FINITE AND INFINITESIMAL FLEXIBILITY OF SEMIDISCRETE SURFACES

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Abstract. In this paper we study infinitesimal and finite flexibility for generic semi-discrete surfaces. We prove that generic 2-ribbon semidiscrete surfaces have one degree of infinitesimal and finite flexibility. In particular we write down a system of differential equations describing isometric deformations in the case of existence. Further we find a necessary condition of 3-ribbon infinitesimal flexibility. For an arbitrary $n \geq 3$ we prove that every generic n-ribbon surface has at most one degree of finite/infinitesimal flexibility. Finally, we discuss the relation between general semidiscrete surface flexibility and 3-ribbon subsurface flexibility. We conclude this paper with one surprising property of isometric deformations of developable semidiscrete surfaces.

Contents

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Introduction

A mapping $f : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}^3$, where the dependence on the continuous parameter is smooth, is called a *semidiscrete surface*. Let us connect $f(t,z)$ with $f(t,z{+}1)$ by segments for all possible pairs (t,z). The resulting surface is a *piecewise ruled surface*.

In this paper we study infinitesimal and finite flexibility for such semidiscrete surfaces. By *isometric deformations* of a semidiscrete surface f we understand deformations that preserves inner geometry of the corresponding ruled surfaces and in addition that preserve all line segments connecting $f(t,z)$ with $f(t,z{+}1)$.

Many questions on discrete polyhedral surfaces have their origins in classical theory of smooth surfaces. Flexibility is not an exception from this rule. The general theory of flexibility of surfaces and polyhedra is discussed in the overview [12] by I. Kh. Sabitov.

In 1890 [1] L. Bianchi introduced a necessary and sufficient condition for the existence of isometric deformations of a surface preserving some conjugate system (i.e., two independent smooth fields of directions tangent to the surface), see also in [5]. Such surfaces can be understood as certain limits of semidiscrete surfaces.

On the other hand, semidiscrete surfaces are themselves the limits of certain polygonal surfaces (or *meshes*). For the discrete case of flexible meshes much is now known. We refer the reader to [2], [10], [8], and [6] for some recent results in this area. For general relations to the classical case see a recent book [3] by A. I. Bobenko and Yu. B. Suris. It is interesting to notice that the flexibility conditions in the smooth case and the discrete case are of a different nature. Currently there is no clear description of relations between them in terms of limits.

The place of the study of semidiscrete surfaces is between the classical and the discrete cases. Main concepts of semidiscrete theory are described by J. Wallner in [13], and [14]. Some problems related to isothermic semidiscrete surfaces are studied by C. Müller in [9]. Semidiscrete surfaces from the viewpoint of parallelity, offsets, and curvatures were studied by J. Wallner and O. Karpenkov in [7].

We investigate necessary condition for existence of isometric deformations of semidiscrete surfaces. To avoid pathological behavior related to noncompactness of semidiscrete surfaces we restrict ourselves to compact subsets of the following type. An *n-ribbon surface* is a mapping

$$f : [a,b] \times \{0,\ldots,n\} \to \mathbb{R}^3, \qquad (t,i) \mapsto f_i(t).$$

We also use the notion

$$\Delta f_i(t) = f_{i+1}(t) - f_i(t).$$

While working with a rather abstract semidiscrete or n-ribbon surface f we keep in mind the two-dimensional piecewise-ruled surface associated to it (see Fig. 1).

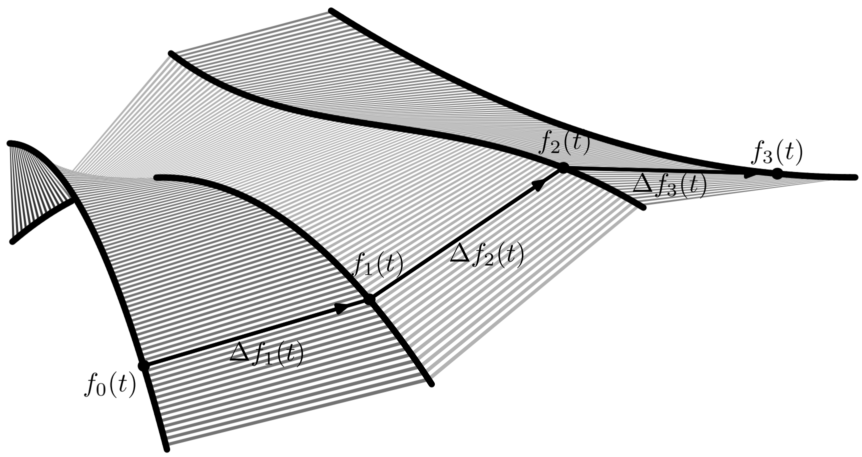

FIGURE 1. A 3-ribbon surface.

Note that, within this paper we traditionally consider t as an argument of a semidiscrete surface f. The time parameter for deformations is λ.

In present paper we prove that every generic 2-ribbon surface (as a ruled surface) is flexible and has one degree of infinitesimal and finite flexibility in the generic case (Theorem 2.3 and Theorem 2.23). This is quite surprising since generic 1-ribbon surfaces have infinitely many degrees of flexibility, see, for instance, in [11], Theorem 5.3.10. We also find a system of differential equations for the deformation of 2-ribbon surfaces (Definition 2.17 and Proposition 2.20). In contrast to that, a generic n-ribbon surface is rigid for $n \geq 3$. For the case $n = 3$ we prove the following statement (see Theorem 3.7 and Remark 3.8).

Infinitesimal flexibility condition.
If a 3-ribbon surface is infinitesimally flexible then the following condition holds:

$$\dot{\Lambda} = (H_2 - H_1)\Lambda,$$

where

$$\Lambda = \frac{(\dot{f}_1, \ddot{f}_1, \Delta f_0)}{(\dot{f}_2, \ddot{f}_2, \Delta f_2)} \frac{(\dot{f}_2, \Delta f_1, \Delta f_2)^2}{(\dot{f}_1, \Delta f_0, \Delta f_1)^2},$$

and

$$H_i(t) = \frac{(\dot{f}_i, \Delta\dot{f}_{i-1}, \Delta f_i) + (\dot{f}_i, \Delta f_{i-1}, \Delta\dot{f}_i)}{(\dot{f}_i, \Delta f_{i-1}, \Delta f_i)}, \quad i = 1, 2.$$

Remark. Throughout this paper we denote the derivative with respect to variable t by the dot symbol.

Further in Theorem 4.4 we state that a generic n-ribbon surface ($n \geq 3$) has at most one degree of finite and infinitesimal flexibility. Finally, we show that a generic n-ribbon surface ($n \geq 4$) is infinitesimally or finitely flexible if and only if all its 3-ribbon subsurfaces are infinitesimally or finitely flexible (see Theorems 4.10 and 4.11). We say a few words in the case of developable semidiscrete surfaces whose finite isometric deformations have additional surprising properties.

Organization of the paper. We start in Section 1 with introduction of necessary notions and definitions. In Section 2 we discuss flexibility of 2-ribbon surfaces. We study infinitesimal flexibility questions for 2-ribbon surfaces in Subsections 2.2 and 2.3. In Subsection 2.2 we give a system of differential equations for infinitesimal flexions, prove the existence of nonzero solutions, and show that all the solutions are proportional to each other. In Subsection 2.3 we define the variational operators of infinitesimal flexion which is studied further in the context of finite flexibility for 2-ribbon surfaces. In Subsection 2.4 we prove that a generic 2-ribbon surface is finitely flexible and has one degree of flexibility. In Section 3 we work with 3-ribbon surfaces. After some preliminary statements of Subsection 3.1 we gives a necessary infinitesimal flexibility condition for 3-ribbon surfaces in Subsection 3.2. In Section 4 we deal with general n-ribbon surfaces for $n \geq 3$. We prove that a generic n-ribbon surface has at most one degree of finite and infinitesimal flexibility in Subsection 4.1. Further after several preparatory statements of Section 4.2 we prove that finite or infinitesimal flexibility of generic n-ribbon surfaces is identified by finite or infinitesimal flexibility of all its 3-ribbon subsurfaces. We conclude the paper with flexibility of developable semidiscrete surfaces in Section 5. In this case isometric deformations have a remarkable geometric property.

1. Necessary notions and definitions

In this section we introduce central notions and definition of the article.

1.1. Differentiable generic semidiscrete surfaces.

We start with several basic definitions.

Definition 1.1. Let $M = (m_0, \ldots, m_n)$ be the $(n+1)$-tuple of non-negative integers. We say that an n-ribbon surface f is a *M-differentiable* if for every $i \in \{0, \ldots, n\}$ and $j \in \{1, \ldots, m_i\}$ there exists a continuous derivative $f_i^{(j)}$.
Denote by $C^{m_0,\ldots,m_n}([a,b],\mathbb{R}^3)$ (or $C^M([a,b],\mathbb{R}^3)$, for short) the Banach space of all M-differentiable n-ribbon surfaces (where $t \in [a,b]$) with the standard norm

$$\rho(f,g) = \max_{i=\{0,\ldots,n\}} \max_{j=\{1,\ldots,m_i\}} \sup_{[a,b]} (f_i^{(j)} - g_i^{(j)}).$$

Remark 1.2. Note that for two non-negative $(n+1)$-tuples $M = (m_0, \ldots, m_n)$ and $K = (k_0, \ldots, k_n)$ satisfying

$$m_0 \geq k_0, \quad \ldots, \quad m_n \geq k_n$$

in holds

$$C^M([a,b],\mathbb{R}^3) \subset C^K([a,b],\mathbb{R}^3).$$

Definition 1.3. We say that an n-ribbon surface f in the space $C^{1,2,2,\ldots,2,1}([a,b],\mathbb{R}^3)$ is *weakly generic* if for every $t\in[a,b]$ and $i=1,\ldots,n-1$ we have

$$(\dot f_i, \Delta f_{i-1}, \Delta f_i)\neq 0.$$

Definition 1.4. We say that an n-ribbon surface f in the space $C^{1,2,2,\ldots,2,1}([a,b],\mathbb{R}^3)$ is *strongly generic* if

— f is weakly generic;

— for every $t\in[a,b]$ and $i=1,\ldots,n-1$ we have

$$\big(\dot f_i(t), \ddot f_i(t), \Delta f_{i-1}(t)\big)\neq 0 \qquad \text{and} \qquad \big(\dot f_i(t), \ddot f_i(t), \Delta f_i(t)\big)\neq 0.$$

1.2. **Isometric semidiscrete surfaces.** Let us now study basic properties of the definition of isometric semidiscrete surfaces.

Definition 1.5. Two n-ribbon surfaces f and g in the space $C^{1,1,\ldots,1}([a,b],\mathbb{R}^3)$ are said to be *isometric* if

$$\begin{cases} |\dot f_i| = |\dot g_i| \\ |\Delta f_i| = |\Delta g_i| \\ \langle \dot f_i, \Delta f_{i-1}\rangle = \langle \dot g_i, \Delta g_{i-1}\rangle \\ \langle \dot f_i, \Delta f_i\rangle = \langle \dot g_i, \Delta g_i\rangle \\ \langle \dot f_i, \dot f_{i+1}\rangle = \langle \dot g_i, \dot g_{i+1}\rangle \end{cases}$$

(for all admissible i and t).

Before to continue let us show that the conditions of Definition 1.5 are precisely the isometric conditions for ruled surfaces. Let f_1 and f_2 be differentiable curves (denote by $\Delta_1 f$ the curve f_2-f_1). Let us define a ruled surface $S(x,t)=xf_1(t)+(1-x)f_2(t)$. To show that the conditions of Definition 1.5 determine integer geometry we prove the following proposition.

Proposition 1.6. *The first fundamental form of the ruled surface $S(x,t)$ is uniquely defined by*

$$|\dot f_1|, \quad |\dot f_2|, \quad |\Delta f_1|, \quad \langle \dot f_1, \Delta f_1\rangle, \quad \langle \dot f_2, \Delta f_1\rangle, \quad \langle \dot f_1, \dot f_2\rangle$$

and vice versa.

Proof. Let us write all the coefficients of the first fundamental form of the surfaces in the coordinates (x,t):

$$\begin{aligned} \left\langle \frac{\partial S}{\partial x}, \frac{\partial S}{\partial x}\right\rangle &= \langle f_1-f_2, f_1-f_2\rangle = |\Delta f_1|^2; \\ \left\langle \frac{\partial S}{\partial x}, \frac{\partial S}{\partial t}\right\rangle &= \big\langle f_1-f_2, x\dot f_1+(1-x)\dot f_2\big\rangle = x\langle \Delta f_1, \dot f_1\rangle + (1-x)\langle \Delta f_1, \dot f_2\rangle; \\ \left\langle \frac{\partial S}{\partial t}, \frac{\partial S}{\partial t}\right\rangle &= \big\langle x\dot f_1+(1-x)\dot f_2(t), x\dot f_1+(1-x)\dot f_2(t)\big\rangle \\ &= x^2|f_1|^2 + 2x(1-x)\langle \dot f_1, \dot f_2\rangle + (1-x)^2|f_2|^2. \end{aligned}$$

As we see, on the one hand the first fundamental form is defined by the above six functions. On the other hand the values of the first fundamental form at $x=0,1/2,1$ defines the values of the above six functions. □

1.3. **Deformations and flexions of semidiscrete surfaces.** We start with the following general definition.

Definition 1.7. A *deformation* of a semidiscrete n-ribbon surface f is a family of n-ribbon surfaces $\{f^\lambda\}$ with parameter λ in the segment $[-\Lambda, \Lambda]$ for some positive Λ such that $f^0 = f$. In this paper we consider only deformations that are continuously differentiable in λ.

Remark 1.8. In this paper λ is the parameter of deformations, while t is the first argument of semidiscrete surfaces.

Let us give a formal definition of deformations that do not change the inner geometry of a surface.

Definition 1.9. We say that a deformation $\{f^\lambda\}$ of a semidiscrete n-ribbon surface f is *isometric* if all the surfaces in the deformation are isometric to each other.

Definition 1.10. Consider a family of functions, vector functions, or semidiscrete surfaces $\gamma = \{w^\lambda\}$ with parameter $\lambda \in [-\varepsilon, \varepsilon]$ for some positive ε, and let $w = w^0$. We say that the derivative

$$\mathcal{D}_\gamma w = \frac{\partial w^\lambda}{\partial \lambda}\Big|_{\lambda=0}$$

is an *infinitesimal deformation* of w.

The infinitesimal deformation of an n-ribbon surface f in $C^M([a,b],\mathbb{R}^3)$ is an element of the tangent space $T_f C^M([a,b],\mathbb{R}^3)$, which is naturally isomorphic to $C^M([a,b],\mathbb{R}^3)$.

Definition 1.11. Consider a deformation $\{f^\lambda\}$ of a semidiscrete n-ribbon surface f in $C^{(1,2,2,\ldots,2,1)}([a,b],\mathbb{R}^3)$. We say that the deformation $\{f^\lambda\}$ is *infinitesimally flexible* if

$$\mathcal{D}_\gamma|\dot{f}_i^\lambda| = 0, \qquad \mathcal{D}_\gamma|\Delta f_i^\lambda| = 0, \qquad \mathcal{D}_\gamma\langle \dot{f}_i^\lambda, \Delta f_{i-1}^\lambda\rangle,$$
$$\mathcal{D}_\gamma\langle \dot{f}_i^\lambda, \Delta f_i^\lambda\rangle = 0, \quad \text{and} \quad \mathcal{D}_\gamma\langle \dot{f}_i^\lambda, \dot{f}_{i+1}^\lambda\rangle = 0$$

(for all admissible i and t).

In fact, infinitesimal flexibility is a property of tangent spaces rather than deformations.

Definition 1.12. We say that a tangent vector $\mathcal{D}f$ at a semidiscrete surface f is an *infinitesimal flexion* if the deformation $\mathcal{D}_\gamma f$ where

$$\gamma(\lambda) = f + \lambda \mathcal{D}f$$

is infinitesimally isometric.
We say that an infinitesimal flexion $\mathcal{D}f$ is a *finite flexion* if there exists an isometric deformation γ with $\gamma(0) = f$ such that $\mathcal{D}_\gamma f = \mathcal{D}f$.

Finally let us determine isometrically nontrivial infinitesimal flexions.

Definition 1.13. An infinitesimal flexion of a weakly generic n-ribbon surface f in $C^{0,1,0}([a,b],\mathbb{R}^3)$ is said to be *isometrically nontrivial* (*trivial*) *at point* (t,i) for some $t \in [a,b]$ and $n \in \{1,\ldots,n-1\}$ if the corresponding infinitesimal deformation of the

angle between the planes spanned by $(\dot{f}_i(t)\Delta f_{i-1}(t))$ and $(\dot{f}_i(t)\Delta f_i(t))$ is nonzero (or zero, respectively).
We say that an infinitesimal flexion of f is *isometrically nontrivial* if it is isometrically nontrivial at least at one point (t, i). Otherwise an infinitesimal inflexion is said to be *isometrically trivial*.
We say that an infinitesimal flexion of f is *strongly isometrically nontrivial* if it is isometrically nontrivial at every point (t, i).

1.4. **Spaces of semidiscrete surfaces with fixed initial position.** In order to calculate the degree of flexibility for a semidiscrete surfaces we should eliminate trivial Euclidean deformations of the surfaces. Let us do this as follows.

Definition 1.14. Denote by

$$C_0^M([a,b],\mathbb{R}^3) \subset C^M([a,b],\mathbb{R}^3)$$

the subset of all *2-ribbon surfaces with fixed initial position*, namely an n-ribbon surface f is in $C_0^M([a,b],\mathbb{R}^3)$ if and only if
— $f_1(0) \in C^M([a,b],\mathbb{R}^3)$;
— $f_1(0) = (0,0,0)$;
— the vector $\dot{f}_1(0)$ is proportional to $(1,0,0)$;
— the vector $\Delta f_0(0)$ has the coordinates $(p,q,0)$.

Remark 1.15. Let Σ denotes all weakly non-generic semidiscrete surfaces. Notice that the set $C_0^M([a,b],\mathbb{R}^3) \setminus \Sigma$ has a natural structure of an 8-fold covering of the quotient space of $C^M([a,b],\mathbb{R}^3) \setminus \Sigma$ by the Euclidean congruence relation. In other words, for every weakly generic M-differentiable semidiscrete surface f there exists exactly eight semidiscrete surfaces that are congruent to f. These 8 surfaces are obtained one from another by 8 symmetries of type

$$(e_1, e_2, e_3) \to (\pm e_1, \pm e_2, \pm e_3).$$

So, on the one hand one can consider any branch of the 8-fold for studying flexibility properties of the original n-ribbon curve. On the other hand the set $C_0^M([a,b],\mathbb{R}^3)$ has a structure of a vector space. For these reasons from now on we prefer to consider the space $C_0^M([a,b],\mathbb{R}^3)$, rather than the quotient space of $C^M([a,b],\mathbb{R}^3) \setminus \Sigma$ by the group of all Euclidean transformation.

Since $C_0^M([a,b],\mathbb{R}^3)$ is a subspace of $C^M([a,b],\mathbb{R}^3)$ we have the induced metric and topology (in particular, $C_0^M([a,b],\mathbb{R}^3)$ is a Banach space), definitions of deformations, isometric deformations, infinitesimal and finite flexions, isometrically trivial and nontrivial infinitesimal flexions in $C_0^M([a,b],\mathbb{R}^3)$.

1.5. **Rigid surfaces. Degrees of flexibility.** We start with the definitions for infinitesimal flexibility.

Definition 1.16. The set of infinitesimal flexions in $C_0^M([a,b],\mathbb{R}^3)$ is a linear space. We say that f has n *degrees of infinitesimal flexibility* if the dimension of the space of infinitesimal flexions is n. If $n = 0$ we say that f is *infinitesimally rigid*.

In the finite case we define only finitely rigid surfaces and surfaces that has one degree of finite flexibility. In order to define finite rigidity we use the following definition.

Definition 1.17. We say that an isometric deformation γ of f in $C_0^M([a,b],\mathbb{R}^3)$ is *regular* at 0 if $\mathcal{D}_\gamma f \neq 0$.

Definition 1.18. We say that an n-ribbon surface f in $C_0^M([a,b],\mathbb{R}^3)$ is *finitely rigid* if the set of regular isometric deformations of f is empty.

Let us finally give the definition of the property to have one degree of finite flexibility. As in infinitesimal case we consider only the space of semidiscrete surfaces with fixed initial position $C_0^M([a,b],\mathbb{R}^3)$. This cancels excess trivial Euclidean rotations of the whole semidiscrete surface. Of course, every finite isometric deformations of a semidiscrete surface with fixed initial position still can be reparametrised, as a result one has another isometric deformation of the surface. So the best thing would be to try to normalize them.

In this paper we consider the following "natural parametrization" of an isometric deformation. It is clear that for every isometric deformation $\{f^\lambda\}$ in $C_0^M([a,b],\mathbb{R}^3)$ we have

$$\mathcal{D}_{f^\lambda}\dot{f}(a)=0, \quad \mathcal{D}_{f^\lambda}\Delta f_0(a)=0, \quad \text{and} \quad \mathcal{D}_{f^\lambda}\Delta f_1(a)=\alpha(\lambda)\dot{f}(a)\times\Delta f_1(a)$$

for some real valued function α.

Definition 1.19. We say that an isometric deformation $\{f^\lambda\}$ is *normalized* if and only if for every admissible values of parameter λ we have $\alpha(\lambda)=1$, where α is the real-valued function defined in the last expression.

In our case by Corollary 2.11 below we have: if $\alpha(\lambda_0)=0$ then $\mathcal{D}_{f^\lambda}f^{\lambda_0}=0$. Hence, there is no regular isometric deformation that preserves the frame at $t=a$. So we can give the following definition.

Definition 1.20. We say that a weakly generic 2-ribbon surface f has *one degree of finite flexibility* if

— f has one degree of infinitesimal flexibility.

— for sufficiently small $\varepsilon>0$ there exists a unique normalized isometric deformation of f defined on $[-\varepsilon,\varepsilon]$.

2. Finite and infinitesimal flexibility of 2-ribbon surfaces

In this section we describe flexions of 2-ribbon surfaces. Such surfaces are defined by three curves f_0, f_1, and f_2. Our main goal here is to prove under some natural genericity assumptions that every 2-ribbon surface is infinitesimally and finitely flexible and has one degree of infinitesimal and finite flexibility. Our first point is to describe the system of differential equations (System A) that determines infinitesimal flexions corresponding to finite flexions and find solutions to this system (see Subsections 2.2). We use it to derive finite flexibility in Theorem 2.3 (also in Subsections 2.2). Further via solutions of System A we define the variational operators of infinitesimal flexion $\mathcal{V}^\pm$ (in Subsection 2.3). Finally, to show finite flexibility of 2-ribbon surfaces we study Lipschitz properties for $\mathcal{V}^\pm$ and prove flexibility Theorem 2.23 (in Subsection 2.4).

2.1. **Basic relations for infinitesimal flexions.** In this small subsection we collect some useful relations.

Proposition 2.1. *Let f be a 2-ribbon surface in $C^{1,2,1}([a,b],\mathbb{R}^3)$. Then for every infinitesimal flexion $\mathcal{D}f$ the following properties hold:*

$$\langle \dot f_1, \mathcal{D}\dot f_1\rangle = 0; \tag{1}$$
$$\langle \dot f_1 - \Delta\dot f_0, \mathcal{D}\dot f_1 - \mathcal{D}\Delta\dot f_0\rangle = 0; \tag{2}$$
$$\langle \dot f_1 + \Delta\dot f_1, \mathcal{D}\dot f_1 + \mathcal{D}\Delta\dot f_1\rangle = 0; \tag{3}$$
$$\langle \Delta f_0, \mathcal{D}\Delta\dot f_0\rangle + \langle \Delta\dot f_0, \mathcal{D}\Delta f_0\rangle = 0; \tag{4}$$
$$\langle \Delta f_1, \mathcal{D}\Delta\dot f_1\rangle + \langle \Delta\dot f_1, \mathcal{D}\Delta f_1\rangle = 0; \tag{5}$$
$$\langle \dot f_1, \mathcal{D}\Delta\dot f_0\rangle + \langle \mathcal{D}\dot f_1, \Delta\dot f_0\rangle = 0; \tag{6}$$
$$\langle \dot f_1, \mathcal{D}\Delta\dot f_1\rangle + \langle \mathcal{D}\dot f_1, \Delta\dot f_1\rangle = 0; \tag{7}$$
$$\langle \mathcal{D}\ddot f_1, \Delta f_0\rangle + \langle \ddot f_1, \mathcal{D}\Delta f_0\rangle = 0; \tag{8}$$
$$\langle \mathcal{D}\ddot f_1, \Delta f_1\rangle + \langle \ddot f_1, \mathcal{D}\Delta f_1\rangle = 0. \tag{9}$$

Remark 2.2. For a semidiscrete or n-ribbon surface f the operations $\mathcal{D}$, Δ, and $\frac{\partial}{\partial t}$ commute, so we do not pay attention to the order of these operations in compositions.

Proof. Equations (1), (2), and (3) follow from the fact that infinitesimal flexions preserve the norms of $\dot f_1$, $\dot f_0 = \dot f_1 - \Delta\dot f_0$, and $\dot f_2 = \dot f_1 + \Delta\dot f_1$ respectively.

The invariance of the lengths of Δf_0 and Δf_1 imply Equations (4), and (5) respectively. They are equivalent to

$$\frac{\partial}{\partial t}\mathcal{D}\langle \Delta f_0, \Delta f_0\rangle = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\mathcal{D}\langle \Delta f_1, \Delta f_1\rangle = 0.$$

Equations (6) and (7) follow from invariance of the angles between the vectors $\dot f_1$ and $\Delta\dot f_0$ and the vectors $\dot f_1$ and $\Delta\dot f_0$.

Let us prove Equation (8). Since the angles between the vectors Δf_0 and $\dot f_1$ are preserved by infinitesimal flexions we have

$$\frac{\partial}{\partial t}\mathcal{D}\langle \dot f_1, \Delta f_0\rangle = 0.$$

Therefore,

$$\langle \mathcal{D}\ddot f_1, \Delta f_0\rangle + \langle \ddot f_1, \mathcal{D}\Delta f_0\rangle + \langle \mathcal{D}\dot f_1, \Delta\dot f_0\rangle + \langle \dot f_1, \mathcal{D}\Delta\dot f_0\rangle = 0.$$

By Equation (6) we have $\langle \mathcal{D}\dot f_1, \Delta\dot f_0\rangle + \langle \dot f_1, \mathcal{D}\Delta\dot f_0\rangle = 0$ and hence

$$\langle \mathcal{D}\ddot f_1, \Delta f_0\rangle + \langle \ddot f_1, \mathcal{D}\Delta f_0\rangle = 0.$$

We have arrived at Equation (8).

Finally Equations (9) is proved by analogy with Equations (8). □

2.2. **Infinitesimal flexibility of 2-ribbon surfaces.** Our main goal for this subsection is to prove the following general theorem

Theorem 2.3. *Let $f \in C_0^{1,2,1}([a,b],\mathbb{R}^3)$ be a weakly generic 2-ribbon surface with fixed initial position. Then f has one degree of infinitesimal flexibility.*

First we write down and investigate a supplementary system of differential equations (System A) which describes infinitesimal flexions of weakly generic 2-ribbon surfaces. We also show the uniqueness of the solution of System A for a given initial data (Proposition 2.6). The remaining part of this subsection is dedicated to the proof of Theorem 2.3 mentioned above. In Proposition 2.7 we show that every infinitesimal flexion satisfies System A. Then in Proposition 2.9 we prove that every solution of System A with certain initial data is an infinitesimal flexion. After that we prove Theorem 2.3.

2.2.1. *System A.* Let

$$
\begin{array}{lll}
G_{11}=\langle \mathcal{D}\dot f_1,\dot f_1\rangle, & G_{12}=\langle \mathcal{D}\dot f_1,\Delta f_0\rangle, & G_{13}=\langle \mathcal{D}\dot f_1,\Delta f_1\rangle,\\
G_{21}=\langle \mathcal{D}\Delta f_0,\dot f_1\rangle, & G_{22}=\langle \mathcal{D}\Delta f_0,\Delta f_0\rangle, & G_{23}=\langle \mathcal{D}\Delta f_0,\Delta f_1\rangle,\\
G_{31}=\langle \mathcal{D}\Delta f_1,\dot f_1\rangle, & G_{32}=\langle \mathcal{D}\Delta f_1,\Delta f_0\rangle, & G_{33}=\langle \mathcal{D}\Delta f_1,\Delta f_1\rangle.
\end{array}
\tag{10}
$$

Denote by *System A* the following system of differential equations

$$
\left\{
\begin{array}{rcl}
\dot G_{11} &=& 0,\\
\dot G_{12} &=& \left(\frac{(\dot f_1,\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\ddot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)G_{12}+\frac{(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{13}-\frac{(\dot f_1,\Delta f_0,\ddot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{23},\\
\dot G_{13} &=& \frac{(\dot f_1,\Delta\dot f_1,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{12}+\left(\frac{(\dot f_1,\Delta f_0,\Delta\dot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\ddot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)G_{13}-\frac{(\dot f_1,\ddot f_1,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{32},\\
\dot G_{21} &=& -\left(\frac{(\dot f_1,\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\ddot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)G_{12}-\frac{(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{13}+\frac{(\dot f_1,\Delta f_0,\ddot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{23},\\
\dot G_{22} &=& 0,\\
\dot G_{23} &=& -\left(\frac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta\dot f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)}-\frac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta f_0)(\Delta\dot f_0,\Delta f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)}+\right.\\
&& \left.\frac{(\dot f_1,\Delta f_0\times\Delta\dot f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2}+\frac{(\dot f_1\times\Delta\dot f_0,\Delta f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2}+\frac{(\Delta\dot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)G_{12}-\\
&& \left(\frac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\dot f_1,\Delta f_0,\Delta f_0\times\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2}\right)G_{13}-\\
&& \left(\frac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)}-\frac{(\dot f_1,\Delta f_0,\dot f_1\times\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2}-\frac{(\dot f_1,\Delta f_0,\Delta\dot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)G_{23},\\
\dot G_{31} &=& -\frac{(\dot f_1,\Delta\dot f_1,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{12}-\left(\frac{(\dot f_1,\Delta f_0,\Delta\dot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\ddot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)G_{13}+\frac{(\dot f_1,\ddot f_1,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{32},\\
\dot G_{32} &=& -\left(\frac{(\Delta f_0,\Delta f_1,\dot f_1\times\Delta f_1)(\dot f_1,\Delta\dot f_1,\Delta f_1)}{|\dot f_1\times\Delta f_1|^2(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\dot f_1,\Delta f_1,\Delta f_1\times\Delta\dot f_1)}{|\dot f_1\times\Delta f_1|^2}\right)G_{12}-\\
&& \left(\frac{(\Delta f_0,\Delta f_1,\dot f_1\times\Delta f_1)(\dot f_1,\Delta f_0,\Delta\dot f_1)}{|\dot f_1\times\Delta f_1|^2(\dot f_1,\Delta f_0,\Delta f_1)}-\frac{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_1)(\Delta\dot f_1,\Delta f_0,\Delta f_1)}{|\dot f_1\times\Delta f_1|^2(\dot f_1,\Delta f_0,\Delta f_1)}+\right.\\
&& \left.\frac{(\dot f_1,\Delta f_1\times\Delta\dot f_1,\Delta f_0)}{|\dot f_1\times\Delta f_1|^2}+\frac{(\dot f_1\times\Delta\dot f_1,\Delta f_1,\Delta f_0)}{|\dot f_1\times\Delta f_1|^2}+\frac{(\Delta\dot f_0,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)G_{13}-\\
&& \left(\frac{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_1)(\dot f_1,\Delta\dot f_1,\Delta f_1)}{|\dot f_1\times\Delta f_1|^2(\dot f_1,\Delta f_0,\Delta f_1)}-\frac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta\dot f_1)}{|\dot f_1\times\Delta f_1|^2}-\frac{(\dot f_1,\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)G_{32},\\
\dot G_{33} &=& 0.
\end{array}
\right.
$$

Remark 2.4. In Proposition 3.2 below we show an explicit formula for the function $G_{23}+G_{32}$, it is Φ in our notation of Section 2.

Note also that $\dot{G}_{12} + \dot{G}_{21} = 0$ and $\dot{G}_{13} + \dot{G}_{31} = 0$ in System A.

Example 2.5. Let us consider a simple example of a 2-ribbon curve where $\dot{f}$, Δf_0, and Δf_1 are all constants. Let us call these surfaces *book-shaped surfaces.* Direct calculations show that

$$\dot{G}_{11} = \dot{G}_{12} = \ldots = \dot{G}_{33} = 0$$

(this happens, since all the summands in the coefficients of System A contain either $\ddot{f}_1$, or $\Delta \dot{f}_0$, or $\Delta \dot{f}_1$ which are all zeroes in our case). Hence all the scalar products of the deformation with vectors $\dot{f}_1, \Delta f_0, \Delta f_1$ do not depend on t. Therefore, every element of every isometric deformations of a book-shaped surface is a book-shaped surface. Here is a typical example of isometric deformation in this class:

$$f_1^\lambda(t) = (t, 0, 0), \quad \Delta f_0^\lambda(t) = (0, 1, 0), \quad \Delta_1^\lambda(t) = (0, \sin\lambda, \cos\lambda).$$

This deformation can be geometrically seen as an opening a museum book with two rigid plastic pages.

In the following proposition we prove that for every single 2-ribbon surface f (not for a deformation) and initial data for G_{ij} at one point $f(t_0)$ System A has a unique solution. Recall that t is an argument of f.

Proposition 2.6. *Let f be a weakly generic 2-ribbon surface in $C^{1,2,1}([a,b], \mathbb{R}^3)$. For every collection of initial data $G_{ij}(a) = c_{ij}$ there exists a unique solution of System A on $[a,b]$.*

Proof. System A is the system of homogeneous linear differential equations with smooth variable coefficients (since $(\dot{f}_1, \Delta f_0, \Delta f_1)$ never vanishes on $[a,b]$) and hence for every collection of initial data it has a unique solution on the segment $[a,b]$. □

2.2.2. *Every infinitesimal flexion satisfies System A.* Let us show the following statement.

Proposition 2.7. *Let f be a weakly generic 2-ribbon surface in $C^{1,2,1}([a,b], \mathbb{R}^3)$. Then for every infinitesimal flexion $\mathcal{D}f$ the functions $G_{11}, G_{12}, \ldots, G_{33}$ satisfy system A.*

We start the proof with the following general lemma.

Lemma 2.8. *For every infinitesimal flexion $\mathcal{D}f$ we have the equalities*

$$G_{11} = G_{22} = G_{33} = 0, \qquad G_{12} + G_{21} = 0, \quad \textit{and} \quad G_{13} + G_{31} = 0.$$

Proof. The functions $|\dot{f}_1|$, $|\Delta f_0|$, and $|\Delta f_1|$ are infinitesimally preserved by infinitesimal flexions, hence G_{11}, G_{22}, and G_{33} vanish.

The invariance of angles between $\dot{f}_1$ and Δf_0, and $\dot{f}_1$ and Δf_1 yield the equations $G_{12} + G_{21} = 0$ and $G_{13} + G_{31} = 0$, respectively. □

Proof of Proposition 2.7. From Lemma 2.8 the functions G_{11}, G_{22}, and G_{33} are zero functions, thus $\dot{G}_{11}$, $\dot{G}_{22}$, and $\dot{G}_{33}$ are zero functions as well.

Let us prove the expression for $\dot{G}_{12}$ and $\dot{G}_{13}$. Note that

$$\dot{G}_{12} = \langle \mathcal{D}\ddot{f}_1, \Delta f_0 \rangle + \langle \mathcal{D}\dot{f}_1, \Delta \dot{f}_0 \rangle.$$

Thus Equations (6) and (8) imply

$$\dot G_{12} = \langle \mathcal{D}\dot f_1, \Delta \dot f_0\rangle - \langle \ddot f_1, \mathcal{D}\Delta f_0\rangle.$$

To obtain the expression for $\dot G_{12}$ rewrite $\Delta \dot f_0$ and $\ddot f_1$ in the basis consisting of vectors $\dot f_1$, Δf_0, and Δf_1.

$$\begin{aligned}
\dot G_{12} &= \langle \mathcal{D}\dot f_1, \Delta \dot f_0\rangle - \langle \ddot f_1, \mathcal{D}\Delta f_0\rangle \\
&= \left(\frac{(\Delta\dot f_0, \Delta f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{11} + \frac{(\dot f_1, \Delta\dot f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{12} + \frac{(\dot f_1, \Delta f_0, \Delta\dot f_0)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{13}\right) \\
&\quad - \left(\frac{(\ddot f_1, \Delta f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{21} + \frac{(\dot f_1, \ddot f_1, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{22} + \frac{(\dot f_1, \Delta f_0, \ddot f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{23}\right) \\
&= \left(\frac{(\dot f_1, \Delta\dot f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)} + \frac{(\ddot f_1, \Delta f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}\right)G_{12} + \frac{(\dot f_1, \Delta f_0, \Delta\dot f_0)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{13} - \frac{(\dot f_1, \Delta f_0, \ddot f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{23}.
\end{aligned}$$

The last equation holds since $G_{11} = 0$, $G_{22} = 0$, and $G_{21} = -G_{12}$.

The same strategy works for the functions $\dot G_{13}$.

Now we study expressions for $\dot G_{21}$ and $\dot G_{31}$. From Lemma 2.8 we know that $G_{21} = -G_{12}$ and $G_{31} = -G_{13}$ and hence $\dot G_{21} = -\dot G_{12}$ and $\dot G_{31} = -\dot G_{13}$. Therefore, the equations for $\dot G_{21}$ and $\dot G_{31}$ are satisfied.

In order to get the expression for $\dot G_{23}$, we first show that the function $(\dot f_1, \Delta f_0, \Delta \dot f_0)$ is an invariant of infinitesimal flexions. Indeed,

$$(\dot f_1, \Delta f_0, \Delta \dot f_0) = (\dot f_1, \Delta f_0, \dot f_1 - \dot f_0) = -(\dot f_1, \Delta f_0, \dot f_0).$$

The vectors $\dot f_0$, $\dot f_1$, and Δf_0 form a rigid frame, hence their triple product is an invariant of infinitesimal flexions. Hence the function $(\dot f_1, \Delta f_0, \Delta \dot f_0)$ is an invariant as well.

The infinitesimal flexion invariance of $(\dot f_1, \Delta f_0, \Delta \dot f_0)$ implies that $\mathcal{D}(\dot f_1, \Delta f_0, \Delta \dot f_0) = 0$. So we get

$$(\mathcal{D}\dot f_1, \Delta f_0, \Delta \dot f_0) + (\dot f_1, \mathcal{D}\Delta f_0, \Delta \dot f_0) + (\dot f_1, \Delta f_0, \mathcal{D}\Delta \dot f_0) = 0.$$

Rewrite

$$\begin{aligned}
(\dot f_1, \Delta f_0, \mathcal{D}\Delta \dot f_0) &= -(\mathcal{D}\dot f_1, \Delta f_0, \Delta \dot f_0) - (\dot f_1, \mathcal{D}\Delta f_0, \Delta \dot f_0) \\
&= -\langle \mathcal{D}\dot f_1, \Delta f_0 \times \Delta \dot f_0\rangle + \langle \mathcal{D}\Delta f_0, \dot f_1 \times \Delta \dot f_0\rangle \\
&= -\tfrac{(\Delta f_0 \times \Delta \dot f_0, \Delta f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{11} - \tfrac{(\dot f_1, \Delta f_0 \times \Delta \dot f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{12} - \tfrac{(\dot f_1, \Delta f_0, \Delta f_0 \times \Delta \dot f_0)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{13} + \\
&\quad \tfrac{(\dot f_1 \times \Delta \dot f_0, \Delta f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{21} + \tfrac{(\dot f_1, \dot f_1 \times \Delta \dot f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{22} + \tfrac{(\dot f_1, \Delta f_0, \dot f_1 \times \Delta \dot f_0)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{23}.
\end{aligned}$$

Second, we have

$$\langle \mathcal{D}\Delta \dot f_0, \Delta f_0\rangle = -\langle \mathcal{D}\Delta f_0, \Delta \dot f_0\rangle = -\tfrac{(\Delta \dot f_0, \Delta f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{21} - \tfrac{(\dot f_1, \Delta \dot f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{22} - \tfrac{(\dot f_1, \Delta f_0, \Delta \dot f_0)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{23}.$$

Third, we get

$$\langle \mathcal{D}\Delta\dot f_0, \dot f_1\rangle = -\langle \mathcal{D}\dot f_1, \Delta\dot f_0\rangle = -\tfrac{(\dot f_1,\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{12} - \tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{13}.$$

Fourth,

$$\langle \mathcal{D}\Delta\dot f_0, \Delta f_1\rangle = \tfrac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}\langle \mathcal{D}\Delta\dot f_0, \dot f_1\rangle + \tfrac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}\langle \mathcal{D}\Delta\dot f_0, \Delta f_0\rangle + \tfrac{(\dot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}(\dot f_1, \Delta f_0, \mathcal{D}\Delta\dot f_0).$$

After the substitution of the four above expressions and simplifications we have

$$\begin{aligned}\langle \mathcal{D}\Delta\dot f_0, \Delta f_1\rangle = &-\Big(\tfrac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta\dot f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)} - \tfrac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta f_0)(\Delta\dot f_0,\Delta f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)} + \\ &\tfrac{(\dot f_1,\Delta f_0\times\Delta\dot f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2} + \tfrac{(\dot f_1\times\Delta\dot f_0,\Delta f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2}\Big)G_{12} - \\ &\Big(\tfrac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)} + \tfrac{(\dot f_1,\Delta f_0,\Delta f_0\times\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2}\Big)G_{13} - \\ &\Big(\tfrac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)} - \tfrac{(\dot f_1,\Delta f_0,\dot f_1\times\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2}\Big)G_{23}.\end{aligned}$$

Further, decomposing the vector $\Delta\dot f_1$ into basis vectors $\dot f_1$, Δf_0, and Δf_1 we get

$$\langle \mathcal{D}\Delta f_0, \Delta\dot f_1\rangle = \tfrac{(\Delta\dot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{21} + \tfrac{(\dot f_1,\Delta\dot f_1,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{22} + \tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{23}.$$

From the last two identities, by substituting $G_{22} = 0$ and $G_{21} = -G_{12}$ (see Lemma 2.8), we obtain the expression for

$$\dot G_{23} = \frac{\partial}{\partial t}\langle \mathcal{D}\Delta f_0, \Delta f_1\rangle = \langle \mathcal{D}\Delta\dot f_0, \Delta f_1\rangle + \langle \mathcal{D}\Delta f_0, \Delta\dot f_1\rangle.$$

The expression for $\dot G_{32}$ is calculated in a similar way. This concludes the proof. □

2.2.3. *Existence of infinitesimal flexions.* Let us prove that every solution of System A with certain initial data determines an infinitesimal flexion.

Proposition 2.9. *Let f be a weakly generic 2-ribbon surface in $C^{1,2,1}([a,b],\mathbb{R}^3)$. Then* (*i*) *For an arbitrary nonzero α there exists a unique tangent vector $\mathcal{D}f$ at f satisfying System A and the boundary conditions*

$$\mathcal{D}\dot f_1(a) = 0, \quad \mathcal{D}\Delta f_0(a) = 0, \quad \textit{and} \quad \mathcal{D}\Delta f_1(a) = \alpha\dot f_1(a)\times\Delta f_1(a).$$

(*ii*) *This tangent vector is an infinitesimal flexion.*

Remark 2.10. Here and below, for a function f defined on $[a,b]$ by $\dot f(a)$ we mean the one-sided derivative at a.

Proof. We start with Proposition 2.9(i). Consider three vectors

$$v_1 = 0, \quad v_2 = 0, \quad \text{and} \quad v_3 = \alpha \dot{f}_1(a) \times \Delta f_1(a).$$

Denote

$$\begin{array}{lll} c_{11} = \langle v_1, \dot{f}_1 \rangle, & c_{12} = \langle v_1, \Delta f_0 \rangle, & c_{13} = \langle v_1, \Delta f_1 \rangle, \\ c_{21} = \langle v_2, \dot{f}_1 \rangle, & c_{22} = \langle v_2, \Delta f_0 \rangle, & c_{23} = \langle v_2, \Delta f_1 \rangle, \\ c_{31} = \langle v_3, \dot{f}_1 \rangle, & c_{32} = \langle v_3, \Delta f_0 \rangle, & c_{33} = \langle v_3, \Delta f_1 \rangle. \end{array}$$

By Proposition 2.6 there exists a unique solution $(G_{11}, G_{12}, \ldots, G_{33})$ satisfying the initial conditions $G_{ij}(a) = c_{ij}$. For every point $t \in [a, b]$ the values $\mathcal{D}\dot{f}_1$, $\mathcal{D}\Delta f_0$, and $\mathcal{D}\Delta f_1$ of the tangent vector $\mathcal{D}f$ are uniquely defined in the basis $(\dot{f}_1, \Delta f_0, \Delta f_1)$ by Equations (10): here we substitute the solution of System A with the initial conditions $G_{ij}(a) = c_{ij}$ to the right hand side of Equations (10). Hence, there exists a unique tangent vector $\mathcal{D}f$ of f satisfying System A and the boundary conditions

$$\mathcal{D}\dot{f}_1(a) = 0, \quad \mathcal{D}\Delta f_1(a) = 0, \quad \text{and} \quad \mathcal{D}\Delta f_0(a) = \alpha \dot{f}_1(a) \times \Delta f_0(a).$$

This concludes the proof of the fist item of the proposition.

Proof of Proposition 2.9(ii). By the definition of an infinitesimal flexion it is enough to check that the following 11 functions are preserved by the infinitesimal deformation:

$$|\dot{f}_i|, \qquad |\Delta f_i|, \qquad \langle \dot{f}_i, \Delta f_{i-1} \rangle, \qquad \langle \dot{f}_i, \Delta f_i \rangle, \quad \text{and} \quad \langle \dot{f}_i, \dot{f}_{i+1} \rangle$$

(for all possible admissible i).

Invariance of $|\dot{f}_1|$, $|\Delta f_0|$, $|\Delta f_1|$, $\langle \dot{f}_1, \Delta f_0 \rangle$, *and* $\langle \dot{f}_1, \Delta f_1 \rangle$.
From System A we have

$$\dot{G}_{11} = 0, \quad \dot{G}_{22} = 0, \quad \dot{G}_{33} = 0, \quad \dot{G}_{21} + \dot{G}_{12} = 0, \quad \dot{G}_{31} + \dot{G}_{13} = 0,$$

and hence the functions

$$\begin{array}{c} \mathcal{D}(|\dot{f}_1|^2) = 2G_{11}; \quad \mathcal{D}(|\Delta f_0|^2) = 2G_{22}; \quad \mathcal{D}(|\Delta f_1|^2) = 2G_{33}; \\ \mathcal{D}\langle \dot{f}_1, \Delta f_0 \rangle = G_{12} + G_{21}, \quad \text{and} \quad \mathcal{D}\langle \dot{f}_1, \Delta f_1 \rangle = G_{31} + G_{13} \end{array}$$

are constant functions. So it is enough to show that they vanish at some point: we show this at point a.

$$\begin{array}{l} \mathcal{D}\langle \dot{f}_1(a), \dot{f}_1(a) \rangle = 2\langle \mathcal{D}\dot{f}_1(a), \dot{f}_1(a) \rangle = 2\langle 0, \dot{f}_1(a) \rangle = 0; \\ \mathcal{D}\langle \Delta f_0(a), \Delta f_0(a) \rangle = 2\langle \mathcal{D}\Delta f_0(a), \Delta f_0(a) \rangle = 2\langle 0, \Delta f_0(a) \rangle = 0; \\ \mathcal{D}\langle \Delta f_1(a), \Delta f_1(a) \rangle = 2\langle \mathcal{D}\Delta f_1(a), \Delta f_1(a) \rangle = 2\langle \alpha \dot{f}_1(a) \times \Delta f_1(a), \Delta f_1(a) \rangle = 0; \\ \mathcal{D}\langle \dot{f}_1(a), \Delta f_0(a) \rangle = \langle \mathcal{D}_\gamma \dot{f}_1(a), \Delta f_0(a) \rangle + \langle \dot{f}_1(a), \mathcal{D}\Delta f_0(a) \rangle = \langle 0, \Delta f_0(a) \rangle + \\ \qquad \langle \dot{f}_1(a), 0 \rangle = 0. \\ \mathcal{D}\langle \dot{f}_1(a), \Delta f_1(a) \rangle = \langle \mathcal{D}\dot{f}_1(a), \Delta f_1(a) \rangle + \langle \dot{f}_1(a), \mathcal{D}\Delta f_1(a) \rangle = \langle 0, \Delta f_0(a) \rangle + \\ \qquad \langle \dot{f}_1(a), \alpha \dot{f}_1(a) \times \Delta f_1(a) \rangle = 0; \end{array}$$

Invariance of $\langle \dot{f}_0, \Delta f_0 \rangle$ *and* $\langle \dot{f}_2, \Delta f_1 \rangle$. Note that

$$\langle \dot{f}_0, \Delta f_0 \rangle = -\frac{1}{2} \frac{\partial}{\partial t} \langle \Delta f_0, \Delta f_0 \rangle + \langle \dot{f}_1, \Delta f_0 \rangle.$$

Hence by the above item we have

$$\mathcal{D}\langle \dot f_0, \Delta f_0\rangle = -\frac{1}{2}\frac{\partial}{\partial t}\mathcal{D}\langle \Delta f_0, \Delta f_0\rangle + \mathcal{D}\langle \dot f_1, \Delta f_0\rangle = -\frac{1}{2}\frac{\partial}{\partial t}(0) + 0 = 0.$$

Similar reasoning shows that $\mathcal{D}\langle \dot f_2, \Delta f_1\rangle = 0$.

Invariance of $\langle \dot f_0, \dot f_1\rangle$ *and* $\langle \dot f_1, \dot f_2\rangle$. Let us prove that $\mathcal{D}\langle \dot f_0, \dot f_1\rangle = 0$. First, note that

$$\langle \mathcal{D}\dot f_0, \dot f_1\rangle = \langle \mathcal{D}\dot f_1, \dot f_1\rangle - \langle \mathcal{D}\Delta \dot f_0, \dot f_1\rangle = -\langle \mathcal{D}\Delta \dot f_0, \dot f_1\rangle = \langle \mathcal{D}\Delta f_0, \ddot f_1\rangle - \frac{\partial}{\partial t}\langle \mathcal{D}\Delta f_0, \dot f_1\rangle.$$

Recall that $\frac{\partial}{\partial t}\langle \mathcal{D}\Delta f_0, \dot f_1\rangle = \dot G_{21} = -\dot G_{12}$. Let us substitute the expression for $\dot G_{12}$ of System A and rewrite $\ddot f_1$ in the basis of vectors $\dot f_1$, Δf_0, and Δf_1. One obtains

$$\begin{aligned}\langle \mathcal{D}\dot f_0, \dot f_1\rangle &= \langle \mathcal{D}\Delta f_0, \ddot f_1\rangle + \dot G_{12}\\ &= \frac{(\dot f_1, \Delta \dot f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}\langle \mathcal{D}\dot f_1, \Delta f_0\rangle + \frac{(\dot f_1, \Delta f_0, \Delta \dot f_0)}{(\dot f_1, \Delta f_0, \Delta f_1)}\langle \mathcal{D}\dot f_1, \Delta f_1\rangle\\ &= \langle \mathcal{D}\dot f_1, \Delta \dot f_0\rangle = -\langle \mathcal{D}\dot f_1, \dot f_0\rangle.\end{aligned}$$

Hence

$$\mathcal{D}\langle \dot f_0, \dot f_1\rangle = \langle \mathcal{D}\dot f_0, \dot f_1\rangle + \langle \mathcal{D}\dot f_1, \dot f_0\rangle = -\langle \mathcal{D}\dot f_1, \dot f_0\rangle + \langle \mathcal{D}\dot f_1, \dot f_0\rangle = 0.$$

Therefore, $\langle \dot f_0, \dot f_1\rangle$ is invariant under the infinitesimal deformation. The proof of the invariance of $\langle \dot f_1, \dot f_2\rangle$ is analogous.

Invariance of $\langle \dot f_0, \dot f_0\rangle$ *and* $\langle \dot f_2, \dot f_2\rangle$. Let us prove that $\mathcal{D}\langle \dot f_0, \dot f_0\rangle = 0$.

$$\mathcal{D}\langle \dot f_0, \dot f_0\rangle = 2\langle \mathcal{D}\dot f_0, \dot f_0\rangle = 2\langle \mathcal{D}\Delta \dot f_0, \Delta \dot f_0\rangle + 2\mathcal{D}\langle \dot f_1, \dot f_0\rangle - 2\langle \mathcal{D}\dot f_1, \dot f_1\rangle.$$

We have already shown that $\mathcal{D}\langle \dot f_1, \dot f_0\rangle = 0$ and $\langle \mathcal{D}\dot f_1, \dot f_1\rangle = 0$. Hence

$$\mathcal{D}\langle \dot f_0, \dot f_0\rangle = 2\langle \mathcal{D}\Delta \dot f_0, \Delta \dot f_0\rangle.$$

We rewrite the last $\Delta \dot f_0$ in the last expression in the basis $\dot f_1, \Delta f_0, \dot f_1 \times \Delta f_0$ and get

$$(\mathcal{D}\Delta \dot f_0, \Delta \dot f_0\rangle = \tfrac{(\Delta \dot f_0, \Delta f_0, \dot f_1 \times \Delta f_0)}{(\dot f_1, \Delta f_0, \dot f_1 \times \Delta f_0)}\langle \mathcal{D}\Delta \dot f_0, \dot f_1\rangle + \tfrac{(\dot f_1, \Delta \dot f_0, \dot f_1 \times \Delta f_0)}{(\dot f_1, \Delta f_0, \dot f_1 \times \Delta f_0)}\langle \mathcal{D}\Delta \dot f_0, \Delta f_0\rangle + \tfrac{(\dot f_1, \Delta f_0, \Delta \dot f_0)}{(\dot f_1, \Delta f_0, \dot f_1 \times \Delta f_0)}(\mathcal{D}\Delta \dot f_0, \dot f_1, \Delta f_0). \tag{11}$$

Let us rewrite $\langle \mathcal{D}\Delta \dot f_0, \dot f_1\rangle$, $\langle \mathcal{D}\Delta \dot f_0, \Delta f_0\rangle$, and $(\mathcal{D}\Delta \dot f_0, \dot f_1, \Delta f_0)$ in terms of $G_{11}, \dots, G_{33}$. First, we have:

$$\langle \mathcal{D}\Delta \dot f_0, \dot f_1\rangle = \langle \mathcal{D}\dot f_0, \dot f_1\rangle = -\langle \mathcal{D}\dot f_1, \dot f_0\rangle = -\langle \mathcal{D}\dot f_1, \Delta \dot f_0\rangle.$$

The second equality holds since we have shown that $\mathcal{D}\langle \dot f_0, \dot f_1\rangle = 0$. If we rewrite $\Delta \dot f_0$ in the basis $\dot f_1, \Delta f_0, \Delta f_1$, we get the following:

$$\langle \mathcal{D}\Delta \dot f_0, \dot f_1\rangle = -\langle \mathcal{D}\dot f_1, \Delta \dot f_0\rangle = -\tfrac{(\dot f_1, \Delta \dot f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{12} - \tfrac{(\dot f_1, \Delta f_0, \Delta \dot f_0)}{(\dot f_1, \Delta f_0, \Delta f_1)}G_{13}.$$

Second, we have

$$\langle \mathcal{D}\Delta\dot f_0, \Delta f_0\rangle = -\langle \mathcal{D}\Delta f_0, \Delta\dot f_0\rangle = \tfrac{(\Delta\dot f_0,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{12} - \tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{23}.$$

Third, with

$$\dot G_{23} - \langle \mathcal{D}\Delta f_0, \Delta\dot f_1\rangle = \langle \mathcal{D}\Delta\dot f_0, \Delta f_1\rangle = \tfrac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}\langle \mathcal{D}\Delta\dot f_0, \dot f_1\rangle + \tfrac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}\langle \mathcal{D}\Delta\dot f_0, \Delta f_0\rangle + \tfrac{(\dot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}(\mathcal{D}\Delta\dot f_0, \dot f_1, \Delta f_0).$$

and the expression for $\dot G_{23}$ of System A we get:

$$(\mathcal{D}\Delta\dot f_0, \dot f_1, \Delta f_0) = -\left(\tfrac{(\dot f_1\times\Delta\dot f_0,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)} + \tfrac{(\dot f_1,\Delta f_0\times\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)G_{12} - \tfrac{(\dot f_1,\Delta f_0,\Delta f_0\times\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{13} + \tfrac{(\dot f_1,\Delta f_0,\dot f_1\times\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)}G_{23}.$$

Finally, we substitute the obtained last three expressions for

$$\langle \mathcal{D}\Delta\dot f_0, \dot f_1\rangle, \quad \langle \mathcal{D}\Delta\dot f_0, \Delta f_0\rangle, \quad \text{and} \quad (\mathcal{D}\Delta\dot f_0, \dot f_1, \Delta f_0)$$

respectively to Expression (11) and arrive at

$$\begin{aligned}\langle \mathcal{D}\Delta\dot f_0, \Delta\dot f_0\rangle = \Big(&-\tfrac{(\Delta\dot f_0,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta f_1)} + \tfrac{(\dot f_1,\Delta\dot f_0,\dot f_1\times\Delta f_0)(\Delta\dot f_0,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta f_1)} -\\ &\tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)(\dot f_1\times\Delta\dot f_0,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta f_1)} - \tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)(\dot f_1,\Delta f_0\times\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta f_1)}\Big)G_{12} +\\ &\Big(-\tfrac{(\Delta\dot f_0,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta f_1)} - \tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)(\dot f_1,\Delta f_0,\Delta f_0\times\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta f_1)}\Big)G_{13} +\\ &\Big(-\tfrac{(\dot f_1,\Delta\dot f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta f_1)} + \tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)(\dot f_1,\Delta f_0,\dot f_1\times\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta f_1)}\Big)G_{23}.\end{aligned}$$

It is clear that the coefficients of G_{13} and G_{23} vanish identically. Let us study the coefficient of G_{12}.

Consider the following mixed product $(\Delta\dot f_0, \Delta\dot f_0, \dot f_1\times\Delta f_0)$, it is identical to zero. Let us rewrite $\Delta\dot f_0$ in the second position of the mixed product in the basis $\dot f_0$, Δf_0, Δf_1. We get the relation

$$\begin{aligned}&\tfrac{(\Delta\dot f_0,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}(\Delta\dot f_0, \dot f_1, \dot f_1\times\Delta f_0) + \tfrac{(\dot f_1,\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}(\Delta\dot f_0, \Delta f_0, \dot f_1\times\Delta f_0)\\ &= -\tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)}(\Delta\dot f_0, \Delta f_1, \dot f_1\times\Delta f_0).\end{aligned}$$

We apply this identity to the first two summands of the coefficient of G_{12} and get the following expression for the coefficient of G_{12}:

$$\tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)(\Delta\dot f_0,\Delta f_1,\dot f_1\times\Delta f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)|\dot f_1\times\Delta f_0|^2} - \tfrac{(\dot f_1,\Delta f_0\times\Delta\dot f_0,\Delta f_1)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)|\dot f_1\times\Delta f_0|^2} - \tfrac{(\dot f_1\times\Delta\dot f_0,\Delta f_0,\Delta f_1)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)|\dot f_1\times\Delta f_0|^2}.$$

We rewrite this as

$$\tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)|\dot f_1\times\Delta f_0|^2}\Big((\Delta\dot f_0, \Delta f_1, \dot f_1\times\Delta f_0) - (\dot f_1, \Delta f_0\times\Delta\dot f_0, \Delta f_1) - (\dot f_1\times\Delta\dot f_0, \Delta f_0, \Delta f_1)\Big).$$

Let us study the expression in the brackets.

$$(\Delta\dot f_0, \Delta f_1, \dot f_1\times\Delta f_0)-(\dot f_1, \Delta f_0\times\Delta\dot f_0, \Delta f_1)-(\dot f_1\times\Delta\dot f_0, \Delta f_0, \Delta f_1) =$$
$$-\big(\Delta\dot f_0\times(\dot f_1\times\Delta f_0) + \dot f_1\times(\Delta f_0\times\Delta\dot f_0) + \Delta f_0\times(\Delta\dot f_0\times\dot f_1), \Delta f_1\big) = (0, \Delta f_1) = 0.$$

The second equality holds by the Jacobi identity. Hence the coefficient of G_{12} is zero. Therefore,

$$\mathcal{D}\langle\dot f_0, \dot f_0\rangle = 2\langle\mathcal{D}\Delta\dot f_0, \Delta\dot f_0\rangle = 0,$$

and $\langle\dot f_0, \dot f_0\rangle$ is invariant under the infinitesimal deformation.

The proof of the invariance of $\langle\dot f_2, \dot f_2\rangle$ repeats the proof for $\langle\dot f_0, \dot f_0\rangle$.

So we have checked the invariance of all the 11 functions in the definition of an infinitesimal flexion. Hence $\mathcal{D}f$ is an infinitesimal flexion. □

Now we have all the ingredients to prove the main theorem of this subsection.

2.2.4. *Conclusion of the proof of Theorem 2.3. Existence.* The existence of an infinitesimal flexion follows directly from Proposition 2.9(i).

Uniqueness. By Proposition 2.7 every infinitesimal flexion satisfies System A. Since we consider 2-ribbon surfaces with fixed initial position, for every non-zero infinitesimal flexion $\mathcal{D}f$ we have:

$$\mathcal{D}\dot f_1(a) = 0, \quad \mathcal{D}\Delta f_0(a) = 0, \quad \text{and} \quad \mathcal{D}\Delta f_1(a) = \alpha\dot f_1(a)\times\Delta f_1(a)$$

for some non-zero α. Hence by Proposition 2.9 this is one of the flexions of Proposition 2.9(i). So the set of infinitesimal flexions is one-dimensional. Since the set is a linear space, it is a line. Hence f has one degree of infinitesimal flexibility. □

Theorem 2.3 together with Proposition 2.9 imply the following.

Corollary 2.11. *Let $f \in C_0^{1,2,1}([a,b], \mathbb{R}^3)$ be a weakly generic 2-ribbon surface with fixed initial position, and let $\mathcal{D}f$ be its infinitesimal flexion satisfying*

$$\mathcal{D}\dot f_1(a) = 0, \quad \mathcal{D}\Delta f_1(a) = 0, \quad \textit{and} \quad \mathcal{D}\Delta f_0(a) = 0,$$

Then $\mathcal{D}f = 0$. □

2.3. **Variational operators of infinitesimal flexions.** Let us fix an orthonormal basis (e_1, e_2, e_3) in $\mathbb{R}^3$. Denote by $\Omega^1_{3\times3}$ the Banach space

$$\big((C^1[a,b])^3\big)^3 \cong (C^1[a,b])^9$$

with the norm

$$\|(h_{11}, h_{12}, \ldots, h_{33})\| = \max_{1\le i,j\le 3}(\max(\sup|h_{ij}|, \sup|\dot h_{ij}|)).$$

Consider the following map

$$Z : C^{1,2,1}([a,b], \mathbb{R}^3) \to \Omega^1_{3\times3},$$

where for a 2-ribbon surface f the image $Z(f)$ in the basis (e_1, e_2, e_3) is defined as

$$\begin{aligned} \dot f_1(t) &= \big(h_{11}(t), h_{12}(t), h_{13}(t)\big),\\ \Delta f_0(t) &= \big(h_{21}(t), h_{22}(t), h_{23}(t)\big),\\ \Delta f_1(t) &= \big(h_{31}(t), h_{32}(t), h_{33}(t)\big). \end{aligned}$$

Note that every 2-ribbon surface f is defined by $\dot f_1$, Δf_0, and Δf_1 up to a translation. So after fixing, say, $f_1(a) = (0,0,0)$ one has a bijection.

We say that a point $h = (h_{11}, h_{12}, \ldots, h_{33})$ in $\Omega^1_{3\times 3}$ is in *general position* if the determinant

$$\det \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \neq 0$$

for every $t \in [a,b]$. This condition obviously corresponds to the weakly genericity condition, i.e., to

$$(\dot f_1, \Delta f_0, \Delta f_1) \neq 0.$$

Denote by Σ_Ω the set of all points h that are not in general position.

Definition 2.12. Denote by $\mathcal{V}^\pm : [0,\Lambda] \times (\Omega^1_{3\times 3} \setminus \Sigma_\Omega) \to \Omega^1_{3\times 3}$ two variational *operators of infinitesimal flexion* in coordinates $(h_{11}, h_{12}, \ldots, h_{33})$:

$$\begin{aligned} \mathcal{V}^\pm_{l-1,m}(\lambda, h) = \; & \frac{(e_m, \Delta f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)} G_{l-1,1}(h) + \frac{(\dot f_1, e_m, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)} G_{l-1,2}(h) + \\ & \frac{(\dot f_1, \Delta f_0, e_m)}{(\dot f_1, \Delta f_0, \Delta f_1)} G_{l-1,3}(h). \end{aligned} \tag{12}$$

for $(1 \leq l, m \leq 3)$. Here $G_{11}(h), G_{12}(h), \ldots, G_{33}(h)$ is a solution of System A at point f with the initial conditions corresponding to

$$\mathcal{D}\dot f_1(a) = 0, \quad \mathcal{D}\Delta f_0(a) = 0, \quad \text{and} \quad \mathcal{D}\Delta f_1(a) = \pm \dot f_1(a) \times \Delta f_1(a),$$

i.e.,

$$\begin{array}{lll} G_{11}(a) = 0, & G_{12}(a) = 0, & G_{13}(a) = 0, \\ G_{21}(a) = 0, & G_{22}(a) = 0, & G_{23}(a) = 0, \\ G_{31}(a) = 0, & G_{32}(a) = \pm(\dot f_1(a), \Delta f_0(a), \Delta f_1(a)), & G_{33}(a) = 0. \end{array} \tag{13}$$

(Here we take "+" sign for $\mathcal{V}^+$ and "−" for $\mathcal{V}^-$.)

Note that both $\mathcal{V}^+$ and $\mathcal{V}^-$ are autonomous operators, they do not depend on time parameter λ.

It is important that the following statement holds.

Proposition 2.13. *Let h be a point of $\Omega^1_{3\times 3}$ in general position and $\lambda \in [0, \Lambda]$. Then we have*

$$\mathcal{V}^\pm(\lambda, h) \in \Omega^1_{3\times 3}.$$

Proof. The proof is straightforward, all functions involved in Expression (12) are continuously differentiable, and hence both $\mathcal{V}^+(\lambda, h)$ and $\mathcal{V}^-(\lambda, h)$ are continuously differentiable. □

Remark 2.14. Let us show in brief how to find the coordinates of the infinitesimal deformation $\mathcal{D}f$ in the basis e_1, e_2, e_3 satisfying

$$\mathcal{D}f_1(a) = 0, \qquad \mathcal{D}\dot{f}_1(a) = 0, \qquad \mathcal{D}\Delta f_0(a) = 0, \quad \text{and} \quad \mathcal{D}\Delta f_1(a) = \dot{f}_1(a)\times\Delta f_1(a).$$

First, one should solve System A with the above initial data, then substitute the obtained solution $(G_{11}, G_{12}, \ldots, G_{33})$ to Equations (12). Now we have the coordinates of $\mathcal{D}\dot{f}_1$, $\mathcal{D}\Delta f_0$, and $\mathcal{D}\Delta f_1$. Having the additional condition $\mathcal{D}f_1(a) = 0$ one can construct $\mathcal{D}f_1$, $\mathcal{D}f_0$, and $\mathcal{D}f_2$:

$$\mathcal{D}f_1(t_0) = \int_a^{t_0} \mathcal{D}\dot{f}_1(t)d(t), \quad \mathcal{D}f_0 = \mathcal{D}f_1 - \mathcal{D}\Delta f_0, \quad \mathcal{D}f_2 = \mathcal{D}f_1 + \mathcal{D}\Delta f_1.$$

Further we will work in the following subspace of $\Omega^1_{3\times3}$. Denote

$$\tilde{\Omega}^1_{3\times3} = \left\{h \in \Omega^1_{3\times3} \middle| h_{12}(a) = h_{13}(a) = h_{23}(a) = 0\right\}.$$

It is clear that $\tilde{\Omega}^1_{3\times3}$ is a Banach space itself.

We have the following important property of $\tilde{\Omega}^1_{3\times3}$.

Proposition 2.15. *For every $\lambda \in [0, \Lambda]$ and $h \in \tilde{\Omega}^1_{3\times3} \setminus \Sigma_\Omega$ the subspace $\tilde{\Omega}^1_{3\times3}$ is an invariant space of the operators $\mathcal{V}^+(\lambda, h)$ and $\mathcal{V}^-(\lambda, h)$.*

Proof. From the conditions

$$\mathcal{D}\dot{f}_1(a) = 0, \quad \text{and} \quad \mathcal{D}\Delta f_0(a) = 0$$

we have $G_{ij}(a) = 0$ for all $i = 1, 2$, and $j = 1, 2, 3$. Hence by Expression (12)

$$\mathcal{V}^\pm_{11}(\lambda, h)(a) = \mathcal{V}^\pm_{12}(\lambda, h)(a) = \ldots = \mathcal{V}^\pm_{23}(\lambda, h)(a) = 0$$

for all $\lambda \in [0, \Lambda]$ and $h \in \Omega^1_{3\times3}$. Therefore, for every $\lambda \in [0, \Lambda]$ and $h \in \tilde{\Omega}^1_{3\times3} \setminus \Sigma_\Omega$ we have $\mathcal{V}^\pm(\lambda, h) \in \tilde{\Omega}^1_{3\times3}$. □

Finally we have the following important statement.

Proposition 2.16. *The map Z is a bijection of $\tilde{\Omega}^1_{3\times3}$ and $C^{1,2,1}_0([a, b], \mathbb{R}^3)$.*

Proof. The inverse map $Z^{-1}(h) = (f_0, f_1, f_2)$ is defined as

$$f_1(t_0) = \int_a^{t_0} \begin{pmatrix} h_{11}(t) \\ h_{12}(t) \\ h_{13}(t) \end{pmatrix} dt, \;\; f_0(t_0) = f_1(t_0) - \begin{pmatrix} h_{21}(t_0) \\ h_{22}(t_0) \\ h_{23}(t_0) \end{pmatrix}, \;\; f_2(t_0) = \begin{pmatrix} h_{31}(t_0) \\ h_{32}(t_0) \\ h_{33}(t_0) \end{pmatrix} - f_1(t_0).$$

at every $t_0 \in [a, b]$. □

2.4. **Finite flexibility of 2-ribbon surfaces.** In Subsection 2.2 we showed that every 2-ribbon surface in general position is infinitesimally flexible and that the space of its infinitesimal flexions is one-dimensional. The aim of this subsection is to show that a weakly generic 2-ribbon surface is finitely flexible and has one degree of finite flexibility.

2.4.1. *Lipschitz condition.* We start with the discussion of the initial value problem for the following two differential equations on the set of all points $\tilde{\Omega}^1_{3\times 3}$ in general position (here λ is the time parameter):

$$\frac{\partial h}{\partial \lambda} = \mathcal{V}^+(\lambda, h) \quad \text{and} \quad \frac{\partial h}{\partial \lambda} = \mathcal{V}^-(\lambda, h). \tag{14}$$

To solve the initial value problem we study local Lipschitz properties for $\mathcal{V}^+$ and $\mathcal{V}^-$.

Definition 2.17. Consider a Banach space E with a norm $|*|_E$, and a positive real number Λ. Let U be a subset of $[0,\Lambda]\times E$. We say that a functional $\mathcal{F} : U \to E$ *locally satisfies a Lipschitz condition* if for every point (λ_0, p) in U there exist a neighborhood V of the point and a constant K such that for every pair of points (λ, p_1) and (λ, p_2) in V the inequality

$$|\mathcal{F}(\lambda, p_1) - \mathcal{F}(\lambda, p_2)|_E \le K|p_1 - p_2|_E$$

holds.

First we verify a Lipschitz condition for the following operator. Define $\mathcal{G} : [0,\Lambda]\times \tilde{\Omega}^1_{3\times 3} \to \tilde{\Omega}^1_{3\times 3}$ by

$$\mathcal{G}_{ij}(\lambda, h) = G_{ij}(h), \quad 1 \le i,j \le 3,$$

where $G_{ij}(h)$ are defined by Equations (10).

Lemma 2.18. *For every point $h \in U$ in general position, there exists a neighborhood V_h of h such that the functional $\mathcal{G}$ locally satisfies a Lipschitz condition in $[0,\Lambda]\times V_h$.*

Proof. Consider a point $h \in U$. The element $(G_{11}, G_{12}, \ldots, G_{33})$ itself satisfies a system of linear differential equations (System A). The coefficients of this system depend only on a point of $\tilde{\Omega}^1_{3\times 3}$. Since the point h is in general position, there exists an integer constant K such that for a sufficiently small neighborhood V_h of h the dependence is K-Lipschitz, i.e., for p and q from V_h every coefficient c of System A satisfies the inequality

$$|c(p) - c(q)| < K\|p - q\|.$$

Hence the solutions for $t \in [a,b]$ satisfy the Lipschitz condition for a fixed initial data on V_h. (This is clear from the fact that the solution of the system with small coefficients $c(p) - c(q)$ will be almost constant, the difference in each coordinate will not be greater than $9(b-a)K\|p-q\|$.) Finally the solution for $t \in [a,b]$ satisfies the Lipschitz condition for a fixed parameter and different initial data on V_h (See Proposition 1.10.1 in [4]). Therefore, for some constants $\overline{K}_l$ we have

$$\sup(|G_{ij}(p) - G_{ij}(q)|) < \overline{K}_{ij}\|p - q\|, \quad 1 \le i,j \le 3.$$

From System A we know that the $\dot{G}_{i,j}$ linearly depend on $G_{11}, G_{12}, \ldots, G_{33}$, therefore, we get the Lipschitz condition for the derivatives: for some constants $\tilde{K}_l$ we have

$$\sup(|\dot{G}_{ij}(p) - \dot{G}_{ij}(q)|) < \tilde{K}_{ij}\|p-q\|, \quad 1 \leq i,j \leq 3.$$

Thus there exists a real number $\hat{K}$ such that for all points p and q in V_h,

$$\|\mathcal{G}(\lambda,p)-\mathcal{G}(\lambda,q)\| = \max_{1\leq i,j\leq 3}\big(\max\big(\sup|G_{ij}(p)-G_{ij}(q)|, \sup|\dot{G}_{ij}(p)-\dot{G}_{ij}(q)|\big)\big) < \hat{K}\|p-q\|.$$

Thus $\mathcal{G}$ satisfies a Lipschitz condition on V_h. Therefore, $\mathcal{G}$ satisfies a Lipschitz condition on $[0,\Lambda]\times V_h$ (since $\mathcal{G}$ is autonomous). □

Lemma 2.18 and Expression (12) directly imply the following statement.

Corollary 2.19. *For every point $h \in U$ in general position and, there exists a neighborhood V_h of h such that both functionals $\mathcal{V}^+$ and $\mathcal{V}^-$ locally satisfy a Lipschitz condition in $[0,\Lambda]\times V_h$.* □

2.4.2. *Existence and uniqueness of solutions.* Let us prove the following general statement.

Proposition 2.20. *Let $h_0 \in \tilde{\Omega}^1_{3\times 3}$ be in general position. Then for sufficiently small positive ε there exists a unique solution γ of the equation*

$$\frac{\partial h}{\partial \lambda} = \mathcal{V}^+(\lambda, h) \tag{15}$$

on $[-\varepsilon,\varepsilon]$, such that $\gamma(0)=h_0$.

We start with the following general lemma.

Lemma 2.21. *Let $h_0 \in \tilde{\Omega}^1_{3\times 3}$ be in general position. A deformation γ with $\gamma(0)=h_0$ is a solutions of Equation (15) if and only if γ satisfies*

$$\frac{\partial \gamma}{\partial \lambda} = \begin{cases} \mathcal{V}^+(\lambda,\gamma(\lambda)) & \text{for all } \lambda\in[0,\Lambda], \\ -\mathcal{V}^-(-\lambda,\gamma(\lambda)) & \text{for all } \lambda\in[-\Lambda,0]. \end{cases}$$

Proof. The proof of the Lemma is straightforward. □

Proof of Proposition 2.20. As we showed in Corollary 2.19, the operators $\mathcal{V}^+$ and $\mathcal{V}^-$ satisfy a Lipschitz condition in some neighborhood of the point $Z^{-1}(f)$. From the general theory of differential equations on Banach spaces (see for instance the first section of the second chapter of [4]) it follows that this condition implies local existence and uniqueness of a solution of the initial value problem for the differential Equations (14). Hence by Lemma 2.21 for a sufficiently small positive ε there exists a unique solution γ of Equation (15) on $[-\varepsilon,\varepsilon]$ satisfying the condition $\gamma(0)=h_0$. □

2.4.3. *Finite flexibility.* The key point for finite flexibility of 2-ribbon surfaces is the following lemma.

Lemma 2.22. *Let $\{f^\lambda\}$, $\lambda\in[-\varepsilon,\varepsilon]$ for some $\varepsilon>0$, be a normalized isometric deformation in the space $C_0^{1,2,1}([a,b],\mathbb{R}^3)$ such that all 2-ribbon surfaces of the family are weakly generic. Let also $\{Z^{-1}(f^\lambda)\}$ be the corresponding deformation in $\tilde{\Omega}^1_{3\times 3}$. Then $\{f^\lambda\}$ is an isometric deformation if and only $\{Z^{-1}(f^\lambda)\}$ satisfies Equation(15) for all $\lambda\in[-\varepsilon,\varepsilon]$.*

Proof. Let $\gamma=\{f^\lambda\}$, $\lambda\in[-\varepsilon,\varepsilon]$, be a normalized isometric deformation in $C_0^{1,2,1}([a,b],\mathbb{R}^3)$ such that all 2-ribbon surfaces of the family are weakly generic. Every normalized deformation satisfies Condition (13) at every point $\lambda\in[-\varepsilon,\varepsilon]$ with the positive choice of the sign.

Since γ is an isometric deformation, $\mathcal{D}_\gamma f^\lambda$ is an infinitesimal flexion at every point $\lambda\in[-\varepsilon,\varepsilon]$. Hence the corresponding functions G^λ_{ij} satisfy system A (by Proposition 2.7). Let us now write $\mathcal{D}_\gamma \dot{f}^\lambda_1$, $\mathcal{D}_\gamma\Delta f^\lambda_0$, and $\mathcal{D}_\gamma\Delta f^\lambda_1$ in the basis e_1, e_2, e_3 using functions G^λ_{ij}. Recall that

$$\mathcal{D}_\gamma \dot{f}^\lambda_1=(\mathcal{D}_\gamma \dot{f}^\lambda_1,\dot{f}^\lambda_1)\dot{f}^\lambda_1+(\mathcal{D}_\gamma \dot{f}^\lambda_1,\Delta f^\lambda_0)\Delta f^\lambda_0+(\mathcal{D}_\gamma \dot{f}^\lambda_1,\Delta f^\lambda_1)\Delta f^\lambda_1=G_{11}\dot{f}^\lambda_1+G_{12}\Delta f^\lambda_0+G_{13}\Delta f^\lambda_1.$$

Hence in the basis (e_1,e_2,e_3) we have

$$\begin{aligned}\frac{\partial \dot{f}^\lambda_1}{\partial\lambda}&=\mathcal{D}_\gamma \dot{f}^\lambda_1\\ &=\sum_{m=1}^{3}\left(\frac{(e_m,\Delta f^\lambda_0,\Delta f^\lambda_1)}{(\dot{f}^\lambda_1,\Delta f^\lambda_0,\Delta f^\lambda_1)}G^\lambda_{11}+\frac{(\dot{f}^\lambda_1,e_m,\Delta f^\lambda_1)}{(\dot{f}^\lambda_1,\Delta f^\lambda_0,\Delta f^\lambda_1)}G^\lambda_{12}+\frac{(\dot{f}^\lambda_1,\Delta f^\lambda_0,e_m)}{(\dot{f}^\lambda_1,\Delta f^\lambda_0,\Delta f^\lambda_1)}G^\lambda_{13}\right)e_m\\ &=\sum_{m=1}^{3}\Big(\mathcal{V}^+_{1,m}\big(\lambda,Z^{-1}(f^\lambda)\big)\Big)e_m.\end{aligned}\tag{16}$$

Similarly we have:

$$\begin{aligned}\frac{\partial\Delta f^\lambda_0}{\partial\lambda}&=\sum_{m=1}^{3}\Big(\mathcal{V}^+_{2,m}\big(\lambda,Z^{-1}(f^\lambda)\big)\Big)e_m,\\ \frac{\partial\Delta f^\lambda_1}{\partial\lambda}&=\sum_{m=1}^{3}\Big(\mathcal{V}^+_{3,m}\big(\lambda,Z^{-1}(f^\lambda)\big)\Big)e_m.\end{aligned}\tag{17}$$

Hence by Definition 2.12 the corresponding derivatives in the space $\tilde{\Omega}^1_{3\times3}$ satisfy:

$$\frac{\partial Z^{-1}(f^\lambda)}{\partial\lambda}=\mathcal{V}^+(\lambda,Z^{-1}(f^\lambda))$$

for every $\lambda\in[-\varepsilon,\varepsilon]$.

Conversely, let $Z^{-1}(f^\lambda)$ satisfy

$$\frac{\partial Z^{-1}(f^\lambda)}{\partial\lambda}=\mathcal{V}^+(\lambda,Z^{-1}(f^\lambda))$$

for every $\lambda\in[-\varepsilon,\varepsilon]$. Then the corresponding $\mathcal{D}_\gamma \dot{f}^\lambda_1$, $\mathcal{D}_\gamma\Delta f^\lambda_0$, and $\mathcal{D}_\gamma\Delta f^\lambda_1$ are defined as in (16) and (17). Hence the correspondent scalar products G^λ_{ij} satisfy System A for

$\lambda \in [-\varepsilon, \varepsilon]$. Thus $\mathcal{D}_\gamma f^\lambda$ is an infinitesimal flexion for every $\lambda \in [-\varepsilon, \varepsilon]$. Hence by $\{f^\lambda\}$ is an isometric deformation on $[-\varepsilon, \varepsilon]$. From construction it follows that $\{f^\lambda\}$ is a normalized deformation. □

Now we prove the following theorem on finite flexibility of weakly generic 2-ribbon surfaces.

Theorem 2.23. *Every 2-ribbon weakly generic semidiscrete surface f in $C_0^{1,2,1}([a,b], \mathbb{R}^3)$ has one degree of finite flexibility.*

Proof. On the one hand, by Lemma 2.22 normalized isometric deformations of f with a fixed initial position are in one-to-one correspondence with Solution of Equation (15) satisfying $\gamma(0) = Z^{-1}(f)$. On the other hand, by Proposition 2.20 for sufficiently small positive ε there exists a unique solution γ of Equation (15) satisfying $\gamma(0) = Z^{-1}(f)$. Hence, there exists a unique normalized isometric deformations of f (with the parameter in $[-\varepsilon, \varepsilon]$ for sufficiently small positive ε). Therefore, f has one degree of finite flexibility. □

Remark 2.24. In fact, one can prove the statement of Theorem 2.23 for the spaces of functions $C_0^{m,m+1,m}([a,b], \mathbb{R}^3)$ for arbitrary $m \geq 1$. We are not going to use this later so we omit the details here. The proofs mostly repeat the ones for the case $m = 1$ shown in details above.

3. Infinitesimal flexibility of 3-ribbon surfaces

In this section we find necessary infinitesimal flexibility condition of 3-ribbon surfaces. For the case of n-ribbon surfaces each 3-ribbon subsurface gives a condition of infinitesimal flexibility.

3.1. **Preliminary statements on infinitesimal flexion of 3-ribbon surfaces.**

In this subsection we prove certain relations that we use further in the proof of the statement on infinitesimal flexibility condition for 3-ribbon surfaces.

Consider the following function

$$\Phi = \langle \Delta f_0, \Delta f_1 \rangle.$$

This function plays a central role in our further description of the infinitesimal flexibility condition of 3-ribbon and n-ribbon surfaces (see Theorem 3.7 and Theorem 4.10). Let $\mathcal{D}\Phi$ be the infinitesimal flexion of Φ. Via the function $\mathcal{D}\Phi$ we describe monodromy conditions for finite flexibility. Proposition 3.2 and Corollary 3.6 deliver necessary tools to describe continuous and discrete parts of the monodromy condition on Φ.

Remark 3.1. In the proofs of the statements of this subsection we fix the flexion of the initial frame at $t = a$ in the following way

$$\mathcal{D}\dot{f}_1 = \mathcal{D}\Delta f_1(t_0) = 0$$

(compare to the space $C_0^M[a,b], \mathbb{R}^3$ where $\mathcal{D}\dot{f}_1(a) = \mathcal{D}\Delta f_0(a) = 0$ instead). This simplifies calculations for the 3-ribbon surfaces, since the fixed bar with endpoints $f_1(a)$ and $f_2(a)$ belongs to the middle strip.

3.1.1. *Continuous shift.* Here we study the dependence of the infinitesimal flexion $\mathcal{D}\Phi$ on the argument t.

Proposition 3.2. (On continuous shift.) *Let f be a weakly generic 2-ribbon surface in $C^{1,2,1}([a,b],\mathbb{R}^3)$. Then for every infinitesimal flexion $\mathcal{D}\Phi$ the following condition holds:*

$$\mathcal{D}\Phi(t_2)=\mathcal{D}\Phi(t_1)\cdot\exp\left(\int\limits_{t_1}^{t_2}\frac{(\dot f_1,\Delta\dot f_0,\Delta f_1)+(\dot f_1,\Delta f_0,\Delta\dot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}dt\right).$$

This is a direct consequence of the next lemma.

Lemma 3.3. *Let f be a weakly generic 2-ribbon surface in $C^{1,2,1}([a,b],\mathbb{R}^3)$, then*

$$\mathcal{D}\dot\Phi=\frac{(\dot f_1,\Delta\dot f_0,\Delta f_1)+(\dot f_1,\Delta f_0,\Delta\dot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\mathcal{D}\Phi.$$

Proof. Note that

$$\begin{aligned}\mathcal{D}\Phi&=\langle\mathcal{D}\Delta f_0,\Delta f_1\rangle+\langle\Delta f_0,\mathcal{D}\Delta f_1\rangle,\qquad\text{and}\\ \mathcal{D}\dot\Phi&=\langle\mathcal{D}\Delta\dot f_0,\Delta f_1\rangle+\langle\mathcal{D}\Delta f_0,\Delta\dot f_1\rangle+\langle\Delta\dot f_0,\mathcal{D}\Delta f_1\rangle+\langle\Delta f_0,\mathcal{D}\Delta\dot f_1\rangle.\end{aligned}$$

Let us prove the statement of the lemma for an arbitrary point t_0. Without loss of generality we fix $\mathcal{D}\dot f_1(t_0)=0$ and $\mathcal{D}\Delta f_1(t_0)=0$ (this is possible since every flexion is isometric to a flexion with such properties and isometries of flexions do not change the functions in the formula of the lemma). Then $\mathcal{D}\Delta f_0(t_0)$ is proportional to $\dot f_1(t_0)\times\Delta f_0(t_0)$, and hence there exists some real number α with

$$\mathcal{D}\Delta f_0(t_0)=\alpha\dot f_1(t_0)\times\Delta f_0(t_0).$$

Thus we immediately get

$$\mathcal{D}\Phi(t_0)=\big\langle\mathcal{D}\Delta f_0(t),\Delta f_1(t)\big\rangle=\alpha\big(\dot f_1(t_0),\Delta f_0(t_0),\Delta f_1(t_0)\big).$$

Let us express the summands for $\mathcal{D}\dot\Phi(t_0)$. We start with $\big\langle\mathcal{D}\Delta\dot f_0(t_0),\Delta f_1(t_0)\big\rangle$. First we note that

(i) $$\Delta f_1=\frac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}\dot f_1+\frac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}\Delta f_0+\frac{(\dot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}\dot f_1\times\Delta\dot f_0.$$

Equation (6) implies

(ii) $$\big\langle\mathcal{D}\Delta\dot f_0(t_0),\dot f_1(t_0)\big\rangle=-\big\langle\mathcal{D}\dot f_1(t_0),\Delta\dot f_0(t_0)\big\rangle=-\big\langle 0,\Delta\dot f_0(t_0)\big\rangle=0.$$

From Equation (4) we have

(iii) $$\big\langle\mathcal{D}\Delta\dot f_0(t_0),\Delta f_0(t_0)\big\rangle=-\big\langle\mathcal{D}\Delta f_0(t_0),\Delta\dot f_0(t_0)\big\rangle=-\alpha\big(\dot f_1(t_0),\Delta f_0(t_0),\Delta\dot f_0(t_0)\big).$$

The function $(\Delta\dot f_0,\dot f_1,\Delta f_0)$ is invariant of an infinitesimal flexion, therefore:

$$(\mathcal{D}\Delta\dot f_0,\dot f_1,\Delta f_0)+(\Delta\dot f_0,\mathcal{D}\dot f_1,\Delta f_0)+(\Delta\dot f_0,\dot f_1,\mathcal{D}\Delta f_0)=0,$$

and hence

$$\text{(iv)} \qquad \begin{aligned}\big\langle \mathcal{D}\Delta \dot f_0(t_0), \dot f_1(t_0)\times\Delta f_0(t_0)\big\rangle &= -\big(\Delta\dot f_0(t_0), \dot f_1(t_0), \mathcal{D}\Delta f_0(t_0)\big)\\ &= -\alpha\big(\Delta\dot f_0(t_0), \dot f_1(t_0), \dot f_1(t_0)\times\Delta f_0(t_0)\big).\end{aligned}$$

Now we decompose $\Delta\dot f_0(t_0)$ in the last formula in the basis of vectors $\dot f_1(t_0)$, $\Delta f_0(t_0)$, and $\Delta f_1(t_0)$:

$$\begin{aligned}\big(\Delta\dot f_0(t_0), \dot f_1(t_0), \dot f_1(t_0)\times\Delta f_0(t_0)\big) = {}& \frac{\big(\dot f_1(t_0),\Delta\dot f_0(t_0),\Delta f_1(t_0)\big)}{\big(\dot f_1(t_0),\Delta f_0(t_0),\Delta f_1(t_0)\big)}\big(\Delta f_0(t_0), \dot f_1(t_0), \dot f_1(t_0)\times\Delta f_0(t_0)\big)+\\ & \frac{\big(\dot f_1(t_0),\Delta f_0(t_0),\Delta\dot f_0(t_0)\big)}{\big(\dot f_1(t_0),\Delta f_0(t_0),\Delta f_1(t_0)\big)}\big(\Delta f_1(t_0), \dot f_1(t_0), \dot f_1(t_0)\times\Delta f_0(t_0)\big).\end{aligned}$$

Therefore, after substitution (i) of Δf_2 we apply (ii), (iii), (iv), and the last expression and get

$$\begin{aligned}\big\langle \mathcal{D}\Delta\dot f_0(t_0), \Delta f_1(t_0)\big\rangle = {}& -\alpha\frac{\big(\dot f_1(t_0),\Delta f_1(t_0),\dot f_1(t_0)\times\Delta f_0(t_0)\big)}{\big(\dot f_1(t_0),\Delta f_0(t_0),\dot f_1(t_0)\times\Delta f_0(t_0)\big)}\big(\dot f_1(t_0), \Delta f_0(t_0), \Delta\dot f_0(t_0)\big)-\\ & \alpha\frac{\big(\dot f_1(t_0),\Delta\dot f_0(t_0),\Delta f_1(t_0)\big)}{\big(\dot f_1(t_0),\Delta f_0(t_0),\dot f_1(t_0)\times\Delta f_0(t_0)\big)}\big(\Delta f_0(t_0), \dot f_1(t_0), \dot f_1(t_0)\times\Delta f_0(t_0)\big)-\\ & \alpha\frac{\big(\dot f_1(t_0),\Delta f_0(t_0),\Delta\dot f_0(t_0)\big)}{\big(\dot f_1(t_0),\Delta f_0(t_0),\dot f_1(t_0)\times\Delta f_0(t_0)\big)}\big(\Delta f_1(t_0), \dot f_1(t_0), \dot f_1(t_0)\times\Delta f_0(t_0)\big)\\ = {}& -\alpha\big(\dot f_1(t_0), \Delta f_1(t_0), \Delta\dot f_0(t_0)\big).\end{aligned}$$

Similar calculations for the summand $\big\langle \Delta f_0(t_0), \mathcal{D}\Delta\dot f_1(t_0)\big\rangle$ (applying Equations (3), (5), and (7) and the conditions $\mathcal{D}\dot f_1(t_0) = 0$ and $\mathcal{D}\Delta f_1(t_0) = 0$) show that

$$\big\langle \Delta f_0(t_0), \mathcal{D}\Delta\dot f_1(t_0)\big\rangle = 0.$$

Further we have

$$\begin{aligned}\big\langle \mathcal{D}\Delta f_0(t_0), \Delta\dot f_1(t_0)\big\rangle &= \alpha\big(\dot f_1(t_0), \Delta f_0(t_0), \Delta\dot f_1(t_0)\big),\\ \big\langle \Delta\dot f_0(t_0), \mathcal{D}\Delta f_1(t_0)\big\rangle &= 0.\end{aligned}$$

Therefore,

$$\mathcal{D}\dot\Phi(t_0) = \alpha\Big(\big(\dot f_1(t_0), \Delta\dot f_0(t_0), \Delta f_1(t_0)\big) + \big(\dot f_1(t_0), \Delta f_0(t_0), \Delta\dot f_1(t_0)\big)\Big),$$

and consequently

$$\mathcal{D}\dot\Phi(t_0) = \frac{\big(\dot f_1(t_0), \Delta\dot f_0(t_0), \Delta f_1(t_0)\big) + \big(\dot f_1(t_0), \Delta f_0(t_0), \Delta\dot f_1(t_0)\big)}{\big(\dot f_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big)}\mathcal{D}\Phi(t_0).$$

Thus Lemma 3.3 holds for all possible values of t_0. □

3.1.2. *Discrete shift.* Every 3-ribbon surface contains two 2-ribbon surfaces as a subsurfaces. Each of them has an infinitesimal flexion $\mathcal{D}\Phi_i$ $(i=1,2)$, where

$$\Phi_1 = \langle \Delta f_0, \Delta f_1 \rangle \qquad \text{and} \qquad \Phi_2 = \langle \Delta f_1, \Delta f_2 \rangle.$$

Let us show the relation between $\mathcal{D}\Phi_1$ and $\mathcal{D}\Phi_2$ for the same values of argument t.

First, in Proposition 3.4 we show a relation for $\mathcal{D}\langle \ddot{f}_1, \ddot{f}_1 \rangle$ and $\mathcal{D}\langle \ddot{f}_2, \ddot{f}_2 \rangle$. Second, in Proposition 3.5 we give a link between $\mathcal{D}\langle \ddot{f}_1, \ddot{f}_1 \rangle$ and $\mathcal{D}\Phi_1$. This will result in the formula of Corollary 3.6 on the relation between $\mathcal{D}\Phi_1$ and $\mathcal{D}\Phi_2$.

We start with a formula expressing $\mathcal{D}\langle \ddot{f}_2, \ddot{f}_2 \rangle$ via $\mathcal{D}\langle \ddot{f}_1, \ddot{f}_1 \rangle$.

Proposition 3.4. *Let f be a strongly generic 3-ribbon surface in $C^{1,2,2,1}([a,b],\mathbb{R}^3)$, and let $\mathcal{D}f$ be its infinitesimal flexion. Then the following equation holds:*

$$\mathcal{D}\langle \ddot{f}_2, \ddot{f}_2 \rangle = \frac{(\dot{f}_2, \ddot{f}_2, \Delta f_1)}{(\dot{f}_1, \ddot{f}_1, \Delta f_1)} \mathcal{D}\langle \ddot{f}_1, \ddot{f}_1 \rangle.$$

Proof. We do calculations at a point t_0 again assuming that $\mathcal{D}\dot{f}_1(t_0)=0$ and $\mathcal{D}\Delta f_1(t_0)=0$ (by choosing an appropriate isometric representative of the deformation). Let us show that $\mathcal{D}\dot{f}_2(t_0)=0$. First, note that

$$\mathcal{D}\dot{f}_2(t_0) = \mathcal{D}\dot{f}_1(t_0) + \mathcal{D}\Delta\dot{f}_1(t_0) = \mathcal{D}\Delta\dot{f}_1(t_0).$$

Secondly we show that the inner products of $\mathcal{D}\Delta\dot{f}_1(t_0)$ and the vectors $\dot{f}_1(t_0)$, $\Delta f_1(t_0)$, and $\dot{f}_1(t_0)\times\Delta f_1(t_0)$ are all zero (this would imply that $\mathcal{D}\Delta\dot{f}_1(t_0)=0$).

From Equation (7) we have

$$\big\langle \mathcal{D}\Delta\dot{f}_1(t_0), \dot{f}_1(t_0)\big\rangle = -\big\langle \mathcal{D}\dot{f}_1(t_0), \Delta\dot{f}_1(t_0)\big\rangle = -\big\langle 0, \Delta\dot{f}_1(t_0)\big\rangle = 0.$$

Further, from Equations (5), we get

$$\big\langle \mathcal{D}\Delta\dot{f}_1(t_0), \Delta f_1(t_0)\big\rangle = -\big\langle \mathcal{D}\Delta f_1(t_0), \Delta\dot{f}_1(t_0)\big\rangle = 0.$$

Finally, from the equation $\mathcal{D}(\dot{f}_1, \Delta f_1, \Delta\dot{f}_1) = 0$ we obtain

$$\begin{array}{l}\big\langle \mathcal{D}\Delta\dot{f}_1(t_0), \dot{f}_1(t_0)\times\Delta f_1(t_0)\big\rangle = \\ -\big(\Delta\dot{f}_1(t_0), \mathcal{D}\dot{f}_1(t_0), \Delta f_1(t_0)\big) - \big(\Delta\dot{f}_1(t_0), \dot{f}_1(t_0), \mathcal{D}\Delta f_1(t_0)\big) = 0.\end{array}$$

Therefore, $\mathcal{D}\Delta\dot{f}_1(t_0)=0$, and hence $\mathcal{D}\dot{f}_2(t_0)=0$.

From Equation (1) and Equation (9) we get

$$\begin{array}{l}\big\langle \mathcal{D}\ddot{f}_1(t_0), \dot{f}_1(t_0)\big\rangle = \frac{\partial}{\partial t}\big\langle \mathcal{D}\dot{f}_1(t_0), \dot{f}_1(t_0)\big\rangle - \big\langle \ddot{f}_1(t_0), \mathcal{D}\dot{f}_1(t_0)\big\rangle = 0 - \big\langle \ddot{f}_1(t_0), 0\big\rangle = 0; \\ \big\langle \mathcal{D}\ddot{f}_1(t_0), \Delta f_1(t_0)\big\rangle = -\big\langle \ddot{f}_1(t_0), \mathcal{D}\Delta f_1(t_0)\big\rangle = -\big\langle \ddot{f}_1(t_0), 0\big\rangle = 0.\end{array}$$

Therefore, for some real number β_1 we have

$$\mathcal{D}\ddot{f}_1(t_0) = \beta_1 \dot{f}_1(t_0)\times\Delta f_1(t_0).$$

By a similar reasoning (since we have shown that $\mathcal{D}\dot{f}_2(t_0)=0$) we get

$$\mathcal{D}\ddot{f}_2(t_0) = \beta_2 \dot{f}_2(t_0)\times\Delta f_1(t_0).$$

Since $\frac{\partial}{\partial t}\big(\mathcal{D}(\dot f_1, \Delta f_1, \dot f_2)\big) = 0$, at point t_0 we have

$$\big(\mathcal{D}\ddot f_1(t_0), \Delta f_1(t_0), \dot f_2(t_0)\big) + \big(\dot f_1(t_0), \Delta f_1(t_0), \mathcal{D}\ddot f_2(t_0)\big) = 0.$$

Hence,

$$\beta_1\big(\dot f_1(t_0)\times\Delta f_1(t_0), \Delta f_1(t_0), \dot f_2(t_0)\big) + \beta_2\big(\dot f_1(t_0), \Delta f_1(t_0), \dot f_2(t_0)\times\Delta f_1(t_0)\big) = 0,$$

and, therefore $\beta_1 = \beta_2$. This implies

$$\mathcal{D}\big\langle \ddot f_1(t_0), \ddot f_1(t_0)\big\rangle = 2\big\langle \mathcal{D}\ddot f_1(t_0), \ddot f(t_0)\big\rangle = 2\beta_1\big(\dot f_1(t_0), \Delta f_1(t_0), \ddot f_1(t_0)\big)$$

and

$$\mathcal{D}\big\langle \ddot f_2(t_0), \ddot f_2(t_0)\big\rangle = 2\beta_1\big(\dot f_2(t_0), \Delta f_1(t_0), \ddot f_2(t_0)\big).$$

The last two formulas imply the statement of Proposition 3.4. □

Now let us relate $\mathcal{D}\langle \ddot f_1, \ddot f_1\rangle$ and $\mathcal{D}\Phi$.

Proposition 3.5. *Let f be a weakly generic 2-ribbon surface in $C^{1,2,1}([a,b],\mathbb{R}^3)$. Then the following identity holds:*

$$\mathcal{D}\langle \ddot f_1, \ddot f_1\rangle = 2\frac{(\dot f_1, \ddot f_1, \Delta f_0)(\dot f_1, \ddot f_1, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)^2}\mathcal{D}\Phi.$$

Proof. We restrict ourselves to the case of a point. Without loss of generality we assume that $\mathcal{D}\dot f_1(t_0) = 0$ and $\mathcal{D}\Delta f_1(t_0) = 0$. So as we have seen before, there exists α such that

$$\mathcal{D}\Delta f_0(t_0) = \alpha \dot f_1(t_0)\times\Delta f_0(t_0)$$

and hence

$$\mathcal{D}\Phi(t_0) = \alpha\big(\dot f_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big).$$

Let us calculate $\mathcal{D}\langle \ddot f_1, \ddot f_1\rangle = 2\langle \mathcal{D}\ddot f_1, \ddot f_1\rangle$. Decompose

$$\ddot f_1 = \frac{(\ddot f_1, \Delta f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}\dot f_1 + \frac{(\dot f_1, \ddot f_1, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}\Delta f_0 + \frac{(\dot f_1, \Delta f_0, \ddot f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}\Delta f_1.$$

Since

$$\big\langle \mathcal{D}\ddot f_1(t_0), \dot f_1(t_0)\big\rangle = 0, \qquad \text{and} \qquad \big\langle \mathcal{D}\ddot f_1(t_0), \Delta f_1(t_0)\big\rangle = 0,$$

we get

$$\mathcal{D}\langle \ddot f_1(t_0), \ddot f_1(t_0)\rangle = 2\frac{\big(\dot f_1(t_0), \ddot f_1(t_0), \Delta f_1(t_0)\big)}{\big(\dot f_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big)}\langle \mathcal{D}\ddot f_1(t_0), \Delta f_0(t_0)\rangle.$$

By Equation (8) we have

$$\big\langle \mathcal{D}\ddot f_1, \Delta f_0\big\rangle = -\big\langle \ddot f_1, \mathcal{D}\Delta f_0\big\rangle.$$

Hence after the substitution of $\mathcal{D}\Delta f_0(t_0)$ in the first summand one gets

$$\big\langle \mathcal{D}\ddot f_1, \Delta f_0\big\rangle = \alpha\big(\dot f_1(t_0), \ddot f_1(t_0), \Delta f_0(t_0)\big) = \frac{\big(\dot f_1(t_0), \ddot f_1(t_0), \Delta f_0(t_0)\big)}{\big(\dot f_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big)}\mathcal{D}\Phi(t_0).$$

Therefore, we obtain

$$\mathcal{D}\langle \ddot{f}_1(t_0), \ddot{f}_1(t_0)\rangle = 2\frac{\big(\dot{f}_1(t_0), \ddot{f}_1(t_0), \Delta f_1(t_0)\big)\big(\dot{f}_1(t_0), \ddot{f}_1(t_0), \Delta f_0(t_0)\big)}{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big)^2}\mathcal{D}\Phi(t_0).$$

Since the statement does not depend on the choice of the basis and invariant under isometries, we get the statement for all the points. □

Let us show a formula of a discrete shift.

Corollary 3.6. (On discrete shift.) *Let f be a strongly generic 3-ribbon surface in $C^{1,2,2,1}([a,b],\mathbb{R}^3)$. Then the following holds:*

$$\mathcal{D}\Phi_2(t) = \frac{\big(\dot{f}_1(t), \ddot{f}_1(t), \Delta f_0(t)\big)}{\big(\dot{f}_2(t), \ddot{f}_2(t), \Delta f_2(t)\big)}\frac{\big(\dot{f}_2(t), \Delta f_1(t), \Delta f_2(t)\big)^2}{\big(\dot{f}_1(t), \Delta f_0(t), \Delta f_1(t)\big)^2}\mathcal{D}\Phi_1(t).$$

Proof. The statement follows directly from Propositions 3.4 and 3.5. □

3.2. **Necessary condition of infinitesimal flexibility.** In this subsection we write down the infinitesimal flexibility monodromy conditions for 3-ribbon surfaces (via continuous shifts of Proposition 3.2 and discrete shifts of Corollary 3.6). Recall that

$$\Lambda(t) = \frac{\big(\dot{f}_1(t), \ddot{f}_1(t), \Delta f_0(t)\big)}{\big(\dot{f}_2(t), \ddot{f}_2(t), \Delta f_2(t)\big)}\frac{\big(\dot{f}_2(t), \Delta f_1(t), \Delta f_2(t)\big)^2}{\big(\dot{f}_1(t), \Delta f_0(t), \Delta f_1(t)\big)^2},$$

and

$$H_i(t) = \frac{(\dot{f}_i(t), \Delta\dot{f}_{i-1}(t), \Delta f_i(t)) + (\dot{f}_i(t), \Delta f_{i-1}(t), \Delta\dot{f}_i(t))}{(\dot{f}_i(t), \Delta f_{i-1}(t), \Delta f_i(t))}, \quad i = 1, 2.$$

Theorem 3.7. *Let f be a strongly generic 3-ribbon surface in $C^{1,2,2,1}([a,b],\mathbb{R}^3)$. If the surface f is infinitesimally flexible then for every $t_1, t_2 \in [a,b]$ we have*

$$\Lambda(t_2)\cdot\exp\Big(\int\limits_{t_1}^{t_2} H_1(t)dt\Big) = \Lambda(t_1)\cdot\exp\Big(\int\limits_{t_1}^{t_2} H_2(t)dt\Big).$$

Remark 3.8. The condition of the proposition can be written in the "almost" equivalent infinitesimal form:

$$\dot{\Lambda} - (H_2 - H_1)\Lambda = 0.$$

Here the left hand side expression is considered as a function in the interval $[a,b]$. The last expression has one disadvantage, $\dot{\Lambda}$ involves the third derivatives of f_1 and f_2, while the expressions in proposition involve only up to the second derivatives.

Proof. Let f be infinitesimally flexible and $\mathcal{D}f$ be its infinitesimal nonzero flexion. On the one side by Corollary 3.6 we get relations between $\mathcal{D}\Phi_1(t_i)$ and $\mathcal{D}\Phi_2(t_i)$ for $i = 1, 2$. On the other side, Proposition 3.2 relates $\mathcal{D}\Phi_i(t_1)$ and $\mathcal{D}\Phi_i(t_2)$ for $i = 1, 2$. These four relations define the monodromy condition for Φ_i that is the condition in the theorem, Therefore, it holds if a surface is infinitesimally flexible. □

Remark 3.9. Let us write a more simple expressions for a surface w defined as

$$\begin{aligned} w_0 &= f_1 - \tfrac{1}{(\dot{f}_1, \Delta f_0, \Delta f_1)} \Delta f_0; \\ w_1 &= f_1; \\ w_2 &= f_2; \\ w_3 &= f_2 + \tfrac{1}{(\dot{f}_2, \Delta f_1, \Delta f_2)} \Delta f_2. \end{aligned}$$

As one can see, all rulings of w (if non-vanished) are parallel to the corresponding rulings of f. In addition the middle strip of f coincides with the middle strip of w.

Notice that

$$\big(\dot{w}_1(t), \Delta w_0(t), \Delta w_1(t)\big) = 1 \qquad \text{and} \qquad \big(\dot{w}_2(t), \Delta w_1(t), \Delta w_2(t)\big) = 1$$

for all arguments t. Hence we have:

$$\begin{aligned} \Lambda &= \frac{\big(\dot{w}_1, \ddot{w}_1, \Delta w_0\big)}{\big(\dot{w}_2, \ddot{w}_2, \Delta w_2\big)}, \\ H_i &= -(\ddot{w}_i, \Delta w_{i-1}, \Delta w_i), \quad i = 1, 2. \end{aligned}$$

Note that this expression holds momentary.

We conclude this subsection with the following open problem.

Problem 1. Find a sufficient condition for infinitesimal/finite flexibility of semidiscrete and 3-ribbon surfaces.

4. Flexibility of n-ribbon surfaces

In this section we study flexibility questions for general case of $n \geq 2$. We show that a strongly generic n-ribbon surface has at most one degree of finite and infinitesimal flexibility (Subsection 4.1). Further we study flexions of combined n-ribbon surfaces (Subsection 4.2). This allows us to prove that finite or infinitesimal flexibility of generic n-ribbon surfaces is identified by finite or infinitesimal flexibility of all its 3-ribbon subsurfaces (Subsection 4.3).

4.1. At most one degree of flexibility for strongly generic n-ribbon surfaces.

In this subsection we prove that all nontrivial infinitesimal flexions of strongly generic n-ribbon surfaces are strongly isometrically nontrivial, and that such surfaces has at most one degree of infinitesimal flexibility.

Let us start with a useful tool to work with isometrically nontrivial flexions.

Lemma 4.1. *An infinitesimal flexion of a weakly generic n-ribbon surface f in the space $C^{0,1,0}([a,b], \mathbb{R}^3)$ is isometrically nontrivial at a point (t,i) (where $i \in [1, \ldots, n-1]$) if and only if*

$$\mathcal{D}\langle \Delta f_{i-1}(t), \Delta f_i(t) \rangle \neq 0.$$

Proof. Since f is weakly generic, the pairs of vectors $(\dot{f}_i, \Delta f_{i-1})$ and $(\dot{f}_i, \Delta f_i)$ span two non-coinciding 2-spaces π_1 and π_2.

Since π_1 and π_2 do not coincide, the condition $\mathcal{D}\langle \Delta f_{i-1}, \Delta f_i\rangle \neq 0$ is equivalent to the fact that the infinitesimal flexion of the angle between π_1 and π_2 is non-zero. Therefore, by Definition 1.13 the last is equivalent to f being isometrically nontrivial at a point (t, i). □

In the next proposition we prove two important preliminary statements.

Proposition 4.2. *Consider $n \geq 2$. Let f be a strongly generic n-ribbon surface in the space $C_0^{1,2,2,\ldots,2,1}([a,b], \mathbb{R}^3)$. Then the following two statements hold.*
(i) Every isometrically nontrivial infinitesimal flexion of f is strongly isometrically nontrivial $($i.e., f is isometrically nontrivial at every point $(t, i))$.
(ii) For every regular isometric deformation γ there exists a locally monotone function ξ such that $\gamma(\xi)$ is a normalized isometric deformation of f in some neighborhood of 0.

Proof. We prove Theorem 4.2(i) by induction in n.
Base of induction. Case $n = 2$. Let $\mathcal{D}f$ be a nontrivial infinitesimal flexion of a weakly generic 2-ribbon surface f in $C_0^{1,2,1}([a,b], \mathbb{R}^3)$. Therefore, there exists t_0 such that $\mathcal{D}\Phi(t_0) \neq 0$. By Proposition 3.2, $\mathcal{D}\Phi(t_0) \neq 0$ implies that $\mathcal{D}\Phi(t) \neq 0$ for every $t \in [a,b]$. Hence, by Lemma 4.1 f is isometrically nontrivial at each point $(t, 1)$. Therefore, f is strongly isometrically nontrivial.

Case $n = 3$. Let $\mathcal{D}f$ be a nontrivial infinitesimal flexion of a generic 3-ribbon surface f. Therefore, there exists a point (t_0, i) such that $\mathcal{D}\Phi_i(t_0) \neq 0$. Without loss of generality we assume that $i = 1$ (the case $i = 2$ is similar).

By the above in case $n = 2$ we have: $\mathcal{D}\Phi_1(t_0) \neq 0$ implies that $\mathcal{D}\Phi_1(t) \neq 0$ for every $t \in [a,b]$. By Corollary 3.6 (and the strongly generic condition for f), for every $t \in [a,b]$ the statement $\mathcal{D}\Phi_1(t) \neq 0$ implies that $\mathcal{D}\Phi_2(t) \neq 0$. Hence, by Lemma 4.1 f is isometrically nontrivial at each point $(t, 1)$ and $(t, 2)$. Therefore, by Definition 1.13 f is strongly isometrically nontrivial.

Step of induction. Consider a strongly generic n-ribbon surface f with $n \geq 4$. Denote

$$f^1 = (f_0, f_1, \ldots, f_{n-1}), \qquad f^2 = (f_1, \ldots, f_{n-1}, f_n) \quad \text{and} \quad f^{12} = (f_1, \ldots, f_{n-1})$$

Let $\mathcal{D}f$ be isometrically nontrivial flexion of f. Without loss of generality we assume that $\mathcal{D}f^1$ is isometrically nontrivial. Hence by the induction assumption $\mathcal{D}f^1$ is strongly isometrically nontrivial. Thus, $\mathcal{D}f^{12}$ is strongly isometrically nontrivial. Since f^{12} is a $(n-2)$-ribbon (with $n \geq 4$), we have that $\mathcal{D}f^2$ is isometrically nontrivial. Then by the induction assumption $\mathcal{D}f^2$ is strongly isometrically nontrivial.

Since $n \geq 3$ and both f^1 and f^2 are strongly isometrically nontrivial, f is strongly isometrically nontrivial as well. This concludes the proof of Proposition 4.2(i).

Let us prove Proposition 4.2(ii). Let $\{f^\lambda\}$ be a regular isometric deformation of f with parameter $\lambda \in [-\Lambda, \Lambda]$. Since f has a fixed initial position, we have:

$$\mathcal{D}_{f^\lambda}\dot{f}_1^{\lambda_0}(a) = 0, \quad \mathcal{D}_{f^\lambda}\Delta f_0^{\lambda_0}(a) = 0, \quad \text{and} \quad \mathcal{D}_{f^\lambda}\Delta f_1^{\lambda_0}(a) = \alpha(\lambda)\dot{f}_1^{\lambda_0}(a) \times \Delta f_1^{\lambda_0}(a)$$

for every $\lambda_0 \in [-\Lambda, \Lambda]$.

Since $\{f^\lambda\}$ is regular, $\mathcal{D}_{f^\lambda} f^0 \neq 0$. Therefore, there exists $\varepsilon > 0$ such that $\mathcal{D}_{f^\lambda} f^{\lambda_0} \neq 0$ for $\lambda_0 \in [-\varepsilon, \varepsilon]$. From Proposition 4.2(i) it follows that for every $\lambda_0 \in [-\varepsilon, \varepsilon]$ the flexion $\mathcal{D}_{f^\lambda} f^{\lambda_0}$ is strongly isometrically nontrivial. Hence

$$\alpha(\lambda) \neq 0 \quad \text{for} \quad \lambda \in [-\varepsilon, \varepsilon].$$

Therefore, $\alpha(\lambda)$ is either a positive function or a negative function on $[-\varepsilon, \varepsilon]$. Denote

$$\varphi(\lambda) = \int_0^\lambda \alpha(\tau) d\tau.$$

The function φ is monotonous on $[-\varepsilon, \varepsilon]$, and hence there exists an inverse function φ^{-1} on that interval. Denote

$$\xi = \begin{cases} \varphi^{-1}, & \text{if } \varphi \text{ is increasing,} \\ -\varphi^{-1}, & \text{if } \varphi \text{ is decreasing.} \end{cases}$$

Choose positive $\hat{\varepsilon}$ such that ξ is defined on $[-\hat{\varepsilon}, \hat{\varepsilon}]$. Then

$$\alpha(\xi(\lambda)) = 1$$

for all $\lambda \in [-\hat{\varepsilon}, \hat{\varepsilon}]$. Hence $\gamma \circ \xi$ is a normalized isometric deformation. □

Now we study degrees of finite and infinitesimal flexibility.

Remark 4.3. To be consistent we mention the case of 2-ribbon surfaces. Let f be a weakly generic 2-ribbon semidiscrete surface in the space $C_0^{1,2,1}([a,b], \mathbb{R}^3)$. Then the following two statements hold.
(i) The surface f has one degree of infinitesimal flexibility (Theorem 2.3).
(ii) The surface f has one degree of finite flexibility (Theorem 2.23).

Let us prove a similar statement for the case of $n \geq 3$.

Theorem 4.4. *Consider $n \geq 3$. Let f be a strongly generic n-ribbon surface in the space $C_0^{1,2,2,\ldots,2,1}([a,b], \mathbb{R}^3)$. Then the following two statements hold.*

(i) *The surface f has at most one degree of infinitesimal flexibility* (*i.e., all infinitesimal isometrically nontrivial flexions are proportional*).

(ii) *The surface f is either finitely rigid or has one degree of finite flexibility.*

Proof. (i) Let us assume the converse. Suppose there are two non-proportional isometrically-nontrivial flexions $\mathcal{D}^1 f$ and $\mathcal{D}^2 f$. By Proposition 4.2(i) both flexions are isometrically nontrivial at $(a, 1)$ Hence there exists α such that the infinitesimal flexion

$$\mathcal{D} f = \mathcal{D}^1 f - \alpha \mathcal{D}^2 f$$

is isometrically trivial at $(a, 1)$. Since $\mathcal{D}^1 f$ and $\mathcal{D}^2 f$ are non-proportional, there exists a point (t, i) at which the flexion $\mathcal{D} f$ is isometrically nontrivial. Hence by Proposition 4.2(i) the infinitesimal flexion $\mathcal{D} f$ is isometrically nontrivial at $(a, 1)$. We arrive at a contradiction.

(*ii*) By Theorem 4.4(i) all infinitesimal flexions are proportional. Hence f has at most one degree of infinitesimal flexibility. If it is zero, then the f is infinitesimally rigid and hence it is finitely rigid.

Let f has one degree of infinitesimal flexibility. If f does not have regular isometric deformations then f is finite rigid. If f has a regular isometric deformation, then f has a normalized isometric deformation. Let us show that there exists at most one normalized isometric deformation of f. Let $\{f^\lambda\}$ be a normalized isometric deformation of f. As before we denote

$$\Phi_i^\lambda = \langle \Delta f_{i-1}^\lambda, \Delta f_i^\lambda \rangle.$$

Notice that for normalized isometric deformations we have:

$$\mathcal{D}_{f^\lambda}(\Phi_0^\lambda(a)) = (\Delta f_0^\lambda(a), \dot{f}_1^\lambda(a), \Delta f_1^\lambda).$$

Therefore

$$\Phi_0^\lambda(a) = \int_0^\lambda (\Delta f_0^\mu(a), \dot{f}_1^\mu(a), \Delta f_1^\mu) d\mu.$$

Hence $\Phi_0^\lambda(a)$ coincides for all normalized isometric deformation of f. Therefore, by Proposition 3.2 and Corollary 3.6 for every (t, i) and every parameter λ the value

$$\Phi_i^\lambda(t)$$

is the same for all normalized isometric deformations. Therefore, by Theorem 2.23 every restriction of an arbitrary normalized isometric deformation $\{f^\lambda\}$ to the deformation of a 2-ribbon subsurface of f does not depend on the choice of the normalized isometric deformation $\{f^\lambda\}$. Hence, all normalized isometric deformations of f coincide. Therefore, f has one degree of finite flexibility. □

Remark 4.5. The strongly genericity condition of Theorem 4.2 is essential. Let us illustrate this with a simple example of a 3-ribbon surfaces which is not strongly generic. Consider

$$f_0(t) = (t, 1, 0); \quad f_1(t) = (t, 0, 0); \quad f_2(t) = (t, 0, 1); \quad f_3(t) = (t, 1, 1);$$

This surface has two distinct isometric deformations:

(i) $f_0^\alpha(t) = (t, 1, 0); \quad f_1^\alpha(t) = (t, 0, 0); \quad f_2^\alpha(t) = (t, 0, 1); \quad f_3^\alpha(t) = (t, \cos\alpha, 1 + \sin\alpha);$

(ii) $f_0^\beta(t) = (t, 1, 0); \quad f_1^\beta(t) = (t, 0, 0);$
$f_2^\beta(t) = (t, \sin\beta, \cos\beta); \quad f_3^\beta(t) = \big(t, \sqrt{2}\sin(\beta + \frac{\pi}{4}), \sqrt{2}\cos(\beta + \frac{\pi}{4})\big);$

The infinitesimal flexions defined by these isometric deformations are not proportional.

4.2. **Flexibility of combined n-ribbon surfaces.** In this subsection we study finite and infinitesimal flexions of combined strongly generic semidiscrete surfaces.

As above, for an arbitrary semidiscrete surface $f = (f_0, f_1, \ldots, f_n)$ we denote

$$f^1 = (f_0, f_1, \ldots, f_{n-1}), \qquad f^2 = (f_1, \ldots, f_{n-1}, f_n) \quad \text{and} \quad f^{12} = (f_1, \ldots, f_{n-1}). \tag{18}$$

4.2.1. *Infinitesimal case.* We start with the infinitesimal case.

Theorem 4.6. (Infinitesimal flexibility of combined semidiscrete surfaces.) *Let $n \geq 4$. Consider a strongly generic n-ribbon semidiscrete surface f in $C_0^{1,2,2,\ldots,2,1}([a,b],\mathbb{R}^3)$. Let surfaces f^1 and f^2 defined by (18) be infinitesimally flexible. Then f is infinitesimally flexible and has precisely one degree of infinitesimal flexibility.*

Proof. Let $\mathcal{D}^1 f^1$ and $\mathcal{D}^2 f^2$ be isometrically nontrivial flexions of f^1 and f^2 respectively. Since f^1 and f^2 are strongly generic $(n-1)$-ribbons surfaces for $n \geq 4$, Theorem 4.2 can be applied. By Theorem 4.2(i) the surfaces f^1 and f^2 are strongly isometrically nontrivial. Hence by Theorem 4.2(i) in case $n > 4$ or by Theorem 2.3 (see Remark 4.3 above) in case $n = 4$ the induced flexions $\tilde{\mathcal{D}}^1 f^{12}$ and $\tilde{\mathcal{D}}^2 f^{12}$ are proportional, i.e, there exists α such that

$$\mathcal{D}^1 f^{12} = \alpha \mathcal{D}^2 f^{12}.$$

Thus the surface f has the combined infinitesimal flexion $\mathcal{D}f$ that induces $\mathcal{D}^1 f^1$ and $\alpha\mathcal{D}^2 f^2$. This flexion is infinitesimally nontrivial, since the induced ones are infinitesimally nontrivial. Hence f has at least one degree of infinitesimal flexibility.

On the other hand by Theorem 4.2(i) the surface f has at most one degree of infinitesimal flexibility. Hence f is infinitesimally flexible and has one degree of infinitesimal flexibility. □

4.2.2. *Finite case.* We start with the following general statement on reparametrisation of deformations.

Proposition 4.7. *Let f be a strongly generic n-ribbon surface ($n \geq 2$) in the space $C^{1,2,2,\ldots,2,1}([a,b],\mathbb{R}^3)$. Consider two regular isometric deformations γ_1 and γ_2 of f. Then there exists a monotonous function ξ such that $\gamma_1(\lambda) = \gamma_2(\xi(\lambda))$ in some small neighborhood of* 0.

Proof. By Proposition 4.2(ii) there exists monotonous functions ξ_1 and ξ_2 such that $\gamma_1 \circ \xi_1$ and $\gamma_2 \circ \xi_2$ are normalized isometric deformations of f. Hence by Theorem 4.4 in case $n \geq 3$ and Theorem 2.23 (see Remark 4.3 above) in case $n = 2$ we have

$$\gamma_1 \circ \xi_1 = \gamma_2 \circ \xi_2$$

in some neighborhood of 0. Set $\xi = \xi_2 \circ \xi_1^{-1}$. The function ξ is the monotonous function such that $\gamma_1(\lambda) = \gamma_2(\xi(\lambda))$ in some small neighborhood of 0. □

Theorem 4.8. (Finite flexibility of combined semidiscrete surfaces.) *Let $n \geq 4$. Consider a strongly generic n-ribbon semidiscrete surface f in $C_0^{1,2,2,\ldots,2,1}([a,b],\mathbb{R}^3)$. Let surfaces f^1 and f^2 defined by (18) be finitely flexible. Then f is finitely flexible and has one degree of finite flexibility.*

Proof. By Theorem 4.4(ii) the surfaces f^1 and f^2 have one degree of finite flexibility. Therefore, there exist unique normalized isometric deformations γ_1 and γ_2 of f^1 and f^2 respectively. They induce two deformations $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ of the $(n-2)$-ribbon surface f^{12}. By Proposition 4.7, since $n-2 \geq 2$, these two deformations locally parameterize the same curve in $C_0^{1,2,2,\ldots,2,1}([a,b],\mathbb{R}^3)$, i.e., in the segment $[-\varepsilon,\varepsilon]$ for some $\varepsilon>0$ there exists a locally increasing function ξ such that $\tilde\gamma_1(\lambda)=\tilde\gamma_2(\xi(\lambda))$.

Now consider the deformation γ of the surface f inducing both isometric deformations γ_1 for f_1 and $\gamma_2\circ\xi$ for f_2 in the segment $[-\varepsilon,\varepsilon]$ for some $\varepsilon>0$. The deformation γ is isometric, since the induced ones are isometric. In addition, γ is normalized, since its restriction to f^1 is a normalized isometric deformation. Hence f is finitely flexible (and not finitely rigid). Therefore, by Theorem 4.2(ii) f has one degree of finite flexibility. □

Remark 4.9. Note that the statements of Theorems 4.6 and 4.8 are no longer true for the case $n=3$. On the one hand every infinitesimally flexible (and, therefore, finitely flexible) 3-ribbon surface should satisfy a condition of Theorem 3.7, and, as it is easy to see, not every strongly generic 3-ribbon surface satisfies it. Hence there are strongly generic rigid 3-ribbon surfaces. On the other hand every 2-ribbon subsurface is weakly generic and hence it is finitely (and, therefore, infinitesimally) flexible. These two statements together contradict to the version of the statement of Theorems 4.6 for the case $n=3$.

4.3. **An n-ribbon surface and its 3-ribbon subsurfaces.** Let us finally describe a relation between finite/infinitesimal flexibility of n-ribbon surfaces and finite/infinitesimal flexibility of all 3-ribbon subsurfaces contained in them.

Theorem 4.10. *Let $n\ge 4$. Consider a strongly generic n-ribbon surface f in the space $C_0^{1,2,2,\ldots,2,1}([a,b],\mathbb{R}^3)$. Then f is infinitesimally flexible* (*and has one degree of infinitesimal flexibility*) *if and only if every 3-ribbon surface contained in the surface is infinitesimally flexible.*

Proof. Let f be infinitesimally flexible. Therefore, there exists an infinitesimal flexion $\mathcal{D}f$ that is isometrically nontrivial. Therefore, by Proposition 4.2(i) the flexion $\mathcal{D}f$ is strongly isometrically nontrivial. Hence all its 3-ribbon surfaces are isometrically nontrivially flexible.

Suppose now that all 3-ribbon subsurfaces in a strongly generic surface f are infinitesimally flexible. We prove that all k-ribbon surfaces are infinitesimally flexible for $k=3,4,\ldots,n$ by induction in k.
Base of induction. The case $k=3$ is tautological.
Step of induction. The k-th statement follows from the $(k-1)$-th by Theorem 4.6.

Hence f is infinitesimally flexible. Therefore, by Theorem 4.4(i) f has one degree of infinitesimal flexibility. □

For the finite flexibility we have the following.

Theorem 4.11. *Let $n\ge 4$. Consider a strongly generic n-ribbon surface f in the space $C_0^{1,2,2,\ldots,2,1}([a,b],\mathbb{R}^3)$. Then this surface is finitely flexible* (*and has one degree of flexibility*) *if and only if every 3-ribbon surface contained in the surface is finitely flexible.*

Remark 4.12. We think of this theorem as of a semidiscrete analogue to the statement of the paper [2] on conjugate nets and all (3×3)-meshes that they contain. In this paper we do not study phenomena related to non-compactness and hence we restrict ourselves to the case of compact n-ribbons surfaces.

Proof. Let f be finitely flexible. Therefore there exists a regular isometric deformation γ of f. Since γ is regular we have $\mathcal{D}_\gamma f \neq 0$. Since every finite flexion is infinitesimal flexion we are in position to apply Proposition 4.2(i). We get that the flexion $\mathcal{D}_\gamma f$ is strongly isometrically nontrivial. Hence the induced isometric deformations of all 3-ribbon surfaces have corresponding nontrivial finite flexions. Therefore, all 3-ribbon surfaces contained in f are finitely flexible.

Suppose that all 3-ribbon subsurfaces in a strongly generic surface f are finitely flexible. Let us prove that every k-ribbon surface in f is finitely flexible for $k = 3, 4, \ldots, n$ by induction in k.
Base of induction. The case $k = 3$ is tautological.
Step of induction. The k-th statement follows from the $(k-1)$-th by Theorem 4.8.

Hence f is finitely flexible. Therefore, by Theorem 4.4(ii) f has one degree of finite flexibility. □

5. Isometric deformation of developable semidiscrete surfaces

Suppose that all ribbons of a semidiscrete surface are developable, i.e., the vectors $\dot{f}_i$, Δf_i, and $\dot{f}_{i+1}$ are linearly dependent. We call such semidiscrete surfaces *developable*. In this section we describe an additional property for flexions of developable semidiscrete surfaces. We start with 2-ribbon surfaces.

Recall that

$$H_1 = \frac{(\dot{f}_1, \Delta\dot{f}_0, \Delta f_1) + (\dot{f}_1, \Delta f_0, \Delta\dot{f}_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)}$$

(as defined on page 28).

Proposition 5.1. *Consider a developable weakly generic 2-ribbon semidiscrete surface f in $C^{1,2,1}([a,b],\mathbb{R}^3)$. Let*

$$\dot{f}_0(t) = a(t)\dot{f}_1(t) + b(t)\Delta f_0(t) \quad \textit{and} \quad \dot{f}_2(t) = c(t)\dot{f}_1(t) + d(t)\Delta f_1(t).$$

Then we have

$$H_1(t) = d(t) - b(t).$$

Proof. First, we have

$$\begin{aligned}(\dot{f}_1, \Delta\dot{f}_0, \Delta f_1) &= (\dot{f}_1, \dot{f}_1 - \dot{f}_0, \Delta f_1) = -(\dot{f}_1, \dot{f}_0, \Delta f_1) = -(\dot{f}_1, a\dot{f}_1 + b\Delta f_0, \Delta f_1)\\ &= -b(\dot{f}_1, \Delta f_0, \Delta f_1).\end{aligned}$$

Secondly, in a similar way we get

$$(\dot{f}_1, \Delta f_0, \Delta\dot{f}_1) = d(\dot{f}_1, \Delta f_0, \Delta f_1).$$

Finally we have

$$H_1 = \frac{(\dot{f}_1, \Delta\dot{f}_0, \Delta f_1) + (\dot{f}_1, \Delta f_0, \Delta\dot{f}_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} = d - b.$$

This concludes the proof. □

This fact gives a surprising corollary concerning the flexion of a 2-ribbon developable surface. Denote by $\alpha(t)$ the angle between $\Delta f_0(t)$ and $\Delta f_1(t)$.

Corollary 5.2. *Let f be a weakly generic 2-ribbon developable surface in $C^{1,2,1}([a,b],\mathbb{R}^3)$. Consider its isometric deformation γ. Let us choose the parameter λ of γ such that $\cos(\alpha(t_0))$ linearly depends on λ. Then for every $t \in [a,b]$ the value $\cos(\alpha(t))$ linearly depends on λ.*

Proof. First of all, notice that

$$|\Delta f_0||\Delta f_1|\cos\alpha = \langle \Delta f_0, \Delta f_1\rangle = \Phi,$$

and hence

$$\cos\alpha = \frac{\Phi}{|\Delta f_0||\Delta f_1|}.$$

By Proposition 3.2 and further by Proposition 5.1 we have

$$\mathcal{D}_\gamma\Phi(t_1) = \mathcal{D}_\gamma\Phi(t_0)\cdot\exp\left(\int_{t_0}^{t_1} H_1(t)dt\right) = \mathcal{D}_\gamma\Phi(t_0)\cdot\exp\left(\int_{t_0}^{t_1}(d(t)-b(t))dt\right).$$

Therefore, the ratio $\mathcal{D}_\gamma\Phi(t_1)/\mathcal{D}_\gamma\Phi(t_0)$ is a nonzero constant that depends entirely on the inner geometry of a 2-ribbon surface, but not on its embedding in $\mathbb{R}^3$. Therefore, the ratio

$$\frac{\cos\alpha(t_1)}{\cos\alpha(t_0)} = \frac{\mathcal{D}_\gamma\Phi(t_1)}{\mathcal{D}_\gamma\Phi(t_0)}\frac{|\Delta f_0(t_0)||\Delta f_1(t_0)|}{|\Delta f_0(t_1)||\Delta f_1(t_1)|}$$

is a nonzero constant that depends entirely on the inner geometry of a 2-ribbon surface but not on its embedding in $\mathbb{R}^3$ as well. This implies the statement of the corollary. □

In fact, Corollary 5.2 implies a similar statement for an isometric deformation of a strongly generic n-ribbon developable surface.

Corollary 5.3. *Consider a strongly generic finitely flexible n-ribbon developable surface of f in $C^{1,2,1}([a,b],\mathbb{R}^3)$. Let γ be a nontrivial isometric deformation of f (i.e., $\mathcal{D}_\gamma f \neq 0$). Then there exists a choice of the parameter λ of the deformation γ, such that for all $t \in [a,b]$ and $i \in \{1,\ldots,n-1\}$ all the cosines of the corresponding angles $\alpha_i(t)$ linearly depend on λ (here $\alpha_i(t)$ denotes the angle between $\Delta f_i(t)$ and $\Delta f_{i+1}(t)$).* □

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References

[1] L. Bianchi, *Sopra alcone nuove classi di superficie e di sistemi tripli ortogonali*, Ann. Matem. v. 18 (1890), pp. 301–358.

[2] A. I. Bobenko, T. Hoffmann, W. K. Schief, *On the integrability of infinitesimal and finite deformations of polyhedral surfaces*, Discrete Differential Geometry, A. I. Bobenko, P. Schröder, J. M. Sullivan, G. M. Ziegler, (eds.), Series: Oberwolfach Seminars, v. 38 (2008), pp. 67–93.

[3] A. I. Bobenko, Yu. B. Suris, *Discrete differential geometry. Integrable structure*, Graduate Studies in Mathematics, 98, American Mathematical Society, Providence, RI, 2008.

[4] H. Cartan, *Calcul différentiel*, Hermann, Paris, 1967.

[5] L. P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Dover Publications, New York, 1960.

[6] O. Karpenkov, *On the flexibility of Kokotsakis meshes*, Geom. Dedicata, v. 147 (2010), no. 1, pp. 15–28.

[7] O. Karpenkov, J. Wallner, *On offsets and curvatures for discrete and semidiscrete surfaces*, Beitr. Algebra Geom., v. 55 (2014), no. 1, pp, 207–228.

[8] A. Kokotsakis, *Über bewegliche Polyeder*, Math. Ann., v. 107 (1932), pp. 627–647.

[9] Ch. Müller, J. Wallner, *Semi-discrete isothermic surfaces*, Results Math., v. 63 (2013), pp. 1395–1407.

[10] H. Pottmann, J. Wallner, *Infinitesimally flexible meshes and discrete minimal surfaces*, Monatshefte Math., v. 153 (2008), pp. 347–365.

[11] H. Pottmann, J. Wallner, *Computational line geometry. Mathematics and Visualization*, Springer-Verlag, Berlin, 2001.

[12] I. Kh. Sabitov, *Local theory of bendings of surfaces, Geometry, III*, Encyclopaedia Math. Sci., v. 48, pp. 179–256, Springer, Berlin, 1992.

[13] J. Wallner, *Semidiscrete surface representations*, In A. Bobenko et al., editors, Discrete Differential Geometry, Oberwolfach Reports. 2009. Abstracts from the workshop held January 12–17, 2009.

[14] J. Wallner, *On the semidiscrete differential geometry of A-surfaces and K-surfaces*, J. Geometry, v. 103 (2012), pp. 161–176.

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