ONE HUNDRED YEARS OF COMPLEX DYNAMICS

MARY REES

The subject of Complex Dynamics, that is, the behaviour of orbits of holomorphic functions, emerged in the papers produced, independently, by Fatou and Julia, almost 100 years ago. Although the subject of Dynamical Systems did not then have a name, the dynamical properties found for holomorphic systems, even in these early researches, were so striking, so unusually comprehensive, and yet so varied, that these systems still attract widespread fascination, 100 years later. The first distinctive feature of iteration of a single holomorphic map f is the partition of either the complex plane or the Riemann sphere into two sets which are totally invariant under f: the Julia set — closed, nonempty, perfect, with dynamics which might loosely be called chaotic — and its complement — open, possibly empty, but, if non-empty, then with dynamics which were completely classified by the two pioneering researchers, modulo a few simply stated open questions. Before the subject re-emerged into prominence in the 1980's, the Julia set was alternately called the Fatou set, but Paul Blanchard introduced the idea of calling its complement the Fatou set, and this was immediately universally accepted.

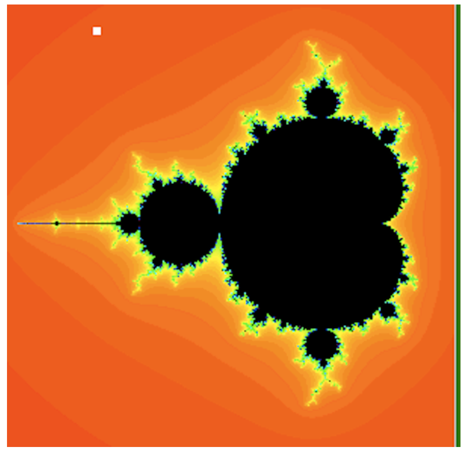
Probably the main reason for the remarkable rise in interest in complex dynamics, about thirty-five years ago, was the parallel with the subject of Kleinian groups, and hence with the whole subject of hyperbolic geometry. A Kleinian group acting on the Riemann sphere is a dynamical system, with the sphere splitting into two disjoint invariant subsets, with the limit set and its complement, the domain of discontinuity, having exactly similar properties to the Julia and Fatou sets. Dennis Sullivan realised that a parallel of a result of Ahlfors [4] solved an outstanding open problem posed by Fatou: whether the Fatou set (in modern parlance) contains any wandering components. I.N. Baker, who worked on transcendental dynamics from the 1950's onwards, had shown that wandering domains are indeed possible in the transcendental setting [11]. Ahlfors had shown that the domain of discontinuity of a finitely generated Kleinian group has only finitely many orbits, and also bounded the number of orbits, by using an argument on dimension using the Measurable Riemann Mapping Theorem [23, 5]. The Fatou set, as it is now called, is analogous to the domain of discontinuity, as, on this set, the family of iterates of the holomorphic function is equicontinuous, in the locally compact topology. Sullivan adapted the argument to solve Fatou's conjecture in the negative [145], and in the process introduced the technique of quasi-conformal deformation into the subject of iteration of a single holomorphic function. The tool was immediately employed by Douady and Hubbard [48, 47] in their groundbreaking work on the dynamics of quadratic polynomials. It was also used a few years later by Shishikura [139] to solve another longstanding open problem of Fatou: obtaining a sharp bound on the number and type of periodic orbits in the Fatou domain, in terms of the degree of the map, or, more accurately, the number of critical values. On a somewhat different tack, Michel Herman, using his work on Arnold's Conjecture for circle diffeomorphisms (which won the Salem prize), found examples of the missing type of periodic Fatou domain: annuli on which the return map is holomorphically conjugate to an irrational rotation [64]. These domains, the possibility of whose existence had been raised by Fatou, are now known, naturally, as Herman rings.

Dennis Sullivan also formulated the Dictionary [145], which translated results and conjectures between the dynamics of holomorphic functions and of Kleinian groups. Of course the "No wandering domains" result was one instance of this. But perhaps the most interesting feature of the dictionary, as Sullivan pointed out, was that key results on one side were parallelled by unsolved conjectures on the other – and this happened in both directions. A very important example of such a parallel concerns hyperbolic and stable maps and groups. Stability and hyperbolicity are both very important concepts in dynamics in general. A dynamical system is *stable* if all systems in a sufficiently small neighbourhood of it — with respect to some suitable topology on which the definition naturally depends — are topologically conjugate to the original. For holomorphic functions, J-stability rather than stability is considered, that is, maps f for which there exist neighbourhoods U_g of the Julia set of g with $g^{-1}(U_g) \subset U_g$, and continuously varying homeomorphisms $\varphi_g : U_f \to U_g$ such that $\varphi_g \circ f = g \circ \varphi_g$ on $f^{-1}(U_f)$. This is simply to take account of the critical points of the holomorphic map, for which the dynamics might vary. The definition of hyperbolicity for invertible dynamical systems, which in a fairly general form is Smale's Axiom A [144], is rather long, and will not be given here. But the idea is to carry over to diffeomorphisms properties of hyperbolic linear maps . A linear map is *hyperbolic* if it has no eigenvalues on the unit circle. For diffeomorphisms, the sums of eigenspaces for eigenvalues of module < 1 (or > 1) translate to invariant *stable* (or *unstable*) foliations. Hyperbolic systems are quite easily shown to be stable, but the converse was not so clear in the 1960's, when, the question was raised as to whether stability and hyperbolicity might be equivalent, and also whether the resulting open set was dense. All of these are true for circle diffeomorphisms. It was realised early that stable systems are not dense in the C^1 space of invertible dynamical systems [143] in dimension three or more. The example was strengthened, in dimension 4, to show non-density of Axiom A. Newhouse showed that Axiom A is not dense in dimension two or greater in the C^r topology for $r \geq 2$. As for the equivalence of the concepts, in the early 1980's Mañé proved that C^1-stable diffeomorphisms are hyperbolic [101].

The definition of hyperbolicity, for iteration of a rational function, is much simpler than the definition for diffeomorphisms. Because of the classical theory of complex dynamics, a rational function can be defined to be hyperbolic if and only if every critical point is attracted to an attractive periodic orbit. For Kleinian groups, the concept of hyperbolicity translates into compactness of a certain manifold (or orbifold) with boundary. This is the quotient of the union of three-dimensional hyperbolic space and the domain of discontinuity on the bounding Riemann sphere by the action of the group. Hyperbolic Kleinian groups are sometimes just called "good groups", and hyperbolicity is generally considered to be "good" behaviour. When Sullivan raised the question in the context of holomorphic dynamics, not long afterwards, he succeeded in proving that stable Kleinian groups are hyperbolic [146]. The corresponding question for any complex variety of rational maps is still unanswered. In the other column of the dictionary, Mañé, Sad and Sullivan [102] showed that stable maps are dense in families of rational maps, with some simply described and well-understood exceptions. At the time, over thirty years ago, the opposite column for Kleinian groups was quite unknown, that is, whether the "good" finitely generated Kleinian groups are dense. As a result of the breakthroughs of recent years, when virtually all outstanding problems concerning hyperbolic manifolds — and, indeed, three-dimensional manifolds in general —- have been solved, density of good groups is also known [26].

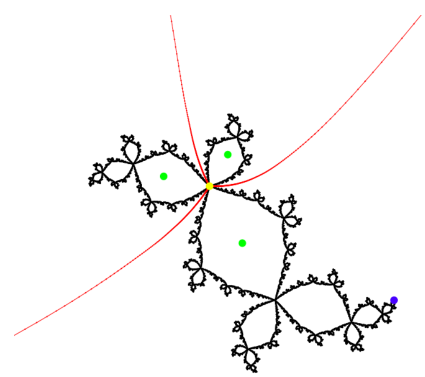
Topological density of hyperbolicity for a parameter space, if true, is one way which might help towards a global understanding of dynamical variation within the parameter space. One can also consider the measurable setting, replacing hyperbolicity by *non-uniform hyperbolicity*, a concept which has been a very important guiding force in dynamics since the 1970's. Non-uniform hyperbolicity is more easily defined in one real or complex dimension for smooth or holomorphic maps by requiring non-uniform exponential growth of derivatives $(f^n)'(x)$ for almost every x. *Pesin theory*, developed for the invertible setting of non-uniform hyperbolicity, where the definition also parallels the definition of hyperbolicity, with "almost everywhere" replacing "everywhere", showed that the property implied strong ergodicity properties, including Bernoulli [117]. The hope was, then, that these properties might hold on a positive measure proportion of parameter space In the invertible case, the original theory developed by Pesin in the 1970's hypothesised a smooth invariant probability measure. Subsequently this condition was expanded to include a larger class of measures known as *Sinai-Bowen-Ruelle* (SBR) measures, and in general dynamics more, and significant, progress, has been made when the non-uniform hyperbolicity has been concentrated on a *strange attractor* which is a proper subset of the underlying manifold, probably of zero measure ([15, 16] for two important, relatively early results in a huge literature). Work in one dimension started this development. Jakobson proved, also in the 1970's, that a positive measure proportion of the *logistic family* (the real quadratic family) supported an absolutely continuous invariant measure [71]. The proof did not give non-uniform hyperbolicity on this positive measure set, but a later proof by Benedicks and Carleson did [14]. A corresponding result for the family of rational maps of any degree was proved in [124]. There have been extensive studies of the conditions needed for non-uniform hyperbolicity [121]. Perhaps the most comprehensive result for the logistic family. or indeed, for any family so far, was obtained by M. Lyubich in "Almost every real quadratic map is either regular or stochastic" [97].

The Lai Sang Young formula [158] relating Hausdorff dimension, entropy and Lyapunov exponents for measures gave rise to considerable development in dynamics, including complex dynamics. Significant impetus was also given to complex dynamics by Makarov's theorem from classical complex analysis, on the Hausdorff dimension of harmonic measure of planar domains, and a sharp bound on the Hausdorff measure gauge function [100]. Both these results had impact in work by Przytycki, Urbanski and Zdunik [122, 123] which connected the harmonic measure to Gibbs measures for some self-similar domains, where even the constant in the gauge function was computed. But Manning's results bounding by 1 Hausdorff dimension of the measure of maximal entropy on the Julia set of any polynomial [103] predated Makarov's result.



The Mandelbrot set [46]

The family of complex quadratic polynomials is the most famous family of complex dynamical systems. This is effectively a family parametrized by $\mathbb{C}$, because, up to affine conjugation, any quadratic polynomial is of the form $f_c(z) = z^2 + c$ for some $c \in \mathbb{C}$. For c sufficiently large, including any c with $|c| > 2$, f_c is hyperbolic, with the orbit of the finite critical point 0 attracted to infinity, and the Julia set is a Cantor set. All these maps are topologically conjugate, in the same hyperbolic component (that is, a connected component in the open set of hyperbolic maps). Attention then focuses on the complement, the Mandelbrot set M, which first drew widespread attention with the pictures of Benoit Mandelbrot. Pictures of the Mandelbrot set are very familiar and have passed into the popular domain. (The set also emerged, at about the same time, in a study of discrete Kleinian groups by Brooks and Mattelski [27].) This creature has a central heart-shaped region M_0, with countably many *limbs* emanating from *roots* on the boundary of M_0. Mathematically, M_0 is the set of c for which f_c has a non-repelling fixed point, which is easily computed as the set of c for which $|1 - \sqrt{1 - 4c}| \leq 1$. Getting a correct mathematical description of the rest of the Mandelbrot set is not so simple. Computer pictures can be misleading, and on this basis, Mandelbrot initially conjectured that M is disconnected. Douady and Hubbard showed that this was not the case and in fact that M is cellular, that is, the complement in $\mathbb{C} \cup \{\infty\}$ is biholomorphic to an open disc. They did this by producing the uniformising map Φ from the complement of M to the complement of the unit disc. In an interlacing formula between dynamical and parameter plane which is typical of dynamical studies, $\Phi(c) = \varphi_c^{-1}(c)$ for the conjugacy φ_c (suitably normalised) satisfying $\varphi_c \circ f_0 = f_c \circ \varphi_c$ in a neighbourhood of ∞ whose image contains the critical value c of f_c, if $c \notin M$. This approach gives a lot more information, which has been a guiding force to research in the subsequent decades. The extraordinary detail obtained by Douady and Hubbard followed from their results that Φ^{-1} has limits along rays in $\{z : |z| > 1\}$ of rational argument, and so does φ_c for $c \in M$ (and for all c, if the rays are suitably interpreted, since in this case the rays are no longer in the set $\{z : |z| > 1\}$). The endpoint of any ray of rational argument for φ_c is a non-attracting periodic point of f_c, and hence in its Julia set. The limit along any such ray for Φ^{-1} is a *parabolic* parameter value c, that is, there is a point z of some period m under f_c such that the *multiplier* $(f_c^m)'(z)$ is of the form $e^{2\pi i\omega}$ for ω real and rational. Endpoints of rays for φ_c might coincide, and similarly for Φ^{-1}. Douady and Hubbard described all possible coincidences. This description provides a topological model for the Mandelbrot set, which is homeomorphic to M if M is locally connected, and also describes the map f_c up to topological conjugacy whenever the Julia set $J(f_c)$ is locally connected. Also, if M is locally connected then hyperbolic maps are dense. The work of Douady and Hubbard was given a very illuminating reinterpretation by W. Thurston [153] in terms of laminations on the unit disc: quadratic invariant laminations and the so-called quadratic minor lamination. The question of whether M is locally connected is still unanswered, and is regarded as one of the most important open questions in Complex Dynamics. It is known as the MLC Conjecture.



Julia set of the "rabbit polynomial" with dynamical rays landing at the α fixed point [99]

The α *fixed point* featured prominently in the exposition of Douady and Hubbard. The equation $z^2+c = z$ has two solutions, except when $c = \frac{1}{4}$ (the cusp of M_0) when there is just one, the parabolic fixed point of $f_{1/4}$. The endpoint of the ray of argument 0 has to be a fixed point. The other fixed point is the α fixed point, and clearly must be repelling, for $c \notin M_0$. Less obviously, for $c \in M \setminus M_0$, the α fixed point is the endpoint of a finite number of rays of rational argument which are cyclically permuted with some rotation p/q, and the rotation determines which limb of M contains c. The root of the limb is the parabolic parameter in ∂M_0 for which the multiplier of the α fixed point is $e^{2\pi i(p/q)}$. This follows from the *Yoccoz inequality*, [68] which estimates the size of limbs in terms of the rotation number, using two different interpretations of modulus of a torus, in an intriguing parallel in method and statement with the *Jorgensen inequality* for Kleinian groups [74]. Almost all of the work on the Mandelbrot set in the last thirty years has involved using the *Yoccoz puzzle*, where, similarly, key methodology is to find bounds on moduli of annuli in terms of other annuli which are defined combinatorially. The Yoccoz puzzle is a sequence of successively finer partitions of successively smaller neighbourhoods of the Julia set, defined for f_c for each c, where the level zero partition is constant on each limb, simply using the α fixed point as boundary, to separate the Julia set into pieces. The finer partitions are made by pulling back the level zero partition under iterates of f_c. There are, correspondingly, successively finer partitions of the parameter space. This forms the Yoccoz *parapuzzle*. The sets in the zero level partition are simply M_0 and the limbs of M. The sets in the n'th level partition of the parapuzzle are the sets on which the n'th level dynamical partition remains topologically the same. Yoccoz used annuli defined by combinatorially by the puzzle to prove local connectivity of the Julia set of any *non-renormalizable* map f_c (a result which can be extended to maps which are not *infinitely renormalizable* by using extensions of the Yoccoz puzzle) and local connectivity of M at any non-renormalizable point. There is a very useful re-working of part of the proof by Shishikura which was exposed by Roesch [135]. Other results followed. Shishikura and Lyubich independently proved measure zero of the Julia set at non-renormalizable points (probably both unpublished). Douady and Hubbard had produced examples showing that the Julia set of a quadratic polynomial need not be locally connected. More, and more detailed, examples were given by Levin [90]. But it is conjectured, and widely believed, that the Mandelbrot set is locally connected – and this would be true if the annulus estimates using the Yoccoz parapuzzle could be extended to the infinitely renormalizable case. The most persistent and successful efforts at extension, probably still ongoing, were made by Jeremy Kahn in collaboration with Lyubich, Avila and others, in a series of papers [76, 77, 78, 79], see also Lyubich and Yampolsky [98]. This extension of methods was probably instrumental in progress in density of hyperbolicity and rigidity results in other families of real polynomials [86], although other results for real polynomials came earlier [28, 85, 91, 92, 93, 94].

The study of renormalizable maps in the quadratic family has been a very important part of the development of complex dynamics. Such maps include those with *Siegel discs*, and *Cremer points*. A Siegel disc is a periodic

Fatou domain on which the return map is holomorphically conjugate to an irrational rotation. Siegel discs occur for polynomials, but Herman rings, mentioned earlier, do not. A Cremer point is a periodic point with multiplier $e^{2\pi i\omega}$, for irrational ω, which is not within a Siegel disc, whose existence was proved by Cremer [43, 108]. The other periodic types of periodic Fatou domain are *attractive domains*, which contain an attractive periodic point, and *parabolic domains*, which contain a periodic point on the boundary, for which the multiplier is a root of unity. The nature of the continued fraction expansion of ω was shown by Yoccoz [157, 104] to determine whether the corresponding periodic point is in a Siegel disc or is a Cremer point: it was shown that a necessary and sufficient condition for a Siegel disk is that ω is a Bryuno number. (A conjecture in that paper was recently proved by Cheraghi and Chéritat [37].) In another result concerning the type of the number, M.Herman had proved [65] that if ω satisfies a Diophantine condition then any periodic orbit of Siegel domain boundaries contains a critical point. In contrast, he showed [66] that there are Siegel domains for quadratic polynomials with boundaries which are quasi-circles (that is, images of round circles under quasi-conformal homeomorphisms) with no critical point in the forward orbit. Buff and Chéritat built on this [29] to show that for each integer n there are quadratic maps with Siegel discs with boundaries which are C^n but not C^{n+1}, with no critical point on the boundary. They also showed, with Avila [7], that there is a dense set of parameter values for which the boundary of the Siegel disc is a Jordan curve. As for Cremer points, Perez- Marco made pioneering studies of the dynamics in these cases, introducing *hedgehogs* by name and concept [114, 115, 116].

Later results on Siegel and Cremer parameters used deep study of parabolic parameters, that is, parameters for which the map has a parabolic domain. Dynamics within parabolic domains are studied using a simple change of coordinate called the *Fatou coordinate* [108]. Techniques for studying dynamics near parabolic points have been very substantially refined in recent decades, perhaps starting with the “Tour de Valse” argument in the Orsay notes [48] used to show landing of periodic rays. Adam Epstein developed a refined index theory for parabolic points and used this to prove that all but one hyperbolic component in the family of hyperbolic quadratic maps are precompact [51]. A significant refinement was developed by Shishikura [140] to show that the Julia sets of some quadratic polynomials, obtained by taking fast-increasing sequences of parabolic parameters (and therefore including some Cremer parameters), have Hausdorff dimension two, and, correspondingly, the boundary of the Mandelbrot set has Hausdorff dimension two. In contrast, C. Petersen proved that the Julia set of a quadratic polynomial with Siegel disc for ω of bounded type is locally connected and of Lebesgue measure zero. McMullen [107] subsequently proved that the Julia set has Hausdorff dimension less than two and proved some results on "space filling" of the Siegel disc under "renormalization" (where this refers to the type of renormalizations used for circle maps, rather than for polynomials). Petersen and Zakeri [120] proved that, for almost all ω the Julia set of $e^{2\pi i\omega}z + z^2$ has zero Lebesgue measure. Buff and Chéritat's production of examples of Siegel and Cremer parameters, and also of infinitely renormalizable parameters, with positive measure Julia sets, had input from the above works, from earlier joint work [30, 31], from Inou-Shishikura's work on renormalisation of parabolic maps [70], and, in particular, from Chéritat's work on parabolic implosion [40, 38, 39]. In a commonly arising situation in mathematics, although there was no direct communication, these results on positive measure were obtained at about the same time that zero measure was proved in the other column of the dictionary. As a corollary [36] of the proof that a three-dimensional hyperbolic manifolds with finitely generated fundamental group (which is a Kleinian group) is tame [2, 33], it was shown that the limit set of the Kleinian group has zero measure: confirmation of the famous Ahlfors Conjecture [3].

Renormalization of maps probably first emerged in dynamics with unimodal interval maps in the work of Feigenbaum [54, 55, 56]. Renormalization simply means restricting to a subset of the dynamical plane and considering the return map to that set, usually in the case when some iterate of the map sends this set to itself. The case when this return map or renormalization bears a resemblance to the original map on the whole dynamical plane is of particular interest.

The original focus of attention was the *period doubling map*, represented in quadratic polynomials by the *Feigenbaum parameter value* for which the returns of 2^n'th iterates of the map to successively smaller sets are the same modulo topological conjugacy – with bounded distortion, as was eventually proved. Feigenbaum noticed that successive period doubling parameters in the quadratic family appeared to form a geometric progression in the limit. He postulated, initially for unimodal maps, the existence of a map which was fixed under renormalization, with the renormalizing map on an appropriate space of maps having a smooth codimension one stable manifold through the fixed map, and an unstable manifold of dimension one. A computer-assisted proof was provided by O. Lanford [87, 88], in part joint with Collet and Eckmann [42] but never, it seems, published in full. Shorter proofs with more analytic input were then given by Lanford [89] and by M. Campanino, H. Epstein and D. Ruelle [34, 35]. Sullivan proved bounded distortion at all scales of the postcritical set for the Feigenbaum parameter polynomial and for other quadratic-like maps [147].

Intense effort has been devoted to obtaining distortion bounds under renormalization in the quadratic family, not just for real polynomials, but complex as well. Obtaining such bounds is probably the key to proving MLC. There have been notable successes in this direction, and involving such bounds. McMullen's book on renormalization [105] proved the non-existence of line-fields on the Julia sets of real quadratic polynomials: not quite enough to prove density of hyperbolicity in the real family as the results of [102] do not apply in this setting. But the study of renormalizable maps did play a part in the proofs of density of hyperbolicity in the real quadratic family, independently, by Graczyk-Swiatek [58, 59], and by Lyubich [95, 96]. The *complex bounds* obtained for the real quadratic family, that is, bounds on distortion of successive renormalizations, obtained by bounds on moduli of suitable annuli, imply that the Julia set of the Feigenbaum polynomial is locally connected [67] and of measure zero, and similarly for other bounded infinite renormalizations in the real quadratic family [72, 105].

In the dictionary between complex dynamics and hyperbolic geometry, infinitely renormalisable polynomials are generally regarded as the analogue of hyperbolic manifolds fibering over the circle [106], and the distortion bounds which have been sought were motivated in part by the proofs of the existence of hyperbolic manifolds fibering over the circle [151, 112]. Another important strand of work initiated in the 1980's, with fundamental connections to hyperbolic geometry, and, in particular, with W.P.Thurston's transforming results in that field, was Thurston's theorem for postcritically finite branched coverings of the sphere [152, 50]. This theorem gives a necessary and sufficient condition, in terms of a natural linear transformation on vector spaces generated by *multicurves*, sets of disjoint closed simple loops, for a postcritically finite branched coverings to be homotopy equivalent, in the appropriate sense (which has always been known as *Thurston equivalence*) to a rational map — which is then unique up to Möbius conjugacy. The statement of the result can be described as a geometrization theorem, the type of theorem with which Thurston transformed three dimensional topology. It is reminiscent of Thurston's hyperbolization theorems: in particular the existence of a hyperbolic manifold in the homeomorphism class of any compact Haken manifold [150]. Such a manifold is unique up to hyperbolic isometry, due to Mostow's Rigidity Theorem. There are parallels in the proofs also, as, in both proofs, Thurston uses an iteration on Teichmüller space which turns out to have a fixed point. So topological types of three-manifolds are replaced by postcritically finite rational maps.

Thurston's theorem for branched coverings, and the proof, opened many avenues. The result indicated a classification of all postcritically finite branched coverings of the sphere, which have come to be known as *Thurston* *maps*, and there has been extensive work on these and on extensions to them [60, 61, 62, 20]. Checking the necessary and sufficient conditions for a map to be Thurston equivalent to a rational map is nontrivial. A simple necessary and sufficient condition for *matings* [47] of quadratic polynomials was conjectured by Douady and proved by Tan Lei [149] (with a more general result in [126]). Thurston's theorem has been generalised in a number of ways. Some hyperbolic components do not contain postcritically finite maps and yet there should be a characterization of those branched coverings which are homotopy equivalent, in the appropriate sense, to hyperbolic rational maps. This was achieved independently, by Zhang Gaofei and and Jiang, Yunping [73] and by Cui Guizhen and Tan Lei [45]. In another direction, [127] considered some cases of generalizations of the original theorem, where just some critical points are constrained. The resulting topological spaces turn out to have very rich topology, usually with infinitely generated fundamental groups, and naturally described up to homotopy equivalence as a union of infinitely many topological spaces with a clearly defined geometric structure, glued together. The component topological spaces are usually either parameter spaces of rational maps, or Thurston equivalence classes of postcritically finite branched coverings or products of these. It was hoped that this would lead to a complete description of variation of dynamics in some parameter spaces of rational maps but so far this has proved difficult even in the first cases [128]. A version of Thurston's Theorem for exponential maps was proved in [69]. In a completely different vein, the concept of monodromy group of a postcritically finite quadratic polynomial was developed [13], which connected with the large field of recurrent, contracting self-similar groups. The proof of Thurston's theorem was developed by Adam Epstein [52] in a way which gave rise to a new concept of index of parabolic fixed points, with a number of applications, including a refinement of the Fatou-Shishikura inequality of [139].

Although the flavour of Thurston's result looks primarily topological and geometric, there is also a connection with a theme which is common in dynamics although, perhaps, not always very explicitly stated: homotopy type determines dynamics up to semiconjugacy. Some results of this type are folklore. A self-map f of the circle of degree d, where $|d| > 1$, is semiconjugate to $z \mapsto z^d$, that is, there is a continuous map $\varphi : S^1 \to S^1$ (of degree 1) such that $\varphi(f(z)) = (\varphi(z))^d$ for all $z \in S^1$. There are results of this type by John Franks for toral automorphisms of hyperbolic type [57] (originally from his thesis). In the case of a Thurston map f which is Thurston equivalent to a rational map g, and where f satisfies some mild local conditions near critical orbits, there is a continuous map $\varphi : S^2 \to S^2$ such that $\varphi \circ f = g \circ \varphi$. This can be developed to describe f up to topological conjugacy [125]. While this is of use, one would hope for more: to find a useful description of all possible Thurston equivalence classes of postcritically finite maps in a way which also gives information about their positions in parameter space. Such a description is provided in the case of quadratic polynomials by Thurston's lamination models [153]. The dynamics of any quadratic polynomial with locally connected Julia set is described up to topological conjugacy by the corresponding *invariant laminations*. Even if the Julia set is not locally connected, the corresponding lamination usually gives some information. The invariant lamination is described by its *minor leaf*. In the postcritically finite case — when the endpoints of the minor leaf have rational arguments — the minor leaf describes the Thurston homotopy type. But the *Quadratic Minor Lamination* as Thurston calls it, whose leaves are minor leaves, provides a model for the Mandelbrot set which is conjecturally homeomorphic to it.

It is an obvious strategy to look for an analogue of this beautifully detailed and complete description in other families of holomorphic maps. Families which have been considered, in different ways, include: polynomials of higher degree; quadratic rational maps; Newton's method maps, especially the rational maps used in Newton's method for finding zeros of polynomials; an families of transcendental maps, with the prime example being the exponential family. Many different considerations come into play. In some cases, such as *unicritical polynomials* (with a single finite critical point) the conjectural description of the Mandelbrot set applies pretty directly. For cubic polynomials in the complement of the connectedness locus, Branner and Hubbard [24, 25] showed that the theory for quadratic polynomials, together with an adaptation of the Yoccoz puzzle again give a complete description modulo MLC. For polynomials of degree d for $d > 2$, the theory of invariant laminations still applies: very effectively [81] (where the first definitions given of laminations might be a little different from Thurston's original one) but there are extra dimensions which cause significant problems (see [63] for example), which have yet to be solved. For parameter space is higher dimensional: of complex dimension $2d - 2$. It is possible to consider slices of parameter space of complex dimension one, and this has yielded interesting information. See [82], for example, for a study near infinity in the parameter space. Study of the whole connectedness locus was soon seen to present significant problems, but significant advances have been made [18, 19], in part by a detailed study of the *main cuboid*, which is the model for the closure of the principal hyperbolic component, with a fixed Fatou domain mapped to itself with degree 3. As for parameter spaces of rational maps: the monotonic character of variation of dynamics which is a feature of polynomial parameter spaces is completely lost, and complete models have to be built with that in mind. At least, that is my view, but probably not universally held, and there were indications that W. Thurston, among others, thought monotonic character could be preserved in his final years before his death in 2012.

Leaving global parameter spaces aside, I believe there is scope for substantial development of local study of parameter spaces. An interesting study near a *type B* (bitransitive) hyperbolic component of rational maps, using *regluing*, was made by Timorin [155]. So far as I know, all applications of generalization of the Yoccoz parapuzzle have essentially been within complex dynamics but there is a general principle involved which suggests that there might even be considerably wider application on a local level. Yoccoz himself pointed out that his puzzle methods did not transfer directly even to unicritical polynomials of degree > 2. After a gap of some years, there were a number of notable achievements on this front [8, 28]. The methods have been applied to real polynomials of any degree [85, 86], and to entire functions [131].

The Yoccoz puzzle for a limb of the Mandelbrot set is an example of a sequence of *Markov partitions*, generated by taking the backward orbit of a single Markov partition, which persists over the entire limb. Markov partitions are ubiquitous in dynamics, where their use in some form goes back nearly a century. Their use has been directed to the analysis of single dynamical systems. Much of the key development was done by Rufus Bowen in the 1970's [21, 22]. (He died in 1978, at the age of 31.) His work concentrates on the case of invertible dynamical systems, including the existence for hyperbolic invertible dynamical systems. These were the main focus of dynamics research at the time. But the definition of Markov partitions for non-invertible dynamical systems, and the results for these, principally for expanding maps and contracting iterated function systems, are generally easier, and even stronger than for invertible systems. What makes the Yoccoz puzzle especially attractive is the simple description of the sets in it, and the persistence over limbs, which gives rise to the parapuzzle for each limb. This means that information can be obtained about, not just single dynamical systems, but the variation of dynamics across each limb.

There has been substantial work on extending methods to other dynamical systems. Significant work on maps of the interval, not confined to complex dynamics, predates Yoccoz' work. Natural partitions for maps of intervals are given by the sets on which a map is monotone – which is a finite partition for smooth maps with finitely many critical points. While these partitions are not usually Markov, simply giving rise to *symbolic dynamics* in general, they generate Markov partitions for the important class of postcritically finite maps, and also give rise to natural puzzles and parapuzzles. Perhaps the best known development was the Milnor-Thurston Kneading Theory for maps of the interval, principally for unimodal maps [109]. In the domain of complex dynamics, early extensions of Yoccoz' work were achieved by Roesch to the family of Newton cubics, and she also extended the use to other specific families [132, 133, 134, 136, 137, 138, 119]. Aspenberg and Yampolsky [6] (see also [154]) used an analogue of the Yoccoz puzzle to study the family of quadratic polynomials with a critical point of period 2 — where all hyperbolic maps are known to be either *matings* or *Wittner captures*. The basic idea appeared, quite non-rigorously, in the 1995 Cornell thesis of Jiaqi Luo. In work intended for further development, I consider a construction, which works only locally, but still in an open neighbourhood of the closure of a hyperbolic component, under fairly general conditions, in [129]. My impression is that it could indeed be possible to give a model for parameter space locally using such a construction.

As indicated at the beginning of this article, a strong motivation for studying complex dynamics is as a microcosm. I have concentrated on one variable and on rational dynamics, as I have no expertise in complex dynamics in several variables, nor of transcendental holomorphic functions, and also for reasons of space. Nor have I said anything about dynamics of rational maps over non-Archimedean fields in place of the complex numbers [83, 84]. All of these these areas have expanded enormously in recent decades. Such an expansion cannot be simply due to mathematicians' inclination to generalise. Dynamical features which are hugely important in general dynamics, and which do not arise in rational complex dynamics, do arise for these. With holomorphic functions in higher dimensions, hyperbolicity can have a nature much more representative of the general picture, with both expanding and contracting directions. At the very least, this has increased the pool of examples which have the potential to to exhibit strange hyperbolic attractors. With transcendental holomorphic dynamics also the variety of non-uniformly hyperbolic behaviour is greater, with the possibility of infinitely many attractive basins (or sinks). In both cases, also, there are strong links with other areas of complex analysis: with potential theory and classical function theory respectively, mirroring the cross-fertilisation between complex dynamics of rational functions and Kleinian groups, and hence with hyperbolic geometry. With the growth of the subject of complex dynamics, the links with adjacent areas have strengthened and have become increasingly intricate, and the big questions in dynamics remain. It would have been impossible to predict these developments 100 years ago. It is also impossible to predict the impetus of complex dynamics over the next century, but the narrative is likely to continue for a while yet.

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