Pairs of closed 3-braids with the same Alexander polynomial

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Introduction.

The knots and links which arise as the closure $\widehat{\beta}$ of some 3-braid $\beta \in B_3$ form a small class among all links, and often have rather restricted properties. A sufficient list of representative braids is given in [Mu] with very few suspected duplications.

The Alexander polynomial $\Delta_{\widehat{\beta}}$ has been calculated [M] for all closed 3-braids $\widehat{\beta}$. It was conjectured there, because of the form of the calculations, that $\widehat{\beta}$ could be recovered from knowledge of Δ or at any rate from the 2-variable Alexander polynomial of the 'complete closure' $\widehat{\beta} \cup L_\beta$, where L_β is the axis of the closed braid $\widehat{\beta}$.

In this note we show that there are many examples of 3-braids α, β with the same Alexander polynomial for their complete closure, while $\widehat{\alpha}$ and $\widehat{\beta}$ are different knots. These examples date from the second author's visit to Zaragoza in 1984. Birman [B2] independently produced another method for constructing such examples, although in general there is more difficulty in guaranteeing that her pairs $\widehat{\alpha}$ and $\widehat{\beta}$ are distinct.

The examples produced, showing that 3-braid closures are rather more varied than at first apparent, have figured importantly in later work with the new 2-variable knot invariant P. For closed 3-braids this invariant P can be found from the Alexander polynomial of the complete closure, although this is not true for $(n+1)$-braids in general [MS], so that the pairs provided easily handled knots having the same P.

These pairs were used [MS] to try to relate the polynomial P for a cable about a knot C to the polynomial for C. This led to the unexpected result, using the simplest of the pairs described here, $\sigma_1^7\sigma_2^{-1}$ and $\sigma_1^{-3}\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2$, that two knots could have the same P, while similarly constructed cables about each had a different P.

Alexander polynomials of closed braids.

Let β be an $(n+1)$-string braid, an element of the braid group B_{n+1}. Write $\widehat{\beta}$ for its closure, and L_β for the axis of the closed braid. Then $\widehat{\beta} \cup L_\beta$ is a link, called in [M] the *complete closure* of β, whose Alexander polynomial is given by $\Delta_{\widehat{\beta} \cup L_\beta}(x,t) = \det(xI - B(t))$, where x, t are represented by meridians of L_β and $\widehat{\beta}$ respectively, oriented as in figure 1, and $B(t)$ is the reduced Burau matrix of β.

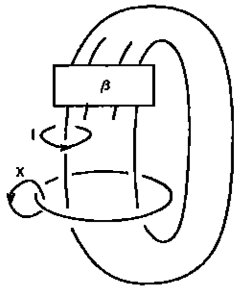

Figure 1

Then

$$\Delta_{\widehat{\beta}\cup L_\beta}(x,t) \approx x^n + f_1(t)x^{n-1} + \ldots + f_{n-1}(t)x + (-t)^{w(\beta)}(-1)^n,$$

where $w(\beta)$ is the algebraic number of crossings in β, since $\det B(t) = (-t)^{w(\beta)}$.

Notation. Here and elsewhere we write $\approx$ to mean equal up to a multiple of $\pm x^r t^s$.

Definition. We shall call this Alexander polynomial, $\det(xI - B(t))$, the *Burau polynomial* of the braid β.

Remark. The Burau polynomial determines the Alexander polynomial of the closure of the braid, and for 3-braids can be recovered from this and the sum $w(\beta)$.

Lemma 1. (Birman [**B1**]) The element $c = \Delta^2_{n+1} \in B_{n+1}$ representing a full twist has reduced Burau matrix $t^{n+1}I$.

We shall use this, and the symmetry of general Alexander polynomials, as shown by Torres [**T**] in the multivariable case, to construct pairs of 3-braids with the same Burau polynomial. At the same time we note the conditions needed for a similar construction on $(n+1)$-braids.

Lemma 2. (Torres [**T**]) The multivariable Alexander polynomial of a link L with m separately coloured components satisfies $\Delta_L(t_1,\ldots,t_m) \approx \Delta_L(t_1^{-1},\ldots,t_m^{-1})$.

Corollary. The polynomial $\Delta_{\widehat{\beta}\cup L_\beta}(x,t) \approx x^n + f_1(t)x^{n-1} + \ldots + f_{n-1}(t)x + (-t)^{w(\beta)}(-1)^n$ satisfies $f_i(t^{-1}) = f_{n-i}(t)(-t)^{-w(\beta)}(-1)^n$ for each i.

Proof: By Lemma 2,

$$\begin{aligned} &x^n + f_1(t)x^{n-1} + \ldots + f_{n-1}(t)x + (-t)^{w(\beta)}(-1)^n \\ &\qquad\approx x^{-n} + f_1(t^{-1})x^{-n+1} + \ldots + f_{n-1}(t^{-1})x^{-1} + (-t)^{-w(\beta)}(-1)^n. \end{aligned}$$

Multiply the right hand side by $(-t)^{w(\beta)}(-1)^n$ and compare coefficients of x^{n-i} on each side. □

Lemma 3. Let β^M be the mirror image of β, i.e. the braid with the signs of all crossings reversed. Then $\Delta_{\widehat{\gamma}\cup L_\gamma}(x,t) \approx \Delta_{\widehat{\beta}\cup L_\beta}(x,t^{-1})$, where $\gamma = \beta^M$.

Proof: The mirror image homeomorphism $h : S^3 - (\widehat{\beta}\cup L_\beta) \to S^3 - (\widehat{\gamma}\cup L_\gamma)$, reflecting in a plane perpendicular to the axis, preserves the orientation of the axis meridian while reversing the orientation of the braid meridians. □

As a consequence we have:

Theorem 1. Let $\beta \in B_{n+1}$, and let $\alpha = c^d\beta^M$. Then $\Delta_{\widehat{\beta}\cup L_\beta}(x,t) \approx \Delta_{\widehat{\alpha}\cup L_\alpha}(x,t)$ if and only if

(1) $\quad n(n+1)d = 2w(\beta) \quad$ and

(2) $\quad f_i(t) = f_i(t^{-1})t^{(n+1)id}, \quad i = 1,\ldots,n-1.$

Proof: By lemma 1, if β^M has Burau matrix $B^M(t)$, then $\alpha = c^d\beta^M$ has matrix $A(t) = t^{(n+1)d}B^M(t)$.

Hence

$$\begin{aligned}\Delta_{\widehat{\alpha}\cup L_\alpha}(x,t) &\approx \det(xI - t^{(n+1)d}B^M(t))\\ &\approx \det(xt^{-(n+1)d}I - B^M(t))\\ &\approx \det(\xi I - B^M(t))\\ &\approx \Delta_{\widehat{\gamma}\cup L_\gamma}(\xi,t),\end{aligned}$$

where $\xi = xt^{-(n+1)d}$ and $\gamma = \beta^M$.

Now $\Delta_{\widehat{\gamma}\cup L_\gamma}(\xi,t) \approx \Delta_{\widehat{\beta}\cup L_\beta}(\xi,t^{-1})$ by lemma 3.

Thus

$$\Delta_{\widehat{\beta}\cup L_\beta}(x,t) \approx \Delta_{\widehat{\alpha}\cup L_\alpha}(x,t)$$

if and only if

$$\Delta_{\widehat{\beta}\cup L_\beta}(x,t) \approx \Delta_{\widehat{\beta}\cup L_\beta}(xt^{-(n+1)d},t^{-1}).$$

Multiply the right hand side by $t^{n(n+1)d}$, and compare coefficients of x^{n-i} to get the condition that

$$\begin{aligned}f_i(t) &= t^{n(n+1)d}t^{-(n+1)d(n-i)}f_i(t^{-1})\\ &= f_i(t^{-1})t^{(n+1)id}.\end{aligned}$$

Condition (1) comes from comparing the constant terms. □

Remark. Condition (1) is simply the requirement that $w(\alpha) = w(\beta)$. It shows that $w(\beta)$ must be a multiple of $n(n+1)/2$ for any choice of d to be possible, and given this, it determines d.

Corollary 1. $\Delta_{\widehat{\beta}\cup L_\beta} = \Delta_{\widehat{\gamma}\cup L_\gamma}$ where $\gamma = \beta^M$ if and only if $w(\beta) = 0$ and $f_i(t) = f_i(t^{-1})$ for each i.

Theorem 2. Where β is a 3-braid, and $\alpha = c^d\beta^M$ we have $\Delta_{\widehat{\beta}\cup L_\beta} = \Delta_{\widehat{\alpha}\cup L_\alpha}$ if and only if $w(\beta) = 6m$ for some m and $d = 2m$.

Proof of Theorem 2: We have $n = 2$ so we require $w(\beta) = 3d$ and $f_1(t) = t^{3d}f_1(t^{-1})$. Now $(-t)^{w(\beta)}f_1(t^{-1}) = f_1(t)$, by corollary to Lemma 2.

Our conditions become $w(\beta) = 3d$ and $(-t)^{w(\beta)} = t^{3d}$. Hence $w(\beta)$ is even, so $d = 2m$, say, and $w(\beta) = 6m$. □

Remark. In the examples that follow we shall use pairs β and $\alpha = c^{2m}\beta^M$ where $w(\beta) = 6m$, having the same Burau polynomial. We can equally use $c^d\beta$ and $c^d\alpha$ for any d, which are in fact also of the same form.

Signature.

We can now construct many pairs α, β of 3-braids with the same Burau polynomial. We can also show that $\widehat{\alpha}$ and $\widehat{\beta}$ are not equivalent in many of these cases by using Murasugi's calculations for their signature, although other methods can frequently be used instead. We shall concentrate on braids in Murasugi's class Ω_6 which represent the vast majority of 3-braids.

Notation. We write $\operatorname{sig}(\beta)$ for the signature of the knot $\widehat{\beta}$.

Proposition. (Murasugi **[Mu, p43,62]**)
If $\beta = \sigma_1^{p_1}\sigma_2^{-q_1}\ldots\sigma_1^{p_r}\sigma_2^{-q_r}$, $p_i, q_i \geq 1$, and d is even, then $\operatorname{sig}(c^d\beta) = 4d + w(\beta)$.
If d is odd, then $\operatorname{sig}(c^d\beta) = 4d + w(\beta) + \varepsilon$, where $\varepsilon = 0, \pm 2$.

Corollary 1. The braids $\beta = \sigma_1^{p_1}\sigma_2^{-q_1}\ldots\sigma_1^{p_r}\sigma_2^{-q_r}$, $p_i, q_i \geq 1$, with $w(\beta) = 6m$ and $\alpha = c^{2m}\beta^M$ have inequivalent closures for $m \neq 0$, but the same Burau polynomial.

Corollary 2. The same is true for the pairs $c^d\beta, c^d\alpha$ for d even; it holds as well for d odd, at least when $|m| > 1$.

Proof of corollary 1: It is enough to show that $\operatorname{sig}(\alpha) \neq \operatorname{sig}(\beta)$.

Now $\operatorname{sig}(\beta) = w(\beta)$ and $\operatorname{sig}(\alpha) = 8m + \operatorname{sig}(\beta^M) = 8m - w(\beta)$. Since $w(\beta) = 6m$ we have $\operatorname{sig}(\alpha) \neq \operatorname{sig}(\beta)$. □

Proof of corollary 2: For d even, $\operatorname{sig}(c^d\beta) - \operatorname{sig}(c^d\alpha) = \operatorname{sig}(\beta) - \operatorname{sig}(\alpha)$.

For d odd, $\operatorname{sig}(c^d\beta) - \operatorname{sig}(c^d\alpha) = \operatorname{sig}(\beta) - \operatorname{sig}(\alpha) + \delta$ where $\delta = \varepsilon_1 - \varepsilon_2$ and each $\varepsilon_i = 0, \pm 2$. Thus $|\delta| \leq 4$.
Now $|\operatorname{sig}(\beta) - \operatorname{sig}(\alpha)| = 4|m|$. □

Examples.

The simplest such examples are $\beta = \sigma_1^p\sigma_2^{-q}$ with $p - q = 6m$.
Then $\alpha = c^{2m}\sigma_1^p\sigma_2^{-q}$ has a different closure for $|m| > 0$, as do the pairs $c^d\sigma_1^p\sigma_2^{-q}$ and $c^{d+2m}\sigma_1^p\sigma_2^{-q}$ under the conditions of corollary 2.

The example with fewest crossings is $\beta = \sigma_1^7\sigma_2^{-1}$ with $\alpha = c^2\sigma_1^{-7}\sigma_2$ conjugate to $\sigma_1^{-3}\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2$. This is the original pair whose 2-cables are studied in **[MS]**.

Remark. In the same vein, $\beta = \sigma_1^{6m+1}\sigma_2^{-1}$ and $\alpha = c^{2m}\sigma_1\sigma_2^{-(6m+1)}$ close to knots with the same polynomial P as the $(2, 6m+1)$ torus knot. It can be shown **[E]** that any closed 3-braid having this as its polynomial P is either a torus knot or the closure of α.

Birman's second class of examples appear more difficult to distinguish; they are generally not of the form discussed here, as her pairs, again with the same Burau polynomial, have the same signature. The examples of this class considered by [MS] could be distinguished by P when applied to their 2-cables, although not by the Jones polynomials of the cables.

Our examples above show that even if we restrict attention to positive 3-braids (which ensures among other things that their closures are fibred) we can still find pairs with the same Burau polynomial, simply by using corollary 2 with d large enough to make both braids positive.

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