Jones Polynomial Invariants for Knots and Satellites

H.R.Morton and P.Strickland

Department of Pure Mathematics
University of Liverpool
PO Box 147
Liverpool L69 3BX.

Abstract.

Results of Kirillov and Reshetikhin on constructing invariants of framed links from the quantum group $SU(2)_q$ are adapted to give a simple formula relating the invariants for a satellite link to those of the companion and pattern links used in its construction. The special case of parallel links is treated first. It is shown as a consequence that any $SU(2)_q$ invariant of a link L is a linear combination of Jones polynomials of parallels of L, where the combination is determined explicitly from the representation ring of SU_2. As a simple illustration Yamada's relation between the Jones polynomial of the 2-parallel of L and an evaluation of Kauffman's polynomial for sublinks of L is deduced.

0 Introduction

In this paper we shall use methods from the paper of Kirillov and Reshetikhin [7] to calculate invariants arising from the quantum group $SU(2)_q$ for links which are constructed as satellites from given companion and pattern links. The methods introduced in [7] allow an explicit tangle-based description, following the ideas of Turaev [16]. Explicit tangle representations were constructed independently by Kluitmann [6], also following the ideas of Turaev, directly from Drinfeld's discussion of $SU(2)_q$[1], while a wider theoretical framework is given by Reshetikhin in [12] and [13], for quantum groups based on a general simple Lie algebra. The general case has been followed up in the tangle setting by Reshetikhin and Turaev [14].

To begin with, we review some of the results of [7], and adapt them to relate invariants of parallels of a link to invariants of the original link. The definitions for the invariants in [7] are based on assignment of irreducible $SU(2)_q$-modules to the components of a link. We show in section 2 that if the definition is extended multilinearly to allow sums or formal linear combinations of modules on each component then the invariant defined by taking the tensor product of two modules on one component, L_1 say, of a link L is the same as the invariant of a simple parallel of L in which two parallel copies of L_1 are used, with the two modules assigned separately to the two copies. We can then relate invariants of general parallels of L to invariants of L itself, using the known decomposition of tensor products, and multilinearity to calculate them in terms of the irreducible module invariants for L. As a consequence, we shall show how knowledge of the bracket polynomial of all possible parallels of a link L is equivalent to knowing all the $SU(2)_q$-invariants for L. In particular, we can readily deduce the result due to Yamada [17], on the Jones polynomial of 2-parallels.

Having done this, we shall present the results on satellite knots in a form which is strikingly reminiscent of the classical Fox formula for Alexander polynomials, and which gives a key role to the Hopf link and its invariants. We give explicit finiteness conditions limiting the number of invariants of the companion and pattern which are needed in calculating a given invariant for the satellite.

1 Review of work of Kirillov and Reshetikhin

In the papers [12] and [13], the foundations of methods to calculate regular isotopy invariants of link diagrams corresponding to finite dimensional representations of quantum groups are set out. The idea is to set up a functor relating tangles to module homomorphisms, following a scheme of Turaev as set out in a talk at the Sussex meeting in 1987, see [14] and [5]. In this scheme oriented tangles correspond to homomorphisms of finite dimensional modules over a given quantum group, in such a way that the natural tangle operations of composition and 'horizontal' adjoining correspond to composition and tensor products of modules. (In order to give meaning to the tensor products, it is essential to use the coproduct for the Hopf algebra structure of the quantum group). Additional relations, including the Yang Baxter equation, allow us to use this correspondence to define link invariants, either by considering $(0,0)$ or $(1,1)$ tangles; this will be made clear later.

The paper [7] takes the programme outlined above, and fills in many details for the case of the quantum group $SU(2)_q$. For any quantum group arising from a simple Lie algebra, it has been proved, [15], that the representations (which will always be taken to be finite dimensional from now on) are in 1-1 correspondence with those of the original Lie algebra. In the case of $su(2)$, there is one representation in every dimension, indexed in [7] by half-integer 'spins'.

Turaev's general construction may be described for $SU(2)_q$ as follows. For each tangle diagram T we assign a finite dimensional irreducible representation of $SU(2)_q$ to each string, or in other words an irreducible module over $SU(2)_q$. We associate a $SU(2)_q$ module to the top of the tangle T by taking the tensor product of the modules assigned to the strings, in order, which meet the top of T. We can similarly associate a tensor product of modules with the bottom of T using the modules of the strings which meet the bottom. The goal is to then produce a module homomorphism, determined by T and the choice of modules on the strings, from the top module to the bottom module, as indicated in figure 1.1.

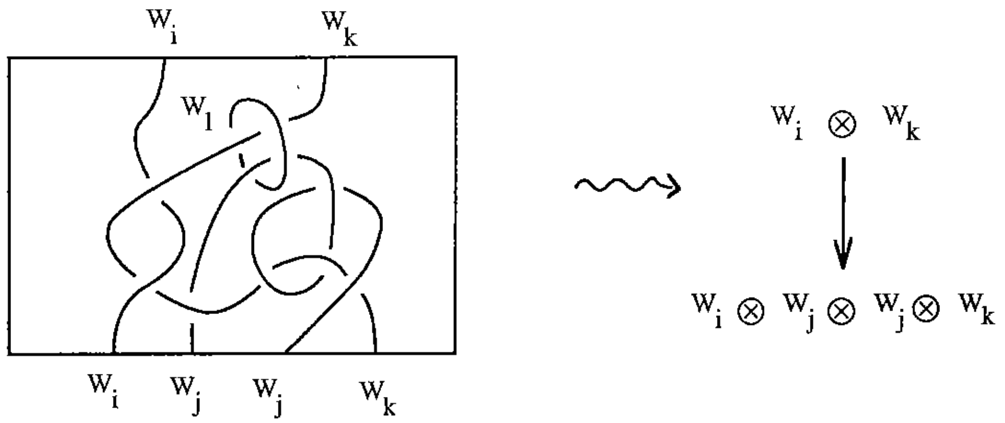

Figure 1.1

These homomorphisms will be chosen functorially so that when two tangles S and T are composed to form the tangle ST, and the assignment of modules to strings is compatible, then the module homomorphisms for S and T compose to give the homomorphism for ST. We also arrange that when tangles are adjoined side by side the homomorphism for the new tangle is the tensor product of the homomorphisms for its constituents. In this general framework the module associated to the top of a tangle where *no* strings are incident is the trivial module, which may be equated with the ground ring or field Λ. In what follows, we take Λ to contain at least the Laurent polynomial ring $\mathbb{Z}[q^{\pm\frac{1}{4}}]$, where the deformation parameter q in the quantum group is regarded as an indeterminate. We may need to extend Λ to contain the field of fractions of this ring, but most of the representations which are used can be chosen with bases so that representing matrices involve only the polynomial ring, and in particular, the link invariants lie in this ring.

Definition of the homomorphisms.

We shall assume that the tangle diagram lies in general position relative to the height function measuring distance above the bottom of the containing rectangle. Thus we assume that there are only a finite number of levels in the tangle where critical points of the diagram occur, being either a local maximum or minimum of the height function, or an apparent simple double point, where two strings cross. We assume also that the critical points occur in distinct levels. We may then decompose the original diagram level by level as the composite of a number of elementary tangles each containing just one critical point. In each elementary tangle all but two of the strings will pass from the top to the bottom without crossing, while the rest of the tangle consists of one of the four tangles shown in figure 1.2.

Figure 1.2

We define the homomorphism for a general tangle T as the composite of the homomorphisms for the elementary pieces. It then remains to define a homomorphism for each elementary tangle, and each assignment of irreducible modules to the strings. We shall write W_i for the irreducible module of dimension i. This is the module referred to in [7] by the half-integer spin $j = \frac{1}{2}(i-1)$. The construction for $SU(2)_q$ in [7], and similarly for other quantum groups, with a little alteration for local extrema, as in [12,13], is then completed simply by a choice of homomorphisms to be used for positive and negative crossings, and the local extrema, U and V. In the case of $SU(2)_q$ the homomorphisms needed for the different string assignments will be

$$
\begin{aligned}
B^{ij} &: W_i \otimes W_j \to W_j \otimes W_i && (+\text{ve crossings}),\\
\left(B^{ji}\right)^{-1} &: W_i \otimes W_j \to W_j \otimes W_i && (-\text{ve crossings}),\\
V_i &: \Lambda \to W_i \otimes W_i && (\frown),\\
U_i &: W_i \otimes W_i \to \Lambda && (\smile).
\end{aligned}
$$

Invariance of the homomorphisms.

The important result in the theory is that the homomorphisms for the elementary tangles can be chosen so that the homomorphism for a tangle T is unchanged when the tangle diagram is altered by any regular isotopy, i.e. any sequence of Reidemeister moves of types II and III leaving the boundary fixed.

In the definition of the homomorphism we have taken account of the height function on the tangle box, and so we must take note of how any regular isotopy of the diagram may affect the height decomposition into elementary tangles. As noted by Turaev, a regular isotopy between diagrams in general position can be arranged to pass through a finite number of critical positions. These occur either where two critical points in distant parts of the diagram interchange level, or where two or three critical points interfere locally. The local interference can be limited to a small number of cases, where the pairs of diagrams immediately before and after the critical position are illustrated in figure 1.3.

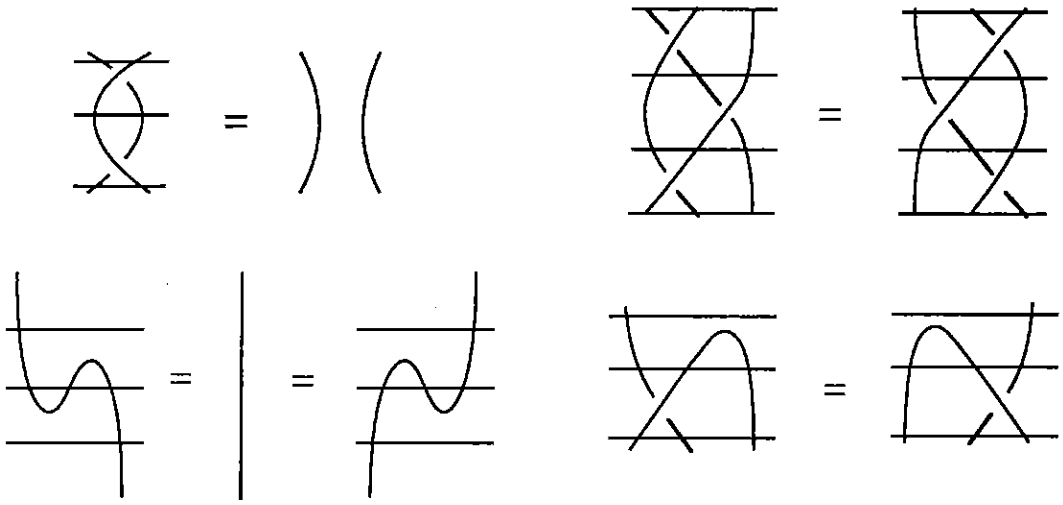

Figure 1.3

The first two of these are the familiar Reidemeister moves II and III. The third arises from creation or annihilation of local extrema, while the last type arises when a crossing moves across a local maximum.

Invariance of the homomorphism for a general tangle T under regular isotopy of T is then guaranteed by proving simply its invariance under these moves. Thus it is enough to show that the homomorphisms defined by the isotopic tangles in figure 1.3 are equal, for all assignments of modules. The horizontal lines in the figure emphasise the decomposition of each tangle into elementary tangles which is used to define the homomorphism.

The construction of B^{ij} in [7], and more generally in [12,13], uses a universal element

$$R \in SU(2)_q \otimes SU(2)_q$$

which automatically gives Reidemeister III invariance, from the Yang-Baxter equation, while Reidemeister II invariance follows from the use of a homomorphism and its inverse on the positive and negative crossings involved. The choice of U_i and V_i is determined up to a scalar, as there is a single trivial summand in the module $W_i \otimes W_i$

for each i, giving U_i by inclusion and V_i by projection. Invariance in the third case is simply a question of choosing the multiple for V_i given the choice for U_i, since the endomorphism of the irreducible module W_i defined by the diagram with the two local extrema must be a scalar multiple of the identity, by Schur's lemma. The fourth case can be shown to follow from the fact that V_i is a module homomorphism, using the universal element R. This is best done by proving the equivalent relation for homomorphisms defined by the tangles in figure 1.4. Full details of these homomorphisms, with explicit bases for the modules, and actions of $SU(2)_q$ are contained in [7].

Figure 1.4

With the invariance of the homomorphism under regular isotopy of the tangle now established, we adopt the term *invariant* of a tangle, with a given assignment of modules to its strings, to mean the homomorphism between the top and bottom modules. In particular, if we are given a link diagram, which we regard as a $(0,0)$-tangle, and make an assignment of modules to its component strings, then the invariant of this tangle will be an endomorphism of the ground ring Λ, which is simply multiplication by some element of Λ. We will call this element of Λ the invariant of the link, for the given assignment of modules to strings. For a given link we shall consider the collection of all these invariants, as the assignment of modules varies, to be the overall $SU(2)_q$-invariant of the link. It depends on the link up to regular isotopy, or equivalently on the link with a choice of framing on each component. We will write $J(L; W_{i_1}, W_{i_2}, \dots, W_{i_r})$ for the invariant of a framed link L where the module W_{i_k}, of dimension i_k is assigned to the kth component L_k.

In general, an endomorphism of any irreducible module will be some scalar multiple of the identity, by Schur's lemma. We then have the following result;

Theorem 1.1. *If we attach any irreducible module of $SU(2)_q$ to the end strings of a $(1,1)$-tangle, the invariant obtained will be a scalar multiple of the identity.*

Corollary 1.2. *Given two such $(1,1)$-tangles T_1, T_2 with invariants;*

$$\tau_p Id_{W_i} : W_i \to W_i, \quad \tau_p \in \Lambda, \ p = 1, 2,$$

the composite tangle $T_1 \circ T_2$ will have invariant $\tau_1 \tau_2 Id_{W_i}$.

This last result can be seen as a result on connected sums of knots and links; for example if the tangles each have a single string to which we assign the module W_2 then the scalar τ_p can be shown to be the bracket polynomial of the knot given by closing the tangle T_p to form a $(0,0)$-tangle. This result shows that the bracket polynomial of a connected sum of two knots is the product of the individual polynomials, provided that they are normalised with the unknot having polynomial 1. In general, if we were to close the tangle T_p to a $(0,0)$-tangle then the new invariant is the scalar τ_p

multiplied by the invariant δ_i of the simple unknotted curve with module W_i. The W_2 invariant of a closed link is then the bracket polynomial, but normalised so that the *empty* link has polynomial 1.

The invariant of the $(1,1)$-tangle consisting of a single positive curl, as shown in figure 1.5 will be some scalar multiple, f_i say, of the identity map on W_i when the module W_i is used on the string. For a negative curl, the scalar multiple of the identity is f_i^{-1}.

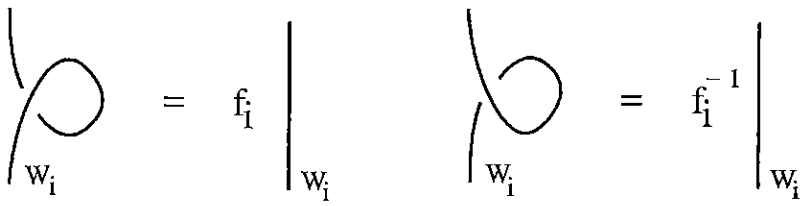

Figure 1.5

It follows that if we insert positive or negative curls in the diagram of a link to change the framing on a component with module W_i, then the effect on the invariant of the framed link is to multiply it by an appropriate power of this 'framing factor' f_i. Hence the invariants for the link with any other framing of its components can be readily found from its invariants with the originally given framing.

We note here the values for f_i as calculated in [7], and also the invariants δ_i for the unknot with crossing-free diagram, and module W_i, which can be calculated from the composite of U_i and V_i. These are:

$$\delta_i = (-1)^{i-1}\frac{s^i - s^{-i}}{s - s^{-1}} = (-1)^{i-1}\left(s^{i-1} + s^{i-3} + \ldots + s^{-i+1}\right)$$

$$f_i = (-1)^{i-1} s^{\frac{i^2-1}{2}}$$

Here $s = \sqrt{q}$; our treatment of right- and left-handed crossings differs from that of [7] at this point as we have replaced their q by q^{-1}. We may view $(-1)^{i-1}\delta_n = [n]_q$ as the q-integer of n; it is a symmetrised version of the Gauss q-integer $1+q+\ldots+q^{n-1}$.

Remark. Invariants for $SU(n)_q$, $n > 2$, can be constructed in a similar way, using oriented tangles. Then on a given string we must use either a module W or its conjugate W^* depending on whether the string is currently oriented from top to bottom, or from bottom to top. The homomorphisms for U and V have to involve the modules $W \otimes W^*$. A choice of scalar must be made at one point to ensure that the invariants δ_W and δ_{W^*} of the unknot oriented in either sense are equal, when using the module W. This ensures that no correction for the rotation number of the oriented diagram is required in passing to an invariant of a framed link.

Direct sum decompositions.

In the particular case of $SU(2)_q$, Jimbo [3] notes that the tensor product of two irreducible modules splits into the direct sum of distinct irreducibles as

$$W_i \otimes W_j \;\cong\; \bigoplus_k W_k, \quad |i-j| < k < i+j, \;\; k = i+j-1 \bmod 2.$$

We shall adopt the notation $k \in i \otimes j$ to mean that W_k is isomorphic to a summand of $W_i \otimes W_j$. Note that this is a symmetric relation for i, j and k, equivalent to their forming the sides of a triangle whose perimeter is odd. There are then inclusion and projection homomorphisms $Y_{i,j}^k : W_k \to W_i \otimes W_j$ and $Y_k^{i,j} : W_i \otimes W_j \to W_k$, defined uniquely up to multiplication by a scalar. These correspond to the classical Clebsch-Gordan coefficients.

Kirillov and Reshetikhin make a choice of these, so that

$$\sum_{k \in i \otimes j} Y_k^{i,j} Y_{i,j}^k = \mathrm{Id}(W_i \otimes W_j) \tag{1}$$

$$\text{and} \quad Y_{i,j}^k Y_k^{i,j} = \mathrm{Id}(W_k) \tag{2}$$

Of course

$$Y_{i,j}^k Y_\ell^{i,j} = 0, \ k \neq \ell, \tag{3}$$

by Schur's lemma. The endomorphisms

$$E_k^{i,j} = Y_k^{i,j} Y_{i,j}^k$$

of $W_i \otimes W_j$ are then orthogonal idempotents which decompose the identity.

Kirillov and Reshetikhin represent the homomorphisms Y diagramatically by 'Y'-shaped pieces of graph as shown in figure 1.6, and consider homomorphisms between modules which can be pictured using these fragments as well as the local maxima and minima and crossings discussed above.

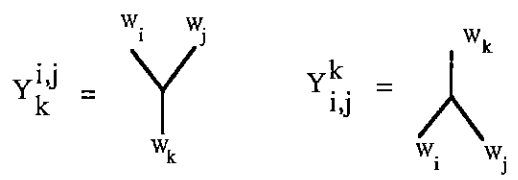

Figure 1.6

A labelled 'tangle-graph' is then allowed, in which strings may meet in 3-valent vertices, and each arc is labelled with some module W_i. As before, the tangle is viewed, with a height function, as representing a module homomorphism from a top module to a bottom module, where the homomorphisms $Y_{i,j}^k$, $Y_k^{i,j}$ are used at vertices of the graph. (We set $Y_{i,j}^k = 0$ if $k \notin i \otimes j$.) The relations (1)-(3) can then be written diagramatically as in figure 1.7.

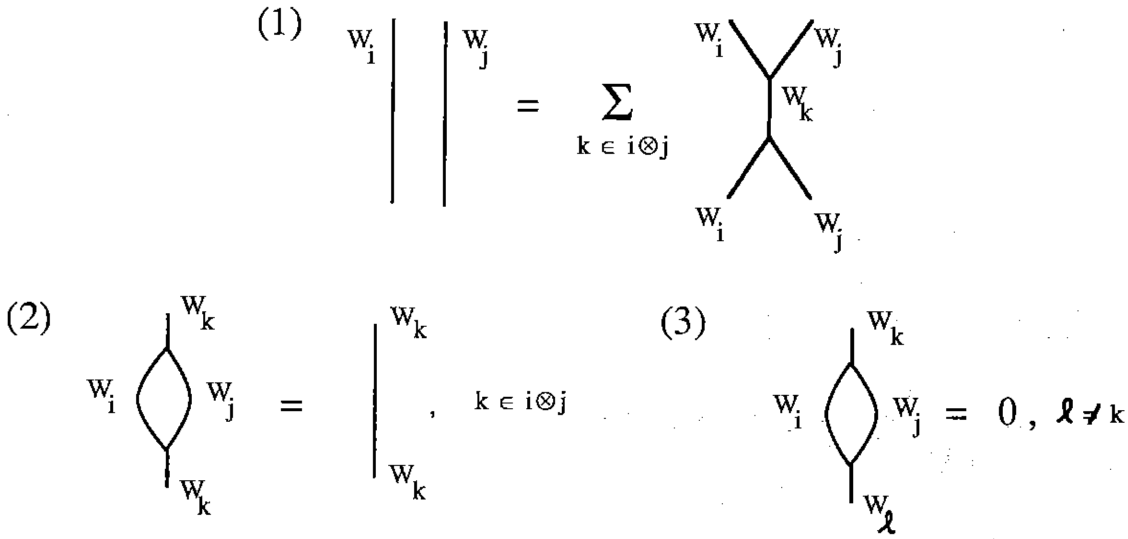

Figure 1.7

We can use relation (1) to start from a link diagram, and then 'fuse' two pieces of parallel string, so as to write the invariant of the link diagram as the sum of certain graph invariants. To make further use of this we shall show how the graph invariants themselves may sometimes be calculated in terms of other link invariants, by manipulation of the graph.

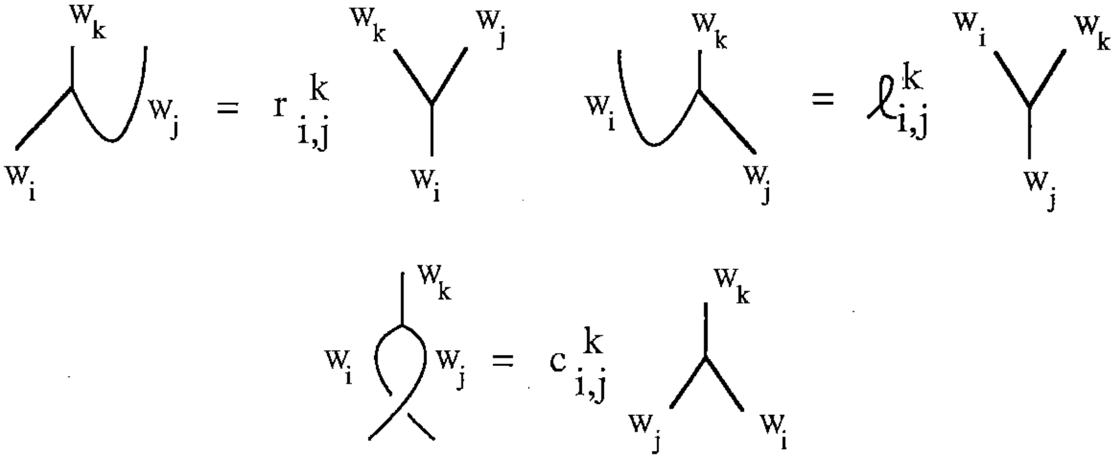

Figure 1.8

This needs just a little care, as the homomorphisms corresponding to a graph diagram will be altered by scalar multiples under certain natural changes of the diagram. In particular, scalars $r^k_{i,j}$, $\ell^k_{i,j}$, $c^k_{i,j}$ are required when $Y^k_{i,j}$ and U_i or B^{ij} are adjoined as shown in figure 1.8. However the vertices will pass unaltered across other strings, as shown in figure 1.9.

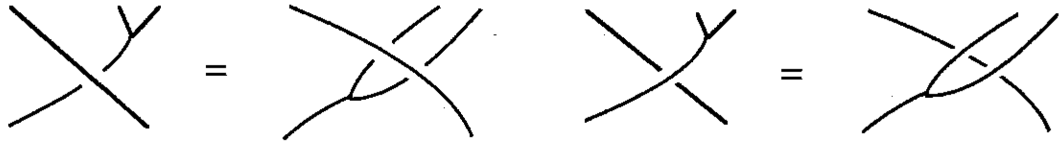

Figure 1.9

Kirillov and Reshetikhin give an explicit choice of basis for each irreducible module W_i, in which the homomorphisms B^{ij}, U_i and V_i are all represented by matrices with entries in $\mathbf{Z}[s^{\pm\frac{1}{2}}]$. There is freedom to alter the choice of each of the homomorphisms $Y^k_{i,j}$, and $Y^{i,j}_k$ by a scalar multiple; the only scalar determined by a particular choice for $Y^k_{i,j}$ is that for $Y^{i,j}_k$. Both U_i and $Y^{i,i}_1$ are homomorphisms from $W_i \otimes W_i$ to the scalars, so one will be a multiple of the other, as will V_i and $Y^1_{i,i}$. They cannot however be chosen to be the same, since $Y^1_{i,i}Y^{i,i}_1 = 1$ while the choice of V_i is dictated by U_i so as to satisfy the diagrammatic relation in figure 3, and we then have $V_iU_i = \delta_i$. In [7] the choice of scalars is made so that

$$\begin{aligned} U_i &= \sqrt{[i]_q}\,Y^{i,i}_1 \\ V_i &= \sqrt{[i]_q}\,Y^1_{i,i} \end{aligned}.$$

Matrices for $Y^{i,i}_1$ and generally $Y^{i,j}_k$ will then require entries in a ring extended to contain appropriate denominators.

The choice of scalar multiples implicit in [7] for $Y^k_{i,j}$ and $Y^k_{j,i}$ ensures that

$$c^k_{j,i} = c^k_{i,j} = \pm\sqrt{\frac{f_k}{f_if_j}}, \qquad \text{where the sign is} \quad (-1)^{\frac{1}{2}(i+j-k-1)}$$

although the only unavoidable relation is that $c^k_{i,j}c^k_{j,i} = \dfrac{f_k}{f_if_j}$. Their choice in determining $Y^k_{j,i}$ in relation to $Y^k_{i,j}$ and other permutations of the indices i,j,k ensures a measure of symmetry for the coefficients $r^k_{i,j}$, $\ell^k_{i,j}$, which appear when 'arms' of an elementary Y are moved up or down, and they have

$$r^k_{i,j} = \ell^k_{j,i} = (-1)^{\frac{1}{2}(k-i-j+1)}\sqrt{\frac{[k]_q}{[i]_q}}.$$

Whatever exact choice of scalar multiple is made for each of the elementary Y fragments, it follows from the definition of $c^k_{i,j}$ as

$$Y^k_{i,j}B^{ij} = c^k_{i,j}Y^k_{j,i}$$

that

$$B^{ij}Y^{j,i}_k = c^k_{i,j}Y^{i,j}_k,$$

by writing $Y^k_{i,j}B^{ij}Y^{j,i}_k$ in two ways as a multiple of $\mathrm{Id}(W_i)$. This also shows that $Y^k_{j,i}(B^{ij})^{-1} = (c^k_{i,j})^{-1}Y^k_{i,j}$ and $(B^{ij})^{-1}Y^{i,j}_k = (c^k_{i,j})^{-1}Y^{j,i}_k$.

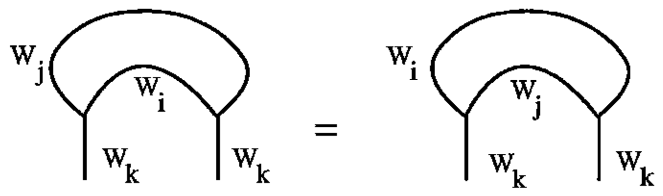

Figure 10

An important consequence is that the two diagrams combining $Y^{i,j}_k$ and $Y^{j,i}_k$ as shown in figure10 determine the same multiple of V_i, as one can be converted to the other by inserting a pair of cancelling half-twists, at the expense of multiplying by suitable constants, seen from the comments above to cancel. With the choices made in [7] they are actually equal to V_i. The similar pair of diagrams with $Y^k_{i,j}$ and $Y^k_{j,i}$ also determine the same multiple of U_i.

2 Parallel invariants

Our first aim in this paper is to give a simple way of understanding and calculating the invariants for parallels of links, in terms of the invariants of the link. To define a parallel, we must choose a framing for each component of the link; we will always use diagrams where the framing is defined by the plane of the given diagram. Write $L^{(p_1,\ldots,p_r)}$ for the link given from $L = L_1 \cup \ldots \cup L_r$ by replacing the ith component L_i by p_i parallel strands, with the same framing as L_i. We shall show how the invariants of $L^{(p_1,\ldots,p_r)}$ can be calculated from those of L itself. The crucial case is that of a 2-parallel where just one of the components has been replaced by two strings, and the others left unaltered. This is because the general case is such a 2-parallel, for example, if $p_1 > 1$ then $L^{(p_1,\ldots,p_r)}$ is given from $L^{(p_1-1,\ldots,p_r)}$ by doubling just one of the strings parallel to L_1.

The invariant $J(L; V_1, V_2, \ldots, V_r)$ has been defined, following [7], for irreducible modules $V_1, \ldots, V_r$. We now extend the definition to the case in which each V_i is the direct sum of irreducible modules.

Definition. Let V_j be a direct sum of irreducible modules

$$V_j = \bigoplus_{i_j=1}^{s_j} W_{i_j}.$$

Define

$$J(L; V_1, V_2, \ldots, V_r) = \sum_{i_1,\ldots,i_r} J(L; W_{i_1}, \ldots, W_{i_r}).$$

We can extend the definition further to allow elements V_i in the representation ring $\mathcal{R}$ of $SU(2)_q$, that is, general linear combinations of irreducible modules. In this way we may view J as giving a linear map $J(L) : \mathcal{R}^{\otimes r} \to \Lambda$ for each framed link L with r components. Since $\mathcal{R}$ is spanned by the elements W_i, the map $J(L)$ is determined by knowledge of $J(L; W_{i_1}, \ldots, W_{i_r}) = L_{i_1 i_2 \ldots i_r}$, say, for all assignments of irreducible modules W_{i_k} to components L_k of L.

We shall pay particular attention to the case of tensor products. With our extended definition we have

$$J(L; W_i \otimes W_j, V_2, \ldots, V_r) = \sum_{k \in i \otimes j} J(L; W_k, V_2, \ldots, V_r),$$

since $W_i \otimes W_j \cong \bigoplus_{k \in i \otimes j} W_k$. Our main result in this section is that we can also calculate this as an invariant of a parallel link to L; as the referee has pointed out, this result also follows implicitly from section 6.4 of [14].

Theorem 2.1. *When irreducible modules W_i, W_j are placed on the two strings parallel to L_1 in the link $L^{(2,1,\ldots,1)}$ then*

$$J(L^{(2,1,\ldots,1)}; W_i, W_j, V_2, \ldots, V_r) = J(L; W_i \otimes W_j, V_2, \ldots, V_r).$$

We illustrate this result in the case of a framed knot K and its 2-parallel $K^{(2)}$ in figure11, where we have

$$J(K^{(2)}; W_i, W_j) = J(K; W_i \otimes W_j).$$

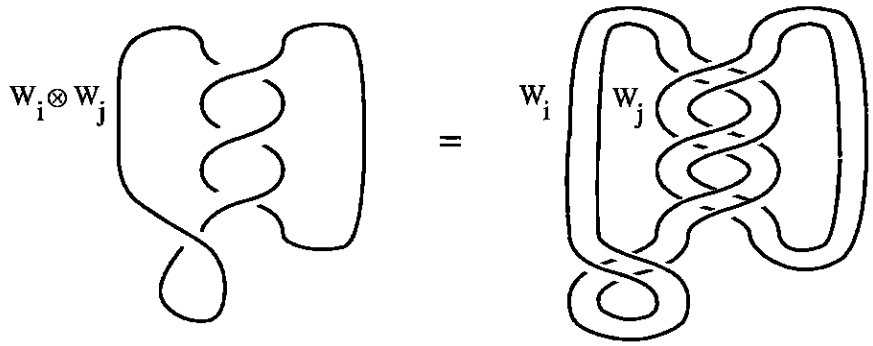

Figure 11

Corollary 2.2. *Theorem 2.1 holds when the irreducible modules W_i and W_j are replaced by any modules.*

Proof: Use the fact that any module is isomorphic to a sum of irreducibles [15], and the definition of J on sums of modules, where the tensor product of the sum is rewritten as the sum of tensor products in the usual way. □

Remark. Theorem 2.1 holds, with a similar proof, when any one of the components L_i is replaced by two parallel strings. We give the proof for $i = 1$ to save unnecessary complication in notation.

Corollary 2.3.

$$\begin{aligned} &J(L^{(p_1,\ldots,p_r)}; V_{1_1}, \ldots, V_{1_{p_1}}, V_{2_1}, \ldots, V_{2_{p_2}}, \ldots, V_{r_1}, \ldots, V_{r_{p_r}}) \\ &\qquad = J(L; \bigotimes_{j=1}^{p_1} V_{1_j}, \bigotimes_{j=1}^{p_2} V_{2_j}, \ldots, \bigotimes_{j=1}^{p_r} V_{r_j}). \end{aligned}$$

Proof: As noted above, the link $L^{(p_1,\ldots,p_r)}$ is the $(2,1,\ldots,1)$ parallel of $L^{(p_1-1,\ldots,p_r)}$, $p_1 > 1$, taking two strings in place of one of the first group of $p_1 - 1$ strings, or a similar 2-parallel for the ith group if $p_i > 1$. The result follows by induction on $(p_1, \ldots, p_r)$, using corollary 2.2. □

Proof of theorem 2.1: It is enough to prove in the case when $V_2, \ldots, V_r$ are irreducible, by the definition of J for sums of modules. Consider a diagram of $L^{(2,1,\ldots,1)}$ with irreducible modules W_i and W_j on the two strings parallel to L_1 and irreducible modules $V_2, \ldots, V_r$ on the other strings. This diagram represents, level by level, a composite of module homomorphisms. At some level in the diagram replace the identity homomorphism $\mathrm{Id}(W_i \otimes W_j)$ represented by the two parallel strands by the sum of the idempotents $\sum\limits_{k \in i \otimes j} E_k^{i,j}$. Now each $E_k^{i,j} = Y_k^{i,j} Y_{i,j}^k$ is represented by the graph shown in figure12, so that the invariant $J(L^{(2,1,\ldots,1)}; W_i, W_j, V_2, \ldots, V_r)$ is the sum for $k \in i \otimes j$ of the invariants of graphs in which the parallel strings have been replaced over a short section by the graph in figure 12, with W_k on the single 'fused' piece of string. The proof of theorem 2.1 will be completed by showing that the invariant for the graph with W_k on the fused part is equal to $J(L; W_k, V_2, \ldots, V_r)$, i.e. the invariant where the two parallel strings have been fused throughout their length, with W_k on the single resulting string.

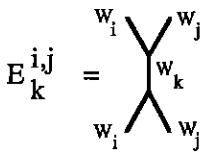

Figure 12

We consider a sequence of graphs, each with two strings parallel to L_1 for part of its length, and a single string for the rest, connected at either end by an elementary Y-fragment. We imagine that we move one of these Y-fragments like a zip around the diagram, to fuse more of the two parallel strings into a single string. We show that as we move the 'zip' from one position to another, keeping modules W_i and W_j on the parallel strings, and W_k on the fused string, the invariant of the graph is unchanged when the zip points the same way (i.e. up or down) in its initial and final positions. When the two positions of the zip are in opposite directions then there is a factor depending only on (i, j, k) to be taken into account. (This factor is 1 for the choices in [7].)

The result will then follow, as we can start the zip-up with the two strings only fused at $E_k^{i,j} = Y_k^{i,j} Y_{i,j}^k$. Treat the upward pointing fragment $Y_k^{i,j}$ as the zip, and move it completely around L_1 to reappear below the downward pointing $Y_{i,j}^k$ as $Y_{i,j}^k Y_k^{i,j}$ while the rest of the parallel strings are replaced by the single string carrying the module W_k. Since the zip still points upwards, the invariant has not altered in the process, and it remains unchanged when we fuse the two strings entirely, using the result that $Y_{i,j}^k Y_k^{i,j} = \mathrm{Id}(W_k)$. The invariant is clearly unaltered in the zip-up process when we move the zip across other strings as in figure 1.9. It remains to analyse the change as we move the zip over a local maximum or minimum of L_1. The homomorphisms A and B given by the diagrams in figure13 are multiples of each other, by a scalar m, depending on i, j and k, as indicated. Thus the invariant of our graph is always multiplied by m as the zip passes over a local maximum, while turning to the right. We shall show in lemma 2.4 that the invariant is also multiplied by m as the zip passes a local maximum while turning to the left, and that it is multiplied by m^{-1} on passing a local minimum in either way. Since the zip passes through equal numbers of maxima and minima if its final position points in the same sense as its initial position, the invariant is then unaltered under any such move, and the result follows.

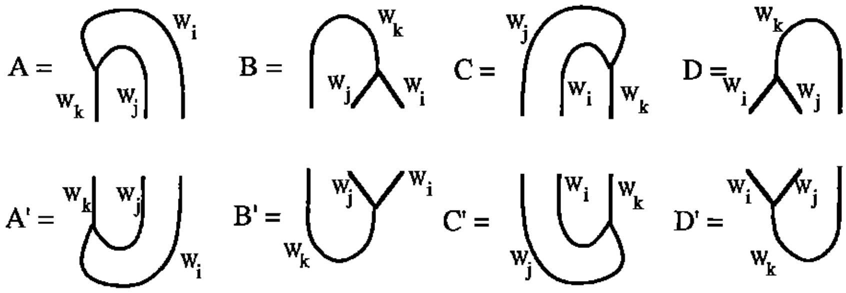

Figure 13

Lemma 2.4. *Suppose that A, B, C, D, A', B', C' and D' are module homomorphisms represented by the diagrams in figure 13, and that $B = mA$. Then $D = mC$ and $B' = m^{-1}A', D' = m^{-1}C'$.*

Proof: We have $D = nC$ for some scalar n. Now the composite

$$A(\mathrm{Id}(W_k) \otimes Y_k^{j,i}) = mB(\mathrm{Id}(W_k) \otimes Y_k^{j,i}) = mV_k$$

as shown in figure14, while

$$C(Y_k^{j,i} \otimes \mathrm{Id}(W_k)) = nD(Y_k^{j,i} \otimes \mathrm{Id}(W_k)) = nV_k.$$

It was noted in section 1 that

$$A(\mathrm{Id}(W_k) \otimes Y_k^{j,i}) = C(Y_k^{j,i} \otimes \mathrm{Id}(W_k)),$$

as seen in figure 1.10, giving the result that $n = m$.

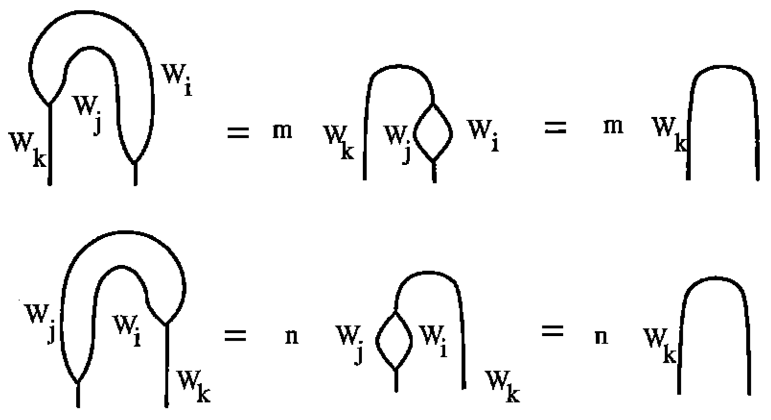

Figure 14

We can show that

$$(A \otimes \mathrm{Id}(W_k))(\mathrm{Id}(W_k) \otimes C') = \mathrm{Id}(W_k),$$

and

$$(B \otimes \mathrm{Id}(W_k))(\mathrm{Id}(W_k) \otimes D') = \mathrm{Id}(W_k),$$

as in figure15. Thus the scalar relating C' and D' must be m^{-1}, and similarly for A' and B'. □

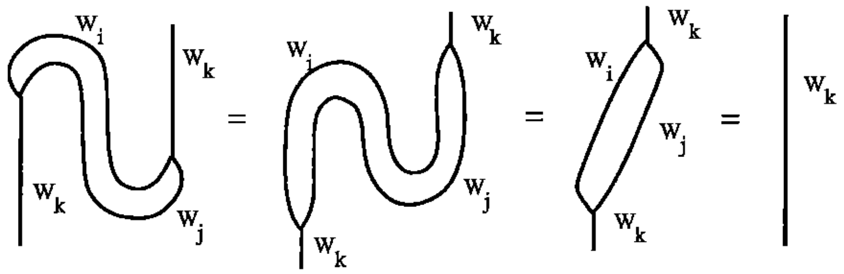

Figure 15

This completes the proof of theorem 2.1. □

By way of example, which will be very useful in the section on satellite links, we shall evaluate the general $SU(2)_q$-invariant $H_{ij} = J(H; W_i, W_j)$ of a 2-crossing diagram of the Hopf link, H. The calculation, which is illustrated in figure16, starts by inserting a positive curl into each component, to present the link, with altered framing, as the 2-parallel of a diagram K for the unknot with a single twist. The two strings may then be fused, to give the invariant as a sum of invariants for K, by theorem 2.1.

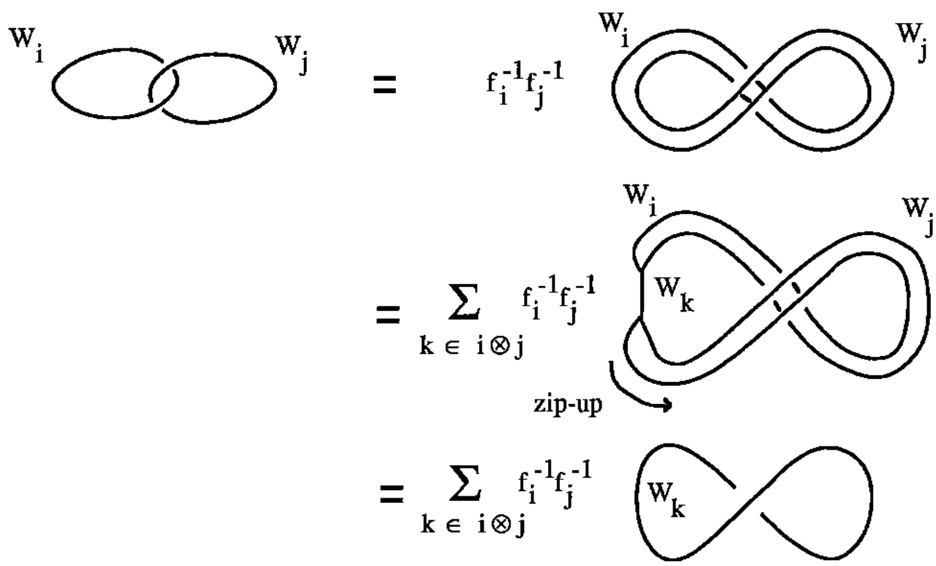

Figure 16

Thus;

$$f_i f_j H_{ij} = J(K^{(2)}; W_i, W_j) = \sum_{k \in i \otimes j} J(K; W_k),$$

where K is the unknot with a single twist. Now $J(K; W_k) = K_k = f_k \delta_k$, the framing factor appearing because of the twist, so

$$H_{ij} = f_i^{-1} f_j^{-1} \sum_{k \in i \otimes j} f_k \delta_k.$$

Then $(s-s^{-1})H_{ij} = f_i^{-1}f_j^{-1}\sum_{k\in i\otimes j} s^{\frac{1}{2}(k^2-1)}(s^k-s^{-k})$

$$= s^{-1}f_i^{-1}f_j^{-1}\sum_{k\in i\otimes j} s^{\frac{1}{2}(k+1)^2} - s^{\frac{1}{2}(k-1)^2}$$

$$= s^{-1}f_i^{-1}f_j^{-1}(s^{\frac{1}{2}(i+j)^2} - s^{\frac{1}{2}|i-j|^2})$$

because $k \in i\otimes j$ increases in steps of 2 from $|i-j|+1$ to $i+j-1$.

So $(s-s^{-1})H_{ij} = (-1)^{i+j}(s^{ij}-s^{-ij})$.

Corollary 2.3 gives us the means of finding the invariants of a parallel of L by taking the product in the representation ring $\mathcal{R}$ of the modules on the parallel strings, and then calculating $J(L)$ with these products assigned to the corresponding components. It is thus appropriate to look at the ring structure of $\mathcal{R}$. Now $\mathcal{R}$ as an algebra over Λ is known to be isomorphic to the representation ring of $SU(2)$; that is a polynomial ring generated by W_2 with the trivial module W_1 as a multiplicative identity. This is already implicit in the results of Jimbo quoted above. An alternative spanning set for $\mathcal{R}$ then consists of the powers $(W_2)^p, p \geq 0$ of the fundamental irreducible module W_2.

The explicit relation between this spanning set and the set of irreducibles is the same as in the classical case. We can use the relation $W_2\otimes W_i \cong W_{i-1}\oplus W_{i+1}$ to write $W_{i+1}\in\mathcal{R}$ inductively as a polynomial in W_2 with integer coefficients. The expression $W_i = \sum_{p=0}^{i-1} a_{ip}(W_2)^p$, with $W_1 = (W_2)^0$, can be found explicitly by the recurrence relation $a_{i+1\,p} = a_{i\,p-1} - a_{ip}$. The matrix $A = (a_{ip}), i\geq 1, j\geq 0$, of coefficients is lower triangular, with $a_{ip} = \begin{cases} 0, & p\geq i \\ 1, & p = i-1 \end{cases}$.

Its inverse $B = (b_{pk}), p\geq 0, k\geq 1$, is also unitriangular, and is given by $(W_2)^p = \sum_{k=1}^{p+1} b_{pk}W_k$ from the direct sum decomposition of $W_2^{\otimes p}$ into irreducibles. The positive integer coefficients b_{pk} are the classical Clebsch-Gordan coefficients for SU_2 satisfying $b_{pk} = b_{p-1\,k-1} + b_{p-1\,k+1}$, and are given by a truncated Pascal triangle construction as illustrated in the following table.

$p\downarrow$ \ $k\rightarrow$	1	2	3	4	5	6	7
0	1						
1	0	1					
2	1	0	1				
3	0	2	0	1			
4	2	0	3	0	1		
5	0	5	0	4	0	1	
6	5	0	9	0	5	0	1

Note that the left-most column of this table gives the Catalan numbers, and that the entries in row p are the dimensions of irreducible representations of the braid group B_p in the p-fold tensor power of W_2.

While the invariants in [7] are viewed in terms of the basis of irreducible representations, we can equally use the powers of W_2 as a basis for $\mathcal{R}$ and then we can evaluate $J(L)$ in terms of simple invariants of parallels of L, using 2.3 in the reverse direction, as follows.

Theorem 2.5. *The complete invariant $J(L)$ for a framed link L determines and is determined by the Jones polynomials $V(L^{(p_1,\ldots,p_r)})$ of all $(p_1,\ldots,p_r)$ parallels of L.*

Proof: The Jones polynomial $V(L)$ of an oriented link L is known to be related to the invariant $L_{22\ldots2}$, where W_2 is used on each component. Explicitly $f_2^w\delta_2 V(L) = L_{22\ldots2}$, where w is the writhe (signed number of crossings) of the framed diagram used to calculate $L_{22\ldots2}$, and V is evaluated at $t = s^{-2}$.

Now $J(L)$ can be calculated for any assignment of elements of $\mathcal{R}$ to the strings once its value is known on elements of the spanning set $(W_2)^p$ of $\mathcal{R}$. Thus $J(L)$ is determined by $J(L;(W_2)^{p_1},\ldots,(W_2)^{p_r})$ for all $(p_1,\ldots,p_r)$. Since $J(L;(W_2)^{p_1},\ldots,(W_2)^{p_r})$ is, up to a known factor, the Jones polynomial of the parallel link $L^{(p_1,\ldots,p_r)}$, by Corollary 2.3, we have established that $J(L)$ is determined by the Jones polynomial for all parallels.

Indeed to find any given $L_{i_1i_2\ldots i_r} = J(L;W_{i_1},\ldots,W_{i_r})$ we need only use the Jones polynomial for parallels $L^{(p_1,\ldots,p_r)}$ with $p_k \le i_k - 1$, following the expression in $\mathcal{R}$ of W_i in terms of $(W_2)^p$.

Conversely the Jones polynomial of the parallel $L^{(p_1,\ldots,p_r)}$ can be calculated, again by corollary 2.3, as a linear combination of the invariants $L_{i_1\ldots i_r}$ with $i_k \le p_k + 1$. $\square$

We complete this section by looking at some consequences.

Let L be the unknot with framing zero, so that it has a diagram with no crossings. Then $L^{(p)}$ is similarly the unlink with p components and zero framing on each. This diagram can be redrawn with the individual components moved well apart, so that

$$J(L^{(p)};V_1,\ldots,V_p) = \prod_{k=1}^{p} J(L;V_r).$$

The map $J(L):\mathcal{R}\to\Lambda$ is then a *ring* homomorphism for this L, by corollary 2.3. So $J(L)$ is determined by its value δ_2 on the generator W_2 of $\mathcal{R}$. Thus $J(L;W_i) = \delta_i = \sum a_{ij}(\delta_2)^j$, where a_{ij} are the coefficients above.

There are no other known knots K for which $J(K) = J(L)$ with L unknotted, nor any for which $J(K)$ is a ring homomorphism. It is not known whether $J(K)$ may be determined for some other choices of K simply by its value on W_2, although it is well known [10] that there are knots K', K'' with $J(K';W_2) = J(K'';W_2)$ but $J(K') \neq J(K'')$. Pairs of mutant knots K', K'' are all known to have $J(K') = J(K'')$, by the result of [11] on the Jones polynomials of their parallels. Thus $K'_i = K''_i$ for mutants for each i.

It seems probable that there is in general some bound k depending on the compexity of two knots K', K'' such that if $K'_i = K''_i$ for all $i \le k$ then $J(K') = J(K'')$. This might possibly follow from some algebraic means of reconstruction all the invariants K'_i given those with $i \le k$, under appropriate complexity assumptions.

Our remark above that $L_{22\ldots2} = f_2^w \delta_2 V(L)$ can be shown by proving that $L_{22\ldots2}$ is Kauffman's bracket polynomial $< L >$, normalised so that the unknot has value δ_2. The proof follows by establishing that the bracket polynomial's skein relation, as shown in figure17, holds for $L_{22\ldots2}$ with $A = s^{\frac{1}{2}}$, where diagrams are interpreted as parts of link diagrams with module W_2 on each string. The skein relation can then be read as a relation between endomorphisms of $W_2 \otimes W_2$, namely $B^{22} = s^{\frac{1}{2}}\,\mathrm{Id} + s^{-\frac{1}{2}} U_2 V_2$, and this last equation is established by writing each side as a linear combination of the idempotents $E_1^{2,2}$ and $E_3^{2,2}$.

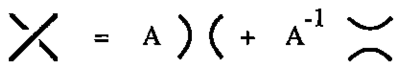

Figure 17

We can make a similar interpretation of the invariant $L_{33\ldots3}$ in terms of Kauffman's Dubrovnik invariant $\mathcal{D}_L(\lambda, z)$ of L as follows.

Theorem 2.6. $L_{33\ldots3} = \mathcal{D}_L(s^4, s^2 - s^{-2})$.

Proof: We shall assume that $\mathcal{D}$ is normalised to take the value 1 on the empty knot, since all the invariants $J(L)$ have this property. It is then enough to show that $L_{33\ldots3}$ satisfies the appropriate skein relations, shown in figure18, with $z = s^2 - s^{-2}$ and $\lambda = s^4$. For $L_{33\ldots3}$ we must regard these relations as referring to parts of link diagrams with W_3 on all strings. Now the framing factor for W_3 is $f_3 = s^4$, so that the second relation is correct with $\lambda = s^4$.

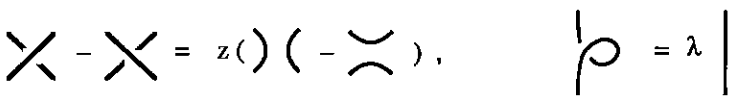

Figure 18

The proof is completed by showing that

$$B^{33} - (B^{33})^{-1} = (s^2 - s^{-2})(\mathrm{Id} - U_3 V_3)$$

as endomorphisms of $W_3 \otimes W_3$.

Write each side in terms of the idempotents $E_1^{3,3}$, $E_3^{3,3}$ and $E_5^{3,3}$ and compare coefficients. We have $\mathrm{Id} = E_1^{3,3} + E_3^{3,3} + E_5^{3,3}$, $E_j^{3,3}(B^{33})^{\pm1} = (c_{3,3}^j)^{\pm1} E_j^{3,3}$ and $U_3 V_3 = \delta_3 E_1^{3,3}$. The required equations are then

$$c_{3,3}^1 - (c_{3,3}^1)^{-1} = (s^2 - s^{-2})(1 - \delta_3) \tag{1}$$

$$c_{3,3}^3 - (c_{3,3}^3)^{-1} = s^2 - s^{-2} \tag{2}$$

$$c_{3,3}^5 - (c_{3,3}^5)^{-1} = s^2 - s^{-2} \tag{3}$$

Now $c_{3,3}^1 = s^{-4}$, $c_{3,3}^3 = -s^{-2}$ and $c_{3,3}^5 = s^2$ as stated in section 1, while $\delta_3 = s^2 + 1 + s^{-2}$, so the equations are satisfied. □

Remark. This is suggested by the result of Turaev [16], who states that the invariant $\mathcal{D}(\sqcup^{\nabla}, \sqcup - \sqcup^{-\infty})$ arises from the fundamental representation of the quantum group of $so(r+1)$ being attached to all strings of a link. We may note that SO_3 has the same Lie algebra as SU_2, and that its fundamental representation is 3-dimensional, so that the link invariant might be expected to be the same as the W_3-invariant.

We can use theorem 2.6 to calculate $L_{33\ldots3}$ in terms of Jones polynomials, and in this way give an alternative proof of a result of Yamada [19]. We remark first that the invariant $J(L; V_1, \ldots, V_r)$ where we have the trivial module $V_i = W_1$ on component i, is the same as the invariant of the sublink L' of L where the i th component is deleted, and the other components have modules V_j as before.Then for $U_j = V_j + W_1$ we have, by multilinearity, that

$$J(L; U_1, \ldots, U_r) = \sum_{L' \subset L} J(L'; \{V_j\})$$

where the sum is taken over all sublinks L' of L with the module V_j assigned to the component L_j provided that this component remains in the sublink L'. An obvious modification can be made for elements $U_j \in \mathcal{R}$ which are linear combinations of V_j and W_1, giving Yamada's operation of summing weighted combinations of invariants over sublinks.

Since $(W_2)^2 = W_1 + W_3 \in \mathcal{R}$ we immediately have that

$$J(L; (W_2)^2, \ldots (W_2)^2) = \sum_{L' \subset L} L'_{33\ldots3}.$$

Now

$$J(L; (W_2)^2, \ldots (W_2)^2) = f_2^w \delta_2 V(L^{(2,\ldots,2)})$$

and $L'_{33\ldots3} = \mathcal{D}_{L'}(s^4, s^2 - s^{-2})$, giving Yamada's relation between V for a 2-parallel of L and $\mathcal{D}$ for its sublinks, including L itself and the empty link. It is equally easy, writing $W_3 = -(W_2)^2 + W_1$, to express $L_{33\ldots3}$ as a sum of Jones polynomials of 2-parallels of sublinks.

3 Satellite links

The parallel construction in the last section is a special case of a *satellite* link formed from a companion link C and patterns P_i for each component C_i of C. The pattern is often described as an $s+1$-component link, where one component is an unknotted *axis*. The complement of the axis is a solid torus, in which the remaining s components form a closed tangle. This is used to replace the interior of a regular neighbourhood of a component of C, as illustrated in figure19. In calculating the invariants of the satellite we shall use the framing which comes from the companion and pattern diagrams. We shall show that, as in the case of the Alexander polynomial, the $SU(2)_q$-invariants of a satellite link S are determined by those of the companion C and patterns P_i. Having done this, we shall give some indication of *how many* of the invariants involved we would need to know in order to calculate one individual invariant of the satellite.

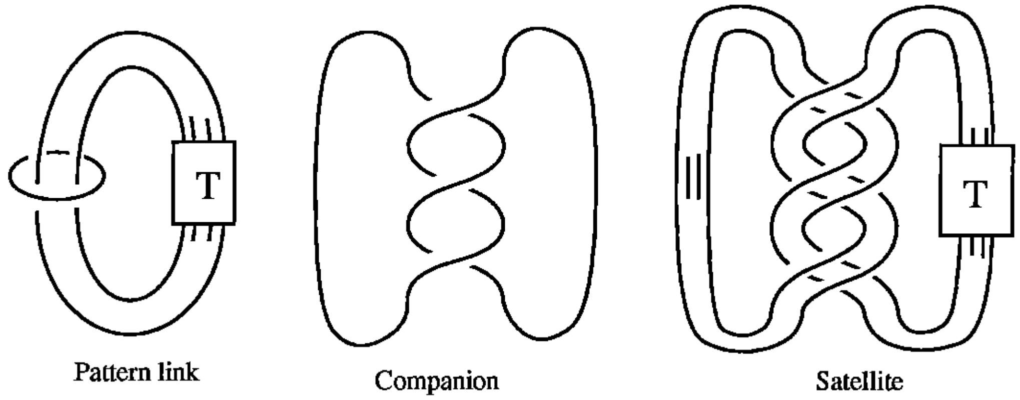

Figure 19

To simplify matters, we shall assume that we only wish to put a non-trivial pattern on one component of the link C, which we shall take to be the first; multiple patterns can then be applied sequentially.

The main result is proved in the next theorem, using the fusion and zip-up methods from section 2. The diagrams occurring as part of the induction involve tangle graphs, as defined in section 1.

Theorem 3.1. *Let T be a tangle-graph with n inputs and outputs, and let modules be assigned to all edges of T so that it represents an endomorphism $\alpha(T)$ of $W_{i_1} \otimes \ldots \otimes W_{i_n}$. Let C be any link diagram, and let S be given from the $(n, 1, \ldots, 1)$ parallel of C by inserting T at some level on the strings parallel to the first component of C. In S assign modules to the edges of T and the strings parallel to the first component of C as in the definition of $\alpha(T)$. Make any choice of modules $V_2, \ldots, V_r$ on the remaining components (if any) of C.*

Then there are coefficients $g^k_{\alpha(T)}$, depending only on $\alpha(T)$ and $k \in i_1 \otimes \ldots \otimes i_n$ such that the diagram S has invariant

$$\sum_{k \in i_1 \otimes \ldots \otimes i_n} g^k_{\alpha(T)} J(C; W_k, V_2, \ldots, V_r).$$

Notation: We write above $k \in i_1 \otimes \ldots \otimes i_n$ to mean that W_k is a summand of $W_{i_1} \otimes \ldots \otimes W_{i_n}$

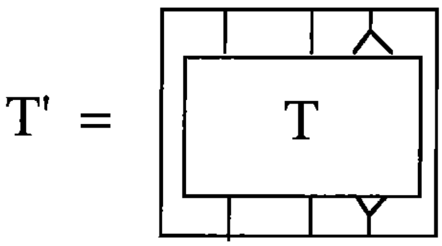

Figure 20

Proof: By induction on n. When $n=1$ we have $\alpha(T)$ as a scalar multiple of $\mathrm{Id}(W_i)$ by Schur's lemma, and this scalar forms the single coefficient $g^k_{\alpha(T)}$.

For $n>1$ we shall fuse the last two of the parallel strings immediately above T. Writing $i=i_{n-1}$ and $j=i_n$, we replace $\mathrm{Id}(W_i\otimes W_j)$ by $\sum_{\ell\in i\otimes j} E^{i,j}_\ell$. The invariant of S is then the sum of invariants of diagrams with two strings fused immediately above T. The invariant of each of these diagrams is unaltered by the zipping up process of section 2, where we take the upward pointing fragment $Y^{i,j}_\ell$ and move it around the last two parallel strings to reappear eventually immediately below T. The new diagram is constructed from C in the same way as S, with $n-1$ strings in place of n and the tangle graph T' in place of T as shown in figure 20. For each $\ell\in i\otimes j$ there is an endomorphism $\alpha_\ell(T')$ of $W_{i_1}\otimes\ldots\otimes W_{i_{n-2}}\otimes W_\ell$ constructed from $\alpha(T)$ and the maps $Y^{i,j}_\ell$ and $Y^\ell_{i,j}$. By induction there are coefficients $g^k_{\alpha_\ell(T')}$ such that the diagram has invariant

$$I_\ell=\sum_{k\in i_1\otimes\ldots\otimes i_{n-2}\otimes \ell} g^k_{\alpha_\ell(T')}J(C;W_k,V_2,\ldots,V_r).$$

The invariant for S is then the sum of these over ℓ, and we can take $g^k_{\alpha(T)}=\sum_{\ell\in i\otimes j} g^k_{\alpha_\ell(T')}$ to complete the proof. □

Apply the theorem with C as the Hopf link, and T as a tangle. Then S is the closure of T together with an unknotted axis forming a pattern link as in figure 3.1. Suppose that the closure of T consists of a single component. Then $\alpha(T)$ will be determined by just one choice of module W_i say, for all the strings in T, and we may write g^k_i in place of $g^k_{\alpha(T)}$. More generally, where the closure of T has s components there will be a choice of $i_1,\ldots,i_s$ in determining $\alpha(T)$, and we could write $g^k_{i_1\ldots i_s}$ for $g^k_{\alpha(T)}$.

Where T has one component in its closure the pattern link P has invariants P_{ij}, from the choice of modules W_j on the axis and W_i on the closure of T. Then $P_{ij}=\sum g^k_i H_{kj}$ by theorem 3.1, where k runs over a finite set determined by i and the number of tangle strings passing through the axis. Any satellite S with pattern link P will have invariants related to its companion C by exactly the same set of coefficients g^k_i. Now given P_{ij} and H_{jk} we can recover the coefficients g^k_i, since for each fixed i there are only a finite number of k with $g^k_i\neq 0$, and we can solve the equations for g^k_i by inverting a truncated matrix $H=(H_{kj})$. We then have;

Theorem 3.2. *The invariants $J(S)$ of a satellite knot S constructed from a companion C with pattern P can be determined from $J(C)$, $J(P)$ and the Hopf link invariants $J(H)$.*

Proof: We can find the coefficients g^k_i, or more generally $g^k_{i_1\ldots i_s}$, from $J(P)$ and $J(H)$ and then we have $J(S)$ from g^k_i and $J(C)$. □

In the case where C and S each have one component the relation can be written in matrix terms as

$$\mathcal{S}=\mathcal{G}\mathcal{C},\ \ \mathcal{P}=\mathcal{G}\mathcal{H},$$

where $\mathcal{G} = (g_i^k)$, $\mathcal{H} = (H_{kj})$, $\mathcal{P} = (P_{ij})$ are matrices and $\mathcal{S} = (S_i)$, $\mathcal{C} = (C_k)$ are column vectors. Although these equations involve infinite matrices they do make sense because $\mathcal{G}$ is row finite. A similar result holds in general, with $\mathcal{S}$, $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{P}$ interpreted as multi-indexed tensors.

To work out the satellite invariant S_i say, where the satellite S has a single component, we note that $g_i^k = 0$ for $k > ni$, where there are n strings passing through the axis in the pattern link P. In calculating the invariants up to S_i we may truncate $\mathcal{G}$ to an $i \times ni$ matrix, $\mathcal{H}$ to an $ni \times ni$ matrix, and the others correspondingly. The truncated Hopf array will be invertible (away from roots of unity) and in terms of the truncated matrices we shall have

$$\mathcal{S} = \mathcal{P}\mathcal{H}^{-1}\mathcal{C}$$

It is interesting to compare this with Fox's formula for Alexander invariants of satellites, in terms of a decomposition of a link exterior as a union of link exteriors [2,9]. In the case of a satellite the exterior is the union of the exteriors of the companion and the pattern link, and their intersection can be taken as a torus, or equally as a thickened torus, which may be viewed as the exterior of the Hopf link.

The result above shows that if we interpret the invariant $J(P)$ of the pattern link as a map from $\mathcal{R}^{\otimes(s+1)}$ to Λ where each factor $\mathcal{R}$ is associated to one boundary component of the link exterior, and choice of longitude and meridian, then $\mathcal{G}$ can be viewed as a map from $\mathcal{R}^{\otimes s}$ to $\mathcal{R}$ in which the component corresponding to the axis has been turned around, using the Hopf link invariant $J(H) : \mathcal{R} \otimes \mathcal{R} \to \Lambda$ as a bilinear form to 'raise' one index. Composition of $\mathcal{G}$ with the invariant of the companion then produces the invariant of the satellite, in a way that corresponds to the construction of the satellite exterior as a union of the other link exteriors.

Remarks. In a more algebraic approach it is natural to split the endomorphism $\alpha(T)$ determined by the tangle T in the pattern into a direct sum of endomorphisms of isotypic submodules isomorphic to $W_k \otimes V_k$ for different k, where V_k is a trivial module of some dimension. Then $\alpha(T)$ determines an endomorphism of each V_k, having $g^k_{\alpha(T)}$ as its trace.

In his work on parallel link invariants Murakami [8] produces irreducible representations of the braid group related to the Jones polynomial of parallels of a given link. These give invariants closely connected to the invariants determined by the quantum group approach, but constructed independently. He then treats the case of satellites from this point of view and gives a character formula for the invariants of a satellite, which may be compared with the formulae here.

Reshetikhin and Turaev [14] have recently formalised the requirements on a Hopf algebra A which permit the definition of a link invariant when any choice of A-module, not necessarily irreducible, is allowed for each component. These 'ribbon Hopf algebras' exist as central extensions of quantum groups for each of the quantum groups associated to the simple Lie groups. The resulting invariants can be proved to behave multilinearly under sums of module. Furthermore the invariant determined by selecting a tensor product of two modules for one component of a link can be directly identified with the invariant given by taking two parallel strings in place of that component with the two modules assigned separately to the two strings. A result about invariants of parallels similar to that in section 2 then follows in this setting, where the proof of multilinearity takes the place of the lemmas in section 2 on zipping up strands past local extrema. We plan to give an account of this and the related behaviour of satellites in a later paper.

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Department of Pure Mathematics
University of Liverpool
PO Box 147
Liverpool L69 3BX
England.