Approximating and reducing bias in 2SLS estimation of dynamic simultaneous equation models

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Abstract

An order O(1/T) approximation is made to the bias in 2SLS estimation of a dynamic simultaneous equation model, building on similar large-T moment approximations for non-dynamic models. The expression is long because it contains two distinct parts: a part due to the simultaneity which is directly related to the Nagar bias and a part due to the dynamics which has many component terms. However, the analytically corrected 2SLS estimators resulting from this approximation perform well in terms of remaining estimation bias. The biases of these estimators are compared with the Quenouille half-sample jackknife and the residual bootstrap for 2SLS in dynamic models, and are found to be competitive. The Monte Carlo and bias approximation also suggest that the bias in estimating endogenous variable coefficients in dynamic simultaneous equation models is non monotonic in the sample size, contrary to the well known theoretical result for static models. The effect of using weaker instruments on our numerical and Monte Carlo results is explored.

Keywords: 2SLS, simultaneous equation model, time series, bias approximation, bias correction, bootstrap, jackknife

1. Introduction

The issues of bias approximation and reduction have been previously addressed in relation to static simultaneous equation models. Recent examples of bias approximation are Hahn & Hausman (2002), Hahn, Hausman & Kuersteiner (2004), Phillips (2007), Iglesias & Phillips (2010), and Bun &

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Windmeijer (2011). On bias reduction see MacKinnon & Davidson (2006), Dahlberg & Blomquist (2006), Davidson & MacKinnon (2007) and Ackerberg & Devereux (2009), who consider the JIVE method and its variants, Iglesias & Phillips (2012) who construct estimators that are unbiased up to orders $O(T^{-1})$ and $O(T^{-2})$, where T is the sample size, and Hsu, Lau, Fung & Ulveling (1986), who assess the bootstrap method due to Freedman (1984) and the standard delete-1 jackknife for static models. The Freedman (1984) method is asymptotically valid in the dynamic setting, and performs well in Ip (1991) for dynamic models. Freedman & Peters (1984a,1984b) use the method to obtain bootstrap estimates of the bias in GLS and 3SLS coefficient estimators, respectively. Freedman & Peters (1984a) also conduct a Monte Carlo simulation study to assess the performance of the bootstrap in estimating standard errors, and MacKinnon (2002) presents Monte Carlo evidence for its use in hypothesis testing in static models. Also in the context of dynamic models, Kiviet & Phillips (1995) present a small-σ approximation to the 2SLS coefficient bias, where, following Kadane (1971), σ is a small scalar multiple of the variance of the structural equation disturbance, and examine its use in bias reduction, showing that certain results for the static model do not carry over to the dynamic case.

Given a sample size T and an estimate $\hat{\alpha}$ of a coefficient vector α, the large-T approach in Nagar (1959) starts by expanding the estimation error as follows:

$$\sqrt{T}(\hat{\alpha}-\alpha) = \sum_{s=1}^{p} \frac{e_s}{T^{\frac{1}{2}(s-1)}} + \frac{r_p}{T^{\frac{1}{2}p}}, \tag{1}$$

where e_s, for $s = 1, ..., p$, and r_p are all $O_p(1)$ as $T \to \infty$. The last term is the remainder in an expansion of $\sqrt{T}(\hat{\alpha}-\alpha)$ to order $O_p(T^{\frac{1}{2}(p-1)})$. In the small-$\sigma$ approach the general expansion is

$$\frac{1}{\sigma}(\hat{\alpha}-\alpha) = \sum_{s-1}^{p} \sigma^{s-1}\dot{e}_p + \sigma^p \dot{r}_p, \tag{2}$$

where $\dot{e}_s$, for $s = 1, ..., p$, and $\dot{r}_p$ are also bounded in probability, this time as σ, the standard deviation of the equation disturbance, tends to zero. The bias is then approximated to order $O(T^{-1})$ or $O(\sigma^2)$ by calculating the first moment of the approximate estimation error in each case.

Kadane (1971) shows that the large-T and small-σ approaches yield essentially equivalent results for the static SEM. In particular, it is shown that the large-T result in Nagar (1959) can be obtained by taking the limit of the small-σ result as $T \to \infty$. Kiviet & Phillips (1989) and Kiviet and Phillips (1993) show that the same is not true in dynamic settings.

A large-T moment approximation for a dynamic simultaneous equation model ("DSEM") is presented here under a Normality assumption, building on the above results for static models and on the small-σ approximations for dynamic models. The simulation experiments in Section 3 investigate the remaining bias and the mean squared error after using this for bias reduction. The performance of the analytically corrected estimator, C2SLS, is compared with the bootstrap method due to Freedman (1984) and the half-sample jackknife in Quenouille (1956). Though Freedman (1984) provides a consistency result for the bootstrap in DSEMs, there is no theoretical result for bootstrap bias correction in this context, though the favourable simulation results in Hsu, Lau, Fung, & Ulveling (1986) for bias-corrected estimation of 2SLS estimation of static models suggest that a correction is likely. Finally, the behaviour of the bias correction numerically is explored as the instruments grow weak, and the three bias correction methods are compared in a situation where the instruments are relatively weak.

The jackknife method considered is due to Quenouille (1956). Dhaene & Jochmans (2010) find that it performs well in terms of bias correction in large-T dynamic panel data modeling with fixed effects. It is referred to as the Quenouille jackknife (QJ) here. Rather than creating subsamples by deleting one observation at a time for each subsample, two subsamples are obtained from the first and second halves of the whole sample with the ordering intact. This has the benefit of retaining the dynamics of the data, and it means that the 2SLS bias does not need to be monotonically decreasing in the sample size for a bias correction to occur. The related delete-d jackknife in Shao (1989) can be applied with $d = \lceil T/2 \rceil$, but it does not retain the dynamics and will not work here.

2. The model and bias approximation

The complete system is assumed to be as follows:

$$YB + Y_{-1}\Lambda + XC = \bar{U}, \tag{3}$$

where Y is a $T \times G$ matrix of observations on G endogenous variables, Y_{-1} is a $T \times G$ matrix of observations on the endogenous variables lagged one time period, X is a $T \times K$ matrix of observations on K stationary exogenous variables and $\bar{U}$ is a $T \times G$ matrix of structural disturbances. The matrices B, Λ and C are of dimension $G \times G$, $G \times G$ and $K \times G$, respectively, while B

is assumed to be non-singular. The rows of $\bar{U}$ are assumed to be normally and independently distributed with zero mean and fixed covariance matrix Σ.

The reduced form of the model is

$$\begin{aligned} Y &= -Y_{-1}\Lambda B^{-1} - XCB^{-1} + \bar{U}B^{-1} \\ &= Y_{-1}\Gamma + X\Pi + \bar{V}, \end{aligned} \tag{4}$$

where $\Gamma = -\Lambda B^{-1}$, $\Pi = -CB^{-1}$ and $\bar{V} = \bar{U}B^{-1}$. Here the rows of $\bar{V}$ are normally distributed with zero mean and covariance matrix $\Omega = (B^{-1})'\Sigma B^{-1}$, and as a stationarity condition it is assumed that the eigenvalues of Γ are inside the unit circle.

It will be assumed that the rows of the Y matrix are generated from a fixed value $Y_{0,.}$ at time $t = 0$ so that by successive substitution the matrix may be separated into stochastic and non-stochastic parts. This is done by noting that the $t - th$ row of Y may be written as

$$y_{t,.} = y_{0,.}\Gamma^t + \sum_{i=1}^{t} X_{i,.}\Pi\Gamma^{t-i} + \sum_{i=1}^{t} \bar{V}_{i,.}\Gamma^{t-i}, \tag{5}$$

where $X_{i,.}$ and $\bar{V}_{i,.}$ are the $i - th$ rows of X and $\bar{V}$, respectively, for $i = 0, 1, ..., t$ and $t = 1, 2, ..., T$. With $Y_{0,.}$ taking a fixed value it is seen that the non-stochastic part of $Y_{t,.}$ is given by the first two terms of (5), while the last term represents the stochastic part. Therefore the following holds for the non-stochastic part:

$$\bar{y}_{t,.} = y_{0,.}\Gamma^t + \sum_{i=1}^{t} X_{i,.}\Pi\Gamma^{t-i}, \tag{6}$$

while the stochastic part is

$$\bar{w}_{t,.} = \sum_{i=1}^{t} \bar{V}_{i,.}\Gamma^{t-i}. \tag{7}$$

Defining a $T \times T$ matrix D as

$$D = \begin{pmatrix} 0 & 0 & . & . & . & . & 0 \\ 1 & 0 & . & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & 0 \\ . & & & & . & . & 0 \\ 0 & . & . & . & . & 1 & 0 \end{pmatrix} \tag{8}$$

where $D^T = 0$ and where D^0 is defined as I_T, the stochastic part of $Y = \bar{Y} + \bar{W}$ is

$$\bar{W} = \sum_{t=1}^{T-1} D^t \bar{V} \Gamma^t + \bar{V} = \sum_{t=0}^{T-1} D^t \bar{V} \Gamma^t. \tag{9}$$

Without loss of generality the estimation of the parameters of the first equation of the system in (3) is considered. This equation is assumed to be over-identified and given by

$$\begin{aligned} y_1 &= Y_2 \beta_1 + LY_1 \lambda_1 + X_1 c_1 + \bar{u}_1 \\ &= R\delta_1 + \bar{u}_1, \end{aligned} \tag{10}$$

where

$$R = [Y_2 : LY_1 : X_1], \quad \delta_1 = (\beta_1', \lambda_1', c_1')' \quad \text{and} \quad \bar{u}_1 = \sigma_1 u_1. \tag{11}$$

Here $Y_1 = (y_1 : Y_2)$ is a $T \times (g+1)$ matrix of observations on $g+1$ included endogenous variables, LY_1 is the one period lagged version of Y_1, X_1 is a $T \times k$ matrix of observations on k included exogenous variables, and σ_1 is the standard deviation of the structural disturbances.

Let

$$\bar{R} = [\bar{Y}_2 : L\bar{Y}_1 : X_1] \quad \text{and} \quad \bar{F} = [\bar{W}_2 : L\bar{W}_1 : 0] \tag{12}$$

denote the non-stochastic and stochastic parts of R, respectively, where use has been made of (9), and where $\bar{F} = \sigma F$, with $F = (W_2 : LW_1 : 0)$. The 2SLS estimator of δ_1 may be written as

$$\begin{aligned} \delta_1^\star &= (\hat{R}'\hat{R})^{-1}\hat{R}' y_1 \\ &= \delta_1 + (\hat{R}'\hat{R})^{-1}\hat{R}'\bar{u}_1, \end{aligned} \tag{13}$$

where $\hat{R} = [\hat{Y}_2 : LY_1 : X_1]$ and $\hat{Y}_2 = L\hat{Y}\hat{\Gamma}_2 + X\hat{\Pi}_2$, and where $\hat{\Gamma}_2$ and $\hat{\Pi}_2$ are obtained from OLS estimation of (4).

As with R, the term $\hat{R}$ may be separated into non-stochastic and stochastic parts:

$$\hat{R} = \bar{R} + (\hat{R} - \bar{R}). \tag{14}$$

The stochastic part can be written as

$$\begin{aligned} \hat{R} - \bar{R} = {} & [L\bar{Y}(\hat{\Gamma}_2 - \Gamma) + X(\hat{\Pi}_2 - \Pi_2) + L\bar{W}\Gamma_2 : L\bar{W}_1 : 0] \\ & + [L\bar{W}(\hat{\Gamma}_2 - \Gamma_2) : 0 : 0]. \end{aligned} \tag{15}$$

The first term in the above is of order $O_p(\sigma)$ while the second is $O_p(\sigma^2)$. The small-σ expansions for the OLS and 2SLS bias in the dynamic case (see Kiviet & Phillips (1995)) are discussed below and given in Theorems 2 and 3.

The 2SLS estimation error is as follows, from (13)

$$\delta_1^\star - \delta_1 = (\hat{R}'\hat{R})^{-1}\hat{R}'\bar{u}_1, \tag{16}$$

and one can rearrange (15) to give

$$\hat{R} = \bar{R} + \Delta_1 + \Delta_2, \tag{17}$$

where

$$\begin{aligned}\Delta_1 = [L\bar{W}\Gamma_2 : L\bar{W}_1 : 0], \quad \text{and} \quad \Delta_2 = [&L\bar{Y}(\hat{\Gamma}_2 - \Gamma_2) + X(\hat{\Pi}_2 - \Pi_2) \\ &+ L\bar{W}(\hat{\Gamma}_2 - \Gamma_2) : 0 : 0].\end{aligned} \tag{18}$$

Using these,

$$\begin{aligned}\hat{R}'\hat{R} = \bar{R}'\bar{R} + E[\Delta_1'\Delta_1] + (\bar{R}'\Delta_1 + \Delta_1'\bar{R}) + (\bar{R}'\Delta_2 + \Delta_2'\bar{R}) + (\Delta_1'\Delta_2 + \Delta_2'\Delta_1) \\ + (\Delta_1'\Delta_1 - E[\Delta_1'\Delta_1]) + \Delta_2'\Delta_2,\end{aligned} \tag{19}$$

where $\bar{R}'\bar{R} + E[\Delta_1'\Delta_1]$ is $O(T)$, $(\bar{R}'\Delta_1 + \Delta_1'\bar{R})$ and $(\Delta_1'\Delta_1 - E[\Delta_1'\Delta_1])$ are $Op(T^{\frac{1}{2}})$, $(\bar{R}'\Delta_2 + \Delta_2'\bar{R})$ and $(\Delta_1'\Delta_2 + \Delta_2'\Delta_1)$ are $Op(1)$, and where $\Delta_2'\Delta_2$ is $O_p(1)$.

Also

$$\hat{R}'\bar{u}_1 = \bar{R}'\bar{u}_1 + \Delta_1'\bar{u}_1 + \Delta_2'\bar{u}_1, \tag{20}$$

where $\bar{R}'\bar{u}_1$ and $\Delta_1'\bar{u}_1$ are $O_p(T^{\frac{1}{2}})$ and $\Delta_2'\bar{u}_1$ is $O_p(1)$.

Defining

$$Q^{\star-1} = \bar{R}'\bar{R} + E[\Delta_1'\Delta_1] \tag{21}$$

and writing the $O_p(T^{\frac{1}{2}})$ component of $\hat{R}'\hat{R}$ as S_1 with the $O_p(1)$ component as S_2, (19) gives

$$\begin{aligned}(\hat{R}'\hat{R})^{-1} = (Q^{\star-1} + S_1 + S_2)^{-1} &= Q^\star(I + S_1Q^\star + S_2Q^\star)^{-1} \\ &= Q^\star - Q^\star S_1 Q^\star + O_p(T^{-\frac{3}{2}}).\end{aligned} \tag{22}$$

Combining (22) with (20) yields

$$\begin{aligned}(\hat{R}'\hat{R})^{-1}\hat{R}'\bar{u}_1 = Q^\star\bar{R}u_1 + Q^\star\Delta_1'\bar{u}_1 + Q^\star\Delta_2'\bar{u}_1 - Q^\star S_1 Q^\star\bar{R}\bar{u}_1 - Q^\star S_1 Q^\star\Delta_1'\bar{u}_1 \\ + o_p(T^{-1}).\end{aligned} \tag{23}$$

The expected value, taken term by term, yields the 2SLS bias. This is presented in Theorem 1 below, where Θ is a vector of all the structural coefficients in (3) along with the parameters in Σ and Ω. The bias expression uses a $G\times(g+k)$ matrix $I_1^\star = \begin{pmatrix} I_g & 0 \\ 0 & 0 \end{pmatrix}$ and a $(G+K)\times G$ matrix $I_2^\star = \begin{pmatrix} I_G \\ 0 \end{pmatrix}$. Moreover, let $Z = (Y_{-1} : X)$, $\bar{Z} = E[Z] = [L\bar{Y} : X]$, $Q_Z = (E[Z'Z])^{-1}$, $\varphi = E[\frac{1}{T}\bar{V}'\bar{u}_1] = \sigma^2\phi$, where ϕ is defined using the decomposition for $\bar{V}$ in Nagar (1959), namely that $\bar{V} = S + \bar{u}_1\phi'$, where $\bar{u}_1$ and S are normally distributed but independent. Additionally, let $\psi = I_1^{\star'}\varphi$, $Q_Z^\star = I_2^{\star'} Q_Z I_2^\star$ and $Q_W = \sum_{t=1}^{T-1}(T-t)\Gamma^{t-1'}\Omega\Gamma^{t-1}$. The following results from Nagar (1959) are used for calulating the expected values of matrix quadratic forms in S.

Lemma 1. *(Nagar (1959)) Given a conformable matrix N with appropriate rank the following hold:*

$$\begin{aligned} E[SNS'] &= tr(C_2^\star N).I \\ E[S'NS] &= \{tr(N).I\}C_2^\star \\ E[SNS] &= N'C_2^\star \\ E[S'N'S'] &= C_2^\star N \end{aligned}$$

where $C_2^\star = \Omega - \sigma^2\phi\phi'$ and $\Omega = E[\frac{1}{T}\bar{V}'\bar{V}]$.

Theorem 1. *To order $O(T^{-1})$ the bias in 2SLS estimation of δ_1 in (10) is $E[\delta_1^\star - \delta_1] = b(Y, Z, \Theta) + o(T^{-1})$ where*

$$\begin{aligned} b(Y,Z,\Theta) = &\\ & - Q^\star\{\bar{R}'\bar{Z}Q_Z\bar{Z}'\bar{R}Q^\star + (tr\{\bar{Z}Q_Z\bar{Z}'\bar{R}Q^\star\bar{R}'\}.I)\}\psi \\ & + Q^\star(tr\{\bar{Z}Q_Z\bar{Z}'\}.I)\psi - Q^\star\sum_{t=1}^{T-1}\{\bar{R}'D^t\bar{R}Q^\star + (tr\{\bar{R}'D^t\bar{R}Q^\star\}.I)\}A'(\Gamma^{t-1})'\varphi \\ & - Q^\star\sum_{t,r=1}^{T-1}\{\bar{R}'D^tD^{r'}\bar{R}Q^\star + (tr\{D^tD^{r'}\bar{R}Q^\star\bar{R}'\}.I)\}(tr\{\Omega\Gamma^{r-1}Q_Z^\star\Gamma^{t-1'}\}.I)\psi \\ & - Q^\star\sum_{t,r=1}^{T-1}(tr\{D^tD^{r'}\bar{Z}Q_ZI_2^\star\Gamma^{t-1'}\Omega\Gamma^{r-1}AQ^\star\bar{R}'\}.I)\psi \\ & - Q^\star\{(tr\{Q_WI_2^{\star'}Q_Z\bar{Z}'\bar{R}Q^\star A'\}.I) + \bar{R}'\bar{Z}Q_ZI_2^\star Q_WAQ^\star + A'Q_WI_2^{\star'}Q_Z\bar{Z}'\bar{R}Q^\star\}\psi \\ & - Q^\star\sum_{r,t=1}^{T-1}(tr\{\bar{Z}Q_Z\bar{Z}'D^tD^{r'}\}.I)\{(tr\{\Omega\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}\}.I) + A'\Gamma^{t-1'}\Omega\Gamma^{r-1}AQ^\star\}\psi \end{aligned}$$

$$
\begin{aligned}
&- Q^\star \sum_{r,t=1}^{T-1} A'\Gamma^{t-1'}\Omega\Gamma^{r-1} I_2^{\star'} Q_Z \bar{Z}' D^t D^{r'} \bar{R} Q^\star \psi \\
&- Q^\star \{A' Q_W Q_Z^\star Q_W A Q^\star + (tr\{Q_W Q_Z^\star Q_W A Q^\star A'\}.I)\}\psi \\
&- Q^\star \bar{R}' \sum_{t,r=1}^{T-1} D^t D^r \bar{R} Q^\star I_1^{\star'} \Omega \Gamma^{t-1} Q_Z^\star \Gamma^{r-1'} \varphi \\
&- Q^\star \sum_{r,t=1}^{T-1} A'\Gamma^{t-1'}\{\Omega I_1^\star Q^\star \bar{R}'(D^{r'} D^{t'} + D^{r'} D^t)\bar{Z} Q_Z I_2^\star \Gamma^{r-1'} \\
&\quad + (tr\{D^{t'} D^r \bar{Z} Q_Z I_2^\star \Gamma^{r-1'} \Omega I_1^\star Q^\star \bar{R}'\}.I)\}\varphi \\
&- Q^\star A' \sum_{r,t=1}^{T-1} \Gamma^{t-1'}(tr\{\Omega I_1^\star Q^\star A' \Gamma^{r-1'}\} tr\{D^{t'} \bar{Z} Q_Z \bar{Z}' D^{r'}\}.I)\varphi \\
&- Q^\star \sum_{r,t,s=1}^{T-1} A'\Gamma^{t-1'}\Omega\Gamma^{s-1} A Q^\star A' \Gamma^{r-1'}\{tr(D^{t'} D^r D^s).I\}\psi \\
&\quad - Q^\star \sum_{r,t,s=1}^{T-1} A'\Gamma^{t-1'} tr(\Omega\Gamma^{r-1} A Q^\star A' \Gamma^{s-1'}) tr(D^{t'} D^r D^{s'})\psi \\
&- Q^\star \sum_{r,t=1}^{T-1} I_1^{\star'} \Omega \Gamma^{r-1}\{Q_Z^\star \Gamma^{t-1'}(tr\{D^t \bar{R} Q^\star R' D^r\}.I) \\
&\quad + A Q^\star A' \Gamma^{t-1'}(tr\{\bar{Z} Q_Z \bar{Z}' D^t D^r\}.I) + A Q^\star \bar{R}' D^{t'} D^{r'} \bar{Z} Q_Z I_2^\star \Gamma^{t-1'}\}\varphi
\end{aligned}
$$

The proof of Theorem 1 appears in Appendix A, and uses the results in Lemma 1 obtained by Nagar under a Normality assumption. Twelve lengthy but routine expected value calculations are collected in Appendix B. It is to be noted that while the result is written in terms of true model parameters and quantities such as $Q_Z = (E[Z'Z])^{-1}$ and $\varphi = E[\frac{1}{T}\bar{V}'\bar{u}_1]$ which require knowledge about the whole system, these can be estimated for the purpose of bias correction without having to specify the whole system, beyond deciding the set of endogenous variables to include in Y and the exogenous variables to include in X; the estimator $\hat{\theta}^{(1)}_{C2SLS}$ in Section 3 simply uses a 2SLS estimate of the first structural equation and an OLS estimate of the reduced form for the system to estimate φ.

The expression in Theorem 1 should reduce to the Nagar (1959) bias approximation in static models when any terms that result from the inclusion of lagged endogenous regressors are removed. This means that a reduction of our result to that for the static case requires the removal of any terms involving the D matrix, including Q_W since the "$T-t$" factor is from a trace of products of D matrices. If the partitions of Z and R that involve lags of the endogenous regressors are removed, then $\bar{Z} = E[Z]$, $Q_Z = (E[Z'Z])^{-1}$, $\bar{R} = [\bar{Y}_2 : X_1]$ and $Q^\star = (\bar{R}'\bar{R})^{-1} = Q$. After removing all terms involving D the expression in Theorem 1 becomes

$$-Q^\star\{\bar{R}'\bar{Z}Q_Z\bar{Z}'\bar{R}Q^\star + (tr\{\bar{Z}Q_Z\bar{Z}'\bar{R}Q^\star\bar{R}'\}.I)\}\psi + Q^\star(tr\{\bar{Z}Q_Z\bar{Z}'\}.I)\psi.$$

These reduce as follows using the above along with the fact that our ψ is the same as Nagar's q when model is static:

$$\begin{aligned}
-Q^\star\{\bar{R}'\bar{Z}Q_Z\bar{Z}'\bar{R}Q^\star\psi &= -(\bar{R}'\bar{R})^{-1}\bar{R}'Z(Z'^{-1}Z'\bar{R}Qq = -Qq \\
-Q^\star\{(tr\{\bar{Z}Q_Z\bar{Z}'\bar{R}Q^\star\bar{R}'\}.I)\}\psi &= -Q\{tr(Z(Z'^{-1}Z'\bar{R}(\bar{R}'\bar{R})^{-1}\bar{R}').I\}q \\
&= -Q\{tr(\bar{R}(\bar{R}'\bar{R})^{-1}\bar{R}').I\}q \\
&= -(g+k)Qq \\
Q^\star(tr\{\bar{Z}Q_Z\bar{Z}'\}.I)\psi &= Q\{tr(\bar{Z}(\bar{Z}'\bar{Z})^{-1}\bar{Z}').I\}q \\
&= KQq.
\end{aligned}$$

This gives the $O(T^{-1})$ bias approximation for the static model:

$$(K-g-k-1)Qq = (L-1)Qq,$$

which agrees with Nagar, where $L = K-g-k$ denotes the degree of overidentification.

The result in Theorem 1 can be compared with the small-σ result in Kiviet & Phillips (1995):

Theorem 2. *(Kiviet & Phillips (1995)) Let* $H_t = \bar{R}'D^t\bar{R}Q + tr\{\bar{R}D^t\bar{R}Q\}.I$, $\varphi = E[\frac{1}{T}\bar{F}'u_1]$, $\psi = E[\frac{1}{T}\bar{V}'u_1]$ *and* $A = (\Gamma_2 : I_1 : 0)$, *where* $I_1 = \begin{pmatrix} I_g + 1 \\ 0 \end{pmatrix}$ *is a* $G \times (g+1)$ *selection matrix and where the other terms are as defined earlier. Then, to order* $O(\sigma^2)$ *the 2SLS bias is*

$$E[\delta_1^\star - \delta_1] = (G+K-2g-k-2)Q\psi - Q\sum_{t=1}^{T-1} H_t(\Gamma^{t-1}A)'\varphi.$$

Kiviet & Phillips (1989) provide an expansion of the estimation error using both methods and note that the small-σ expression will not contain all of the $O(T^{-1})$ terms while the large-T expression will contain all the $O(\sigma^2)$ terms. A result for the OLS estimator is given below:

Theorem 3. *(Kiviet & Phillips (1995)) To order $O(\sigma^2)$ the OLS bias is*

$$E[\delta_1^\star - \delta_1] = (T - 2g - k - 2)Q\psi - Q\sum_{t=1}^{T-1} H_t(\Gamma^{t-1}A)'\varphi.$$

Kiviet & Phillips (1995) show that a weighted average of OLS and 2SLS due to Sawa (1973a), see also Sawa (1973b), which is unbiased to order $O(\sigma^2)$ in static SEMs, is not unbiased to this order in the dynamic case. Note that the expression for the small-σ bias expansion of 2SLS is very similar to the expression for OLS. The terms involving dynamics are identical in Theorems 2 and 3, and the first terms correspond to the 2SLS and OLS bias approximations for the static model in Sawa (1973a). The Sawa estimator eliminates the simultaneity component of the bias, but cannot remove bias introduced by the dynamics.

3. Bias reduction simulations

The bias approximation in Theorem 1 is applicable to systems containing G equations. Its performance is considered here, along with the bootstrap and jackknife, in correcting the bias in 2SLS estimation of equation (10) in a two-equation model where Y_2 contains one endogenous variable:

$$y_1 = y_2\beta_1 + y_{1,-1}\lambda_1 + X_1c_1 + \bar{u}_1. \tag{24}$$

The ith element of δ_1 is denoted in the following by θ, and its 2SLS estimator by $\hat{\theta}_{2SLS}$.

The analytically bias-corrected estimator is denoted by $\hat{\theta}_{C2SLS}$, and is defined as follows, where e_i is a vector with one in position i and zeros elsewhere:

Definition 1. *The C2SLS estimator of θ is given by*

$$\hat{\theta}_{C2SLS} = \hat{\theta}_{2SLS} - e_i'\hat{b}(Y, Z, \Theta). \tag{25}$$

where $\hat{b}(Y, Z, \Theta)$ is an estimate of the true bias $b(Y, Z, \Theta)$.

A *Matlab* implementation of the analytical correction is available from the corresponding author. Monte Carlo results are presented below for two different ways of estimating $b(Y, Z, \Theta)$, which correspond to two different ways of estimating $\varphi = E[\bar{V}'\bar{u}_1]$. The first uses $\hat{\varphi}^{(1)} = \frac{1}{T}\hat{\bar{V}}'\hat{\bar{u}}_1$ where $\hat{\bar{V}}$ and

$\hat{\bar{u}}_1$ are residuals from OLS estimation of the reduced form and from 2SLS estimation of the structural equation, and the estimator that results from this is denoted by $\hat{\theta}^{(1)}_{C2SLS}$. The second method, which is denoted by $\hat{\theta}^{(2)}_{C2SLS}$, has $\hat{\varphi}^{(2)} = (\hat{B}^{-1})'(\hat{\sigma}^2, \hat{\sigma}_{12})'$, and uses estimates of the second structural equation. It is based on

$\frac{1}{T}E[\bar{V}'\bar{u}_1] = (B^{-1})'(\sigma^2, \sigma_{12})'$, where σ^2 and σ_{12} are estimated by $\hat{\sigma}^2 = \frac{1}{T}\hat{\bar{u}}_1'\hat{\bar{u}}_1$ and $\hat{\sigma}_{12} = \frac{1}{T}\hat{\bar{u}}_1'\hat{\bar{u}}_2$, and where B is estimated by 2SLS. The 2SLS estimator has moments up to the order of overidentification L, therefore the mean and MSE of $\hat{\theta}^{(2)}_{C2SLS}$ requires at least $L = 2$ and $L = 4$, respectively, for both Equation 1 and 2, because of the use of $\hat{\sigma}^2$ and $\hat{\sigma}_{12}$ which are obtained using the 2SLS estimates. The product of estimated terms in $\hat{\varphi}^{(2)}$ is difficult to analyse. We also recall that the relationship between moment existence and order of overidentification for 2SLS, see for example Kinal (1980), was obtained for static models, and it has yet to be shown that it applies to the dynamic case. The standard degrees of freedom correction is made in both cases when estimating the reduced form covariance matrix. Bias corrections such as the one here can potentially be iterated, as discussed in MacKinnon & Smith (1998), see in particular their equation (16), though convergence would not be guaranteed in the present setting and conditions for this would need to be established.

The bootstrap method due to Freedman requires pseudodata $y_1^\star$, $y_{1,-1}^\star$ and $y_2^\star$ to be generated iteratively from the 2SLS estimate of (24),

$$y_1 = y_2\hat{\beta}_1 + y_{1,-1}\hat{\lambda}_1 + X_1\hat{c}_1 + \hat{\bar{u}}_1 \tag{26}$$

in conjunction with the OLS estimated reduced form for y_2,

$$y_2 = \hat{\gamma}_2 y_{1,-1} + X\hat{\pi}_2 + \hat{\bar{v}}_2. \tag{27}$$

The disturbances are resampled from the rows of $(\hat{\bar{u}}_1, \hat{\bar{v}}_2)$, and these resample rows are denoted by $(\hat{\bar{u}}_1^\star, \hat{\bar{v}}_2^\star)$. Using a starting value for y_1, which here is the first observation in the sample, one can generate $(y_2^\star)_1$ from (27) then $(y_1^\star)_1$ from (26), and $(y_2^\star)_2$ from (27). Continuing in this way gives the full vectors $y_1^\star$, $y_{1,-1}^\star$ and $y_2^\star$.

The vector $y_2^\star$ is regressed on $(y_{1,-1}^\star : X)$ to obtain fitted values $\hat{y}_2^\star$, then $y_1^\star$ is regressed on $(\hat{y}_2^\star : y_{1,-1}^\star : X)$ to give bootstrap 2SLS replicates $\hat{\beta}_{1,b}^\star$, $\hat{\lambda}_{1,b}^\star$ and $\hat{c}_{1,b}^\star$. The mean of the bootstrap estimates is denoted by $\hat{\theta}_{\tilde{b}} = \frac{1}{B}\sum_{b=1}^{B}\hat{\theta}_b^\star$. The following defines the bias-corrected bootstrap for our model:

Definition 2. *The bootstrap bias-corrected estimator $\hat{\theta}_b$ is given by*

$$\hat{\theta}_b = 2\hat{\theta}_{2SLS} - \hat{\theta}_{\tilde{b}}. \tag{28}$$

The following defines the QJ estimator when the sample size is even, which is sufficient for our purpose.

Definition 3. *(Assuming even T) Let $\hat{\theta}_{2SLS}$ be the 2SLS estimator of θ_1 as before, based on all T observations. Let $\hat{\theta}^1_{1,2SLS}$ and $\hat{\theta}^2_{1,2SLS}$ be the 2SLS estimators using only the first and second $T/2$ observations in the sample, respectively. The Quenouille jackknife estimator of θ is then*

$$\hat{\theta}_Q = 2\hat{\theta}_{2SLS} - \left(\frac{\hat{\theta}^1_{2SLS} + \hat{\theta}^2_{2SLS}}{2}\right) \tag{29}$$

There will be a bias correction from application of the QJ in (29) so long as 2SLS estimation over the full sample is roughly half as biased as 2SLS estimation over each half-sample. The standard delete-1 jackknife, in contrast, requires the estimator being jackknifed to have a bias that is monotonically decreasing in sample size, something that is not guaranteed for the 2SLS estimator of the dynamic simultaneous equation system. In the context of static models, Owen (1976) shows that the bias and MSE of the 2SLS estimator of the endogenous variable coefficients are monotonically non-increasing in the sample size, while Ip & Phillips (1998) find that the same is not true for 2SLS estimators of exogenous variable coefficients. What has been shown for endogenous variable coefficients in static models, moreover, does not necessarily carry over to dynamic settings, in particular the bias in the endogenous variable coefficient estimates may not be monotonically decreasing in sample size.

3.1. Estimation with L=2

Two models are considered where $L = 2$ for the equation being estimated: Model 1 and Model 2. The Model 1 coefficient matrices are as follows:

$$B = \begin{pmatrix} 1 & -0.40 \\ -\beta_1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\text{and } C = \begin{pmatrix} -c_{11} & -c_{12} & 0 & 0 & 0 & -c_{13} \\ -0.20 & 0 & -0.30 & 0.20 & -0.80 & 0 \end{pmatrix}',$$

where $\beta_1 = 0.20$, $c_{11} = 0.80$, $c_{12} = 0.30$, $c_{13} = 0.50$ and $\lambda_1 = 0.10$. The maximum eigenvalue of Γ is $\tau = 0.11$. A matrix of six exogenous variables $X = (x_1, ..., x_5, x_6)$ is used in the following, where x_1 is a constant and the others are realisations from a Gaussian autoregressive processes with mean zero and autoregressive coefficient 0.9.

The reduced form of the structural model is given in (4), where $\bar{V} = (\bar{v}_1, \bar{v}_2)$ is a $T \times 2$ matrix of reduced form disturbances. These are generated using a matrix P from a Cholesky factorisation of Ω, so that

$$\begin{pmatrix} \bar{v}_{1,t} \\ \bar{v}_{2,t} \end{pmatrix} = P \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}, \tag{30}$$

where $\epsilon_{1,t}$ and $\epsilon_{2,t}$ denote the standardised disturbances. The distribution of $\epsilon_t = (\epsilon_{1,t}, \epsilon_{2,t})'$ has mean 0 and covariance matrix I, and is i.i.d. Normal. The distribution of the structural disturbances can be recovered from

$$B'\bar{v}_t = \bar{u}_t \Rightarrow \bar{u}_t \overset{iid}{\sim} (0, \Sigma), \tag{31}$$

where $\Sigma = B'\Omega B$. The structural covariance matrix is as follows:

$$\Sigma = \begin{pmatrix} 4 & -2 \\ -2 & 5 \end{pmatrix},$$

which implies the following reduced form covariance matrix:

$$\Omega = \begin{pmatrix} 4.02 & 0.52 \\ 0.52 & 4.78 \end{pmatrix}.$$

Figure 1 below plots the Monte Carlo simulated 2SLS bias and the $O(T^{-1})$ approximate bias for a number of sample sizes. The two are very close, and it is clear that 2SLS estimation of the endogenous variable coefficient β_1 is not monotonically non increasing in the sample size, something that would be the case for estimation of a static model.

Figure 1: Approximate vs Simulated Bias in 2SLS estimation

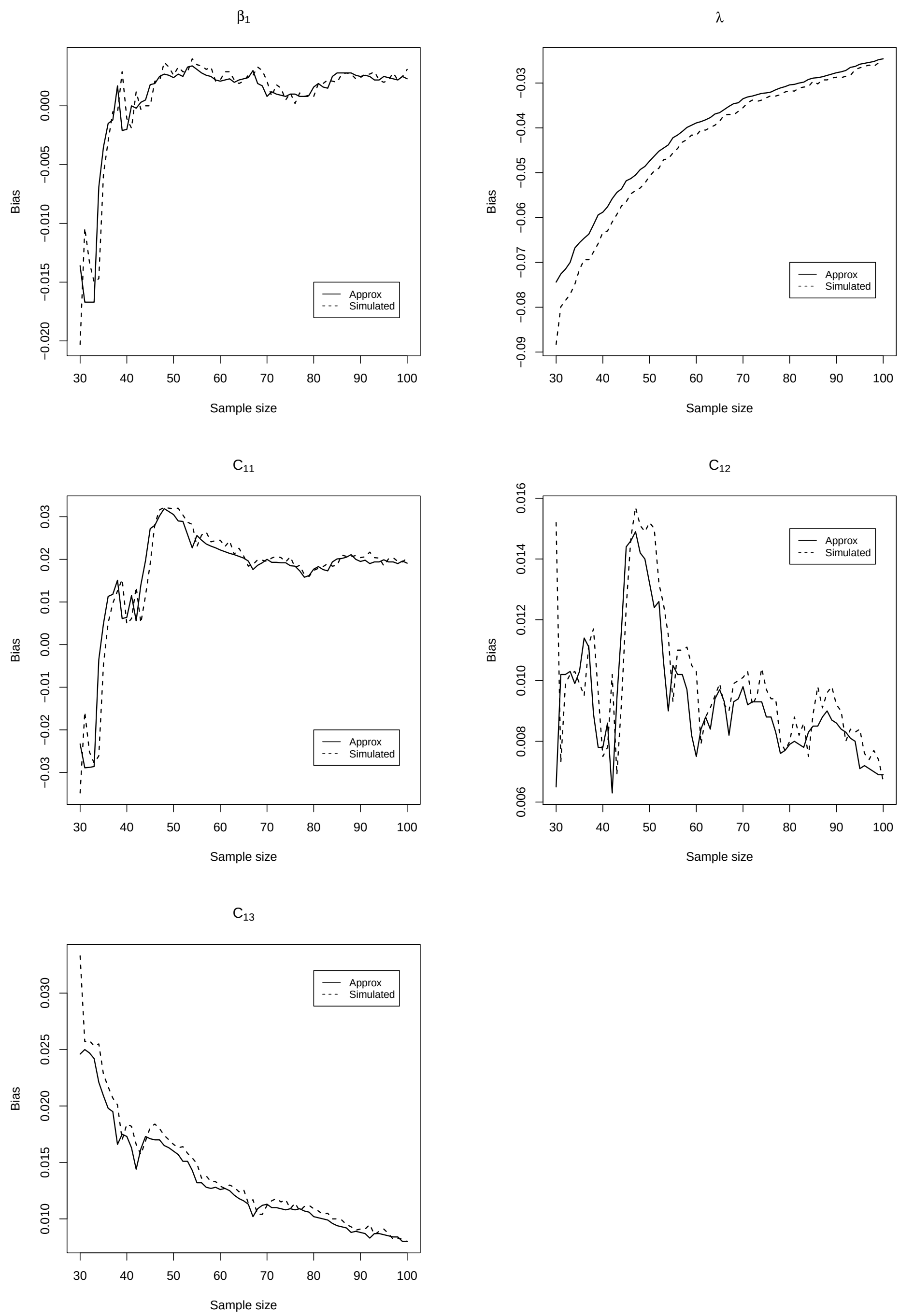


Table 1 presents empirical % bias and RMSE values for the parameterisation above at a sample size of $T = 50$. The studies by Hsu, Lau, Fung, & Ulveling (1986) on static models and Ip (1991) on both static and dynamic models, cited earlier, used 200 and 500 Monte Carlo replications respectively when providing evidence for the 2SLS bootstrap bias correction. The number of bootstrap replicates considered were 150 and 300 by Hsu, Lau, Fung, & Ulveling (1986), and 200 by Ip (1991). It was found in our own simulation experiments that a relatively large number of replications were required, especially in cases with moderate or large 2SLS bias, and for the parameterisations and exogenous data that are considered. 100,000 or more Monte Carlo replications are used throughout when computing bias and mean squared error values in this section, which enables an accurate comparison of the three bias correction methods. 199 bootstrap replicates were used when obtaining the bias-corrected bootstrap. It can be seen from Table 1 that the 2SLS bias varies substantially across the parameters, with relatively small biases of around 3.4-4.4% for the exogenous variable coefficients c_{11}, c_{12} and c_{13}, 1.8% for the endogenous variable coefficient β_1, and 51% for the lagged endogenous variable coefficient λ_1.

Both C2SLS$^{(1)}$ and C2SLS$^{(2)}$ do well in terms of bias reduction for the model considered here, though C2SLS$^{(2)}$ is the only method that managed to reduce the bias for every coefficient. C2SLS$^{(1)}$ and C2SLS$^{(2)}$ also have reduced values for the empirical RMSE, though with $L = 2$ the estimator C2SLS$^{(2)}$ will not have a second moment, as mentioned earlier. We report the the empirical RMSE values anyway, as in Hahn, Hausman & Kuersteiner (2004) for the LIML estimator under the conventional normalisation where LIML does not have finite sample moments. It is worth noting that LIML does have finite sample moments under an alternative normalisation, though, as shown in Anderson (2010). See also Fuller (1977). Though the bootstrap has inflated RMSE values for estimation of λ_1 and c_{13} in Model 1, this is not observed in the other models that follow, except to a much lesser extent in Model 5, where there is a slight increase in RMSE for each parameter. The Quenouille Jackknife has inflated RMSE throughout, as was the case in Orcutt and Winokur (1969) and Liu-Evans and Phillips (2012) for estimation of autoregressive models.

Table 1. Model 1 % bias and RMSE, $T = 50$

		2SLS	QJ	Boot	C2SLS[1]	C2SLS[2]
% bias	β_1	1.760	8.282	-2.670	-2.790	0.9730
	λ_1	-50.70	-8.229	-7.043	-15.17	-13.89
	c_{11}	4.173	4.251	0.2588	0.6933	1.111
	c_{12}	4.448	0.7100	1.203	2.226	1.223
	c_{13}	3.379	-0.02648	0.7600	1.238	0.8096
RMSE	β_1	0.1786	0.2817	0.1735	0.1734	0.1755
	λ_1	0.1316	0.1461	0.4268	0.1270	0.1284
	c_{11}	0.3817	0.5215	0.3676	0.3666	0.3674
	c_{12}	0.2209	0.2484	0.2146	0.2191	0.2190
	c_{13}	0.1306	0.1489	0.4237	0.1284	0.1280

The original 2SLS bias in estimation of β_1 is small in Model 1, and this could make bias correction for it difficult, particularly given the much larger bias in estimation of λ_1; it may also explain the inflated RMSE values for the bootstrap. Model 2 below, and Models 3-5 in the next subsection, have original 2SLS biases of around 10-20% in absolute terms. Similar comparisons are made between the various bias correction methods. The Model 2 coefficient matrices are

$$B = \begin{pmatrix} 1 & -0.31 \\ -\beta_1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\text{and } \; C = \begin{pmatrix} -c_{11} & -c_{12} & 0 & 0 & 0 & -c_{13} \\ -0.31 & 0 & -0.47 & -0.16 & -0.20 & 0 \end{pmatrix}',$$

where $\beta_1 = -0.43$, $c_{11} = 0.44$, $c_{12} = 0.40$, $c_{13} = 0.05$ and $\lambda_1 = 0.59$. The maximum eigenvalue of Γ is $\tau = 0.52$.

Table 2. Model 2 % bias and RMSE, $T = 50$

		2SLS	QJ	Boot	C2SLS[1]	C2SLS[2]
% bias	β_1	15.23	-9.081	-1.698	4.544	6.234
	λ_1	-13.31	-4.683	-3.324	-5.659	-5.180
	c_{11}	11.91	-2.178	1.5165	4.305	4.495
	c_{12}	9.687	3.371	-0.2232	3.224	3.895
	c_{13}	14.27	-15.40	1.002	4.538	4.833
RMSE	β_1	0.2901	0.5753	0.2740	0.2796	0.2710
	λ_1	0.1471	0.1964	0.1297	0.1287	0.1274
	c_{11}	0.4085	0.6465	0.3725	0.3789	0.3764
	c_{12}	0.2532	0.2966	0.2363	0.2451	0.2415
	c_{13}	0.1262	0.1603	0.1214	0.1219	0.1216

The bootsrap does particularly well here, and is the least biased in every case while also reducing the 2SLS RMSE. The Quenouille Jackknife and C2SLS methods reduce the bias, except the Quenouille Jackknife in one case.

3.2. Estimation with L=4

A further three models are considered, Models 3, 4, and 5, where $L = 4$ for the equation being estimated. An additional two exogenous variables are added to Equation 2, so that $X = x_1, x_2, \ldots, x_8$. The estimator C2SLS$^{(2)}$ does not have a second moment here still, as the order of overidentification is only $L = 2$ for the second structural equation. The coefficient matrices in our simulations are as follows:

Model 3

$$B = \begin{pmatrix} 1 & -1.44 \\ -\beta_1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} -c_{11} & -c_{12} & 0 & 0 & 0 & -c_{13} & 0 & 0 \\ -0.11 & 0 & -0.38 & -1.08 & 0.82 & 0 & -1.31 & 0.67 \end{pmatrix}'$$
$$\beta_1 = 0.2, c_{11} = 0.85, c_{12} = 0.68, c_{13} = 0.67, \lambda_1 = 0.63, \text{ and } \tau = 0.88$$

Model 4

$$B = \begin{pmatrix} 1 & -0.73 \\ -\beta_1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} -c_{11} & -c_{12} & 0 & 0 & 0 & -c_{13} & 0 & 0 \\ 0.08 & 0 & 1.33 & 0.38 & -0.41 & 0 & 0.43 & 0.72 \end{pmatrix}'$$
$$\beta_1 = 0.40, c_{11} = 0.62, c_{12} = 0.24, c_{13} = 0.20, \lambda_1 = 0.54, \text{ and } \tau = 0.76$$

Model 5

$$B = \begin{pmatrix} 1 & -1.19 \\ -\beta_1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} -c_{11} & -c_{12} & 0 & 0 & 0 & -c_{13} & 0 & 0 \\ 1.23 & 0 & -0.08 & -0.35 & 1.22 & 0 & -0.38 & -0.20 \end{pmatrix}'$$
$$\beta_1 = 0.43, c_{11} = -0.76, c_{12} = -1.47, c_{13} = 0.78, \lambda_1 = 0.38, \text{ and } \tau = 0.78$$

Estimation by Quenouille Jackknife is practically unbiased in some cases, for example the bias is just 0.257% in estimation of λ_1 in Model 3, and there are even smaller remaining biases for Model 5. In estimation of c_{12} and c_{13} in Model 4, though, it does not reduce the bias as much as the bootstrap and C2SLS methods. Moreover, the RMSE is inflated in every case. The bootstrap performs well across Models 3 and 4, with both reduced bias and RMSE in estimation of each parameter. The C2SLS methods also reduce the bias, but not by as much, and the RMSE is marginally higher overall than with the bootstrap. Model 5 tells a different story, with the bootstrap reducing bias but not by as much as C2SLS, and with the RMSE slightly inflated when compared with 2SLS and C2SLS.

Table 3. Model 3 % bias and RMSE, $T = 50$

		2SLS	QJ	Boot	C2SLS[1]	C2SLS[2]
% bias	β_1	19.44	1.291	-4.280	7.844	6.128
	λ_1	-10.34	0.257	2.233	-4.119	-3.239
	c_{11}	-14.63	4.718	2.177	-6.021	-4.850
	c_{12}	-13.08	-1.520	2.347	-5.401	-4.193
	c_{13}	-9.953	-0.289	2.069	-4.200	-3.328
RMSE	β_1	0.08017	0.09433	0.07459	0.07677	0.07715
	λ_1	0.1230	0.1405	0.1138	0.1154	0.1155
	c_{11}	0.4393	0.5145	0.4040	0.4208	0.4196
	c_{12}	0.2919	0.3214	0.2874	0.2948	0.2971
	c_{13}	0.1850	0.2143	0.1829	0.1827	0.1838

Table 4. Model 4 % bias and RMSE, $T = 50$

		2SLS	QJ	Boot	C2SLS[1]	C2SLS[2]
% bias	β_1	10.86	-0.2902	-2.779	4.204	4.956
	λ_1	-11.86	3.566	2.162	-4.988	-5.188
	c_{11}	10.38	-3.388	-2.078	4.120	3.940
	c_{12}	-13.22	-12.18	-2.028	-7.719	-6.406
	c_{13}	11.59	6.512	-1.470	4.951	4.904
RMSE	β_1	0.08608	0.1251	0.07699	0.07694	0.07957
	λ_1	0.11085	0.1361	0.09587	0.09580	0.09845
	c_{11}	0.39892	0.5924	0.3508	0.3711	0.3703
	c_{12}	0.22870	0.2422	0.2178	0.2244	0.2241
	c_{13}	0.12141	0.1386	0.1189	0.1191	0.1194

Table 5. Model 5 % bias and RMSE, $T = 50$

		2SLS	QJ	Boot	C2SLS^1	C2SLS^2
% bias	β_1	13.05	-0.1020	-10.38	5.683	5.245
	λ_1	-18.53	-0.6889	14.40	-8.123	-7.411
	c_{11}	-12.61	1.243	8.973	-5.590	-5.110
	c_{12}	-10.49	1.037	8.685	-4.627	-4.360
	c_{13}	-14.95	-0.007210	11.47	-6.676	-6.223
RMSE	β_1	0.09646	0.1224	0.1037	0.09444	0.09541
	λ_1	0.1150	0.1375	0.1222	0.1111	0.1120
	c_{11}	0.3784	0.4272	0.3809	0.3750	0.3748
	c_{12}	0.3490	0.4416	0.3885	0.3534	0.3567
	c_{13}	0.2301	0.2799	0.2517	0.2302	0.2323

It seems clear that the C2SLS methods are competitive with the bootstrap and jackknife. The C2SLS^1 estimator reduces bias in all but one case, β_1 in Model 1, where the bootstrap and jackknife corrections also don't seem to work. The C2SLS^2 estimator has a reduced bias in every case, perhaps due to its use of overidentifying information. The RMSE of the C2SLS^1 estimator compares well with 2SLS in every case in our experiments. Moreover, no substantial improvement in bias reduction has been observed from using C2SLS$^{(2)}$ over C2SLS$^{(1)}$, indeed sometimes it is worse. While the success of C2SLS$^{(1)}$ may depend on the correct specification of the reduced form for Y_2, the potential for mispecification to reduce the bias correction performance also seems greater in the case of C2SLS$^{(2)}$, where each structural equation needs to be estimated. The stronger requirement of having all equations overidentified, and to a higher order, makes it unlikely that an investigator would prefer C2SLS$^{(2)}$ over C2SLS$^{(1)}$, and it is therefore not investigated further.

3.3. Estimation with weak instruments

When instrumental variables are weak, it is well known that the performance of 2SLS can be very poor. Moreover, it has been shown by Hahn & Hausman (2002), Hahn, Hausman & Kuersteiner (2004) for a static model that the performance of 2SLS moment approximations can also be poor. Moreira, Porter, & Suarez (2004) provide examples where the bootstrap and higher order Edgeworth expansion are valid in weak instrument cases, but it is still unknown how the bootstrap will work in the present context of bias correction for 2SLS in dynamic models. Moreover, while the standard delete-1 jackknife method appears to work well in weak instruments cases in static models, see Hahn & Hausman (2002), Hahn, Hausman & Kuersteiner

(2004), it is unknown how the Quenouille Jackknife, chosen for its potential performance in dynamic models, will fare when there are weak instruments.

All three bias correction methods performed quite well for Model 4 in terms of bias correction, particularly the bootstrap and Quenouille Jackknife. Estimation by the bootstrap and C2SLS also resulted in a RMSE reduction. In order to consider cases with weaker instruments, new models are formed starting from Model 4 by shrinking the reduced form coefficients towards zero while keeping the endogenous variable coefficients and the structural covariance matrix unchanged in B and Σ, respectively. Multiplying Γ and Π by a constant $s \in (0,1)$ while holding B fixed is equivalent to multplying Λ and C by s. As a measure of instrument weakness the expected R^2 from the first stage regression is used, namely $\rho^2 = 1 - \mathbb{E}[y_2' M y_2 / y_2' N y_2]$ where $M = I - Z(Z'Z)Z'$, $N = I - \frac{1}{n}\iota\iota'$, and where ι is a $T \times 1$ vector of ones.

Figure 2 below plots the bias approximation in Theorem 1 and the simulated bias for the different ρ^2 values achieved by using different values of s. Smaller values of ρ correspond to smaller values of s; a value of $\rho^2 = 0.20$ was achieved using $s = 0.1$. Over the interval of ρ^2 values that have been used in the figure, the bias approximation seems to do quite well in terms of its closeness to the true bias. It is also suggested by the figures that the bias approximation begins to break down as ρ^2 is moved below 0.2 or, equivalently, as s is reduced below 0.1, and this is indeed the case. For sufficiently small values of s, the true bias can be in the thousands, and the approximate bias is severely overstated.

Table 4 presents bias and RMSE values for 2SLS, QJ, bootstrap and C2SLS estimation of the following model where $s = 0.1$:

*Model 4**

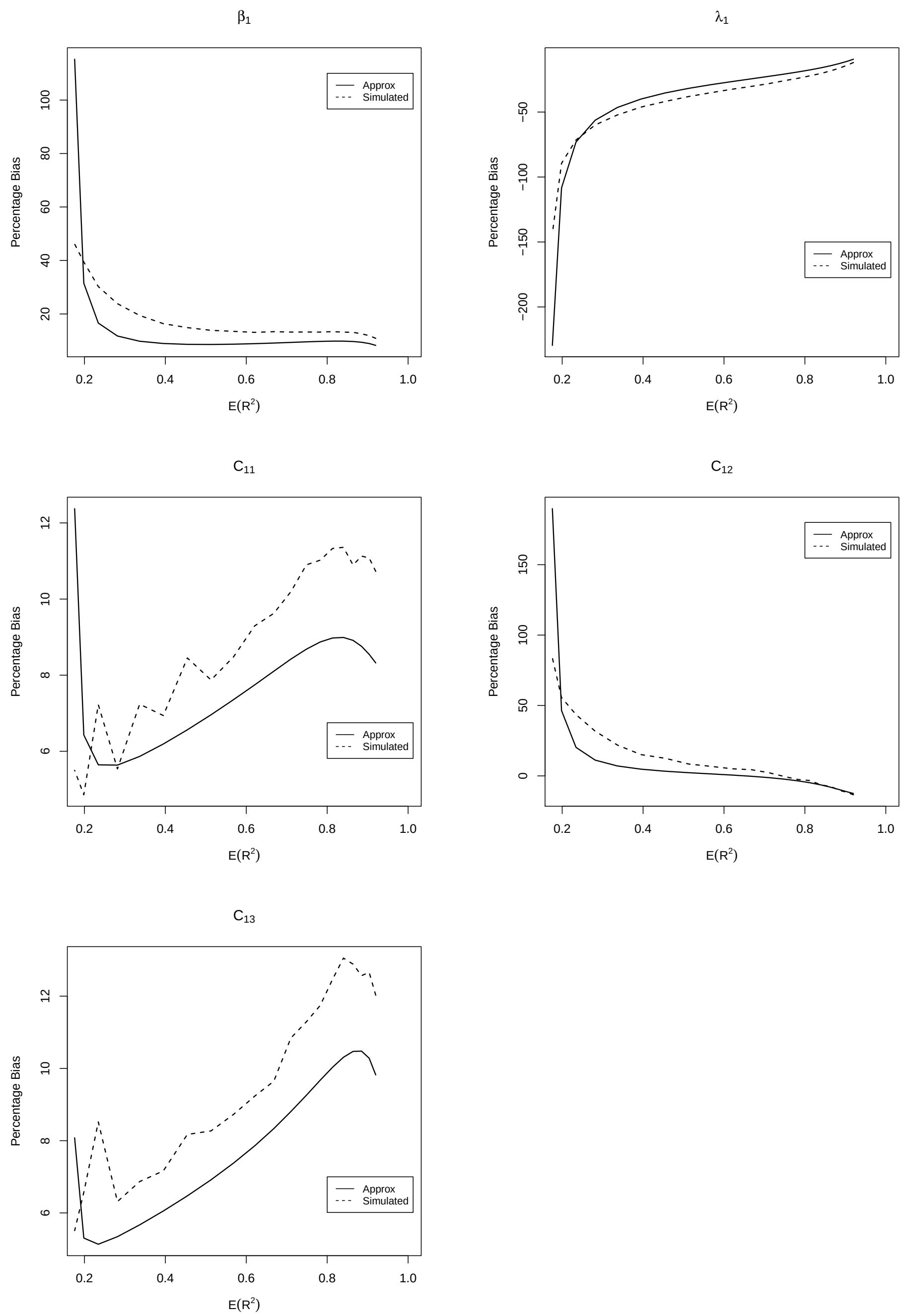
$$B = \begin{pmatrix} 1 & -0.73 \\ -\beta_1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} -c_{11} & -c_{12} & 0 & 0 & 0 & -c_{13} & 0 & 0 \\ 0.08s & 0 & 1.33s & 0.38s & -0.41s & 0 & 0.43s & 0.72s \end{pmatrix}'$$

$$\beta_1 = 0.40, c_{11} = 0.62s, c_{12} = 0.24s, c_{13} = 0.20s, \lambda_1 = 0.54s, \text{ and } \tau = 0.076$$

Model 4* is a case with weaker instruments where the bias approximation still appears to work, and this is reflected in the results for bias correction, which strongly favour C2SLS over the other methods. The bootstrap also

Figure 2: Approximate vs Simulated Bias given varying levels of instrument quality



corrects the bias in this case to some degree, but does not perform nearly as well as C2SLS. The QJ corrects the bias in estimation of two out of five coefficients, and substantially increases the bias in two cases. All three methods inflate the RMSE, though for the bootstrap this increase in RMSE is not substantial.

Table 6. Model 4* % bias and RMSE, $T = 50$

		2SLS	QJ	Boot	C2SLS[1]
% bias	β_1	38.90	30.63	-17.39	-1.182
	λ_1	-89.97	-24.21	-39.65	-43.68
	c_{11}	4.737	18.90	-1.100	-0.2420
	c_{12}	62.41	89.25	-27.34	0.3050
	c_{13}	4.371	5.17	-4.165	-0.9410
RMSE	β_1	0.3752	0.6427	0.3677	0.6131
	λ_1	0.1362	0.1751	0.1406	0.1652
	c_{11}	0.3791	0.5124	0.3957	0.4575
	c_{12}	0.2336	0.2998	0.2390	0.2907
	c_{13}	0.1271	0.1630	0.1333	0.1604

Though the main objective of this section has been to investigate the ability to use the result in Theorem 1 for bias correction, and to compare this with other approaches, it may also be interesting to see how the densities and confidence sets for 2SLS and C2SLS compare in cases where the instruments are strong verses cases where the instruments are relatively weak. Figure 3 depicts the estimated densities and 95% confidence sets for 2SLS and C2SLS estimation of the Model 4 equation 1 structural coefficients. A Gaussian kernel with bandwidth parameter 0.5 was used throughout, and a lower number of Monte Carlo replications, 10000, was sufficient. It can be seen that the confidence sets are marginally larger for C2SLS in the case of β_1 and λ, and marginally smaller for c_{11}, c_{12} and c_{13}.

Figure 4 does the same for 2SLS and C2SLS estimation of Model 4* coefficients, and the picture changes somewhat. As in Figure 3, the location of the distribution appears substantially better in the case of C2SLS, though the spread is worse. There does not appear to be a clear winner between the bootstrap and C2SLS based on the results in Models 1-5 and 4*, though it may be concluded that both perform well overall, while the Quenouille jackknife fares less well.

Figure 3: C2SLS vs 2SLS densities, Model 4 ($\rho^2 = 0.92$)

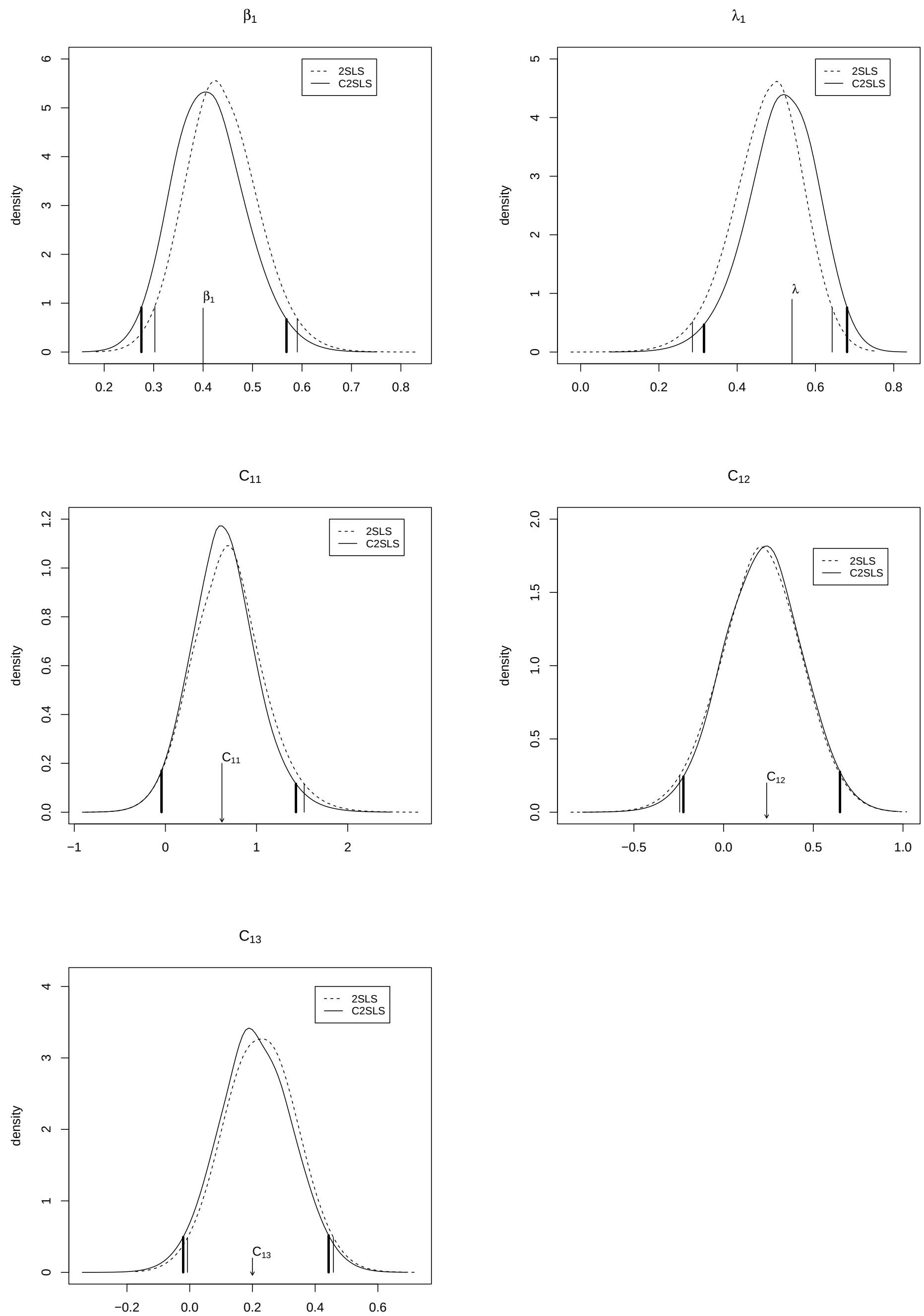
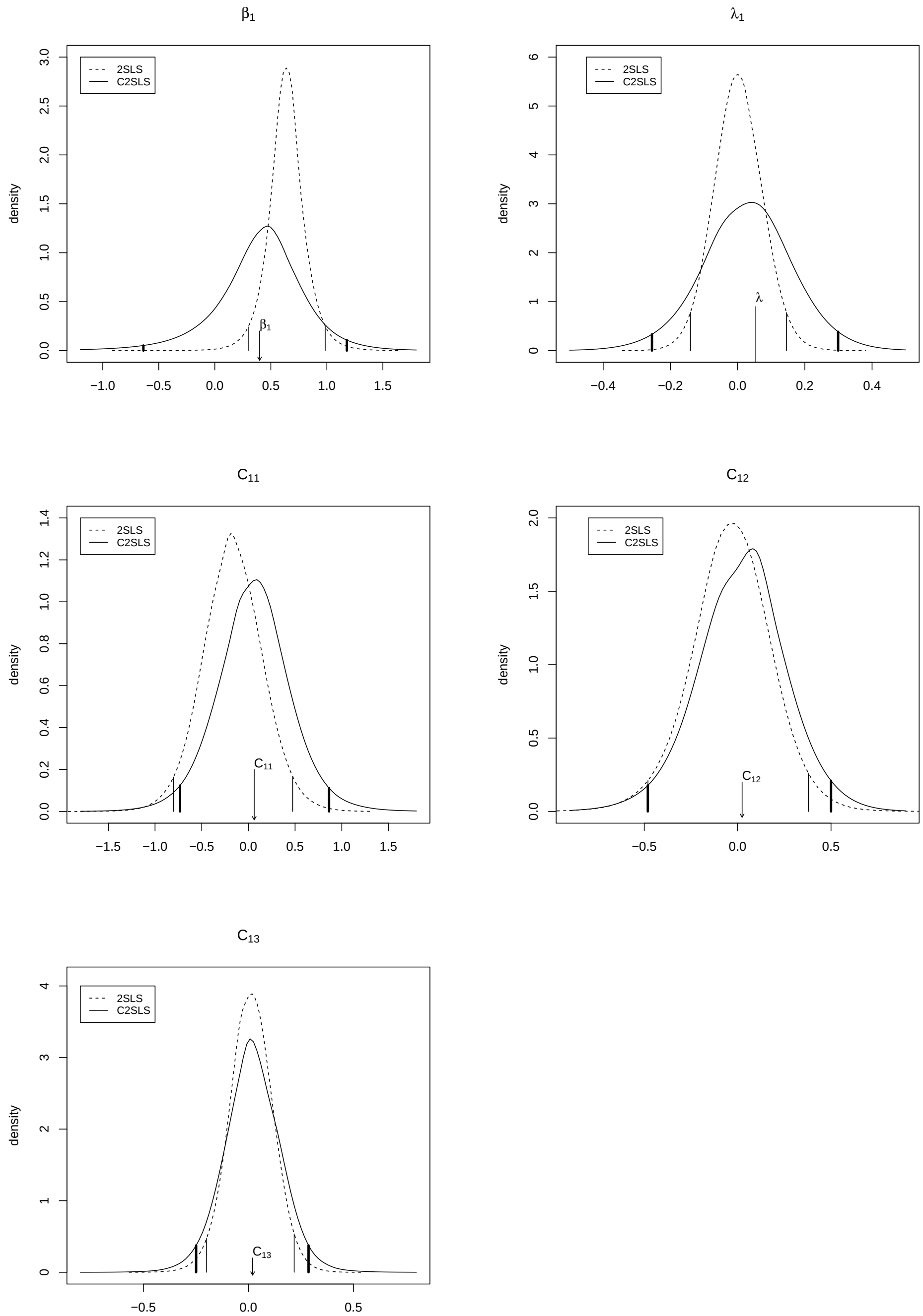


Figure 4: C2SLS vs 2SLS densities, Model 4* ($\rho^2 = 0.20$)



4. Conclusion

The $O(T^{-1})$ bias approximation for $2SLS$ in Nagar (1959) is one of the best known approximations in Econometrics and it has led not only to a better understanding of simultaneous equation bias in static models but has also provided a means of overcoming the problem through the development of bias corrected estimators; however it is invalid when the model includes lagged dependent endogenous variables anywhere in the system which is clearly a significant limitation. There has been some earlier work noted, based on the small-σ expansion, which has the attraction of relative simplicity, but it is well known that such an approach does not often work well in dynamic models; in particular the approximations may be less accurate than those provided by the more complex large-T expansion. It has now proved possible to extend the Nagar approximation to the dynamic case based on the large-T approach which represents a significant advance. The bias has two distinct parts; a part due to simultaneity and a part which derives from the dynamics of the equation. While the bias expression is long primarily due to the dynamic part, it has proved possible to use the approximation for practical bias reduction and the resulting bias corrected estimator is competitive with both the bootstrap and the Quenouille jackknife on a bias criterion. In addition it was found to be better overall in terms of RMSE, as there is no inflation of the 2SLS RMSE. An example has also been presented where the bias in an endogenous variable coefficient does not appear to be monotonic in the sample size, suggesting that the result in Owen (1976) for static models does not go through for dynamic models.

The asymptotic approximation provided here has been obtained under a Normality assumption and has used a Nagar expansion methodology. The methodology in Phillips (2000) for 2SLS estimation of static models, or Bao and Ullah (2007), see also Rilstone, Srivastava and Ullah (1996), for GMM estimation of dynamic models, could potentially be used to extend the work here to non Normal settings; the validity assumptions for these approaches, stated in terms of smoothness conditions and the moments of model errors, are also perhaps more acceptable than for the Nagar method.

Appendix A

Calculating $E[Q^\star \bar{R}'\bar{u}_1]$

It is clear that $E[Q^\star \bar{R}'\bar{u}_1] = 0$

Calculating $E[Q^\star \Delta_1'\bar{u}_1]$

It is clear that $E[Q^\star \Delta_1'\bar{u}_1] = 0$ since $E[W'L'\bar{u}_1] = 0$

Calculating $E[Q^\star \Delta_2'\bar{u}_1]$

$E[Q^\star \Delta_2'\bar{u}_1] = Q^\star E[\Delta_2'\bar{u}_1]$. Note that

$$\begin{aligned}
Q^\star \Delta_2'\bar{u}_1 &= Q^\star\{\bar{Z}(Z'Z)^{-1}Z'[\bar{V}_2 : 0 : 0] + [L\bar{W}I_2^{\star'}(Z'Z)^{-1}Z'\bar{V}_2 : 0 : 0]\}'\bar{u}_1 \\
&= Q^\star\{[\bar{V}_2 : 0 : 0]'Z(Z'Z)^{-1}\bar{Z}' + [\bar{V}_2 : 0 : 0]'Z(Z'Z)^{-1}I_2^\star \bar{W}'L'\}'\bar{u}_1 \\
&= \{I_1^{\star'}\bar{V}'\bar{Z}(E[Z'Z])^{-1}\bar{Z}' + I_1^{\star'}\bar{V}'\bar{Z}(E[Z'Z])^{-1}I_2^\star \bar{W}'L'\}\bar{u}_1 \\
&\quad + Q^\star I_1^{\star'}\bar{V}'\bar{W}^\star(E[Z'Z])^{-1}\bar{Z}'\bar{u}_1 \\
&\quad + o_p(T^{-1}),
\end{aligned}$$

since

$$\begin{pmatrix} \hat{\Gamma}_2 - \Gamma_2 \\ \hat{\Pi}_2 - \Pi_2 \end{pmatrix} = (Z'Z)^{-1}Z'\bar{V}_2 = (E[Z'Z])^{-1}\bar{Z}'\bar{V}_2 + (E[Z'Z])^{-1}\bar{W}^{\star'}\bar{V}_2 + o_p(T^{-1/2}).$$

This gives

$$\begin{aligned}
E[Q^\star \Delta_2'\bar{u}_1] &= Q^\star I_1^{\star'}E[\bar{V}'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{u}_1] \\
&\quad \text{(since the expectation of the last two terms is zero)} \\
&= Q^\star I_1^{\star'}E[(S + \bar{u}_1\phi')'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'u] \\
&= Q^\star(tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}'I_1^{\star'}\}.I)(\sigma^2\phi).
\end{aligned}$$

Calculating -$E[Q^\star S_1 Q^\star \bar{R}'\bar{u}_1]$

To order $o(T^{-1})$,

$$\begin{aligned}
-E[Q^\star S_1 Q^\star \bar{R}'\bar{u}_1] &= -E[Q^\star\{(\bar{R}'\Delta_1 + \Delta_1'\bar{R}) + (\bar{R}'\Delta_2 + \Delta_2'\bar{R}) + (\Delta_1'\Delta_1 + \Delta_2'\Delta_1) \\
&\quad + (\Delta_1'\Delta_1 - E[\Delta_1'\Delta_1])\}Q^\star \bar{R}'\bar{u}_1] \\
&\quad \text{(ignoring the } o_p(T^{-1}) \text{ part of } Q^\star S_1 Q^\star \bar{R}'\bar{u}_1).
\end{aligned}$$

This leads to

$$
\begin{aligned}
&-E[Q^\star S_1 Q^\star \bar{R}'\bar{u}_1] = \\
&\quad -E[Q^\star(\bar{R}'L\bar{W}A + A'\bar{W}'L'\bar{R})Q^\star \bar{R}'\bar{u}_1] \\
&\quad -E[Q^\star \bar{R}'[L\bar{Y}(\hat{\Gamma}_2-\Gamma_2) + X(\hat{\Pi}_2-\Pi_2) + L\bar{W}(\hat{\Gamma}_2-\Gamma_2) : 0 : 0]Q^\star \bar{R}'\bar{u}_1] \\
&\quad -E[Q^\star[L\bar{Y}(\hat{\Gamma}_2-\Gamma_2) + X(\hat{\Pi}_2-\Pi_2) + L\bar{W}(\hat{\Gamma}_2-\Gamma_2) : 0 : 0]'\bar{R}Q^\star \bar{R}'\bar{u}_1] \\
&\quad -E[Q^\star[L\bar{W}\Gamma_2 : L\bar{W}_1 : 0]'[L\bar{Y}(\hat{\Gamma}_2-\Gamma_2) + X(\hat{\Pi}_2-\Pi_2) + L\bar{W}(\hat{\Gamma}_2-\Gamma_2) : 0 : 0]Q^\star \bar{R}'\bar{u}_1] \\
&\quad -E[Q^\star[L\bar{Y}(\hat{\Gamma}_2-\Gamma_2) + X(\hat{\Pi}_2-\Pi_2) + L\bar{W}(\hat{\Gamma}_2-\Gamma_2) : 0 : 0]'[L\bar{W}\Gamma_2 : L\bar{W}_1 : 0]Q^\star \bar{R}'\bar{u}_1]
\end{aligned}
$$

Using

$$
\begin{pmatrix} \hat{\Gamma}_2 - \Gamma_2 \\ \hat{\Pi}_2 - \Pi_2 \end{pmatrix} = (Z'Z)^{-1}Z'\bar{V}_2 = (E[Z'Z])^{-1}\bar{Z}'\bar{V}_2 + (E[Z'Z])^{-1}\bar{W}^{\star\prime}\bar{V}_2 + o_p(T^{-1/2})
$$

and

$$
\hat{\Gamma}_2 - \Gamma_2 = I_2^{\star\prime}\{(E[Z'Z])^{-1}\bar{Z}'\bar{V}_2 + (E[Z'Z])^{-1}\bar{W}^{\star\prime}\bar{V}_2\} + o_p(T^{-1/2})
$$

the non-zero terms become

$$
\begin{aligned}
&-E[Q^\star \bar{R}'L\bar{W}AQ^\star \bar{R}'\bar{u}_1] - E[Q^\star A'\bar{W}'L'\bar{R}Q^\star \bar{R}'\bar{u}_1] \\
&-E[Q^\star \bar{R}'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{V}I_1^\star Q^\star \bar{R}'\bar{u}_1] - E[Q^\star \bar{R}'L\bar{W}I_2^{\star\prime}(E[Z'Z])^{-1}\bar{W}^{\star\prime}\bar{V}I_1^\star Q^\star \bar{R}'\bar{u}_1] \\
&-E[Q^\star I_1^{\star\prime}\bar{V}'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star \bar{R}'\bar{u}_1] - E[Q^\star I_1^{\star\prime}\bar{V}'\bar{W}^\star(E[Z'Z])^{-1}I_2^\star \bar{W}'L'\bar{R}Q^\star \bar{R}'\bar{u}_1] \\
&-E[Q^\star[L\bar{W}\Gamma_2 : L\bar{W}_1 : 0]'\bar{Z}(E[Z'Z])^{-1}\bar{W}^{\star\prime}\bar{V}I_1^\star Q^\star \bar{R}'\bar{u}_1 \\
&-E[Q^\star I_1^{\star\prime}\bar{V}'\bar{W}^\star(E[Z'Z])^{-1}\bar{Z}'L\bar{W}AQ^\star \bar{R}'\bar{u}_1].
\end{aligned}
$$

These are calculated in (a)-(i) below. For these calculations it is noted that

$$
\bar{W} = \sum_{t=0}^{T-1} D^t\bar{V}\Gamma^t \quad \text{and} \quad L\bar{W} = \sum_{t=1}^{T-1} D^t\bar{V}\Gamma^{t-1},
$$

where $\bar{V} = S + \bar{u}_1\phi'$.

(a)

$$
\begin{aligned}
-E[Q^\star \bar{R}' L \bar{W} A Q^\star \bar{R}' \bar{u}_1] &= -E[Q^\star \bar{R}' \sum_{t=1}^{T-1} D^t \bar{V} \Gamma^{t-1} A Q^\star \bar{R}' \bar{u}_1] \\
&\quad - E[Q^\star \bar{R}' \sum_{t=1}^{T-1} D^t \bar{u}_1 \phi' \Gamma^{t-1} A Q^\star \bar{R}' \bar{u}_1] \\
&= -E[Q^\star \bar{R}' \sum_{t=1}^{T-1} D^t \bar{u}_1 \bar{u}_1' \bar{R} Q^\star A' (\Gamma^{t-1})' \phi] \\
&\quad - Q^\star \bar{R}' \sum_{t=1}^{T-1} D^t \bar{R} Q^\star A' (\Gamma^{t-1})' (\sigma^2 \phi)
\end{aligned}
$$

(b)

$$
\begin{aligned}
-E[Q^\star A' \bar{W}' L' \bar{R} Q^\star \bar{R}' \bar{u}_1] &= -E[Q^\star A' \sum_{t=1}^{T-1} \Gamma^{t-1'} \bar{V}' (D^{t'} \bar{R} Q^\star \bar{R}' \bar{u}_1] \\
&= -E[Q^\star A' \sum_{t=1}^{T-1} \Gamma^{t-1'} \phi \bar{u}_1' D^{t'}] \bar{R} Q^\star \bar{R}' \bar{u}_1 \\
&= -\sigma^2 Q^\star A' \sum_{t=1}^{T-1} \Gamma^{t-1'} \phi tr\{\bar{R}' D^{t'} \bar{R} Q^\star\} \\
&= -\sigma^2 Q^\star A' \sum_{t=1}^{T-1} \Gamma^{t-1'} \phi tr\{\bar{R}' D^{t'} \bar{R} Q^\star\} \\
&= -Q^\star A' \sum_{t=1}^{T-1} \Gamma^{t-1'} (tr\{\bar{R}' D^t \bar{R} Q^\star\}.I)(\sigma^2 \phi)
\end{aligned}
$$

(c)

$$
\begin{aligned}
-E[Q^\star \bar{R}' \bar{Z} (E[Z'Z])^{-1} \bar{Z}' \bar{V} I_1^\star Q^\star \bar{R}' \bar{u}_1] &= -E[Q^\star \bar{R}' \bar{Z} (E[Z'Z])^{-1} \bar{Z}' \bar{u}_1 \phi' I_1^\star Q^\star \bar{R}' \bar{u}_1] \\
&= -Q^\star \bar{R}' \bar{Z} (E[Z'Z])^{-1} \bar{Z}' \bar{R} Q^\star I_1^{\star'} (\sigma^2 \phi)
\end{aligned}
$$

(d)

$$
\begin{aligned}
&- E[Q^\star \bar{R}' L \bar{W} I_2^{\star'} (E[Z'Z])^{-1} \bar{W}^{\star'} \bar{V} I_1^\star Q^\star \bar{R}' \bar{u}_1] = \\
&\qquad - E[Q^\star \bar{R}' \sum_{t=1}^{T-1} D^t \bar{V} \Gamma^{t-1} I_2^{\star'} (E[Z'Z])^{-1} \times \begin{pmatrix} \sum_{r=1}^{T-1} \Gamma^{r-1'} \bar{V}' D^{r'} \\ 0 \end{pmatrix} \bar{V} I_1^\star Q^\star \bar{R}' \bar{u}_1]
\end{aligned}
$$

In the final section of this Appendix, it is shown in **Note 1** that

$$E[\bar{V}\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\begin{pmatrix}\sum_{r=1}^{T-1}\Gamma^{r-1'}\bar{V}'D^{r'}\\ 0\end{pmatrix}\bar{V}I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1]$$
$$= \sigma^2 D^{r'}\bar{R}Q^{\star}I_1^{\star'}\phi tr\{\Omega[\Gamma^{r-1}:0](E[Z'Z])^{-1}I_2^{\star}\Gamma^{t-1'}\}$$
$$+ \sigma^2 D^{r}\bar{R}Q^{\star}I_1^{\star'}\Omega\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\phi,$$

and therefore the final expression for (d) is

$$-Q^{\star}\bar{R}'\sum_{t,r=1}^{T-1}D^tD^{r'}\bar{R}Q^{\star}(tr\{\Omega[\Gamma^{r-1}:0](E[Z'Z])^{-1}I_2^{\star}\Gamma^{t-1'}\}.I)I_1^{\star'}(\sigma^2\phi)$$
$$-Q^{\star}\bar{R}'\sum_{t,r=1}^{T-1}D^tD^{r}\bar{R}Q^{\star}I_1^{\star'}\Omega\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}(\sigma^2\phi),$$

(e)

$$-E[Q^{\star}I_1^{\star'}\bar{V}'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^{\star}\bar{R}'\bar{u}_1] = -E[Q^{\star}I_1^{\star'}\phi\bar{u}_1'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^{\star}\bar{R}'\bar{u}_1]$$
$$= -Q^{\star}I_1^{\star'}\phi\sigma^2 tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}\bar{R}Q^{\star}\bar{R}'\}$$
$$= -Q^{\star}(tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^{\star}\bar{R}'\}.I)I_1^{\star'}(\sigma^2\phi),$$

(f)

$$-E[Q^{\star}I_1^{\star'}\bar{V}'\bar{W}^{\star'}(E[Z'Z])^{-1}I_2^{\star}\bar{W}'L'\bar{R}Q^{\star}\bar{R}'\bar{u}_1]$$
$$= -E[Q^{\star}I_1^{\star'}\bar{V}'(\sum_{t=1}^{T-1}D^t\bar{V}\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star}\bar{W}'L'\bar{R}Q^{\star}\bar{R}'\bar{u}_1]$$
$$= -E[Q^{\star}I_1^{\star'}\bar{V}'(\sum_{t=1}^{T-1}D^t\bar{V}\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star}\sum_{r=1}^{T-1}\Gamma^{r-1'}\bar{V}'D^{r'}\bar{R}Q^{\star}\bar{R}'\bar{u}_1].$$

In in the final section of this Appendix, it is shown in **Note 2** that

$$E[\bar{V}'D^t\bar{V}\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\bar{V}'D^{r'}\bar{R}Q^{\star}\bar{R}'\bar{u}_1]$$
$$= \sigma^2 tr\{\Omega\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\}tr\{D^tD^{r'}\bar{R}Q^{\star}\bar{R}'\}\phi$$
$$+ \sigma^2 tr\{D^t\bar{R}Q^{\star}R'D^r\}\Omega\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{t-1'}\phi$$

and therefore the final expression for (f) is

$$- Q^\star \sum_{r,t=1}^{T-1} (tr\{\Omega\Gamma^{t-1} I_2^{\star'} (E[Z'Z])^{-1} I_2^\star \Gamma^{r-1'}\}.I)(tr\{D^t D^{r'} \bar{R} Q^\star \bar{R}'\}.I) I_1^{\star'} (\sigma^2 \phi)$$

$$- Q^\star I_1^{\star'} \sum_{r,t=1}^{T-1} \Omega\Gamma^{r-1} I_2^{\star'} (E[Z'Z])^{-1} I_2^\star \Gamma^{t-1'} (tr\{D^t \bar{R} Q^\star R' D^r\}.I)(\sigma^2 \phi)$$

(g)

$$- E[Q^\star A' \bar{W}' L' \bar{Z} (E[Z'Z])^{-1} \bar{W}^{\star'} \bar{V} I_1^\star Q^\star \bar{R}' \bar{u}_1]$$
$$= -E[Q^\star A' \sum_{t=1}^{T_1} \Gamma^{t-1'} \bar{V}' D^{t'} \bar{Z} (E[Z'Z])^{-1} \begin{pmatrix} \sum_{r=1}^{T-1} \Gamma^{r-1'} \bar{V}' D^{r'} \\ 0' \end{pmatrix} \bar{V} I_1^\star Q^\star \bar{R}' \bar{u}_1].$$

In in the final section of this Appendix, it is shown in **Note 3** that

$$\bar{V}' D^{t'} \bar{Z} (E[Z'Z])^{-1} \begin{pmatrix} \sum_{r=1}^{T-1} \Gamma^{r-1'} \bar{V}' D^{r'} \\ 0' \end{pmatrix} \bar{V} I_1^\star Q^\star \bar{R}' \bar{u}_1$$
$$= \Omega\Gamma^{r-1} I_2^{\star'} (E[Z'Z])^{-1} \bar{Z}' D^t D^{r'} \bar{R} Q^\star I_1^{\star'} (\sigma^2 \phi)$$
$$+ \Omega I_1^\star Q^\star \bar{R}' D^{r'} D^{t'} \bar{Z} (E[Z'Z])^{-1} I_2^\star \Gamma^{r-1'} (\sigma^2 \phi)$$

and therefore the final expression for (g) is

$$- Q^\star A' \sum_{r,t=1}^{T-1} \Gamma^{t-1'} \Omega\Gamma^{r-1} I_2^{\star'} (E[Z'Z])^{-1} \bar{Z}' D^t D^{r'} \bar{R} Q^\star I_1^{\star'} (\sigma^2 \phi)$$

$$- Q^\star A' \sum_{r,t=1}^{T-1} \Gamma^{t-1'} \Omega I_1^\star Q^\star \bar{R}' D^{r'} D^{t'} \bar{Z} (E[Z'Z])^{-1} I_2^\star \Gamma^{r-1'} (\sigma^2 \phi)$$

(h)

$$- E[Q^\star A' \bar{W}' L' L \bar{W} I_2^{\star'} (E[Z'Z])^{-1} \bar{Z}' \bar{V} I_1^\star Q^\star \bar{R}' \bar{u}_1]$$
$$= -E[Q^\star A' \sum_{t=1}^{T-1} \Gamma^{t-1'} \bar{V}' D^{t'} \sum_{r=1}^{T-1} D^r \bar{V} \Gamma^{r-1'} I_2^{\star'} (E[Z'Z])^{-1} \bar{Z}' \bar{V} I_1^\star Q^\star \bar{R}' \bar{u}_1]$$

In in the final section of this Appendix, it is shown in **Note 4** that

$$\bar{V}' D^{t'} D^r \bar{V} \Gamma^{r-1'} I_2^{\star'} (E[Z'Z])^{-1} \bar{Z}' \bar{V} I_1^\star Q^\star \bar{R}' \bar{u}_1$$
$$= (tr\{D^{t'} D^r \bar{Z} (E[Z'Z])^{-1} I_2^\star \Gamma^{r-1'} \Omega I_1^\star Q^\star \bar{R}'\}.I)(\sigma^2 \phi)$$
$$+ \Omega\Gamma^{r-1} I_2^{\star'} (E[Z'Z])^{-1} \bar{Z}' \bar{R} Q^\star (tr\{D^{t'} D^r\}.I) I_1^{\star'} \sigma^2 \phi$$
$$+ \Omega I_1^\star Q^\star \bar{R}' D^{r'} D^t \bar{Z} (E[Z'Z])^{-1} I_2^\star \Gamma^{r-1'} (\sigma^2 \phi)$$

and therefore the final expression for (h) is

$$- Q^\star A' \sum_{t,r=1}^{T-1} \Gamma^{t-1'}(tr\{D^{t'} D^r \bar{Z}(E[Z'Z])^{-1} I_2^\star \Gamma^{r-1'} \Omega I_1^\star Q^\star \bar{R}'\}.I)(\sigma^2 \phi)$$

$$- Q^\star A' \sum_{t,r=1}^{T-1} \Gamma^{t-1'} \Omega \Gamma^{r-1} I_2^{\star'} (E[Z'Z])^{-1} \bar{Z}' \bar{R} Q^\star (tr\{D^{t'} D^r\}.I) I_1^{\star'} \sigma^2 \phi$$

$$- Q^\star A' \sum_{t,r=1}^{T-1} \Gamma^{t-1'} \Omega I_1^\star Q^\star \bar{R}' D^{r'} D^t \bar{Z}(E[Z'Z])^{-1} I_2^\star \Gamma^{r-1'} (\sigma^2 \phi)$$

(i)

$$- E[Q^\star I_1^{\star'} \bar{V}' \bar{W}^\star (E[Z'Z])^{-1} \bar{Z}' L \bar{W} A Q^\star \bar{R}' \bar{u}_1]$$

$$- E[Q^\star I_1^{\star'} \bar{V}' (\sum_{t=1}^{T-1} D^t \bar{V} \Gamma^{t-1} : 0)(E[Z'Z])^{-1} \bar{Z}' \sum_{r=1}^{T-1} D^r \bar{V} \Gamma^{r-1} A Q^\star \bar{R} \bar{u}_1]$$

In in the final section of this Appendix, it is shown in **Note 5** that

$$E[\bar{V}'(D^t \bar{V} \Gamma^{t-1} : 0)(E[Z'Z])^{-1} \bar{Z}' D^r \bar{V} \Gamma^{r-1} A Q^\star \bar{R} \bar{u}_1]$$
$$= -(tr\{D^t D^{r'} \bar{Z}(E[Z'Z])^{-1} I_2^\star \Gamma^{t-1'} \Omega \Gamma^{r-1} A Q^\star \bar{R}'\}.I)(\sigma^2 \phi)$$
$$- \Omega \Gamma^{r-1} A Q^\star \bar{R}' \sigma^2 D^{t'} D^{r'} \bar{Z}(E[Z'Z'])^{-1} I_2^\star \Gamma^{t-1'} (\sigma^2 \phi)$$

and therefore the final expression for (i) is

$$- \sum_{r,t=1}^{T-1} Q^\star (tr\{D^t D^{r'} \bar{Z}(E[Z'Z])^{-1} I_2^\star \Gamma^{t-1'} \Omega \Gamma^{r-1} A Q^\star \bar{R}'\}.I) I_1^{\star'} (\sigma^2 \phi)$$

$$- \sum_{r,t=1}^{T-1} Q^\star I_1^{\star'} \Omega \Gamma^{r-1} A Q^\star \bar{R}' \sigma^2 D^{t'} D^{r'} \bar{Z}(E[Z'Z'])^{-1} I_2^\star \Gamma^{t-1'} (\sigma^2 \phi)$$

Calculating -$E[Q^\star S_1 Q^\star \bar{R}' \bar{u}_1]$

$$E[Q^\star S_1 Q^\star \Delta_1' \bar{u}_1] = -E[Q^\star \bar{R}' \bar{Z}(E[Z'Z])^{-1} \bar{W}^{\star'} \bar{V} I_1^\star Q^\star A' \bar{W}' L' \bar{u}_1]$$
$$- E[Q^\star I_1^{\star'} \bar{V}' \bar{W}^\star (E[Z'Z])^{-1} \bar{Z}' \bar{R} Q^\star A' \bar{W}' L' \bar{u}_1]$$
$$- E[Q^\star A' \bar{W}' L' \bar{Z}' (E[Z'Z])^{-1} \bar{Z}' \bar{V} I_1^\star Q^\star A' \bar{W}' L' \bar{u}_1]$$
$$- E[Q^\star A' \bar{W}' L' L \bar{W} L' L \bar{W} I_2^{\star'} (E[Z'Z])^{-1} \bar{W}^{\star'} \bar{V} I_1^\star Q^\star A' \bar{W}' L' \bar{u}_1]$$
$$- E[Q^\star \{A' \bar{W}' L' L \bar{W} A - E[A' \bar{W}' L' L \bar{W} A]\} Q^\star A' \bar{W}' L' \bar{u}_1]$$
$$- E[Q^\star I_1^{\star'} \bar{V}' \bar{Z}(E[Z'Z])^{-1} \bar{Z}' L W A Q^\star A' \bar{W}' L' \bar{u}_1]$$
$$- E[Q^\star I_1^{\star'} \bar{V}' \bar{W}^\star (E[Z'Z])^{-1} I_2^\star \bar{W}' L' L \bar{W} A Q^\star A' \bar{W}' L' \bar{u}_1]$$

These are calculated in (a')-(g') below.

(a')

$$
\begin{aligned}
&- E[Q^\star \bar{R}'\bar{Z}(E[Z'Z])^{-1}\bar{W}^{\star\prime}\bar{V}I_1^\star Q^\star A'\bar{W}'L'\bar{u}_1] \\
&\qquad = -E[Q^\star \bar{R}'\bar{Z}(E[Z'Z])^{-1}\sum_{t=1}^{T-1}\begin{pmatrix}\Gamma^{t-1'}\bar{V}'D^{t'}\\ 0'\end{pmatrix}\bar{V}I_1^\star Q^\star A'\sum_{r=1}^{T-1}\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1] \\
&\qquad = -E[Q^\star \bar{R}'\bar{Z}(E[Z'Z])^{-1}\sum_{t=1}^{T-1}\begin{pmatrix}\Gamma^{t-1'}\\ 0'\end{pmatrix}\bar{V}'D^{t'}\bar{V}I_1^\star Q^\star A'\sum_{r=1}^{T-1}\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1]
\end{aligned}
$$

In the final section of this Appendix, it is shown in **Note 6** that

$$E[\bar{V}'D^{t'}\bar{V}I_1^\star Q^\star A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1] = (tr\{D^{t'}D^r\}.I)\Omega\Gamma^{r-1}AQ^\star I_1^{\star\prime}(\sigma^2\phi)$$

and therefore the final expression for (a') is

$$-Q^\star \bar{R}'\bar{Z}(E[Z'Z])^{-1}\sum_{r,t=1}^{T-1} I_2^\star \Gamma^{t-1'}(tr\{D^{t'}D^r\}.I)\Omega\Gamma^{r-1}AQ^\star I_1^{\star\prime}(\sigma^2\phi)$$

(b')

$$
\begin{aligned}
&- E[Q^\star I_1^{\star\prime}\bar{V}'\bar{W}^\star(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\bar{W}'L'\bar{u}_1] \\
&\qquad = -E[Q^\star I_1^{\star\prime}\bar{V}(\sum_{t=1}^{T-1}D^t\bar{V}\Gamma^{t-1} : 0)(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\sum_{r=1}^{T-1}\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1]
\end{aligned}
$$

In in the final section of this Appendix, it is shown in **Note 7** that

$$
\begin{aligned}
&E[\bar{V}(D^t\bar{V}\Gamma^{t-1} : 0)(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1] \\
&\qquad = tr\{D^tD^{r'}\}(tr\{\Omega\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}\}.I)(\sigma^2\phi)
\end{aligned}
$$

and therefore the final expression for (b') is

$$-Q^\star\sum_{r,t=1}^{T_1}(tr\{D^tD^{r'}\}.I)(tr\{\Omega\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}\}.I)I_1^{\star\prime}(\sigma^2\phi)$$

(c')

$$
\begin{aligned}
&- E[Q^\star A'\bar{W}'L'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{V}I_1^\star Q^\star A'\bar{W}'L'\bar{u}_1] \\
&\qquad = -E[Q^\star A'\sum_{t=1}^{T-1}\Gamma^{t-1'}\bar{V}'D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{V}I_1^\star Q^\star A'\sum_{r=1}^{T-1}\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1].
\end{aligned}
$$

In in the final section of this Appendix, it is shown in **Note 8** that

$$E[\bar{V}'D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{V}I_1^{\star}Q^{\star}A'\sum_{r=1}^{T-1}\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1]$$
$$= tr\{\Omega I_1^{\star}Q^{\star}A'\Gamma^{r-1'}\}(tr\{D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^{r'}\}.I)(\sigma^2\phi)$$
$$+ (tr\{D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^{r}\}.I)\Omega\Gamma^{r-1}AQ^{\star}I_1^{\star'}(\sigma^2\phi)$$

and therefore the final expression for (c') is

$$-Q^{\star}A'\sum_{r,t=1}^{T-1}\Gamma^{t-1'}(tr\{\Omega I_1^{\star}Q^{\star}A'\Gamma^{r-1'}\}.I)(tr\{D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^{r'}\}.I)(\sigma^2\phi)$$

(d')

$$- E[Q^{\star}A'\bar{W}'L'L\bar{W}I_2^{\star'}(E[Z'Z])^{-1}\bar{W}^{\star'}\bar{V}I_1^{\star}Q^{\star}A'\bar{W}'L'\bar{u}_1]$$
$$= -E[Q^{\star}A'E[\bar{W}'L'L\bar{W}]I_2^{\star'}(E[Z'Z])^{-1}\bar{W}^{\star'}\bar{V}I_1^{\star}Q^{\star}A'\bar{W}'L'\bar{u}_1]$$

to order $O_p(T^{-1})$. Therefore, to order $O_p(T^{-1})$,

$$- E[Q^{\star}A'\bar{W}'L'L\bar{W}I_2^{\star}(E[Z'Z])^{-1}\bar{W}^{\star'}\bar{V}I_1^{\star}Q^{\star}A'\bar{W}'L'\bar{u}_1]$$
$$= -E[Q^{\star}A'E[\bar{W}'L'L\bar{W}]I_2^{\star'}(E[Z'Z])^{-1}\begin{pmatrix}\sum_{t=1}^{T-1}\Gamma^{t-1'}\bar{V}'D^{t'}\\ 0'\end{pmatrix}\bar{V}I_1^{\star}Q^{\star}A'\sum_{t-1}^{T-1}\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1]$$

In in the final section of this Appendix, it is shown in **Note 9** that

$$E[\bar{V}'D^{t'}\bar{V}I_1^{\star}Q^{\star}A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1] = \Omega\Gamma^{r-1}AQ^{\star}(tr\{D^{t'}D^{r}\}.I)I_1^{\star'}(\sigma^2\phi)$$

and therefore the final expression for (d') is

$$-Q^{\star}A'E[\bar{W}'L'L\bar{W}]I_2^{\star'}(E[Z'Z])^{-1}\sum_{r,t=1}^{T-1}I_2^{\star}\Gamma^{t-1'}\Omega\Gamma^{r-1}AQ^{\star}(tr\{D^{t'}D^{r}\}.I)I_1^{\star'}(\sigma^2\phi)$$

(e')

$$- E[Q^{\star}A'\bar{W}'L'L\bar{W}AQ^{\star}A'\bar{W}'L'\bar{u}_1] =$$
$$- E[Q^{\star}A'\sum_{r,t,s=1}^{T-1}\Gamma^{t-1'}\bar{V}'D^{t'}D^{r}\epsilon\Gamma^{r-1}AQ^{\star}A'\Gamma^{s-1'}\bar{V}'D^{s'}\bar{u}_1].$$

In in the final section of this Appendix, it is shown in **Note 10** that

$$
\begin{aligned}
&E[\bar{V}'D^{t'}D^r\epsilon\Gamma^{r-1}AQ^\star A'\Gamma^{s-1'}\bar{V}'D^{s'}\bar{u}_1] = \\
&\qquad \Omega\Gamma^{s-1}AQ^\star A'\Gamma^{r-1'}\{tr(D^{t'}D^rD^s).I\}(\sigma^2\phi) \\
&\qquad + tr(\Omega\Gamma^{r-1}AQ^\star A'\Gamma^{s-1'})tr(D^{t'}D^rD^{s'})(\sigma^2\phi),
\end{aligned}
$$

and therefore the final expression for (e') is

$$
\begin{aligned}
&-Q^\star\sum_{r,t,s=1}^{T-1}A'\Gamma^{t-1'}\Omega\Gamma^{s-1}AQ^\star A'\Gamma^{r-1'}\{tr(D^{t'}D^rD^s).I\}(\sigma^2\phi) \\
&\quad -Q^\star\sum_{r,t,s=1}^{T-1}A'\Gamma^{t-1'}tr(\Omega\Gamma^{r-1}AQ^\star A'\Gamma^{s-1'})tr(D^{t'}D^rD^{s'})(\sigma^2\phi)
\end{aligned}
$$

(f')

$$
\begin{aligned}
&-E[Q^\star I_1^{\star'}\bar{V}'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'L\bar{W}AQ^\star A'\bar{W}'L'\bar{u}_1] \\
&\qquad\qquad = -E[Q^\star I_1^{\star'}\bar{V}'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\sum_{r,t}^{T-1}D^t\bar{V}\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1].
\end{aligned}
$$

In in the final section of this Appendix, it is shown in **Note 11** that

$$
\begin{aligned}
&E[\bar{V}'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^t\bar{V}\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1] \\
&\qquad = (tr\{\Omega\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}\}.I)(tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^tD^{r'}\}.I)(\sigma^2\phi) \\
&\qquad + \Omega\Gamma^{r-1}AQ^\star A'\Gamma^{t-1'}(tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^tD^r\}.I)(\sigma^2\phi)
\end{aligned}
$$

and therefore the final expression for (f') is

$$
\begin{aligned}
&-\sum_{r,t=1}^{T-1}Q^\star(tr\{\Omega\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}\}.I)(tr\{\bar{Z}(E[Z'Z])^{-1}I_2^\star\bar{W}'L'L\bar{W}AQ^\star A'\bar{W}'L'\bar{u}_1\}.I)I_1^{\star'}(\sigma^2\phi) \\
&\qquad -\sum_{r,t=1}^{T-1}Q^\star I_1^{\star'}\Omega\Gamma^{r-1}AQ^\star A'\Gamma^{t-1'}(tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}D^tD^r\}.I)(\sigma^2\phi)
\end{aligned}
$$

(g')

$$
\begin{aligned}
&-E[Q^\star I_1^{\star'}\bar{V}'\bar{W}^\star(E[Z'Z])^{-1}I_2^{\star'}\bar{W}'L'L\bar{W}AQ^\star A'\bar{W}'L'\bar{u}_1] \\
&\qquad\qquad -E[Q^\star I_1^{\star'}\bar{V}'\bar{W}^\star(E[Z'Z])^{-1}I_2^{\star'}E[\bar{W}'L'L\bar{W}]AQ^\star A'\bar{W}'L'\bar{u}_1]
\end{aligned}
$$

to order $O_p(T^{-1})$. Therefore, to order $O_p(T^{-1})$,

$$- E[Q^\star I_1^{\star'} \bar{V}' \bar{W}^\star (E[Z'Z])^{-1} I_2^{\star'} \bar{W}' L' L \bar{W} A Q^\star A' \bar{W}' L' \bar{u}_1]$$
$$= -E[Q^\star I_1^{\star'} \bar{V}' (\sum_{t=1}^{T-1} D^t \bar{V} \Gamma^{t-1} : 0)(E[Z'Z])^{-1} I_2^{\star'} E[\bar{W}' L' L \bar{W}] A Q^\star A' \sum_{r=1}^{T-1} \Gamma^{r-1'} \bar{V}' D^{r'} \bar{u}_1].$$

In in the final section of this Appendix, it is shown in **Note 12** that

$$\bar{V}'(D^t \bar{V} \Gamma^{t-1} : 0)(E[Z'Z])^{-1} I_2^{\star'} E[\bar{W}' L' L \bar{W}] A Q^\star A' \Gamma^{r-1'} \bar{V}' D^{r'} \bar{u}_1$$
$$= tr\{\Omega(\Gamma^{t-1} : 0)(E[Z'Z])^{-1} I_2^\star E[W' L' L W] A Q^\star A' \Gamma^{r-1'}\}(tr\{D^t D^{r'}\}.I)(\sigma^2 \phi)$$

and therefore the final expression for (g') is

$$- \sum_{r,t=1}^{\star} Q^\star (tr\{\Omega(\Gamma^{t-1} : 0)(E[Z'Z])^{-1} I_2^\star E[W' L' L W] A Q^\star A' \Gamma^{r-1'}\}(tr\{D^t D^{r'}\}.I) I_1^{\star'} (\sigma^2 \phi)$$

Rearranging for the final expression
In the following all the expectations calculations for the terms in equation (23) are added together, in the order that they appear. Recall that $Q_Z = (E[Z'Z])^{-1}$.

$$Q^\star (tr\{\bar{Z} Q_Z \bar{Z}'\}.I) I_1^{\star'} (\sigma^2 \phi)$$
$$- Q^\star \bar{R}' \sum_{t=1}^{T-1} D^t \bar{R} Q^\star A' (\Gamma^{t-1})' (\sigma^2 \phi)$$
$$- Q^\star A' \sum_{t=1}^{T-1} \Gamma^{t-1'} (tr\{\bar{R}' D^t \bar{R} Q^\star\}.I)(\sigma^2 \phi)$$
$$- Q^\star \bar{R}' \bar{Z} Q_Z \bar{Z}' \bar{R} Q^\star I_1^{\star'} (\sigma^2 \phi)$$
$$- Q^\star \bar{R}' \sum_{t,r=1}^{T-1} D^t D^{r'} \bar{R} Q^\star (tr\{\Omega \Gamma^{r-1} I_2^{\star'} Q_Z I_2^\star \Gamma^{t-1'}\}.I)(\sigma^2 I_1^{\star'} \phi)$$
$$- Q^\star \bar{R}' \sum_{t,r=1}^{T-1} D^t D^r \bar{R} Q^\star I_1^{\star'} \Omega \Gamma^{t-1} I_2^{\star'} Q_Z I_2^\star \Gamma^{r-1'} (\sigma^2 \phi)$$
$$- Q^\star (tr\{\bar{Z} Q_Z \bar{Z}' \bar{R} Q^\star \bar{R}'\}.I) I_1^{\star'} (\sigma^2 \phi)$$
$$- Q^\star \sum_{r,t=1}^{T-1} (tr\{\Omega \Gamma^{t-1} I_2^{\star'} Q_Z I_2^\star \Gamma^{r-1'}\}.I)(tr\{D^t D^{r'} \bar{R} Q^\star \bar{R}'\}.I) I_1^{\star'} (\sigma^2 \phi)$$

$$- Q^\star I_1^{\star'} \sum_{r,t=1}^{T-1} \Omega\Gamma^{r-1} I_2^{\star'} Q_Z I_2^\star \Gamma^{t-1'} (tr\{D^t \bar{R} Q^\star R' D^r\}.I)(\sigma^2\phi)$$

$$- Q^\star A' \sum_{r,t=1}^{T-1} \Gamma^{t-1'} \Omega\Gamma^{r-1} I_2^{\star'} Q_Z \bar{Z}' D^t D^{r'} \bar{R} Q^\star I_1^{\star'} (\sigma^2\phi)$$

$$- Q^\star A' \sum_{r,t=1}^{T-1} \Gamma^{t-1'} \Omega I_1^\star Q^\star \bar{R}' D^{r'} D^{t'} \bar{Z} Q_Z I_2^\star \Gamma^{r-1'} (\sigma^2\phi)$$

$$- Q^\star A' \sum_{t,r=1}^{T-1} \Gamma^{t-1'} (tr\{D^{t'} D^r \bar{Z} Q_Z I_2^\star \Gamma^{r-1'} \Omega I_1^\star Q^\star \bar{R}'\}.I)(\sigma^2\phi)$$

$$- Q^\star A' \sum_{t,r=1}^{T-1} \Gamma^{t-1'} \Omega\Gamma^{r-1} I_2^{\star'} Q_Z \bar{Z}' \bar{R} Q^\star (tr\{D^{t'} D^r\}.I) I_1^{\star'} (\sigma^2\phi)$$

$$- Q^\star A' \sum_{t,r=1}^{T-1} \Gamma^{t-1'} \Omega I_1^\star Q^\star \bar{R}' D^{r'} D^t \bar{Z} Q_Z I_2^\star \Gamma^{r-1'} (\sigma^2\phi)$$

$$- Q^\star \sum_{t,r=1}^{T-1} (tr\{D^t D^{r'} \bar{Z} Q_Z I_2^\star \Gamma^{t-1'} \Omega\Gamma^{r-1} A Q^\star \bar{R}'\}.I) I_1^{\star'} (\sigma^2\phi)$$

$$- Q^\star \sum_{t,r=1}^{T-1} I_1^{\star'} \Omega\Gamma^{r-1} A Q^\star \bar{R}' \sigma^2 D^{t'} D^{r'} \bar{Z} Q_Z I_2^\star \Gamma^{t-1'} (\sigma^2\phi)$$

$$- Q^\star \bar{R}' \bar{Z} Q_Z \sum_{r,t=1}^{T-1} I_2^\star \Gamma^{t-1'} (tr\{D^{t'} D^r\}.I) \Omega\Gamma^{r-1} A Q^\star I_1^{\star'} (\sigma^2\phi)$$

$$- Q^\star \sum_{r,t=1}^{T-1} (tr\{D^t D^{r'}\}.I)(tr\{\Omega\Gamma^{t-1} I_2^{\star'} Q_Z \bar{Z}' \bar{R} Q^\star A' \Gamma^{r-1'}\}.I) I_1^{\star'} (\sigma^2\phi)$$

$$- Q^\star A' \sum_{r,t=1}^{T-1} \Gamma^{t-1'} (tr\{\Omega I_1^\star Q^\star A' \Gamma^{r-1'}\} tr\{D^{t'} \bar{Z} Q_Z \bar{Z}' D^{r'}\}.I)(\sigma^2\phi)$$

$$- Q^\star A' \sum_{r,t=1}^{T-1} \Gamma^{t-1'} (tr\{D^{t'} \bar{Z} Q_Z \bar{Z}' D^r\}.I) \Omega\Gamma^{r-1} A Q^\star I_1^{\star'} (\sigma^2\phi)$$

$$- Q^\star A' E[\bar{W}' L' L \bar{W}] I_2^{\star'} Q_Z \sum_{r,t=1}^{T-1} I_2^\star \Gamma^{t-1'} \Omega\Gamma^{r-1} A Q^\star (tr\{D^{t'} D^r\}.I) I_1^{\star'} (\sigma^2\phi)$$

$$- Q^\star \sum_{r,t,s=1}^{T-1} A'\Gamma^{t-1'}\Omega\Gamma^{s-1}AQ^\star A'\Gamma^{r-1'}\{tr(D^{t'}D^rD^s).I\}(\sigma^2\phi)$$

$$- Q^\star \sum_{r,t,s=1}^{T-1} A'\Gamma^{t-1'}tr(\Omega\Gamma^{r-1}AQ^\star A'\Gamma^{s-1'})tr(D^{t'}D^rD^{s'})(\sigma^2\phi)$$

$$- Q^\star \sum_{r,t=1}^{T-1}(tr\{\Omega\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}\}.I)(tr\{\bar{Z}Q_Z\bar{Z}'D^tD^{r'}\}.I)I_1^{\star'}(\sigma^2\phi)$$

$$- Q^\star I_1^{\star'} \sum_{r,t=1}^{T-1}\Omega\Gamma^{r-1}AQ^\star A'\Gamma^{t-1'}(tr\{\bar{Z}Q_Z\bar{Z}'D^tD^r\}.I)(\sigma^2\phi)$$

$$- \sum_{r,t=1}^{T-1} Q^\star(tr\{\Omega\Gamma^{t-1}I_2^{\star'}Q_ZI_2^{\star'}E[W'L'LW]AQ^\star A'\Gamma^{r-1'}\}.I)(tr\{D^tD^{r'}\}.I)I_1^{\star'}(\sigma^2\phi)$$

Next recall that $\varphi = \sigma^2\phi$, $\psi = I_1^{\star'}\varphi$, $(\Gamma^{r-1} : 0) = \Gamma^{r-1}I_2^{\star'}$, $I_2^{\star'}Q_ZI_2^\star = Q_Z^\star$, and $Q_W = \sum_{t=1}^{T-1}(T-t)\Gamma^{t-1'}\Omega\Gamma^{t-1}$. Also, note that $tr\{D^{t'}D^r\} = T - t$ when $t = r$ and 0 otherwise. The terms above can then be written (in the same order) as follows:

$$Q^\star(tr\{\bar{Z}Q_Z\bar{Z}'\}.I)\psi$$

$$- Q^\star\bar{R}' \sum_{t=1}^{T-1} D^t\bar{R}Q^\star A'(\Gamma^{t-1})'\varphi$$

$$- Q^\star A' \sum_{t=1}^{T-1}\Gamma^{t-1'}(tr\{\bar{R}'D^t\bar{R}Q^\star\}.I)\varphi$$

$$- Q^\star\bar{R}'\bar{Z}Q_Z\bar{Z}'\bar{R}Q^\star\psi$$

$$- Q^\star\bar{R}' \sum_{t,r=1}^{T-1} D^tD^{r'}\bar{R}Q^\star(tr\{\Omega\Gamma^{r-1}Q_Z^\star\Gamma^{t-1'}\}.I)\psi$$

$$- Q^\star\bar{R}' \sum_{t,r=1}^{T-1} D^tD^r\bar{R}Q^\star I_1^{\star'}\Omega\Gamma^{t-1}Q_Z^\star\Gamma^{r-1'}\varphi$$

$$- Q^\star(tr\{\bar{Z}Q_Z\bar{Z}'\bar{R}Q^\star\bar{R}'\}.I)\psi$$

$$- Q^\star \sum_{r,t=1}^{T-1}(tr\{\Gamma^{r-1'}\Omega\Gamma^{t-1}Q_Z^\star\}.I)(tr\{D^tD^{r'}\bar{R}Q^\star\bar{R}'\}.I)\psi$$

$$
\begin{aligned}
&- Q^\star I_1^{\star'} \sum_{r,t=1}^{T-1} \Omega\Gamma^{r-1} Q_Z^\star \Gamma^{t-1'} (tr\{D^t \bar{R} Q^\star R' D^r\}.I)\varphi \\
&- Q^\star A' \sum_{r,t=1}^{T-1} \Gamma^{t-1'} \Omega\Gamma^{r-1} I_2^{\star'} Q_Z \bar{Z}' D^t D^{r'} \bar{R} Q^\star \psi \\
&- Q^\star A' \sum_{r,t=1}^{T-1} \Gamma^{t-1'} \Omega I_1^\star Q^\star \bar{R}' D^{r'} D^{t'} \bar{Z} Q_Z I_2^\star \Gamma^{r-1'} \varphi \\
&- Q^\star A' \sum_{t,r=1}^{T-1} \Gamma^{t-1'} (tr\{D^{t'} D^r \bar{Z} Q_Z I_2^\star \Gamma^{r-1'} \Omega I_1^\star Q^\star \bar{R}'\}.I)\varphi \\
&- Q^\star A' Q_W I_2^{\star'} Q_Z \bar{Z}' \bar{R} Q^\star \psi \\
&- Q^\star A' \sum_{t,r=1}^{T-1} \Gamma^{t-1'} \Omega I_1^\star Q^\star \bar{R}' D^{r'} D^t \bar{Z} Q_Z I_2^\star \Gamma^{r-1'} \varphi \\
&- Q^\star \sum_{t,r=1}^{T-1} (tr\{D^t D^{r'} \bar{Z} Q_Z I_2^\star \Gamma^{t-1'} \Omega\Gamma^{r-1} A Q^\star \bar{R}'\}.I)\psi \\
&- Q^\star \sum_{t,r=1}^{T-1} I_1^{\star'} \Omega\Gamma^{r-1} A Q^\star \bar{R}' D^{t'} D^{r'} \bar{Z} Q_Z I_2^\star \Gamma^{t-1'} \varphi \\
&- Q^\star \bar{R}' \bar{Z} Q_Z I_2^\star Q_W A Q^\star \psi \\
&- Q^\star (tr\{Q_W I_2^{\star'} Q_Z \bar{Z}' \bar{R} Q^\star A'\}.I)\psi \\
&- Q^\star A' \sum_{r,t=1}^{T-1} \Gamma^{t-1'} (tr\{\Omega I_1^\star Q^\star A' \Gamma^{r-1'}\} tr\{D^{t'} \bar{Z} Q_Z \bar{Z}' D^{r'}\}.I)\varphi \\
&- Q^\star A' \sum_{r,t=1}^{T-1} \Gamma^{t-1'} (tr\{D^{t'} \bar{Z} Q_Z \bar{Z}' D^r\}.I) \Omega\Gamma^{r-1} A Q^\star \psi \\
&- Q^\star A' Q_W Q_Z^\star Q_W A Q^\star \psi \\
&- Q^\star \sum_{r,t,s=1}^{T-1} A' \Gamma^{t-1'} \Omega\Gamma^{s-1} A Q^\star A' \Gamma^{r-1'} \{tr(D^{t'} D^r D^s).I\}\psi \\
&- Q^\star \sum_{r,t,s=1}^{T-1} A' \Gamma^{t-1'} tr(\Omega\Gamma^{r-1} A Q^\star A' \Gamma^{s-1'}) tr(D^{t'} D^r D^{s'})\psi
\end{aligned}
$$

$$
\begin{aligned}
&- Q^\star \sum_{r,t=1}^{T-1} (tr\{\Omega\Gamma^{t-1} A Q^\star A' \Gamma^{r-1'}\}.I)(tr\{\bar{Z} Q_Z \bar{Z}' D^t D^{r'}\}.I)\psi \\
&- Q^\star I_1^{\star'} \sum_{r,t=1}^{T-1} \Omega\Gamma^{r-1} A Q^\star A' \Gamma^{t-1'} (tr\{\bar{Z} Q_Z \bar{Z}' D^t D^r\}.I)\varphi \\
&- Q^\star (tr\{Q_W Q_Z^\star Q_W A Q^\star A'\}.I)\psi
\end{aligned}
$$

The terms are then rearranged slightly.

Appendix B

Note 1

$$
\begin{aligned}
&E[\bar{V}\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}\begin{pmatrix}\Gamma^{r-1\prime}\bar{V}'D^{r\prime}\\ 0\end{pmatrix}\bar{V}I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \\
&= E[\bar{u}_1\phi'\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1\prime}\phi\bar{u}_1'D^{r\prime}\bar{u}_1\phi'I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \qquad (32)\\
&+ E[S\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1\prime}S'D^{r\prime}\bar{u}_1\phi'I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \qquad (33)\\
&+ E[\bar{u}_1\phi'\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1\prime}S'D^{r\prime}SI_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \qquad (34)\\
&+ E[S\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1\prime}\phi\bar{u}_1'D^{r\prime}SI_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \qquad (35)
\end{aligned}
$$

(32) is calculated as follows:

$$
\begin{aligned}
&E[\bar{u}_1\bar{u}_1'\bar{R}Q^{\star\prime}I_1^{\star\prime}\phi\phi'\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}I_2^{\star}\begin{pmatrix}\Gamma^{r-1\prime}\\ 0\end{pmatrix}\phi\bar{u}_1'D^{r\prime}\bar{u}_1] \\
&= \sigma^4\{tr(D^r).I + D^r + D^{r\prime}\}\bar{R}Q^{\star\prime}I_1^{\star\prime}\phi\phi'\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1\prime}\phi \\
&= \sigma^4\{D^r + D^{r\prime}\}\bar{R}Q^{\star\prime}I_1^{\star\prime}\phi\phi'\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1\prime}\phi
\end{aligned}
$$

(33) is calculated as follows:

$$
\begin{aligned}
&E[S\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}\begin{pmatrix}\Gamma^{r-1\prime}\\ 0'\end{pmatrix}S'D^{r\prime}\bar{u}_1\bar{u}_1'\bar{R}Q^{\star\prime}I_1^{\star\prime}\phi] \\
&= (tr\{C_2^{\star}\Gamma^{t-1}I_2^{\star\prime}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1\prime}\}D^{r\prime}\sigma^2\bar{R}Q^{\star\prime}I_1^{\star\prime}\phi,
\end{aligned}
$$

using $SNS' = tr(C_2^{\star}N).I$ from Lemma 1.
(34) is calculated as follows:

$$
\begin{aligned}
&E[\bar{u}_1\bar{u}_1'\bar{R}Q^{\star\prime}I_1^{\star\prime}S'D^rS\begin{pmatrix}\Gamma^{r-1\prime}\\ 0\end{pmatrix}'(E[Z'Z])^{-1}I_2^{\star}\Gamma^{t-1\prime}\phi] \\
&= \sigma^2\bar{R}Q^{\star}I_1^{\star\prime}E[S'D^rS]\Gamma^{r-1}I_2^{\star\prime}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{t-1\prime}\phi \\
&= \sigma^2\bar{R}Q^{\star}I_1^{\star\prime}\{tr(D^r)\}C_2^{\star}\Gamma^{r-1}I_2^{\star\prime}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{t-1\prime}\phi
\end{aligned}
$$

using $E[S'NS] = \{tr(N)\}.C_2^{\star}$ from Lemma 1.
(35) can be written as follows, recalling that S and $\bar{u}_1$ are independent, and recalling from Lemma 1 that $SNS = N'C_2^{\star}$:

$$
E[SD^r\bar{u}_1\phi'\Gamma^{r-1}I_2^{\star\prime}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{t-1\prime}C_2^{\star}I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1].
$$

Then

$$
\begin{aligned}
&E[SD^r\bar{u}_1\phi'\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{t-1'}C_2^\star I_1^\star Q^\star\bar{R}'\bar{u}_1]\\
&= E[D^r\bar{u}_1\bar{u}_1'\bar{R}Q^\star I_1^{\star'}C_2^\star\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\phi]\\
&= \sigma^2 D^r\bar{R}Q^\star I_1^{\star'}C_2^\star\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\phi\\
&= \sigma^2 D^r\bar{R}Q^\star I_1^{\star'}\Omega\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\phi\\
&- \sigma^4 D^r\bar{R}Q^\star I_1^{\star'}\phi\phi'\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\phi
\end{aligned}
$$

Putting these together gives

$$
\begin{aligned}
E[\bar{V}\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\begin{pmatrix}\Gamma^{r-1'}\bar{V}'D^{r'}\\ 0\end{pmatrix}\bar{V}I_1^\star Q^\star\bar{R}'\bar{u}_1]&\\
= \sigma^2 D^{r'}\bar{R}Q^\star I_1^{\star'}\phi tr\{\Omega\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{t-1'}\}&\\
+ \sigma^2 D^r\bar{R}Q^\star I_1^{\star'}\Omega\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\phi,&
\end{aligned}
$$

Note 2

$$
\begin{aligned}
&E[\bar{V}'D^t\bar{V}\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\bar{V}'D^{r'}\bar{R}Q^\star\bar{R}'\bar{u}_1]\\
&= E[\phi\bar{u}_1'D^t\bar{u}_1\phi'\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{R}Q^\star\bar{R}'\bar{u}_1] \quad (36)\\
&\quad + E[\phi\bar{u}_1'D^tS\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}S'D^{r'}\bar{R}Q^\star\bar{R}'\bar{u}_1] \quad (37)\\
&\quad + E[S'D^t\bar{u}_1\phi'\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}S'D^{r'}\bar{R}Q^\star\bar{R}'\bar{u}_1] \quad (38)\\
&\quad + E[S'D^tS\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{R}Q^\star\bar{R}'\bar{u}_1] \quad (39)
\end{aligned}
$$

(36) becomes

$$
\phi\phi'\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}P_2'\Gamma^{r-1'}\phi tr\{(D^t+D^{t'})D^{r'}\bar{R}Q^\star\bar{R}'\}
$$

(37) becomes

$$
\begin{aligned}
&\sigma^2 tr\{\Omega\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\}tr\{D^tD^{r'}\bar{R}Q^\star\bar{R}'\}\phi\\
&- \sigma^4 tr\{\phi\phi'\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\}tr\{D^tD^{r'}\bar{R}Q^\star\bar{R}'\}\phi
\end{aligned}
$$

(38) becomes

$$
\begin{aligned}
&\sigma^2 tr(D^t\bar{r}Q^\star\bar{R}'D^r)\Omega\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{t-1'}\phi\\
&- \sigma^4 tr(D^t\bar{r}Q^\star\bar{R}'D^r)\phi\phi'\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^\star\Gamma^{t-1'}\phi
\end{aligned}
$$

(39) becomes zero, and putting these together gives

$$E[\bar{V}'D^t\bar{V}\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\bar{V}'D^{r'}\bar{R}Q^{\star}\bar{R}'\bar{u}_1]$$
$$= \sigma^2 tr\{\Omega\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\}tr\{D^tD^{r'}\bar{R}Q^{\star}\bar{R}'\}\phi$$
$$+ \sigma^2 tr\{D^t\bar{R}Q^{\star}R'D^r\}\Omega\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{t-1'}\phi$$

Note 3

$$E[\bar{V}'D^{t'}\bar{Z}(E[Z'Z])^{-1}\begin{pmatrix}\Gamma^{r-1'}\\ 0'\end{pmatrix}\bar{V}'D^{r'}\bar{V}I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1]$$
$$= E[\phi\bar{u}_1'D^{t'}\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1\phi'I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \quad (40)$$
$$+ E[\phi\bar{u}_1'D^{t'}\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}S'D^{r'}SI_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \quad (41)$$
$$+ E[S'D^{t'}\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}S'D^{r'}\bar{u}_1\phi'I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \quad (42)$$
$$+ E[S'D^{t'}\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}SI_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \quad (43)$$

(40) becomes

$$\sigma^4\phi'I_1^{\star}Q^{\star}\bar{R}'D^rD^{t'}\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\phi\phi'$$
$$+\sigma^4\phi'I_1^{\star}Q^{\star}\bar{R}'D^{r'}D^t\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\phi\phi'$$

(41) is zero
(42) becomes

$$\sigma^2\Omega\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}Z'D^tD^{r'}\bar{R}Q^{\star}I_1^{\star'}\phi$$
$$- \sigma^4\phi\phi'I_1^{\star}Q^{\star}\bar{R}'D^rD^{t'}\bar{Z}(EZ'Z)^{-1}I_2^{\star}\Gamma^{t-1'}\phi$$

(43) becomes

$$\sigma^2\Omega I_1^{\star}Q^{\star}\bar{R}D^{r'}D^{t'}\bar{Z}(EZ'Z)^{-1}I_2^{\star}\Gamma^{r-1'}\phi$$
$$- \sigma^4\phi\phi'I_1^{\star}Q^{\star}\bar{R}D^{r'}D^{t'}\bar{Z}(EZ'Z)^{-1}I_2^{\star}\Gamma^{r-1'}\phi.$$

These reduce to

$$E[\bar{V}'D^{t'}\bar{Z}(E[Z'Z])^{-1}\begin{pmatrix}\sum_{r=1}^{T-1}\Gamma^{r-1'}\bar{V}'D^{r'}\\ 0'\end{pmatrix}\bar{V}I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1]$$
$$= \Omega\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'D^tD^{r'}\bar{R}Q^{\star}I_1^{\star'}(\sigma^2\phi)$$
$$+ \Omega I_1^{\star}Q^{\star}\bar{R}'D^{r'}D^{t'}\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}(\sigma^2\phi)$$

Note 4

$$
\begin{aligned}
&E[\bar{V}'D^{t'}D^r\bar{V}\Gamma^{r-1'}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{V}I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \\
&= E[\phi\bar{u}_1'D^{t'}D^r\bar{u}_1\phi'\Gamma^{r-1'}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{u}_1\phi'I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \qquad (44)\\
&\quad + E[\phi\bar{u}_1'D^{t'}D^rS\Gamma^{r-1'}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'SI_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \qquad (45)\\
&\quad + E[S'D^{t'}D^rS\Gamma^{r-1'}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{u}_1\phi'I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \qquad (46)\\
&\quad + E[S'D^{t'}D^r\bar{u}_1\phi'\Gamma^{r-1'}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'SI_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \qquad (47)
\end{aligned}
$$

(44) becomes

$$
\begin{aligned}
&\sigma^4\phi tr[\frac{1}{2}\{D^{t'}D^r + D^{r'}D^t\}]tr\{\bar{R}Q^{\star}I_1^{\star'}\phi\phi'\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\} \\
&+ 2\sigma^4\phi tr[\frac{1}{2}\{D^{t'}D^r + D^{r'}D^t\}\bar{R}Q^{\star}I_1^{\star'}\phi\phi'\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}']
\end{aligned}
$$

(45) becomes

$$
\begin{aligned}
&\sigma^2\phi tr[D^{t'}D^r\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\Omega I_1^{\star}Q^{\star}\bar{R}'] \\
&- \sigma^4\phi tr[D^{t'}D^r\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\phi\phi'I_1^{\star}Q^{\star}\bar{R}']
\end{aligned}
$$

(46) becomes

$$
\begin{aligned}
&\sigma^2 tr\{D^{t'}D^r\}\Omega\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^{\star}I_1^{\star'}\phi \\
&- \sigma^4 tr\{D^{t'}D^r\}\phi\phi'\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^{\star}I_1^{\star'}\phi
\end{aligned}
$$

(47) becomes

$$
\begin{aligned}
&\sigma^2\Omega I_1^{\star}Q^{\star}\bar{R}'D^{r'}D^t\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\phi \\
&- \sigma^2\phi\phi'\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'D^{t'}D^r\bar{R}Q^{\star}I_1^{\star}\phi
\end{aligned}
$$

Putting these together gives

$$
\begin{aligned}
&E[\bar{V}'D^{t'}D^r\bar{V}\Gamma^{r-1'}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{V}I_1^{\star}Q^{\star}\bar{R}'\bar{u}_1] \\
&\qquad = (tr\{D^{t'}D^r\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}\Omega I_1^{\star}Q^{\star}\bar{R}'\}.I)(\sigma^2\phi) \\
&\qquad\quad + \Omega\Gamma^{r-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^{\star}(tr\{D^{t'}D^r\}.I)I_1^{\star'}\sigma^2\phi \\
&\qquad\quad + \Omega I_1^{\star}Q^{\star}\bar{R}'D^{r'}D^t\bar{Z}(E[Z'Z])^{-1}I_2^{\star}\Gamma^{r-1'}(\sigma^2\phi)
\end{aligned}
$$

Note 5

$$
\begin{aligned}
&E[\bar{V}'(D^t\bar{V}\Gamma^{t-1}:0)(E[Z'Z])^{-1}\bar{Z}'D^r\bar{V}\Gamma^{r-1}AQ^\star\bar{R}\bar{u}_1] \\
&= E[\phi\bar{u}_1'D^t\bar{u}_1\phi'\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'D^r\bar{u}_1\phi'\Gamma^{r-1}AQ^\star\bar{R}\bar{u}_1] \qquad (48)\\
&\quad + E[\phi\bar{u}_1'D^tS\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'D^rS\Gamma^{r-1}AQ^\star\bar{R}\bar{u}_1] \qquad (49)\\
&\quad + E[S'D^tS\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'D^r\bar{u}_1\phi'\Gamma^{r-1}AQ^\star\bar{R}\bar{u}_1] \qquad (50)\\
&\quad + E[S'D^t\bar{u}_1\phi'\Gamma^{t-1}I_2^\star(E[Z'Z])^{-1}\bar{Z}'D^rS\Gamma^{r-1}AQ^\star\bar{R}\bar{u}_1] \qquad (51)
\end{aligned}
$$

(48) becomes

$$2\phi\sigma^4 tr\{\frac{1}{2}(D^t+D^{r'})D^{r'}\bar{Z}(E[Z'Z])^{-1}I_2^\star\Gamma^{t-1'}\phi\phi'\Gamma^{r-1}AQ^\star\bar{R}'\}$$

(49) becomes

$$
\begin{aligned}
&\sigma^2\phi tr\{D^tD^{r'}\bar{Z}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\Omega\Gamma^{r-1}AQ^\star\bar{R}'\} \\
&\qquad - \sigma^4\phi tr\{D^tD^{r'}\bar{Z}(E[Z'Z])^{-1}I_2^\star\Gamma^{r-1'}\phi\phi'\Gamma^{r-1}AQ^\star\bar{R}'\}
\end{aligned}
$$

(50) is zero
(51) becomes

$$
\begin{aligned}
&\sigma^2\Omega\{D^t\bar{R}Q^\star A'\Gamma^{r-1'}\}'D^{r'}\bar{Z}(E[Z'Z])^{-1}I_2^\star\Gamma^{t-1'}\phi \\
&- \sigma^2\phi\phi\{D^t\bar{R}Q^\star A'\Gamma^{r-1'}\}'D^{r'}\bar{Z}(E[Z'Z])^{-1}I_2^\star\Gamma^{t-1'}\phi
\end{aligned}
$$

using $E[W'N'W'] = C_2^\star N$ (see Lemma 1). Putting these together gives

$$
\begin{aligned}
&E[\bar{V}'D^t\bar{V}\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'D^r\bar{V}\Gamma^{r-1}AQ^\star\bar{R}\bar{u}_1] \\
&\qquad = -(tr\{D^tD^{r'}\bar{Z}(E[Z'Z])^{-1}I_2^\star\Gamma^{t-1'}\Omega\Gamma^{r-1}AQ^\star\bar{R}'\}.I)(\sigma^2\phi) \\
&\qquad\quad - \Omega\Gamma^{r-1}AQ^\star\bar{R}'\sigma^2D^{t'}D^{r'}\bar{Z}(E[Z'Z'])^{-1}I_2^\star\Gamma^{t-1'}(\sigma^2\phi)
\end{aligned}
$$

Note 6

$$
\begin{aligned}
&E[\bar{V}'D^{t'}\bar{V}I_1^\star Q^\star A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1] \\
&= E[\phi\bar{u}_1'D^{t'}\bar{u}_1\phi'I_1^\star Q^\star A'\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \qquad (52)\\
&\quad + E[\phi\bar{u}_1'D^{t'}SI_1^\star Q^\star A'\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \qquad (53)\\
&\quad + E[S'D^{t'}SI_1^\star Q^\star A'\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \qquad (54)
\end{aligned}
$$

$$+ E[S'D^{t'}\bar{u}_1\phi' I_1^\star Q^\star A'\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \quad (55)$$

(52) becomes

$$\sigma^4\phi\phi' I_1^\star Q^\star A'\Gamma^{r-1'}\phi[2tr\{\frac{1}{2}(D^t + D^{t'})D^{r'}\}]$$

(53) is zero
(54) is zero
(55) becomes

$$\sigma^2 tr(D^{t'}D^r)\Omega\Gamma^{r-1}AQ^\star I_1^{\star'}\phi$$
$$- \sigma^4 tr(D^{t'}D^r)\phi\phi'\Gamma^{r-1}AQ^\star I_1^{\star'}\phi$$

Putting these together gives

$$E[\bar{V}'D^{t'}\bar{V}I_1^\star Q^\star A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1] = (tr\{D^{t'}D^r\}.I)\Omega\Gamma^{r-1}AQ^\star I_1^{\star'}(\sigma^2\phi)$$

Note 7

$$E[\bar{V}D^t\bar{V}\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1]$$
$$= E[\phi\bar{u}_1'D^t\bar{u}_1\phi'\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \quad (56)$$
$$+ E[\phi\bar{u}_1'D^tS\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \quad (57)$$
$$+ E[SD^t\bar{u}_1\phi'\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \quad (58)$$
$$+ E[SD^tS\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \quad (59)$$

(56) becomes

$$\sigma^4\phi\phi'\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}\phi tr\{\frac{1}{2}(D^t + D^{t'})D^{r'}\}$$

(57) and (58) are zero.
(59) becomes

$$\sigma^2\phi tr(D^tD^{r'})tr\{\Omega\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}\}$$
$$- \sigma^4\phi tr(D^tD^{r'})tr\{\phi\phi'\Gamma^{t-1}I_2^{\star'}(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}\}$$

Putting these together gives

$$E[\bar{V}(D^t\bar{V}\Gamma^{t-1} : 0)(E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1]$$

$$= tr\{D^t D^{r'}\}(tr\{\Omega\Gamma^{t-1} I_2^{\star'} (E[Z'Z])^{-1}\bar{Z}'\bar{R}Q^\star A'\Gamma^{r-1'}\}.I)(\sigma^2\phi)$$

Note 8

$$E[\bar{V}'D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{V}I_1^\star Q^\star A' \sum_{r=1}^{T-1}\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1]$$

$$= E[\phi\bar{u}_1'D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{u}_1\phi' I_1^\star Q^\star A' \sum_{r=1}^{T-1}\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \quad (60)$$

$$+ E[\phi\bar{u}_1'D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'SI_1^\star Q^\star A' \sum_{r=1}^{T-1}\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \quad (61)$$

$$+ E[S'D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{u}_1\phi' I_1^\star Q^\star A' \sum_{r=1}^{T-1}\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \quad (62)$$

$$+ E[S'D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'SI_1^\star Q^\star A' \sum_{r=1}^{T-1}\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \quad (63)$$

(60) becomes

$$\sigma^4\phi\phi' I_1^\star Q^\star A'\Gamma^{r-1'}\phi tr\{(D^r + D^{r'})D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\}$$

(61) becomes

$$\sigma^2\phi tr(\Omega I_1^\star Q^\star A'\Gamma^{r-1'})tr\{D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^{r'}\}$$
$$- \sigma^4\phi tr(\phi\phi' I_1^\star Q^\star A'\Gamma^{r-1'})tr\{D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^{r'}\}$$

(62) becomes

$$\sigma^2 tr\{D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^r\}\Omega\Gamma^{r-1}AQ^\star I_1^{\star'}\phi$$
$$- \sigma^4 tr\{D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^r\}\phi\phi'\Gamma^{r-1}AQ^\star I_1^{\star'}\phi$$

(63) is zero. Putting these together gives

$$E[\bar{V}'D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'\bar{V}I_1^\star Q^\star A' \sum_{r=1}^{T-1}\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1]$$
$$= tr\{\Omega I_1^\star Q^\star A'\Gamma^{r-1'}\}(tr\{D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^{r'}\}.I)(\sigma^2\phi)$$

$$+ (tr\{D^{t'}\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^{r}\}.I)\Omega\Gamma^{r-1}AQ^{\star}I_1^{\star'}(\sigma^2\phi)$$

Note 9

$$\begin{aligned}
&E[\bar{V}'D^{t'}\bar{V}I_1^{\star}Q^{\star}A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1] \\
&= E[\phi\bar{u}_1'D^{t'}\bar{u}_1\phi'I_1^{\star}Q^{\star}A'\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \quad (64)\\
&\quad + E[\phi\bar{u}_1'D^{t'}SI_1^{\star}Q^{\star}A'\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \quad (65)\\
&\quad + E[S'D^{t'}\bar{u}_1\phi'I_1^{\star}Q^{\star}A'\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \quad (66)\\
&\quad + E[S'D^{t'}SI_1^{\star}Q^{\star}A'\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \quad (67)
\end{aligned}$$

(64) becomes

$$\sigma^4\phi\phi'I_1^{\star}Q^{\star}A'\Gamma^{r-1'}\phi tr\{(D^t + D^{t'})D^{r'}\}$$

(65) is zero
(66) becomes

$$\sigma^2 tr(D^{t'}D^r)\Omega\Gamma^{r-1}AQ^{\star}I_1^{\star'}\phi - tr(D^{t'}D^r)\phi\phi'\Gamma^{r-1}AQ^{\star}I_1^{\star'}\phi$$

(67) is zero.
Simplifying gives

$$E[\bar{V}'D^{t'}\bar{V}I_1^{\star}Q^{\star}A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1] = \Omega\Gamma^{r-1}AQ^{\star}(tr\{D^{t'}D^r\}.I)I_1^{\star'}(\sigma^2\phi)$$

Note 10

$$\begin{aligned}
&E[\bar{V}'D^{t'}D^r\bar{V}\Gamma^{r-1}AQ^{\star}A'\Gamma^{s-1'}\bar{V}'D^{s'}\bar{u}_1] \\
&= E[\phi\bar{u}_1'D^{t'}D^r\bar{u}_1\phi'\Gamma^{r-1}AQ^{\star}A'\Gamma^{s-1'}\phi\bar{u}_1'D^{s'}\bar{u}_1] \quad (68)\\
&\quad + E[\phi\bar{u}_1'D^{t'}D^rS\Gamma^{r-1}AQ^{\star}A'\Gamma^{s-1'}S'D^{s'}\bar{u}_1] \quad (69)\\
&\quad + E[S'D^{t'}D^r\bar{u}_1\phi'\Gamma^{r-1}AQ^{\star}A'\Gamma^{s-1'}S'D^{s'}\bar{u}_1] \quad (70)\\
&\quad + E[S'D^{t'}D^rS\Gamma^{r-1}AQ^{\star}A'\Gamma^{s-1'}\phi\bar{u}_1'D^{s'}\bar{u}_1] \quad (71)
\end{aligned}$$

(68) becomes

$$\sigma^4\phi\phi'\Gamma^{r-1}AQ^{\star}A'\Gamma^{s-1'}\phi tr\{(D^s + D^{s'})D^{t'}D^r\}.$$

(69) becomes

$$\sigma^2\phi tr(\Omega\Gamma^{r-1}AQ^{\star}A'\Gamma^{s-1'})tr(D^{t'}D^rD^{s'})$$

$$-\sigma^4\phi tr(\phi\phi'\Gamma^{r-1}AQ^\star A'\Gamma^{s-1'})tr(D^{t'}D^rD^{s'}).$$

(70) becomes

$$tr(D^{t'}D^rD^s)\Omega\Gamma^{s-1}AQ^\star A'\Gamma^{r-1'}\sigma^2\phi$$
$$-\sigma^4 tr(D^{t'}D^rD^s)\phi\phi'\Gamma^{s-1}AQ^\star A'\Gamma^{r-1'}\phi.$$

(71) is zero.

Therefore

$$E[\bar{V}'D^{t'}D^r\epsilon\Gamma^{r-1}AQ^\star A'\Gamma^{s-1'}\bar{V}'D^{s'}\bar{u}_1] =$$
$$\Omega\Gamma^{s-1}AQ^\star A'\Gamma^{r-1'}\{tr(D^{t'}D^rD^s).I\}(\sigma^2\phi)$$
$$+tr(\Omega\Gamma^{r-1}AQ^\star A'\Gamma^{s-1'})tr(D^{t'}D^rD^{s'})(\sigma^2\phi)$$

Note 11

$$E[\bar{V}'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^t\bar{V}\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1]$$
$$= E[\phi\bar{u}_1'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^t\bar{u}_1\phi'\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \quad (72)$$
$$+E[\phi\bar{u}_1'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^tS\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \quad (73)$$
$$+E[S'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^t\bar{u}_1\phi'\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \quad (74)$$
$$+E[S'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^tS\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \quad (75)$$

(72) becomes

$$\sigma^4\phi\phi'\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}\phi tr\{(D^r+D^{r'})\bar{Z}(E[Z'Z])^{-1}\bar{Z}D^t\}$$

(73) becomes

$$\sigma^2\phi tr(\Omega\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'})tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^tD^{r'}\}$$
$$-\sigma^4\phi tr(\phi\phi'\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'})tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^tD^{r'}\}$$

(74) becomes

$$\sigma^2 tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^tD^r\}\Omega\Gamma^{r-1}AQ^\star A'\Gamma^{t-1'}\phi$$
$$-\sigma^4 tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^tD^r\}\phi\phi'\Gamma^{r-1}AQ^\star A'\Gamma^{t-1'}\phi$$

(75) is zero.
Simplifying gives

$$E[\bar{V}'\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^t\bar{V}\Gamma^{t-1}AQ^\star A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1]$$

$$
\begin{aligned}
&= (tr\{\Omega\Gamma^{t-1}AQ^{\star}A'\Gamma^{r-1'}\}.I)(tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^tD^{r'}\}.I)(\sigma^2\phi) \\
&\quad + \Omega\Gamma^{r-1}AQ^{\star}A'\Gamma^{t-1'}(tr\{\bar{Z}(E[Z'Z])^{-1}\bar{Z}'D^tD^r\}.I)(\sigma^2\phi)
\end{aligned}
$$

Note 12

$$E[\bar{V}'(D^t\bar{V}\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star'}E[\bar{W}'L'L\bar{W}]AQ^{\star}A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1]$$

$$= E[\bar{u}_1\phi'(D^t\bar{u}_1\phi'\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star'}E[\bar{W}'L'L\bar{W}]AQ^{\star}A'\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \tag{76}$$

$$+ E[\bar{u}_1\phi'(D^tS\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star'}E[\bar{W}'L'L\bar{W}]AQ^{\star}A'\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \tag{77}$$

$$+ E[S'(D^tS\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star'}E[\bar{W}'L'L\bar{W}]AQ^{\star}A'\Gamma^{r-1'}\phi\bar{u}_1'D^{r'}\bar{u}_1] \tag{78}$$

$$+ E[S'(D^t\bar{u}_1\phi'\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star'}E[\bar{W}'L'L\bar{W}]AQ^{\star}A'\Gamma^{r-1'}S'D^{r'}\bar{u}_1] \tag{79}$$

(76) becomes

$$\sigma^4\phi\phi'(\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star}E[W'L'LW]AQ^{\star}A'\Gamma^{r-1'}\phi tr\{(D^t+D^{t'})D^{r'}\}$$

(77) becomes

$$
\begin{aligned}
&\sigma^2\phi tr\{\Omega(\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star}E[W'L'LW]AQ^{\star}A'\Gamma^{r-1'}\}tr(D^tD^{r'}) \\
&- \sigma^4\phi tr\{\phi\phi'(\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star}E[W'L'LW]AQ^{\star}A'\Gamma^{r-1'}\}tr(D^tD^{r'})
\end{aligned}
$$

(78) is zero.
(79) will have $tr(D^tD^r)$ as a factor and is therefore zero.
Simplifying gives

$$
\begin{aligned}
&E[\bar{V}'(D^t\bar{V}\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star'}E[\bar{W}'L'L\bar{W}]AQ^{\star}A'\Gamma^{r-1'}\bar{V}'D^{r'}\bar{u}_1] \\
&\qquad = tr\{\Omega(\Gamma^{t-1}:0)(E[Z'Z])^{-1}I_2^{\star}E[W'L'LW]AQ^{\star}A'\Gamma^{r-1'}\}(tr\{D^tD^{r'}\}.I)(\sigma^2\phi)
\end{aligned}
$$

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