3]*P[3,2]+
    Pnn[1,4]*P[4,2]
Pn[1,3]<-Pnn[1,1]*P[1,3]+Pnn[1,2]*P[2,3]+Pnn[1,3]*P[3,3]+
    Pnn[1,4]*P[4,3]
Pn[1,4]<-Pnn[1,1]*P[1,4]+Pnn[1,2]*P[2,4]+Pnn[1,3]*P[3,4]+
    Pnn[1,4]*P[4,4]

Pn[2,1]<-Pnn[2,1]*P[1,1]+Pnn[2,2]*P[2,1]+Pnn[2,3]*P[3,1]+
    Pnn[2,4]*P[4,1]
Pn[2,2]<-Pnn[2,1]*P[1,2]+Pnn[2,2]*P[2,2]+Pnn[2,3]*P[3,2]+
    Pnn[2,4]*P[4,2]
Pn[2,3]<-Pnn[2,1]*P[1,3]+Pnn[2,2]*P[2,3]+Pnn[2,3]*P[3,3]+
    Pnn[2,4]*P[4,3]
Pn[2,4]<-Pnn[2,1]*P[1,4]+Pnn[2,2]*P[2,4]+Pnn[2,3]*P[3,4]+
    Pnn[2,4]*P[4,4]

Pn[3,1]<-Pnn[3,1]*P[1,1]+Pnn[3,2]*P[2,1]+Pnn[3,3]*P[3,1]+
    Pnn[3,4]*P[4,1]
Pn[3,2]<-Pnn[3,1]*P[1,2]+Pnn[3,2]*P[2,2]+Pnn[3,3]*P[3,2]+
    Pnn[3,4]*P[4,2]
Pn[3,3]<-Pnn[3,1]*P[1,3]+Pnn[3,2]*P[2,3]+Pnn[3,3]*P[3,3]+
    Pnn[3,4]*P[4,3]
Pn[3,4]<-Pnn[3,1]*P[1,4]+Pnn[3,2]*P[2,4]+Pnn[3,3]*P[3,4]+
    Pnn[3,4]*P[4,4]

Pn[4,1]<-Pnn[4,1]*P[1,1]+Pnn[4,2]*P[2,1]+Pnn[4,3]*P[3,1]+
    Pnn[4,4]*P[4,1]
Pn[4,2]<-Pnn[4,1]*P[1,2]+Pnn[4,2]*P[2,2]+Pnn[4,3]*P[3,2]+
    Pnn[4,4]*P[4,2]
Pn[4,3]<-Pnn[4,1]*P[1,3]+Pnn[4,2]*P[2,3]+Pnn[4,3]*P[3,3]+
    Pnn[4,4]*P[4,3]
```

```
Pn[4,4]<-Pnn[4,1]*P[1,4]+Pnn[4,2]*P[2,4]+Pnn[4,3]*P[3,4]+
    Pnn[4,4]*P[4,4]
BGn<-BQn[,,runsx]/Pn
dQn<-array(NA, dim=c(4,4,runs-1))
dQn[1,1,] <- diff(AQn[1,1,]*BGn[1,1])
dQn[1,2,] <- diff(AQn[1,2,]*BGn[1,2])
dQn[1,3,] <- diff(AQn[1,3,]*BGn[1,3])
dQn[1,4,] <- diff(AQn[1,4,]*BGn[1,4])
dQn[2,1,] <- diff(AQn[2,1,]*BGn[2,1])
dQn[2,2,] <- diff(AQn[2,2,]*BGn[2,2])
dQn[2,3,] <- diff(AQn[2,3,]*BGn[2,3])
dQn[2,4,] <- diff(AQn[2,4,]*BGn[2,4])
dQn[3,1,] <- diff(AQn[3,1,]*BGn[3,1])
dQn[3,2,] <- diff(AQn[3,2,]*BGn[3,2])
dQn[3,3,] <- diff(AQn[3,3,]*BGn[3,3])
dQn[3,4,] <- diff(AQn[3,4,]*BGn[3,4])
dQn[4,1,] <- diff(AQn[4,1,]*BGn[4,1])
dQn[4,2,] <- diff(AQn[4,2,]*BGn[4,2])
dQn[4,3,] <- diff(AQn[4,3,]*BGn[4,3])
dQn[4,4,] <- diff(AQn[4,4,]*BGn[4,4])
dQn<-array(c(dQn,0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0),dim=
    c(4,4,runs))

Mn[1,1]<-AQn[1,1,runs]*BGn[1,1]-sum(rev(AH[1,1,])*dQn
    [1,1,])
Mn[1,2]<-AQn[1,2,runs]*BGn[1,2]-sum(rev(AH[1,2,])*dQn
    [1,2,])
Mn[1,3]<-AQn[1,3,runs]*BGn[1,3]-sum(rev(AH[1,3,])*dQn
    [1,3,])
Mn[1,4]<-AQn[1,4,runs]*BGn[1,4]-sum(rev(AH[1,4,])*dQn
    [1,4,])
```

```
Mn[2,1]<-AQn[2,1,runs]*BGn[2,1]-sum(rev(AH[1,1,])*dQn
    [2,1,])
Mn[2,2]<-AQn[2,2,runs]*BGn[2,2]-sum(rev(AH[1,2,])*dQn
    [2,2,])
Mn[2,3]<-AQn[2,3,runs]*BGn[2,3]-sum(rev(AH[1,3,])*dQn
    [2,3,])
Mn[2,4]<-AQn[2,4,runs]*BGn[2,4]-sum(rev(AH[1,4,])*dQn
    [2,4,])

Mn[3,1]<-AQn[3,1,runs]*BGn[3,1]-sum(rev(AH[1,1,])*dQn
    [3,1,])
Mn[3,2]<-AQn[3,2,runs]*BGn[3,2]-sum(rev(AH[1,2,])*dQn
    [3,2,])
Mn[3,3]<-AQn[3,3,runs]*BGn[3,3]-sum(rev(AH[1,3,])*dQn
    [3,3,])
Mn[3,4]<-AQn[3,4,runs]*BGn[3,4]-sum(rev(AH[1,4,])*dQn
    [3,4,])

Mn[4,1]<-AQn[4,1,runs]*BGn[4,1]-sum(rev(AH[1,1,])*dQn
    [4,1,])
Mn[4,2]<-AQn[4,2,runs]*BGn[4,2]-sum(rev(AH[1,2,])*dQn
    [4,2,])
Mn[4,3]<-AQn[4,3,runs]*BGn[4,3]-sum(rev(AH[1,3,])*dQn
    [4,3,])
Mn[4,4]<-AQn[4,4,runs]*BGn[4,4]-sum(rev(AH[1,4,])*dQn
    [4,4,])
F2<-F2+F2n
M<-M+Mn
print(c(seno,(times+1)))
AQnn<-AQn
```

```
BQnn<-BQn
Pnn<-Pn
if (mean(Mn)<1e-9 && times>40){break}
}
infsum1<-sum(M[1,1:4])*pi1+sum(M[2,1:4])*pi2+sum(M[3,1:4])*pi3+sum(M[4,1:4])*pi4
infsum2<-F2[1,1]*pi1+F2[2,1]*pi2+F2[3,1]*pi3+F2[4,1]*pi4
resultinfsum1<-c(resultinfsum1,infsum1)
resultinfsum2<-c(resultinfsum2,infsum2)
print(c(seno,infsum1,infsum2))
}
```

And finally obtain the price of N-CAT bond.

```
value=priceCIR*(0.5+0.5*resultinfsum1-0.25*resultinfsum2)
library(lattice)
wireframe(value ~ myGrid$t * myGrid$D,
 scales = list(arrows = FALSE),
  xlab = "T(years)", ylab = "D(US$ million)", zlab="V(US$)",
  drape = TRUE, colorkey = TRUE,
 screen = list(z = -45, x = -60),
)
```

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