

# KNOTS, SATELLITES AND QUANTUM GROUPS

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## 1. Introduction

These notes are an expanded version of a series of lectures given in ICTP Trieste and at Universidad Complutense, Madrid in April and May 2009.

They are primarily an account of knot and link invariants derived from the Homfly polynomial by the use of satellites. They are based mainly on work by myself and former students at Liverpool University. Much of this can be found in greater detail through the Liverpool University knot theory publications list, including the doctoral theses of Aiston, Lukac and Hadji.

I refer readers also to an extended expository article<sup>11</sup> where a similar approach, focused more on satellite invariants for the Jones polynomial, is adopted. This article contains earlier work on the Homfly invariants, which culminated in two papers with Aiston.<sup>2,13</sup>

More recent papers are those on the Murphy operators<sup>9,10</sup> and work with Lukac and Hadji<sup>3,7,14</sup>, where the technique of using the meridian maps has been developed and refined. The latest paper with Manchon<sup>15</sup> gathers together results about the Homfly skein of the annulus, which is the prime tool for organising satellite invariants.

### 1.1. *Setting the scene*

We are all able to tie a knot in a piece of rope. The pictures in figure 1 show some examples.

What exactly do we mean though when we say that a piece of rope is knotted?

Let us first stop the knots escaping by joining up the ends of the rope,

2



Fig. 1.

as in figure 2. Compare what happens in the three cases.



Fig. 2.

In the first case we get a simple, or ‘unknotted’ circle, while in the second case we have a circle with what appears to be a knot in it.

Let us say that the rope is *knotted* if no possible manipulation of it will result in the unknotted circle. We do not allow cutting and rejoining.

The third example can clearly be undone by a little manipulation to form the simple circle, so again the rope is unknotted.

We model this notion of a knot mathematically by referring to a closed curve in  $\mathbf{R}^3$  as a knot, with the special case of the simple circle, lying say as the unit circle in a plane, known as the *trivial knot* or *unknot*. Knot theory in the mathematical sense is then the study of closed curves in space.

We call two knots *equivalent* if one can be manipulated, without passing one strand through another, to become the other knot. I give a more formal technical description of this below, but essentially anything is allowed which could be done with a rather stretchable piece of rope. The one manoeuvre which must be excluded is the analogue of the bachelor’s technique for ignoring knots on a piece of cotton – pull it so tight that you can hardly see it! Using this technique on a curve with no physical thickness would get rid of any knot.



Fig. 3. Bachelor’s unknotting

We would like to know for a start if there are any knots which are not equivalent to the trivial knot. If so, are there lots of different knots, and how might we distinguish between them?

It is easy to imagine that you have been given two knots and by a little patient work you manage to manipulate one to look like the other, e.g. the first and third knots in figure 2. What happens though if you find that even after a lot of trying you can't make them look the same – does it follow that the knots are inequivalent, or have you just not been dextrous enough? There is clearly a problem here, and something else will be needed, as there is no way that failure to manipulate can show that it is actually impossible to do so.

It should be realised that the question of how the rope is knotted isn't an intrinsic question about the rope alone, but rather a matter of how the rope is placed in space. Every closed loop of rope looks the same to an ant inside the rope. Some of the techniques developed for the study of knots have proved fruitful in other 'placement problems', i.e. in studying the different ways in which one particular geometric object, here a closed curve, may lie inside a larger one.

## 1.2. *Background*

The idea of looking at knotted and unknotted closed curves goes back to Gauss and beyond. Kelvin had some idea of trying to relate different types of atoms to knotted curves in the ether; this was taken up by a Scottish physicist Tait, who set out to enumerate all possible different knots in the hope of tallying them against different atoms. His lists of knots soon showed that the task of systematically enumerating all knots was hopelessly complicated; among other problems there are infinitely many. It is still true today that no practical framework exists for producing a comprehensive list, although Thistlethwaite has devised a fairly good means of handling the simpler knots. Various mathematicians in the 1920s and 1930s developed methods to show up a number of general properties shared by all knots, using some very elegant geometrical techniques and exploiting the growing interplay between algebra and this style of geometry. From this period has come the Alexander polynomial, and interpretations of it, as well as group theoretic invariants. Much more recently knot theory and theoretical physics have again had close contacts.

**Definition 1.1.** A *knot* is a simple closed curve  $K \subset \mathbf{R}^3$  or in  $S^3$ .

**Definition 1.2.** The *complement* of  $K$  is  $S^3 - K$ .

We shall only deal with *tame* knots, e.g. smooth or polygonal curves, and we assume that  $K$  has a solid torus neighbourhood  $V$  with

$$(V, K) \cong (S^1 \times D^2, S^1 \times \{0\}).$$

This is like insisting on using a piece of rope, although one whose exact thickness will not matter.

It is often convenient to deal with  $S^3 - \text{int}V = \text{ext}K$ , the *exterior* of  $K$ , which is a *compact* 3-manifold with boundary  $\partial(\text{ext}K) = \partial V \cong \text{torus } S^1 \times S^1$ .

From the point of view of topological invariants there is not much difference between  $S^3 - K$ ,  $\text{ext}K$  and  $S^3 - V$ .

**Definition 1.3.** Knots  $K_0$  and  $K_1$  are *homeomorphic* if there exists a homeomorphism  $h : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  such that  $h(K_0) = K_1$ .

If  $h$  is orientation preserving we can deform  $K_0$  to  $K_1$  through a family of knots  $K_t = h_t(K_0)$ . We shall call  $K_0$  and  $K_1$  *equivalent* when they are related in this way. (The term *ambient isotopic* is also used.)

Conversely a 1-parameter sliding of a neighbourhood  $V$  of  $K_0$  to one of  $K_1$  through  $\mathbf{R}^3$  can be extended to such a family  $h_t$  of homeomorphisms, and models quite well the physical notion of equivalence by manipulation of a closed loop of rope.

We then have the result, by composing with a reflection if necessary, that two knots  $K_0$  and  $K_1$  are homeomorphic if and only if  $K_0$  is equivalent to  $K_1$  or its mirror-image.

**Remark 1.1.** Some knots, for example the trefoil, are not equivalent to their mirror image, while others such as the figure-eight knot are.

### 1.3. Knot diagrams and moves

For our subsequent analysis we concentrate on tame knots, i.e. knots equivalent to finite polygonal curves or equally to regular smooth curves.

Diagrams connected by a sequence of Reidemeister's moves, seen in figure 4, represent equivalent knots.

The converse is also true.

**Theorem 1.1 (Reidemeister).** *If two diagrams represent equivalent knots then one diagram can be converted to the other by a finite sequence of Reidemeister moves, along with isotopy (deformation) of the image within the projection plane.*

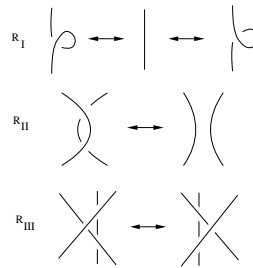


Fig. 4. Reidemeister's moves

#### 1.4. Links and linking number

We may enlarge our scope slightly and look, as Gauss did, not just at a single closed curve but at several at once.

**Definition 1.4.** A *link* of  $r$  components is a collection  $L = L_1 \cup L_2 \cup \dots \cup L_r$  of  $r$  closed non-intersecting curves.

When  $r = 1$  we have a knot. In the case  $r = 2$  we can very simply associate an integer with a link, which is the same for every equivalent link. This is called the *linking number* of the two components.

To define the linking number  $\text{lk}(L_1, L_2)$  we must first choose an orientation of each of the components, which we note on a diagram of the link by drawing arrows on the curves. Now look at one diagram of the link and consider only the crossings where  $L_1$  crosses over  $L_2$ . Each of these crossings  $c_i$  can be given a sign  $\varepsilon_i = \pm 1$ , according to a conventional choice. The sum of these signs  $\sum \varepsilon_i$  is unaltered when the diagram is changed by Reidemeister moves. For crossings of  $L_1$  over  $L_2$  are not affected by moves I and III, while if there are any involved in a move of type II they occur as a pair with opposite sign, so that the sum is unchanged.

Reidemeister's theorem holds also for links. We may then set  $\text{lk}(L_1, L_2) = \sum \varepsilon_i$  for any choice of diagram.

**Proposition 1.1.**  $\text{lk}(L_2, L_1) = \text{lk}(L_1, L_2)$ .

**Proof.** To calculate  $\text{lk}(L_2, L_1)$  we must count the crossings of  $L_2$  over  $L_1$  in some diagram. Start with a diagram in which we count the crossings  $c_i$  of  $L_1$  over  $L_2$ . If we turn this diagram over and view it from the other side we get a new diagram of the link in which the crossings  $c_i$  become the crossings of  $L_2$  over  $L_1$ . Each crossing, viewed from the other side has the same sign as it had initially, so the sum needed to calculate  $\text{lk}(L_2, L_1)$  from

this diagram is identical to the sum calculating  $\text{lk}(L_1, L_2)$  in the original diagram.  $\square$

### 1.5. Framed links

Framed links are made from pieces of ribbon rather than rope, so that each component has a preferred annulus neighbourhood. Combinatorially they can be modelled by diagrams in  $S^2$  up to  $R_{II}$  and  $R_{III}$ , excluding  $R_I$ , by use of the ‘blackboard framing’ convention. The ribbons are determined by taking parallel curves on the diagram.

Reidemeister moves  $II$  and  $III$  on a diagram give rise to isotopic ribbons. Any apparent twists in a ribbon can be flattened out using Reidemeister  $I$ .

Oriented link diagrams  $D$  have a *writhe*  $w(D)$  which is the sum of the signs of all crossings. This is unchanged by moves  $II$  and  $III$ .

For a framed knot the writhe is sometimes called its ‘self-linking number’, which is independent of the orientation of the diagram. Generally a framing of a link is determined by a choice of writhe for each component.

### 1.6. Satellites

A *satellite* of a framed knot  $K$  is determined by choosing a diagram  $Q$  in the standard annulus, and then drawing  $Q$  on the annular neighbourhood of  $K$  determined by the framing, to give the satellite knot  $K * Q$ . We refer to this construction as *decorating  $K$  with the pattern  $Q$*  (see figure 5).

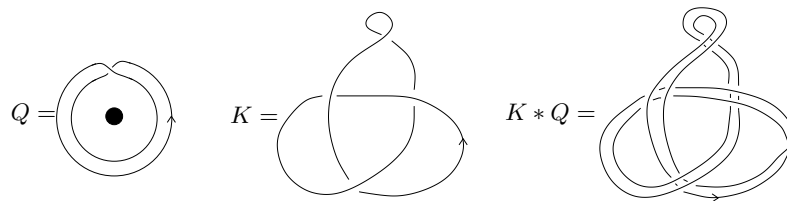


Fig. 5. Satellite construction

It is often possible to use satellites with some fixed choice of pattern  $Q$  in comparing two framed knots  $K$  and  $K'$ . When  $K$  and  $K'$  are equivalent then  $K * Q$  and  $K' * Q$  are equivalent. If we can find some knot invariant  $I$  for which  $I(K * Q) \neq I(K' * Q)$  we can conclude that  $K'$  is not equivalent to  $K$ .

The Conway polynomial  $\nabla(K)$  is not useful in this context, since

$$\nabla(K * Q) = \nabla(K' * Q)$$

for every choice of  $Q$ , if  $\nabla(K) = \nabla(K')$ .

This limitation does not hold in general. In particular the extension of the Conway polynomial known as the Homfly polynomial will often give useful extra information when applied to satellites.

**Remark 1.2.** The use of satellites is sometimes known as *cabling*. I prefer to reserve the term ‘cable’ for satellites where the pattern  $Q$  is based on some  $(p, q)$  torus knot.

## 2. Homfly invariants

In 1984 V.F.R.Jones constructed a new invariant of oriented links  $V_L(t) \in \mathbf{Z}[t^{\pm\frac{1}{2}}]$ , which turned out to have the property that

$$t^{-1}V_{L_+} - tV_{L_-} = (\sqrt{t} - 1/\sqrt{t})V_{L_0}$$

where the links

$$L_+ = \begin{array}{c} \nearrow \\ \searrow \end{array}, \quad L_- = \begin{array}{c} \nwarrow \\ \swarrow \end{array}, \quad L_0 = \begin{array}{c} \nearrow \\ \nearrow \end{array} \quad \left( \begin{array}{c} \nearrow \\ \searrow \end{array} \right)$$

differ only as shown. This was quickly extended to a 2-variable invariant  $P_L(v, z) \in \mathbf{Z}[v^{\pm 1}, z^{\pm 1}]$ , with the property that

$$v^{-1}P_{L_+} - vP_{L_-} = zP_{L_0}.$$

The name ‘Homfly polynomial’ has come to be attached to  $P$ , being the initial letters of six of the eight people involved in this further development. The polynomial  $P$  contains both the Conway/Alexander polynomial, and Jones’ invariant, and can be shown to contain more information in general than both of these taken together. We have

$$\begin{aligned} P(1, z) &= \nabla(z) \\ P(1, s - s^{-1}) &= \Delta(s^2) \\ P(s^2, s - s^{-1}) &= V(s^2) \\ P(s, s - s^{-1}) &= \pm 1 \end{aligned}$$

The skein relation (1) can readily be shown to determine  $P$  and  $V$  once its value on the trivial knot is given. It has been usual to take  $P = 1$  on the trivial knot, although in some recent applications a different normalisation can be more appropriate.

Given the existence of  $V$  and  $P$  we can then make some calculations. For example, the unlink with two components has

$$P = \frac{v^{-1} - v}{z},$$

$$V(s^2) = -(s + s^{-1}),$$

while the Hopf link with linking number  $+1$  has

$$P = vz + (v^{-1} - v)v^2z^{-1},$$

$$V(s^2) = s^3 - s - (s + s^{-1})s^4 = -s(1 + s^4).$$

The Hopf link with linking number  $-1$  has

$$P = -v^{-1}z + (v^{-1} - v)v^{-2}z^{-1},$$

$$V(s^2) = -s^{-1}(1 + s^{-4}).$$

This illustrates the general feature that for the mirror image  $\overline{L}$  of a link  $L$ , (where the signs of all crossings are changed), we have  $P_{\overline{L}}(v, z) = P_L(v^{-1}, -z)$  and so  $V_{\overline{L}}(s^2) = V_L(s^{-2})$ . It is thus quite possible to use  $V$  in many cases to distinguish a knot from its mirror-image, while there will be no difference in their Conway polynomials. It is worth noting that although there are still knots which cannot be distinguished from each other by  $P$  in spite of being inequivalent, no non-trivial knot has so far been found for which  $P = 1$ , or even  $V = 1$ .

The original Homfly polynomial is invariant under *all* Reidemeister moves, but there is a convenient version which is an invariant of a framed oriented link

In its most adaptable form,  $P_L(v, s)$ , it lies in the ring

$$\Lambda = \mathbf{Z}[v^{\pm 1}, s^{\pm 1}, (s^r - s^{-r})^{-1}], r > 0.$$

Its defining characteristics are the local skein relations.

$$(1) \quad \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} - \begin{array}{c} \nearrow \nearrow \\ \nwarrow \nwarrow \end{array} = (s - s^{-1}) \left( \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right) \left( \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right)$$

$$(2) \quad \begin{array}{c} \uparrow \\ \circlearrowleft \end{array} = v^{-1} \left. \begin{array}{c} \uparrow \\ \phantom{\circlearrowleft} \end{array} \right\}, \quad \begin{array}{c} \uparrow \\ \circlearrowright \end{array} = v \left. \begin{array}{c} \uparrow \\ \phantom{\circlearrowright} \end{array} \right\}.$$

These relate the invariants of links whose diagrams differ only locally as shown.

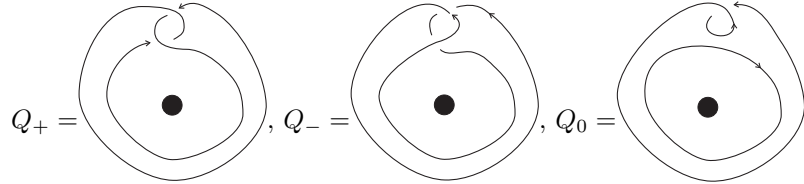
They are enough to allow its recursive calculation from simpler diagrams in terms of the value for the unknot.

The unframed version for  $L$ , invariant under all Reidemeister moves, is given by  $v^{w(D)}P_L(v, s)$  where  $D$  is a diagram for the framed link.



The local nature of the skein relations between invariants allows us to make a useful simplification in studying them.

Compare for example three patterns  $Q_{\pm}$  and  $Q_0$ .



The framed Homfly invariants of  $K * Q_{\pm}$  and  $K * Q_0$  then satisfy

$$P(K * Q_+) - P(K * Q_-) = (s - s^{-1})P(K * Q_0).$$

Since  $K * Q_-$  is the unknot for any  $K$ , this relates the invariants of the Whitehead double  $K * Q_+$  of  $K$  and those of its reverse parallel.

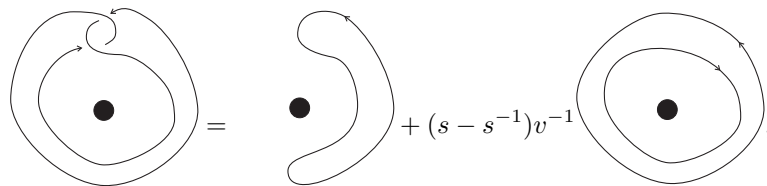
More generally, consider the linear space  $\mathcal{C}$  of  $\Lambda$ -linear combinations of diagrams in the annulus (up to  $R_{II}, R_{III}$ ) and impose the local relations

$$(1) \quad \begin{array}{c} \nearrow \searrow \\ \swarrow \nearrow \end{array} - \begin{array}{c} \nearrow \swarrow \\ \nwarrow \nearrow \end{array} = (s - s^{-1}) \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array}$$

$$(2) \quad \begin{array}{c} \curvearrowright \\ \uparrow \end{array} = v^{-1} \begin{array}{c} \uparrow \\ \curvearrowright \end{array}, \quad \begin{array}{c} \curvearrowleft \\ \uparrow \end{array} = v \begin{array}{c} \uparrow \\ \curvearrowleft \end{array}.$$

Decorating  $K$  by an element  $\sum a_i Q_i$  of  $\mathcal{C}$  gives a well-defined Homfly invariant  $\sum a_i P(K * Q_i)$  since the skein relations are respected when the Homfly polynomials of the satellites are compared.

We could summarise our calculation above by saying that in the skein  $\mathcal{C}$  we have

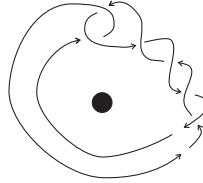


and hence

$$P(K * Q_+) = P(\text{unknot}) + (s - s^{-1})vP(\text{reverse parallel}).$$

The space  $\mathcal{C}$ , called the *Homfly skein of the annulus*, then gives a more effective parameter space for satellite invariants, as we only need to know the pattern as an element of  $\mathcal{C}$ .

For example, any of the twist patterns



is a linear combination of the reverse parallel and the trivial pattern, so the Homfly polynomial of any twisted double can be found from the reverse parallel.

We will return to look at more details of  $\mathcal{C}$  later. For now, I will look at a further skein formulation which results in interesting models of certain algebras.

### 3. General Homfly skein theory

For a surface  $F$  with some designated input and output boundary points the (linear) Homfly skein of  $F$  is defined as linear combinations of framed oriented diagrams in  $F$ , up to Reidemeister moves II and III, modulo the skein relations

$$\begin{aligned}
 (1) \quad & \left( \begin{array}{c} \nearrow \\ \searrow \end{array} \right) - \left( \begin{array}{c} \nwarrow \\ \swarrow \end{array} \right) = (s - s^{-1}) \left( \begin{array}{c} \nearrow \\ \nearrow \end{array} \right) \left( \begin{array}{c} \searrow \\ \searrow \end{array} \right), \\
 (2) \quad & \left( \begin{array}{c} \uparrow \\ \text{loop} \end{array} \right) = v^{-1} \left( \begin{array}{c} \uparrow \end{array} \right), \quad \left( \begin{array}{c} \downarrow \\ \text{loop} \end{array} \right) = v \left( \begin{array}{c} \downarrow \end{array} \right).
 \end{aligned}$$

It is an immediate consequence that

$$\left( \begin{array}{c} \uparrow \\ \text{circle} \end{array} \right) = \delta \left( \begin{array}{c} \uparrow \end{array} \right),$$

where  $\delta = \frac{v^{-1} - v}{s - s^{-1}} \in \Lambda$ . The coefficient ring  $\Lambda$  is taken as  $Z[v^{\pm 1}, s^{\pm 1}]$ , with denominators  $\{r\} = s^r - s^{-r}, r \geq 1$ .

We have already met the skein of the annulus,  $\mathcal{C}$ .

In the skein of  $\mathbf{R}^2$  or  $S^2$  every diagram  $D$  is equivalent to a multiple of the trivial diagram  $\bigcirc$ . Explicitly,

$$D = P(D) \bigcirc$$

where  $P(D)$  is the framed Homfly polynomial of  $D$ .

As a general rule, geometric operations induce linear maps on the corresponding skeins. For example, given a framed knot  $K$  there is a linear map  $K* : \mathcal{C} \rightarrow \mathcal{S}(\mathbf{R}^2)$  induced by  $Q \mapsto K * Q$ .

The skein of the rectangle with  $m$  inputs at the top and  $m$  outputs at the bottom is denoted by  $H_m$ . Elements are represented by combinations of diagrams in the rectangle made up of  $m$  arcs joining the input and output points, and possibly some further closed curves. Such diagrams are known as  $m$ -tangles.

A simple example of an  $m$ -tangle is an  $m$ -braid, while another important  $m$ -tangle is the tangle

$$T^{(m)} = \left[ \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \end{array} \right] .$$

### 3.1. Composition

Putting one  $m$ -tangle above another defines an associative product with identity.

**Theorem 3.1.** *The set of invertible tangles consist of the  $m$ -braids, which form the braid group  $B_m$ .*

Artin's braid group  $B_m$  has a presentation in terms of elementary braids,  $\{\sigma_i\}, i = 1, \dots, m - 1$  satisfying the braid relations

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i - j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \end{aligned}$$

where

$$\sigma_i = \left[ \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \end{array} \right] .$$

Tangle composition induces a product in the skein  $H_m$ . This skein is spanned by a finite set of  $m$ -braids. One such spanning set consists of the  $m!$  'totally descending' braids in which the  $m$  arcs of the tangle are numbered from the bottom left, and each crossing is met first as an overcrossing on going along the arcs in order. These braids are sometimes termed 'positive permutation braids', and they each realise one of the permutations of the endpoints.

Then  $H_m$  forms a finite-dimensional algebra, with a presentation on generators  $\{\sigma_i\}, i = 1, \dots, m - 1$  satisfying the braid relations

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i - j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \end{aligned}$$

and the quadratic relations  $\sigma_i^2 = (s - s^{-1})\sigma_i + 1$ , which result from the skein relation

$$\sigma_i - \sigma_i^{-1} = (s - s^{-1})\text{Id}.$$

The resulting algebra is also known as the Hecke algebra  $H_m(z)$ , when  $z = s - s^{-1} = \{1\}$  and the coefficients are extended to  $\Lambda$ . The Hecke algebra  $H_m$  can be also seen as the group algebra of Artin's braid group  $B_m$  generated by the elementary braids  $\sigma_i$ ,  $i = 1, \dots, m - 1$ , modulo the further quadratic relation  $\sigma_i^2 = z\sigma_i + 1$ .

In the special case  $z = 0$  the Hecke algebra reduces to the group algebra of the symmetric group,  $\mathbf{C}[S_m]$ , with  $\sigma_i$  becoming the transposition  $(i \ i+1)$ .

### 3.2. Closure

The closure map from  $H_m$  to  $\mathcal{C}$  is the  $\Lambda$ -linear map induced by considering the closure  $\hat{T}$  of a tangle  $T$  in the annulus (see figure 6). The image of this map is denoted by  $\mathcal{C}_m$ .

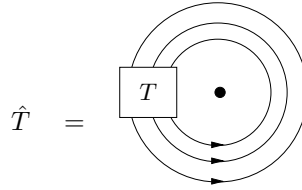


Fig. 6. The closure map

## 4. The skein of the annulus

The skein  $\mathcal{C}$  of the annulus has been used formally for some time as a parameter space for the Homfly satellite invariants of a knot.

It has a product structure induced at the level of diagrams by placing one annulus outside another. This defines a bilinear product under which  $\mathcal{C}$  becomes an algebra. This algebra is clearly commutative (lift the inner annulus up and stretch it so that the outer one will fit inside it).

**Remark 4.1.** The Kauffman bracket skein of the annulus gives a quotient of this algebra which is combinatorially convenient to use when working with the specialisation to the Jones polynomial.

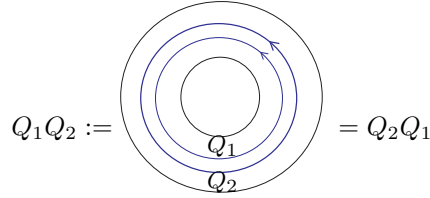


Fig. 7. The product  $Q_1Q_2$

Turaev<sup>17</sup> showed that  $\mathcal{C}$  is freely generated as an algebra by the elements  $A_m$ , which are the closures of the  $m$ -braids  $\sigma_{m-1} \cdots \sigma_2 \sigma_1 \in H_m$ , and the elements  $A_m^*$ , with the reverse orientation. The identity element in the algebra is represented by the empty diagram in the annulus.

So the pattern  $Q_+$  for the Whitehead double is a linear combination of 1 and  $A_1A_1^*$  in this notation.

The linear subspace  $\mathcal{C}_m$  spanned by the closures of  $m$ -tangles has a linear basis of monomials in  $\{A_i\}$  with total weight  $m$ , where  $A_i$  has weight  $i$ . There are  $p(m)$  of these, where  $p(m)$  is the number of partitions of  $m$ . Thus  $\mathcal{C}_3$  is spanned by  $A_1^3, A_1A_2$  and  $A_3$ .

Although the satellite invariants of a knot  $K$  behave additively under addition of patterns, there is no relation between the invariants with patterns  $Q_1, Q_2$  and  $Q_1Q_2$ . It may then happen that there are  $p(m)$  independent invariants of a knot arising from decorations in  $\mathcal{C}_m$ .

In the interests of relating these to other invariants it is good to work with a rather different basis for  $\mathcal{C}_m$ , and indeed for the whole skein  $\mathcal{C}$ , which has the advantage of behaving well when the framing of the knot is changed.

For example when an extra twist is added to the framing of a knot  $K$  to form  $K'$  the satellite  $K' * A_1^2$  becomes  $K * Q$  with  $Q = v^{-2}A_1^2 + zv^{-2}A_2$  in the skein  $\mathcal{C}_2$  so that  $P(K' * A_1^2) = v^{-2}P(K * A_1^2) + v^{-2}zP(K * A_2)$ .

The two basis elements  $Q_1 = A_1^2 + sA_2$  and  $Q_2 = A_1^2 - s^{-1}A_2$  are much better for framing changes, in the sense that  $P(K' * Q_1) = v^{-2}s^2P(K * Q_1)$  while  $P(K' * Q_2) = v^{-2}s^{-2}P(K * Q_2)$

The framing change map is illustrated in figure 8 by its effect on the 2-parallel element  $(A_1)^2$ .

The framing change map is connected to the element in the Hecke algebra  $H_m$  represented by the full twist  $\Delta_m^2$  on  $m$  strings. This braid commutes with all  $m$ -braids, and hence represents an element in the centre of the Hecke algebra  $H_m$ .

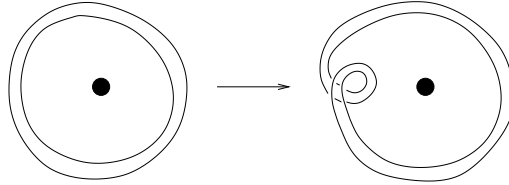
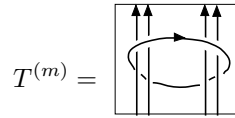


Fig. 8. The framing change map on a 2-parallel

A further important central element of  $H_m$  is represented by the tangle



consisting of an oriented meridian curve around  $m$  parallel strings.

Closely related to the elements  $T^{(m)}$  are the meridian maps  $\varphi, \bar{\varphi} : \mathcal{C} \rightarrow \mathcal{C}$  in the skein of the annulus.

#### 4.1. Meridian maps

The meridian map  $\varphi : \mathcal{C} \rightarrow \mathcal{C}$  is induced by including a single meridian curve around a diagram  $Q$  in the thickened annulus to give the diagram shown in figure 9.

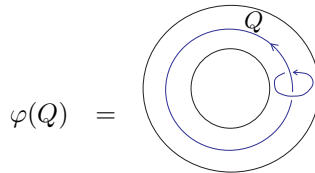


Fig. 9. The meridian map

The map  $\bar{\varphi}$  is given similarly, using the opposite orientation on the meridian curve.

When  $Q$  is the closure  $Q = \hat{T}$  of an  $m$ -tangle  $T$  then  $\varphi(Q)$  is the closure of  $T^{(m)}T$ .

The subspace  $\mathcal{C}_m$  spanned by closed  $m$ -tangles is invariant under the meridian map  $\varphi$ . The map  $\varphi|_{\mathcal{C}_m}$  has  $p(m)$  distinct eigenvalues, one for each partition of  $m$ , with a 1-dimensional space of eigenvectors for each eigenvalue.

While  $\mathcal{C}_m$  is also invariant under the framing change map, this map has fewer distinct eigenvalues than  $\varphi$  for  $m \geq 6$ . The eigenvectors for  $\varphi$  are also eigenvectors for the framing change map, and indeed the basis given above for  $\mathcal{C}_2$  consisted of eigenvectors for  $\varphi$ .

#### 4.2. Partitions

Partitions are widely used in descriptions of irreducible representations of the symmetric groups.

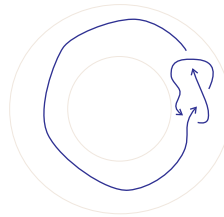
A partition  $\lambda$  of  $m$  into  $k$  parts  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$  can be represented combinatorially by a *Young diagram* with  $m$  cells arranged in  $k$  rows. Successive rows have  $\lambda_1, \lambda_2, \dots, \lambda_k$  cells starting from a fixed left-hand end.

**Theorem 4.1 (Hadji, Morton).** *There is a basis for the skein  $\mathcal{C}$  consisting of eigenvectors  $Q_{\lambda, \mu}$  of the meridian map  $\varphi$ . Here  $\lambda$  and  $\mu$  run through the set of all partitions, and the corresponding eigenvalues  $s_{\lambda, \mu}$  are all distinct.*

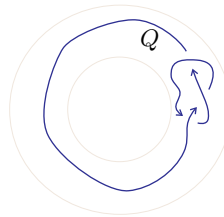
The basis  $Q_{\lambda, \mu}$  is thus very natural, and it shows up in many different ways.

For example the basis vectors are then also eigenvectors for any other linear endomorphism of  $\mathcal{C}$  which commutes with  $\varphi$ . These include  $\bar{\varphi}$  and the framing change map.

A further example is given by drawing a given knot  $K$  as the closure of a 1-tangle in the annulus.

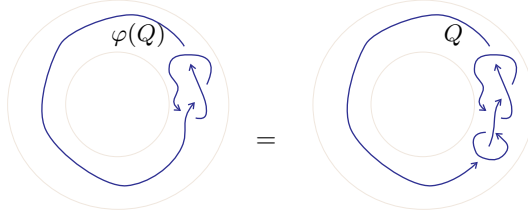


Decorate this with a pattern  $Q$  to get a diagram for  $K * Q$  in the annulus,



inducing a map  $T_K : \mathcal{C} \rightarrow \mathcal{C}$ .

Now  $T_K$  commutes with  $\varphi$ ,



$$\text{so } T_K(Q_{\lambda,\mu}) = a(K, \lambda, \mu)Q_{\lambda,\mu}.$$

**Theorem 4.2 (Morton).** *The eigenvalues  $a(K, \lambda, \mu) \in \Lambda$  are integral in  $\Lambda$ , and are the ratio of the Homfly invariants*

$$\frac{P(K * Q_{\lambda,\mu})}{P(\bigcirc * Q_{\lambda,\mu})}.$$

**4.3. Branching rules in  $\mathcal{C}$ .**

The basis  $Q_{\lambda,\mu}$  for  $\mathcal{C}$  also behaves well under the product operation, namely the product of two basis elements is always a non-negative integer combination of basis elements. These can be found explicitly by combinatorial formulae from classical work with partitions.

Besides the identity element in  $\mathcal{C}$ , which is represented by the empty diagram, and forms the basis element  $Q_{\lambda,\mu}$  with  $|\lambda| = |\mu| = 0$ , the simplest basis elements are the single oriented core curves  $A_1$  and  $A_1^*$ . These represent  $Q_{1,\phi}$  with  $|\lambda| = 1, |\mu| = 0$  and  $Q_{\phi,1}$  respectively.

The branching rules for these can be summarised as

$$Q_{\lambda,\mu} = A_1 Q_{\lambda,\mu} + \sum_{\rho \in \lambda^+} Q_{\rho,\mu} + \sum_{\nu \in \mu^-} Q_{\lambda,\nu}.$$

Here  $\lambda^+$  is the set of partitions given from the Young diagram of  $\lambda$  by adding one further cell, and  $\lambda^-$  is the set of partitions given by removing a single cell.

**5. Symmetric functions and the skein of the annulus**

This section is concerned primarily with the subspace  $\mathcal{C}^+$  of the skein of the annulus which is spanned by the elements  $Q_{\lambda,\mu}$  with  $\mu$  empty, or equally by



the subspaces  $\mathcal{C}_m$  spanned by the closures of directly oriented  $m$ -tangles. We write  $Q_\lambda := Q_{\lambda, \phi}$  for the spanning basis elements.

An algebraic model of  $\mathcal{C}^+$  that fits particularly well with these basis elements  $\{Q_\lambda\}$  and also connects with the ideas of quantum group representations is that of symmetric functions.

### 5.1. Symmetric functions

We consider polynomials in  $N$  commuting variables  $x_1, \dots, x_N$  which are unchanged by permutation of the variables. The most familiar are the *elementary symmetric functions*

$$e_m = \sum_{i_1 < i_2 < \dots < i_m} x_{i_1} x_{i_2} \dots x_{i_m}.$$

These appear as the coefficients of the polynomial

$$E(t) = \prod_{i=1}^N (1 + x_i t) = 1 + e_1 t + \dots + e_m t^m + \dots$$

The *complete symmetric functions* are the coefficients of

$$H(t) = \prod_{i=1}^N \frac{1}{1 - x_i t} = 1 + h_1 t + \dots + h_m t^m + \dots$$

The generating series for these two sets of functions satisfy the relation  $E(t)H(-t) = 1$ .

Other familiar symmetric functions are the *power sums*  $P_m = x_1^m + \dots + x_N^m$ .

A classical result says that every symmetric integer polynomial in  $x_1, \dots, x_N$  is an integer polynomial in  $\{e_m\}$  and also in  $\{h_m\}$ . Indeed the polynomial is independent of the number of variables  $N$  for large enough  $N$ . For example  $p_2 = e_1^2 - 2e_2$  for  $N > 1$ .

There is an extensive body of literature about symmetric functions. They occur in the representation theory of symmetric groups and the related representation theory of unitary groups. One substantial reference is the book of Macdonald<sup>8</sup>.

The *character* of a representation of the unitary group  $U(N)$  is the trace of the representing matrices considered as a function on the representation of the diagonal matrices (which form a maximal torus in the group). The trace gives a function on diagonal matrices which is symmetric in the entries. Characters add under sum of representations, and multiply under tensor product.

The irreducible representations correspond to certain symmetric functions called the *Schur* functions. The Schur functions  $s_\lambda(\mathbf{x})$  of degree  $m$  form a basis for all degree  $m$  symmetric polynomials in  $\mathbf{x} = (x_1, \dots, x_N)$ , and they correspond directly with the partitions  $\lambda$  of  $m$ . By the general result above each Schur function can be expressed as a polynomial in the elementary symmetric functions  $\{e_m\}$ , or the complete symmetric functions  $\{h_m\}$ . The functions  $e_m$  and  $h_m$  themselves are Schur functions, corresponding to the partitions of  $m$  into a single column or row respectively.

In the skein of the annulus a choice of elements to represent the complete symmetric functions  $\{h_m\}$  can be made in such a way that the resulting Schur polynomial  $s_\lambda$  represents the basis element  $Q_\lambda$ <sup>7</sup>. The interpretation of  $\mathcal{C}^+$  as symmetric functions based on this choice of representatives for  $\{h_m\}$  then leads to a natural role for  $\{Q_\lambda\}$  as the Schur functions. It allows the known formulae for products of Schur functions to tell us how to write a product of basis elements  $Q_\lambda Q_\rho$  as a sum of basis elements. It also suggests a relation to the irreducible representations of the unitary groups. It is striking that the elements representing the power sums also play a significant role in satellite constructions and have satisfying geometric representatives,<sup>10</sup>

## 5.2. Construction of the basis elements

The elements  $h_m$  and  $e_m$  can be constructed readily in terms of the simplest idempotents of the Hecke algebra  $H_m$ .

The element  $h_m \in \mathcal{C}_m$ , which is taken to represent the complete symmetric function of degree  $m$ , is the closure of the element  $\frac{1}{\alpha_m} a_m \in H_m$  where

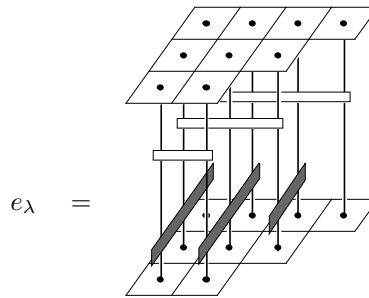
$$a_m = \sum_{\pi \in S_m} s^{l(\pi)} \omega_\pi$$

is one of the two basic quasi-idempotent elements of  $H_m$ . Here  $\omega_\pi$  is the positive permutation braid associated to the permutation  $\pi \in S_m$  with length  $l(\pi)$ , which is the writhe of the braid  $w_\pi$ . The scalar  $\alpha_m$  is given by the equation  $a_m a_m = \alpha_m a_m$ .<sup>2,7,9</sup> Using the other quasi-idempotent

$$b_m = \sum_{\pi \in S_m} (-s)^{-l(\pi)} \omega_\pi$$

in a similar way determines the element  $e_m$  which represents the elementary symmetric function. These elements are related by the power series equation  $H(t)E(-t) = 1$ . These two idempotents arise from the 1-dimensional representations of  $H_m$  in which  $\sigma_i \rightarrow s$  or  $\sigma_i \rightarrow s^{-1}$  respectively.

Aiston's view of the elements  $Q_\lambda$  is in a more 3-dimensional context of combinations of diagrams in a solid torus, rather than an annulus. We show below a diagrammatic view of a linear combination  $e_\lambda$  of 3-dimensional braids, whose endpoints lie on the cells of a Young diagram  $\lambda$ , rather than in the conventional straight line. In this illustration  $m = 9$  and  $\lambda$  is the partition 4, 3, 2.



Here  $e_\lambda$  should be regarded as an element of the Homfly skein of  $D^2 \times I$ , with endpoints at the top and bottom on the template  $\lambda$ , and with some implicit choice of parallel for each strand to determine a framing. The white boxes, following the rows of  $\lambda$ , contain the braid combination  $a_j$  when the box has length  $j$ , while the grey boxes, following the columns, similarly contain combinations  $b_j$ . The whole combination will be denoted by  $e_\lambda$ . (The notation  $e_\lambda$  is used in<sup>1</sup> for a closely related element of the Hecke algebra  $H_{|\lambda|}$  given by making a specific arrangement of the  $|\lambda|$  endpoints in a straight line.) In either context the element  $e_\lambda$  can naturally be composed with itself, and satisfies the relation  $e_\lambda^2 = \alpha_\lambda e_\lambda$  for some scalar  $\alpha_\lambda \in \Lambda_1$ . Aiston defines the element  $Q_\lambda$  by

$$Q_\lambda = \frac{1}{\alpha_\lambda} \hat{e}_\lambda,$$

where the closure of  $e_\lambda$  is an element of the skein  $\mathcal{C}$ .

In defining  $Q_\lambda$  in this way we have to ensure that the coefficient ring includes denominators  $\alpha_\lambda$ . There is an explicit formula

$$\alpha_\lambda = \prod_{x \in \lambda} s^{c(x)} \frac{s^{h(x)} - s^{-h(x)}}{s - s^{-1}}$$

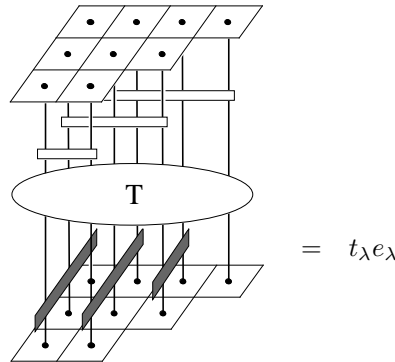
for  $\alpha_\lambda$  as a product over the cells  $x$  of  $\lambda$ .

In this formula the *content*,  $c(x)$ , of a cell  $x$  in position  $(i, j)$  in a Young diagram for  $\lambda$  is  $c(x) = j - i$ . The *hook length*,  $h(x)$ , of the cell  $x$  is the

total number of cells immediately to its right and immediately below it in the Young diagram.

Thus the denominators in  $Q_\lambda$  are indeed of the form  $s^k - s^{-k}$ , where the largest value of  $k$  is the largest hook length of any cell. This occurs for the cell in position  $(1, 1)$ , at the top left of  $\lambda$ .

One striking feature<sup>2</sup> of the elements  $e_\lambda$  is their ‘internal stability’, namely that if any tangle  $T$  is inserted in  $D^2 \times I$  between the white and the grey boxes, as shown schematically here, the resulting element of the skein is just some scalar multiple  $t_\lambda e_\lambda$  of  $e_\lambda$ .



The fact that  $e_\lambda^2 = \alpha_\lambda e_\lambda$  for some  $\alpha_\lambda \in \Lambda$  is an immediate consequence of this, although we need to know also that  $\alpha_\lambda \neq 0$  in order to construct  $Q_\lambda$ .

An important case is when  $T$  is the complete right-hand curl on  $|\lambda|$  strings. The resulting scalar  $f_\lambda \in \Lambda$  is known as the *framing factor* for  $\lambda$ . When the invariant  $P(L; \dots, Q_\lambda, \dots)$  is calculated with one component of the link  $L$  decorated by  $Q_\lambda$ , and the framing on that component is increased by 1, keeping the decorations of all other components unchanged, then the value of the invariant is multiplied by  $f_\lambda$ . This can be readily seen because the two invariants to be compared can be calculated from diagrams which differ only in having  $e_\lambda$  with or without the full curl inside it as one part of the complete diagram. A direct skein theory calculation<sup>2</sup> gives a cell-based formula

$$f_\lambda = v^{-|\lambda|} s^{n_\lambda}, \text{ where } n_\lambda = 2 \sum_{x \in \lambda} c(x),$$

twice the sum of the content of the cells.

Besides the full curl, another important central element of the Hecke algebra  $H_m$  is the element  $T^{(m)}$ . The internal stability ensures that plac-

ing  $T^{(m)}$  inside  $e_\lambda$  results in a multiple  $s_\lambda e_\lambda$ . Then the closure of  $e_\lambda$  is an eigenvector of the meridian map with eigenvalue  $s_\lambda$ , and can therefore be identified with one of the basis elements  $Q_{\lambda,\mu}$ , up to a scalar. This is the argument adopted by Lukac<sup>7</sup> to identify his element  $Q_\lambda$ , originally constructed in terms of Schur functions as a determinant of a matrix with entries drawn from the elements  $\{h_k\}$ , with Aiston's element constructed from the idempotent  $e_\lambda$ .

## 6. Unitary quantum invariants

Quantum groups give rise to 1-parameter invariants  $J(K; W)$  of an oriented framed knot  $K$  depending on a choice of finite dimensional module  $W$  over the quantum group, following constructions of Turaev and others<sup>2,17,19</sup>. This choice is referred to as *colouring*  $K$  by  $W$ , and can be extended for a link to allow a choice of colour for each component.

Fix a natural number  $N$ . When we colour  $K$  by a finite dimensional module  $W$  over the quantum group  $sl(N)_q$ , its invariant  $J(K; W)$  depends on one variable  $s$ . The invariant  $J$  is linear under direct sums of modules and all the modules over  $sl(N)_q$  are semi-simple, so we can restrict our attention to the irreducible modules  $V_\lambda^{(N)}$ . For  $sl(N)_q$  these are indexed by partitions  $\lambda$  with at most  $N$  parts, without distinguishing two partitions which differ in some initial columns with  $N$  cells.

To help in our comparison between Homfly satellite invariants and quantum invariants of  $K$  we write  $P(K; Q)$  for  $P(K * Q)$  and more generally  $P(L; Q_1, Q_2, \dots, Q_k)$  for the Homfly polynomial of a link  $L$  when its components are decorated by  $Q_1, \dots, Q_k$  respectively.

**Theorem 6.1 (Comparison theorem).** (1) *The  $sl(N)_q$  invariant for the irreducible module  $V_\lambda^{(N)}$  is the Homfly invariant for the knot decorated by  $Q_\lambda$  with  $v = s^{-N}$ , suitably normalised as in.<sup>6</sup> Explicitly,*

$$P(K; Q_\lambda)|_{v=s^{-N}} = x^{k|\lambda|^2} J(K; V_\lambda^{(N)})$$

where  $k$  is the writhe of  $K$ , and  $x = s^{1/N}$ .

- (2) *Each invariant  $P(K; Q)|_{v=s^{-N}}$  is a linear combination of quantum invariants  $\sum c_\alpha J(K; W_\alpha)$ .*  
 (3) *Each  $J(K; W)$  is a linear combination of Homfly invariants*

$$\sum d_j P(K; Q_j)|_{v=s^{-N}}.$$

**Remark 6.1.**

- In the special case when  $N = 2$  we can interpret quantum invariants of  $K$  in terms of Kauffman bracket satellite invariants, using the skein of the annulus based on the Kauffman bracket relations. This simpler skein is a quotient of the algebra  $\mathcal{C}$ . More generally the  $sl(N)_q$  invariants depend only on a quotient of the algebra  $\mathcal{C}$  for each  $N$ .
- The 2-variable invariant  $P(K; Q)$  can be recovered from the specialisations  $P(K; Q)|_{v=s^{-N}}$  for sufficiently many  $N$ .
- If the pattern  $Q$  is a closed braid on  $m$  strings then we only need use partitions  $\lambda \vdash m$ , since  $\mathcal{C}_m$  is spanned by  $\{Q_\lambda\}_{\lambda \vdash m}$ . Conversely, to realise  $J(K; V_\lambda^{(N)})$  with  $\lambda \vdash m$  we can use closed  $m$ -braid patterns.

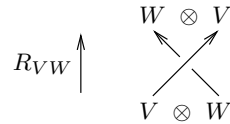
**6.1. Basic constructions of quantum invariants**

A quantum group  $\mathcal{G}$  is an algebra over a formal power series ring  $\mathbf{Q}[[\hbar]]$ , typically a deformed version of a classical Lie algebra. We write  $q = e^\hbar, s = e^{\hbar/2}$  when working in  $sl(N)_q$ . A finite dimensional module over  $\mathcal{G}$  is a linear space on which  $\mathcal{G}$  acts.

Crucially,  $\mathcal{G}$  has a coproduct  $\Delta$  which ensures that the tensor product  $V \otimes W$  of two modules is also a module. It also has a *universal R-matrix* (in a completion of  $\mathcal{G} \otimes \mathcal{G}$ ) which determines a well-behaved module isomorphism

$$R_{VW} : V \otimes W \rightarrow W \otimes V.$$

This has a diagrammatic view indicating its use in converting coloured tangles to module homomorphisms.



A braid  $\beta$  on  $m$  strings with permutation  $\pi \in S_m$  and a colouring of the strings by modules  $V_1, \dots, V_m$  leads to a module homomorphism

$$J_\beta : V_1 \otimes \dots \otimes V_m \rightarrow V_{\pi(1)} \otimes \dots \otimes V_{\pi(m)}$$

using  $R_{V_i, V_j}^{\pm 1}$  at each elementary braid crossing. The homomorphism  $J_\beta$  depends *only on the braid  $\beta$  itself*, not its decomposition into crossings, by the Yang-Baxter relation for the universal  $R$ -matrix.

When  $V_i = V$  for all  $i$  we get a module homomorphism  $J_\beta : W \rightarrow W$ , where  $W = V^{\otimes m}$ . Now any module  $W$  decomposes as a direct sum  $\bigoplus (W_\mu \otimes V_\mu^{(N)})$ , where  $W_\mu \subset W$  is a linear subspace consisting of the *highest weight vectors* of type  $\mu$  associated to the module  $V_\mu^{(N)}$ . Highest

weight subspaces of each type are preserved by module homomorphisms, and so  $J_\beta$  determines (and is determined by) the restrictions  $J_\beta(\mu) : W_\mu \rightarrow W_\mu$  for each  $\mu$ , where  $\mu$  runs over partitions with at most  $N$  parts.

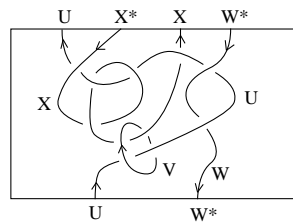
If a knot (or one component of a link)  $K$  is decorated by a pattern  $T$  which is the closure of an  $m$ -braid  $\beta$ , then its quantum invariant  $J(K * T; V)$  can be found from the endomorphism  $J_\beta$  of  $W = V^{\otimes m}$  in terms of the quantum invariants of  $K$  and the restriction maps  $J_\beta(\mu) : W_\mu \rightarrow W_\mu$  by the formula

$$J(K * T; V) = \sum c_\mu J(K; V_\mu^{(N)}) \tag{1}$$

with  $c_\mu = \text{tr } J_\beta(\mu)$ . This formula follows from lemma II.4.4 in.<sup>18</sup> We set  $c_\mu = 0$  when  $W$  has no highest weight vectors of type  $\mu$ .

More generally the methods of Reshetikhin and Turaev allow the quantum groups  $\mathcal{G} = SU(N)_q$  to be used to represent oriented tangles whose components are coloured by  $\mathcal{G}$ -modules as  $\mathcal{G}$ -module homomorphisms. One additional feature is needed, namely the use of the dual module  $V^*$  defined by means of the antipode in  $\mathcal{G}$ , (an antiautomorphism of  $\mathcal{G}$  which is part of its structure as a Hopf algebra). When the components of the tangle are coloured by modules the tangle itself is represented by a homomorphism from the tensor product of the modules which colour the strings at the bottom to the tensor product of the modules which colour the strings at the top, provided that the string orientations are inwards at the bottom and outwards at the top. The dual module  $V^*$  comes into play in place of  $V$  when an arc of the tangle coloured by  $V$  has an output at the bottom or an input at the top.

For example, the  $(4, 2)$ -tangle below, when coloured as shown, is represented by a homomorphism  $U \otimes W^* \rightarrow U \otimes X^* \otimes X \otimes W^*$ .



It is possible to build up the definition so that consistently coloured tangles are represented by the appropriate composite homomorphisms, starting from a definition of the homomorphisms for the elementary oriented tangles. Two cases, depending on the orientation, must be considered for both the local maximum and the local minimum, and a little care is needed here

to ensure consistency. The final result is a definition of a homomorphism which is invariant when the coloured tangle is altered by  $R_{II}$  and  $R_{III}$ . When applied to an oriented  $k$ -component link diagram  $L$  regarded as an oriented  $(0, 0)$ -tangle it gives an element  $J(L; V_1, \dots, V_k) \in \Lambda = \mathbf{Q}[[\hbar]]$  for each colouring of the components of  $L$  by  $\mathcal{G}$ -modules, which is an invariant of the framed oriented link  $L$ .

The construction is simplified in the case of  $sl(2)_q$  by the fact that all modules are isomorphic to their dual, and so orientation of the strings plays no role.

The quantum group invariants based on  $sl(3)_q$  also admit a combinatorial simplification due to Kuperberg to allow an easier diagrammatic calculation of them. At the same time the quantum group itself is straightforward enough to make it possible to work directly with some of the smaller dimensional modules,<sup>12,16</sup>

## 7. Manifold invariants

Following work of Reshetikhin and Turaev, in response to ideas of Witten, there are increasingly sophisticated ways to construct invariants of oriented 3-dimensional manifolds based on quantum groups, and correspondingly on knot invariants such as the Homfly satellite invariants. The basic principles come from the original paper of Reshetikhin and Turaev, adapted at various times to give easier details in special cases, notably the case of the quantum  $SU(2)$  invariants, as for example in<sup>11</sup>.

### 7.1. Surgery presentation

The strategy is to present the manifold  $M$  by *surgery on a framed link*  $L$  with  $k$  components. This means that  $M$  is given by removing a neighbourhood of  $M$  from  $S^3$  to give a manifold with  $k$  torus boundary components and then reattaching a solid torus to each of the  $k$  boundary tori in a way determined by the framing. The resulting manifold  $M = M(L)$  depends on the choice of  $L$ . Any other link  $L'$  which also determines the same manifold  $M$  is related to  $L$  by a sequence of *Kirby moves* and their inverses.

These can be summarised as operations on framed link diagrams regarded in some way as a satellite of the unknot  $U_0$  with framing 0. Then we can replace a link  $L = U_0 * Q$  by  $U_{\pm 1} * A_1 Q$ , where  $U_{\pm 1}$  is the unknot with framing  $\pm 1$  and  $A_1 Q$  is the decoration  $Q$  with one extra parallel strand.

The strategy for finding invariants of  $M$  is then to identify knot invariants of framed links which are unchanged by the two basic Kirby moves.



### 7.2. *Manifolds with boundary*

The whole setting of manifold invariants is extended to include manifolds with boundary, regarded as cobordisms between two subsets of their boundary components. The wider setting envisages a standard vector space for each boundary component, associating a vector space to the incoming and outgoing boundary, with a linear map between them determined by the manifold itself, in such a way that pasting together manifolds corresponds to composition of linear maps. This is sometimes termed a ‘modular functor’ or ‘topological quantum field theory’ (TQFT).

Associated to the empty boundary component is the 1-dimensional space of scalars. A closed manifold then produces a linear map from scalars to scalars, in other words a scalar.

The exterior of a link  $L$ , with  $k$  torus boundary components thought of as the incoming boundary, and empty outgoing boundary, fits in to this general scheme. We could take the skein  $\mathcal{C}$  (or  $\mathcal{C}^+$ ) as the linear space associated to a torus and use the Homfly satellite invariants of  $L$  to provide a linear map from the  $k$ -fold tensor product of  $\mathcal{C}$  to the scalars. One difficulty in trying to extend this to give a TQFT is that there is no immediate candidate for handling outgoing torus boundaries and hence no scope for gluing manifolds together along torus boundary components.

All the same, it suggests that what might be needed when attaching a solid torus to a boundary component would be to find an invariant for a solid torus, regarded as having empty incoming boundary and a torus as outgoing boundary. According to the proposed scheme we would need a linear map from the scalars to the linear space  $\mathcal{C}$ , which simply means the choice of one preferred element  $\Omega$ , say, of  $\mathcal{C}$ . The resulting scalar for the manifold given by attaching a solid torus to each boundary component of the exterior of  $L$  would then be the evaluation of the satellite invariant of  $L$  where each component is decorated by  $\Omega$ , giving  $P(L : \Omega, \dots, \Omega)$  as the invariant of the manifold  $M(L)$ .

Although it is *not* possible to find such a universal element  $\Omega$  to carry through this plan it turns out that *a restricted version of this idea, sketched below, can be made to work.*

### 7.3. *Evaluation of knot invariants*

The comparison between  $sl(N)_q$  invariants and Homfly satellite invariants uses the evaluation of the Homfly invariants by putting  $v = s^{-N}$ . This is the ring homomorphism  $s_N : \Lambda \rightarrow \mathbf{Z}[s^{\pm 1}, (s^r - s^{-r})^{-1}]$  with  $s_N(v) =$

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$s^{-N}, s_N(s) = s.$

For the trivial knot  $U$  write

$$\delta(Q) = P(U; Q) \in \Lambda.$$

Now  $\delta(Q_1 Q_2) = \delta(Q_1)\delta(Q_2)$ . There is a nice formula

$$\delta(Q_\lambda) = \prod_{x \in \lambda} \frac{v^{-1}s^{c(x)} - vs^{-c(x)}}{s^{h(x)} - s^{-h(x)}}. \quad (2)$$

Since  $s_N(\delta(Q_\lambda)) = J(U; V_\lambda^{(N)})$  is the ‘quantum dimension’ of the module  $V_\lambda^{(N)}$  it is common to call  $\delta(Q)$  the *quantum dimension* of  $Q \in \mathcal{C}$ .

It follows from (2) that  $s_N(\delta(Q_\lambda)) = 0$  if  $\lambda$  has more than  $N$  rows. It is then also true that  $s_N(P(L; \dots, Q_\lambda, \dots)) = 0$  if  $\lambda$  has more than  $N$  rows.

To find the  $s_N$  evaluation of a Homfly satellite invariant we then only need to know its value for decorations with at most  $N$  rows. This can be simplified further, as decorations by  $\lambda$  and  $\lambda'$  give the same  $s_N$  evaluation when  $\lambda$  and  $\lambda'$  differ by a number of columns with exactly  $N$  cells in each.

For example in calculating the  $s_2$  evaluation (to get the Jones polynomial) we only need to use decorations with one row.

#### 7.4. Level invariants for manifolds

Following Witten, Reshetikhin and Turaev we can use quantum group invariants to get a sequence of manifold invariants, along the general lines proposed above.

Choose  $N \geq 2$  and a further positive integer  $l$ , termed the *level*.

Write

$$\Omega_{N,l} = \sum_{\lambda \in (N-1,l)} \delta(Q_\lambda) Q_\lambda,$$

where  $(N-1, l)$  is the finite set of partitions with at most  $N-1$  rows and at most  $l$  columns.

Take  $s \in \mathbf{C}$  to satisfy  $s^{2(l+N)} = 1$ .

**Theorem 7.1.** *The evaluation  $s_N(P(L; \Omega_{N,l}, \dots, \Omega_{N,l})) \in \mathbf{C}$  is an invariant of the manifold  $M(L)$ , up to a normalising factor depending on the linking numbers of  $L$ .*

In this form the result is shown by Aiston<sup>1</sup> with some refinements by Lukac;<sup>7</sup> a brief account can also be found in,<sup>13</sup> where the more natural use of dual modules in places makes for a slightly easier argument. In this

context, having chosen  $N$ , the Young diagram of the partition  $\lambda^*$  dual to  $\lambda$  is the complement of the diagram of  $\lambda$  in an  $N \times \lambda_1$  rectangle.

The amended definition for  $\Omega_{N,l}$  is

$$\Omega_{N,l} = \sum_{\lambda \in (N-1,l)} \delta(Q_{\lambda^*}) Q_{\lambda}.$$

Since  $s_N(\delta(Q_{\lambda^*})) = s_N(\delta(Q_{\lambda}))$ , the result above is unaltered.

The main technical fact needed about  $\Omega_{N,l}$  is that the product

$$S\Omega_{N,l} = \delta(S)\Omega_{N,l}$$

for any  $S \in \mathcal{C}^+$ , modulo elements of an ideal which contribute 0 to the  $s_N$  evaluation when  $s^{2(N+l)} = 1$ .

**Proof.**

To show Kirby move invariance, when we change  $L = U_0 * Q$  to  $U_{\pm 1} * A_1 Q$ , decorate the components of the diagram  $Q$  in the annulus by  $\Omega_{N,l}$  to determine an element  $S \in \mathcal{C}^+$ .

We need to compare  $P(U_0; S)$  and  $P(U_{\pm 1}; S\Omega_{N,l})$ . Now  $P(U_0; S) = \delta(S)$  and

$$P(U_{\pm 1}; S\Omega_{N,l}) = P(U_{\pm 1}; \delta(S)\Omega_{N,l}) = \delta(S)P(U_{\pm 1}; \Omega_{N,l}),$$

after evaluation.

The factors  $c_{\pm} = P(U_{\pm 1}; \Omega_{N,l})$  are dealt with by the normalisation.  $\square$

**Remark 7.1.** When evaluating invariants under  $s_N$  with the additional restriction that  $s^{2(l+N)} = 1$  it is possible to replace  $\mathcal{C}^+$  as the decorating space by the finite dimensional space spanned by  $\{Q_{\lambda}\}, \lambda \in (N-1, l)$  in a straightforward way, since the space can be interpreted as a Verlinde algebra, given by factoring out a suitable ideal from  $\mathcal{C}^+$ .

In fact this space can be interpreted as the ring of polynomials in  $e_1, \dots, e_{N-1}$  modulo the ideal generated by the polynomials  $h_{l+1}, \dots, h_{l+N-1}$  written as polynomials in the elementary symmetric function, setting  $e_N = 1, e_m = 0, m > N$ .

In the case  $N = 2$  we only have to use polynomials in  $e_1 = A_1$  as decoration when evaluating satellite invariants, in other words linear combinations of parallels of our given link will provide all the satellite invariants needed.

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