Continuity of Quadratic Matings

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by

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Contents

1 Introduction

1.1 Some brief history of complex dynamics study

The study of complex dynamics was started by Schröder [Sch], Koenigs [Koe], Böttcher [Bot], Lattès [Lat], Carathéodory [Car], Fatou [Fat1] [Fat2][Fat3], Julia [Jul1] [Jul2] and many others. D. S. Alexander wrote a detailed history about the early research up to Julia [Ale]. After personal computers were introduced into this field, the research came into explosion in the 1980s. There were many exciting results during this period. Douady and Hubbard showed that the Mandelbrot set M is closed, bounded and connected [DH2] [Dou2]. The exterior of M is isomorphic to the exterior of $\bar{\mathbb{D}}$ (the closed unit disc) under a conformal map Φ_M [DH2]. They also developed the theory of renormalization and polynomial-like mappings to explain the occurrence of Mandelbrot-like sets in various families of rational maps [DH1]. The mating structure was also invented in this article by them. Later on, the theory of captures and tunings was developed. Douady also investigated some continuity and discontinuity problems on Julia sets and filled Julia sets of degree d polynomials under Hausdorff metric [Dou1]. Sullivan [Sul] showed that there are no wandering Fatou domains for rational maps, which answers a question of Fatou and Julia, by introducing the theory of quasiconformal mappings [Ahl] into complex dynamics. Mañé, Sad and Sullivan developed the powerful λ lemma [MSS] to link the geometric properties of the sets in some parameter spaces with those of sets on the dynamical planes. Later on McMullen showed that Mandelbrot-like sets are dense in the bifurcation locus for holomorphic families of any rational maps [McM1]. He also developed Douady and Hubbard's theory of renormalization in [McM2]. At about the same time when the rational dynamics were studied intensively, the research of holomorphic and meromorphic dynamics, or even some general quasi-conformal dynamics, for example, the exponential maps, $\lambda e^z, \lambda \in \mathbb{C} \setminus \{0\}$ and quasi-regular maps, was also in fast development. Higher dimensional complex dynamics also got attentions of many mathematicians in recent decades. It turns out that people want to iterate more and more general functions or function systems now, new phenomenons and patterns are expected on them, of course.

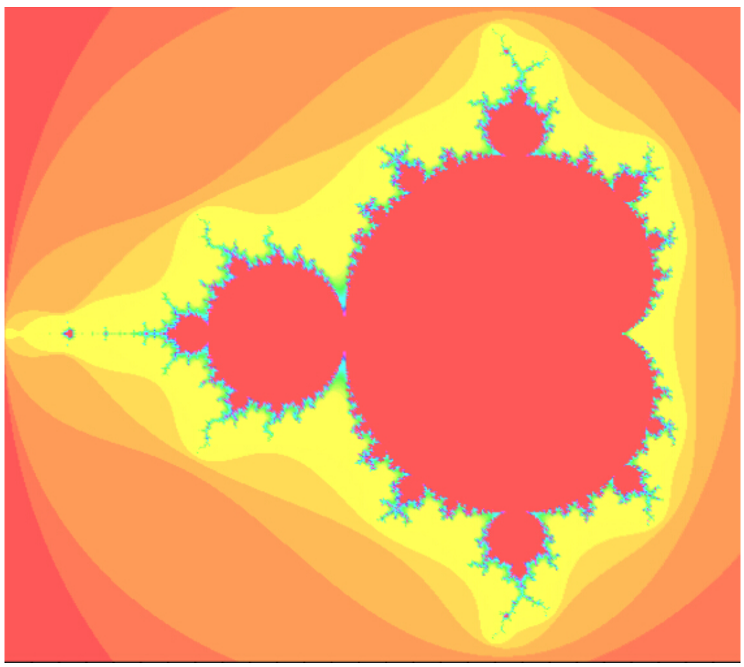

Figure 1: The Mandelbrot set

The main problem considered in this thesis is continuity of matings. Mating is a construction to create a rational map from two polynomials of the same degree. The construction can be considered up to Thurston equivalence (which is a type of homotopy equivalence, see the definition in Section 2.3) or up to topological conjugacies, or even some particular conformal conjugacies. Now we want to put the continuity problem into context.

1.2 History of the mating construction and the current research frontiers

As mentioned before, matings first appeared in the work of Douady and Hubbard [DH1] in 1985. They are there as a particular example on which their theory of polynomial-like mappings can be applied to show the existence of Mandelbrot-like sets in some rational parameter space. After this people began to ask whether a mating of two arbitrary polynomials exists or not, and if it exists, is the quadratic rational map which is Thurston equivalent to it unique? The existence problem of matings between quadratic hyperbolic polynomials was solved by Tan Lei in her thesis. Mary Rees had an unpublished proof earlier [Ree1]. They confirmed a conjecture of Douady and Hubbard, that the mating of two critically finite quadratic polynomials exists and is unique up to Möbius conjugacies as long as they are not in conjugate limbs of the Mandelbrot set M. In fact it is possible to produce a topological model (from the lamination models for polynomials) for any hyperbolic critically finite rational map which is a mating up to Thurston equivalence, see Definition 2.7.1. This was done in [Ree2] and was extended to other critically finite rational maps which are matings by Shishikura [Shi2]. Using the topological models for matings, the concept can be extended to maps which are not critically finite, so one can consider matings of general hyperbolic quadratic polynomials.

Later on there were results about matings of non-hyperbolic ones, where the mating is defined up to some conformal conjugacies, see Definition 2.7.2. For example, there were results on mating Siegel quadratic polynomials in [YZ] and results on mating non-renormalizable quadratic polynomials with the so called basilica polynomial in [AY]. The latter result was suggested by J. Luo in his thesis [Luo] in 1995.

The uniqueness usually follows from Thurston's criterion. Thurston's criterion (Theorem 2.1, [Tan1]) gives necessary and sufficient conditions for a critically finite branched covering to be Thurston equivalent to a rational map. Except for some simple examples (those ones having Thurston obstructions [Thu1]), the rational map is unique up to Möbius conjugacies. Levy proved in his thesis that a critically finite degree 2 branched covering map of the sphere is Thurston equivalent to a rational map as long as it does not have Levy cycles [Lev]. There are Lattès examples for which the rational maps of matings are not unique by K. Pilgrim in [Mil2].

Thurston equivalence between a critically finite branched covering and a critically finite rational map implies semiconjugacy under mild adjustment on the definition of the branched covering near its periodic critical points (see Section 2.4). If $f \simeq g$ (we use $\simeq$ to denote Thurston equivalence), where g is a critically finite rational map, f is a critically finite branched covering, then there is a continuous map $\theta : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, such that

$$\theta \circ f = g \circ \theta.$$

θ maps critical orbits to critical orbits and is a topological conjugacy near any periodic critical orbit (see for example [Ree2, Section 1.5]).

In 1997 Adam Epstein gave some examples of non-continuous quadratic matings in [Eps], which tells us that in general, the mating structure is not continuous in the parameter space of two variables. As there are two variables in the mating structure, the continuity problem can be considered in one or two variables. Epstein's counterexamples depend on variation of the two variables simultaneously, while the continuity in one variable has more chance to be true, at least in some special cases as we will show in the thesis.

P. Haïssinsky and Tan Lei proved a continuity result along simple pinching paths of geometrically finite polynomials with connected Julia sets and attracting points [HT]. As a generalization of Cui's techniques of pinching and plumbing [Cui] from geometrically finite rational maps to weakly hyperbolic maps, they can show the continuity of matings with each of the two polynomial variables g_0, g_1 going along two paths with the two endpoints f_0, f_1 being two parabolic parameters. f_0 and f_1 are on the boundary of the two hyperbolic components of M containing g_0, g_1.

Matings of n-polynomials (we use the terminology for polynomials of degree n) for $n \geq 3$ have also been studied, for example, [ST] and [Che]. In [ST], Shishikura and Tan demonstrated that some cubic matings may have Thurston obstructions, although they do not have Levy cycles. In [Che], Chéritat gave a pair of non-matable polynomials without Levy cycles. These results indicate some difference between quadratic matings and higher dimensional matings.

The mating structure shows its power in exploring properties of the parameter slice $Per_2(0)$, see [AY] [Tim2]. Maps on the boundary of the unbounded components of $\mathbb{C} \setminus M_a$ (M_a is the non-escape locus of $Per_2(0)$, see Section 10) are classified in [Tim1]. D. Dudko [Dud] showed that the non-escape locus M_a of $Per_2(0)$ is locally connected, which can "almost" be homeomorphically described as $M\setminus$ the $\frac{1}{2}$-limb. The latter result is a refined conjecture of Ben Wittner [Wit].

There are some generalizations of the notion to more general contexts. For example, one can consider matings between some Fuchsian groups or between certain polynomial maps and Fuchsian groups, see [Bul1] and [Bul2].

The data used to construct a mating is therefore a pair of polynomials (of the same degree). The data can be more combinatorial in the case of quadratic polynomials with connected Julia sets. There are various continuity problems that can be considered. In the thesis we are trying to answer a question of Mary Rees: given a critically finite quadratic polynomial f_q associated with some odd denominator rational $q \in (0,1)$, a sequence of critically finite quadratic polynomials f_{p_n} associated with odd denominator rationals $p_n \in (0,1)$ (the association will be explained in Section 2.2) such that $\lim\limits_{n\to\infty} p_n = p$ is also an odd denominator rational in $(0,1)$, does the sequence

$$f_{p_n} \amalg f_q$$

converges or not in some suitable rational parameter slice up to Möbius conjugacies? If it converges, what does the limit map look like? Note that since all the hyperbolic quadratic polynomials z^2+c are in some hyperbolic components with their centers being critically periodic quadratic polynomials [DH2], all the critically periodic quadratic polynomials are encoded by some odd denominator rationals in $(0,1)$ [Thu2]. Combined with Yoccoz's control of size of limbs of M [Hub], our control of size of the mating components on the parameter slice and quasiconformal deformation techniques, we can get more general continuity result on matings of hyperbolic quadratic polynomials parametrised by $z^2+c, c \in \mathbb{C}$. We are able to prove the continuity in several sub-cases of the problem in the thesis, and would like to conjecture the continuity holds for the general problem (see Conjecture 4.1.1). There are some by-results obtained in the research for the problem. All the continuity results are collected in Section 4.

2 Basic concepts

In this section the basic concepts and notations used in the thesis will be introduced.

2.1 Quadratic polynomials

Douady and Hubbard had made a comprehensive study of the dynamics of quadratic polynomials and the Mandelbrot set. Up to affine conjugacy, every quadratic polynomial can be written uniquely in the form

$$f_c(z) = z^2 + c,\ c \in \mathbb{C}.$$

Therefore the dynamical study of quadratic polynomials can be restricted to polynomials of this form.

Now we use $J(f_c)$ to represent the Julia set of f_c, $K(f_c)$ is the filled Julia set. The Mandelbrot set M is the set $\{c : J(f_c) \text{ is connected}\}$. All polynomials f_c for $c \notin M$ are globally quasi-conformally conjugate. A hyperbolic component of M^o is a connected component of the set M^o. The first results of Douady and Hubbard are that M is closed, bounded and connected. They were also able to show that these hyperbolic components are simply connected, and each such component is characterized by a pair of odd-denominator rationals in $(0, 1)$ (refer to Section 2.2 for details of the characterization). The exterior of M is conformally equivalent to the exterior of the unit disc [DH2].

For a degree d polynomial $f(z) = a_d z^d + \cdots + a_0, d > 1, a_d \neq 0$, we have

Lemma 2.1.1. *$J(f)$ is either connected or has infinitely many connected components.*

Proof. Considering the super-attractive fixed point at ∞. Change the coordinates by $\frac{1}{z}$, we get

$$g(z) = \frac{1}{f(\frac{1}{z})} = \frac{1}{a_d \frac{1}{z_d} + \cdots + a_0} = \frac{z^d}{a_d + \cdots + a_0 z^d}.$$

If $d > 1$, then $g'(0) = 0$ and $g^{(j)}(0) = 0$ for $1 \leq j < d$ while $g^{(d)}(0) = \frac{d!}{a_d} \neq 0$. This can be proved by induction or by expanding $g(z)$ around 0 as a Taylor series. By Böttcher's Theorem, there exists a conformal map Φ near 0 with $\Phi(0) = 0$ such that

$$\Phi \circ g \circ \Phi^{-1}(z) = z^d$$

Now let $\phi(z) = \frac{1}{\Phi(\frac{1}{z})}$, then

$$\phi \circ f \circ \phi^{-1}(z) = z^d \tag{1}$$

near ∞. $\phi(z)$ is defined on some neighbourhood $U \supset \{\infty\}$ with

$$\phi(U) = D_r = \{z : |z| > r > 1\}.$$

If no finite critical value is in U then

$$f : f^{-1}(U) \setminus \{\infty\} \to U \setminus \{\infty\}$$

is a covering map of degree d, we can extend ϕ onto a larger domain $\phi(f^{-1}(U)) = D_{\frac{1}{r^d}}$, then consider whether there is a finite critical value in $f^{-1}(U)$. We can continue this process to extend the domain $\{z : |z| > r > 1\}$ to its largest $D_{r_0} = \{z : |z| > r_0 \geq 1\}$ on which ϕ^{-1} is a conformal map which satisfies (1), until we hit a finite critical value. Let $V = \phi^{-1}(D_{r_0})$. Let $\mathcal{A}$ denote the super-attractive basin of f containing ∞. We have the following commutative diagram of conformal isomorphisms

$$\begin{array}{ccc} V & \xrightarrow{f} & f(V) \\ \uparrow{\scriptstyle \phi^{-1}} & & {\scriptstyle \phi}\downarrow \\ D_{r_0} & \xrightarrow{z^d} & D_{r_0^d} \end{array}$$

Now we claim that there must be at least one critical point of f on ∂V which interrupts the extension of ϕ^{-1} on larger discs. We show this by contradiction. Note that obviously there are no critical points of f in V, or else this would imply that ϕ^{-1} has critical points in D_{r_0}, which will contradict the fact of the conformality of ϕ^{-1}.

Now suppose that there are no critical points on ∂V. For any point $w_0 \in \partial D_{r_0}$, choose a small neighbourhood W of w_0 on which $\phi^{-1} : W \to \phi^{-1}(W)$ is defined as

$$w \to f^{-1} \circ \phi^{-1}(w^d).$$

Note that on this domain f^{-1} is well defined as there are no critical value on $\partial f(V)$. Then these small neighbourhoods together with D_{r_0} form a neighbourhood of D_{r_0} on which we can conformally extend ϕ^{-1} to larger open discs $D_{r_0'}$ with $r_0' > r_0$. This is a contradiction on our maximum assumption on r_0. So there must be a critical point of f on ∂V.

Now there are two possibilities. If some finite critical value $c \in f^{-n}(U)$ for some $n \in \mathbb{N}$, then the set $f^{-(n+1)}(\bar{\mathbb{C}} \setminus U)$ has $m \geq 2$ connected components depending on the order of the critical point, $m \in \mathbb{N}$. Inductively, $f^{-(n+j)}(\bar{\mathbb{C}} \setminus U)$ has at least m^j connected components, $j \in \mathbb{N}$. So $\cup_{k\geq 0} f^{-k}(U) = \bar{\mathbb{C}} \setminus \cap_{k\geq 0} f^{-k}(\bar{\mathbb{C}} \setminus U)$ is the complement of a set with infinitely many connected components. Consequently the Julia set $J(f) = \partial \cup_{k\geq 0} f^{-k}(U)$ is also a set with infinitely many connected components. If no finite critical value is in $f^{-k}(U)$ for any k, then $\cup_{k\geq 0} f^{-k}(U)$ is simply connected and $J(f) = \partial \cup_{k\geq 0} f^{-k}(U)$ is connected.

□

Remark 2.1.2. *The conclusion holds for all rational maps in fact, See Corollary 4.15 [Mil1] for a proof.*

In the degree two case, for each hyperbolic component of M, there is a unique map in it which is critically periodic, often called the center of the hyperbolic component. There is a combinatorial model, called the lamination model [Thu2] to describe the topological dynamics of it. We can associate a pair of odd denominator rationals in $(0, 1)$ to each hyperbolic component (therefore to each critically periodic 2-polynomial) where these pairs are all distinct and disjoint, refer to Section 2.2. Every odd denominator rational occurs in one such pair.

This association allows a complete dynamical description for hyperbolic quadratic polynomials and many others, although we are mainly concerned with hyperbolic quadratic polynomials. Moreover, this association gives a conjectural description of the Mandelbrot set, see Theorem 11.7.0b [Thu2]. As far as it is related to hyperbolic components, if f_n is a sequence of hyperbolic quadratic polynomials with associated odd denominator rationals p_n tending to an odd denominator rational p as $n \to \infty$, where p is associated with a hyperbolic quadratic polynomial f, then

$$\lim_{n\to\infty} f_n = f_0,$$

where f_0 is the parabolic map on the boundary of the hyperbolic component containing f, and the multiplier of f_0^k at the parabolic cycle is 1 (k is period of the attractive cycle of f). This can be deduced from the work of Yoccoz [Yoc] as described by Hubbard in [Hub]. In fact there are many other results of this type contained in [Hub]. Our type of continuity results which relates the Mandelbrot set and its combinatorial model can be compared with many others' results.

2.2 The external rays and lamination models for quadratic polynomials

There is a very explicit description of critically finite quadratic polynomials in the Mandelbrot set using the map $z \to z^2$ on the unit circle and rational numbers in $[0, 1]$. Let f_c be a critically periodic quadratic polynomial of the form $f_c(z) = z^2 + c$. Then there exists a continuous map $\psi_c = \phi^{-1}$ in which ϕ is defined in the proof of Lemma 2.1.1 (the inverse Böttcher coordinate)

$$\psi_c : \{z : |z| > 1\} \to \bar{\mathbb{C}} \setminus K(f_c)$$

satisfying

$$\psi_c(z^2) = f_c \circ \psi_c(z) \quad \forall\ |z| > 1.$$

ψ_c extends from $\{z : |z| > 1\}$ to $\{z : |z| \geq 1\}$ if the Julia set $J(f_c)$ is locally connected by Theorem 2.2.5. ψ_c is holomorphic on $\{z : |z| > 1\}$ and continuous on $\{z : |z| \geq 1\}$. In addition,

$$\psi_c(\{z : |z| = 1\}) = J(f_c).$$

Now define the external ray $\gamma(t)$ of f_c with angle $t, t \in [0, 1)$ as

$$\gamma(t) = \psi_c(\{re^{2\pi it} : 1 < r < \infty\}).$$

For an arbitrary quadratic polynomial (in fact an arbitrary polynomial), there are some well known landing properties about the external rays as following. They are all from [Mil1]. Let $\mathbb{D} = \{z : |z| < 1\}, S^1 = \{z : |z| = 1\}$.

Theorem 2.2.1. *Every periodic external ray lands at a periodic point which is either repelling or parabolic. If t is rational but not periodic, then the ray of angle t lands at a point which is eventually periodic but not periodic.*

Theorem 2.2.2. *For almost every point $e^{2\pi it}$ of the unit circle S^1 with respect to Lebesgue measure, the external ray of angle t lands on $J(f_c)$. However, if we fix any particular point $z_0 \in \partial J(f_c)$, then the set of angles of which the external rays land on z_0 has Lebesgue measure zero.*

Theorem 2.2.3. *Every repelling or parabolic periodic point is the landing point of at least one periodic ray.*

Theorem 2.2.4. *If one periodic ray lands on z_0, then only finitely many rays land on z_0. These rays are all periodic of the same period (which may be larger than the period of z_0).*

The following result is a consequence of results of Marie Torhorst and Constantin Carathéodory. It provides us with quite efficient techniques to judge whether a ray lands or not [Mil1].

Theorem 2.2.5. *For any given f_c with connected Julia set, the following four conditions are equivalent.*

- *Every external ray $\gamma(t)$ lands on a point $\kappa(t)$ which depends continuously on the angle t.*

- *The Julia set $J(f_c)$ is locally connected.*

- *The filled Julia set $K(f_c)$ is locally connected.*

- *The inverse Böttcher map $\psi_c : \mathbb{C} \setminus \bar{\mathbb{D}} \to \mathbb{C} \setminus K(f)$ extends continuously over the boundary $\partial\mathbb{D}$, and this extended map carries each $e^{2\pi it} \in \partial\mathbb{D}$ to $\kappa(t) \in J(f_c)$.*

Furthermore, whenever these conditions are satisfied, the resulting map $\kappa : \mathbb{R}\backslash\mathbb{Z} \to J(f_c)$ *satisfies the semiconjugacy identity*

$$\kappa(2t) = f_c(\kappa(t))$$

and maps the circle $\mathbb{R}\backslash\mathbb{Z}$ *onto the Julia set* $J(f_c)$.

This makes whether the Julia set of a polynomial is locally connected or not a very important character for $J(f_c)$, and similarly for the Mandelbrot set M. The existence of a quadratic polynomial with non-locally connected Julia set is proved in [Sor]. Milnor gave a way to get a quadratic polynomial with non-locally connected Julia set by tunings in [Mil4]. G. Levin [Levin] gave a class of infinitely renormalizable quadratic polynomials with non-locally connected Julia sets, while the Mandelbrot set is locally connected at such parameters. There is a discussion on the local connectivity of Julia sets of polynomials and rational maps by A. Dezotti and P. Roesch in [DR].

As for the hyperbolic maps, we have [Mil1]

Theorem 2.2.6. *If the Julia set of a hyperbolic map is connected, then it is locally connected.*

Theorem 2.2.7. *If* U *is a simply connected Fatou component for a hyperbolic map, then the boundary* ∂U *is locally connected.*

Either one of the two theorems implies local connectivity of Julia set of a hyperbolic map in M. Theorem 2.2.5 guarantees ψ_c extends to $\partial\mathbb{D}$ for maps with locally connected Julia sets, which enables us to define the following combinatorial structure for these quadratic polynomials.

Now we introduce the lamination models for quadratic polynomials with locally connected Julia sets. Let

$$L_c = \{l : l \text{ is a chord in } \mathbb{D}$$
$$\text{in the boundary of the convex hull of } \psi_c^{-1}(z), z \in J(f_c)\}.$$

A chord l is called a leaf of the lamination, while a connected component of $\mathbb{D} \setminus L_c$ is called a gap. L_c is an invariant lamination. This means that L_c is a closed set and L_c is forward and backward invariant . Forward invariant means that if l is a chord in L_c with end points z_1 and z_2, then the chord joining z_1^2 and z_2^2 is also in L_c. We denote this chord by l^2. This condition is empty when $z_1 = -z_2$, that is, when $z_1^2 = z_2^2$. Backward invariant means that if l is a chord in L_c, then the two chords l_1, l_2 with $l_1^2 = l_2^2 = l$ are both in L_c.

Let μ_c be the chord or leaf of L_c such that $\psi_c(\mu_c)$ is the point of least possible period in the boundary of the Fatou component F_1 of f_c which contains $f_c(0)$, and such that μ_c is not separated from $\psi_c^{-1}(\partial F_1)$ by any other leaf of L_c. The end points of μ_c are $e^{2\pi i p}$ and $e^{2\pi i \bar{p}}$, where p and $\bar{p}$ are both odd denominator rationals in $[0, 1]$ of the same period under $x \to 2x$ mod 1, and also of the same period as 0 under f_c (the only exception to this is when $c = 0$, for which the lamination L_0 is empty). Now denote the lamination L_c by $L_p = L_{\bar{p}}$.

The lamination L_p then can be used to describe completely the dynamics of f_c according to [Thu2]. $\mu_c = \mu_p$ ($\mu_{\bar{p}}$) is the minor leaf of L_p ($L_{\bar{p}}$). It is the image of the longest leaf in L_p ($L_{\bar{p}}$).

There is a critically finite branched covering map $s_p : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that

$$s_p(z) = z^2 \text{ for } |z| \geq 1,$$
$$s_p(L_p) = L_p.$$

s_p maps gaps to gaps. Note that ψ_c can be extended to all of $\bar{\mathbb{C}}$ to satisfy

$$\psi_c \circ s_p(z) = f_c \circ \psi_c(z) \quad \forall \ |z| \geq 1.$$

Moreover, s_p can be chosen so that ψ_c is a homeomorphism on the preimages of each Fatou component. In this way, s_p is uniquely determined, up to topological conjugacy for every L_p. We can also choose s_p such that 0 is the finite critical point and periodic under s_p, of the same period as 0 under f_c. We call a minor leaf which is not separated from 0 by other minor leaves a *minimal minor leaf* [Thu2]. More can be found in [Ree2] about the relationship between L_p, the lamination map s_p and the quadratic critically finite polynomial which is Thurston equivalent to s_p (see below).

2.3 Thurston equivalence

Thurston equivalence for critically finite branched coverings is a type of homotopy equivalence. Let

$$X(f) = \{f^n(c) : c \text{ is any critical point of } f,\ n > 0\}$$

be the post-critical set for a C^1 map $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$.

Definition 2.3.1. *Let f and g be two critically finite branched coverings. $X(f)$ and $X(g)$ are the post-critical sets. Then f and g are Thurston equivalent if there are orientation-preserving homeomorphisms χ_1 and $\chi_2 : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that*

$$\chi_1(X(f)) = X(g),\ \chi_2(X(f)) = X(g),$$

$$\chi_1 \sim \chi_2 \ \textit{rel}\ X(f),$$

$$\chi_1 \circ f = g \circ \chi_2.$$

We use $\sim$ to denote the isotopy between two homeomorphisms. The branched coverings f_c and s_p described in Section 2.2 are Thurston equivalent.

While most people are more familiar with this definition, there is another description which is equivalent to this definition. Since we will use it in the following to identify the condition of a critically periodic map being Thurston equivalent to a mating, we also present it here.

Definition 2.3.2. *Let f and g be two critically finite branched coverings with post-critical sets $X(f)$ and $X(g)$, then f and g are Thurston equivalent if there exists a homotopy $h_t: \bar{\mathbb{C}} \to \bar{\mathbb{C}}, t \in [0,1]$ of critically finite branched coverings, and an orientation preserving homeomorphism φ, such that*

$$h_0 = \varphi \circ f \circ \varphi^{-1},$$
$$h_1 = g,$$
$$X(h_t) = X(g) \text{ for any } t \in [0,1].$$

This is denoted as $f \simeq_\varphi g$ in [Ree2]. For the equivalence of the two definitions, see Section 1.4, [Ree2].

In the last section, we saw how to obtain two odd denominator rationals $p, \bar{p} \in [0,1]$ from a critically periodic quadratic polynomial f_c. Conversely, given any odd denominator rational $p \in [0,1]$, there is a critically periodic quadratic polynomial f_c such that $f_c \simeq s_p$. Moreover, f_c is unique if we choose it to be of the form $f_c(z) = z^2 + c$. Thus there is a 2 to 1 map from the set of odd denominator rationals in $[0,1]$ to critically periodic quadratic polynomials of the form z^2+c, since p and $\bar{p}$ correspond to the same polynomial. We shall show in our Theorem 3.5.1 that this map admits a certain kind of continuity.

2.4 Thurston equivalence implies semiconjugacy

In Definition 2.3.2 for Thurston equivalence between critically finite branched coverings, by Section 4.1 [Ree2], for a critically finite rational map f and a critically finite branched covering g with $f \simeq_\varphi g$, let φ be a topological conjugacy between f and g in neighbourhoods of orbits of critical points of f and g. Then there is a well-defined sequence of homeomorphisms which satisfies

$$\varphi \circ g^n = f^n \circ \theta^{(n)}$$

with $\theta^{(0)} = \varphi$. Moreover,

$$\lim_{n\to\infty} \theta^{(n)} = \theta$$

exists, and satisfies

$$\theta \circ g = f \circ \theta$$

which gives a semiconjugacy between the two maps being Thurston equivalent.

For the proof of the convergence of the sequence $\theta^{(n)}$, see P48-49 [Ree2]. θ needs not to be a homeomorphism, but is a surjective map on $\bar{\mathbb{C}}$. θ is a homeomorphism from some neighbourhood of the forward orbits of the critical points of f to a neighbourhood of the forward orbits of the critical points of g. There are some particular properties on the map θ when g is a lamination map, see Section 4 in [Ree2]. This sequence of homeomorphisms $\theta^{(n)}$ and the semiconjugacy θ will be used in Section 8.

2.5 Thurston's criterion

Now we want to introduce *Thurston's criterion* for judging whether a critically finite branched covering is Thurston equivalent to a rational map, which is closely related with the definition of mating. The following definitions are from [Tan1].

Definition 2.5.1. *Let F be a post-critically finite branched covering of degree $d \geq 2$ on $\bar{\mathbb{C}}$.*

(1) A simple closed curve γ is non-peripheral *if $\gamma \cap X(F) = \emptyset$ and each connected component of $\bar{\mathbb{C}} \setminus \gamma$ contains at least two points of $X(F)$.*

(2) A system of curves $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ is called a multicurve*, if it consists of simple closed, non-peripheral, disjoint, and non-homotopic (rel $X(F)$) curves.*

(3) To each multicurve Γ we associate a non-negative matrix $F_\Gamma = (f_{ij})_{n\times n}$ by: for $\gamma_i, \gamma_j \in \Gamma$, if none of the connected components of $F^{-1}(\Gamma_j)$ is homotopic (rel $X(F)$) to γ_i, set $f_{ij} = 0$; if, say, $\delta_{1j}, \delta_{2j}, \cdots, \delta_{kj}$ are the connected components of $F^{-1}(\gamma_j)$ homotopic to γ_i rel $X(F)$, set

$$f_{ij} = \Sigma_{p=1}^{k}(1/deg(F : \delta_{pj} \to \gamma_j)).$$

(4) We denote by $\lambda(\Gamma)$ or $\lambda(F_\Gamma)$ the leading eigenvalue of F_Γ. It is well defined since F_Γ is a non-negative matrix.

(5) A multicurve Γ is called F-invariant *if for each $\gamma_j \in \Gamma$, each connected component of $F^{-1}(\gamma_j)$ is either peripheral, or homotopic to a curve of γ (rel X(F)).*

(6) A F-invariant *multicurve Γ is called a* Thurston obstruction *if $\lambda(F_\Gamma) \geq 1$.*

Now we can state Thurston's criterion relying on the former definitions.

Theorem 2.5.2. *A post-critically finite branched covering F of degree $d \geq 2$ with hyperbolic orbifold [McM2] is Thurston equivalent to a rational map G if and only if there are no Thurston obstructions for F. Moreover, G is unique up to conjugacy by Möbius transformations.*

There is a proof for the theorem in [DH3].

2.6 Tuning

The notion of tuning was also invented by Douady and Hubbard [Dou2] [Dou3]. We only give the definition here by our lamination maps. The tuning of s_p by s_t, denoted by $s_p \vdash s_t$, is the branched covering s_r, (p, t, r are all odd denominator rationals in (0,1)) up to Thurston equivalence, where the lamination L_r is obtained from L_p and L_t as follows.

Let G_p be the image of the central gap of the lamination L_p. $L_p \subset L_r$ and all leaves of $L_r \setminus L_p$ are in the grand (forward and backward) orbit of $\bar{G_p}$. Note that $\mu_p \in \partial\bar{G_p}$ and G_p is separated from 0 by μ_p. Since L_r is invariant, as long as leaves in $\bar{G_p}$ are defined, we can get the other leaves by forward and backward iterations of the map $z \to z^2$ on S^1. Now we define $L_r \cap \bar{G_p}$, this is determined by L_t as follows. There is a continuous map $\psi : \mathbb{D} \to G_p$, such that

$$\psi(L_t) = \bar{G_p} \cap L_r,$$
$$\psi(z^2) = s_p^m \circ \psi(z),$$
$$\psi \circ s_t = s_p^m \circ \psi \text{ on } G_p,$$

in which m is the period of 0 under s_p. ψ is a homeomorphism on the gap G_p, moreover, ψ^{-1} extends continuously to map $\bar{G_p}$ to $\bar{\mathbb{D}}$, map leaves on ∂G_p to points.

The notion of tuning is used in [Dev] to compute angles of some rays landing on M.

2.7 Mating

Matings of polynomials give a way to describe the dynamics of some rational maps, and in some cases, to obtain a partial model of some parameter slices. Computer pictures of some rational parameter slices [Tim2] show the appearance of some Mandelbrot-like sets, often with some particular limbs truncated. This must have been part of the motivation for the study of mating.

Mating can be regarded as a topological structure or as a structure with at least some conformal information. It can in some cases be used more generally than for post-critically finite polynomials. However, this thesis mainly restricts to the case of critically periodic polynomials and the definition of mating used is a topological one, based on the idea of Thurston equivalence. As a critically finite branched covering, up to Thurston equivalence, we can define the mating of any two critically periodic quadratic polynomials. As we are defining the mating up to Thurston equivalence (at least initially), we may start from two critically finite branched coverings instead of polynomials. These covering maps, although themselves not being critically periodic quadratic polynomials, are Thurston equivalent to them.

Now we directly present the definition using the lamination map s_p with respect to an odd denominator rational $p \in (0,1)$. Essentially we use formal mating in [Tan1] by Thurston equivalence while the definition of conformal mating in [AY] is a finer version. As mentioned in [YZ], these definitions are presumably equivalent.

Definition 2.7.1. *Let p and q be any two odd denominator rationals in $[0,1]$. We define the formal mating of s_p with s_q, which is denoted by $s_p \amalg s_q$, to be:*

$$s_p \amalg s_q(z) = \begin{cases} s_p(z) & for \quad |z| \leq 1 \\ (s_q(z^{-1}))^{-1} & for \quad |z| \geq 1 \end{cases} \tag{2}$$

Thus $s_p \amalg s_q$ is a critically periodic branched covering on $\bar{\mathbb{C}}$. Now if there is a rational map R, such that $s_p \amalg s_q$ is Thurston equivalent to R, then we say s_p and s_q are matable and call R a mating of s_p and s_q.

This definition can be generalized to degree $d > 2$ laminations. For degree d polynomials we can still construct invariant laminations for these polynomials. For an invariant lamination L of degree $d \geq 2$, if a leaf $l \in L$ has endpoints z_1, z_2, then there is a leaf $l^d \in L$ with endpoints z_1^d, z_2^d unless $z_1^d = z_2^d$. The d-preimages of each leaf $l \in L$ are all in L. Each gap G of L has a finite number of preimages which are mapped under branched coverings to G. The sum of the degrees of these maps is d. If L is an invariant lamination of degree d with no critical leaves, all finite sided gaps are mapped forward homeomorphically. Then we can define a critically finite branched covering s_L of degree d such that

$$s_L(L) = L,\ s_L(z) = z^d \text{ on } S^1.$$

If L_1, L_2 are two invariant laminations of degree d, then we can define

$$s_{L_1} \amalg s_{L_2}(z) = \begin{cases} s_{L_1}(z) & for \quad |z| \leq 1 \\ (s_{L_2}(z^{-1}))^{-1} & for \quad |z| \geq 1 \end{cases} \tag{3}$$

Although we mainly work with this definition related with lamination maps, we still want to give the definition in another way as our Theorem 4.2.2 follows this definition. Now we present the following definition of conformal mating relying on external rays to substitute (2) from [AY].

Consider two monic $d \geq 2$ polynomials $f_1(z) = z^d + \cdots$ and $f_2(z) = z^d + \cdots$ with locally connected Julia sets. Let

$$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty e^{2\pi i t} : t \in [0,1]\}$$

be the blow up of $\bar{\mathbb{C}}$ at ∞. For each $f_i, i = 1, 2$, define

$$f_i(\infty e^{2\pi i t}) = \infty e^{4\pi i t}$$

on $\tilde{\mathbb{C}}_i$.

Let $\tilde{\mathbb{C}}_1 = \tilde{\mathbb{C}}_2 = \tilde{\mathbb{C}}$, let $\tilde{\mathbb{C}}_1 \uplus \tilde{\mathbb{C}}_2$ be the disjoint union. Let $\Sigma = \tilde{\mathbb{C}}_1 \uplus \tilde{\mathbb{C}}_2 / \sim_\infty$, where $\infty e^{2\pi i t_1} \sim_\infty \infty e^{2\pi i t_2}$ as long as $t_1 = -t_2$ with $t_i \in [0, 1)$ for $i = 1, 2$. By restricting on each $\tilde{\mathbb{C}}_i$ to be f_i we get a well defined map F on Σ. Recall that in Section 2.2 there exists a continuous map $\psi_i : \{z : |z| > 1\} \to \bar{\mathbb{C}} \setminus K(f_i)$ for $i = 1, 2$, which is easily extended onto $\{z : |z| > 1\} \cup \{\infty e^{2\pi it} : t \in [0, 1]\}$. Now define the external ray $\kappa_i(t) = \psi_i(\{re^{2\pi it} : 1 < r \leq \infty e^{2\pi it}\})$ of angle t on $\tilde{\mathbb{C}}_i$ (hence on Σ) for $i = 1, 2$. Let $\overline{\kappa_i(t)}$ be the closure of the external ray $\kappa_i(t)$ of angle t on each $\tilde{\mathbb{C}}_i$. Define an equivalent relationship $\sim_\kappa$ on Σ:

$x \sim_\kappa y$ as long as there exists a finite sequence of closed external rays $\{\kappa_{i_j}\bar{(}t_j) : j = 1, \cdots, k, i_j = 1, 2\}$ such that

$$\overline{\kappa_{i_j}(t_j)} \cap \overline{\kappa_{i_{j+1}}(t_{j+1})} \neq \emptyset \text{ for } 1 \leq j \leq k-1 \text{ and } x \in \overline{\kappa_{i_1}(t_1)}, y \in \overline{\kappa_{i_k}(t_k)}.$$

Definition 2.7.2. *Now if $\Sigma/\sim_\kappa$ is a 2-sphere, then we say f_1 and f_2 are topologically matable. If there exists a rational map $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ and a pair of semiconjugacies $\psi_i : K(f_i) \to \bar{\mathbb{C}}, i = 1, 2$, such that*

$$R \circ \psi_i = \psi_i \circ f_i$$

and

$$\psi_i \text{ is conformal on } K(f_i)^o = int(K(f_i)),\ i = 1, 2,$$

then we say f_1 and f_2 are conformally matable. R is called a conformal mating of f_1 and f_2, denote this by $R \approx f_1 \amalg f_2$.

There is obviously a relationship that conformally matable $\Rightarrow$ topologically matable.

2.8 Tan Lei's Theorem

Tan Lei's Theorem is derived from a special case of Thurston's Theorem for critically finite branched coverings. Thurston's criterion gives necessary and sufficient conditions for a critically finite branched covering to be Thurston equivalent to a rational map. This theorem can, of course, be applied to critically finite branched coverings that are matings, and Tan Lei's Theorem [Tan1] (also proved by Mary Rees in her manuscript [Ree1]) adapts this condition to a very simple one. We state it only in the critically periodic case.

Tan Lei's Theorem. *let p and q be two odd denominator rationals, then $s_p \amalg s_q$ is Thurston equivalent to a rational map if and only if μ_p and μ_{-q} are not separated from 0 in $\bar{\mathbb{D}} = \{z : |z| \leq 1\}$ by the same minimal minor leaf.*

Note that the condition on μ_p and μ_{-q} are not separated from 0 in $\bar{\mathbb{D}}$ by the same minimal minor leaf (or say that μ_p and μ_q are not in conjugate limbs of the lamination model) is equivalent to say that, the polynomials which are Thurston equivalent to s_p and s_q are not in conjugate limbs of the Mandelbrot set.

2.9 Recognising matings and the lamination maps up to Thurston equivalence

In this section we prove a lemma on recognizing the conditions of a critically periodic branched covering being a mating. We will restrict to the degree 2 case.

Lemma 2.9.1. *Let f be a critically periodic branched covering of degree 2. Then f is Thurston equivalent to a mating $s_p \amalg s_q$ if and only if there exists a closed loop $\gamma \in \bar{\mathbb{C}} \setminus X(f)$ which separates the two critical orbits of f, such that $f^{-1}(\gamma)$ is connected and isotopic to γ in $\bar{\mathbb{C}} \setminus X(f)$ and $f: \ f^{-1}(\gamma) \to \gamma$ preserves orientation.*

Proof. The idea of the proof is from [Exa], we only give a sketch here.

By the first definition of Thurston equivalence, we can assume $\gamma = f^{-1}(\gamma) = S^1$ and $f(z) = z^2$ on the unit circle S^1, or we can find a map

$g \simeq f$ with these properties. Consider the two maps $f^{\triangle}$ and $f_{\triangle}$ such that

$$f^{\triangle}(z) = \begin{cases} f(z) & for \quad |z| \leq 1 \\ z^2 & for \quad |z| \geq 1, \end{cases}$$

$$f_{\triangle}(z) = \begin{cases} z^2 & for \quad |z| \leq 1 \\ f(z) & for \quad |z| \geq 1. \end{cases}$$

$f^{\triangle}$ and $f_{\triangle}$ are critically periodic branched coverings with fixed critical points at ∞ for $f^{\triangle}$ and 0 for $f_{\triangle}$. By a folklore result (known as Levy's Theorem since he considered the problem in his thesis), $f^{\triangle}$ and $f_{\triangle}$ do not have any Thurston obstructions, that is, they satisfy the condition for being Thurston equivalent to rational maps. Since they have fixed critical points of maximum multiplicity (at ∞ and 0), they are Thurston equivalent to quadratic polynomials. So there are odd denominator rationals $p,\ q \in (0,1)$ such that $f^{\triangle} \simeq s_p$ and $f_{\triangle} \simeq s_q$. We omit the details but the homotopy f^t between s_p and $f^{\triangle}$ can be chosen so that $f^t(z) = z^2$ on S^1 for all t, so is the homotopy f_t between s_q and $f_{\triangle}$. Now the combination of the two homotopies gives a new homotopy $\tilde{f}_t$ between f and $s_p \amalg s_q$:

$$\tilde{f}_t(z) = \begin{cases} f^t(z) & for \quad |z| \leq 1 \\ f_t(z) & for \quad |z| \geq 1 \end{cases}$$

So $f \simeq s_p \amalg s_q$ by Definition 2.3.2.

For the other direction, if $f \simeq s_p \amalg s_q$, then there are orientation-preserving homeomorphisms χ_1 and $\chi_2 : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that

$$\chi_1(X(f)) = X(s_p \amalg s_q),\ \chi_2(X(f)) = X(s_p \amalg s_q),$$

$$\chi_1 \sim \chi_2 \text{ rel } X(f),$$

$$\chi_1 \circ f = (s_p \amalg s_q) \circ \chi_2.$$

In this case choose the loop $\gamma = \chi_2^{-1}(S^1)$, one can check that the requirements on γ in the theorem are satisfied. $\square$

The closed loop γ is called an *invariant circle* for f (of course not unique). We will use this lemma to show the critically periodic maps we get are matings in Section 8.

2.10 Identifying the map s_p associated with a particular mating

Suppose we have a degree two critically periodic branched covering f with a periodic critical point c_1 of period m and an invariant circle γ. We already know that $f \simeq s_p \amalg s_q$ for some odd denominator rational p by Lemma 2.9.1, with p being periodic of period m under the map $z \to 2z$ mod 1. Then how do we find this p? One way is to use a sequence of arcs joining $X(f)$ to the invariant circle γ as showing in the following lemma.

Lemma 2.10.1. *Let f be as above, and let $p \in (0,1)$ be an odd denominator rational. There exists a family of non-self-intersection arcs ξ_j joining $s_p^j(0)$ with $e^{2\pi i 2^{j-1}p}$ for $0 \le j < m$ which is disjoint from $L_p \cup S^1$ except for the endpoint $e^{2\pi i 2^{j-1}p}$ (when $j = 0$ replace $e^{2\pi i 2^{-1}p}$ by $e^{2\pi i 2^{m-1}p}$) such that $\xi_{j+1} = s_p(\xi_j)$. There exists a non-self-intersection arc ξ_0'' joining the point $x \in S^1$ with 0 where $s_p(x) = e^{2\pi i p}$ but $x \ne e^{2\pi i 2^{m-1}p}$ and ξ_0'' does not intersect $L_p \cup S^1$ besides at x. Now suppose there exists an arc ζ_j joining $f^j(c_1)$ with γ for $0 \le j < m$ satisfying $\zeta_{j+1} = f(\zeta_j)$. Let $\zeta_j' \subset f^{-1}(\zeta_{j+1})$ be the arc joining $f^j(c_1)$ with $f^{-1}(\gamma)$ for $0 \le j < m-1$ and $\zeta_{m-1}' \subset f^{-1}(\zeta_0)$. Let $\zeta_0'' = (f^{-1}(\zeta_1) \setminus \zeta_0) \cup \{c_1\}$. Suppose that there are two orientation-preserving homeomorphisms Θ and Θ' such that*

$$\Theta(f^{-1}(\gamma)) = \gamma,\ \Theta(\zeta_j') = \zeta_j \text{ for } 0 \le j < m,$$

$$\Theta'(f^{-1}(\gamma)) = S^1,\ \Theta'(\zeta_j') = \xi_j \text{ for } 0 \le j < m,$$

$$\Theta'(\zeta_0'') = \xi_0''.$$

Then we have

$$f \simeq s_p \amalg s_q$$

where q is also an odd denominator rational in $(0, 1)$.

Proof. Write $\chi_1 = \Theta' \circ \Theta^{-1}$, then $\chi_1(\gamma) = S^1$. Without loss of generality assume that $s_p(\xi_j) = \xi_{j+1}$ for $0 \leq j < m-1$, $s_p(\xi_{m-1}) = \xi_0$. We can also assume that $s_p(z) = s_p(-z)\ \forall\ z \in \mathbb{C}$ and $\xi_0'' = -\xi_0$. Θ' maps critical orbit c_1 of f to critical orbit of s_p. Then there is a homeomorphism χ_2 of $\mathbb{C}$ such that

$$\chi_1 \circ f = s_p \circ \chi_2 \tag{4}$$

Actually there are two such kind of homeomorphisms $z \to \chi_2(z)$ and $z \to -\chi_2(z)$ because $s_p(z) = s_p(-z)$.

Now $\chi_1(\gamma) = S^1$, so $\chi_1 \circ f \circ \chi_2^{-1} = s_p$. Then

$$\chi_2^{-1}(S^1) = f^{-1} \circ \chi_1^{-1} \circ s_p(S^1) = f^{-1}(\gamma),$$
$$\chi_2^{-1}(\xi_j) \subset f^{-1} \circ \chi_1^{-1} \circ s_p(\xi_j) = f^{-1} \circ \chi_1^{-1}(\xi_{j+1}) = f^{-1}(\zeta_j),$$

when $j = m-1$ replace $j+1$ by 0.

Note that we also have $\chi_2^{-1}(\xi_0'') = \chi_2^{-1}(-\xi_0) \subset f^{-1}(\xi_1)$, so either $\chi_2^{-1}(\xi_0) = \zeta_0'$ or $\chi_2^{-1}(-\xi_0) = \zeta_0'$. Replacing χ_2 by $-\chi_2$ if necessary, we can assume

$$\chi_2^{-1}(\xi_0) = \zeta_0'.$$

Then $\chi_2^{-1}(s_p^{-1}(\xi_1) \setminus \xi_0) = f^{-1}(\zeta_1) \setminus \zeta_0'$. Now $s_p^{-1}(\xi_1)$ separates $\bar{\mathbb{D}}$ into 2 halves and $f^{-1}(\zeta_1)$ separates the disc containing c_1 bounded by $f^{-1}(\gamma)$ into 2 halves. We know that there exists a homeomorphism Θ' which maps ξ_0 and $s_p^{-1}(\xi_1)$ to the same sets as χ_2^{-1}. So Θ' and χ_2^{-1} must also both map ξ_j to ζ_j' for $1 \leq j < m$, in particular

$$\chi_2(f^j(c_1)) = s_p^j(0)$$

for $0 \le j < m$. So $f \simeq s_p \amalg h$ for some degree two branched covering h which has a fixed critical point and the other critical point periodic. So h is Thurston equivalent to a critically periodic degree two polynomial (from Levy's Theorem mentioned in section 1.2) and hence $h \simeq s_q$ for some odd denominator rational q. Now we get $f \simeq s_p \amalg s_q$ as required. Figure 2 shows an example of the curves γ, $f^{-1}(\gamma)$, ζ_j, ζ_j', ξ_j, ζ_0'', ξ_0'' with $m = 3$ (of period 3). $\square$

The technique will be used later in Section 9 to recover the mating components.

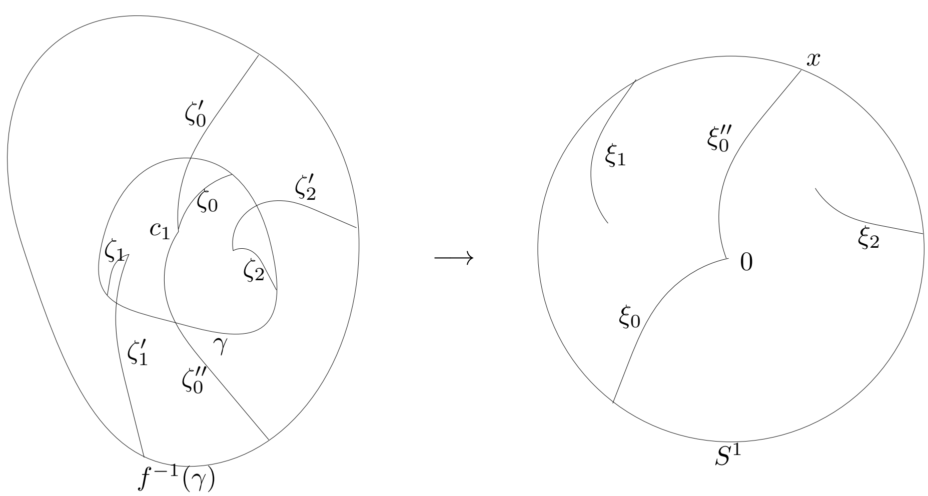

Figure 2: An example on getting the s_p map with $m = 3$

2.11 Conjugacy on Julia sets between R and R_* on $Per_{m'}(0)$

Now we prove a proposition which will be used in Section 8 on the conjugacy of dynamics between the "center" R of a hyperbolic component and its "root" R_* on their Julia sets in the parameter slice $Per_{m'}(0)$.

Proposition 2.11.1. *Let R be the critically periodic quadratic rational map with two critical points $c_1(R), c_2(R)$ of period m and m', let R_* be on the boundary of the hyperbolic component of R with one periodic critical point $c_2(R_*)$ of period m', with the parabolic periodic point v of period m described in Section 4. $c_1(R_*)$ is attracted to v under iterations of R_*^m (we will simply write these symbols $c_1(R), c_2(R), c_1(R_*), c_2(R_*)$ as c_1, c_2 if there is no confusion about which dynamical planes these points are on). Then there is a homeomorphism $\varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, such that*

$$\varphi(J(R)) = J(R_*)$$

and

$$\varphi \circ R = R_* \circ \varphi \text{ on } J(R).$$

On dynamics of the hyperbolic map R, because c_1 is of period m, there is one and only one repelling periodic point x' on the boundary of the Fatou component containing $c_1(R)$ of period dividing m. As the first step for proving the proposition, we show the following lemma.

Lemma 2.11.2. *There exists a flow of homeomorphisms $\varphi_t : \bar{\mathbb{C}} \to \bar{\mathbb{C}}, t \in [0, \infty)$, such that $\varphi_t \to \varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ uniformly as $t \to \infty$. φ is a continuous map which satisfies*

$$\varphi \circ R_* = R \circ \varphi$$

on $J(R_*)$. Moreover, φ is a homeomorphism from $\bar{\mathbb{C}} \setminus J(R_*)$ to $\bar{\mathbb{C}} \setminus J(R)$.

Proof. Let $M_3 = \{v, ..., R_*^{m-1}(v)\}$ be the parabolic cycle. We start with a neighbourhood U_0 of $\bigcup_{i=0}^{m-1} \bigcup_{k=0}^{\infty} R_*^{km+i}(\alpha) \cup M_3$, where α is an arc joining $c_1(R_*)$ with $R_*^m(c_1(R_*))$. We can choose the neighbourhood U_0 to satisfy

$$\bar{U_0} \subset R_*^{-1}(Int(U_0))$$

A similar set V_0 can be chosen on the dynamical plane of R, that is, neighbourhoods of $\bigcup_{i=0}^{m-1} R^i(\beta)$, in which β is an arc from $c_1(R)$ to x'. $R^m(\beta) = \beta$ and $\beta \cap J(R) = \{x'\}$. We also require that

$$\bar{V_0} \subset R^{-1}(Int(V_0)).$$

Now choose a homeomorphism $\varphi_0 : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ isotopic to the identity such that

$$\varphi_0(U_0) = V_0.$$

Then define a homeomorphism $\varphi_1 : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ via the map φ_0 by

$$\varphi_0 \circ R = R_* \circ \varphi_1$$

on $\bar{\mathbb{C}} \setminus R_*^{-1}(U_0)$. Let $\varphi_1 = \varphi_0$ on U_0. Let $\varphi_t, t \in [0,1]$ be an isotopy between φ_0 and φ_1 which is constant on U_0. Define the homeomorphisms $\varphi_{t+n}, t \in [0,1], n \in \mathbb{N}$ inductively by

$$\varphi_{t+n} \circ R_* = R \circ \varphi_{t+n+1} \tag{5}$$

on $\bar{\mathbb{C}} \setminus R_*^{-n}(U_0)$, and

$$\varphi_{t+n} = \varphi_{t+n+1} \tag{6}$$

on $R_*^{-n}(U_0)$. Then $\varphi_t \circ R_*^n = R^n \circ \varphi_{t+n}$ on $J(R_*)$. So

$$d(\varphi_{t+n+1}(z), \varphi_{t+n}(z)) = d(S \circ \varphi_{t+1} \circ R_*^n(z), S \circ \varphi_t \circ R_*^n(z))$$

where S is a local inverse of R^n suitably chosen. R expands uniformly on a neighbourhood of $J(R)$, say the expanding factor is $\frac{1}{\lambda}, 0 < \lambda < 1$. Then S contracts by the factor λ^n. So

$$d(\varphi_{t+n+1}(z), \varphi_{t+n}(z)) \le C\lambda^n \sup_{w \in U_0} d(\varphi_t(w), \varphi_{t+1}(w)) \le C_1\lambda^n$$

for $t \in [0,1]$ and $z \in R_*^{-n}(U_0)$. This forces φ_{t+n} to converge uniformly on $J(R_*)$. In particular, φ_n converges uniformly on $J(R_*)$ to a continuous map $\varphi : J(R_*) \to J(R)$. Convergence on $\bar{\mathbb{C}} \setminus J(R_*)$ is guaranteed by (6), as for any point $z \in \bar{\mathbb{C}} \setminus J(R_*)$, there is some integer n_0 such that $z \in (\bar{\mathbb{C}} \setminus R_*^{-n}(U_0))$ for all $n > n_0$. This means $\varphi_{t+n}(z)$ will be constant in n for all $t \in [0,1]$ and $n > n_0$. Moreover, by (5) , set $t = 0$ now, let $n \to \infty$, we get

$$\varphi \circ R_* = R \circ \varphi \tag{7}$$

on $J(R_*)$. $\square$

Now we continue to prove Proposition 2.11.1 by showing that the map φ we get in the last lemma is in fact a homeomorphism on $J(R_*)$.

Proof of Proposition 2.11.1:

Proof. We already have the map $\varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that $\varphi \circ R_* = R \circ \varphi$ on $J(R_*)$. Now we show that it is injective on $J(R_*)$. First note that $\varphi^{-1}(z)$ is connected for all $z \in \bar{\mathbb{C}}$. To see this, note that

$$\varphi^{-1}(z) = \bigcap_{\delta_n > 0} \varphi^{-1}\{|w - z| \leq \delta_n\}$$

where δ_n is any positive sequence with $\lim_{n\to\infty} \delta_n = 0$. Since $\varphi_n \to \varphi$ uniformly, by suitable choice of δ_n,

$$\varphi_n^{-1}\{|w - z| \leq \tfrac{\delta_n}{2}\} \subset \varphi_n^{-1}\{|w - z| \leq \delta_n\} \subset \varphi_n^{-1}\{|w - z| \leq 2\delta_n\}.$$

By restricting to a subsequence of n, we can assume

$$\varphi_{n+1}^{-1}\{|w - z| \leq 2\delta_{n+1}\} \subset \varphi_n^{-1}\{|w - z| \leq \tfrac{\delta_n}{2}\}.$$

So $\varphi^{-1}(z) = \bigcap_{\delta_n > 0} \varphi^{-1}\{|w - z| \leq \dfrac{\delta_n}{2}\}$, as the intersection of a decreasing sequence of connected sets, is connected.

Suppose $\varphi^{-1}(z_0)$ is not a point for some $z_0 \in J(R)$. We can assume $\varphi^{-1}(z_0) \subset J(R_*)$ because $\varphi(\bar{\mathbb{C}} \setminus J(R_*)) = \bar{\mathbb{C}} \setminus J(R)$ and φ is a homeomorphism on $\bar{\mathbb{C}} \setminus J(R_*)$ by Lemma 2.11.2. Let $P(R_*) = \overline{X(R_*)}$, where $X(R_*)$ is the post-critical set. We will show that the lift of a subset of $R_*^n(\varphi^{-1}(z_0))$ to the universal cover of $\bar{\mathbb{C}} \setminus P(R_*)$ has diameter tending to ∞. To achieve this we use the fact that R_* is expanding with respect to the Poincaré metric.

Let W be a small neighbourhood of the parabolic cycle such that $\overline{W} \cap J(R_*) \subset R_*(W)$, and $diam(\varphi^{-1}(z_0) \setminus W) \geq \delta_0 > 0$. The diameter is considered in the Poincaré metric. Now consider the set $R_*^n(\varphi^{-1}(z_0)) \setminus W$ for all n and lift it to the universal cover of $\bar{\mathbb{C}} \setminus P(R_*)$. Let $\widetilde{A}$ be the covering space of a set A, $\pi : \widetilde{\bar{\mathbb{C}} \setminus P(R_*)} \to \bar{\mathbb{C}} \setminus P(R_*)$ be the covering map.

Let X_n be a component of $\pi^{-1}(R_*^n(\varphi^{-1}(z_0) \setminus W)) = \pi^{-1}(\varphi^{-1}(R^n(z_0) \setminus W))$ for each n. Now we claim that either X_n intersects two components of $\pi^{-1}(W)$ or $diam(X_n) \to \infty$ as $n \to \infty$. The second possibility is a contradiction because if $\tilde{\varphi} : \widetilde{\bar{\mathbb{C}} \setminus P(R_*)} \to \widetilde{\bar{\mathbb{C}} \setminus P(R)}$ is a lift of φ, then $\tilde{\varphi}$ is also a uniform limit of homeomorphisms and $\tilde{\varphi}^{-1}(z)$ has bounded Poincaré diameter for all $z \in \widetilde{\bar{\mathbb{C}} \setminus P(R)}$.

The first possibility can not hold for arbitrarily small neighbourhood W of the parabolic cycle. If it does, then we can take limits of $\varphi^{-1}(R^n(z_0))$ and obtain a point z_1 such that $\varphi^{-1}(z_1)$ contains two points which are both mapped to the parabolic cycle. But φ is injective on the parabolic cycle, so $R_* \circ \varphi^{-1}(z_1) = \varphi^{-1} \circ R(z_1)$ would then be non-contractible or have interior, both of which are impossible.

Now we show that for some $\lambda > 1$, either $diam(X_{k+1}) \geq \lambda diam(X_k)$ for any $0 \leq k < n$ or X_k intersects two components of $\pi^{-1}(W)$ for some $0 \leq k < n$. Suppose inductively that X_{k-1} does not intersect two components of $\pi^{-1}(W)$ for any $1 \le k \le n$. If X_{k-1} intersects two components of $\pi^{-1} \circ R_*^{-1}(W)$, let $\tilde{R_*}$ be the lift of R_*, then $X_k = \tilde{R_*}(X_{k-1})$ intersects two components of $\pi^{-1}(W)$, we are done. So suppose X_{k-1} does not intersect two components of $\pi^{-1} \circ R_*^{-1}(W)$ for any $1 \le k \le n$. Then let $\lambda > 1$ be such that

$$d(\tilde{R_*}(x_1), \tilde{R_*}(x_2)) \ge \lambda d(x_1, x_2) \ \forall \ x_1, x_2 \notin \pi^{-1}(W)$$

where d denotes the Poincaré metric. Then

$$diam(\tilde{R_*}(X_{k-1})) \ge \lambda diam(X_{k-1})$$

i.e.

$$diam(X_n) \ge \lambda diam(X_{n-1})$$

as we expected. So the claim holds in this case.

□

Remark 2.11.3. *By the techniques of pinching and plumbing, Cui and Tan can get global semi-conjugacy between R_* and R, which is a conjugacy on their Julia sets, although R_* is not J-stable according to [MSS] and [McM2]. One can refer to [CT] for more incisive results.*

3 Yoccoz puzzle

The Yoccoz puzzle is a specific example of Markov partitions for complex dynamical systems. A Markov partition is a way to connect the dynamical systems to symbolic ones. The idea is used but not restricted to the study of dynamical systems. For more interesting discussions of the idea of partitions, see [AW] [Sin] [Bow].

In the following we first give the definition of Yoccoz puzzle for a quadratic polynomial $f_c(z)$ with connected Julia set $J(f_c)$, then quote some results of J. C. Yoccoz using the Yoccoz puzzle [Yoc] without proof.

3.1 The definition

Let $f_c(z) = z^2 + c$ be a quadratic polynomial with the filled Julia set $K(f_c)$ connected, so the finite critical point will be confined in $K(f_c)$. We know from Böttcher's Theorem that there is a holomorphic change of coordinates $\psi_c : \bar{\mathbb{C}} \setminus \mathbb{D} \to \bar{\mathbb{C}} \setminus K(f)$, such that

$$f_c \circ \psi_c = \psi_c \circ z^2.$$

The map f_c has two fixed points. One is of combinatorial rotation number 0 with a unique fixed external ray landing on it. The other one is repelling with combinatorial rotation number $\frac{m_1}{m_2} \in (0,1), m_1, m_2 \in \mathbb{N}$ if c is out of the main cardioid of M. The latter one is called the α fixed point in [Hub]. Let $|\psi_c| = h_c$ be the Green function, $\{R_{\theta_1}, R_{\theta_2}, \cdots, R_{\theta_q}\}, q \in \mathbb{N}$ be the external rays landing on the α fixed point, in which $\{\theta_1, \theta_2, \cdots, \theta_q\}$ are the angles. The initial graph Γ_0 is formed by the potential line $\{z : h_c(z) = R_0 > 1\} = H_0$ and part of the truncated rays R_{θ_i} (the part inside $U_0 = \{z : h_c(z) \leq R_0, R_0 > 1\}$). Define $U_n = f_c^{-1}(U_{n-1})$ and $\Gamma_n = f_c^{-1}(\Gamma_{n-1})$. The sequence $U_0 \supset U_1 \supset U_2 \cdots$ together with the graph Γ_n are called the puzzle partition of f_c. The closure of a component of $U_n \setminus \Gamma_n$ is called a depth (level) n puzzle piece.

A nested sequence of puzzle pieces

$$x = (X_0 \supset X_1 \supset X_2 \cdots)$$

is called an end of the puzzle. $I(x) = \bigcap_n X_n(x)$ is called the impression of the end x.

Let $x_b = (B_0 \supset B_1 \supset B_2 \cdots),\ c \in B_i\ \forall\ i \in \mathbb{N}$ be the critical value end, $x_c = (C_0 \supset C_1 \supset C_2 \cdots),\ 0 \in C_i\ \forall\ i \in \mathbb{N}$ be the critical end. We say the polynomial f_c is *combinatorially non-recurrent* if the orbit of the critical value end x_b never enters C_N for some $N \in \mathbb{N}$, or else we say f_c is *combinatorially recurrent.*

3.2 The parameter Yoccoz puzzle

Now we define the parameter Yoccoz puzzle (parapuzzle) relying on the Yoccoz puzzle. According to [Hub], for every $\frac{r_1}{r_2}$ limb of the main cardioid, let $W_{\frac{r_1}{r_2}}$ be the wake which is bounded by the two rays landing at the parameter

$$c_{\frac{r_1}{r_2}} = \frac{e^{2\pi i \frac{r_1}{r_2}}}{2}(1 - \frac{e^{2\pi i \frac{r_1}{r_2}}}{2})$$

on ∂M. For every $c \in W_{\frac{r_1}{r_2}}$, we have defined the Yoccoz puzzle in the previous subsection. Now we use U_n^c to denote a level n partition piece of the polynomial $f_c = z^2 + c$, $\Gamma_n^c = \partial U_n^c$.

Definition 3.2.1. *The n level parameter Yoccoz puzzle MU_n of the $\frac{r_1}{r_2}$ limb of M in the parameter space is the region*

$$MU_n = \{c : c \in U_n^c\}.$$

These regions together with their boundaries

$$M\Gamma_n = \partial MU_n = \{c : c \in \Gamma_n^c\}$$

are called the parameter puzzle.

The parapuzzle provides ways to transfer some geometric properties from the dynamical plane to the parameter plane by certain techniques, see Proposition 13.2 [Hub].

3.3 The renormalization theory

The renormalization theory in complex dynamics is closely related with Douady and Hubbard's theory of polynomial-like mappings [DH1].

Definition 3.3.1. *Let U', U be open subsets of $\mathbb{C}$ isomorphic to discs, $U' \subset U$ is relatively compact in U. A holomorphic map $f : U' \to U$ is called a polynomial-like mapping of degree d if it is proper of degree d.*

For a polynomial-like map, we can still define its Julia set $J(f)$ and filled Julia set $K(f)$ as

$$K(f) = \{z \in U' : f^n(z) \in U' \ \forall \ n \in \mathbb{N}\},$$

$$J(f) = \partial K(f).$$

There are some concrete examples of polynomial-like maps in [DH1] and [McM2].

Now we begin to introduce the notion of renormalization. We focus on a quadratic polynomial f_c. Let $f_c(z) = z^2 + c$ be a quadratic polynomial with connected Julia set. We say f_c^n is *quadratic renormalizable* [McM2] for some positive integer $n > 1$ if there are open discs $U' \subset U \subset \mathbb{C}$ such that the critical point

$$0 \in U'$$

and

$$f_c^n : U' \to U$$

is a polynomial-like map of degree 2 with connected Julia set. Note that the choices of U', U may not be unique for f_c^n to be quadratic renormalizable. If there are infinitely many $n \in \mathbb{N}$ such that f_c^n is renormalizable (quadratic renormalizable), then we say f_c is *infinitely renormalizable*, or else we will say f_c is finitely renormalizable. Infinitely renormalizable polynomials turn out to have more complicated dynamics than finitely renormalizable ones.

The renormalization theory not only links Julia sets of large scales and smaller scales, but also plays an important role in the local connectivity of the Julia sets and the Mandelbrot set M by the results (e) and (f) in the following [Hub].

3.4 Some results proved using Yoccoz puzzles

The following results related with the puzzle and parapuzzle partitions are from [Yoc]. They are demonstrated in [Hub] by Hubbard.

(a) If f_c is not renormalizable, the impression of each end of its puzzle is a point.

(b) If f_c is renormalizable, then the ends of its puzzle which are preimages of the critical end have impressions which are homeomorphic to $K(f_{c_1})$ for some quadratic polynomial $f_{c_1} = z^2 + c_1$ with $K(f_{c_1})$ connected. The impressions of the other ends are points.

(c)For any piece X of the puzzle, the intersection $K(f_c) \cap X$ is connected.

(d)If a polynomial f_c is non-recurrent, then all its ends are points. In particular, $K(f_c)$ is locally connected.

(e)If $c \in M$ is not infinitely renormalizable, and does not have an indifferent periodic point, then $K(f)$ is locally connected.

(f)If $c \in M$ is not infinitely renormalizable, then M is locally connected at the point c.

The conclusion (e) is obtained by an enriched puzzle, see [Hub, Section 11]. By comparing the moduli of annuli on the parameter space with moduli of the corresponding annuli on the dynamical plane (see

[Hub, Section 13]), the local connectivity of the Julia set can be transferred to the parameter plane, that is, for a non-infinitely renormalizable parameter $c \in M$, c has a basis of connected neighbourhoods in M.

D. Schleicher gave a proof on local connectivity of the Multibrot sets at the Misiurewicz points [Schl1] by introducing fibers [Schl2] of compact connected and full sets in $\mathbb{C}$. The idea of fibers, in Schleicher's words, is to show shrinking of puzzle pieces without using specific puzzles.

The set of infinitely renormalizable parameters at which M is locally connected is dense by Yunping Jiang [Jia].

3.5 A convergence result proved using the parameter Yoccoz puzzle

The following result describes the distribution of the quadratic polynomials Thurston equivalent to the lamination maps s_p with odd denominator rationals $p \in (0,1)$. The statement uses lamination language.

Theorem 3.5.1. *Let p_n, p be odd denominator rationals in $[0,1]$, $n \in \mathbb{N}$. $p_n \to p$ as $n \to \infty$, $s_{p_n} \simeq f_{c_n}(z)$, $s_p \simeq f_c(z)$, where s_{p_n}, s_p are degree two critically finite branched coverings associated with the laminations L_{p_n}, L_p (refer to [Ree2]), $f_{c_n}(z) = z^2 + c_n$ and $f_c(z) = z^2 + c$ are quadratic polynomials, $c_n, c \in \mathbb{C}$. Then*

$$\lim_{n\to\infty} c_n = c'$$

where $c' \in M$ is the quadratic parabolic parameter being the root of the wake on the boundary of the hyperbolic component containing c.

The proof is given in Section 5.

Remark 3.5.2. *This result follows directly from many others' results, for example, the local connectivity of M at a parabolic parameter, see*

[Yoc], [Tan2] and local connectivity of M at every point on the boundary of a hyperbolic component (Schleicher's proof is for the Multibrot sets), see [Schl1]. But there are still some reasons for us to present our proof here. One is that the proof shows patterns of the convergence of the polynomials with respect to the odd denominator rationals, the other is that there is something new in the proof of local connectivity of M at the primitive parabolic point.

4 Main results

Before giving the main results, we first want to pose a conjecture on continuity of 2-matings (we refer to d as the degree for the notation d-matings). Our main results mainly deal with sub-cases of it. As the cases of hyperbolic matings should be dealt differently from the non-hyperbolic cases, we would like to state the conjecture in two cases.

4.1 The conjectures on one-parameter continuity of 2-matings

The conjecture on one-parameter continuity of hyperbolic 2-matings:

Conjecture 4.1.1. *Let $p_n \in (0,1)$ be a sequence of odd denominator rationals such that $\lim_{n\to\infty} p_n = t$, in which $t \in (0,1)$ is a real number. Let $q \in (0,1)$ be an odd denominator rational. Assume furthermore that μ_{p_n} and μ_q are not in conjugate limbs of the lamination model of M for any $n \in \mathbb{N}$ and $e^{2\pi it}$ and μ_q are not in conjugate limbs of the lamination model of M. Let R_n be the rational map on some parameter slice $Per_{m'}(0)$ such that $R_n \simeq s_{p_n} \amalg s_q$. Let m' be the period of q under the doubling map $x \to 2x$ mod $\mathbb{Z}$. Then*

$$\lim_{n\to\infty} R_n = R_t$$

with R_t being a rational map on the parameter slice $Per_{m'}(0)$ containing R_n.

In particular, if t is also an odd denominator rational and μ_t and μ_q are not in conjugate limbs of the lamination model of M, then R_t is the parabolic map on the boundary of the hyperbolic component containing the rational map $R \simeq s_t \amalg s_q$, with one parabolic cycle corresponding to the orbit of p and one superattractive cycle corresponding to the orbit of q under the doubling map.

All the rational maps R_n, R, R_t are in the rational parameter space with one of their critical points c_2 being periodic of the same period with the period of q under the doubling map $x \to 2x \mod \mathbb{Z}$. This parameter slice is denoted by $Per_{m'}(0)$ in [Tim2]. Without special declaration we always stay in this parameter space in the remainder of the thesis for rational maps. As mentioned before, this conjecture, combined with the results of control on sizes of the mating limbs in M_a (see explanation for the notation in Section 10) will imply the general continuity result on 2-matings with one parameter fixed. The following continuity conjecture implies Conjecture 4.1.1.

The general one-parameter continuity conjecture on 2-matings:

Conjecture 4.1.2. *Let $f_{c'}(z) = z^2 + c'$ and $f_c(z) = z^2 + c$ be two conformally matable quadratic polynomials such that $R_c \approx f_{c'} \amalg f_c$ on some parameter slice $Per_{m'}(0)$. Let $c_n \in \mathbb{C}$ such that $\lim_{n\to\infty} c_n = c$ and $f_{c_n} = z^2 + c_n$ is conformally matable with $f_{c'}$ for any $n \in \mathbb{N}$, $R_{c_n} \approx f_{c'} \amalg f_{c_n}$ on the parameter slice $Per_{m'}(0)$ containing R_c. Then*

$$\lim_{n\to\infty} R_{c_n} = R_c.$$

This conjecture is quite ambitious, considering the matings between both hyperbolic and non-hyperbolic ones. The matability between non-hyperbolic ones is not guaranteed, although we do have some positive results [YZ] [AY] on this. Our Theorem 8.2.1 deals with the conjecture in the case c is not renormalizable and $c' = -1$ as a corollary of methods from [AY].

4.2 Main results on continuity of matings

The following is a collection of main results on continuity of 2-matings that we get throughout the thesis.

Theorem 4.2.1. *Let p and q be odd denominator rationals in $(0,1)$ such that $s_p \amalg s_q$ is Thurston equivalent to a rational map R, that is, the conditions of Tan Lei's Theorem are satisfied. Let m, m' be the period of p, q separately under the doubling map: $x \to 2x \bmod 1$. Let $p' \in (0,1)$ satisfy $2^s p' = p \bmod 1$ for some large positive integer s. For some fixed positive integer r large enough, define a sequence of odd denominator rationals*

$$p_n = p + \frac{2^{sm}(p' - p)}{2^{(s+r+2n)m} - 1} \tag{8}$$

in $(0,1)$, $n \in \mathbb{N}$. Let $R_n \simeq s_{p_n} \amalg s_q$ be the rational map in the quadratic rational parameter slice $Per_{m'}(0)$ containing R, then

$$\lim_{n \to \infty} R_n = R_*$$

where R_* is a parabolic map on the boundary of the hyperbolic component containing the rational map $R \simeq s_p \amalg s_q$ described in Section 6.

This means that for n large enough, we can find degree two critically periodic rational maps which are Thurston equivalent to $s_{p_n} \amalg s_q$ in an arbitrary small neighbourhood of the parabolic map R_*. The geometric significance of the sequence p_n in equation (8) is explained in Section 9.

In order to state the next theorem, we introduce the R_a family (refer to [AY]):

$$R_a(z) = \frac{a}{z^2 + 2z}, \qquad a \in \mathbb{C}. \tag{9}$$

It is an example of the parameter slice $Per_2(0)$.

Theorem 4.2.2. *Let $c \in M$ be a parameter outside the $\frac{1}{2}$-limb of M, such that $f_c = z^2 + c$ is non-renormalizable and without non-repelling periodic cycles. Moreover, suppose that the critical point 0 is not in the backward orbit of the α fixed point of f_c. Let $c_n \in M, n \in \mathbb{N}$ be a sequence of hyperbolic parameters such that $c_n \to c$ as $n \to \infty$. Then there exist unique $a(c), a(c_n) \in \mathbb{C}$, such that*

$$R_{a(c)} \approx (f_c \amalg f_{-1}),\ R_{a(c_n)} \approx (f_{c_n} \amalg f_{-1})$$

in the R_a family. Moreover,

$$\lim_{n\to\infty} a(c_n) = a(c).$$

Remark 4.2.3. *The existence of $a(c), a(c_n)$ is guaranteed by AY's Theorem in Section 10.1 and Tan Lei's Theorem in Section 2.8 (in fact Tan Lei's Theorem is for all the critically finite quadratic polynomials, and can be easily extended to hyperbolic ones). The condition that 0 is not in the backward orbit of the α fixed point of f_c rules out exactly the intersection of the parameter puzzle pieces MU_n with M, and these parameters are countable.*

The following theorem deals with the case when all the minor leaves μ_{p_n} are in some small copy of the Mandelbrot set bounded by some minimal minor leaf μ_r and all closures of Fatou components of the rational map $R \simeq s_p \amalg s_q$ are disjoint. Recall that μ_p (μ_q) is the minor leaf in the lamination L_p (L_q), refer to Section 2.2.

Theorem 4.2.4. *Let p_n, p, r, q all be odd denominator rationals in $(0, 1)$, $n \in \mathbb{N}$. Suppose $p_n \to p$ as $n \to \infty$ such that the minor leaves μ_{p_n} are all in the small copy of the Mandelbrot set bounded by μ_r, μ_p and μ_q are not in conjugate limbs of the lamination model for M, and the closures of Fatou components of $R \simeq s_p \amalg s_q$ are all disjoint. Let $R_n \simeq s_{p_n} \amalg s_q$, and R_* be the parabolic rational map on the boundary of the hyperbolic component containing the rational map $R \simeq s_p \amalg s_q$ described in Section 6. Then*

$$\lim_{n\to\infty} R_n = R_*.$$

The following theorem deals with the case when all closures of Fatou components of the rational map $R \simeq s_p \amalg s_q$ are disjoint, which includes Theorem 4.2.4 as a special case.

Theorem 4.2.5. *Let p_n, p, q all be odd denominator rationals in $(0, 1)$ such that $\lim_{n\to\infty} p_n = p$ and suppose that μ_p and μ_q are not in conjugate limbs of the lamination model for M. Let R_n, R be the rational maps such that $R_n \simeq s_{p_n} \amalg s_q$, $R \simeq s_p \amalg s_q$ and closures of all Fatou components of R are disjoint. Let R_* be the rational map on the boundary of the hyperbolic component containing R, with one parabolic cycle (corresponding to p) and one superattractive cycle (corresponding to q). If $\varphi_*^{-1}(\varphi_*(p))$ contains at most three points (refer to Section 12.7 for the map φ_*), then*

$$\lim_{n\to\infty} R_n = R_*.$$

We will use the idea of Markov partitions (see Section 12.3) for the rational maps near R_* to deal with it, which avoids the parabolic cycle on the boundary of any partition element. According to Lemma 12.3.2 [Ree4], the Markov partitions persists for maps in a small neighbourhood of R_* on the parameter slice $Per_{m'}(0)$.

5 Proof of Theorem 3.5.1: convergence of the polynomials Thurston equivalent to the lamination maps s_p

In this section, we will show the dependence between odd denominator rationals and the corresponding 2-polynomials assumes some continuity, that is, Theorem 3.5.1.

5.1 Control of the size of limbs of M

Our tool of the proof is centred on Proposition 4.2 [Hub]. Now for a hyperbolic component $V \subset M$, let

$$\psi_V : \bar{\mathbb{D}} \to \bar{V}$$

be the interior parametrization by the multipliers. Let

$$c_V = \psi_V(1), c_{V,t} = \psi_V(e^{2\pi it}), t \in \mathbb{Q}.$$

There are two external rays of M landing on c_V. These two rays together with c_V cut the parameter space $\mathbb{C}$ into two parts. Now define the part of $\mathbb{C}$ disjoint from V bounded by the two rays to be a *wake* W_{c_V} of the hyperbolic component V. We call c_V the root of the wake, and call the intersection

$$L_{V,\frac{r_1}{r_2}} = M \cap W_{c_V}, r_1, r_2 \in \mathbb{N}$$

the $\frac{r_1}{r_2}$ limb of V. Then (Proposition 4.2 [Hub])

Proposition 5.1.1. *(a) Every point of M in the wake of c_V is either in V or in one of the limbs of V.*
(b) There exists a function $\eta_V : \mathbb{N} \to \mathbb{R}$ with $\eta_V(r_2) \to 0$ as $r_2 \to \infty$, such that

$$diam(L_{V,\frac{r_1}{r_2}}) \leq \eta_V(r_2).$$

The proof uses the Yoccoz inequality, one can refer to P478 [Hub] for the proof.

5.2 Proof of Theorem 3.5.1

As the situation in the satellite case is different from the primitive case, we will prove the theorem separately in the two situations.

Proof. **The Satellite Case.**

In this case the wake $W_{\mathcal{P}_p}$ (see the following explanation) is attached on another hyperbolic component. We prove this by orbit portrait theory, especially, we rely on Theorem 1.2 [Mil2].

We use l_p to denote the minor leaf in the lamination L_p with one end point $e^{2\pi ip}$, l_p^2 to denote the leaf of its image under z^2 in $\bar{\mathbb{D}}$ and so on. Use $P(p)$ to denote the period of any odd denominator rational p under the doubling map. Then $l_p, l_p^2, \cdots, l_p^{P(p)}$ form a repelling cycle of period dividing $P(p)$ after collapsing the leaves and polygon gaps in the lamination L_p (the period can even be 1, in which case $l_p, l_p^2, \cdots, l_p^{P(p)}$ form a polygon gap). For convenience of notations, we assume after collapsing leaves and polygon gaps in the lamination L_p, $l_p, l_p^2, \cdots, l_p^{P(p)}$ form a repelling cycle of period $P(p)$. The argument can be applied to cases of periods dividing $P(p)$. Now denote the points corresponding to $l_p, l_p^2, \cdots, l_p^{P(p)}$ after collapsing the leaves and polygon gaps by $z_1^p, z_2^p, \cdots, z_{P(p)}^p$. Denote the other endpoint of l_p by $e^{2\pi i\bar{p}}$ for $\bar{p} \in (0, 1)$. Then the portrait associated with the repelling orbit

$$\mathcal{O}_p = \{z_1^p, z_2^p, \cdots, z_{P(p)}^p\}$$

of f_c is

$$\mathcal{P}_p = \{\{p, \bar{p}\}, \{2p, 2\bar{p}\}, \cdots, \{2^{P(p)}p, 2^{P(p)}\bar{p}\}\}.$$

The valence $\upsilon_p = 2$ (number of external rays landing on a point in the repelling cycle). As l_p is the minor leaf, we can define the angular width (refer to [Mil2]) of a sector S to be the length of the open arc $\mathcal{I}_S$ consisting of all the angles $t \in \mathbb{R}/\mathbb{Z}$ with $\mathcal{R}_t \subset S$. Here we use $\mathcal{R}_t$ to denote the external ray of angle $t \in [0, 1)$. Then the sector bounded by l_p will have the least angular width compared with other sectors bounded by $l_p^2, \cdots, l_p^{P(p)}$. $[p, \bar{p}]$ (in the case $p < \bar{p}$) or $[\bar{p}, p]$ (in the case $p > \bar{p}$) is called the characteristic arc of the orbit portrait $\mathcal{P}_p$ in [Mil2].

Now by Theorem 1.2 [Mil2], the two corresponding rays $\mathcal{R}_p^M$, $\mathcal{R}_{\bar{p}}^M$ land on the parabolic parameter c' which has a parabolic cycle of the same period $P(p)$ with portrait $\mathcal{P}_p$. Denote the wake cut by the two rays $W_{\mathcal{P}_p}$, so c' is the root of the wake. Denote the hyperbolic component in $W_{\mathcal{P}_p}$ with c' on its boundary by U. We will show that c_n must tend to c' as $n \to \infty$.

Now following the same process, the repelling periodic orbit

$$\mathcal{O}_{p_n} = \{z_1^{p_n}, z_2^{p_n}, \cdots, z_{P(p_n)}^{p_n}\}$$

of f_{c_n} has portrait

$$\mathcal{P}_{p_n} = \{\{p_n, \bar{p_n}\}, \{2p_n, 2\bar{p_n}\}, \cdots, \{2^{P(p_n)}p_n, 2^{P(p_n)}\bar{p_n}\}\}$$

while $[p_n, \bar{p_n}]$ (in the case $p_n < \bar{p_n}$) or $[\bar{p_n}, p_n]$ (in the case $p_n > \bar{p_n}$) is the characteristic arc. The two parameter rays $\mathcal{R}_{p_n}^M$, $\mathcal{R}_{\bar{p_n}}^M$ land at a parabolic parameter which cut a wake $W_{\mathcal{P}_{p_n}}$ containing c_n. Now consider the relative positions of the minors l_p, l_{p_n} on the closed unit disc $\bar{\mathbb{D}}$ and the wakes $W_{\mathcal{P}_p}$, $W_{\mathcal{P}_{p_n}}$ on M.

There are two possible choices for the relative positions of the sequence p_n and the two points $p, \bar{p}$. Either the sequence p_n is between p and $\bar{p}$ or it is not. First we will assume that all p_n are between p and $\bar{p}$. In the general case we can split the sequence p_n into two sub-sequences such that one is between p and $\bar{p}$ while the other is not. Under this assumption $c_n \in \bar{U} \cup W_{\mathcal{P}_p}$ for all n. We first consider the case $\bar{p} < p$ and p_n strictly increases to p.

Whatever the choices are, p_n and $\bar{p_n}$ both will tend to p as every point in the wake $W_{\mathcal{P}_p}$ is either in $\bar{U}$ or in one of the limbs of U (Proposition 5.1.1). The relative positions of all the leaves l_n can be quite complicated, though all of them tend to the point $e^{2\pi i p}$, Figure 3 shows a possible relative positions of the leaves l_{p_n} and l_p with subscripts $1 \leq n \leq 11$.

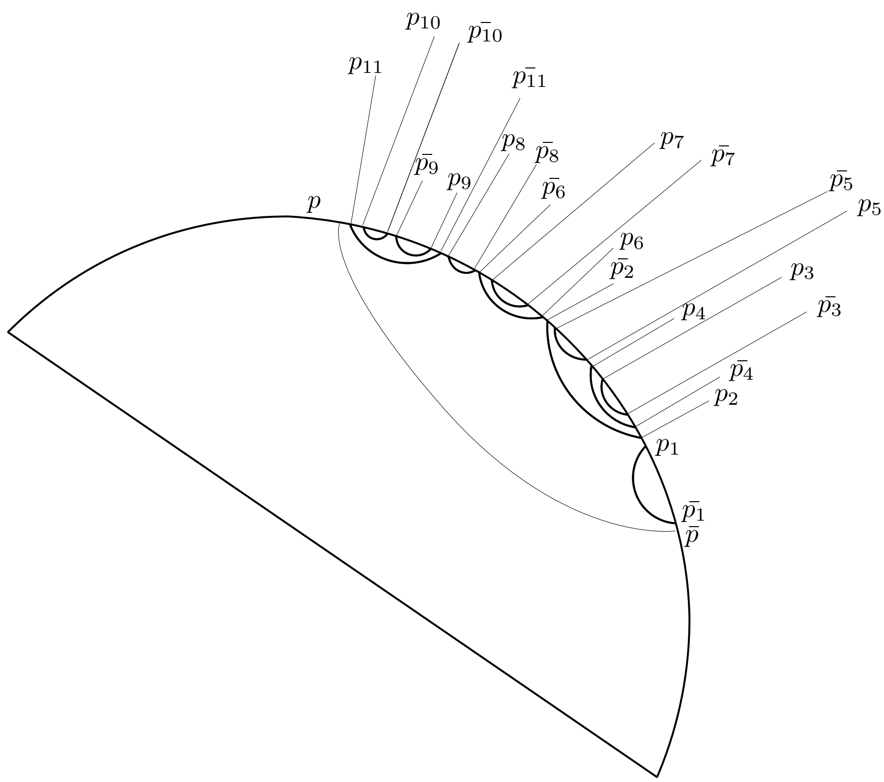

Figure 3: A possible relative positions of the leaves l_{p_n} and l_p with subscripts $1 \leq n \leq 11$ in the case $\bar{p} < p$, in order to draw the arcs clearly length of the minors has been magnified

Note that p_n and $\bar{p_n}$ are between t_n and $\bar{t_n}$, where $e^{2\pi i t_n}$ and $e^{2\pi i \bar{t_n}}$ are the endpoints of a leaf l_{t_n} on the boundary of the gap bounded by l_p and separated from 0 by l_p. So

$$t_n \to p \text{ as } n \to \infty.$$

Assuming without loss of generality that

$$\bar{t_n} < p_n < t_n.$$

Then $\mathcal{R}^M_{t_n}$ and $\mathcal{R}^M_{\bar{t_n}}$ land on a common point $h_n \in \partial M$ such that

$$h_n \to c' \text{ as } n \to \infty.$$

Now let $W_{\mathcal{P}_{t_n}} \cap M = L_{U,t_n}$ be the limbs. According to Proposition 5.1.1,

$$diam(L_{U,t_n}) \to 0 \text{ as } n \to \infty,$$

so it must be the case that the whole limb

$$L_{U,t_n} \to c' \text{ as } n \to \infty$$

or else h_n will not tend to c'. Then $c_n \to c'$ as $n \to \infty$ follows from the fact that

$$c_n \in L_{U,t_n}.$$

Figure 4 shows a possible distribution of the wakes and external rays of M corresponding to distributions of $p, \bar{p}, p_n, \bar{p_n}$ in Figure 3.

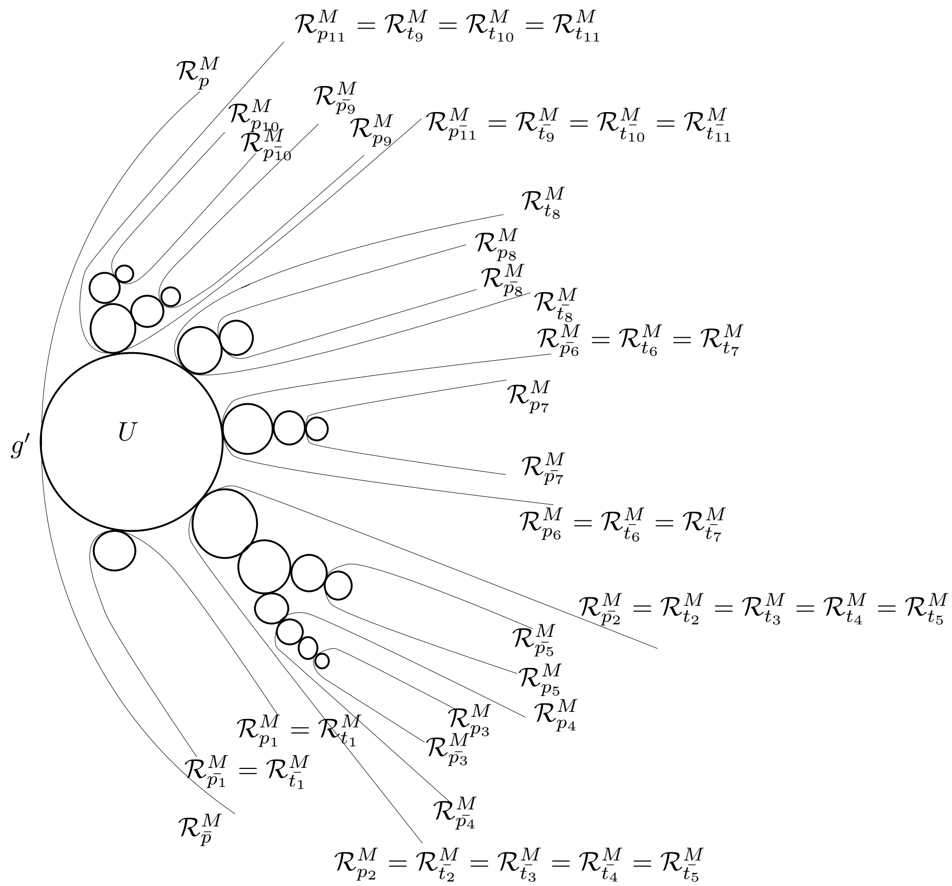

Figure 4: A possible distribution of the wakes and external rays of M corresponding to Figure 3 in the case $\bar{p} < p$, in order to draw the arcs clearly area of the wakes have been magnified

Now suppose $\bar{p} > p$ and the sequence $p_n \notin [p, \bar{p}]$. Using similar arguments as before, the whole limb

$$L_{U,p_n} \to c',$$

however, not from interior of $W_{\mathcal{P}_p}$, but from exterior of it. Here we denote by U the hyperbolic component attached to c' and not contained in the wake $W_{\mathcal{P}_p}$ (the former U component is the one in the wake $W_{\mathcal{P}_p}$). The following two pictures show a possible relative positions of the leaves

l_{p_n} and l_p on $\bar{\mathbb{D}}$ with subscripts $1 \le n \le 12$ and the corresponding distributions of the wakes and external rays of M.

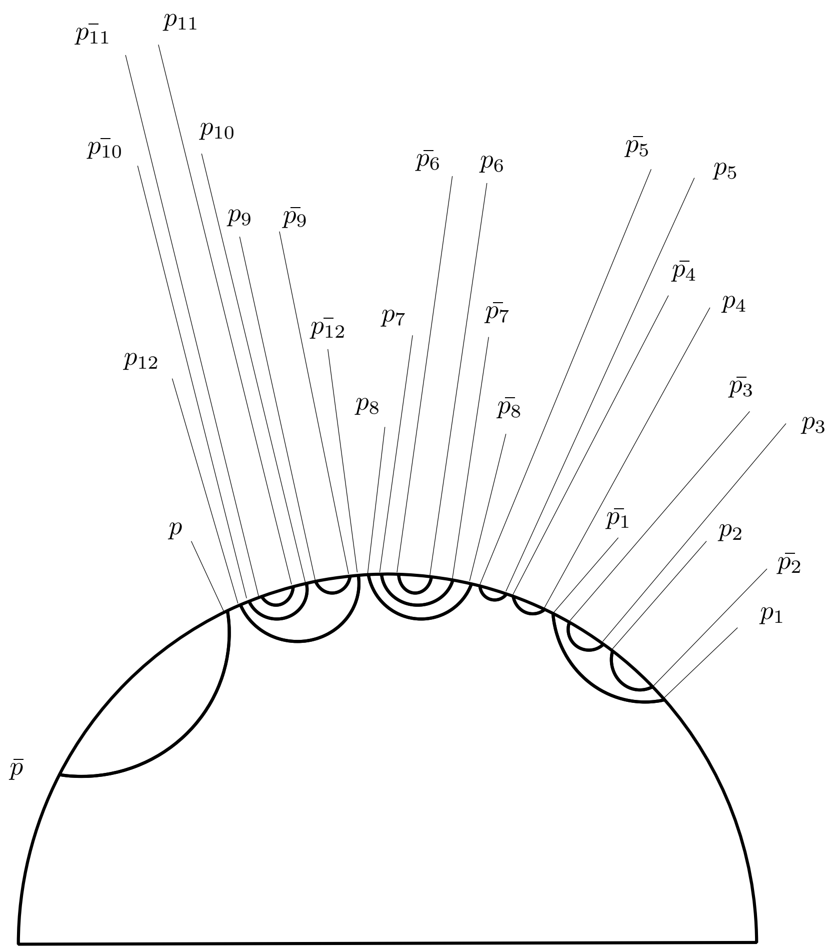

Figure 5: A possible relative positions of the leaves l_{p_n} and l_p on $\bar{\mathbb{D}}$ with subscripts $1 \le n \le 12$ in the case $\bar{p} > p$, in order to draw the arcs clearly length of the minors have been magnified

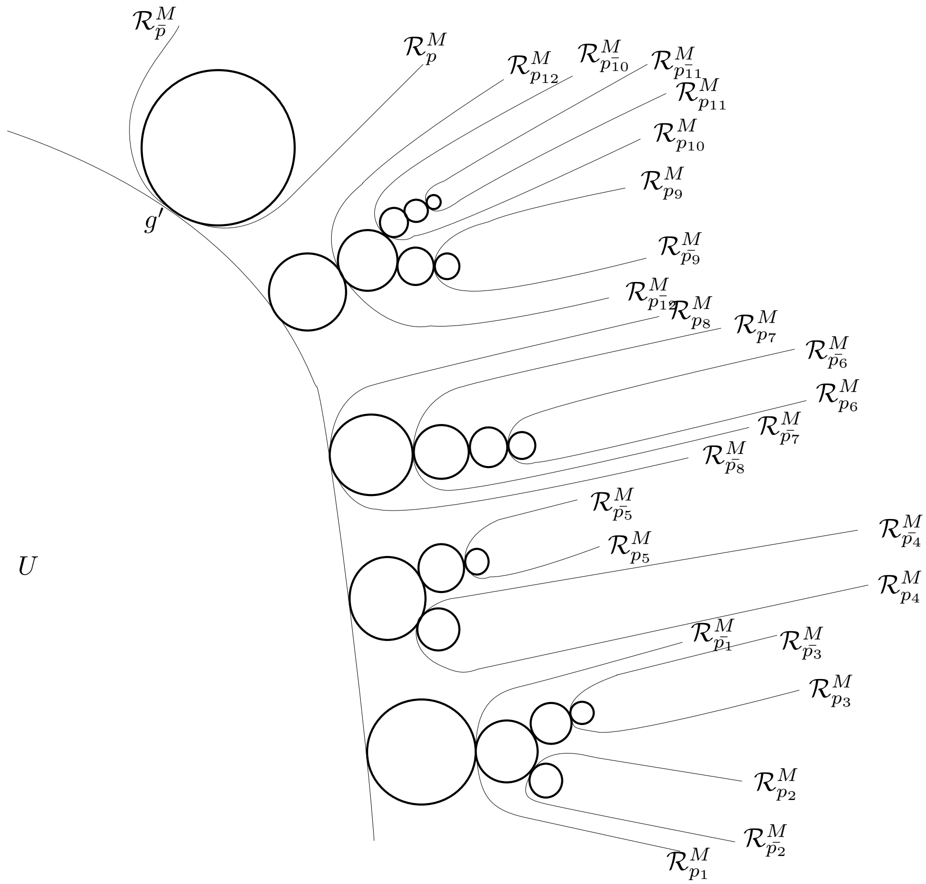

Figure 6: A possible distribution of the wakes and external rays corresponding to Figure 5 on M in the case $\bar{p} > p$, in order to draw the arcs clearly area of the wakes have been magnified

$\square$

Remark 5.2.1. *If one does not require p_n strictly increase to p, more mixing patterns of relative positions of the leaves and wakes are allowed, however, the convergence still holds.*

Proof. **The Primitive Case.**

In this case the wake $W_{\mathcal{P}_p}$ is not attached on another hyperbolic component. We distinguish two cases. First, if every p_n is out of the interval

$[\bar{p}, p]$ (by this expression we are assuming $\bar{p} < p$, the case $\bar{p} > p$ is similar), according to the wake description of M near the primitive parabolic parameter c' [Mil2, Theorem 1.2] and the control of size of limbs of a hyperbolic component of M [Hub, Proposition 4.2], let U be the hyperbolic component in $W_{\mathcal{P}_p}$ which contains c' on its boundary, c' is the root of the wake $W_{\mathcal{P}_p}$, then the convergence of c_n goes as the satellite case.

Now the main case to consider is when p_n is in $[\bar{p}, p]$, or else split the sequence into two sub sequences, one is out of the interval $[\bar{p}, p]$, the other one is in the interval. In order to deal with convergence in this situation we will need the following definition of extremal length, which is similar to the definition of modulus of annulus. One can refer to P10 [Ahl] for more interesting material of the notion.

For a topological disk X with two assigned opposite edges l, l', let Γ denote the union of arcs in X which connect l with l'. For a conformal metric ρ on X, denote the length of an arc $\gamma \in \Gamma$ by $L_\rho(\gamma)$, denote the area of X by $A_\rho(X)$. Then we call

Definition 5.2.2. $Mod(X) = \sup\limits_{\rho} \dfrac{\left(\inf\limits_{\gamma\in\Gamma} L_\rho(\gamma)\right)^2}{A_\rho(X)}$

the modulus of X with two assigned edges l, l' (in the cases where the assigned edges are obvious we omit them).

We want to present the Grötzsch inequality here. It relates the modulus of an annulus with its conformal-homotopy embeddings.

Grötzsch inequality 5.2.3. *Let E be an open annulus, E_j be a (finite or infinite) sequence of open annulus. Let $\varphi_j : E_j \to E$ be conformal mappings which are homotopy equivalences, with disjoint images. Then*

$$\textstyle\sum_{j=0}^{\infty} Mod(E_j) \leq Mod(E)$$

One is recommended to Proposition 5.4 [BH] for a proof.

Now by the construction of the parapuzzle partition introduced in Section 3.2, the two rays of M, $\mathcal{R}_p^M$ and $\mathcal{R}_{\bar{p}}^M$ both land on $c' \in \partial M$, the parabolic parameter. Let $\mathcal{R}_p^M$ and $\mathcal{R}_{\bar{p}}^M$ encounter the boundary of the parapuzzle piece $MX_n(c')$ (the level n parapuzzle piece containing the parameter c') at the two points a_n, a_n'. Then

$$h_M(a_n) = h_M(a_n') = \frac{R}{2^n}, R > 1.$$

Now we get a nested sequence of sector shaped domains. Every domain is bounded by parts of the two rays $\mathcal{R}_p^M$, $\mathcal{R}_{p'}^M$, say r_n, r_n', and the arc joining a_n and a_n' as part of the boundary of $MX_n(c')$, denote this arc by l_n and denote the closed sector area bounded by r_n, r_n' and l_n by MB_n. l_n is piecewise smooth. Denote the closed topological disc bounded by l_n, l_{n+1} and part of $\mathcal{R}_p^M, \mathcal{R}_{p'}^M$ by MC_n. So MC_n is the closure of $MB_n \backslash MB_{n+1}$. First we prove the following lemma, which will be true for more general nested sequences with piecewise smooth boundaries.

Let C_n ($n \geq 0$) be a sequence of closed topological rectangles with piecewise smooth boundaries such that C_n and C_{n+1} share a connected common part l_n of boundaries and $C_i \cap C_j = \emptyset$ for $|i-j| \geq 2$. All the C_n lie between two smooth topological arcs I_+ and I_- intersecting each other at x. Let $I_<$ be the arc connecting the other two endpoints of I_+ and I_-. Let B_n be the closure of the set $\cup_{j=n}^{\infty} C_j$. Since $\cup_{n=0}^{\infty} C_n$ is bounded, then

Lemma 5.2.4. *If* $\sum_{n=0}^{\infty}(Mod(C_n)) = \infty$, *then*

$$diam(B_n) \to 0$$

as $n \to \infty$, *and*

$$\cap_{n=0}^{\infty} B_n = x.$$

Proof. There is a conformal mapping f from B_0^o to the upper half disc $D_+ = \{z : |z| < 1\} \cap \{z : Im\ z > 0\}$. As B_0 has piecewise smooth boundaries, f extends to the boundary of B_0 by the smooth Riemann mapping theorem. We require

$$f(x) = 0, f(I_+) = [-1, 0], f(I_-) = [0, 1]$$

and

$$f(I_<) = \{z : |z| = 1\} \cap \{z : Im\ z > 0\}.$$

Figure 7 shows a sketch of the correspondence between the two structures under f up to $n = 4$.

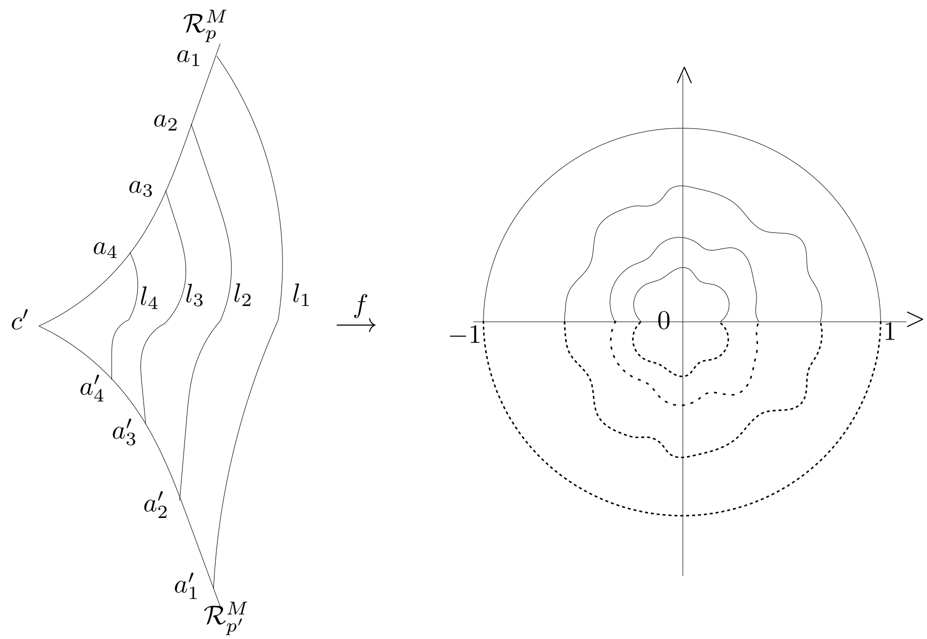

Figure 7: The conformal mapping f

Now we have

$$Mod(f(C_n)) = Mod(C_n)$$

because the modulus is preserved under conformal mapping. Denote by

$$A_n = f(C_n) \cup f(\bar{C}_n) \text{ (the complex conjugate of } f(C_n)),$$

A_n is an annulus. Moreover

$$Mod(A_n) = \tfrac{1}{2} Mod(f(C_n)) = \tfrac{1}{2} Mod(C_n).$$

Now if $\sum_{n=1}^{\infty}(Mod(C_n)) = \infty$, then

$$\sum_{n=1}^{\infty}(Mod(A_n)) = \infty.$$

Denote the closed disc bounded by the curve $f(l_n) \bigcup f(\bar{l}_n)$ by B'_n. According to Proposition 6.1 [Hub],

$$diam(B'_n) \to 0 \text{ as } n \to \infty$$

and

$$\cap_{n=1}^{\infty} B'_n$$

is a single point, which forces

$$diam(B_n) \to 0 \text{ as } n \to \infty$$

and

$$\cap_{n=1}^{\infty} B_n = x.$$

□

Remark 5.2.5. *This lemma is a sector-shaped evolution of Proposition 6.1 in [Hub]. One can imagine that generalizations to other specific shaped sequences will also be possible and quite useful.*

Apply Lemma 5.2.4 to our case with $C_n = MC_n$ and $B_n = MB_n$, we see that if $\sum_{n=0}^{\infty} Mod(MC_n) = \infty$, then

$$\lim_{n\to\infty} diam(MB_n) \to 0 \text{ and } \cap_{n=1}^{\infty}(MB_n) = c'.$$

Now we consider the corresponding picture on the dynamical plane. Let $\phi_{c'}$ be the Böttcher map of c'. $\Phi_M : \mathbb{C} \setminus M \longrightarrow \mathbb{C} \setminus \bar{\mathbb{D}}$ is the conformal isomorphism in [DH2]. According to Proposition 12.4 [Hub],

$$C_n = \phi_{c'}^{-1} \circ \Phi_M(MC_n)$$

form a new topological disc sequence inside the sector shaped area

$$B_0 = \phi_{c'}^{-1} \circ \Phi_M(MB_0).$$

B_0 still has parts of the two external rays of angle p and $\bar{p}$ as part of its boundary on the dynamical plane of $f_{c'}$, the same with the equipotentials. Moreover, $(f_{c'})^{-k}$ maps C_n conformally onto C_{n+k}, where k is the period of critical point of f_c. Without loss of generality suppose $Mod(C_1) > 0$, then

$$Mod(C_{kj+1}) = Mod(C_1) > 0,\ j \in \mathbb{Z}.$$

So we have

$$\sum_{n=1}^{\infty} Mod(C_n) = \infty.$$

Now we want to transfer the divergent results on sums of the modulus from the dynamical plane of $f_{c'}$ to the parameter plane. Now consider the situation that c' is in p/q limb of the main cardioid of M. Choose a neighbourhood V of the root of the limb $M_{p/q}$, $p, q \in \mathbb{N}^+$ as small as possible such that c' is not in V. We have the following links:

Lemma 5.2.6. *There exists a constant $C' > 0$ depending only on p/q and V, such that for each contributing disc C_{kj+1} on the dynamical plane, the corresponding topological disc MC_{kj+1} sitting on the parameter plane has modulus at least C'.*

Proof. According to the definition of MC_{kj+1}, if we can find a conformal metric μ, such that $L_\mu(\gamma_j)$ has lower bounds for any curve γ_j in MC_{kj+1} joining l_{kj+1} and l_{kj+2}, $A_\mu(MC_{kj+1})$ has upper bounds (both bounds do not depend on j), then things are done. To achieve this, use the metric

$$\mu = \mu_1 + \mu_2$$

in the proof of Proposition 13.2 [Hub].

$$\mu_1 = 2^{n+\nu}|d\log\Phi_M|$$

is a metric on MC_n (in fact we only care about the cases $n = kj+1$). Notice that because $\log\circ\Phi_M$ maps MC_n to a rectangle

$$\{(r',\theta) : \frac{R}{2^{n+1}} < r' < \frac{R}{2^n}, p < \theta < \bar{p}\},$$

$R > 1$, we multiply 2^n so that the area (length) of MC_{kj+1} (γ_j) can be competitive with the area (length) of MC_{kj+1} (γ_j) under μ_2.

Choose $r > 0$ such that for all $c' \in M_{p/q}$, the disc $D_r(\alpha[c'])$ is contained in $U_0[c']$ and does not intersect the $q-1$ non-critical pieces of depth 1 in the critical piece $C_0[c']$. Then choose $r_1 < r$ and set

$$A = \inf_{c\in M_{p/q}\setminus V} \inf_{i=1,2} \inf_{\{z\in R_c(\theta_i)\bigcap(\mathbb{C}\setminus D_{r_1})\}} h_c(z),$$

$A > 0$. Now if γ_j is a curve in MC_{kj+1} joining l_{kj+1} and l_{kj+2} which starts from a potential less than A, $L_{\mu_1}(\gamma_j)$ can be quite small, but in this case γ_j must cross a small neighbourhood of $l_{kj+1}\bigcap M$, so we introduce a second measure μ_2 supported on this region to ensure that $L_{\mu_2}(\gamma_j)$ is bounded below in the case that γ_j stays in a small neighbourhood of $l_{kj+1}\cap M$. μ_2 is chosen to let $L_{\mu_2}(\gamma_j)$ be the length in the usual conformal metric on $\mathbb{C}$ of a path between $\{z : |z| = r_1\}$ and $\{z : |z| = r\}$ for suitable r_1 and r. We define μ_2 on each component U (in our case there is in fact only one component) of $F_i^{-1}(D_r) \cap MX_i(c')$, where

$$F_i(c) = f_c^i(c) - \alpha(c),\ f_c = z^2 + c.$$

Define μ_2 on each U to be dF_i on the image of s_U, a section of a function $\phi_i : c \to \Phi(c, F_i(c))$, which is an analytic branch of $\log(z - \alpha[c'])$.

Now $L_\mu(\gamma_j)$ is bounded below from $\inf\{A, l_{p/q}, r - r_1, \frac{\log R}{2}\}$, while $A_\mu(MC_{kj+1})$ is bounded above by a constant depending only on c' from lemma 13.7 and Lemma 13.8 [Hub], and this proves our Lemma. $\square$

Remark 5.2.7. *This is a corollary of Proposition 13.2 [Hub] by its proof. The crucial point is the selection of the metric μ_2 to control the modulus of MC_{kj+1}. One can refer to the proof of Proposition 13.2 [Hub] for the origin of the metric.*

Corollary 5.2.8. $\sum_{n=1}^{\infty} Mod(MC_n) = \infty.$

Proof. This is from Lemma 5.2.6 because

$$Mod(MC_{kj+1}) > C'$$

for some fixed $C' > 0$ and all $j \in \mathbb{Z}^+$. $\square$

By Corollary 5.2.8 and Lemma 5.2.4,

$$diam(MB_n) \to 0 \text{ as } n \to \infty,$$

and

$$\cap_{n=1}^{\infty}(MB_n) = c'.$$

Now note that

$$c_n \in MB_{N_n}$$

for some $N_n \in \mathbb{N}$ and $N_n \to \infty$ as $n \to \infty$. This completes the proof of $\lim_{n\to\infty} c_n = c'$ in the primitive case.

$\square$

Remark 5.2.9. *Now if $r \in [0,1)$ is an arbitrary real number not being an odd denominator rational and the external ray of angle r for M lands on a non-renormalizable parameter $c' \in \partial M$, for a sequence of odd denominator rationals p_n such that $\lim_{n\to\infty} p_n = r$, by Yoccoz's control of size of Limbs and the orbit portrait theory, convergence of the corresponding critically periodic polynomials $f_{c_n}(z) = z^2 + c_n$ Thurston equivalent to s_{p_n} still holds (this can also be deduced from Schleicher's theory of triviality of fibers of M at c' in [Schl1])*

$$\lim_{n\to\infty} c_n = c'.$$

However, the limit map $f_{c'}(z) = z^2 + c'$ is no longer a parabolic map. For example, if r is an even denominator rational, $f_{c'}(z)$ will be a Misiurewicz map.

6 Outline of proof of Theorem 4.2.1: on the particular sequence of odd denominator rationals

Suppose that p is of period m, p_n is of period m_n, q is of period m' under the doubling map $z \to 2z$ on $\mathbb{R}\backslash\mathbb{Z}$. We know that on the boundary of the hyperbolic component containing R on $Per_{m'}(0)$, there is a unique parabolic map R_*, which has the following properties:

- Let v be the point in the parabolic cycle of R_* which attracts the critical point c_1 under iterations of R_*^m, then $R_*^m(v) = v$. The period of v divides m.
- R_* has two critical points c_1 and c_2, of which c_1 is attracted to the parabolic cycle while c_2 is periodic of period m'.
- R_*^m has multiplier 1 at the parabolic fixed point v.

Theorem 4.2.1 is proved in three steps. First we get some particular critically finite quadratic rational maps R_n through parabolic perturbations around R_* (the techniques are from [DH2]), then we prove these maps are matings by finding invariant circles for them, at last we recover the particular convergent sequence p_n from R_n in equation (8). Remember that in the whole process we stay in the parameter slice of quadratic rational maps $Per_{m'}(0)$ containing R. One is recommended to [Tim2] for the structure of hyperbolic components on $Per_{m'}(0)$.

7 Parabolic perturbation for critically periodic maps

The technique of parabolic perturbation started from Douady and Hubbard [DH2]. Later Lavaurs [Lav] and Shishikura [Shi3] developed their theory of Écalle cylinders. The theory was then applied or generalised by Lavaurs [Lav], Shishikura [Shi2], P. Haïssinsky [Hai], Tan Lei [Tan2], Buff and Chéritat [BC], Buff, Écalle and Epstein [BEE] to get many beautiful results.

Let $\tau(w) = -1\backslash w$, f is a parabolic map of the form $f(z) = z + z^2 + ...$. A perturbed function f_α has two new fixed point near 0, by moving either one of them to 0, we can assume $f_\alpha = e^{2\pi i\alpha}z + z^2 + O(z^3)$, where α is a small number close to 0. Let

$$F_\alpha(w) = \tau^{-1} \circ f \circ \tau(w).$$

Denote the critical point attracted by the parabolic point 0 of f by c_1, the corresponding critical point of the perturbed map f_α by $c_1(\alpha)$. The following theorem tells us that one can expect to find a critically periodic map f_{α_n} with its critical point $c_1(\alpha_n)$ being periodic of period $2n + r + s$ (n, r, s are all in $\mathbb{N}^+$).

Theorem 7.0.10. *For n large enough, there exist α_n, ϵ_n depending on n, $n\alpha_n = \frac{1}{2} + \epsilon_n$, $\lim_{n\to\infty} \epsilon_n = 0$, two positive integers r, s, a point $v_{-s}(\alpha_n) \in f_{\alpha_n}^{-s}(c_1(\alpha_n))$ in the backward orbit of $c_1(\alpha_n)$ under f_{α_n} and in the domain of the local inverse of f_{α_n} defined near 0 with $Arg(v_{-s}(\alpha_n)) \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, such that*

$$f_{\alpha_n}^n(f_{\alpha_n}^r(c_1(\alpha_n))) = f_{\alpha_n}^{-n}(v_{-s}(\alpha_n)) \tag{10}$$

where $f_{\alpha_n}^{-1}$ denotes the branch such that $f_{\alpha_n}^{-1}(0) = 0$ in the formula.

The theorem is proved in terms of F_{α_n} in the following theorem. Now use $C_1(\alpha_n)$ to denote the corresponding critical point of F_{α_n}, we have

Theorem 7.0.11. *For n large enough, there exist α_n, ϵ_n depending on n, $n\alpha_n = \frac{1}{2} + \epsilon_n$, $\lim_{n\to\infty} \epsilon_n = 0$, two positive integers r, s, a point $w_{-s}(\alpha_n) \in F_{\alpha_n}^{-s}(C_1(\alpha_n))$ in the backward orbit of $C_1(\alpha_n)$ under F_{α_n} and in the domain of the local inverse of F_{α_n} defined near ∞ with $Arg(w_2(\alpha_n)) \in [\frac{3\pi}{4}, \frac{5\pi}{4}]$, such that*

$$F_{\alpha_n}^n(F_{\alpha_n}^r(C_1(\alpha_n))) = F_{\alpha_n}^{-n}(w_{-s}(\alpha_n)) \tag{11}$$

where $F_{\alpha_n}^{-1}$ denotes the branch such that $F_{\alpha_n}^{-1}(\infty) = \infty$.

The main idea of the proof is estimating the main terms of the forward and backward orbits of the critical point $C_1(\alpha)$ under perturbation, then use the argument principle to collapse them near a small neighbourhood of ∞.

7.1 Estimate the forward iterations

Lemma 7.1.1. *For $|w|$ large enough, we have*

$$F_\alpha(w) = e^{-2\pi i\alpha}w + e^{-4\pi i\alpha} + \frac{A}{w} + \frac{B}{w^2} + O(\frac{1}{w^3}).$$

Proof. By the definition, $F_\alpha(w) = \tau^{-1} \circ f_\alpha \circ \tau(w)$, so

$$\begin{aligned}
F_\alpha(w) &= \frac{1}{\frac{e^{2\pi i\alpha}}{w} - \frac{1}{w^2} + O(\frac{1}{w^3})} \\
&= e^{-2\pi i\alpha} w \left(\frac{1}{1 - \left[\frac{e^{-2\pi i\alpha}}{w} + O(\frac{1}{w^2})\right]} \right) \\
&= e^{-2\pi i\alpha} w (1 + \frac{e^{-2\pi i\alpha}}{w} + \frac{A'}{w^2} + \frac{B'}{w^3} + O(\frac{1}{w^4})) \\
&= e^{-2\pi i\alpha} w + e^{-4\pi i\alpha} + \frac{A}{w} + \frac{B}{w^2} + O(\frac{1}{w^3}).
\end{aligned}$$

□

In fact A and B both depend on α but are bounded for small α.

By introducing the transformation τ we map 0 to ∞. The forward orbit of critical point $c_1 = c_1(0)$ of $f = f_0$ tends to 0 from the direction of real axis on the left half plane, while on the w plane the modulus of points on the orbit of $C_1(\alpha)$ under F_α gets successively larger at beginning of the iterations. Long term behaviour is hard to predict, but at the beginning the map tends to have an increase of $e^{-4\pi i\alpha}$, which approximates 1 when α is close to 0, just like the Fatou coordinate [Mil1] for the parabolic map.

Because the perturbations that can create super attractive cycles are in the domain $\{\alpha : Arg(\alpha) \in (-\pi\backslash 4, \pi\backslash 4) \bigcup (3\pi\backslash 4, 5\pi\backslash 4)\}$, so we assume α in this range in the following.

By Lemma 7.1.1 we can easily get the following iterating expression:

$$\begin{aligned}
F_\alpha^n(w) = \quad & e^{-2\pi in\alpha} w + e^{-4\pi i\alpha} \frac{1 - e^{-2\pi in\alpha}}{1 - e^{-2\pi i\alpha}} + \sum_{j=1}^{n} \frac{e^{-2\pi i(j-1)\alpha} A}{F_\alpha^{n-j}(w)} \\
& + \sum_{j=1}^{n} \frac{e^{-2\pi i(j-1)\alpha} B}{(F_\alpha^{n-j}(w))^2} + \textit{higher order terms}.
\end{aligned} \tag{12}$$

Now let

$$P = e^{-2\pi i n\alpha}w,\ Q = e^{-4\pi i\alpha}\frac{1-e^{-2\pi i n\alpha}}{1-e^{-2\pi i\alpha}},$$

$$S = \sum_{j=1}^{n}\frac{e^{-2\pi i(j-1)\alpha}A}{F_\alpha^{n-j}(w)} + \sum_{j=1}^{n}\frac{e^{-2\pi i(j-1)\alpha}B}{(F_\alpha^{n-j}(w))^2} + \textit{higher order terms}.$$

We will show that $P+Q$ (more precisely, Q) is the dominant term and S is of less order. We resort to induction to bound S.

Lemma 7.1.2. *Denote $w = Re(w) + iIm(w)$. There exist three small numbers δ_1, δ_2 and δ_3 such that for any $\alpha, w \in \mathbb{C}, k \in \mathbb{N}$ (k is not fixed) satisfying $|\alpha| < \delta_1$, $k^2|\alpha| < \delta_2$, $|w| > \frac{1}{\delta_3}$,w has positive real part and $Re(w) \gg |Im(w)|$, the following holds:*

$$|F_\alpha^l(w)| \geq |w| + \frac{l}{2} \tag{13}$$

for all $l \leq k$.

Proof. For $l = 0$, the inequality obviously holds. Now suppose the inequality (13) holds for $l = j' \leq k-1$, we want to prove it holds for $j'+1 \leq k$.

First, when $n = j'+1$,

$$\begin{aligned} P+Q &= (\cos(2\pi(j'+1)\alpha) - i\sin(2\pi(j'+1)\alpha))\,(Re(w)+iIm(w)) \\ &\quad + (1+O(\alpha))\,\frac{1-\cos(2\pi(j'+1)\alpha)+i\sin(2\pi(j'+1)\alpha)}{2\pi i\alpha + O(\alpha^2)} \\ &= \cos(2\pi(j'+1)\alpha)Re(w) + \sin(2\pi(j'+1)\alpha)Im(w) \\ &\quad + \frac{\sin(2\pi(j'+1)\alpha)}{2\pi\alpha} + i\cos(2\pi(j'+1)\alpha)Im(w) \\ &\quad + i\left(-\sin(2\pi(j'+1)\alpha)Re(w) + \frac{\cos(2\pi(j'+1)\alpha)-1}{2\pi\alpha}\right) \\ &\quad + O(\alpha^2). \end{aligned}$$

Choose δ_1 and δ_2 small enough, so $k|\alpha| < k^2|\alpha| < \delta_2$ will be small enough, such that $\sin(2\pi(j'+1)\alpha)$ will be close enough to 0 and $\cos(2\pi(j'+$

$1)\alpha)$ will be close enough to 1. Thus the main terms in the above expression will be

$$\cos(2\pi(j'+1)\alpha)Re(w)+\frac{\sin(2\pi(j'+1)\alpha)}{2\pi\alpha}+i\cos(2\pi(j'+1)\alpha)Im(w) \quad (14)$$

Note that $\dfrac{\cos(2\pi(j'+1)\alpha)-1}{2\pi\alpha}$ can be controlled because its first term will be less than $O(k^2|\alpha|)$. Now we deal with the term

$$S=\sum_{j=1}^{j'+1}\frac{e^{-2\pi i(j-1)\alpha}A}{F_\alpha^{j'+1-j}(w)}+\sum_{j=1}^{j'+1}\frac{e^{-2\pi i(j-1)\alpha}B}{(F_\alpha^{j'+1-j}(w))^2}+higher\ order\ terms.$$

Suppose now $|e^{-2\pi i(j-1)\alpha}|<M$ for all $j-1\leq k$. By assumption (13) holds for all integers $\leq j'$. Choose δ_3 small enough such that $MA\delta_3<\frac{1}{100}$. So the dominant term of S will be controlled by $\frac{j'+1}{100}$, whose influence can be easily offset by the term $\dfrac{\sin(2\pi(j'+1)\alpha)}{2\pi\alpha}$ in $P+Q$ which approximates $j'+1$ and is of the same sign (positive) with $\cos(2\pi(j'+1)\alpha)Re(w)$ if we choose δ_1 small enough. In other words, it can contribute to $|F_\alpha^{j'+1}(w)|$ of enough amount to make it increase by $\frac{3}{4}(j'+1)$.

By all the analysis just now for $P+Q$ and S we can guarantee that

$$|F_\alpha^{j'+1}(w)|>|w|+\tfrac{j'+1}{2}$$

for suitable choices of δ_1,δ_2 *and* δ_3. By mathematical induction this proves the lemma. □

From now on we will assume $n\alpha_n=\frac{1}{2}+\epsilon_n$, in which ϵ_n is a small complex number depending on n and $\lim\limits_{n\to\infty}\epsilon_n=0$. By this expression we can see that when n is a large integer α_n is quite close to $\frac{1}{2n}$, a real number. So we sometimes will treat it as a small real number, actually all the things hold for small complex number ϵ_n with α_n substituted by $|\alpha_n|$. Now we show that

Theorem 7.1.3. *Let $|w| > \frac{1}{\delta_3}$, $Re(w) \gg Im(w)$. For n large enough, ϵ_n a small complex number depending on n and $\lim_{n\to\infty} \epsilon_n = 0$, $n\alpha_n = \frac{1}{2} + \epsilon_n$, then for all $k|\alpha_n| < \frac{3}{4}$, we have*

$$|F^k_{\alpha_n}(w)| > \frac{1}{2\pi}k. \tag{15}$$

This gives us quite efficient technique to bound the R term before. In order to prove this we still need several lemmas.

Lemma 7.1.4. *For term $Q = e^{-4\pi i\alpha_n}\dfrac{1-e^{-2\pi ik\alpha_n}}{1-e^{-2\pi i\alpha_n}}$, choose δ_1 small enough, we have*

$$|Q| > \frac{\sqrt{2}}{2\pi}k \tag{16}$$

for all $k|\alpha_n| < \frac{3}{4}$ and $|\alpha_n| < \delta_1$.

Proof. In the proof we treat α_n as a small real number, all the things are true for small complex numbers close to $\frac{1}{2n}$. First note that

$$\begin{aligned}|Q| &= (1+O(\alpha_n))\frac{1-e^{-2\pi ik\alpha_n}}{1-e^{-2\pi i\alpha_n}}\\ &= (1+O(\alpha_n))\frac{1-\cos(2\pi k\alpha_n)+i\sin(2\pi k\alpha_n)}{2\pi i\alpha_n+O(\alpha_n^2)}\\ &= (1+O(\alpha_n))(1+O(\alpha_n^2))\frac{1-\cos(2\pi k\alpha_n)+i\sin(2\pi k\alpha_n)}{2\pi i\alpha_n}.\end{aligned}$$

Let $S = \dfrac{1-\cos(2\pi k\alpha_n)+i\sin(2\pi k\alpha_n)}{2\pi i\alpha_n}$. Now consider the situation in two cases. First assume $k\alpha_n < \dfrac{1}{4}$, then

$$\frac{\sin(2\pi k\alpha_n)}{2\pi\alpha_n} > \frac{2}{\pi}\frac{2\pi k\alpha_n}{2\pi\alpha_n} = \frac{2}{\pi}k, \tag{17}$$

so obviously (16) holds in this case. When $\frac{1}{4} \le k\alpha_n < \frac{3}{4}$,

$$\begin{aligned}|S| = \frac{\sqrt{(1-\cos(2\pi k\alpha_n))^2+(\sin(2\pi k\alpha_n))^2}}{2\pi\alpha_n} &= \frac{\sqrt{2-2\cos(2\pi k\alpha_n)}}{2\pi\alpha_n}\\ > \frac{\sqrt{2}}{2\pi\frac{2}{3n}} = \frac{3\sqrt{2}}{4\pi}n &> \frac{\sqrt{2}}{2\pi}k\end{aligned}$$

The last inequality holds because $k < \dfrac{\frac{3}{4}}{\alpha_n} < \dfrac{\frac{3}{4}}{\frac{1}{2n}} = \frac{3}{2}n$, so $n > \frac{2}{3}k$.

$\square$

Now we can use these lemmas to prove Theorem 7.1.3.

Proof of Theorem 7.1.3

Proof. Remember $F^k_{\alpha_n}(w) = P + Q + S$. First note that (11) holds for all $k^2\alpha_n < \delta_2$ by Lemma 7.1.2 if we choose n large enough such that $|\alpha_n| < \delta_1$. We will use induction to prove the case

$$J = \{k : \; k^2\alpha_n \geq \delta_2 \; and \; k\alpha_n < \tfrac{3}{4}\}.$$

Note that all $k \in J$ satisfy $k \geq \sqrt{\frac{\delta_2}{\alpha}} > \sqrt{n\delta_2}$. Suppose now (15) holds for some $k \in J$ and numbers less than it, we will deduce (15) holds for $k+1 \in J$.

For P, Choose n large enough such that $|P| < M|w| < \frac{\sqrt{2}-1}{4\pi}\sqrt{n\delta_2} \leq \frac{\sqrt{2}-1}{4\pi}(k+1)$. As for S, choose n large enough (so k large enough because $k > \sqrt{n\delta_2}$) such that

$$\begin{aligned} |S| &= |\sum_{j=1}^{k+1} \frac{e^{-2\pi i(j-1)\alpha_n}A}{F^{k+1-j}_{\alpha_n}(w)} + \sum_{j=1}^{k+1} \frac{e^{-2\pi i(j-1)\alpha_n}B}{(F^{k+1-j}_{\alpha_n}(w))^2} + higher\ order\ terms \\ &< MA\sum_{j=1}^{k+1} \frac{1}{\frac{1}{2\pi}j} < 2\pi MA \sum_{j=1}^{k+1}\frac{1}{j} < 2\pi MA(\ln k + a) < \frac{\sqrt{2}-1}{4\pi}(k+1). \end{aligned}$$

In the former inequality we use the approximation $\displaystyle\sum_{j=1}^{k+1}\frac{1}{j} < \ln k + a$ for some fixed number a. In fact it can be shown that the approximation is $\displaystyle\lim_{k\to\infty}\sum_{j=1}^{k}\frac{1}{j} = \ln k + b$ where b is called the Euler constant.

Recall from Lemma 7.1.4 that for $k+1 \in J$, $|Q| > \frac{\sqrt{2}}{2\pi}(k+1)$, Now

$$|F^{k+1}_{\alpha_n}(w)| = |P + Q + S|$$

$$> |Q| - |P| - |S| > \frac{1}{2\pi}(k+1) + \frac{\sqrt{2}-1}{4\pi}(k+1) - |P| + \frac{\sqrt{2}-1}{4\pi}(k+1) - |S|$$
$$> \tfrac{1}{2\pi}(k+1).$$

So (15) is true for $k+1$. By induction we prove the theorem. □

7.2 Estimate the backward iterations

We can deduce completely similar results for $f_\alpha^{-1}(z) = e^{-2\pi i\alpha}z - e^{-6\pi i\alpha}z^2 + O(z^3)$ and $F_\alpha^{-1}(w) = \tau^{-1} \circ f_\alpha^{-1} \circ \tau(w)$. We will not prove them again but will list them in the following.

Lemma 7.2.1. *For $|w|$ large enough, we have*

$$F_\alpha^{-1}(w) = e^{2\pi i\alpha}w - e^{-2\pi i\alpha} + \tfrac{A}{w} + \tfrac{B}{w^2} + O(\tfrac{1}{w^3}),$$

$$\begin{aligned} F_\alpha^{-n}(w) = \quad & e^{2\pi in\alpha}w - e^{-2\pi i\alpha}\frac{1-e^{2\pi in\alpha}}{1-e^{2\pi i\alpha}} + \sum_{j=1}^{n}\frac{e^{2\pi i(j-1)\alpha}A}{F_\alpha^{-(n-j)}(w)} \\ & + \sum_{j=1}^{n}\frac{e^{2\pi i(j-1)\alpha}B}{(F_\alpha^{-(n-j)}(w))^2} + \textit{higher order terms.} \end{aligned} \tag{18}$$

Here $F_\alpha^{-1}(w)$ is the certain branch with $F_\alpha^{-1}(\infty) = \infty$, $F_\alpha^{-n}(w)$ is the n times iteration of $F_\alpha^{-1}(w)$. Without special declaration the following F_α^{-1} or F_α^{-n} mean the same. Now denote $F_\alpha^{-n}(w) = P' + Q' + S'$, in which

$$P' = e^{2\pi in\alpha}w,\ Q' = -e^{-2\pi i\alpha}\frac{1-e^{2\pi in\alpha}}{1-e^{2\pi i\alpha}},$$

$$S' = \sum_{j=1}^{n}\frac{e^{2\pi i(j-1)\alpha}A}{F_\alpha^{-(n-j)}(w)} + \sum_{j=1}^{n}\frac{e^{2\pi i(j-1)\alpha}B}{(F_\alpha^{-(n-j)}(w))^2} + \textit{higher order terms}$$

Lemma 7.2.2. *There exist three small numbers δ_1, δ_2 and δ_3 such that for any $\alpha, w \in \mathbb{C}, k \in \mathbb{N}$ (k is not fixed) satisfying $|\alpha| < \delta_1$,$k^2|\alpha| <$*

δ_2,$|w| > \frac{1}{\delta_3}$,*w has negative real part and* $|Re(w)| \gg |Im(w)|$, *the following holds:*

$$|F_\alpha^{-l}(w)| \geq |w| + \frac{l}{2} \tag{19}$$

for all $l \leq k$.

Lemma 7.2.3. *For term* $Q' = -e^{-2\pi i\alpha_n}\dfrac{1-e^{2\pi in\alpha_n}}{1-e^{2\pi i\alpha_n}}$, *choose* δ_1 *small enough, we have*

$$|Q'| > \tfrac{\sqrt{2}}{2\pi}k$$

for all $k|\alpha_n| < \frac{3}{4}$ *and* $|\alpha_n| < \delta_1$.

Theorem 7.2.4. *For* n *large enough,* ϵ_n *a small complex number depending on* n *and* $\lim_{n\to\infty} \epsilon_n = 0$, $n\alpha_n = \frac{1}{2} + \epsilon_n$, *then for all* $k|\alpha_n| < \frac{3}{4}$,

$$|F_{\alpha_n}^{-k}(w)| > \frac{1}{2\pi}k. \tag{20}$$

We use the same collection of symbols $A, B, \delta_1, \delta_2, \delta_3, k, n$ for convenience.

Note that the critical point $c_1(0)$ is absorbed by 0, the parabolic point of f_0. We denote the perturbed critical point corresponding to $c_1(0)$ by $c_1(\alpha)$, denote $C_1(\alpha) = \tau(c_1(\alpha))$. Because $c_1(\alpha)$ and $f_\alpha(z)$ depend continuously on α, we can suppose that after r-th iterations $|f_\alpha^r(c_1(\alpha))| < \delta_3$, in other words, $|F_\alpha^r(C_1(\alpha))| > \frac{1}{\delta_3}$. We can also require that

$$Re(f_\alpha^r(c_1(\alpha))) < 0,\ |Re(f_\alpha^r(c_1(\alpha)))| \gg |Im(f_\alpha^r(c_1(\alpha)))|,$$

i.e.

$$Re(F_\alpha^r(C_1(\alpha))) > 0,\ |Re(F_\alpha^r(C_1(\alpha)))| \gg |Im(F_\alpha^r(C_1(\alpha)))|.$$

This is because $f_0^n(c_1(0))$ converges to 0 from the direction of the real axis and $c_1(\alpha), f_\alpha^r(c_1(\alpha))$ depend continuously on α. Then we can use

Theorem 7.1.3 to do analysis of the following iterations on $F^r_\alpha(C_1(\alpha))$. Denote $w_r(\alpha) = F^r_\alpha(C_1(\alpha))$.

The following lemma provides us with preparation to deal with the backward orbit of $F^{-s}_\alpha(C_1(\alpha))$ for some suitable integer s.

Lemma 7.2.5. *Given $\delta_3 > 0, \Delta > 0$, there exists an integer s, such that if $|\alpha_n| < \delta_1$ for some sufficiently small number $\delta_1 > 0$, we have*

$$|f^{-s}_{\alpha_n}(c_1(\alpha_n))| < \delta_3,\ Re(f^{-s}_{\alpha_n}(c_1(\alpha_n))) > 0,$$
$$|Re(f^{-s}_{\alpha_n}(c_1(\alpha_n)))| > \Delta|Im(f^{-s}_{\alpha_n}(c_1(\alpha_n)))|,$$

for a suitable point $f^{-s}_{\alpha_n}(c_1(\alpha_n))$ in the backward orbit of $c_1(\alpha_n)$, which means that on w plane we have

$$|F^{-s}_{\alpha_n}(C_1(\alpha_n))| > \tfrac{1}{\delta_3},\ Re(F^{-s}_{\alpha_n}(C_1(\alpha_n))) < 0,$$
$$|Re(F^{-s}_{\alpha_n}(C_1(\alpha_n)))| > \Delta|Im(F^{-s}_{\alpha_n}(C_1(\alpha_n)))|.$$

The inverse branches here are different from notations before.

Proof. We only prove this on z plane, and first we prove it for $f_0(z) = f(z)$, then pass it to f_{α_n} by continuity.

By standard parabolic attracting and repelling petals theory [Mil1] for $f(z)$, we can find a repelling petal V (open) which contains part of the boundary (in Julia set of f) of the Fatou component in which all the points are attracted to point 0. Denote one of the points in this part by a_1. Note that 0 is in the Julia set $J(f)$ of $f(z)$, so the backward orbit of 0 is dense in $J(f)$. Then there exists s_1 and a point $b_1 \in \{f^{-s_1}(0)\} \subset V$ which is close to a_1. Moreover, there exists s_2 such that $f^{s_2}(c_1)$ is close to 0, so $f^{s_2}(b_1)$ will also be close to a_1, which means that $f^{-s_1+s_2}(c_1) \in V$. Then $\lim\limits_{n\to\infty} f^{-s_1+s_2-n}(c_1) = 0$ and the convergence is from the direction of the real axis because $f^{s_2}(b_1)$ is in the repelling petal. Now choose n_0 large enough such that

$$|f^{s_2-n_0}(b_1)| < \delta_3,\ Re(f^{s_2-n_0}(b_1)) > 0,$$
$$|Re(f^{s_2-n_0}(b_1))| > \Delta|Im(f^{s_2-n_0}(b_1))|.$$

Now write

$$f^{-s}(c_1) = f^{s_2-n_0}(b_1)$$

in which $s = n_0 + s_1 - s_2$ for the suitable point $f^{-s}(c_1)$.

By now we have proved the case for $f(z)$. Now because $c_1(\alpha)$ and $f_\alpha(z)$ depend continuously on α, which means that if we confine α_n in a small neighbourhood $|\alpha_n| < \delta_1$, we will also get

$$|f_{\alpha_n}^{-s}(c_1(\alpha_n))| < \delta_3,\ Re(f_{\alpha_n}^{-s}(c_1(\alpha_n))) > 0,$$
$$|Re(f_{\alpha_n}^{-s}(c_1(\alpha_n)))| > \Delta|Im(f_{\alpha_n}^{-s}(c_1(\alpha_n)))|.$$

for the suitable point $f_{\alpha_n}^{-s}(c_1(\alpha_n))$ in the backward orbit of $c_1(\alpha_n)$. □

7.3 Colliding the forward and backward iterations around the parabolic cycle: proof of Theorem 7.0.11

Now denote by $w_{-s}(\alpha_n) = F_{\alpha_n}^{-s}(C_1(\alpha_n))$ in Lemma 7.2.5 such that

$$|w_{-s}(\alpha_n)| > \tfrac{1}{\delta_3},\ Re(F_{\alpha_n}^{-s}(C_1(\alpha_n))) < 0,$$
$$|Re(F_{\alpha_n}^{-s}(C_1(\alpha_n)))| > \Delta|Im(F_{\alpha_n}^{-s}(C_1(\alpha_n)))|$$

for a large number Δ. Recall $w_r(\alpha_n) = F_{\alpha_n}^{r}(C_1(\alpha_n))$. Now we are able to prove Theorem 7.0.11.

Proof of Theorem 7.0.11:

Proof. Since $\lim_{n\to\infty} \epsilon_n = 0$, choose n large enough such that $|n\alpha_n| = |\frac{1}{2} + \epsilon_n| < \frac{3}{4}$. Recall that

$$\begin{aligned} F^n_{\alpha_n}(w) &= e^{-2\pi i n\alpha_n} w + e^{-4\pi i\alpha_n}\frac{1-e^{-2\pi i n\alpha_n}}{1-e^{-2\pi i\alpha_n}} + \sum_{j=1}^{n}\frac{e^{-2\pi i(j-1)\alpha_n}A}{F^{n-j}_{\alpha_n}(w)} \\ &\quad + \sum_{j=1}^{n}\frac{e^{-2\pi i(j-1)\alpha_n}B}{(F^{n-j}_{\alpha_n}(w))^2} + \textit{higher order terms} \\ &= P+Q+S. \end{aligned}$$

For r large enough, $w_r(\alpha_n)$ has entered our required domain. Now apply Lemma 7.1.4 and Theorem 7.1.3 on $F^n_{\alpha_n}(w_r(\alpha_n))$, we have

$$|P| \le M|w_r(\alpha_n)|,\ |Q| \ge \tfrac{1}{2\pi}n,\ |S| \le 2\pi MA\sum_{k=1}^{n}\frac{1}{k} \le \ln n + a.$$

So when n is large enough Q will dominate the other two terms. Now we give a finer estimate on Q for later use.

$$\begin{aligned} Q = \ & (1+O(\alpha_n))\frac{1-e^{-2\pi i n\alpha_n}}{2\pi i\alpha_n + O(\alpha_n^2)} \\ & = (1+O(\alpha_n))(1+O(\alpha_n))\frac{1-e^{-2\pi i n(\frac{1}{2}+\epsilon_n)}}{2\pi i\alpha_n} \\ & \sim \frac{1+e^{-2\pi i\epsilon_n}}{2\pi i\alpha_n} \\ & = \frac{2-2\pi i\epsilon_n + O(\epsilon_n^2)}{2\pi i\alpha_n} \end{aligned} \tag{21}$$

Similar results hold for $F^{-n}_{\alpha_n}(w_{-s}(\alpha_n))$, Q' will also dominate P' and S',

$$Q' \sim \frac{1+e^{2\pi i\epsilon_n}}{2\pi i\alpha_n} = \frac{2+2\pi i\epsilon_n + O(\epsilon_n^2)}{2\pi i\alpha_n} \tag{22}$$

Now consider the equation

$$F^n_{\alpha_n}(w_r(\alpha_n)) = F^{-n}_{\alpha_n}(w_{-s}(\alpha_n)),$$

i.e.

$$P+Q+S = P'+Q'+S'$$

with respect to ϵ_n. Because $Q-Q'$ dominates other terms, according to argument principle, in order to get a solution ϵ_n in a small neighbourhood of 0 we only need to consider the equation $Q=Q'$. By (21) in Q the term

$\dfrac{2-2\pi i\epsilon_n}{2\pi i\alpha_n}$ dominates $\dfrac{O(\epsilon_n^2)}{2\pi i\alpha_n}$, while by (22) in Q' $\dfrac{2+2\pi i\epsilon_n}{2\pi i\alpha_n}$ dominates $\dfrac{O(\epsilon_n^2)}{2\pi i\alpha_n}$. Again by argument principle we we only need to consider the following equation:

$$\frac{2-2\pi i\epsilon_n}{2\pi i\alpha_n} = \frac{2+2\pi i\epsilon_n}{2\pi i\alpha_n}$$

Obviously this equation has a unique solution 0. This guarantees a solution ϵ_n for $F^n_{\alpha_n}(w_r(\alpha_n)) = F^{-n}_{\alpha_n}(w_{-s}(\alpha_n))$ in a small neighbourhood of 0.

□

Remark 7.3.1. *We can analyse the order of ϵ_n by the following equation:*

$$\frac{1+e^{-2\pi i\epsilon_n}}{2\pi i\alpha_n} - \frac{1+e^{2\pi i\epsilon_n}}{2\pi i\alpha_n} + O(\ln n) + c_1 = 0 \tag{23}$$

Note that

$$\begin{aligned}(23) &\iff e^{-2\pi i\epsilon_n} - e^{2\pi i\epsilon_n} + 2\pi i\alpha_n(O(\ln n) + c_1) = 0\\ &\iff e^{4\pi i\epsilon_n} - \frac{2\pi i}{2n}(O(\ln n) + c_1)e^{2\pi i\epsilon_n} - 1 = 0\end{aligned}$$

Solve the last equation we get

$$e^{2\pi i\epsilon_n} \sim \frac{c_2\frac{\ln n}{n} + \sqrt{(c_2\frac{\ln n}{n})^2 + 4}}{2} \sim \frac{c_2\frac{\ln n}{n} + 2(1+\frac{1}{2}(\frac{c_2}{2}\frac{\ln n}{n})^2)}{2} \sim 1 + \frac{c_2}{2}\frac{\ln n}{n} + O((\frac{\ln n}{n})^2).$$

So

$$|\epsilon_n| \sim \frac{1}{2\pi i}\ln(1 + \frac{c_2}{2}\frac{\ln n}{n} + O((\frac{\ln n}{n})^2)) \sim \frac{1}{2\pi i}\frac{c_2}{2}\frac{\ln n}{n} \sim O(\frac{\ln n}{n}).$$

Considering that $|S| \leq O(\ln n)$, this computation shows that ϵ_n is at most of order $\dfrac{\ln n}{n}$, which coincides with $\lim\limits_{n\to\infty}\epsilon_n = 0$.

Now apply Theorem 7.0.10 to R^m_*. By a Möbius change of coordinate we can assume $v = 0$ and $R^m_*(z) = z + z^2 + O(z^3)$ around 0. c_1 is still the critical point of R_* which is attracted to 0 under R_*^m. Then by 7.0.10, for n large enough, there are two integers r, s, a point $|v_{-sm}(\alpha_n)| = |v_{-sm}(R_n)| = |R_n^{-sm}(c_1(\alpha_n))| < \delta_3$ (the subscript of v_{-s} is changed to $-sm$ now in order to coordinate with the times of iterations of R_n), such that the critical point $c_1(\alpha_n) = c_1(R_n)$ of R_n (a perturbation of R_*) is periodic of period $(r+s+2n)m$, that is

$$R_n^{nm}(R_n^{rm}(c_1(\alpha_n))) = R_n^{-nm}(v_{-sm}(\alpha_n))$$

where R_n^{-m} is the branch $R_n^{-m}(0) = 0$.

8 The critically periodic maps R_n are matings

In this section we show that R_n is Thurston equivalent to the mating of two critically periodic degree two covering maps by Lemma 2.9.1. We achieve this by finding an *invariant circle* for R_n. First recall the definition of an invariant circle.

Definition 8.0.2. *An invariant circle γ for a critically finite degree two branched covering map, say g, is a simple closed loop which satisfies:*

- *γ separates the two critical orbits of g.*
- *$g^{-1}(\gamma)$ is connected.*
- *$g^{-1}(\gamma)$ is isotopic to γ in $\bar{\mathbb{C}} \backslash X(g)$, in which $X(g)$ means post-critical set of g plus the two critical points.*
- *$g : g^{-1}(\gamma) \to \gamma$ preserves orientation.*

8.1 Links between various dynamical planes

Let $v_{-sm+1}(\alpha_n) = R_n(v_{-sm}(\alpha_n))$. Define $v_{-sm}(R_*) = R_*^{-sm}(c_1)$ according to continuity with respect to $v_{-sm}(\alpha_n) = R_n^{-sm}(c_1(\alpha_n))$. Let $v_{-sm+1}(R_*) = R_*(v_{-sm}(R_*))$. Now denote

$$M_1^{(j)} = \{R_*^k(v_{-sm+1}(R_*)) : -j \leq k < \infty\}$$

where R_*^{-k} denotes the inverse branch of R_*^k which preserves the parabolic cycle. If we can find an "invariant circle" for R_*, which separates $M_1^{(j)}, j \in \mathbb{N}$ large enough and $M_2 = \{R_*^k(c_2(R_*)) : k \in \mathbb{N}\}$ ($\#M_2$ is finite), we can get an invariant circle for R_n by continuity. Let $M_3 = \{R_*^i(0) : 0 \leq i \leq m-1\}$ be the parabolic cycle.

By Proposition 2.11.1, there exists a homeomorphism $\varphi : J(R) \to J(R_*)$ such that

$$\varphi \circ R = R_* \circ \varphi \tag{24}$$

φ can be extended to $\mathbb{C}$ and map Fatou components of R to corresponding Fatou components of R_*. In addition, we can guarantee $\varphi \circ R = R_* \circ \varphi$ on the complement of the Fatou components containing the forward orbit of $c_1(R)$. As for the images of the Fatou components containing the forward orbit of $c_1(R)$, we can choose $\varphi(R^i(c_1(R))) = R_*^i(\varphi(c_1))$ for $0 \leq i \leq m-1$. Write $s_{p,q} = s_p \amalg s_q$ from now on.

As $s_{p,q}$ is Thurston equivalent to R, according to [Ree2], section 1.5, there is a sequence of homeomorphisms $\theta^{(n)} : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, such that

$$\lim_{n\to\infty} \theta^{(n)} = \theta \tag{25}$$

uniformly. The limit continuous map $\theta : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ satisfies

$$\theta \circ s_{p,q} = R \circ \theta. \tag{26}$$

Moreover, $\theta(S^1) = J(R)$, in fact, $\theta(S^1 \bigcup L_p \bigcup L_q) = J(R)$.

We would like to say more about this sequence $\theta^{(n)}$. $\theta^{(0)}$ and $\theta^{(1)}$ are the homeomorphisms χ_1 and χ_2 in Definition 2.3.1. $\theta^{(0)}$ is an orientation-preserving homeomorphism which sends neighbourhoods of the forward orbits in $X(s_{p,q})$, say, $U(s_{p,q})$, to neighbourhoods of $X(R)$, say, $U(R)$. Moreover, we have

$$R^n \circ \theta^{(n)} = \theta^{(0)} \circ s_{p,q}^n, \tag{27}$$

$$\theta^{(0)} = \theta^{(n)} \text{ on } U(s_{p,q}),$$

$$R \circ \theta^{(n+1)} = \theta^{(n)} \circ s_{p,q}, \tag{28}$$

$$\theta^{(n)} = \theta^{(n+1)} \text{ on } s_{p,q}^{-n} U(s_{p,q}),$$

$$\theta^{(n)} \sim \theta^{(n+1)} \text{ rel } s_{p,q}^{-n} U(s_{p,q})$$

for all n. Here we also make restriction on $\theta^{(0)}$ such that $\theta^{(0)}(S^1)$ does not intersect the closure of the Fatou components containing the forward orbits of the critical point $c_1(R)$.

Now let $\psi^{(n)} = \varphi \circ \theta^{(n)}, \psi = \varphi \circ \theta$. Then

$$\lim_{n\to\infty} \psi^{(n)} = \psi \tag{29}$$

uniformly.

8.2 An invariant circle for R_n

We know that S^1 is an invariant circle for $s_{p,q}$, and we have the link ψ between the dynamical planes of $s_{p,q}$ and R_*, so we hope $\psi(S^1)$ will preserve some properties of the invariant circle for $s_{p,q}$. However, it is not a simple closed loop as ψ is not injective, so we turn to $\psi^{(n)}(S^1)$. Note that there may be problems around the accumulation points of the forward and backward iterations of c_1, that is, the parabolic cycle M_3. We will show in the following that given $j' \in \mathbb{N}$, for n large enough, $\psi^{(n)}(S^1)$ not only separates $M_1^{(j')} = \{R_*^k(v_{-sm+1}(R_*)) : -j' \leq k < \infty\}$ and M_2, but also can be isotoped away from a small neighbourhood of the parabolic cycle M_3.

Theorem 8.2.1. *Given a small real number $\epsilon > 0$, for n large enough, $\psi^{(n)}(S^1)$ is isotopic to a simple closed curve γ in $\bar{\mathbb{C}} \setminus (M_2 \cup M_1^{(n-sm)})$. γ is disjoint from the ϵ-neighbourhood of M_3 and separates M_2 from $M_1^{(n-sm)} = \{R_*^j(v_{-sm+1}(R_*)) : -(n-sm) \leq j < \infty\}$. Moreover, $R_*^{-1}(\gamma)$ is connected and isotopic to γ in $\bar{\mathbb{C}} \setminus (M_2 \cup M_1^{(n-sm)})$.*

Proof. Define $v_{-sm+1}(s_{p,q}) = (\psi^{(n)})^{-1}(v_{-sm+1}(R_*))$ for any $n \geq sm - 1$. Define $v_{-sm+1}(R) = \theta(v_{-sm+1}(s_{p,q}))$. Note that $\theta(v_{-sm+1}(s_{p,q})) = \theta^{(l)}(v_{-sm+1}(s_{p,q})) = \theta^{(sm-1)}(v_{-sm+1}(s_{p,q}))$ for all $l \geq sm - 1$ because $\theta^{(l)} = \theta^{(l+1)}$ on $(s_p \amalg s_q)^{-l}U(s_{p,q})$ and $v_{-sm+1}(s_{p,q}) \in s_{p,q}^{-l}(U(s_{p,q}))$ for $l \geq sm - 1$. Taking limits we have $v_{-sm+1}(R_*) = \psi(v_{-sm+1}(s_{p,q}))$. We also have

$$\psi(v_{-sm+1}(s_{p,q})) = \psi^{(n)}(v_{-sm+1}(s_{p,q})) = \psi^{(sm-1)}(v_{-sm+1}(s_{p,q}))$$

for all $n \geq sm - 1$. The iterations of $v_{-sm+1}(s_{p,q})$ under $s_{p,q}^{-m}$ converge to a point u_p on the leaf μ_p in the lamination L_p from some fixed orientation. $s_{p,q}^{-m}$ is the particular backward branch which preserves the leaf μ_p. Without specific declaration we always mean this branch by using the symbol $s_{p,q}^{-m}$. Similarly, on the dynamical plane of R_*, $R_*^{sm-1}(c_1(R_*)) = v_{-sm+1}(R_*)$. The backward iterations of $v_{-sm+1}(R_*)$ under R_*^{-m} (where this denotes the local inverse of R_*^m fixing its parabolic point) converge to $R_*(0)$ from some fixed orientation, which denotes the local inverse of R_*^m fixing the parabolic point. In the following we will not distinguish literally but simply write the symbols $v_{-sm}, v_{-sm+1}, c_1, c_2$ if there is no confusion about which dynamical planes these points are on.

Now consider two sequences of $k+1$ points. The first is on the dynamical plane of $s_{p,q}$,

$$s_{p,q}^{-km}(v_{-sm+1}), \cdots, s_{p,q}^{-2m}(v_{-sm+1}), s_{p,q}^{-m}(v_{-sm+1}), v_{-sm+1}.$$

Connect $s_{p,q}^{-jm}(v_{-sm+1})$ with $s_{p,q}^{-(j-1)m}(v_{-sm+1})$ by a straight line ξ_j inside the open unit disc D for $1 \le j \le k$. ξ_j will approximate some fixed direction as j becomes large because $s_{p,q}^{-jm}(v_{-sm+1})$ converge to u_p from this fixed direction. Connect u_p with $s_{p,q}^{-km}(v_{-sm+1})$ by a straight line ξ' inside D. Now consider another sequence of $k+1$ points on dynamical plane of R_*,

$$R_*^{-km}(v_{-sm+1}), \cdots, R_*^{-2m}(v_{-sm+1}), R_*^{-m}(v_{-sm+1}), v_{-sm+1}.$$

Connect $R_*^{-jm}(v_{-sm+1})$ with $R_*^{-(j-1)m}(v_{-sm+1})$ by a straight line segment, say, ζ_j for $1 \le j \le k$. These line segments will also be close to some fixed direction for large j because $R_*^{-jm}(v_{-sm+1})$ converge to $R_*(0)$ from this direction. Connect $R_*(0)$ with $R_*^{-km}(v_{-sm+1})$ by a straight line segment ζ'. R_*^{-jm} are the particular jm backward iterations preserving the parabolic cycle M_3.

Note that if v_{-sm+1} is sufficiently close to u_p, that is, if s is sufficiently large, $\psi(v_{-sm+1}(s_{p,q})) = v_{-sm+1}(R_*)$ is within some small neighbourhood of $R_*(0) = \psi(\mu_p)$, that is, within the domain of the branch R_*^{-1} preserving M_3. We can guarantee that $\zeta_j \sim R_*^m(\zeta_{j+1})$ for all $1 \le j \le k$. Since $\psi(\mu_p) = R_*(0)$ and ψ is continuous, it follows that

$$\psi(s_{p,q}^{-jm}(v_{-sm+1}(s_{p,q}))) = R_*^{-jm}(v_{-sm+1}(R_*)),$$

which implies

$$\psi^{(n)}(s_{p,q}^{-jm}(v_{-sm+1}(s_{p,q})) = R_*^{-jm}(v_{-sm+1}(R_*))$$

for any $n \ge jm + sm - 1$. To get the last equality, one can show by induction that

$$\theta^{(n)}(s_{p,q}^{-jm}(v_{-sm+1}(s_{p,q}))) = R^{-jm}(v_{-sm+1}(R))$$

for any $n \geq jm + sm - 1$, then both sides compose with φ.

Denote by $\psi^{(n)}(\xi_j) = \sigma_j, 1 \leq j \leq k, \psi^{(n)}(\xi') = \sigma'$. The former equality shows that σ_j (ξ') has same endpoints with ζ_j (σ') on dynamical plane of R_*. We claim that they are in fact isotopic to each other in $\bar{\mathbb{C}} \setminus R_*^{-n}(X(f_*))$ for any $n \geq jm + sm - 1$. This is proved in Proposition 8.2.2 below. In fact we prove more than this.

Now we continue to prove our theorem. As S^1 does not intersect any ξ_j, $\psi^{(n)}(S^1)$ does not intersect σ_j, by the isotopy this means $\psi^{(n)}(S^1)$ does not intersect the sequence ζ_j. As k can be any large enough integer, this shows that all the points $R_*^{-jm}(v_{-sm+1}), j \in \mathbb{N}$ are in the same domain of the two domains separated by $\psi^{(n)}(S^1)$.

For points converging to $R_*(0)$ from the other direction, that is, $R_*^{jm+1}(c_1), j \in \mathbb{N}$, remember that we require $\theta^{(0)}(S^1)$ does not intersect the closure of the Fatou components containing the forward orbits of the critical point $c_1(R)$, so $\psi^{(0)}(S^1) = \varphi \circ \theta^{(0)}(S^1)$ does not intersect the closure of the Fatou components containing $R_*^j(c_1), j \in \mathbb{N}$. So by (27), $\psi^{(n)}(S^1)$ will avoid the Fatou components containing $R_*^j(c_1)$ for all n, then $\psi^{(n)}(S^1)$ separates $M_1^{(n-sm)}$ and M_2 for n large enough. It is a simple closed loop quite close to $J(R_*)$ according to (28). The part of $\psi^{(n)}(S^1)$ around the parabolic cycle M_3 is confined in the cusp area formed by the boundary of the Fatou component of R_* containing $R_*^j(c_1), j \in \mathbb{N}$.

Now for a small $\epsilon > 0$, choose a homeomorphism $\chi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, $\chi = Id$ outside a small neighbourhood U of M_3 (this neighbourhood U should contain ϵ neighbourhood of M_3), such that $\chi \circ \psi^{(n)}(S^1)$ is out of ϵ-neighbourhood of M_3 and $\chi \circ \psi^{(n)}(S^1)$ still separates $M_1^{(n-sm)}$ and M_2. Now denote $\chi \circ \psi^{(n)}(S^1) = \gamma$. Since $\gamma \sim \psi^{(n)}(S^1)$, so $R_*^{-1}(\gamma) \sim R_*^{-1}(\psi^{(n)}(S^1)) = \psi^{(n+1)}(S^1)$, so $\gamma \sim R_*^{-1}(\gamma)$ and they both separate

$M_1^{(n-sm)}$ from M_2. All the isotopies are in $\bar{\mathbb{C}} \setminus (M_1^{n-sm} \cup M_2)$.

$\square$

Proposition 8.2.2. *$\sigma_j = \psi^{(n)}(\xi_j)$ is isotopic to ζ_j, $\sigma' = \psi^{(n)}(\xi')$ is isotopic to ζ' for $n \geq jm + sm - 1$, $1 \leq j \leq k$ in $\bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$ according to the notations before.*

Proof. We have said in Section 2.2, the definition of the lamination map s_p, that it is defined up to topological conjugacy. However, in order to prove the isotopy in this set $\bar{\mathbb{C}} \setminus (M_2 \cup M_1^{(n-sm)} \cup \{R_*(0)\})$, we need to make sure that all $\psi^{(n)}$ map the point u_p to $R_*(0)$ as ψ. To achieve this we require all the points in the leaf μ_p are fixed under s_p^m. Choose $\theta^{(0)}$ to map u_p to the point of period m on the boundary of the attractive basin of $R(c_1)$ (we use the same notation c_1 to denote the critical point of R corresponding to the c_1 critical point of R_*). Now define

$$Y(s_{p,q}) = \{s_{p,q}^i(0) : i \geq 0\} \cup \{s_{p,q}^i(\infty) : i \geq 0\} \cup \{s_{p,q}^i(u_p) : i \geq 0\}.$$

In addition to satisfy (27) and (28), we can assure that

$$\theta^{(i+1)} \sim \theta^{(i)} \text{ rel } Y(s_{p,q}) \text{ for all } i \geq 1.$$

To do this we choose $\theta^{(0)}$ such that $\psi^{(0)}(u_p) = \varphi \circ \theta^{(0)}(u_p) = R_*(0)$, so $\psi^{(n)}(u_p) = \varphi \circ \theta^{(n)}(u_p) = R_*(0)$ for all n. First we show that

$$R_*^m(\sigma_j) \sim \sigma_{j-1}$$

in $\bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\}) = \bar{\mathbb{C}} \setminus (\{R_*^j(v_{-sm+1}) : -(n-sm) \leq j < \infty\} \cup M_2 \cup \{R_*(0)\})$.

According to (28), we have $\theta^{(n)} \circ s_{p,q}^m = R^m \circ \theta^{(n+m)}$. Composing both sides on the left by φ, we get

$$\psi^{(n)} \circ s_{p,q}^m = \varphi \circ R^m \circ \theta^{(n+m)}.$$

As $\varphi \circ R^m = R_*^m \circ \varphi$ outside the Fatou components containing the forward orbit of $c_1(R)$, so

$$\psi^{(n)} \circ s_{p,q}^m(\xi_j) = R_*^m \circ \psi^{(n+m)}(\xi_j).$$

Note that $s_{p,q}^m(\xi_j) \sim \xi_{j-1}$, and $\psi^{(n)}(\xi_j) \sim \psi^{(n+m)}(\xi_j)$ because $\psi^{(n)} \sim \psi^{(n+k)}$ rel $s_{p,q}^{-n}(X(s_{p,q}))$ for any $k \geq 0$. Then

$$R_*^m(\sigma_j) = R_*^m(\psi^{(n)}(\xi_j)) \sim R_*^m(\psi^{(n+m)}(\xi_j)) = \psi^{(n)} \circ (s_{p,q})^m(\xi_j) \sim \psi^{(n)}(\xi_{j-1}) = \sigma_{j-1}.$$

All the isotopies are in $\bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$.

Now choose ϵ_1 such that $\zeta_i \cap \{z : |z - R_*(0)| < \epsilon_1\} = \emptyset, i = 1, 2$. Then choose an integer n_1 such that $\psi^{(n)}(\xi_j) \subset \{z : |z - R_*(0)| < \epsilon_1\}$ for all $j \leq n_1$, choose $\epsilon_2 < \epsilon_1$ such that

$$\psi^{(n)}(\xi') = \sigma' \subset \{z : |z - R_*(0)| < \epsilon_2\}$$

for n large enough. Theses can be done because $\lim_{n\to\infty} \psi^{(n)} = \psi$ uniformly. Then choose $n_1 < n_2 < km + sm - 1$ such that for all $jm + sm - 1 \leq n_2$,

$$\psi^{(n)}(\xi_j) \cap \{z : |z - R_*(0)| \leq \epsilon_2\} = \emptyset.$$

We know that $\psi^{(n)}(\xi_j) = \sigma_j$ has the same endpoints with ζ_j, so for $n_1 < jm + sm - 1 \leq n_2$, σ_j and ζ_j both lie in the annulus $\{z : \epsilon_1 < |z - R_*(0)| < \epsilon_2\}$. Then they must be isotopic to each other in $\bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$. If this is not true, consider the first non-trivial intersection of σ_j with some ζ_i for $im + sm - 1 \leq n_1$. Because $R_*^m(\sigma_j) \sim \sigma_{j-1}$, and $R_*^m(\sigma_j)$ is a translation of σ_j inside a small neighbourhood of the parabolic cycle, so σ_j will have more intersections with σ_{j-1} besides the endpoint. This is impossible because ξ_j only has one intersection with ξ_{j-1} (the end point) and $\psi^{(n)}$ are all homeomorphisms. Figure 3 shows the only possible picture for σ_j and two impossible pictures of σ_j which have been excluded by our proof. Now we have proved $\sigma_j \sim \zeta_j$ in $\bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$ for $n_1 \leq jm + sm - 1 \leq n_2$.

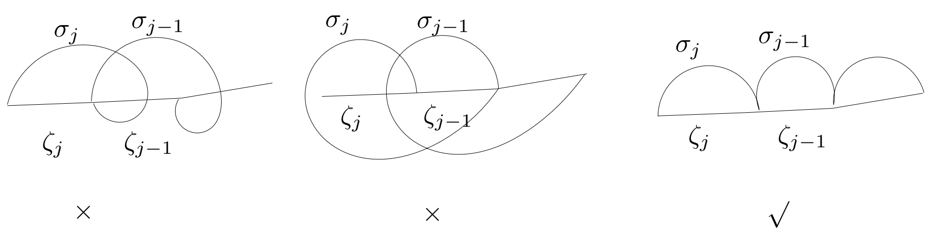

Figure 8: Possible and impossible pictures for σ_j

Now for $j \leq k$ and $jm + sm - 1 > n_2$, suppose $j' < j$ is an integer such that $(j - j')m + sm - 1 \leq n_1$. Since

$$R_*^{-j'm}(\sigma_{j-j'}) \sim \sigma_j \text{ in } \bar{\mathbb{C}} \setminus R_*^{-j'm}(M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\}),$$

$$R_*^{-j'm}(\zeta_{j-j'}) \sim \zeta_j \text{ in } \bar{\mathbb{C}} \setminus R_*^{-j'm}(M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\}),$$

$$\sigma_{j-j'} \sim \zeta_{j-j'} \text{ in } \bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$$

and $R_*^{-j'm}(M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\}) \supset (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$, so $\sigma_j \sim \zeta_j$ in $\bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$ for $jm + sm - 1 > n_2$. Now for j such that $jm + sm - 1 < n_1$, suppose j'' is a positive integer such that $n_1 \leq (j'' + j)m + sm - 1 \leq n_2$. Since

$$\sigma_{j''+j} \sim \zeta_{j''+j} \text{ in } \bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\}),$$

$$R_*^{j''m}(\sigma_{j''+j}) \sim \sigma_j \text{ in } \bar{\mathbb{C}} \setminus R_*^{j''m}(M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\}),$$

$$R_*^{j''m}(\zeta_{j''+j}) \sim \zeta_j \text{ in } \bar{\mathbb{C}} \setminus R_*^{j''m}(M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$$

and $R_*^{j''m}(M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\}) \subset M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\}$, so

$$\sigma_j \sim \zeta_j \text{ in}$$
$$\bar{\mathbb{C}} \setminus R_*^{j''m}(M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\}) = \bar{\mathbb{C}} \setminus (M_1^{(n-sm-j''m)} \cup M_2 \cup \{R_*(0)\}).$$

However, for n large enough and $jm + sm - 1 < n_1$, we will have

$$\sigma_j \cap \{R_*^{-(n-sm-i)}(v_{-sm+1}) : 0 \le i \le j''m\} = \emptyset,$$
$$\zeta_j \cap \{R_*^{-(n-sm-i)}(v_{-sm+1}) : 0 \le i \le j''m\} = \emptyset.$$

So in fact $\sigma_j \sim \zeta_j$ in $\bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$ for $jm + sm - 1 < n_1$.

At last, we show that $\psi^{(n)}(\xi') = \sigma'$ is isotopic to ζ' in $\bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$ for $jm+sm-1 > n_1$. This is because $\psi^{(n)}(\xi')$ is out of the Fatou components containing the parabolic cycle M_3, and $\psi^{(n)}(\xi')$ does not intersect $\psi^{(n)}(\xi_j) = \sigma_j$ except for the end point of $\psi^{(n)}(\xi_k) = \sigma_k$. Remember $\psi^{(n)}(\xi') = \sigma' \subset \{z : |z - R_*(0)| < \epsilon_2\}$, and we have proved that $\sigma_j \sim \zeta_j$ in $\bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$ for $1 \le j \le k$, so the only possibility is that $\sigma' \sim \zeta'$ in $\bar{\mathbb{C}} \setminus (M_1^{(n-sm)} \cup M_2 \cup \{R_*(0)\})$. □

We have got a simple closed loop γ, now we prove that it can serve as an invariant circle for R_n by the virtue of Theorem 8.2.1.

Corollary 8.2.3. *R_n is a mating, moreover,*

$$R_n \simeq s_{p'_n} \amalg s_{q'} \tag{30}$$

where p'_n and q' are odd denominator rationals in $(0,1)$.

Proof. We prove this by showing that γ is an invariant circle for R_n, which satisfies all the four requirements at the beginning of this section. First, γ separates $M_1^{(n'-sm)}$ (in the superscript the symbol n is changed to n' now in order to avoid confusion with the subscript n in R_n) and M_2. Secondly, $R_*^{-1}(\gamma)$ is connected. Thirdly, $R_*^{-1}(\gamma)$ is isotopic to γ in $\bar{\mathbb{C}} \setminus (M_1^{(n'-sm)} \cup M_2)$. These are all from Theorem 8.2.1. At last, $R_* : R_*^{-1}(\gamma) \to \gamma$ preserves orientation because it does so near the parabolic cycle. Recall the process of perturbation of R_* to get R_n. By continuity, $\{R_n^j(c_1(R_n)) : j \in \mathbb{N}\} \cup \{R_n^j(c_2(R_n)) : j \in \mathbb{N}\}$ is close to $M_1^{(n_0-sm)} \cup M_2$

for a finite number n_0. So if α_n is small enough, γ is isotopic to $R_n^{-1}(\gamma)$ in $\bar{\mathbb{C}} \setminus X(R_n)$ and both of them separate $\{R_n^j(c_1(R_n)) : j \in \mathbb{N}\}$ from $\{R_n^j(c_2(R_n)) : j \in \mathbb{N}\}$. Now we can conclude that γ is an invariant circle for R_n if n' is large enough (the critical orbit $\{R_n^j(c_1(R_n)) : j \in \mathbb{N}\}$ is finite). So according to Lemma 2.9.1, R_n is Thurston equivalent to a mating of two critically periodic degree two coverings, say, $s_{p'_n}$ and $s_{q'}$, which are again Thurston equivalent to two critically periodic degree two polynomials separately. p'_n and q' are both odd denominator rationals in $(0, 1)$. $\square$

9 Recover the odd denominator rational sequence p_n from R_n

In this section we prove Theorem 4.2.1. The basic idea is, given p, q, we already get a sequence of critically periodic maps $R_n \simeq s_{p'_n} \amalg s_{q'}$ around R_*, where R_* is on the boundary of the hyperbolic component containing $R \simeq s_p \amalg s_q$. It is obvious that $q' = q$ because the dynamics around finite critical orbit M_2 of R_* is inherited completely from s_q, and the perturbation is on $M_1^{(\infty)}$ which does not affect M_2. Now there is a reason that we can expect $p'_n = p_n$ for a certain choice of p_n tending to p considering R_n. We recover p_n from R_n by Lemma 2.10.1.

Now we have three planes, two of which are dynamical planes of s_p and s_{p_n}. There are lamination descriptions on their dynamics. The third one is the dynamical plane of R_*, on which we depend to get R_n. Recall in the last section there is a map $\psi : S^1 \to J(R_*)$ such that $\psi \circ z^2 = R_* \circ \psi$ ($s_{p,q} = z^2$ on S^1). Recall that $\psi(e^{2\pi i p}) = \psi(e^{2\pi i \bar{p}}) = R_*(0)$. The orbit of 0, as critical point of s_p, corresponds to the orbit of $e^{2\pi i p}$ under z^2 on S^1, because $s_{p_n}(0)$ is in the closed gap containing $e^{2\pi i p}$ on its boundary

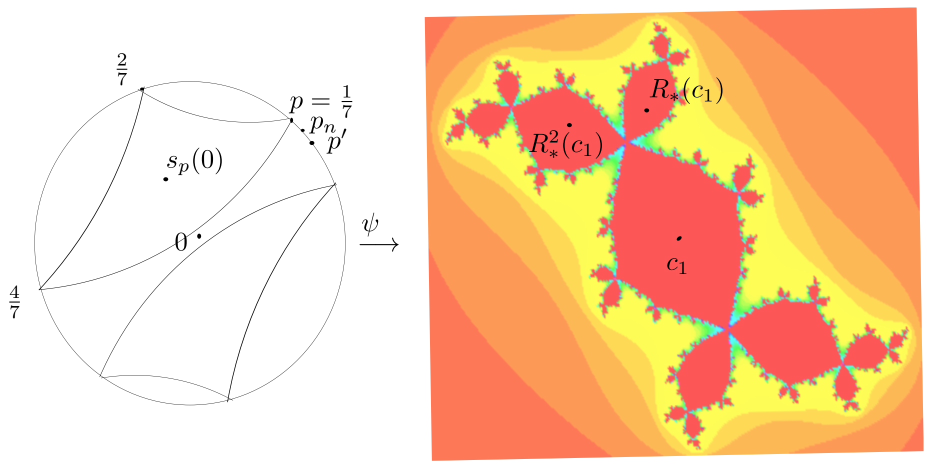

Figure 9: Links between dynamical planes of $s_{\frac{1}{7}}$ and R_* through ψ with assumed positions of p', p_n

for any odd denominator rational p. Figure 9 gives an example of $p = \frac{1}{7}$ with $m = 3$. Only parts of the laminations and Fatou components are shown.

Recall $p \in (0, 1)$ is of period m under the doubling map, that is, $2^m p = p \mod \mathbb{Z}$. Similar to the process of colliding the backward and forward iterations of c_1 in the δ_3 neighbourhood of 0 around the parabolic cycle of R_*, for $s \in \mathbb{N}$ large enough choose p' close to p, $p' < p$, such that $2^{sm} p' = p \mod \mathbb{Z}$, $v_{-sm+1}(s_{p,q}) \in s_{p,q}^{-(sm-1)}(0)$ is in the closed gap containing $e^{2\pi i p'}$ on its boundary. Since $|v_{-sm}(R_*)| = |v_{-sm}(R_*) - 0| < \delta_3$, R_* is conformal around a small neighbourhood of the parabolic cycle M_3, we can assume

$$|v_{-sm+1}(R_*) - R_*(0)| = |R_*(v_{-sm}(R_*)) - R_*(0)| < \delta_3.$$

$\psi(e^{2\pi i p'})$ is on the boundary of the Fatou component containing $v_{-sm+1}(R_*)$. $r \in \mathbb{N}$ is large enough such that $|R_*^{rm}(c_1)| < \delta_3$.

Since $c_1(R_n)$ as critical point of R_n has period $(s+r+2n)m$, so we expect p_n is of period $(s+r+2n)m$. Because $p_n \to p$ and p' is close to p, we want

$$\bar{\delta}_n = 2^{(r+2n)m}(p_n - p) - (p' - p) \bmod \mathbb{Z}$$

being quite small. Apply 2^{sm} to both sides gives

$$2^{sm}\bar{\delta}_n = p_n - 2^{sm}p' = p_n - p \bmod \mathbb{Z}.$$

Choose $\bar{\delta}_n = \dfrac{p_n - p}{2^{sm}}$, we have

$$2^{(r+2n)m}(p_n - p) = p' - p + \frac{p_n - p}{2^{sm}} \bmod \mathbb{Z}.$$

Choose $p_n - p = \dfrac{p' - p}{2^{(r+2n)m} - 2^{-sm}} = \dfrac{2^{sm}(p' - p)}{2^{(s+r+2n)m} - 1} \bmod \mathbb{Z}$, then let

$$p_n = p + \frac{2^{sm}(p' - p)}{2^{(s+r+2n)m} - 1}.$$

For the above sequence of odd-denominator rationals p and p_n in $(0,1)$, $n \in \mathbb{N}$, parameters s, p', r, m are all determined by p, now we are ready to prove our Main Theorem by Lemma 2.10.1.

Proof of the Main Theorem:

Proof. We know that for critically periodic maps, the pattern of their critical orbits determines the dynamics completely up to topological conjugacy. We will prove that, 0, the critical point of s_{p_n}, has exactly the same pattern under s_{p_n} as c_1, the critical point of R_n, under R_n. This resemblance enables us to construct the arcs ξ_j and ζ_j in Lemma 2.10.1 and the two orientation preserving homeomorphisms Θ and Θ'. From this we can show $p'_n = p_n$.

First note that $R_*^{-j}(c_1)$ are completely determined by $2^{-j-1}p$ for $0 \leq j \leq sm$, by assigning $R_*^{-j}(c_1)$ the j-th backward iteration of c_1 in the Fatou component containing $\psi(e^{2\pi i 2^{-j-1}p})$ on its boundary. Remember that we choose p' near p such that $2^{sm}p' = p \mod \mathbb{Z}$. As $s_{p_n}(0)$ is in the gap containing $e^{2\pi i p_n}$ on its boundary, every iteration (forward and backward) of 0 is related to an iteration of $e^{2\pi i p_n}$ under z^2 on S^1, so in fact we will prove that $e^{2\pi i p_n}$ behaves the same under z^2 as c_1 under R_n. An important feature of $R_n(c_1)$ under iterations of R_n is that $R_n^{1-sm-km}(c_1), k \in \mathbb{N}$ and $R_n^{1+(r+k)m}(c_1), k \in \mathbb{N}$ approach $R_*(0)$ along the two fixed directions as k goes to n, then collide with each other when $k = n$ around $R_*(0)$. In the following we will see that for some integer a_k, $e^{2\pi i 2^{-sm-km}(p_n+a_k)}$ and $e^{2\pi i 2^{(r+k)m}p_n}$ also approach some point from two fixed directions on S^1.

Now we begin to construct the ordered arcs and the two homeomorphisms Θ and Θ' required in Lemma 2.10.1. We assume $p' < p$ from now on, the case $p' > p$ is similar.

First, backward iterate p_n for sm times, choose $2^{-sm}(p_n + a_0) = 2^{-sm}(p + p_n - p + a_0) = p' + \dfrac{p_n - p}{2^{sm}}$ close to p' on S^1 for some $a_0 \in \mathbb{Z}$, then continue to backward iterate $2^{-sm}p_n$ for km times,

$$2^{-sm-km}(p_n + a_k) = 2^{-km}(p' + a_k + \frac{p_n - p}{2^{sm}}) =$$
$$2^{-km}(p + a_k + p' - p) + \frac{p_n - p}{2^{sm+km}}$$
$$= p + \frac{p' - p}{2^{km}} + \frac{p_n - p}{2^{sm+km}} = p + \frac{p' - p}{2^{km}} + \frac{p' - p}{2^{km}(2^{(s+r+2n)m} - 1)} =$$
$$p + \frac{2^{(s+r+2n)m}(p' - p)}{2^{km}(2^{(s+r+2n)m} - 1)}.$$

where $a_k \in \mathbb{Z}$ is chosen so that $2^{-km}(p + a_k) = p$. Since $p' - p < 0$, it is a sequence gradually increasing to $p + \dfrac{2^{(s+r+n)m}(p' - p)}{2^{(s+r+2n)m} - 1}$ as k goes to n. Draw an arc $\beta_{1-sm-km}$ in each gap which joins $s_{p_n}^{1-sm-km}(0)$ with $e^{2\pi i 2^{-sm-km}(p_n + a_k)}$, $0 \leq k \leq n$, such that $\beta_{1-sm-km} = s_{p_n}^m(\beta_{1-sm-(k+1)m})$, $0 \leq k \leq n-1$. Then push forward this group of arcs to other $m-1$ groups $\{\beta_{1-sm-km+j} : 0 \leq k \leq n\}$ for $0 < j < m$ by $s_{p_n}^j$, that is, define $\beta_{1-sm-km+j} = s_{p_n}^j(\beta_{1-sm-km})$ for $0 \leq k \leq n$ and $0 < j < m$.

Now forward iterate $2^{rm}p_n$ for km times, we get

$$2^{(r+k)m}p_n = 2^{(r+k)m}(p + p_n - p) = p + 2^{(r+k)m}(p_n - p) = \\ p + \frac{2^{(r+k+s)m}(p' - p)}{2^{(s+r+2n)m} - 1} \bmod \mathbb{Z}.$$

The sequence gradually decreases to $p + \dfrac{2^{(s+r+n)m}(p' - p)}{2^{(s+r+2n)m} - 1}$ as k goes to n. When $k = n$, the two sequences collide with each other around p, considering that $n \in \mathbb{N}$ is a quite large integer. Note that we have already got an arc in the gap containing $s_{p_n}^{1-sm-nm}(0) = s_{p_n}^{(r+n)m+1}(0)$ (with $e^{2\pi i 2^{-sm-nm}p_n} = e^{2\pi i 2^{(r+n)m}p_n}$ on its boundary). Now denote $\beta_{(r+n)m+1} = \beta_{1-sm-nm}$. Draw an arc $\beta_{(k+r)m+1}$ in each corresponding gap joining $s_{p_n}^{(k+r)m+1}(0)$ with $e^{2\pi i 2^{(k+r)m}p_n}$, $0 \leq k \leq n$, such that $\beta_{(k+r+1)m+1} = s_{p_n}^m(\beta_{(k+r)m+1})$, $0 \leq k \leq n-1$. Then push forward this group of arcs to other $m-1$ groups $\{\beta_{(k+r)m+1+j} : 0 \leq k \leq n-1\}$, $0 < j < m$ by $s_{p_n}^j$, that is, $\beta_{(k+r)m+1+j} = s_{p_n}^j(\beta_{(k+r)m+1})$.

Draw the other arcs β_{j_-} (β_{j_+}) for $1 - s \leq j_- \leq 0$ ($0 \leq j_+ \leq rm$) such that $\beta_{j_-+1} = s_{p_n}(\beta_{j_-})$ ($\beta_{j_++1} = s_{p_n}(\beta_{j_+})$). Note that the arc β_0 deduced from β_{j_-} will be the same one as deduced from β_{j_+} because $\beta_{(r+n)m+1} = \beta_{1-sm-nm}$.

Now construct the corresponding arcs on the dynamical plane of R_n following similar process. Let z_p be the point $\psi^{(l)}(e^{2\pi i p})$ for l being an integer large enough under the isotopy in Theorem 8.2.1 on γ. First draw arcs $\eta'_{1-sm-km}$ joining $R_n^{1-sm-km}(c_1)$ with points z_{-s-km} around the point $R_n^{-1}(z_p)$ on $R_n^{-1}(\gamma)$ in the region $D_{R_n^{-1}(\gamma)}$ bounded by $R_n^{-1}(\gamma)$. We require these arcs do not intersect each other and the sequence of points z_{-sm-km} is of the same cyclic order as $e^{2\pi i 2^{-sm-km} p_n}$ on S^1. To achieve this, we can first draw arcs in the corresponding Fatou components (corresponding to gaps of the L_p lamination) on the dynamical plane of R_* joining $R_*^{1-sm-km}(c_1)$ with $\psi(e^{2\pi ip})$, then isotope it to an arc joining $R_n^{1-sm-km}(c_1)$ with a point z_{-s-km}. This arc will be close to the first because $R_n \to R_*$ as $n \to \infty$. Push forward the group of arcs to other $m-1$ groups $\{\eta'_{1-sm-km+j} : 0 \leq k \leq n\}$, $0 < j < m$ by R_n^j, that is, define $\eta'_{1-sm-km+j} = R_n^j(\eta'_{1-sm-km})$ for $0 \leq k \leq n$ and $0 < j < m$. Let $\eta'_{(r+n)m+1} = \eta'_{1-sm-nm}$, draw arcs $\eta'_{(r+k)m+1}$ in $D_{R_n^{-1}(\gamma)}$ joining $R_n^{(r+k)m+1}(c_1)$ with points $z_{(r+k)m}$ around the point $R_n^{-1}(z_p)$ by the same isotopy method. Again make sure these arcs do not intersect each other and the arcs already existing. The sequence of points $z_{(r+k)m}$ should be of the same cyclic order on $R_n^{-1}(\gamma)$ as $e^{2\pi i 2^{(k+r)m} p_n}$ on S^1. Push forward the group of arcs to other $m-1$ groups $\{\eta'_{(r+k)m+1+j} : 0 \leq k \leq n\}$, $0 < j < m$ by R_n^j, that is, define $\eta'_{(r+k)m+1+j} = R_n^j(\eta'_{(r+k)m+1})$ for $0 \leq k \leq n$ and $0 < j < m$. Choose the other arcs η'_{j_-} and η'_{j_+} for $1 - sm \leq j_- \leq 0$ and $0 \leq j_+ \leq rm$ suitably such that all the endpoints $\{z_j : 1 - sm - nm \leq j \leq (n+r)m + 1\}$ of these arcs $\{\eta'_{j+1} : -sm - nm \leq j \leq (n+r)m\}$ are of the same cyclic order as $\{e^{2\pi i 2^j p_n} : 1 - sm - nm \leq j \leq (n+r)m + 1\}$ on S^1.

By these choices of the arcs η'_j we can get an orientation preserving homeomorphism Θ', such that

$$\Theta'(R_n^{-1}(\gamma)) = S^1,$$

$$\Theta'(\eta_0'') = \beta_0'',$$

$$\Theta'(\eta_j') = \beta_j,\ 1 - sm - nm \leq j \leq (n+r)m + 1,$$

where $\beta_0'' = \{s_{p_n}^{-1}(\beta_1) \setminus \beta_0\} \cup \{0\}$, $\eta_0'' = \{R_n^{-1}(\eta_1') \setminus \eta_0'\} \cup \{c_1\}$. Now let $\eta_{j+1} = R_n(\eta_j')$ for $-sm - nm \leq j \leq (n+r)m$. There is an orientation preserving homeomorphism Θ which is isotopic to the identity, such that

$$\Theta(R_n^{-1}(\gamma)) = \gamma,$$

$$\Theta(\eta_j') = \eta_j,\ 1 - sm - nm \leq j \leq (n+r)m + 1.$$

This is because the map $R_n : R_n^{-1}(\gamma) \to \gamma$ satisfies the four properties of an invariant circle (refer to Definition 8.0.2). The order of the sequence z_j is preserved on $R_n^{-1}(\gamma)$.

Now apply Lemma 2.10.1 with the family of arcs $\xi_j, \zeta_j, \zeta_j', \xi_0'', \zeta_0''$ substituted by the family $\beta_j, \eta_j, \eta_j', \beta_0'', \eta_0''$, we see that for this particular sequence p_n, $R_n \simeq s_{p_n} \amalg s_q$.

□

Remark 9.0.4. *The proof relies heavily on comparing traces of critical points on the two dynamical planes of s_{p_n} and R_n, for which the links comes from the bridge between the dynamical planes of s_p and R_*. The techniques of perturbation confine our result to be limited, and these will have to be modified if we are to get more general results.*

Remark 9.0.5. *For $R_*^m(z) = z + z^n + ...$, $n \geq 3$, that is, parabolic maps with more than 2 petals, one can get critically periodic maps by perturbing R_* from any two attracting and repelling directions. There are many choices before the backward and forward orbits converge to the parabolic cycle from some fixed direction. Every choice offers us a different critically periodic map. By a similar process one can show that these maps* *are also matings, so the continuity result is true for the cases that,* R_*^m *has more than 2 petals.*

10 Bubble rays puzzle partitions and parapuzzle partitions for some 2-rational maps

10.1 Matability of the basilica with non-renormalizable quadratic polynomials

When dealing with 2-rational maps, one hopes that the Yoccoz puzzle method for 2-polynomials can be transplanted into the 2-rational maps case. Luo [Luo] first proposed this idea in his thesis. Then M. Aspenberg and M. Yampolsky carried this idea out to the family

$$R_a = \frac{a}{z^2 + 2z}$$

to show the conformal matability of the basilica polynomial f_{-1} with an arbitrarily non-renormalizable parameter c not in the $\frac{1}{2}$-limb of M [AY]. The following picture from [AY] shows structure of the hyperbolic components on the parameter space R_a. The black area is the mating components M_a, which is called the non-escape locus in [Dud]. This and subsequent figures from [AY] are reproduced with kind permission of the authors.

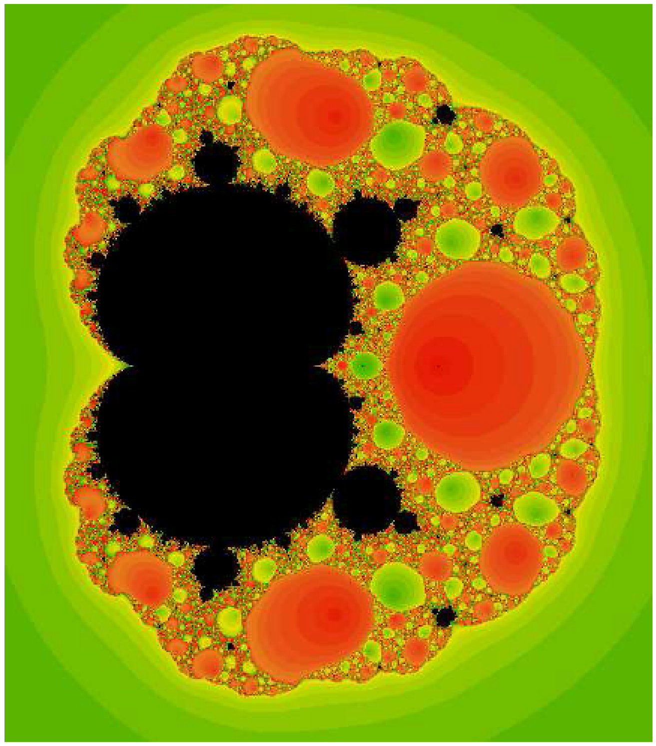

Figure 10: The parameter space for the R_a family. The picture is from [AY].

Aspenberg and Yampolsky showed the following theorem (Main Theorem [AY]):

AY's Theorem. *For a non-renormalizable parameter c not in the $\frac{1}{2}$-limb of M such that $f_c(z) = z^2 + c$ does not have a non-repelling periodic orbit, f_c and $f_{-1} = z^2 - 1$ are conformally matable, and their mating is unique up to Möbius changes of coordinates.*

By certain adaptation, the method should work for f_{-1} substituted by a starlike map and c being a finitely renormalizable quadratic polynomial [DH1] with only repelling periodic cycles not in the $\frac{1}{2}$-limb of M. In the following section we gradually introduce the notions of their bubble rays method.

10.2 Bubble rays puzzle partitions for f_{-1}, R_a and a landing theorem

Note that on the dynamical plane of the 2-rational maps R_a, we no longer have the external rays. As a substitute, we use the so called bubble rays. We first define it for f_{-1}, then define it similarly for R_a. Now let B_0 and B_{-1} be the corresponding Fatou components of f_{-1} containing 0 and -1.

Definition 10.2.1. *A bubble of the filled Julia set $K(f_{-1})$ is a Fatou component F of the filled Julia set. The generation of a bubble F is the smallest non-negative integer $n = Gen(F)$ for which $f_{-1}^n(F) = B_0$. The center of a bubble F is the preimage $f_{-1}^{-Gen(F)}(0) \cap F$.*

Definition 10.2.2. *A bubble ray $\mathcal{B}$ of f_{-1} is a collection of bubbles $\cup_{k=0}^{\infty} F_k$ with $Gen(F_k) < Gen(F_{k+1})$ such that for each k the intersection $\bar{F}_k \cap \bar{F}_{k+1} = \{x_k\}$ is a single point.*

Note that this concept is well-defined due to the following lemma [AY].

Lemma 10.2.3. *For two different bubbles F_b and F_c of f_{-1} such that neither of them is the attracting basin of ∞, then one of the following holds*

- $\bar{F}_b \cap \bar{F}_c = \emptyset$.
- $\bar{F}_b \cap \bar{F}_c$ *is a single point, which is a pre-fixed point for f_{-1}.*

Now analogously we define bubbles and bubble rays for R_a. Let A_∞ be the Fatou component of R_a containing ∞.

Definition 10.2.4. *A bubble F of R_a is a Fatou component in the set $\cup_{k=0}^{\infty} R_a^{-k}(A_\infty)$. The generation of a bubble F is the smallest non-negative $n = Gen(F)$ for which $R_a^n(F) = A_\infty$. The center of a bubble F is the preimage $R_a^{-Gen(F)}(\infty) \cap F$.*

Definition 10.2.5. *A bubble ray $\mathcal{B}$ of R_a is a collection of bubbles $\cup_{k=0}^{\infty} F_k$ with $Gen(F_k) < Gen(F_{k+1})$ such that for each k the intersection $\bar{F_k} \cap \bar{F_{k+1}} = \{x_k\}$ is a single point.*

There are similar properties for the Fatou components of R_a as Lemma 10.2.3 to guarantee the well definition of bubble rays for R_a.

As the place of external rays is taken by bubble rays now, we can reconstruct lots of similar structures compared with 2-polynomials for the 2-rational maps R_a. There are all kinds of landing lemmas in [AY] for these bubble rays. One can define the angles for a bubble ray $\mathcal{B}$ of f_{-1} being inverse of the angles of the external rays which land at the point of $J(f_{-1})$ where $\mathcal{B}$ lands. Then through a conjugacy between bubble rays for f_{-1} and R_a one can define angles of bubble rays for R_a. One can also define orbit portraits for these bubble rays and characteristic arcs for the orbit portraits as in [Mil2]. There are several landing Lemmas in Chapter 6 [AY] for f_{-1} and R_a to guarantee the well definition of angles above. For more details, see Chapter 4 and 6 of [AY].

Having the definitions of bubble rays on the dynamical planes, now we can define parabubbles (or called capture components) and parabubble rays on the parameter space of the family $R_a = \frac{a}{z^2+2z}$. These parabubble rays also cut out similar wake structures as for M. We present them in the following theorem (Proposition 6.10 and Lemma 6.11 of [AY]).

Theorem 10.2.6. *Let a_0 be such that R_{a_0} has a parabolic fixed point z_0 with eigenvalue $R'_{a_0}(z_0) = e^{2\pi i \frac{p}{q}}, (p,q) = 1$. Denote $\mathcal{O} = \{\{\theta_1, ..., \theta_q\}\}$ the orbit portrait for the bubble rays landing at z_0, and let $I = [t-, t+]$ be its characteristic arc. Then the corresponding parabubble rays with angles $t+$ and $t-$ land on a_0. They cut out an open set in the complex plane, called the bubble wake $W = W(t-, t+)$ such that $a \in W$ if and only if*

R_a exhibits the repelling orbit portrait $\mathcal{O} = \mathcal{O}(t-, t+)$.

The following wake picture is from [AY].

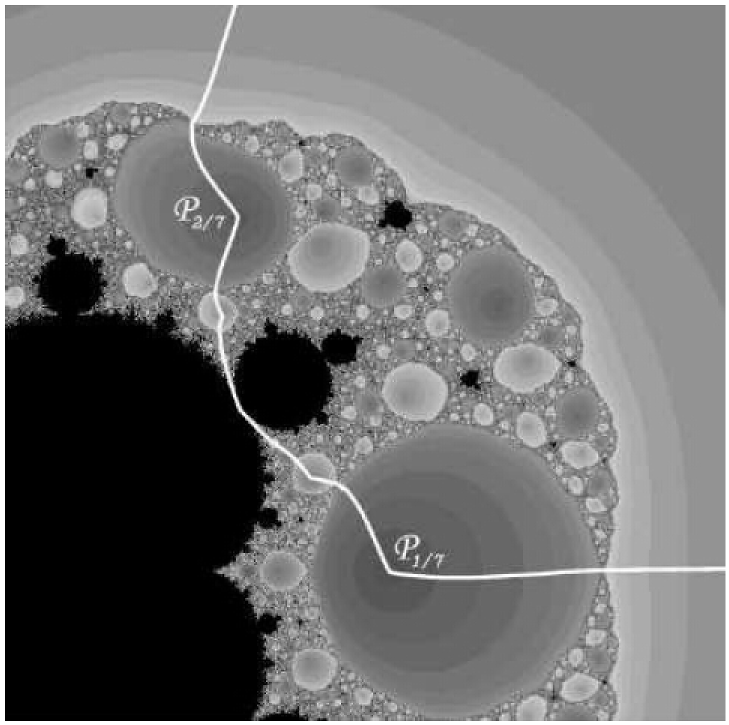

Figure 11: The parameter wake $W(\frac{1}{7}, \frac{2}{7})$. The picture is from [AY].

11 Proof of Theorem 4.2.2: continuity of matings between f_{-1} and parameters near an arbitrary non-renormalizable quadratic polynomial following Aspenberg and Yampolsky

In this section we prove Theorem 4.2.2 on continuity of conformal matings from [AY]'s existence theory of the corresponding conformal matings.

First we give a notion originating in [AY]. It measures the difference between two dynamical systems with puzzle partitions. Since we have defined Yoccoz puzzle for quadratic polynomials and bubble rays puzzle for the R_a family, we will present the definition on the maps $\{f_c = z^2+c : c \in M\} \cup \{R_a : a \in \mathbb{C}\}$.

Definition 11.0.7. *For two maps F and G in $\{f_c = z^2 + c : c \in M\} \cup \{R_a : a \in \mathbb{C}\}$, we say they have the same combinatorics of puzzle up to depth n, if there exists an orientation preserving continuous surjection $\psi : \mathbb{C} \to \mathbb{C}$, such that*

- *ψ maps distinct puzzle pieces P_k^i of depth $k \leq n$ of F homeomorphically to distinct puzzle pieces Q_k^j of G.*
- *ψ maps the critical piece of depth $k \leq n$ of F to the critical piece of depth $k \leq n$ of G.*
- *ψ respects the dynamics, that is,*

$$P_{k-1}^i = F(P_k^j) \text{ if and only if } \psi(P_{k-1}^i) = G(\psi(P_k^j)).$$

When $F, G \in \{f_c = z^2 + c : c \in M\}$ or $F, G \in \{R_a : a \in \mathbb{C}\}$, the map $\psi : \mathbb{C} \to \mathbb{C}$ can be chosen to be a homeomorphism on $\mathbb{C}$.

In order to prove the theorem we still need a lemma which shows us that the puzzle partitions vary continuously with respect to parameters in M.

Lemma 11.0.8. *For any $n_1 \in \mathbb{N}$, there exists $N(n_1) \in \mathbb{N}$, such that for $n > N(n_1)$, f_{c_n} has the same combinatorics of puzzle up to depth n_1 with f_c, moreover, c_n (critical value of f_{c_n}) and c (critical value of f_c) stay in some particular puzzle piece of depth n_1.*

Proof. We will prove stronger results by mathematical induction, that is, we will prove that, for any $n_1 \in \mathbb{N}$, there exists $\delta_{n_1} > 0$, such that for all the quadratic polynomials $f_{c'} = z^2 + c'$ with parameters $c' \in D_c^{\delta_{n_1}} \cap M$

(we use the symbol $D_c^{\delta_{n_1}}$ to denote the disc centred on c with radius δ_{n_1}), f_c has the same combinatorics of puzzle up to depth n_1 with $f_{c'}$ and the critical value c, c' of f_c and $f_{c'}$ stay in some particular puzzle piece of depth n_1.

The base step follows from Theorem 1.2 [Mil2], that is, there is a wake (containing c) in which all the maps in it has a repelling fixed point with orbit portrait given by the the orbit portrait of the α fixed point of f_c.

Assume when $n_1 = n$ the result holds, that is, there exists $\delta_n > 0$, such that for all maps $f_{c'} = z^2 + c'$ with $c' \in D_c^{\delta_n} \cap M$, f_c has the same combinatorics of puzzle up to depth n as $f_{c'}$, c' stays in some particular puzzle piece of depth n. By definition, there exists an orientation preserving homeomorphism $\psi_{c'}^n : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, such that:

(a) $\psi_{c'}^n$ maps distinct puzzle pieces $P_k^{i,c}$ of depth $k \le n$ of f_c to corresponding puzzle pieces $P_k^{i,c'}$ of depth $k \le n$ of f_c'. Here we number the finitely many puzzle pieces $P_k^{i,c}$ of f_c of depth k by integer $i \in \mathbb{N}$, and do the same thing with $f_{c'}$.

(b) for all $k \le n$, we have $\psi_{c'}^n$ maps depth k critical puzzle piece of f_c to depth k critical puzzle piece of $f_{c'}$.

(c) $\psi_{c'}^n$ respects the dynamics, that is,

$$P_{k-1}^{i,c} = f_c(P_k^{i,c}) \text{ if and only if } \psi_{c'}^n(P_{k-1}^{i,c}) = f_{c'}(P_k^{i,c}) \text{ for } k \le n.$$

For the $n_1 = n + 1$ case, we make a finer restriction on $\psi_{c'}^n : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ to co-ordinate the puzzle pieces of depth $n+1$. Denote the α fixed point of f_c by α_c. α_c is used to form the base Yoccoz puzzle partition of f_c. Note that unless for some integer l, $f_c^l(0) \in \cup_{k=0}^{\infty} f_c^{-k}(\alpha_c)$, c will always stay in the interior of the puzzle pieces of any depth. Suppose this is not the case, and consider the set $f_c^{-1}\{\cup_{k=0}^{n} f_c^{-k}(\alpha_c)\}$ together with the external rays landing on them. Label these points as $\alpha_{n+1,c}^i$, $1 \le i \le n_2$.

Now choose $\delta_{n+1} < \delta_n$, such that for any map $f_{c'}$ with $c' \in D_c^{\delta_{n+1}}$, $\alpha^i_{n+1,c'}$ ($1 \leq i \leq n_2$) stays in the corresponding puzzle piece $\psi^n_{c'}(P^c_n(\alpha^i_{n+1,c}))$, in which $P^c_n(\alpha^i_{n+1,c})$ denotes the depth n puzzle piece of f_c containing $\alpha^i_{n+1,c}$. Furthermore, one should guarantee δ_{n+1} small enough such that c' stays in the corresponding puzzle piece which varies continuously from $P^c_{n+1}(c)$ (depth $n+1$ puzzle piece of f_c containing c). This can be done because by assumption $c \in (P^c_{n+1}(c))^o$ (interior of $P^c_{n+1}(c)$).

Now we only need to consider the map $\psi^n_{c'}$ on $P^c_n(\alpha^i_{n+1,c})$ (depth n puzzle piece of f_c containing the point $\alpha^i_{n+1,c}$). Modify $\psi^n_{c'}$ on $P^c_n(\alpha^i_{n+1,c})$ to get a new map $\psi^{n+1}_{c'}$, such that for any $c' \in D_c^{\delta_{n+1}}$, $\psi^{n+1}_{c'}$ satisfies (a), (b) and (c) for $k \leq n+1$. This can be done because according to Lemma 2.1 [Mil2], the cyclic order of external rays landing on some point is preserved by the map $f^{-1}_{c'}$. Now the lemma follows by induction.

$\square$

Now we can prove Theorem 4.2.2 by virtue of Lemma 11.0.8.

Proof. We use the Yoccoz puzzle for quadratic polynomials f_c and f_{c_n} in [Hub] and bubble rays puzzle for $R_{a(c_n)}$ and $R_{a(c)}$ in [AY]. Notice from Chapter 9 [AY] that f_c has the same combinatorics of the puzzle up to depth ∞ with $R_{a(c)}$, f_{c_n} has the same combinatorics of the puzzle up to depth ∞ with $R_{a(c_n)}$, while both critical points stay in some particular puzzle pieces. We use the same notation Δ^c_n, $n \in \mathbb{N}$ as in [AY] to denote the depth n parameter puzzle piece on the a plane in which all the maps have the same combinatorics of puzzle up to depth n with f_c (refer to Proposition 7.11 , [AY]). That is,

$\Delta^c_n = \{R_a : R_a$ has the same combinatorics of puzzle up to depth n and $-a$ is contained in the particular puzzle piece of depth n which varies isotopically from the depth n puzzle piece of $R_{a(c)}$ containing its critical point $-a(c)\}$.

Now by Lemma 11.0.8, for any $n_1 \in \mathbb{N}$, there exists $N(n_1)$ large enough such that for $n > N(n_1)$, f_{c_n} has the same combinatorics of puzzle up to depth n_1 with f_c and the critical points stay in the particular puzzle pieces as $\lim_{n\to\infty} c_n \to c$. Then $R_{a(c_n)} \in \Delta^c_{n_1}$. Now let $n_1 \to \infty$, then

$$diam(\Delta_{n_1}) \to 0$$

by Theorem 9.1 [AY]. It follows that $\lim_{n\to\infty} R_{a(c_n)} = R_{a(c)}$. $\square$

12 The case of Fatou components with disjoint closures: proof of Theorem 4.2.5

12.1 An example $s_{\frac{3}{7}} \amalg s_{\frac{3}{31}}$

Now consider the case when all the Fatou components of R have disjoint closures, e.g. $p = \frac{3}{7}, q = \frac{3}{31}$. We first show that

Lemma 12.1.1. *Closures of Fatou components of the map $R \simeq s_{\frac{3}{7}} \amalg s_{\frac{3}{31}}$ are pairwise disjoint.*

We need Lemma 3.3 from [Ree3] presented as the following lemma in our proof.

Lemma 12.1.2. *Let f be of quadratic rational hyperbolic type IV (means f has two critically periodic points with disjoint orbits), then one of the following occurs.*

(1) f is Möbius conjugate to a critically periodic polynomial.

(2) The closure of any Fatou component is a closed topological disc. The closures of any two Fatou components U_1 and U_2 of f intersect in at most one point. If $\bar{U_1}$ and $\bar{U_2}$ do intersect at x, and U_1 and U_2 are

both periodic, then x is periodic with period $\max(m_1, m_2)$*, where m_i is the minimum period of points on ∂U_i under f.*

Proof of Lemma 12.1.1:

Proof. First, observe the orbit of the minor leaves (we use $[r, s], r, s \in \mathbb{Q}$ to represent the leaf with endpoints $e^{2\pi ir}$ and $e^{2\pi is}$ on S^1) in the two laminations:

$s_{\frac{3}{7}} : [\frac{3}{7}, \frac{4}{7}] \to [\frac{6}{7}, \frac{1}{7}] \to [\frac{2}{7}, \frac{5}{7}]$,

$s_{\frac{3}{31}} : [\frac{3}{31}, \frac{4}{31}] \to [\frac{6}{31}, \frac{8}{31}] \to [\frac{12}{31}, \frac{16}{31}] \to [\frac{24}{31}, \frac{1}{31}] \to [\frac{17}{31}, \frac{2}{31}]$.

Notice that by Lemma 12.1.2, $m_1 = 1$ or 3, $m_2 = 1$ or 5 in this case. If two Fatou components U_i of $R \simeq s_{\frac{3}{7}} \amalg s_{\frac{3}{31}}$ do intersect, in case $m_1 = 1$ this means $\partial U_1 \cap R(\partial U_1) \cap R^2(\partial U_1)$ is a single fixed point and every other point in ∂U_1 has period divisible by an integer ≥ 3. Similarly in case $m_2 = 1$, $\partial U_2 \cap R(\partial U_2) \cap R^2(\partial U_2) \cap R^3(\partial U_2)$ is a single fixed point and every other point in ∂U_2 has period divisible by an integer ≥ 5. So the possible intersection between ∂U_1 and ∂U_2 is a fixed point and this would have to be the equivalent class of $[\frac{3}{7}, \frac{4}{7}]$ and $[\frac{3}{31}, \frac{4}{31}]$. They would have to be joined by finitely many leaves. In the combination of the two laminations $L_{\frac{3}{7}}$ and $L_{\frac{3}{31}}^{-1}$, two leaves can intersect only if they have a common endpoint. Both endpoints of a leaf have the same period, so $[\frac{3}{7}, \frac{4}{7}]$ and $[\frac{3}{31}, \frac{4}{31}]$ can not be joined by finitely many leaves because these leaves have endpoints of different periods. So the closures of Fatou components are all disjoint.

$\square$

12.2 Notations and outline strategy for the proof of Theorem 4.2.5

In this section we will prove Conjecture 4.1.1 in the case when all Fatou components of R_* (or equivalently of R) have disjoint closures, that is, Theorem 4.2.5. There is currently an extra hypothesis in our Lemma 12.7.4 on $\varphi_*^{-1}(\varphi_*(p))$, quite mild, or a technical problem, which could hopefully be removed later on. We use the idea of Markov partitions for R_n and R_*, which do not have the parabolic cycle on the boundary of any partition element.

Let M_{x_*} be the copy of the Mandelbrot set containing x_* as its cusp. There is a corresponding combinatorial copy of the Mandelbrot set in the unit disc bounded by μ_p. If μ_{p_n} is in this combinatorial copy of the Mandelbrot set for all n, then $R_n \in M_{x_*}$. Let L_p be the invariant lamination with minor leaf μ_p. Let G_p be the minor gap which is the gap bounded from 0 by μ_p with μ_p on its boundary. s_{p_n} is a tuning of s_p if $L_{p_n} \supset L_p$, which means $\mu_{p_n} \subset \bar{G}_p$, $G_{p_n} \subset G_p$. All leaves of $L_{p_n} \setminus L_p$ are in the orbit of $\bar{G}_p$. This means that there is $t_n \in \mathbb{Q}$ and a lamination L_{t_n} whose minor leaf is μ_{t_n} with an endpoint $e^{2\pi i t_n}$. There is a continuous map $\psi : \bar{\mathbb{D}} \to G_p$, such that

$$\psi(L_{t_n}) = \bar{G}_p \bigcap L_{p_n},$$
$$\psi(z^2) = s_p^m \circ \psi(z),$$
$$\psi \circ s_{t_n} = s_p^m \circ \psi \text{ on } G.$$

ψ is a homeomorphism on G_p. Moreover, ψ^{-1} extends continuously to map $\bar{G}_p$ to $\bar{\mathbb{D}}$, map leaves on ∂G_p to points. Then $R_n \to R_*$ as $n \to \infty$ by the theory of polynomial-like mappings because $s_{p_n} \simeq s_p \vdash s_{t_n}$ where $t_n \to 0$ as $n \to \infty$.

The tuning case, that is, Theorem 4.2.4 is proved in Section 12.4. Let

ϕ_n be a semiconjugacy between $s_{p_n} \amalg s_q$ and R_n by Section 2.4, that is,

$$\phi_n \circ (s_{p_n} \amalg s_q) = R_n \circ \phi_n.$$

Let ϕ, ϕ_* be semiconjugacies such that

$$\phi \circ (s_p \amalg s_q) = R \circ \phi,$$

$$\phi_* \circ (s_p \amalg s_q) = R_* \circ \phi_*. \tag{31}$$

ϕ_* exists by Proposition 2.11.1. $\phi_*(S^1) = J(R_*)$. $\phi(e^{2\pi ip})$ is in the interior of a set of $\mathcal{P}_0(R)$ (the initial partition pieces for R). $\phi_*(e^{2\pi ip})$ is in the interior of a set of $\mathcal{P}_0(R_*)$, so for n large enough, $\phi(e^{2\pi ip_n})$ and $\phi_*(e^{2\pi ip_n})$ are both in the interior of some sets of $\mathcal{P}_0(R)$ and $\mathcal{P}_0(R_*)$.

Now let

$$\mathcal{P}_n(R_*) = \bigvee_{i=0}^{n} R_*^{-i}(\mathcal{P}_0(R_*)) = \{\cap_{i=0}^{n} P_{j_i} : P_{j_i} \in R_*^{-i}(P), P \in \mathcal{P}_0(R_*)\},$$

$$\mathcal{P}_n(R) = \bigvee_{i=0}^{n} R^{-i}(\mathcal{P}_0(R)).$$

Some of the ends of partitions in $\bigvee_{i=0}^{\infty} R_*^{-i}(P_0(R_*))$ do not shrink to points. For example there is a sequence $\{P_n\}$ with $P_n \in \mathcal{P}_n(R_*)$ such that as $n \to \infty$, P_n tends to the Fatou component with $\phi_*(e^{2\pi ip})$ on its boundary. Here P_n is the set in $\mathcal{P}_n(R_*)$ which contains the Fatou component with $\phi_*(e^{2\pi ip})$ on its boundary.

In the following Section 12.5 we will construct a sequence of closed nested topological discs $B_n(R_*), n \in \mathbb{N}$, with $\phi_*(e^{2\pi ip})$ on their boundaries. None of these discs is a single partition piece in $\mathcal{P}_n$, but they will be constructed with boundaries in the union of the boundary of U_* and of the graph $\cup_n G_n$. They are closures of the union of some partition pieces of level greater than some integer N. Here, G_n is the graph such that the partition pieces in $\mathcal{P}_n$ are the closures of components of $\bar{\mathbb{C}} \setminus G_n$. We can show that the sequence $B_n(R_*)$ shrinks to the point $\phi_*(e^{2\pi i p})$ as $n \to \infty$ by Lemma 5.2.4. Then we transfer the shrinking from the dynamical plane of R_* to some induced nested sets MB_n on the parameter plane by some method. At last we show the rational map $R_n \simeq s_{p_n} \amalg s_q$ is in some set MB_{N_n} such that $N_n \to \infty$ as $n \to \infty$. This is the idea of the whole proof.

12.3 Persistent Markov partitions for rational maps near R_*

Now denote by v the point in the parabolic cycle of R_* of period m such that

$$\lim_{n\to\infty} R_*^{mn+1}(c_1) = v$$

and hence $R_*^m(v) = v$. Using the partition introduced in the last section, one can see that the nested partition pieces of all levels containing v tend to the Fatou component U_* with v on its boundary. The set in the intersection of all levels of dynamical partitions containing $R_*(c_1)$ is a copy of the Mandelbrot set by the theory of polynomial-like mappings [DH1], as we shall see in Section 12.4.

The base partition graph

$$G_0(R_x) = \bigcup_{P\in\mathcal{P}_0(R_x)} \partial P$$

varies continuously to $G_0(R_*)$ in a neighbourhood of R_* according to Lemma 12.3.2. For each n, $G_n(R_x)$ varies continuously to $G_n(R_*)$ in a neighbourhood of R_*. These neighbourhoods are getting smaller and smaller as $n \to \infty$.

Let X be the parameter slice $Per_{m'}(0)$ containing the parameter x_* which parametrizes the rational map R_*. One can take the example of the parameter a-plane in [AY] as a special example for $Per_2(0)$. Now we quote two results (Corollary 1.2 and Lemma 2.1) from [Ree4] which will be used to form our base partition $\mathcal{P}_0(R_*)$ and persistent partitions $\mathcal{P}_0(R_x)$ for x in a small neighbourhood of x_* on X.

Theorem 12.3.1. *Let $f : \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$ be a rational map with connected Julia set J, such that the forward orbit of each critical point is attracted to an attractive or parabolic periodic orbit, the closure of any Fatou component is a closed topological disc, and all of these are disjoint. Then there exists a graph $G \subset \bar{\mathbb{C}}$ such that the following hold.*

- *$G \subset f^{-1}(G)$.*
- *G does not intersect the closure of any Fatou component.*
- *All components of $\bar{\mathbb{C}} \setminus G$ are topological discs.*
- *Any component of $\bar{\mathbb{C}} \setminus G$ contains at most one periodic Fatou component of f.*
- *The boundary of any component of $\bar{\mathbb{C}} \setminus G$ is a quasi-circle.*

In particular, the set of closures of components of $\bar{\mathbb{C}} \setminus G$ is a Markov partition for f.

By this theorem the closures of components of $\bar{\mathbb{C}} \setminus G$ gives the foundation level partition of a Markov partition for R_*, which we call $\mathcal{P}_0$. The condition on rational maps with disjoint closures of Fatou components for existence of the Markov partitions imposes a restriction on our result. The Markov partition persists for nearby maps R_x by the following lemma (Lemma 2.1 [Ree4]).

Lemma 12.3.2. *Let f be a rational map. Let $G \subset \bar{\mathbb{C}}$ be a graph, and U a connected closed neighbourhood of G such that the following hold.*

- $G \subset f^{-1}(G)$.
- *U is disjoint from the set of critical values of f.*
- *U contains the component of $f^{-1}(U)$ containing G, and, for some $N > 0$, U^o contains the component of $f^{-N}(U)$ containing G.*

Then for all g sufficiently close to f in the uniform topology, the properties above hold with g replacing f and a graph $G(g)$ isotopic to the graph $G = G(f)$ above, varying continuously with g.

In particular, these properties hold for nearby g, if f is a rational map such that the forward orbit of every critical point is attracted to an attractive or parabolic periodic orbit, the closures of any two periodic Fatou components are disjoint, and G is a graph with the properties above, which is also disjoint from the closure of any periodic Fatou component.

Let U_* be the Fatou component containing $R_*(c_1(R_*))$ on its boundary. Now write

$$W_i = W_i(R_*)$$

for the set in $\mathcal{P}_i = \mathcal{P}_i(R_*)$ which contains U_*. Thus $W_{i+1}(R_*) \subset W_i(R_*)$ and $W_{i+j}(R_*)$ is a component of $R_*^{-jm}(W_i(R_*))$. Now we introduce part of Corollary 1.2 of [Ree4], by perturbing the graph G of [Ree4] near U_* to a different graph in $\cup_{n\geq 0} R_*^{-n}(G)$, if necessary.

Let XW_i be the set of parameters x such that $\partial G_i(R_x)$ varies isotopically for nearby maps and $R_x(c_1(x)) \in W_i(R_x)$. Thus, $x_* \in XW_i$ for all i, and there is a homeomorphism

$$h_x^i : G_i(R_*) \cup \{R_*(c_1(R_*))\} \to G_i(R_x) \cup \{R_x(c_1(R_x))\}$$

for $x \in XW_i$, which satisfies

$$h_x^i \circ R_* = R_x \circ h_x^i$$

on $G_i(R_*)$.

Theorem 12.3.3. *(from Theorem 3.2 of [Ree4])*

$$x \to h_x^{-i}(R_x(c_1(R_x)))$$

is a homeomorphism from ∂XW_i to $W_i(R_*)$. Consequently XW_i is a closed topological disc for all i, and $XW_{i+1} \subset int(XW_i)$.

As a consequence of the theorem, $\cap_{i \geq 0} XW_i$ is a compact set.

12.4 The case with a bounded minor leaf: proof of Theorem 4.2.4 following the Mandelbrot-like family theory

When all the minor leaves are in a small copy of the Mandelbrot set, that is, the quadratic polynomials $f_n \simeq s_{p_n}$ are all tunings and R_* has Fatou components of disjoint closures, we can deal with the continuity problem by Douady and Hubbard's theory of Mandelbrot-like families [DH1].

Proof of Theorem 4.2.4:

Proof. Note that

$$(R_*^m, int(W_m(R_*)), int(W_0(R_*)))$$

is a polynomial-like mapping in the sense of [DH1], that is,

$$W_m(R_*) \subset int(W_0(R_*))$$

and

$$R_*^m : W_m(R_*) \to W_0(R_*)$$

is a degree two branched covering. For $x \in XW_m$,

$$(R_x^m, int(W_m(R_x)), int(W_0(R_x)))$$

is also a polynomial-like mapping. The Mandelbrot-like set MR_* is $\cap_{i\geq 0}XW_i$, which is compact by Theorem 12.3.3. The map

$$\chi : int(XW_m) \to \mathbb{C}$$

in [DH1] is continuous by Proposition 14 of [DH1] and holomorphic over M, by Theorem 4 of [DH1]. By construction, this map χ is such that, if $R_x \simeq s_r \amalg s_q$ for some tuning s_r of s_p, then $x \in MR_*$ and $\chi(x) = c$ where $f_c \simeq s_r$, $f_c(z) = z^2 + c$. As commented in Section 4 of [DH1], if MR_* is compact, then $\chi : MR_* \to M$ is a ramified cover of some degree. But χ is injective restricted to the critically periodic maps in MR_* , because these are precisely those maps R_x with $R_x \simeq s_r \amalg s_q$ for an odd denominator rational r such that s_r is a tuning of s_p. So χ is of degree one on MR_* and a homeomorphism from MR_* to M. Now let $\lim_{n\to\infty} p_n = p$ with s_{p_n} being a tuning of s_p. Let $f_n \simeq s_{p_n}$ where $f_n \to f_*$ as $n \to \infty$, in which f_* is the parabolic map on the boundary of the hyperbolic component containing $f \simeq s_p$ with its parabolic point of the least possible period. Then

$$R_n = \chi^{-1}(f_n) \to \chi^{-1}(f_*) = R_*$$

as $n \to \infty$. □

12.5 Construction and shrinking of the nested sets $B_n(R_*)$ on the dynamical plane

First consider the Fatou component U_* of R_*. We distinguish elements of the graph $G(R_*) = \cup_n G_n(R_*)$ by *lateral* or *vertical* ones as the following. There are arcs on the graph which are parts of some closed loop in $G(R_*)$ enclosing U_* at some level j. We call them the (level j) *lateral* edges of the graph $G(R_*)$. Denote the collection of them by Γ_L. Denote the set of $\mathcal{P}_0(R_*)$ containing U_* by $P_0(R_*)$. The edges which are not part of any closed loop enclosing U_* at any level are called the *vertical* edges. Denote the collection by Γ_V.

Notice that there is a homeomorphism $\phi : S^1 \to \partial U_*$ such that

$$\phi(z^2) = R_*^m \circ \phi(z).$$

Choose some periodic points on S^1 near 1, for example, $e^{\pm\frac{2\pi i}{99}}$. We claim that (this statement is for the particular number $\frac{1}{99}$, in fact it works for any odd denominator rational number)

Lemma 12.5.1. *There are arcs γ_+ and γ_- composed of edges in $\Gamma_L \cup \Gamma_V$, such that*

$$\gamma_\pm \subset \bigcup_n G_n(R_*),\ \gamma_+ \subset R_*^{99m}(\gamma_+), \gamma_- \subset R_*^{99m}(\gamma_-),$$

γ_+ lands on $\phi(e^{\frac{2\pi i}{99}})$ while γ_- lands on $\phi(e^{-\frac{2\pi i}{99}})$.

Proof. First, choose a vertex $y_{0,+}$ on some edge in Γ_L. Let $\omega_j, j \geq 0$ be the closed loop composed of edges in Γ_L of G_{jm} surrounding U_*. $\omega_0 = \partial P_0(R_*)$. $R_*^m(\omega_{j+1}) = \omega_j$ and the map $R_*^m : \omega_{j+1} \to \omega_j$ is a covering map of degree 2. ω_{j+1} separates ∂U_* from ω_j.

Now choose a vertical edge $\alpha_{0,+}$ between ω_0 and ω_1 in the repelling petal of v without loss of generality. $R_*^{-jm}(\alpha_{0,+})$ has 2^j components joining ω_j to ω_{j+1}. Define the arc

$$\beta_{0,+} \subset (\cup_{j=0}^{98} R_*^{-jm}(\alpha_{0,+})) \cup (\cup_{j=0}^{98} \omega_j)$$

according to the following rules. First observe the orbit of $e^{\frac{2\pi i}{99}}$ under the doubling map (we omit $e^{\frac{2\pi i}{99}}$ as there will be no ambiguity):

$1 \to 2 \to 4 \to 8 \to 16 \to 32 \to 64 \to 29 \to 58 \to 17 \to 34 \to 68 \to 37 \to 74 \to 49 \to 98 \to 97 \to 95 \to 91 \to 83 \to 67 \to 35 \to 70 \to 41 \to 82 \to 65 \to 31 \to 62 \to 25 \to 50 \to 1.$

Join $\alpha_{0,+}$ with the sets $R_*^{-jm}(\alpha_{0,+})$ according to the orbit of $e^{\frac{2\pi i}{99}}$ under the doubling map. Note that $R_*^{-m}(\alpha_{0,+})$ splits the region between ω_1 and ω_2 into 2 parts, and $R_*^{-jm}(\alpha_{0,+})$ splits the region between ω_j and ω_{j+1} into 2^j parts. The 2^j components of $R_*^{-jm}(\alpha_{0,+})$ between ω_j and ω_{j+1} are ordered in the same way as the points $e^{2\pi i \frac{k}{2^j}}, 0 \leq k < 2^j$ on S^1, so choose $\alpha_{j,+}$ to be the component of $R_*^{-99jm}(\alpha_{0,+})$ which corresponds to the nearest point of $\frac{k}{2^j}$ with respect to $\frac{2^{99-j}}{99} \mod 1$ for $0 \leq k < 2^j$. Let $\beta_{0,+} \subset (\cup_{j=0}^{98}\alpha_{j,+}) \cup (\cup_{j=0}^{98}\omega_j)$ and $\beta_{0,+}$ joins $y_{0,+}$ and a point $y_{1,+} \in R_*^{-99m}(y_{0,+})$. Then inductively define $\beta_{j+1,+}$ to be the component of $R_*^{-99m}(\beta_{j,+})$ which shares an endpoint with $\beta_{j,+}$ for $j \geq 0$. Repeat this process infinitely many times. Now let

$$\gamma_+ = \cup_{j=0}^{\infty}\beta_{j,+},$$

where $\beta_{j,+} \subset \cup_{j=0}^{98}G_{j+k}(R_*)$ and $R_*^{99m}(\beta_{j+1,+}) = \beta_{j,+}$. We claim that $diam(\beta_{j,+}) \to 0$ as $j \to \infty$ because since $\beta_{j,+}$ is bounded away from the critical orbits, it is contained in a small open set W on which $\{R_*^{-km} : k \in \mathbb{N}\}$ is a normal family when j is large enough. So $diam(R_*^{-km}(W)) \to 0$ as $k \to \infty$, which implies $\lim_{j\to\infty} diam(\beta_{j,+}) = 0$. Since any accumulation point is fixed by R_*^{99m}, they must accumulate on a single periodic point, which is $\phi(e^{\frac{2\pi i}{99}})$. The arc

$$\gamma_- = \cup_{j=0}^{\infty}\beta_{j,-}$$

lands on $\phi(e^{-\frac{2\pi i}{99}})$ where $\beta_{j,-} \subset \cup_{j=0}^{98}G_{j+k}(R_*)$ and $R_*^{99m}(\beta_{j+1,-}) = \beta_{j,-}$ is constructed similarly with $\beta_{j,+}$. So $\beta_{j+1,\pm}$ is determined by $\beta_{j,\pm}$. Denote the two endpoints of $\beta_{j,\pm}$ by $y_{j,\pm}$ and $y_{j+1,\pm}$. All the $\beta_{j,\pm}$ are determined by $\beta_{0,\pm}$.

$\square$

Now we begin to construct the nested sequence of closed topological discs $B_j(R_*)$. From now on suppose v is a primitive parabolic point which has exactly one attracting petal and one repelling petal for simplicity.

As we mainly care about the dynamics near the parabolic point, and the Fatou component U_* is the limit of all levels of the partition pieces which contains it, we can suppose both γ_+ and γ_- are in the ϵ-neighbourhood of the parabolic point v, $\beta_{j,+}$ and $\beta_{j,-}$ are on the same level of the graph.

In order to construct the sequence, denote the shorter arc on ∂U_* joining $\phi(e^{\frac{2\pi i}{99}})$ and $\phi(e^{-\frac{2\pi i}{99}})$ by $\gamma_{0,<}$. It is an arc containing the parabolic point v in its interior. Write $\gamma_{j,<} = R_*^{-jm}(\gamma_{0,<}), j \in \mathbb{N}$, and choose R_*^{-m} the inverse branch such that $R_*^{-m}(v) = v$ (in the following without special declaration we always mean this branch by R_*^{-m}). Obviously the two points $\phi(e^{\frac{2\pi i}{99}})$ and $\phi(e^{-\frac{2\pi i}{99}})$ converge to v from two sides under iterations of R_*^{-m}, so $\gamma_{j,<}$ contains v in its interior for all $j \in \mathbb{N}$. Denote the shorter arc on ω_0 joining the two end points of $\beta_{0,+}$ and $\beta_{0,-}$ on ω_0 by $\gamma_{0,>}$. Now consider the dynamics of R_*^{-m} in the topological disc $B_0(R_*)$ bounded by $\gamma_+, \gamma_-, \gamma_{0,<}$ and $\gamma_{0,>}$. Suppose $B_0(R_*)$ is in the ϵ-neighbourhood of v. Then let

$$B_{j+1}(R_*) = R_*^{-(j+1)m}(B_0(R_*)) \subset R_*^{-jm}(B_0(R_*)) = B_j(R_*) \tag{32}$$

for $j \in \mathbb{N}$.

Now we want to use Lemma 5.2.4 to show the shrinking of the sequence $B_n(R_*)$ to the point v. For the nested sequence $B_{j+1}(R_*) \subset B_j(R_*), j \in \mathbb{N}$, apply Lemma 5.2.4 by letting $B_n = B_n(R_*)$, $C_n = C_n(R_*)$ be the closure of the set $B_n(R_*) \setminus B_{n+1}(R_*)$, $n \in \mathbb{N}$. Both $C_n(R_*)$ and $B_n(R_*)$ are unions of some partition pieces whose levels are greater than some integer N, moreover,

$$C_{n+1}(R_*) = R_*^{-m}(C_n(R_*)), n \in \mathbb{N}, n \geq 1.$$

Then $Mod(C_n(R_*)) = c > 0$ is a constant, which implies

$$\sum_{n=0}^{\infty} Mod(C_n(R_*)) = \infty. \tag{33}$$

So $B_n(R_*)$ converges to v by Lemma 5.2.4.

12.6 Convergence of the nested sets XB_n on the parameter plane X

Now apply Lemma 12.3.2 to $f = R_*$. Following partitions on the dynamical planes we define partitions on the parameter plane X (in fact only in a small neighbourhood of R_* is enough for us). Denote by x_* the parameter corresponding to the map R_*, and by x the parameter corresponding to the map R_x on the parameter plane X. What we do is similar to the definition of the partitions of parameter space of the $\frac{p}{q}$ limb of M in Part III, [Hub]. By the persistent property of the Markov partitions for maps near R_*, for a depth n partition piece $P^j_{n,x_*}, n, j \in \mathbb{N}$, there is a corresponding one $P^j_{n,x}$ for R_x depending continuously on x by lemma 12.3.2. Now define

$$XP^j_{n,x_*} = \{x \in X : R_x(c_1(x)) \in P^j_{n,x}\},$$

and

$$\partial XP^j_{n,x_*} = \{x \in X : R_x(c_1(x)) \in \partial P^j_{n,x}\}.$$

$\cup_{n,j\in\mathbb{N}} XP^j_{n,x_*}$ and $\cup_{n,j\in\mathbb{N}} \partial XP^j_{n,x_*}$ form a partition in a neighbourhood of x_* on X plane.

In order to show the convergence on the X plane, we construct a nested sequence of topological discs XB_n. As the Markov partitions persist for maps near R_*, the curve γ_+ corresponds to a curve $\gamma_{+,x}$ depending continuously on x in a neighbourhood of x_*. For simplicity we still denote this curve by γ_+ if there are no ambiguity, so for γ_- and $\gamma_{0,>}$. Now let

$$X\gamma_{j,+} = \{x \in X : R_x(c_1(x)) \in R_x^{-jm}(\gamma_+) = \gamma_{j,+}\}, j \in \mathbb{N},$$

$$X\gamma_{j,-} = \{x \in X : R_x(c_1(x)) \in R_x^{-jm}(\gamma_-) = \gamma_{j,-}\}, j \in \mathbb{N},$$

$$X\gamma_{j,>} = \{x \in X : R_x(c_1(x)) \in R_x^{-jm}(\gamma_{0,>}) = \gamma_{j,>}\}, j \in \mathbb{N}.$$

Denote by M_x the closure of the mating components on X, then $X\gamma_{j,+}$ is a ray landing on a point $x_{j,+} \in M_x$. By the theory of polynomial-like mappings in [DH1] as explained in Section 12.4, there is a homeomorphism

$$\Phi_{x_*} : M \to M_{x_*}$$

where $x_* \in M_{x_*} \subset M_x$ and the homeomorphism has the following properties.

- Φ_{x_*} maps hyperbolic components to hyperbolic components.
- Φ_{x_*} maps critically finite maps to critically finite maps.
- Φ_{x_*} maps parabolic maps to parabolic maps. Moreover, If $R_{x_1} \simeq (s_p \vdash s_{r_1}) \amalg s_q$, $s_{r_1} \simeq f_{a_1}(z) = z^2 + a_1$, then

$$\Phi_{x_*}(a_1) = x_1.$$

$x_{j,\pm}$ are the images of the endpoints of the ray of argument $\pm\frac{2\pi}{2^j 99}$ under Φ_{x_*}. Yoccoz showed that there are only finitely many limbs of M with diameters greater than a fixed number ϵ. Let $t_{j,\pm}$ be the roots of the limbs containing the endpoints of the rays of argument $\pm\frac{2\pi}{2^j 99}$, let

$$\Phi_{x_*}(t_{j,+}) = x'_{j,+},\ \Phi_{x_*}(t_{j,-}) = x'_{j,-}.$$

C. Peterson and P. Roesch (Theorem 4.21 [PR]) showed (there is also an unpublished proof by Johannes Riedl in his Ph.D thesis [Rie]) that there is an arc in M joining $t_{j,+}$ and the endpoint of the ray of argument $\frac{2\pi}{2^j 99}$. Similarly, there is an arc in M joining $t_{j,-}$ and the endpoint of the ray of argument $-\frac{2\pi}{2^j 99}$. By Yoccoz's result diameters of these rays $\to 0$ as $j \to \infty$.

Now define $X\gamma'_{j,+}$ and $X\gamma'_{j,-}$ to be the images of these arcs under Φ_{x_*}. Thus $X\gamma'_{j,+}$ is an arc in M_{x_*} joining $x_{j,+}$ and $x'_{j,+}$, $X\gamma'_{j,-}$ is an arc in M_{x_*} joining $x_{j,-}$ and $x'_{j,-}$. Define $X\gamma''_j$ to be the image under Φ_{x_*} of the arc on the boundary of the main cardioid between $t_{j,+}$ and $t_{j,-}$. Since Φ_{x_*} is a homeomorphism, the diameters of $X\gamma'_{j,+}$, $X\gamma'_{j,-}$ and $X\gamma''_j$ tend to 0 as $j \to \infty$, so

$$\lim_{j\to\infty} diam(X\gamma'_{j,+} \cup X\gamma'_{j,-} \cup X\gamma''_j) = 0.$$

Figure 12 shows positions of the j and $j+1$-th parameter curves mentioned above.

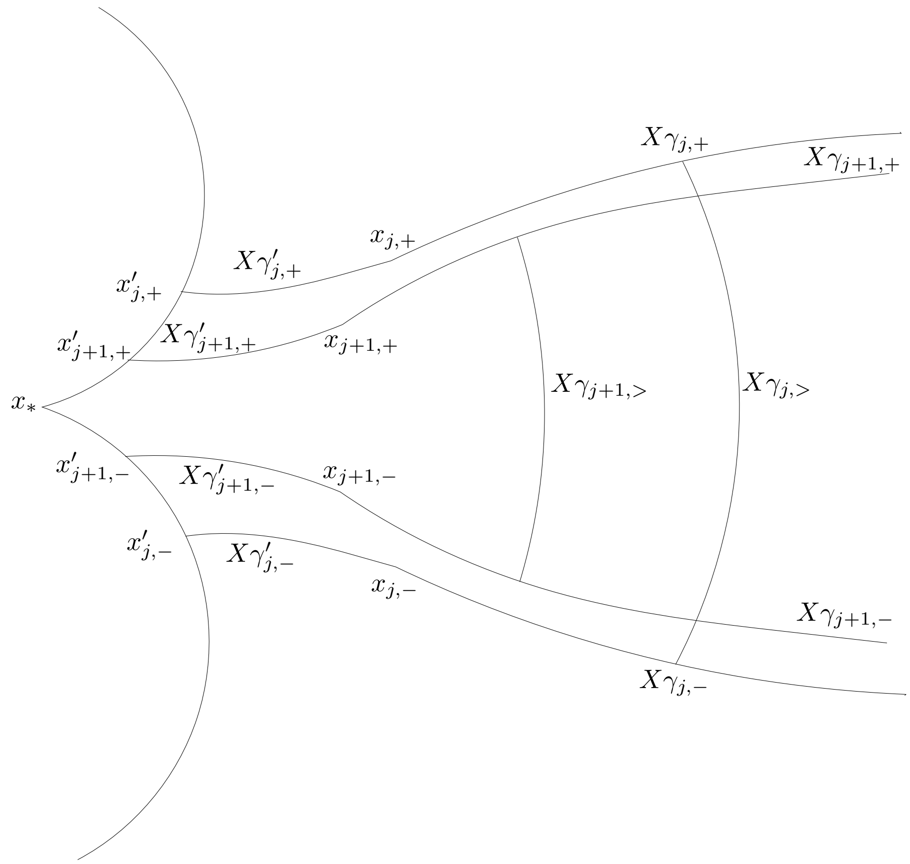

Figure 12: The parameter curves on X plane

Now we have

Theorem 12.6.1. $\lim_{j\to\infty} diam(X\gamma_{j,+} \cup X\gamma_{j,-} \cup X\gamma_{j,>}) = 0$

To show the theorem we first show that

Lemma 12.6.2. *Given $c > 0$, there exists $K = K_c$, such that the following holds. Let $A \subset \mathbb{C}$ be an annulus of modulus $> c$. Then there is a closed loop $\gamma \subset A$ which is homotopic to A, such that if for $x \in \gamma$, r_x denotes the radius of the largest disc centred at x and contained in A, then*

$$\frac{r_x}{diam(\gamma)} \geq K$$

for any $x \in \gamma$.

Proof. We use the Koebe distortion theorem. This theorem says that if $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and injective, then

$$\tfrac{1-r}{(1+r)^3} \leq \tfrac{|f'(z)|}{|f'(0)|} \leq \tfrac{1+r}{(1-r)^3} \ \forall |z| \leq r$$

and

$$(\tfrac{1-r}{1+r})^4 \leq \tfrac{|f'(z)|}{|f'(w)|} \leq (\tfrac{1+r}{1-r})^4 \ \forall |z|, |w| \leq r.$$

In particular, we can apply this to our case with $r = \frac{1}{2}$. Now suppose that $A \subset \mathbb{C}$ has modulus $c > 0$. The round annulus $\{z : 1 < |z| < e^{2\pi c}\}$ is conformally equivalent to A. The holomorphic map

$$F : \{z : 1 < |z| < e^{2\pi c}\} \to A$$

is uniquely determined up to composition with a rotation $z \to e^{i\theta} z$. Let $\gamma_0 = \{z : |z| = \frac{1+e^{2\pi c}}{2}\}$, let $\gamma = F(\gamma_0)$. Apply the Koebe distortion theorem to discs D centred on γ_0 and of radius $\frac{e^{2\pi c}-1}{2} = r_c$ with $f = F|_D$. f is univalent on the disc of radius r_c with centre $z_D \in \gamma_0$. It follows that

$$(\tfrac{1-r}{1+r})^4 \leq \tfrac{|f'(z)|}{|f'(w)|} \leq (\tfrac{1+r}{1-r})^4$$

for any $|z - z_D| \leq \lambda r_c, |w - z_D| \leq \lambda r_c$.

We can cover γ_0 by discs of radius r_c such that the distance between centres of consecutive discs is $\leq \frac{r_c}{3}$. γ_0 has radius $r_c + 1$ and length $2\pi(r_c + 1)$. So if N is the first integer $\geq \frac{2\pi(r_c+1)}{r_c}$, then γ_0 can be covered by N discs with centres on γ_0 of radius r_c and with consecutive centres of distance $\leq \frac{r_c}{3}$. Then every point in γ_0 is in one of these discs centred on γ_0 of radius r_c with distance $\leq \frac{r_c}{3}$ from the center. So for any $z \in \gamma_0$, there exists D_i with center z_i such that

$$|z - z_i| \leq \tfrac{r_c}{3}$$

and

$$\{w : |w - z| \leq \tfrac{r_c}{3}\} \subset \{z : |z - z_i| \leq \tfrac{2r_c}{3}\}.$$

So $\{w : |w - z| \leq \frac{r_c}{3}, z \in \gamma_0\} \subset \cup_{i=1}^{N}\{w : |w - z_i| \leq \dfrac{2r_c}{3}\}$. Then for any $w_1, w_2 \in \cup_{i=1}^{N}\{w : |w - z_i| \leq \dfrac{2r_c}{3}\}$, we have

$$|z_i - z_{i+1}| \leq \frac{r_c}{3}, \frac{1}{3^4} \leq \frac{|F'(z_i)|}{|F'(z_{i+1})|} \leq 3^4.$$

So for all $1 \leq i \leq j \leq N$ we have

$$(\frac{3}{2})^{-4|j-i|} \leq \frac{|F'(z_i)|}{|F'(z_{i+1})|} \leq (\frac{3}{2})^{4|j-i|}.$$

Then for any $w_1 \in D_i$ and $w_2 \in D_j$ with $|w_1 - z_i| \leq \frac{2r_c}{3}, |w_2 - z_j| \leq \frac{2r_c}{3}$, we have

$$3^{-4|j-i|-8} \leq \frac{|F'(w_1)|}{|F'(w_2)|} \leq 3^{4|j-i|+8}.$$

In particular this is true for all $w_1, w_2 \in \{w : |w - z| \leq \frac{r_c}{3}$ for some $z \in \gamma_0\}$. Then we get the constant K_c. $\square$

Now we continue to prove 12.6.1.

Proof of Theorem 12.6.1:

Proof. First, $x_{j,+} \to x_*$ and $x_{j,-} \to x_*$ as $j \to \infty$. This is because the endpoints of the rays of arguments $\pm\frac{2\pi}{2^j 99}$ tend to the parameter x_* as $j \to \infty$ (by Yoccoz's result), apply the homeomorphism Φ_{x_*}, it suffices to shows that

$$Mod(XB_j \setminus (X\gamma_{j+1,+} \cup X\gamma_{j+1,-} \cup X\gamma_{j+1,>}))$$

is bounded from 0 for all j. Now let γ be the closed loop in Lemma 12.6.2, let A be the annulus $XB_j \setminus (X\gamma_{j+1,+} \cup X\gamma_{j+1,-} \cup X\gamma_{j+1,>})$. Then

$$diam(\gamma) \geq$$
$$diam(X\gamma_{j+1,+} \cup X\gamma_{j+1,-} \cup X\gamma_{j+1,>}) - Max\{|x_{j+1,-} - x_*|, |x_{j+1,+} - x_*|\}.$$

For at least one $x \in \gamma$, $r_x \leq |x_{j,+} - x_{j+1,+}|$, so the theorem follows. $\square$

Now we get a nested sequence of closed topological discs $XB_j, j \geq 0$ bounded by

$$X\gamma_{j,+} \cup X\gamma_{j,-} \cup X\gamma_{j,>} \cup X\gamma_{j,+}{}' \cup X\gamma_{j,-}{}' \cup X\gamma_j{}''.$$

Let XC_j be the closure of $XB_j \setminus XB_{j+1}$. For all $x \in int(XB_j)$, $\gamma_{j,+} \cup \gamma_{j,-} \cup \gamma_{j,>}$ varies holomorphically with x, for all $x \in int(XB_{j+1})$, $\gamma_{j+1,+} \cup \gamma_{j+1,-} \cup \gamma_{j+1,>}$ varies holomorphically with x. Now we prove

Theorem 12.6.3. *The modulus*

$$Mod(\bar{\mathbb{C}} \setminus (\gamma_{j,+} \cup \gamma_{j,-} \cup \gamma_{j,>} \cup \gamma_{j+1,+} \cup \gamma_{j+1,-} \cup \gamma_{j+1,>}))$$

is bounded from 0*, and it follows immediately that*

$$Mod(XB_j \setminus (X\gamma_{j,+} \cup X\gamma_{j,-} \cup X\gamma_{j,>} \cup X\gamma_{j+1,+} \cup X\gamma_{j+1,-} \cup X\gamma_{j+1,>})) > C > 0$$

for some constant C*.*

Proof. Let

$$\gamma_{j,+}(x) \cup \gamma_{j,-}(x) \cup \gamma_{j,>}(x) \cup \gamma_{j+1,+}(x) \cup \gamma_{j+1,-}(x) \cup \gamma_{j+1,>}(x) = \Gamma_j(x),$$
$$X\gamma_{j,+} \cup X\gamma_{j,-} \cup X\gamma_{j,>} \cup X\gamma_{j+1,+} \cup X\gamma_{j+1,-} \cup X\gamma_{j+1,>} = X\Gamma_j.$$

First we construct a map

$$\varphi : U \setminus \Gamma_j(x_*) \to XB_j \setminus X\Gamma_j$$

s.t. $U \setminus \Gamma_j(x_*)$ has modulus bounded from 0 for some topological disc U. Note that

$$\Gamma_j(x_*) = \gamma_{j,+} \cup \gamma_{j,-} \cup \gamma_{j,>} \cup \gamma_{j+1,+} \cup \gamma_{j+1,-} \cup \gamma_{j+1,>}.$$

We will make φ of bounded distortion. Note that

$$h_j^x : \Gamma_j(x_*) \to \Gamma_j(x)$$

varies holomorphically with respect to x.

We are going to show that there is an injective quasi-conformal map

$$H_j : XB_j \setminus XB_{j+1} \to B_j(x_*) \setminus B_{j+1}(x_*)$$

with bounded distortion K on their interiors. The closure of $B_j(x_*) \setminus B_{j+1}(x_*)$ has modulus bounded from 0. Note that

$$h_j^x : B_j(x_*) \setminus B_{j+1}(x_*) \to B_j(x) \setminus B_{j+1}(x),\ x \notin XB_{j+1}$$

is $q-c$ of distortion k_1. Moreover the family h_j^x depend holomorphically on x. h_j^x are defined inductively by

$$R_x^{jm} \circ h_j^x = h_0^x \circ R_x^{jm} \tag{34}$$

where

$$h_0^x : B_0(x_*) \setminus B_1(x_*) \to B_0(x) \setminus B_1(x)$$

is a q-c map of bounded distortion. h_0^x conjugates $\partial B_0(x_*)$ to $\partial B_0(x)$ for $x \in XB_0$. By equation (34), h_j^x conjugates $\partial B_j(x_*)$ to $\partial B_j(x)$ for $x \in XB_j$.

Now let

$$H_j(x) = (h_j^x)^{-1}(R_x(c_1(x))). \tag{35}$$

One can see from (34) that all h_j^x have the same distortion for $j \geq 0$ as R_x^{jm} is holomorphic. $H_j(x)$ satisfies

$$|\frac{\bar{\partial} H_j}{\partial H_j}|(x) = |\frac{\bar{\partial} h_j}{\partial h_j}|(R_x(c_1(x))). \tag{36}$$

To see this, let $F(x), G(x)$ be any two complex function with $x = x_1 + ix_2$, we need to compute $\partial(F \circ G(x))$ and $\bar{\partial}(F \circ G(x))$. In order to do this, write

$$F = u(x_1, x_2) + iv(x_1, x_2), DF = \begin{pmatrix} u_{x_1} & u_{x_2} \\ v_{x_1} & v_{x_2} \end{pmatrix}.$$

Then

$$\partial F(x) = \tfrac{1}{2}(F_{x_1} - iF_{x_2}) = \tfrac{1}{2}(u_{x_1} - v_{x_2} + i(v_{x_1} + u_{x_2}))$$

identifies with the matrix

$$\tfrac{1}{2}\begin{pmatrix} u_{x_1} + v_{x_2} & u_{x_2} - v_{x_1} \\ v_{x_1} - u_{x_2} & u_{x_1} + v_{x_2} \end{pmatrix}.$$

Similarly, $\bar{\partial} F(x) = \frac{1}{2}(F_{x_1} + iF_{x_2}) = \frac{1}{2}(u_{x_1} + v_{x_2} + i(v_{x_1} - u_{x_2}))$ identifies with the matrix

$$\tfrac{1}{2}\begin{pmatrix} u_{x_1} - v_{x_2} & -u_{x_2} - v_{x_1} \\ v_{x_2} + u_{x_1} & u_{x_1} - v_{x_2} \end{pmatrix}.$$

The map $z \to \bar{z}$ identifies with

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so $DF = \partial F + \bar{\partial} F J$. From this we can work out that

$$D\partial(F \circ G) = \partial F(G(z))\partial G(z) + \bar{\partial} F(G(z)) J \bar{\partial} G(z) J,$$
$$D\bar{\partial}(F \circ G) = \partial F(G(z))\bar{\partial} G(z) J + \bar{\partial} F(G(z)) J \partial G(z) J.$$

Then by (35), $H_j \circ (h_j^x) = R_x^m(c_1(x))$, so

$$\bar{\partial} H_j J \partial h_j^x + \partial H_j \bar{\partial} h_j^x J = 0,$$
$$\bar{\partial} H_j J \partial h_j^x J + \partial H_j \bar{\partial} h_j^x = 0,$$

so

$$\frac{\bar{\partial} H_j}{\partial H_j} = -\frac{\bar{\partial} h_j^x}{J \partial h_j^x J}. \tag{37}$$

By taking modulus both sides one gets (36). So in order to bound $||H_j|| = |\frac{\bar{\partial} H_j(x)}{\partial H_j(x)}|$, we only need to bound $|\frac{\bar{\partial} h_0}{\partial h_0}|$. The distortion of

$$h_0^x : B_0(x_*) \setminus B_1(x_*) \to B_0(x) \setminus B_1(x)$$

is bounded. In fact the distortion of

$$h_0^x : (\mathbb{C}, \Gamma_0(x_*)) \to (\mathbb{C}, \Gamma_0(x))$$

is bounded, so the theorem follows.

□

Combining Theorem 12.6.1, Theorem 12.6.3 and Lemma 5.2.4 we have

Theorem 12.6.4. *The whole set XB_j shrinks to x_* as $j \to \infty$, that is*

$$\lim_{j \to \infty} diam(XB_j) = 0.$$

Now define

$$H(x) : XB_0 \to B_0(x_*)$$

by $H(x) = H_j(x)$ for $x \in XB_j \setminus XB_{j+1}$ for later use.

12.7 Proof of Theorem 4.2.5

We write S_m for the two-valued local inverse of R_*^m on P_0 with $S_m(P_0) \subset P_0$. We identify the unit circle with $\mathbb{R}/\mathbb{Z}$ and with the unit interval $[0,1]/(0 \sim 1)$ with 0 and 1 identified. We write

$$T(x) = 2x \mod 1$$

for $x \in [0,1]/(0 \sim 1)$. We assume that the closures of Fatou components of R_* are disjoint (which implies that the same is true for R). p is of period m under T. We write $L = L_p \cup L_q^{-1}$ and $\sim_L$ for the associated equivalence class on S^1. We write T_m for the local degree one inverse of T^m, defined in a neighbourhood of p, with $T_m(p) = p$.

Since $R \simeq s_p \amalg s_q$, there is a continuous map

$$\varphi_* : [0,1]/(0 \sim 1) \to J(R_*),$$

where $J(R_*)$ denotes the Julia set of R_*, such that

$$\varphi_* \circ T(x) = R_* \circ \varphi_*(x) \ \forall x \in [0,1]/(0 \sim 1).$$

Considering (31), we have

$$\varphi_*(x) = \phi_*(e^{2\pi i x}) \ \forall x \in [0,1]/(0 \sim 1).$$

Let p_n be a sequence of odd denominator rationals with $\lim_{n\to\infty} p_n = p$. Let R_n be the sequence of rational maps (uniquely determined up to Möbius conjugacies) with $R_n \simeq s_{p_n} \amalg s_q$ which are parametrised by parameters on the X plane. Recall that we aim to show that $R_n \to R_*$ as $n \to \infty$ (assuming some suitable Möbius conjugacies of R_n). We assume that s_{p_n} is not a tuning of s_p for any n by Theorem 4.2.4.

Now let Q_0 be the union of the components of $\varphi_*^{-1}(P_0)$ which intersect $\varphi_*^{-1}(\partial U_*)$. We have $Q_0 \subset (0,1)$. Assuming P_0 is a sufficiently small neighbourhood of U_*, Q_0 is a neighbourhood of $p \in (0,1)$. By the assumptions on the size of P_0, Q_0 is disjoint from $T^i(Q_0)$ for $0 < i < m$, and $T_m|_{Q_0}$ is of degree two. We write I_0 for the component of Q_0 containing p, so that I_0 is a closed interval with p in its interior. The set $\varphi_*^{-1}(\varphi_*(p))$ is a finite subset of S^1, which is the $\sim_L$-equivalence class of p. Let $I_{0,j}$ (for $1 \leq j \leq k$ for some $k \geq 2$) be the components of $\varphi_*^{-1}(P_0)$ which contain the points of $\varphi_*^{-1}(\varphi_*(p))$. We can number them so that there is a leaf of $L_p \cup L_q^{-1}$ connecting the points of $\varphi_*^{-1}(\varphi_*(p))$ in $I_{0,j}$ and $I_{0,j+1}$ for each $1 \leq j < k$. Then $I_0 = I_{0,j_0}$ for some $1 \leq j_0 \leq k$. The intervals are lined up in this way because the Fatou components of R_* have disjoint closures, and for the same reason the points of $\varphi_*^{-1}(\varphi_*(p))$ all have the same period as the set itself.

Since P_0 is a set in the Markov partition for R_*, we have that for any j, any component of $T^{-i}(\cup_{j=1}^{k} I_{0,j})$ is either contained in $\cup_{j=1}^{k} I_{0,j}$ or has interior disjoint from $\cup_{j=1}^{k} I_{0,j}$. Again, by assuming P_0 sufficiently small, we can assume that $T_m|_{\cup_{j=1}^{k} I_{0,j}}$ is a homeomorphism, and write T_m for the local inverse of T^m with $T_m(\varphi_*^{-1}(\varphi_*(p))) = \varphi_*^{-1}(\varphi_*(p))$.

Discarding the first part of the sequence if necessary, we can assume that $p_n \in T_m(I_0)$ for all n, which implies that $\varphi_*(p_n) = S_m(P_0)$for all n.

Now s_{p_n} not being a tuning of s_p is equivalent to p_n not being in the closure of the minor gap of L_p, and also to $\varphi_*(p_n)$ not being in ∂U_*. This means that there is a least $N_n \geq 1$ such that

$$\varphi_*(p_n) \in S_m^{N_n}(P_0),\ \varphi_*(p_n) \notin S_m^{N_n+1}(P_0).$$

Obviously N_n depends on n and

$$\lim_{n\to\infty} N_n = \infty. \tag{38}$$

By definition, $\varphi_*(p_n) \in S_m^j(P_0)$ for $0 \leq j \leq N_n$, and $p_n \notin T_m^{N_n+1}(I_0)$.

Lemma 12.7.1. *$p_n \in T_m^j(I_0)$ for $0 \leq j \leq N_n$.*

Proof. By assumption we have $p_n \in T_m(I_0)$, considering the case $j = 1$. Now we prove the general case $j \leq N_n$ by induction on j. So suppose that $1 \leq j < N_n$ and $p_n \in T_m^j(I_0)$ but $p_n \notin T_m^{j+1}(I_0)$. From $\varphi_*(p_n) \in S_m^{j+1}(P_0)$ we deduce that $p_n \in T^{-m}(T_m^j(I_0)) \cap Q_0 \subset T^{-m}(I_0) \cap Q_0$. But $T^{-m}(I_0) \cap Q_0$ has exactly two components, and from the assumption that $T_m|_{I_0}$ is a homeomorphism, the component which is not $T_m(I_0)$ is disjoint from I_0. So we have a contradiction and hence $p_n \in T_m^{j+1}(I_0)$, as required. □

Now write $P_1 = P_1(x_*)$ for the set in $\mathcal{P}_{m(N_n+1)}$ which contains $\varphi_*(p_n)$. There is an even integer $i_1 \geq 0$ such that

$$\varphi_*^{-1}(S_m^{i_1}(P_0)) \subset \varphi_*^{-1}(P_0)$$

and

$$\varphi_*^{-1}(\varphi_*(\cup_{j=1}^t T_m^{i_1/2}(I_{0,j}))) \subset \cup_{j=1}^t I_{0,j}.$$

Lemma 12.7.2. *If i_1 is sufficiently large, then there is $i_1' \leq i_1/4$ such that for any set $P \in \mathcal{P}_{i_1}(R_*)$ which intersects $T_m^{2i_1'}(\varphi_*(I_0))$, $P \neq S_m^{i_1}(P_0)$, we have*

$$\varphi_*^{-1}(P) \subset \varphi_*^{-1}(\varphi_*(T_m^{i_1'}(I_0))) \subset \cup_{j=1}^k I_{0,j}.$$

Proof. Given $\epsilon > 0$, we can choose N and t_1 such that all but N sets in the partition $\mathcal{P}_{t_1}(R_*)$ have diameters $< \epsilon$, and those that are not of diameter $< \epsilon$ are within ϵ neighbourhoods of the Fatou components. Then we can choose i_1' so that any set of $\mathcal{P}_{t_1}(R_*)$, apart from the one containing U_*, which intersects $T_m^{i_1'}(\varphi_*(I_0))$, is of diameter $< \epsilon$. Then we can choose ϵ so that any point w within 2ϵ neighbourhood of $\varphi_*(p)$ has $\varphi_*^{-1}(w) \subset \cup_{j=1}^k I_{0,j}$. Finally, we choose $i_1 > 4i_1'$. □

We assume from now on (as we may do, for p_n sufficiently close to p) that $N_n > i_1$. We write

$$P_1' = S_m^{N_n - i_1}(P_0),$$
$$I_{1,j} = T_m^{N_n - i_1}(I_{0,j}).$$

Then by Lemma 12.7.1 and 12.7.2 we have

$$\varphi_*^{-1}(P_1) \subset \varphi_*^{-1}(\varphi_*(I_{1,j_0})) \subset \bigcup_{j=1}^k I_{1,j} \subset \varphi_*^{-1}(P_1'), \tag{39}$$

and of course $P_1 \subset P_1'$.

There are subsets XP_1 and XP_1' on the parameter space X, such that for $x \in XP_1'$, the sets $\partial P_1(x)$ and $\partial P_1'(x)$ vary holomorphically and $R_x(c_1(x)) \in P_1'(x)$ for $x \in XP_1'$, while

$$XP_1 = \{x \in XP_1' : R_x(c_1(x)) \in P_1(x)\}.$$

Thus $I_{1,j}$ contains a point of $\varphi_*^{-1}(\varphi_*(p))$ for each $1 \leq j \leq k$. Let $k_1 > i_1 m_1$ be the least integer for which $R_*^{k_1}(\varphi_*(p_n)) \in P_1$, and write S_{k_1} for the local inverse of $R_*^{k_1}$ which maps $R_*^{k_1}(\varphi_*(p_n))$ to $\varphi_*(p_n)$. Then $S_{k_1}(P_1) \subset S_{k_1}(P_1') \subset P_1$ and $\cap_{i\geq 0} S_{k_1}^i(P_1)$ contains a single point of period k_1. Moreover, since $R_*^i(\varphi_*(p_n)) \notin P_1$ for $i_1 m < i < k_1$ and $R_*^i(S_{k_1}(P_1'))$ is a set in $\mathcal{P}_{m(N_n-i_1)+k_1-i}$, we have

$$R_*^i(S_{k_1}(P_1')) \cap P_1' = \emptyset \text{ for } i_1 m < i < k_1. \tag{40}$$

Since $P_1 \subset S_m^{N_n}(P_0) \setminus S_m^{N_n+1}(P_0)$, we also have

$$R_*^j(P_1) \cap P_1 = \emptyset \text{ for } 0 < j \leq i_1 m, \tag{41}$$

and hence

$$R_*^i(S_{k_1}(P_1')) \cap P_1 = \emptyset \text{ for } 0 < i \leq i_1 m. \tag{42}$$

So

$$\cap_{i\geq 0} S_{k_1}^i(P_1) = \cap_{i\geq 0} S_{k_1}^i(P_1') = \{z_1\}$$

for a point z_1 of period k_1 under R_*.

Let T_{k_1} be the local inverse of T^{k_1} defined on $\cup_{j=1}^k I_{1,j} \subset \varphi_*^{-1}(P_1')$ such that $T_{k_1}(T^{k_1}(p_n)) = p_n$ and extend T_{k_1} to map the union of leaves of $L_p \cup L_q^{-1}$ connecting points of $\varphi_*^{-1}(\varphi_*(T^{k_1}(p_n)))$ to the union of leaves of $L_p \cup L_q^{-1}$ connecting points of $\varphi_*^{-1}(\varphi_*(p_n))$.

Lemma 12.7.3.

$$\varphi_* \circ T_{k_1} = S_{k_1} \circ \varphi_* \textit{ on } \cup_{j=1}^k I_{1,j}, \tag{43}$$

and $\varphi_*^{-1}(z_1) \subset \cup_{j=1}^k I_{1,j}$ *is a finite set satisfying*

$$T^{k_1}(\varphi_*^{-1}(z_1)) = \varphi_*^{-1}(z_1), T^i(\varphi_*^{-1}(z_1)) \cap \varphi_*^{-1}(z_1) = \emptyset \textit{ for } 0 < i < k_1. \tag{44}$$

Moreover, if $y \in \varphi_*^{-1}(z_1)$ and $p' \in \varphi_*^{-1}(\varphi_*(p))$ with y, p' in $I_{1,j}$ for some j, then $T^i(y)$ is not between p' and y in $I_{1,j}$ for any $0 < i < k_1$.

Proof. (43) follows directly from $\varphi_* \circ T = R_* \circ \varphi_*$ and $T_{k_1}(T^{k_1}(p_n)) = p_n$. (44) follows from (42). From (40) we obtain

$$T^i(\cup_{j=1}^k I_{1,j}) \cap \cup_{j=1}^k I_{1,j} = \emptyset \text{ for } mi_1 < i < k_1, \tag{45}$$

so $T^i(y) \notin I_{1,j}$ for $i_1 m < i < k_1$. But for $0 < i \leq i_1 m$, $T^i(y) \notin I_{1,j}$ unless $m|i$. For $i = lm$ for some $0 < l \leq i_1$, $T^i(y) - p' = 2^{lm}(y - p')$ and $T^i(y)$ is not between p' and y. □

Now $\varphi_*^{-1}(z_1)$ has one point in each of the intervals $T_{k_1}(I_{1,j})$, and one of these intervals $T_{k_1}(I_{1,j})$ contains p_n, which is contained in I_{1,j_0} , the same interval which contains p. We write y_1 for the point of $\varphi_*^{-1}(z_1)$ in $T_{k_1}(I_{1,j_1})$. Thus we have $y_1 \in I_{1,j_0}$.

Lemma 12.7.4. *Suppose that $\varphi_*^{-1}(\varphi_*(p))$ contains at most three points, then y_1 either has period k_1 or period $2k_1$, and is not separated in I_{1,j_0} from p by any point in its forward orbit.*

Proof. The number of points in $\varphi_*^{-1}(\varphi_*(p))$ is k, since precisely one point of $\varphi_*^{-1}(\varphi_*(p))$ is contained in each of the intervals $I_{1,j}$ for $1 \leq j \leq k$. We also have $I_{1,j} \subset I_{0,j}$. So $k \leq 3$ by assumption. If $k = 1$ the result is immediate, because then $\varphi_*^{-1}(z_1) = \cap_{i \geq 0} T_{k_1}^i(I_{1,1})$ is the single point y_1.

Now suppose that $k = 2$. We can assume without loss of generality that $j_0 = 1$. If $T_{k_1}(I_{1,1}) \subset I_{1,1}$ then $\cap_{i \geq 0} T_{k_1}^i(I_{1,1}) = \{y_1\} \in \varphi_*^{-1}(z_1)$ is of period k_1 and the proof is finished by Lemma 12.7.3. If this is not true then $T_{k_1}(I_{1,1}) \subset I_{1,2}$, $T_{k_1}(I_{1,2}) \subset I_{1,1}$ and $\cap_{i \geq 0} T_{k_1}^i(I_1 \cup I_2)$ contains two points, one in each of the intervals $I_{1,1}$ and $I_{1,2}$, in the same periodic orbit of period $2k_1$. We write y_1 for the point in the orbit in $I_{1,1}$. Then

$$T^2_{k_1}(I_{1,1}) \subset T_{k_1}(I_{1,2}) \subset I_{1,1}$$

and

$$T^2_{k_1}(I_{1,2}) \subset T_{k_1}(I_{1,1}) \subset I_{1,2}.$$

By Lemma 12.7.3, $T^i(y_1)$ does not separate p from y_1 for $0 < i < k_1$, and $T_{k_1}(y_1) \in I_{1,2}$. Then $T^i(I_{1,1}) \cap T^j(I_{1,2}) = \emptyset$ for all $0 \leq i, j \leq i_1 m$ and $T^i(T_{k_1}(I_2)) \cap I_{1,1} = \emptyset$ for $mi_1 \leq i < k_1$, that is,

$$T^i(T^2_{k_1}(I_{1,1})) \cap I_{1,1} = \emptyset \text{ for } k_1 < i < 2k_1.$$

So the Lemma is proved in the case $k = 2$.

Now suppose that $k = 3$. If $T_{k_1}(I_{1,j_0}) \subset I_{1,j_0}$, then $\cap_{i \geq 0} T^i_{k_1}(I_{1,j_0}) = \{y_1\}$ is a point of period k_1 and the proof is finished as before. If $T_{k_1}(I_{1,j_0}) \cap I_{1,j_0} = \emptyset$ and $j_1 = j_0 \pm 1$ then we must have $T_{k_1}(I_{j_0}) \subset I_{j_1}$ and the proof can proceed as before. If $j_1 = j_0 \pm 2$ then we can assume without loss of generality that $j_0 = 1$, then any leaf of $L_p \cup L_q^{-1}$ between $T_{k_1}(I_{1,3})$ and $T_{k_1}(I_{1,2})$ must be short and we have $T_{k_1}(I_{1,2} \cup I_{1,3}) \subset I_{1,1}$, and hence $T_{k_1}(I_{1,1}) \subset I_{1,1} \cup I_{1,2}$. Then we obtain a point y_1 of period k_1 or $2k_1$ as in the case $k = 2$, depending on whether $T_{k_1}(I_{1,1}) \subset I_{1,1}$ or $T_{k_1}(I_{1,1}) \subset I_{1,2}$. □

If p_n is in the combinatorial copy of the Mandelbrot set, then $\mu_{p_n} \subset \bar{G}_p$. Now suppose $\mu_{p_n} \not\subseteq \bar{G}_p$, that is, s_{p_n} is not a tuning of s_p. We want to show that $R_n \in XP_1 \subset XB_{N_n}$ for $N_n \in \mathbb{N}$. This will suffice to show the convergence of R_n because $XB_j \to x_*$ as $j \to \infty$.

Now we claim that

Theorem 12.7.5. *$\varphi_*(p_n) \in P_1(x_*)$ implies $R_n \in XP_1 \subset XB_{N_n}$.*

Proof. We consider the set XP_1. The local inverse $S_{k_1,x}$ of $R_x^{k_1}$ is 2 valued for $x \in XP_1$ because $R_x(c_1) \in P_1$, but

$$R_x^i \circ S_{k_1,x}(P_1) \cap P_1 = \emptyset$$

for $0 < i < k_1$. So $S_{k_1,x}(P_1)$ varies continuously for $x \in XP_1$. So $X(S_{k_1,x}(P_1)) = \{x \in X : R_x(c_1(x)) \in S_{k_1,x}(P_1)\}$ is well defined.

There is a map $H_{x,y} : \partial P_1(x) \to \partial P_1(y)$ which extends to a conjugacy between $\cup_{i \geq 0} R_x^i(\partial P_1(x))$ and $\cup_{i \geq 0} R_y^i(\partial P_1(y))$ for any $x, y \in XP_1$. Then

$$y \to H_{x,y}(R_y^{k_1+1}(c_1))$$

maps $\partial X(S_{k_1,x}(P_1))$ to $\partial P_1(x)$ and is of degree 1. So

$$y \to H_{x,y}(R_y^{k_1+1}(c_1)) - H_{x,y}(R_y(c_1))$$

is of degree 1. Then there must be $x_1 \in X(S_{k_1,x}(P_1))$ such that

$$R_{x_1}^{k_1+1}(c_1) = R_{x_1}(c_1)$$

i.e.

$$R_{x_1}^{k_1}(c_1) = c_1$$

for some $x_1 \in X(S_{k_1,x}(P_1)) \subset XP_1$.

First suppose y_1 is of period k_1. Let $\gamma = \gamma_2 * \gamma_1$ be a path from $s_p(0)$ to $e^{2\pi i y_1}$, in which γ_1 is a path from $s_p(0)$ to $e^{2\pi i p}$ in the gap of L_p containing $s_p(0)$ apart from the endpoint $e^{2\pi i p}$, γ_2 is a path on S^1 from $e^{2\pi i p}$ to $e^{2\pi i y_1}$. Let ζ_1 be a path in the gap of of L_{y_1} from the periodic pre-image of $e^{2\pi i y_1}$ to 0. Then we have

$$s_{y_1} \amalg s_q \simeq \sigma_{\zeta_1}^{-1} \circ \sigma_\gamma \circ (s_p \amalg s_q). \tag{46}$$

This works because $(\sigma_\gamma \circ s_p)^{-1}(L_{y_1}) \supset L_{y_1}$ up to isotopy. The path γ does not cross any leaf in the forward orbit of the minor leaf μ_{y_1} of L_{y_1}.

There is a path ω in XP_1' from x_* to x_1 because XP_1' is a connected open set. This path can be chosen so that there is a conjugacy between $(R_x, \cup_{0 \leq i} R_x^i(S_{k_1,x}(\partial P_1(x))))$ and $(R_*, \cup_{0 \leq i} R_{x_*}^i(S_{k_1,x_*}(\partial P_1(x_*))))$ for all x

in this path. Denote this conjugacy by h_x because it extends the original conjugacy $h_x^j = h_x^{N_n}$. $\omega(0) = x_*, \omega(1) = x_1, h_{\omega(0)} = Id$.

$\alpha(t) = h_{\omega(t)}^{-1}(R_{\omega(t)}(c_1(\omega(t))))$ is a path in the dynamical plane of R_* with one endpoint at $R_*(c_1(x_*))$ and the other at $\varphi_{x_*}(e^{2\pi i y_1})$. Let $\beta(t) = h_{\omega(t)}^{-1}(c_1(\omega(t)))$, then

$$R_{x_1} \simeq \sigma_\beta^{-1} \circ \sigma_\alpha \circ R_*. \tag{47}$$

We want α to be homotopic to $\varphi_{x_*}(\gamma)$ and β to be isotopic to $\varphi_{x_*}(\zeta_1)$. This means we want that $\varphi_{x_*}^{-1}(P_1'(x_*))$ contains the arc S_{p,y_1} on S^1 between $e^{2\pi i p}$ and $e^{2\pi i y_1}$, which is implied by $\varphi_{x_*}(I_{1,j_0}) \subset P_1'(x_*)$. This follows from (39). Then

$$R_{x_1} \simeq \sigma_{\zeta_1}^{-1} \circ \sigma_\gamma \circ (s_p \amalg s_q). \tag{48}$$

In order to get this, by (21)(46)(47), we only need to show

$$\sigma_\beta^{-1} \circ \sigma_\alpha \circ R_* \simeq \sigma_{\zeta_1}^{-1} \circ \sigma_\gamma \circ (s_p \amalg s_q). \tag{49}$$

To get this, recall that by [Ree1], φ_{x_*} is the limit of a sequence of homeomorphisms $\varphi_k, k \in \mathbb{N}$. We can choose the sequence φ_k so that φ_k fixes $e^{2\pi i p}$ and $e^{2\pi i y_1}$ for all sufficiently large k given n. So if α is homotopic to $\varphi_{x_*}(\gamma)$ *rel* endpoints and β is isotopic to $\varphi_{x_*}(\zeta_1)$ *rel* endpoints, then α is homotopic to $\varphi_k(\gamma)$ *rel* endpoints and β is isotopic to $\varphi_k(\zeta_1)$ *rel* endpoints for all sufficiently large k. Then

$$\begin{aligned} & \varphi_k \circ \sigma_{\zeta_1}^{-1} \circ \sigma_\gamma \circ (s_p \amalg s_q) \circ \varphi_k^{-1} \\ = \ & (\varphi_k \circ \sigma_{\zeta_1}^{-1} \circ \sigma_\gamma \circ \varphi_k^{-1}) \circ \varphi_k \circ (s_p \amalg s_q) \circ \varphi_k^{-1} \\ = \ & \sigma_{\varphi_k(\zeta_1)}^{-1} \circ \sigma_{\varphi_k(\gamma)} \circ (\varphi_k \circ (s_p \amalg s_q) \circ \varphi_k^{-1}) \\ \simeq \ & \sigma_\beta^{-1} \circ \sigma_\alpha \circ R_*. \end{aligned}$$

The last equivalence holds because for k large enough, $\varphi_k \circ (s_p \amalg s_q) \circ \varphi_k^{-1}$ is uniformly close to $(R_* \circ \varphi_k) \circ \varphi_k^{-1} = R_*$.

If y_1 has period $2k_1$, then replace R_{x_1} by the tuning of period 2 that is in the copy M_{x_1} of the Mandelbrot set, still call the map R_{x_1} and again we have

$$R_{x_1} \simeq s_{y_1} \amalg s_q.$$

Now similar to the process of Lemma 12.7.4, if $y_1 \neq p_n$, then we have to continue the construction with points y_i and parameters x_i for $1 \leq i \leq l$ until some integer $l \in \mathbb{N}$ with $p_n = y_l$ and

$$s_{y_i} \amalg s_q \simeq R_{x_i} \tag{50}$$

with $x_j \in XP'_j \subset XP'_1$ for $1 \leq j \leq l$. As before, Thurston equivalence implies semiconjugacy, we will have $\varphi_j : S^1 \to J(R_{x_j})$ with

$$\varphi_j \circ (s_{y_j} \amalg s_q) = R_{x_j} \circ \varphi_j \tag{51}$$

on S^1. For $i > 2$, we might also need w_i and t_i such that R_{w_i} is a tuning of R_{x_i} and s_{t_i} is a tuning of s_{y_i} with $R_{w_i} \simeq s_{t_i} \amalg s_q$. We will have that $R_{x_{i+1}}$ is not a tuning of R_{x_i} and $w_i \neq t_{i+1}$ for $i \leq l-2$, but $w_{l-1} = x_l$ and $t_{l-1} = y_l$ are possible.

Now define $x_0 = w_0 = x_*$. We will construct integers k_j for $0 \leq j \leq l$ with $k_0 = m$ and r_i for $1 \leq j \leq l$, with $k_j > r_j k_{j+1}$ for $1 \leq j \leq l$. We define

$$n_j = N_n m + r_2 k_1 + \cdots + r_j k_{j-1} \text{ for } 1 \leq j \leq l.$$

For $x = x_j$ and $x = w_j$ we will have sets

$$P'_t(R_x) \in \mathcal{P}_{n_t - i_1 m}(R_x) \text{ for } 1 \leq t \leq j+1,$$
$$P_t(R_x) \in \mathcal{P}_{n_t + k_t}(R_x) \text{ for } 0 \leq t \leq j+1.$$

and $R_x(c_1(x)) \in P_j(R_x)$. The notation will make sense because $G_t(x_j) \cup \{R_{x_j}(c_1), R_{x_j}(c_2)\}$ and $G_t(w_j) \cup \{R_{w_j}(c_1), R_{w_j}(c_2)\}$ will be isotopic for all

$t \geq 0$, and $G_t(x_j) \cup \{R_{x_j}(c_2)\}$ and $G_t(x_{j+1}) \cup \{R_{w_j}(c_2)\}$ will be isotopic for all $t \leq n_{j+1} + k_{j+1}$, but this isotopy does not extend to map $R_{x_j}(c_2)$ to $R_{x_{j-1}}(c_2)$. We also assume inductively that we have a sequence of decreasing intervals $I_t \in S^1$ for $0 \leq t \leq j$ with

$$y_j \in I_j$$

and

$$\varphi_j^{-1}(P_t(R_{x_j})) \subset \varphi_j^{-1}(\varphi_j(I_t)) \subset P_t'(R_{x_j}) \tag{52}$$

for $1 \leq t \leq j$. We will always have

$$R_x(c_2) \in P_{j+1}'(R_x) \subset P_j(R_x) \subset P_j'(R_x).$$

There is a two-valued local inverse T_{k_j} defined on $\varphi^{-1}(\varphi_j(I_j))$ for $j \geq 1$ with

$$T_{k_j}(\varphi^{-1}(\varphi_j(I_j))) \subset \varphi^{-1}(\varphi_j(I_j)).$$

Inductively, $P_j(R_{x_j})$ is contained in a component of $R_{x_j}^{-k_j}(P_j'(R_{x_j}))$ such that $R_{x_j}^n(c_1) \notin P_j'(R_{x_j})$ for $0 < n < k_j$ but $R_{x_j}^{k_j}(c_1) \in P_j(R_{x_j}) \subset P_j'(R_{x_j})$. Also, $\varphi_j(y_j) \in P_j(R_{x_j})$ and $\varphi_j(p_n) \in P_j(R_{x_j})$. We then define $S_{k_j,x}$ for $x = x_{j-1}$ and $x = x_j$, to be the two-valued local inverse of $R_x^{k_j}$ which maps $P_j'(R_x)$ to a set containing $P_j(R_x)$. For $x = x_j, R_x(c_1) \in S_{k_j,x}^t(P_j'(R_x))$ for all $t \geq 0$ and hence $S_{k_j,x}$ is two-valued on $S_{k_j}^t(P_j'(R_x))$ for all $t \geq 0$ (while for $x = x_{j-1}$, this is only true for $t = 0$). We then define r_{j+1} to be the largest integer, which is ≥ 1, with $\varphi_j(p_n) \in S_{k_j,x}^{r_{j+1}}(P_j'(R_x))$. Then define

$$P_{j+1}'(R_x) = S_{k_j,x}^{r_{j+1}}(P_j'(R_x))$$

and

$$I_{j+1}'' = T_{k_j}^{r_{j+1}}(I_j)$$

for some two-valued local inverse T_{k_j} of T^{k_j}. Let I'_{j+1} be the union of I''_{j+1} and any complementary components between two components of I''_{j+1} in the same component of I_j. It follows from the inductive hypothesis that $P'_j(R_x) \in \mathcal{P}_{n_j - i_1 m}$ (which is true for $j = 1$) that $P'_{j+1}(R_x) \in \mathcal{P}_{n_j+1-i_1 m}(R_x)$. Then we define k_{j+1} to be the least integer $t > 0$ such that $R^t_{x_j}(\varphi_j(p_n)) \in P'_{j+1}(R_{x_j})$. Define $P_{j+1}(R_{x_j})$ to be the set in $\mathcal{P}_{n_{j+1}+k_{j+1}}(R_{x_j})$ which contains $\varphi_j(p_n)$. It is also the component of $R^{(1-r_{j+1})k_j - k_{j+1}}_{x_j}(P_j(R_{x_j}))$ which contains $\varphi_j(p_n)$ and is contained in a component $P''_{j+1}(R_{x_j})$ of $R^{-k_{j+1}}_{x_j}(P'_{j+1}(R_{x_j}))$.

We also have a periodic point $z_{j+1} \in P_{j+1}(R_{x_j})$ of period k_{j+1} under R_{x_j}, and thus $z_{j+1} \in \varphi_j(I'_{j+1})$, and there is $y_{j+1} \in I'_{j+1}$ with $\varphi_{j+1}(y_{j+1}) = z_{j+1}$. For $j \geq 1$, y_{j+1} might not be in the same component of I'_{j+1} as y_j. But every component of $\varphi_j(I'_{j+1})$ intersects $\partial U_*(R_{x_j})$ in a non-empty open set. So there is some t_j, which is periodic under T in the same component of I_{j+1} as y_{j+1}, with $\varphi_j(t_j) \in \partial U_*(R_{x_j})$ periodic under R_{x_j} . We write I_{j+1} for this component of I'_{j+1}. Of course, s_{t_j} is a tuning of s_{y_j} , and there is $w_j \in XP_j$ such that R_{w_j} is a tuning round c_1 of R_{x_j} and $R_{w_j} \simeq s_{t_j} \amalg s_q$. Then (52) follows from the definitions and from (39).

We then have a connected set XP'_{j+1} containing x_j such that $P'_{j+1}(x) \supset P''_{j+1}(x) \supset P_{j+1}(x)$ are defined for all $x \in XP'_{j+1}$, with $R_x(c_1) \in P'_{j+1}(R_x)$ for all $x \in XP'_{j+1}$. We also have sets

$$XP''_{j+1} = \{x \in XP'_{j+1} : R_x(c_1) \in P''_{j+1}(R_x)\},$$
$$XP_{j+1} = \{x \in XP'_{j+1} : R_x(c_1) \in P_{j+1}(R_x)\},$$

and

$$XP_{j+1} \subset XP''_{j+1} \subset XP'_{j+1}$$

on the parameter plane X. Then as in the case $j = 0$ we can find $x_{j+1} \in XP_{j+1}$ such that $c_1(x_{j+1})$ is of period k_{j+1} under $R_{x_{j+1}}$ and we get (50), using the fact that y_{j+1} and t_j are both in the same component I_{j+1} of I'_{j+1}. We also get y_{j+1} of period k_{j+1} or $2k_{j+1}$. By similar process as the $j = 1$ case, we can deduce that $R_{x_{j+1}} \simeq s_{y_{j+1}} \amalg s_q$ or $R_{x'_{j+1}} \simeq s_{y_{j+1}} \amalg s_q$ where $R_{x'_{j+1}}$ is a period two tuning of $R_{x_{j+1}}$.

Finally we get a parameter $y_l = p_n$ and $x_l \in XB_{N_n}$ after l steps such that

$$R_{x_l} \simeq s_{y_l} \amalg s_q = s_{p_n} \amalg s_q. \tag{53}$$

□

Now considering (38), Theorem 4.2.5 follows as a corollary of Theorem 12.7.5 and 12.6.4.

13 References

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