

Four loop renormalization of  $\phi^3$  theory in six dimensions

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June, 2015.

**Abstract.** We renormalize six dimensional  $\phi^3$  theory in the modified minimal subtraction ( $\overline{\text{MS}}$ ) scheme at four loops. From the resulting  $\beta$ -function, anomalous dimension and mass anomalous dimension we compute four loop critical exponents relevant to the Lee-Yang edge singularity and percolation problems. Using resummation methods and information on the exponents of the relevant two dimensional conformal field theory we obtain estimates for exponents in dimensions 3, 4 and 5 which are in reasonable agreement with other techniques for these two problems. The renormalization group functions for the more general theory with an  $O(N)$  symmetry are also computed in order to obtain estimates of exponents at various fixed points in five dimensions. Included in this  $O(N)$  analysis is the full evaluation of the mass operator mixing matrix of anomalous dimensions at four loops. We show that its eigen-exponents are in agreement with the mass exponents computed at  $O(1/N^2)$  in the non-perturbative large  $N$  expansion.

# 1 Introduction.

Recently there has been renewed interest in analysing the  $\phi^3$  scalar quantum field theory which is perturbatively renormalizable in six spacetime dimensions. This interest is primarily due in the main to the modern development of the original conformal bootstrap method, [1, 2, 3, 4, 5, 6, 7], to study the fixed point structure of field theories in a nonperturbative way, [8, 9, 10, 11]. One of the aims is, for example, to ascertain whether a conformal window exists and if so for what range of parameters of the symmetry group. Such studies are not limited to two dimensions where the structure of the conformal group being infinite dimensional, is fundamentally different to the finite dimensional conformal group in  $d > 2$  where  $d$  is the spacetime dimension. Instead models exhibiting conformal symmetry in three, four and higher dimensions are of interest. One motivation for such studies rested in part on applications to dualities in higher spin AdS/CFT's [12, 13] as well as model building beyond the Standard Model. Indeed in terms of gauge theories the original study in quantum chromodynamics (QCD) of [14] suggested that for a certain range of the number of quarks there could be a nontrivial fixed point in strictly *four* dimensions. Such a fixed point, which is known as the Banks-Zaks fixed point, may have a connection with the chiral symmetry phase transition, [14]. Another motivation for studying conformal properties in higher dimensional theories rests in trying to generalize properties of theorems such as the two dimensional  $c$ -theorem, [15], to analogues in three and four dimensions. Though to have some insight into such extensions one has to be aware of the fixed point structure of the underlying quantum field theory. In this respect there has been interest in tackling this problem in scalar  $\phi^3$  theory in six and lower dimensions. See, for example, [16, 17, 18, 19] for recent in depth studies of the  $O(N)$  symmetric  $\phi^3$  theory. For instance, the fixed point structure has been comprehensively studied perturbatively to three loops in [18, 20]. This has also subsequently been extended to the theory with an  $Sp(N)$  symmetry in [21].

Those works exploit the renormalization of the theory from a generation ago, [22, 23, 24], when modern multiloop computational techniques were not available. One highlight of [18, 20] was the estimate of the conformal window. It is possible to compute order by order in perturbation theory the value of  $N$  as a function of the spacetime dimension for which the stable infrared fixed point ceases to exist. This critical value is denoted by  $N_{cr}$ . Evaluating the expression for  $N_{cr}$  in five dimensions the one loop result of [25, 18] was that the leading order value is  $N_{cr} = 1038$ . Indeed the first examination of the conformal window in six dimensional  $O(N)$   $\phi^3$  theory was given in [25]. In [20] the three loop analysis reduced this to  $N_{cr} = 64$ . Such a large reduction gives credence to the hope that the value can be reduced further. Indeed there appears to be support for this in conformal bootstrap analyses, [26, 27, 28], and moreover the series for  $N_{cr}$  in [20] appears to be convergent. However, partly in order to resolve this but mainly for applications to other problems, the primary aim of this article is to extend the work of [22, 23, 24] to *four* loops. This is now possible given the advances in methods to evaluate high loop massless Feynman diagrams and in particular 2-point functions in the period since [23, 24]. Moreover, while this will involve a large amount of integration such a renormalization could not proceed in the absence of powerful symbolic manipulation languages as well as powerful computing resources. Therefore, we will not only provide a comprehensive analysis of the four loop structure of the renormalization group functions of  $\phi^3$  theory in the modified minimal subtraction ( $\overline{MS}$ ) scheme, but we will also outline the computational algorithm used. In this respect we will provide the basic four loop massless 2-point function Feynman integrals which are central to the calculations to allow others to extend the programme to other models.

In indicating our analysis will be extensive this not only means that we will focus on the  $O(N)$  symmetric theory but also simpler models where there is no or different symmetry. This is because of the connection of these theories to condensed matter or statistical physics problems.

The simplest such case is the theory with one scalar field. It is of interest because of the relation to the Lee-Yang edge singularity problem, [29]. When the coupling constant of the single field theory is purely imaginary, [29], then the critical exponent  $\sigma$ , which is determined at the  $d$ -dimensional Wilson-Fisher fixed point, is the main quantity of physical interest. It can be determined via the renormalization group functions of the theory. The exponent  $\sigma$  has been estimated by different techniques such as high temperature series and Monte Carlo methods for the discrete dimensions less than six. Several references to such work are [30, 31, 32], for instance. More recently the conformal bootstrap programme have been applied to it, [33]. With these evidently more powerful methods giving reasonable agreement it is therefore timely to extend the  $\epsilon$ -expansion analysis of critical exponents to  $O(\epsilon^4)$  where we will use  $d = 6 - 2\epsilon$  throughout. Not only will this improve the estimates given in [23, 24] but we will follow the method of [32] where constrained Padé approximants were used. Central to this idea is the exploitation of the properties of the underlying *two* dimensional conformal field theory. There the corresponding critical exponents are known exactly. This information is used like a boundary condition on the Padé approximant and, as will be evident, will significantly improve exponent estimates for large values of  $\epsilon$ . It will transpire that this analytic perturbative approach will give exponent estimates which are not unreasonable in comparison with numerically intense alternatives. While these comments have been driven by the Lee-Yang edge singularity connection to  $\phi^3$  theory, they will equally apply to a similar, in terms of the underlying quantum field theory, but different physical problem. This is the percolation problem which can be formulated in continuum  $\phi^3$  theory by a multiplet of scalar fields with a group valued coupling constant, [34]. In the appropriate replica limit the critical exponents at the Wilson-Fisher fixed point equate to the 1-state Potts model and hence percolation. We will provide estimates for a large set of critical exponents in the dimensions between two and six. Again it will be the case that the estimates are in keeping with other approaches.

That we draw attention to these applications in the same context as the recent  $O(N)$   $\phi^3$  theory studies is not unconnected. As outlined in [20] the  $O(N)$  theory has to have a connectivity with an underlying two dimensional conformal field theory in respect of the various critical points which emerge. If ultimately the  $d$ -dimensional theory can be identified with such a two dimensional theory then it will open the possibility of extended constrained Padé approximants to the exponents of the various fixed points of the  $O(N)$  theory. Though it is not currently clear what the application to a physical problem is at the moment. One other motivation in [18] was to make the connection of  $\phi^3$  theory with  $O(N)$   $\phi^4$  theory in the dimension range  $4 < d < 6$  at the Wilson-Fisher fixed point via the large  $N$  expansion in  $d$ -dimensions. It is well known that in  $2 < d < 4$  the  $O(N)$   $\phi^4$  theory is in the same universality class as the  $O(N)$  nonlinear  $\sigma$  model and the three dimensional Heisenberg ferromagnet. Above four dimensions it turns out that at the Wilson-Fisher fixed point the  $d$ -dimensional theory is in the same universality class as one of the  $O(N)$   $\phi^3$  fixed points. So using the large  $N$  expansion we can evaluate the four loop  $d$ -dimensional critical exponents and compare them with those known from the explicit large  $N$  expansion, [35, 36, 37, 38]. This will provide a nontrivial check on our perturbative results. Indeed we will compute the full mixing matrix for the mass operators in the  $O(N)$  case and show the subtlety in connecting the mass eigen-critical exponents with the two separate mass critical exponents in the  $d$ -dimensional  $O(N)$   $\phi^4$  theory. In this context it is interesting to note that the determination of the exponent  $\eta$  at  $O(1/N^3)$ , [37], was originally evaluated as a function of  $d$  by using a  $d$ -dimensional conformal bootstrap method but in an analytic as opposed to the modern numerical approach.

The article is organized as follows. We summarize the relevant background to performing the four loop renormalization of the basic six dimensional  $\phi^3$  theory in section 2. The notation used throughout is given there as well as the technical details of how the computation was organized.

In particular we emphasise that all the basic renormalization constants can be extracted purely from an evaluation of the 2-point function. Results of this method are summarized in the subsequent section where the renormalization group functions are recorded for various symmetry configurations. Included in this are theories with  $SU(N_c)$  symmetry which were examined as early toy examples of the strong interactions. Sections 4 and 5 provide details respectively of the results for the Lee-Yang edge singularity and percolation problems. The final main problem we analyse is the full four loop structure of the  $O(N)$  version of  $\phi^3$  theory which is provided in section 6. Conclusions are given in section 7. In addition two appendices are provided. The first gives the  $\epsilon$  expansions of the relevant basic but nontrivial integrals we needed at four loops. The second extends the three loop results for various values of  $N$  given in the appendix A of [28] to four loops but also includes the mass eigen-critical exponents for each fixed point as an expansion in  $\epsilon$ .

## 2 Background.

In this section we outline the technical aspects of the four loop renormalization of six dimensional  $\phi^3$  theory. We will consider the theory in different guises depending on the particular application required to extract, for instance, critical exponents. These will depend essentially on different decorations of the underlying Lagrangian, which is

$$L = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{6} \phi^3 \quad (2.1)$$

with various symmetry groups. Here  $g$  is the coupling constant which is dimensionless in six dimensions. For example, both the Lee-Yang edge singularity and percolation problems can be accommodated with the more general theory, [23, 24],

$$L = \frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{g}{6} d^{ijk} \phi^i \phi^j \phi^k \quad (2.2)$$

where there are a multiplet of fields  $\phi^i$  and a group theory tensor which is totally symmetric in its indices in addition to a coupling constant  $g$ . The Lagrangians for theories with more than one set of fields will be discussed in later sections but the underlying calculational procedure is effectively the same as that introduced here. For (2.2) we use the same assumptions as [23, 24] for the renormalizability of the Lagrangian. Briefly this reduces to two observations. First, for 2-point self-energy graphs, including subgraphs, the product of two coupling tensors satisfies

$$d^{i_1 i_3 i_4} d^{i_2 i_3 i_4} = T_2 \delta^{i_1 i_2} \quad (2.3)$$

where we use a different and more systematic notation here compared to [23, 24] for what corresponds to the group theory Casimirs. The second assumption in [23, 24] is that the product of coupling tensors is such that if the product has three free indices then it is proportional to  $d^{ijk}$  itself. In this way (2.2) will clearly be renormalizable. Given this it might be thought that there is a sizeable number of group invariants which can appear at high loop order. This is not the case as it transpires that to four loops the following Casimirs suffice to write the renormalization group functions in a concise form. These are

$$\begin{aligned} d^{ii_1 i_2} d^{j i_1 i_3} d^{k i_2 i_3} &= T_3 d^{ijk} \quad , \quad d^{ii_1 i_2} d^{j i_3 i_4} d^{k i_5 i_6} d^{i_1 i_3 i_5} d^{i_2 i_4 i_6} = T_5 d^{ijk} \\ d^{ii_1 i_2} d^{j i_3 i_4} d^{k i_5 i_6} d^{i_1 i_3 i_7} d^{i_2 i_5 i_8} d^{i_4 i_6 i_9} d^{i_7 i_8 i_9} &= T_{71} d^{ijk} \\ d^{ii_1 i_2} d^{j i_3 i_4} d^{k i_5 i_6} d^{i_1 i_3 i_7} d^{i_2 i_5 i_8} d^{i_4 i_8 i_9} d^{i_6 i_7 i_9} &= T_{72} d^{ijk} \\ d^{ii_1 i_2} d^{j i_3 i_4} d^{k i_5 i_{12}} d^{i_1 i_5 i_6} d^{i_2 i_7 i_8} d^{i_3 i_9 i_{12}} d^{i_4 i_7 i_{10}} d^{i_6 i_8 i_{11}} d^{i_9 i_{10} i_{11}} &= T_{91} d^{ijk} \end{aligned}$$

$$\begin{aligned}
d^{ii_1i_2} d^{jj_3i_4} d^{ki_{11}i_{12}} d^{i_1i_5i_6} d^{i_2i_7i_8} d^{i_3i_5i_9} d^{i_4i_7i_{10}} d^{i_6i_8i_{11}} d^{i_9i_{10}i_{12}} &= T_{92} d^{ijk} \\
d^{ii_1i_2} d^{jj_3i_4} d^{ki_6i_{12}} d^{i_1i_5i_6} d^{i_2i_7i_8} d^{i_3i_5i_9} d^{i_4i_7i_{10}} d^{i_8i_{11}i_{12}} d^{i_9i_{10}i_{11}} &= T_{93} d^{ijk} \\
d^{ii_1i_2} d^{jj_3i_4} d^{ki_5i_{12}} d^{i_1i_5i_6} d^{i_2i_7i_8} d^{i_3i_9i_{12}} d^{i_4i_{10}i_{11}} d^{i_6i_7i_{10}} d^{i_8i_9i_{11}} &= T_{94} d^{ijk} \\
d^{ii_1i_2} d^{jj_3i_4} d^{ki_8i_{12}} d^{i_1i_5i_6} d^{i_2i_7i_8} d^{i_3i_5i_9} d^{i_4i_{10}i_{11}} d^{i_6i_7i_{10}} d^{i_9i_{11}i_{12}} &= T_{95} d^{ijk} \\
d^{ii_1i_2} d^{jj_3i_4} d^{ki_{11}i_{12}} d^{i_1i_3i_5} d^{i_2i_6i_7} d^{i_4i_6i_8} d^{i_5i_9i_{10}} d^{i_7i_9i_{11}} d^{i_8i_{10}i_{12}} &= T_{96} d^{ijk} \\
d^{ii_1i_2} d^{jj_3i_4} d^{ki_5i_{12}} d^{i_1i_5i_6} d^{i_2i_7i_8} d^{i_3i_9i_{12}} d^{i_4i_7i_{10}} d^{i_6i_{10}i_{11}} d^{i_8i_9i_{11}} &= T_{97} d^{ijk} \\
d^{ii_1i_2} d^{jj_3i_4} d^{ki_6i_{12}} d^{i_1i_5i_6} d^{i_2i_7i_8} d^{i_3i_5i_9} d^{i_4i_7i_{10}} d^{i_8i_9i_{11}} d^{i_{10}i_{11}i_{12}} &= T_{98} d^{ijk} \\
d^{ii_1i_2} d^{jj_3i_4} d^{ki_{11}i_{12}} d^{i_1i_5i_6} d^{i_2i_7i_8} d^{i_3i_5i_9} d^{i_4i_7i_{10}} d^{i_6i_{10}i_{11}} d^{i_8i_9i_{12}} &= T_{99} d^{ijk} .
\end{aligned} \tag{2.4}$$

where we have included the previous invariants up to three loops of [23, 24]. So at most there are nine new group invariants at four loops in principle. Though it will turn out that while these nine correspond to different graph topologies the four loop Casimirs,  $T_{9m}$ , do not all have distinct values. The syntax here is that the first label on  $T_n$  corresponds to the number of tensors  $d^{ijk}$  in the product and the second label distinguishes between different invariants. If the first label is even then it arises in a 2-point function while an odd label indicates an invariant which will appear in the coupling constant renormalization. Labelling the actual group indices in each product in (2.4) with subscripts allows one to straightforwardly construct the underlying topologies graphically. Each paired index corresponds to a propagator and these will join at vertices defined by each coupling tensor. To make contact with the notation used in [23, 24] for the invariants up to three loops, we note that  $T_2 = \alpha$ ,  $T_3 = \beta$ ,  $T_5 = \gamma$ ,  $T_{71} = \delta$  and  $T_{72} = \lambda$  when one observes that the defining graphs given in Figure 1 of [23] appear in reversed mirror image. We prefer the notation  $T_n$  here to avoid confusion with the standard notation of various critical exponents which also use Greek letters.

Although the renormalization of both (2.1) and (2.2) are the same we will focus the technical discussion on the former as the extension to tensor couplings is not onerous. For (2.1) we have to determine the wave function and coupling constant renormalization constants. Once these have been established then the determination of the mass anomalous dimension follows as a corollary even though the Lagrangian is massless. Indeed we work with massless fields throughout as this renders the four loop  $\beta$ -function accessible with a relatively minimal amount of computation. The first stage is to compute the 2-point and 3-point functions of (2.1). The Feynman graphs are generated using the QGRAF package, [39]. There are respectively 1, 2, 10 and 64 one, two, three and four loops diagrams to evaluate to the simple pole in  $\epsilon$  in dimensional regularization which we use throughout. Indeed we note that the index labelling in (2.4) is based on the QGRAF output. We note that throughout our convention is  $d = 6 - 2\epsilon$ . For the 3-point vertex the respective numbers are 1, 7, 56 and 540 which is an order of magnitude increase at four loops compared to the 2-point case. However, we will compute the coupling constant renormalization constant for (2.1) purely from the 2-point graphs by exploiting certain properties of the specific field theory. It is based on the following observation. If for the moment one considers the massive extension of (2.1) then the propagator can be formally expanded in powers of  $m^2$  where  $m$  is the mass via

$$\frac{1}{[k^2 - m^2]} = \frac{1}{k^2} + \frac{m^2}{(k^2)^2} + O(m^4) . \tag{2.5}$$

The first term on the right hand side of course corresponds to the massless theory. However, the second term represents the zero momentum insertion of the unit operator on a propagator. Diagrammatically for the self energy renormalization this corresponds to a 2-point function with a zero momentum insertion but more importantly this term would correspond to a 3-point function graph where one of the external legs has a nullified momentum. In other words this would be equivalent to a graph contributing to the coupling constant renormalization. As

we are computing in dimensional regularization and only interested in the  $\overline{\text{MS}}$  renormalization scheme the coupling constant renormalization constant can be correctly extracted from this nullified external momentum configuration. Indeed in four dimensional gauge theories this is the standard procedure for three loop renormalization [40, 41]. One concern is that nullifying an external momentum in (2.1) could introduce unwanted infrared divergences which would be indistinguishable from ultraviolet ones in dimensional regularization. Indeed in four dimensions if a propagator was present in a massless graph which had the form of the second term of (2.5) then it would be infrared singular. That it is not an issue for (2.1) is because in six dimensions such a propagator is by contrast infrared safe even in scalar field theories unlike in four dimensions. Therefore, this observation radically reduces the number of Feynman graphs to be computed. One downside is that there are more integrals to determine but this is surmountable as we will indicate later. One concern which may arise with the expansion approach is that symmetry factors in the 2-point function may not be set in such a way that after expansion the term relating to the vertex has the incorrect factor. We have checked that it is the case to three loops. We illustrate this at one loop. There the basic one loop self-energy graph has a symmetry factor of  $\frac{1}{2}$  but the one loop 3-point vertex has a unit symmetry factor. However, using (2.5) there are two  $O(m^2)$  terms which cancel the factor of  $\frac{1}{2}$  so that everything tallies. One final point resides in the determination of the renormalization of the mass operator  $\frac{1}{2}\phi^2$  in (2.1). It transpires that in this theory the mass and coupling constant renormalization are equivalent. This is implicit in the algorithm we discussed for the determination of the latter and has already been noted in, for instance, [23, 24]. So in (2.1) no extra computation has to be carried out. However, for (2.2) the mass and coupling constant renormalization constants are different. This is evident in the algorithm we have introduced. Both can be determined using our method of only computing the graphs for the 2-point function by extending (2.5) and mapping each massless propagator with

$$\frac{\delta^{ij}}{k^2} \mapsto \frac{\delta^{ij}}{k^2} + \frac{m_1^2 \delta^{ij}}{(k^2)^2} + \frac{m_2^2 g d^{ijk_e}}{(k^2)^2} \quad (2.6)$$

after the graphs have been generated with QGRAF. We have included the group theory structure on the propagators and  $m_i^2$  are parameters of the dimension of a mass. These parameters are included as a label to distinguish which term is which when one comes to extract the various wave function, mass and coupling constant renormalization constants in the sum of all the contributions to our 2-point function. That involving  $m_1$  corresponds to the mass operator. The final term of (2.6) with label  $m_2$  corresponds to the zero momentum insertion for the renormalization of the coupling constant  $g$ . The index  $k_e$  in (2.6) is the index associated with the third leg of the 3-point function and is a fixed free external index.

Having discussed the method to isolate the necessary graphs and integrals contributing to the wave function, coupling constant and mass operator renormalization constants we now discuss the method used to determine their divergent part. The approach is to use the Laporta algorithm, [42]. This is a systematic application of integration by parts to establish towers of relations between integrals which are then solved algebraically to express all integrals in terms of a relatively small set of basic integrals which are called masters. The values for these have to be determined directly as integration by parts can no longer be used. Once these are found then the integration algorithm is complete and we run an automatic determination of the respective Green's functions to find the underlying renormalization constants. Crucial to this is the symbolic manipulation language FORM, [43], and its threaded version TFORM, [44], which we use extensively to handle the large amounts of algebra. This arises since the reduction to masters produces relations where the coefficients are large rational polynomials in  $d$ . Not only have these to be expanded in powers of  $\epsilon$  and the masters substituted but the denominators may contain factors of  $(d-6)$  which are termed spurious poles. Thus not only have the polynomial coefficients to be expanded to higher order in  $\epsilon$  but some masters need to be evaluated to a precision beyond

the simple pole. Even for the simple scalar theory, (2.1), where there are no tensor integrals certain graphs become tedious to evaluate. For the present four loop computation we have used the REDUZE package, [45]. One useful feature is that the output relations between integrals can be readily converted to FORM input notation and thus included as a module within the automatic FORM evaluation.

The final ingredient in the algorithm is the explicit values for the master integrals. This requires a method other than integration by parts and we have adapted several approaches which have been used for similar problems in other contexts such as [45, 46]. In addition we were able to exploit a feature of REDUZE which means that we could do this systematically. By this we mean the following. When the relation between integrals are solved in REDUZE the algorithm has some internal criterion for determining which integrals are to be used as the masters in terms of which all other integrals are expressed. This is not always the best choice for the calculation of interest. For instance, some of the masters may be simple self-energy graphs where each subgraph is itself a lower loop self-energy graph or products of lower loop graphs. Such integrals are straightforward to determine and are retained in the set of masters. However, more difficult integrals remain such as those which are primitively divergent and it is their simple and only pole which will be needed. The feature of REDUZE which one exploits is that one can specify a set of integrals which the package identifies as the masters. This is done after the initial reduction has been determined and a database constructed, [46]. Therefore, we have chosen a set of basic master integrals which, aside from those which are simple to evaluate, are finite using Weinberg's theorem, [47]. This technique has been elaborated on recently in [48]. While this may seem to resolve the computational algorithm one has to be careful. This is because of the problem of spurious poles in  $\epsilon$  which means that not only the leading term of these finite integrals are required, but sometimes also several terms in powers of  $\epsilon$ . However, one also has to have some information on a master choice to be able to solve for these and other required coefficients.

To do this we use the known four loop self-energy master integrals given in [46]. These were determined by application of the glue-and-cut method of *five* loop primitive massless vacuum diagrams to varying orders in the  $\epsilon$  expansion. The results are consistent with a subsequent independent numerical sector decomposition evaluation given in [49]. Analytic evaluations were also developed thereafter in [50] to even higher powers in the  $\epsilon$  expansion compared to [46]. The only problem is that those masters were computed near *four* and not six dimensions which is the dimension we require them for to complete our master integral determination. This is achieved by using the method of [51, 52]. It allows one to relate a  $d$ -dimensional integral to a set of integrals in  $(d + 2)$ -dimensions. The latter set will always be of the same topology as the original lower dimensional one but with increased propagator powers. Indeed if the  $d$ -dimensional integral has  $P$  propagators then the  $(d + 2)$ -dimensional integrals will have  $(P + L)$  propagators by simple power counting where  $L$  is the number of loops. In our case if the four dimensional integral is a known master then it can be related to the as yet undetermined six dimensional integral. Though to do this one has to use the REDUZE database to effect the reduction to the unknown master and a set of previously evaluated masters. In other words if one builds the system of master evaluation from integrals with low numbers of propagators then one can move up the tower of unknown masters systematically until all the ones required for the renormalization have been found. In the list of masters given in [46] there is enough information in terms of the  $\epsilon$  expansion there in  $(4 - 2\epsilon)$  dimensions to ascertain all the Feynman graphs contributing to the full renormalization of (2.2) to four loops. To assist with future work we have recorded the values of certain integrals in  $(6 - 2\epsilon)$  dimensions in Appendix A. Rather than present the ones used in our efforts, so that others are not restricted to our particular basis of masters, we have presented the values for the same topologies given in [46]. These integrals were denoted by  $M_i$

and we have given several to quite a high order in  $\epsilon$ . This was partly because of the spurious pole problem. As a simple check on this evaluation of the four loop masters we have followed the same procedure at three loops using the three loop four dimensional masters given in [46]. When these were included in the automatic three loop renormalization we correctly reproduced the results of [22, 23, 24]. While a simple check we also had to do this exercise anyway as we needed higher terms in  $\epsilon$  at three loops since each three loop integral will be multiplied by counterterms. Hence they contribute to the four loop renormalization constants. Finally to effect the renormalization within the automatic computation we follow the method of [41]. In this one computes the Green's function in terms of the bare coupling constant. Then after all the graphs have been summed the bare variable is rescaled by the coupling constant renormalization constant. This systematically introduces the appropriate counterterms automatically and the overall remaining divergence in the sum is fixed by the associated unknown renormalization constant. One advantage of this approach is that it avoids the subtraction at the level of individual diagrams which is tedious and not possible to encapsulate easily in an automatic symbolic manipulation algorithm. Finally, with this we have provided all the computational pieces to fully renormalize six dimensional  $\phi^3$  theory in the  $\overline{\text{MS}}$  scheme at four loops.

### 3 Results.

We are now in a position to formally record the renormalization group functions at four loops for various formulations of six dimensional  $\phi^3$  theory. Throughout all our results will be in the  $\overline{\text{MS}}$  scheme and we have included an electronic file data with the main results of the article. For (2.2) we have

$$\begin{aligned}
\beta(g) = & [-T_2 + 4T_3] \frac{g^3}{8} + [-11T_2^2 + 66T_2T_3 - 108T_3^2 - 72T_5] \frac{g^5}{288} \\
& + [-821T_2^3 + 6078T_2^2T_3 - 12564T_2T_3^2 + 2592\zeta_3T_2T_5 - 9288T_2T_5 - 11664T_3^3 \\
& - 51840\zeta_3T_3T_5 + 61344T_3T_5 + 20736T_{71} + 62208\zeta_3T_{72} - 20736T_{72}] \frac{g^7}{41472} \\
& + [-20547T_2^4 + 1728\zeta_3T_2^3T_3 + 185774T_2^3T_3 - 31104\zeta_3T_2^2T_3^2 - 510960T_2^2T_3^2 \\
& + 127008\zeta_3T_2^2T_5 - 23328\zeta_4T_2^2T_5 - 285336T_2^2T_5 + 373248\zeta_3T_2T_3^3 - 437472T_2T_3^3 \\
& - 2716416\zeta_3T_2T_3T_5 + 559872\zeta_4T_2T_3T_5 + 2744064T_2T_3T_5 + 124416\zeta_3T_2T_{71} \\
& - 622080\zeta_5T_2T_{71} + 1005696T_2T_{71} + 2457216\zeta_3T_2T_{72} - 559872\zeta_4T_2T_{72} \\
& - 1005696T_2T_{72} - 1866240\zeta_3T_3^4 + 3005424T_3^4 + 8315136\zeta_3T_3^2T_5 - 1866240\zeta_4T_3^2T_5 \\
& - 6763392T_3^2T_5 - 5474304\zeta_3T_3T_{71} + 14929920\zeta_5T_3T_{71} - 7755264T_3T_{71} \\
& - 6096384\zeta_3T_3T_{72} + 2239488\zeta_4T_3T_{72} + 7755264T_3T_{72} + 1306368\zeta_3T_5^2 \\
& - 1321920T_5^2 - 7464960\zeta_3T_{91} + 7464960\zeta_5T_{91} - 746496T_{91} - 7464960\zeta_3T_{98} \\
& + 7464960\zeta_5T_{98} + 1492992\zeta_3T_{99} - 1866240\zeta_5T_{99} - 3732480\zeta_3T_{92} \\
& + 3732480\zeta_5T_{92} + 10450944\zeta_3T_{93} - 14929920\zeta_5T_{93} - 2239488\zeta_3T_{94} \\
& + 746496T_{94} + 10450944\zeta_3T_{95} - 14929920\zeta_5T_{95} - 7464960\zeta_3T_{96} + 7464960\zeta_5T_{96} \\
& + 5225472\zeta_3T_{97} - 7464960\zeta_5T_{97}] \frac{g^9}{1492992} + O(g^{11}) \tag{3.1}
\end{aligned}$$

and

$$\gamma_\phi(g) = -\frac{T_2}{12}g^2 + [-11T_2 + 24T_3] \frac{T_2g^4}{432}$$



$$\begin{aligned}
& + \left[ -821T_2^2 + 3222T_2T_3 - 3060T_3^2 + 2592\zeta_3T_5 - 4536T_5 \right] \frac{T_2g^6}{62208} \\
& + \left[ -2283T_2^3 + 576\zeta_3T_2^2T_3 + 12428T_2^2T_3 - 6912\zeta_3T_2T_3^2 - 14872T_2T_3^2 + 14112\zeta_3T_2T_5 \right. \\
& \quad - 2592\zeta_4T_2T_5 - 20928T_2T_5 + 41472\zeta_3T_3^3 - 43392T_3^3 - 87552\zeta_3T_3T_5 \\
& \quad - 10368\zeta_4T_3T_5 + 122496T_3T_5 + 13824\zeta_3T_{71} - 69120\zeta_5T_{71} + 59904T_{71} \\
& \quad \left. + 34560\zeta_3T_{72} + 31104\zeta_4T_{72} - 59904T_{72} \right] \frac{T_2g^8}{248832} + O(g^{10}) \tag{3.2}
\end{aligned}$$

where  $\zeta_z$  is the Riemann zeta function. As a check we have reproduced the three loop results of [23, 24] and moreover, the non-simple poles in  $\epsilon$  in the four loop renormalization constants correctly emerge. Their residues depend on the poles in  $\epsilon$  at the previous three loops. A comment on our conventions is in order at the outset in relation to other papers. As is usual, [23, 24], we have absorbed the common factor of  $S(d)/(2\pi)^d$ , where  $S(d)$  is the surface area of the  $d$ -dimensional unit sphere, into  $g^2$ . This factor plays no role in the values of critical exponents. Also in comparison with [23, 24] our renormalization group functions are defined with an overall factor of 2 different. This will be our convention throughout this and later sections. Finally, to map our results to [23, 24] and later to [20, 21] the sign of  $g^2$  needs to be reversed.

For the mass operator

$$\mathcal{O} = \frac{1}{2}\phi^i\phi^i \tag{3.3}$$

the anomalous dimension is

$$\begin{aligned}
\gamma_{\mathcal{O}}(g) &= -\frac{T_2}{2}g^2 + [-T_2 + 24T_3] \frac{T_2g^4}{48} \\
& + \left[ -380T_2^2 + 432\zeta_3T_2T_3 + 711T_2T_3 - 864\zeta_3T_3^2 - 1170T_3^2 - 756T_5 \right] \frac{T_2g^6}{1728} \\
& + \left[ -34560\zeta_3T_2^3 + 42635T_2^3 + 261792\zeta_3T_2^2T_3 - 69984\zeta_4T_2^2T_3 + 364812T_2^2T_3 \right. \\
& \quad - 544320\zeta_3T_2T_3^2 + 419904\zeta_4T_2T_3^2 - 1244160\zeta_5T_2T_3^2 - 200088T_2T_3^2 \\
& \quad + 69984\zeta_3T_2T_5 + 23328\zeta_4T_2T_5 - 505872T_2T_5 + 3825792\zeta_3T_3^3 - 559872\zeta_4T_3^3 \\
& \quad - 2488320\zeta_5T_3^3 - 134784T_3^3 - 2975616\zeta_3T_3T_5 - 466560\zeta_4T_3T_5 + 2488320\zeta_5T_3T_5 \\
& \quad + 2376000T_3T_5 + 870912\zeta_3T_{71} - 1866240\zeta_5T_{71} + 1638144T_{71} + 684288\zeta_3T_{72} \\
& \quad \left. + 559872\zeta_4T_{72} + 2177280\zeta_5T_{72} - 1638144T_{72} \right] \frac{T_2g^8}{746496} + O(g^{10}) . \tag{3.4}
\end{aligned}$$

The three loop piece agrees with [23, 24] and we note that we have followed the convention used there to include the wave function renormalization constant as part of the operator renormalization constant. We have checked that  $\gamma_{\mathcal{O}}(g)$  is equivalent to the  $\beta$ -function when the expressions are reduced to the single coupling theory (2.1). In the conventions of [23, 24] the relevant relation is

$$\frac{3}{2}\gamma_{\phi}(g) - \gamma_{\mathcal{O}}(g) = \beta(g) \tag{3.5}$$

for (2.1). For completeness we reproduce the independent renormalization group functions for (2.1) which are

$$\begin{aligned}
\beta(g) &= \frac{3}{8}g^3 - \frac{125}{288}g^5 + 5[2592\zeta_3 + 6617] \frac{g^7}{41472} \\
& + [-4225824\zeta_3 + 349920\zeta_4 + 1244160\zeta_5 - 3404365] \frac{g^9}{1492992} + O(g^{11}) \\
\gamma_{\phi}(g) &= -\frac{1}{12}g^2 + \frac{13}{432}g^4 + [2592\zeta_3 - 5195] \frac{g^6}{62208}
\end{aligned}$$

$$+ [10080\zeta_3 + 18144\zeta_4 - 69120\zeta_5 + 53449] \frac{g^8}{248832} + O(g^{10}) \quad (3.6)$$

which will be relevant to the Lee-Yang edge singularity problem.

As a final application of the renormalization it is interesting to consider the situation when the  $\phi^3$  theory is endowed with  $SU(N_c)$  symmetry. Such theories were considered at two and three loops in [53, 54] as simple models of the strong interactions in four dimensions and  $N_c$  denotes the number of colours. The motivation was partly due to the six dimensional theory being asymptotically free and so it could have parallel properties to QCD. Indeed this was in part the starting point for the study in [25]. The other motivation in [53, 54] rested in the idea that the ultraviolet properties of one theory could be regarded as being driven by the infrared behaviour of another. So the aim in [53, 54] was to determine the relevant  $d$ -dimensional critical exponents. In light of the recent development of the conformal bootstrap method which aims at examining the fixed point properties of various scalar theories, we will therefore extend the results of [53, 54] to four loops. This is in order to provide complementary data for future bootstrap studies. In [53, 54] two  $\phi^3$  theories with underlying  $SU(N_c)$  symmetry were considered. The first scenario was when the tensor  $d^{ijk}$  is identified as the totally symmetric rank 3 tensor in  $SU(N_c)$ . In this case the group invariants become

$$\begin{aligned} T_2 &= \frac{[N_c^2 - 4]}{N_c} \quad , \quad T_3 = \frac{1}{2N_c}[N_c^2 - 12] \quad , \quad T_5 = -\frac{4}{N_c^2}[N_c^2 - 10] \\ T_{71} &= \frac{1}{8N_c^3}[N_c^2 - 8][N_c^4 - 8N_c^2 + 256] \quad , \quad T_{72} = -\frac{1}{2N_c^3}[N_c^4 - 68N_c^2 + 528] \\ T_{91} &= [N_c^8 - 20N_c^6 + 352N_c^4 - 5120N_c^2 + 26880] \frac{1}{16N_c^4} \\ T_{92} &= T_{96} = T_{98} = [3N_c^6 - 16N_c^4 - 896N_c^2 + 7296] \frac{1}{4N_c^4} \\ T_{93} &= T_{95} = T_{97} = [N_c^6 + 16N_c^4 - 1024N_c^2 + 7104] \frac{1}{4N_c^4} \\ T_{94} &= -[N_c^6 - 64N_c^4 + 1216N_c^2 - 6784] \frac{1}{4N_c^4} \\ T_{99} &= [5N_c^6 - 72N_c^4 - 640N_c^2 + 7680] \frac{1}{4N_c^4} \end{aligned} \quad (3.7)$$

where we have made use of the properties of  $d^{ijk}$  given in [55] when the indices  $i$  take values in the adjoint representation. In this instance the renormalization group functions are

$$\begin{aligned} \beta(g) &= [N_c^2 - 20] \frac{g^3}{8N_c} + [-5N_c^4 + 496N_c^2 - 5360] \frac{g^5}{288N_c^2} \\ &+ [211N_c^6 + 62208\zeta_3 N_c^4 - 27132N_c^4 - 20736\zeta_3 N_c^2 + 1220688N_c^2 - 4396032\zeta_3 \\ &\quad - 9272896] \frac{g^7}{41472N_c^3} \\ &+ [-870048\zeta_3 N_c^8 + 1321920\zeta_5 N_c^8 - 327893N_c^8 + 14427072\zeta_3 N_c^6 + 559872\zeta_4 N_c^6 \\ &\quad - 31570560\zeta_5 N_c^6 + 8142840N_c^6 - 155416320\zeta_3 N_c^4 - 11384064\zeta_4 N_c^4 \\ &\quad + 421770240\zeta_5 N_c^4 - 112740480N_c^4 + 1477343232\zeta_3 N_c^2 - 35831808\zeta_4 N_c^2 \\ &\quad - 1950842880\zeta_5 N_c^2 + 1264882304N_c^2 - 7029669888\zeta_3 + 791285760\zeta_4 \\ &\quad - 995328000\zeta_5 - 5761837824] \frac{g^9}{1492992N_c^4} + O(g^{11}) \\ \gamma_\phi(g) &= [-N_c^2 + 4] \frac{g^2}{12N_c} + [N_c^4 - 104N_c^2 + 400] \frac{g^4}{432N_c^2} \end{aligned}$$

$$\begin{aligned}
& + [25N_c^6 - 10368\zeta_3 N_c^4 + 17196N_c^4 + 145152\zeta_3 N_c^2 - 296592N_c^2 - 414720\zeta_3 \\
& \quad + 909632] \frac{g^6}{62208N_c^3} \\
& + [5472\zeta_3 N_c^8 - 8640\zeta_5 N_c^8 + 2277N_c^8 - 92160\zeta_3 N_c^6 + 15552\zeta_4 N_c^6 + 172800\zeta_5 N_c^6 \\
& \quad - 57760N_c^6 + 804096\zeta_3 N_c^4 + 393984\zeta_4 N_c^4 - 3317760\zeta_5 N_c^4 + 2122784N_c^4 \\
& \quad - 4018176\zeta_3 N_c^2 - 7133184\zeta_4 N_c^2 + 28753920\zeta_5 N_c^2 - 22816768N_c^2 \\
& \quad + 7704576\zeta_3 + 21233664\zeta_4 - 70778880\zeta_5 + 60416256] \frac{g^8}{248832N_c^4} + O(g^{10}) \\
\gamma_{\mathcal{O}}(g) = & [-N_c^2 + 4] \frac{g^2}{2N_c} + [11N_c^4 - 184N_c^2 + 560] \frac{g^4}{48N_c^2} \\
& + [-317N_c^6 + 1728\zeta_3 N_c^4 + 8664N_c^4 - 27648\zeta_3 N_c^2 - 90960N_c^2 + 82944\zeta_3 \\
& \quad + 245504] \frac{g^6}{1728N_c^3} \\
& + [547344\zeta_3 N_c^8 - 855360\zeta_5 N_c^8 + 362939N_c^8 - 14211072\zeta_3 N_c^6 + 839808\zeta_4 N_c^6 \\
& \quad + 20995200\zeta_5 N_c^6 - 8790080N_c^6 + 166116096\zeta_3 N_c^4 - 12503808\zeta_4 N_c^4 \\
& \quad - 170449920\zeta_5 N_c^4 + 118589664N_c^4 - 943681536\zeta_3 N_c^2 + 64198656\zeta_4 N_c^2 \\
& \quad + 423014400\zeta_5 N_c^2 - 812965376N_c^2 + 1886257152\zeta_3 - 110481408\zeta_4 \\
& \quad - 89579520\zeta_5 + 1824079616] \frac{g^8}{746496N_c^4} + O(g^{10}) \tag{3.8}
\end{aligned}$$

where the two loop expressions for  $\beta(g)$  and  $\gamma_{\phi}(g)$  are in agreement with [53] when converted to the conventions used there. For large  $N_c$  the  $\beta$ -function is an alternating series. One interesting case is when  $N_c = 4$  when the  $\beta$ -function has a Banks-Zaks type fixed point, [14], in strictly six dimensions since

$$\begin{aligned}
\beta^{SU(4)}(g) &= -\frac{1}{8}g^3 + \frac{9}{32}g^5 + [6480\zeta_3 + 2417] \frac{g^7}{1536} \\
& \quad + [-3053376\zeta_3 - 58320\zeta_4 + 4786560\zeta_5 - 364729] \frac{g^9}{55296} + O(g^{11}) \\
\gamma_{\phi}^{SU(4)}(g) &= -\frac{1}{4}g^2 - \frac{7}{48}g^4 + [-48\zeta_3 + 43] \frac{g^6}{256} \\
& \quad + [18864\zeta_3 + 10368\zeta_4 - 46080\zeta_5 + 21907] \frac{g^8}{9216} + O(g^{10}) \\
\gamma_{\mathcal{O}}^{SU(4)}(g) &= -\frac{3}{2}g^2 + \frac{9}{16}g^4 + 3[2\zeta_3 - 7] \frac{g^6}{8} \\
& \quad + [336384\zeta_3 + 55728\zeta_4 - 338400\zeta_5 + 335503] \frac{g^8}{9216} + O(g^{10}). \tag{3.9}
\end{aligned}$$

The second case considered in [53, 54] was when the symmetry group is  $SU(3) \times SU(3)$ . We summarize the relevant background to the construction of that Lagrangian, as it is more involved than the previous case, before detailing the extraction of our results. The main ingredient is that  $\phi^i$  becomes a complex field and the Lagrangian (2.2) is extended to group to

$$L = \partial_{\mu} \bar{\phi}^i \partial^{\mu} \phi^i + \frac{g}{6} d^{ijk} \left( \phi^i \phi^j \phi^k + \bar{\phi}^i \bar{\phi}^j \bar{\phi}^k \right). \tag{3.10}$$

To implement the  $SU(3) \times SU(3)$  symmetry the coupling tensor is defined by

$$d^{ijk} = \epsilon_{\alpha\beta\gamma} \epsilon_{\lambda\chi\xi} T_{\alpha\lambda}^i T_{\beta\chi}^j T_{\gamma\xi}^k \tag{3.11}$$

where the Greek letter subscripts take the values 1, 2 or 3 and  $\epsilon_{\alpha\beta\gamma}$  is the Levi-Civita symbol. The matrices  $T^i$  are related to the unit matrix and the  $SU(3)$  group generators. In other words  $0 \leq i \leq 8$  and

$$T_{\alpha\beta}^0 = \sqrt{\frac{3}{2}}\delta_{\alpha\beta} \quad (3.12)$$

with the remaining eight objects corresponding to the  $SU(3)$  group generators. The normalization of  $T_{\alpha\beta}^0$  is chosen, [53, 54], so that for the nine objects spanning the group we have

$$\text{Tr}(T^i T^j) = \frac{1}{2}\delta^{ij} . \quad (3.13)$$

With this convention we have the simple rule for the full set of  $T^i$  with  $0 \leq i \leq 8$  that

$$T_{\alpha\beta}^i T_{\gamma\lambda}^i = \frac{1}{2}\delta_{\alpha\lambda}\delta_{\gamma\beta} . \quad (3.14)$$

These relations are crucial in evaluating the products of the coupling tensors within the renormalization group function determination. We note that unlike [53, 54] our convention is to base the definition of the  $SU(3) \times SU(3)$  tensor  $d^{ijk}$  on the  $SU(3)$  group generators themselves rather than the Gell-Mann matrices used in [53, 54]. With this construction the fields are in the  $[3, \bar{3}] + [\bar{3}, 3]$  representation of  $SU(3) \times SU(3)$ .

To renormalize (3.10) we constructed the relevant set of QGRAF diagrams to three loops. As noted in [54] the number of Feynman diagrams is substantially less than that of the real scalar theory (2.1). This is because not all the basic topologies survive when the field is complex. It is simple to see this as each line of a graph must have an arrow representing the charge. In addition from (3.10) the two vertices are represented by the convergence or divergence of three directed lines where all lines either have the arrows directed to the point or directed away from the point. Therefore, to see which topologies survive from the real theory one merely takes the topology and endows its lines with arrows consistent with these rules. If this results in a graph with any vertex where not all arrows converge or diverge then it is absent or excluded. It transpires, for instance, [54], that any graph with a subgraph with an odd number of propagators will immediately vanish. We have checked this by carrying out the full three loop renormalization explicitly with the Lagrangian (3.10). Moreover, the renormalization group functions to this order are the same as (3.1) and (3.2) but with  $T_3 = 0$  and  $T_{72} = 0$ . The first invariant corresponds to the basic one loop triangle 3-point function. So it is clearly absent, [54]. Given this observation with regard to the renormalization group functions already determined then the four loop expressions will be given by excluding those invariants in (3.1) and (3.2) which correspond to absent topologies. It is straightforward to check that of the set given in (2.4) only  $T_{94}$  passes the test and the remaining four loop invariants are excluded from the renormalization group functions. At this stage we have only argued for the consequences of the renormalization group functions when one has a  $\phi^3$  theory of the form (3.10). For the  $SU(3) \times SU(3)$  case we need the explicit values of the non-zero invariants  $T_2$ ,  $T_5$ ,  $T_{71}$  and  $T_{94}$ . Using the properties for this group discussed above we find the values

$$T_2 = \frac{1}{2} \quad , \quad T_5 = 0 \quad , \quad T_{71} = \frac{1}{32} \quad , \quad T_{94} = \frac{1}{256} \quad (3.15)$$

in our conventions. We note that, [54], while the two loop nonplanar topology corresponding to  $T_5$  passes the arrow test it actually vanishes for the specific  $SU(3) \times SU(3)$  group. For other groups  $T_5$  may be non-zero. These result in

$$\beta(g) = -\frac{1}{16}g^3 - \frac{11}{1152}g^5 + \frac{4363}{331776}g^7$$

$$\begin{aligned}
& + [-4032\zeta_3 - 5760\zeta_5 + 10279] \frac{g^9}{884736} + O(g^{11}) \\
\gamma_\phi(g) &= -\frac{1}{24}g^2 - \frac{11}{1728}g^4 - \frac{821}{497664}g^6 \\
& + [1152\zeta_3 - 5760\zeta_5 + 4231] \frac{g^8}{1327104} + O(g^{10}) \\
\gamma_{\mathcal{O}}(g) &= -\frac{1}{4}g^2 - \frac{1}{192}g^4 - \frac{95}{3456}g^6 \\
& + [183168\zeta_3 - 466560\zeta_5 + 452171] \frac{g^8}{11943936} + O(g^{10}) \tag{3.16}
\end{aligned}$$

in the  $\overline{\text{MS}}$  scheme.

The *two* loop expressions given in (3.16) are in accord with [53, 54]. That they do not agree with the three loop results is because (3.10) was renormalized in a different scheme. The scheme of [54] is what is now termed the momentum subtraction scheme which was developed later in [56] for QCD. Briefly the coupling constant renormalization constant is defined, [54], by requiring that there are no corrections beyond the tree term when the vertex function is evaluated at the fully symmetric point where the values of the square of the external momentum of each leg are equivalent. For (3.10) when the symmetry group is  $SU(3) \times SU(3)$  the constraint from the group theory meant only one three loop graph needed to be determined at this subtraction point. Moreover, only the residue of the simple pole was needed for the three loop momentum subtraction  $\beta$ -function. The finite part would be necessary for the four loop correction. That  $\gamma_\phi(g)$  agrees at two loop is perhaps somewhat surprising given that only the leading term of this is independent of the renormalization scheme. In (3.10) the agreement at two loop appears to be purely coincidental rather as deriving from some property of the two renormalization schemes. However, we do actually have a check on the three loop computation which comes via the critical exponents computed in [53, 54] at the Wilson-Fisher fixed point. While the renormalization group functions are central to the evolution of Green's functions for a range of scales, the critical exponents at a fixed point are renormalization group invariants. So they have the *same* value in all schemes. In [53, 54] the  $d$ -dimensional momentum subtraction scheme renormalization group functions were provided. In other words the finite parts of the underlying momentum subtraction scheme renormalization constants manifest themselves as  $O(\epsilon)$  terms in  $\beta(g)$  and  $\gamma_\phi(g)$ . If one were merely interested in purely six dimensions the  $O(\epsilon)$  terms would have been absent in the expressions given in [53, 54]. More crucially, if they were excluded from a Wilson-Fisher fixed point analysis the derived exponents would not be renormalization group invariants. By contrast, in  $\overline{\text{MS}}$  there are no  $O(\epsilon)$  terms in the coefficients of  $d$ -dimensional renormalization group functions aside from the  $O(g)$  term of  $\beta(g)$  which defines the dimension of  $g$  in  $d$ -dimensions. This is because by definition in the  $\overline{\text{MS}}$  there are no finite parts in the renormalization constants. Therefore, from our  $\overline{\text{MS}}$  expressions (3.16) we have

$$\begin{aligned}
\omega &= 2\epsilon - \frac{22}{9}\epsilon^2 - \frac{4847}{81}\epsilon^3 + [-326592\zeta_3 - 466560\zeta_5 + 422035] \frac{\epsilon^4}{1458} + O(\epsilon^5) \\
\eta &= \frac{2}{3}\epsilon + \frac{32}{3}\epsilon^3 + 16[18\zeta_3 + 1] \frac{\epsilon^4}{9} + O(\epsilon^5) \tag{3.17}
\end{aligned}$$

where there is no  $O(\epsilon^2)$  term in  $\eta$ . Comparing the three loop exponents with the corresponding expressions given in [53, 54] we find exact agreement. This is reassuring since the original three loop computation and the present one were in different schemes. However, using the  $\overline{\text{MS}}$  scheme, where we were able to simplify the computation of the divergence structure of the 3-point functions, has allowed us to proceed to a higher loop order for these exponents than would be currently possible in a momentum subtraction scheme.

## 4 Lee-Yang edge singularity.

Equipped with the renormalization group functions we can now study several problems at a new order in the perturbative expansion. For the first major application of our results we turn to the Lee-Yang edge singularity problem which as elucidated in [29] is related to  $\phi^3$  theory but with a purely imaginary coupling constant. In terms of the results in the previous section this requires setting  $d^{ijk} = i$  in the initial Lagrangian to produce the values

$$T_2 = -1, \quad T_3 = -1, \quad T_5 = 1, \quad T_{7i} = -1, \quad T_{9i} = 1. \quad (4.1)$$

Alternatively one can use the mapping  $g \rightarrow ig$  in (3.6). The key quantity of interest for the Lee-Yang problem is the critical exponent  $\sigma$  which is related to the anomalous dimension of  $\phi$  of (2.1) through the hyperscaling law

$$\sigma = \frac{[d - 2 + \eta]}{[d + 2 - \eta]} \quad (4.2)$$

in  $d$ -dimensions. With  $d = 6 - 2\epsilon$  determining the critical coupling constant in  $d$ -dimensions and expanding the  $\phi$  field anomalous dimension at that point we find

$$\begin{aligned} \eta = & -\frac{2}{9}\epsilon - \frac{172}{729}\epsilon^2 + 2[15552\zeta_3 - 8375]\frac{\epsilon^3}{59049} \\ & + [-2783808\zeta_3 + 3779136\zeta_4 - 2799360\zeta_5 - 3883409]\frac{\epsilon^4}{4782969} + O(\epsilon^5) \end{aligned} \quad (4.3)$$

where the terms to  $O(\epsilon^3)$  are in agreement with [24] but expressed in our conventions for  $d$ .

$d$	$\eta$		$\sigma$	
	3 loop	4 loop	3 loop	4 loop
5	-0.1450	-0.1545	0.3996	0.3977
4	-0.3173	-0.3824	0.2664	0.2534
3	-0.4981	-0.6805	0.0913	0.0562
2	-0.6826	-1.0484	-0.1458	-0.2077
1	-0.8691	-1.4860	-0.4831	-0.5542

Table 1. Critical exponents  $\eta$  and  $\sigma$  estimates using Padé approximants.

$d$	$\eta$		$\sigma$	
	3 loop	4 loop	3 loop	4 loop
5	-0.1468	-0.1529	0.3992	0.3980
4	-0.3280	-0.3702	0.2642	0.2558
3	-0.5239	-0.6446	0.0862	0.0630
2	-0.7281	-0.9742	-0.1540	-0.1958
1	-0.9377	-1.3583	-0.4921	-0.5411

Table 2. Critical exponents  $\eta$  and  $\sigma$  estimates using Padé-Borel method.

As the Lee-Yang singularity problem stretches across dimensions to  $d = 1$  one has to be careful in using the perturbative expansion for large values of  $\epsilon$ . Therefore, to gain estimates for  $\sigma$  we have used Padé and Padé-Borel resummation for  $\eta$  and then evaluated  $\sigma$  through

the scaling law. The results for both are given in Tables 1 and 2 where we have used a  $[2, 1]$  approximant at three loops and  $[3, 1]$  at four loops. We have included the three loop results of [24] for comparison. From Table 1 it is evident that for dimensions close to 6 convergence appears to be present from three to four loops for  $\sigma$ . The large discrepancy in  $\eta$  estimates down to  $d = 4$  seems to get washed out in the scaling law. A similar feature is apparent in Table 2 for the Padé-Borel application. Though the convergence if anything appears marginally improved. The main problem is that the exact values of  $\sigma$  at  $d = 1$  and 2 are not emerging which are  $-\frac{1}{2}$  and  $-\frac{1}{6}$  respectively. Indeed if anything the four loop estimates in both Tables for these values is worse than the three loop ones. This might have been expected as naively setting a value for a parameter to be of order 2 in a summed perturbative expansion will mean the larger  $O(\epsilon^4)$  term will dominate. One way to handle this is to use a constrained Padé as discussed in [32]. In that method the two exact values for  $\sigma$  are included in the derivation of the rational polynomial of the Padé approximant. We have carried this out for the four loop estimate of  $\sigma$  which is given by

$$\begin{aligned} \sigma = & \frac{1}{2} - \frac{1}{6}\epsilon - \frac{79}{972}\epsilon^2 + [15552\zeta_3 - 10445]\frac{\epsilon^3}{157464} \\ & + [-2503872\zeta_3 + 3779136\zeta_4 - 2799360\zeta_5 - 4047533]\frac{\epsilon^4}{25509168} + O(\epsilon^5). \end{aligned} \quad (4.4)$$

The results are given in Table 3 where we have reproduced the constrained three loop  $[3, 2]$  results of [32] and given our four loop  $[4, 2]$  Padé estimates. The constraints are included for completeness. Results from other methods are included for comparison. These include a strong coupling expansion [32], as well as two Monte Carlo methods which are based on critical behaviour in problems seemingly unrelated to the Lee-Yang singularity problem. These are termed (lattice) animals and fluids with the former originating in polymers in a solvent and the latter related to pressure in fluids where there is a repulsive core, [30, 31]. The final column in Table 3 are recent results from a conformal bootstrap analysis, [33]. With the inclusion of the exact results for low dimension in the Padé approximant not only is there better convergence for  $d = 3$  from three to four loops but there is remarkable agreement with the values from [32]. For the other methods the four loop estimates lie within error bars except compared to the  $d = 3$  value for the fluids method. In light of this it is worth commenting on why we presented Tables 1 and 2 in the first place. This is partly to make contact with [24]. More crucially given that we will be using Padé approximants later for other problems down to low dimensions it is important to be aware of the potential limitations of the technique in the absence of known exact two dimensional conformal field theory exponents. As exact results are known for  $d = 1$  and 2 here we can gauge how far off estimates may be for these dimensions. The  $d = 3$  values for  $\sigma$  in Tables 1 and 2 are perhaps not reliable but those for  $d = 4$  and 5 are in keeping with those of the other methods listed in Table 3.

$d$	3 loop	4 loop	Ref [32]	Ref [30]	Ref [31]	Ref [33]
5	0.3989	0.3981	0.401(9)	0.402(5)	0.40(2)	0.4105(5)
4	0.2616	0.2584	0.258(5)	0.2648(15)	0.261(12)	0.2685(1)
3	0.0785	0.0747	0.076(2)	0.0877(25)	0.080(7)	0.085(1)
2	- 0.1667	- 0.1667	- 0.166(5)	- 0.161(8)	- 0.165(6)	- 0.1664(5)
1	- 0.5000	- 0.5000	-	-	-	-

Table 3. Critical exponent  $\sigma$  estimates using constrained Padé approximant and comparison with [30, 31, 32, 33].

## 5 Percolation.

We now turn to the application of the renormalization to the percolation problem which requires the evaluation of the tensors (2.4) for a specific configuration. The percolation problem is described by the replica limit in the  $(N + 1)$ -state Potts model [34] which in the case of (2.2) corresponds to a special designation of the coupling tensor  $d^{ijk}$ . A straightforward way of mapping to the Potts model was provided in [57] which involves a set of vectors,  $e_\alpha^i$ . These  $(N + 1)$  vectors describe the vertices of an  $N$ -dimensional tetrahedron and allow one to decompose  $d^{ijk}$  as

$$d^{ijk} = \sum_{\alpha=1}^{N+1} e_\alpha^i e_\alpha^j e_\alpha^k . \quad (5.1)$$

In order to represent the tetrahedron the vectors must satisfy the following relations,

$$\sum_{\alpha=1}^{N+1} e_\alpha^i e_\alpha^j = 0 \quad , \quad \sum_{\alpha=1}^{N+1} e_\alpha^i e_\alpha^j = (N + 1)\delta^{ij} \quad (5.2)$$

for sums over the  $(N + 1)$ -dimensional label and

$$\sum_{i=1}^N e_\alpha^i e_\beta^i = (N + 1)\delta_{\alpha\beta} - 1 \quad (5.3)$$

for summations over the original indices denoted by  $i$ . It is the form of the final relation which means that the evaluation of the underlying tensors (2.4) requires special care. In [24] a diagrammatic method was outlined to handle the lower rank tensors. However, we have written a FORM routine to reproduce the evaluations given in [24] and then applied it to the cases  $T_{9m}$  for  $1 \leq m \leq 9$ . Such a systematic path seems more appropriate since the diagrammatic method is tedious at three loops as it involves seven sums but manageable for only two tensors. At four loops the nine summations for nine independent tensors is not straightforward. For arbitrary  $N$  we have

$$\begin{aligned} T_2 &= (N + 1)^2[N - 1] \quad , \quad T_3 = (N + 1)^2[N - 2] \\ T_5 &= [(N + 1)^2 - 6(N + 1) + 10](N + 1)^2 \\ T_{71} &= [(N + 1)^3 - 9(N + 1)^2 + 29(N + 1) - 32](N + 1)^6 \\ T_{72} &= [(N + 1)^2 - 6(N + 1) + 11][N - 2](N + 1)^6 \\ T_{91} &= [(N + 1)^3 - 9(N + 1)^2 + 30(N + 1) - 35][N - 2](N + 1)^8 \\ T_{92} &= T_{96} = T_{98} = [(N + 1)^3 - 9(N + 1)^2 + 30(N + 1) - 38][N - 2](N + 1)^8 \\ T_{93} &= T_{95} = T_{97} = [(N + 1)^3 - 9(N + 1)^2 + 30(N + 1) - 37][N - 2](N + 1)^8 \\ T_{94} &= [(N + 1)^4 - 12(N + 1)^3 + 57(N + 1)^2 - 125(N + 1) + 106](N + 1)^8 \\ T_{99} &= [(N + 1)^2 - 5(N + 1) + 10][N - 2][N - 3](N + 1)^8 \end{aligned} \quad (5.4)$$

where we present the expressions in the same format as [24] and have included the known values from [24] for completeness but with  $T_{72}$  factorized further. Although there are two instances of three tensors giving the same value for this tetrahedron configuration, it is clear from the underlying Feynman diagram defining the tensors that the graphs themselves are topologically distinct. This should not be a surprise since in QCD, for example, when one examines high loop diagrams different topologies can have the same combination of colour group Casimirs multiplying them. Taking the  $N \rightarrow 0$  replica limit gives the values we require for the evaluation of the critical exponents for the percolation problem. We have

$$\begin{aligned} T_2 &= -1 \quad , \quad T_3 = -2 \quad , \quad T_5 = 5 \quad , \quad T_{71} = -11 \quad , \quad T_{72} = -12 \\ T_{91} &= 26 \quad , \quad T_{92} = 32 \quad , \quad T_{93} = 30 \quad , \quad T_{94} = 27 \quad , \quad T_{99} = 36 \end{aligned} \quad (5.5)$$



for the independent tensors.

With these values we have computed various critical exponent in powers of  $\epsilon$  to  $O(\epsilon^4)$ . Using

$$\begin{aligned}
\beta(g) &= -\frac{\epsilon}{2}g - \frac{7}{8}g^3 - \frac{671}{288}g^5 + \left[ -\frac{414031}{41472} - \frac{93}{16}\zeta_3 \right] g^7 \\
&\quad + \left[ -\frac{121109}{1728}\zeta_3 + \frac{651}{64}\zeta_4 - \frac{595}{12}\zeta_5 - \frac{84156383}{1492992} \right] g^9 + O(g^{11}) \\
\gamma_\phi(g) &= \frac{1}{12}g^2 + \frac{37}{432}g^4 + \left[ \frac{29297}{62208} - \frac{5}{24}\zeta_3 \right] g^6 \\
&\quad + \left[ \frac{225455}{82944} + \frac{233}{864}\zeta_3 + \frac{33}{32}\zeta_4 - \frac{55}{18}\zeta_5 \right] g^8 + O(g^{10}) \\
\gamma_{\mathcal{O}}(g) &= \frac{1}{2}g^2 + \frac{47}{48}g^4 + \left[ \frac{3709}{864} + \frac{3}{2}\zeta_3 \right] g^6 \\
&\quad + \left[ \frac{18486131}{746496} + \frac{20027}{864}\zeta_3 - \frac{33}{32}\zeta_4 + \frac{15}{2}\zeta_5 \right] g^8 + O(g^{10}) \tag{5.6}
\end{aligned}$$

they are

$$\begin{aligned}
\eta &= -\frac{2}{21}\epsilon - \frac{824}{9261}\epsilon^2 + 4[290304\zeta_3 - 93619]\frac{\epsilon^3}{4084101} \\
&\quad + 2[286336512\zeta_3 + 384072192\zeta_4 - 1493614080\zeta_5 - 103309103]\frac{\epsilon^4}{1801088541} + O(\epsilon^5) \\
\eta_{\mathcal{O}} &= -\frac{4}{7}\epsilon - \frac{710}{3087}\epsilon^2 + [925344\zeta_3 - 235495]\frac{\epsilon^3}{1361367} \\
&\quad + [603983520\zeta_3 + 1224230112\zeta_4 - 5334336000\zeta_5 - 157609181]\frac{\epsilon^4}{1200725694} + O(\epsilon^5) \\
\gamma &= 1 + \frac{2}{7}\epsilon + \frac{565}{3087}\epsilon^2 + [-925344\zeta_3 + 408997]\frac{\epsilon^3}{2722734} \\
&\quad + [-933950304\zeta_3 - 1224230112\zeta_4 + 5334336000\zeta_5 + 302378687]\frac{\epsilon^4}{2401451388} + O(\epsilon^5) \\
\nu &= \frac{1}{2} + \frac{5}{42}\epsilon + \frac{589}{9261}\epsilon^2 + [-1614816\zeta_3 + 716519]\frac{\epsilon^3}{16336404} \\
&\quad + [344397667 - 1344827232\zeta_3 - 2136401568\zeta_4 + 10028551680\zeta_5]\frac{\epsilon^4}{14408708328} + O(\epsilon^5) \\
\omega &= 2\epsilon - \frac{1342}{441}\epsilon^2 + [62496\zeta_3 + 40639]\frac{\epsilon^3}{7203} \\
&\quad + [248046624\zeta_4 - 702654624\zeta_3 - 1209116160\zeta_5 - 317288185]\frac{\epsilon^4}{19059138} + O(\epsilon^5) . \tag{5.7}
\end{aligned}$$

The exponents  $\eta$ ,  $\eta_{\mathcal{O}}$  and  $\omega$  are obtained from the corresponding renormalization group function and  $\gamma$  and  $\nu$  are deduced from the scaling relations

$$\eta_{\mathcal{O}} = \nu^{-1} - 2 + \eta \quad , \quad \gamma = (2 - \eta)\nu . \tag{5.8}$$

With these we have repeated the exercise of the previous section to obtain Padé and Padé-Borel estimates of various critical exponents following the method of [24]. There estimates were found for  $\eta$  and  $\eta_{\mathcal{O}}$  and then values for  $\gamma$  and  $\nu$  were obtained from the scaling laws. Our results are contained in Tables 3 to 7 with those at three loops agreeing with [24] and included for comparison with the four loop estimates. The three loop estimates for  $\omega$  are in accord with those given in [58].

Overall a similar feature emerges as for the Lee-Yang edge singularity estimates in that down to four dimensions there is reasonable convergence but below this the results are not as

$d$	$\eta$	$\eta_{\mathcal{O}}$	$\gamma$	$\nu$	$\omega$
5	- 0.0569	- 0.3097	1.1773	0.5723	0.7910
4	- 0.1186	- 0.6319	1.4250	0.6726	1.5155
3	- 0.1812	- 0.9566	1.7812	0.8166	2.2326
2	- 0.2442	- 1.2822	2.3328	1.0395	2.9473

Table 4. Critical exponent estimates using Padé approximants to three loop expressions.

$d$	$\eta$	$\eta_{\mathcal{O}}$	$\gamma$	$\nu$	$\omega$
5	- 0.0594	- 0.3192	1.1834	0.5746	0.7085
4	- 0.1338	- 0.6885	1.4764	0.6919	1.1590
3	- 0.2215	- 1.1048	1.9893	0.8955	1.4775
2	- 0.3222	- 1.5678	3.0782	1.3256	1.7150

Table 5. Critical exponent estimates using Padé approximants to four loop expressions.

reliable. This situation was ameliorated by exploiting known results in two dimensions and using this as a constraint or boundary condition on the Padé approximant. Therefore, we have followed this procedure again and constructed constrained Padé approximants to  $\nu$ ,  $\gamma$  and  $\omega$  from their respective exact two dimensional values of  $\frac{4}{3}$ ,  $\frac{43}{18}$  and 2, [59]. The results of this exercise are given in Table 8. There the two dimensional values of  $\eta$  and the exponent  $\beta$  agree with their known exact values. The latter exponent as well as  $\sigma$  and  $\tau$ , which are deduced from hyperscaling laws from the previous columns in the table, are included for comparison with results from other methods. In this respect Table I of [60] gives a comprehensive summary of estimates for these exponents. To compare we have included the results of the computation of  $\nu$ ,  $\gamma$ ,  $\eta$  and  $\beta$ , [60], in Table 9 which used a high temperature series method. Though it is worth noting that a more recent study, [61], has obtained estimates for  $\gamma$  which are equivalent to the those of the high temperature series of [60]. In [61] the series was extended beyond the 15th order of [60]. Examining the results given in Tables 8 and 9 on the whole the constrained Padé estimates are in reasonable consistency with the central values of [60]. Perhaps more significantly the three dimensional estimates from the perturbative approach are in line not only with [60] but results from other methods as is evident from Table I of [60]. There the perturbative estimate of 0.34(4) was quoted for  $\beta$ , for example, but that of Table 8 is more in line with other methods now. One exponent not covered by the summary table of [60] is the correction to scaling exponent  $\omega$ . Two studies of  $\omega$  using Monte Carlo methods are given in [62] and [58] which give results in three and four dimensions respectively. These are 1.62(13) and 1.13(10). Other estimates in three dimensions are 1.61(5), [63], and 1.77(13), [64]. Our estimates in Table 8 are in remarkable agreement in three dimensions and within the error of [58] in four dimensions. With the results in Tables 8 and 9 and the close tally it gives support to the earlier observation that the usual Padé approximant is only really competitive down to four dimensions. Below that the summation fluctuates as is apparent from the four loop estimates and does not capture the exact two dimensional picture. Hence results down to four dimensions should only be considered. For the remaining two exponents  $\sigma$  and  $\tau$  we note that the estimates for  $\tau$  are in good agreement over all dimensions in comparison with Monte Carlo estimates given in Table I of [61]. An estimate for  $\sigma$  of around 0.452-0.454 is given there too but only in three dimensions. So our value is below the central value. The remaining two columns in Table 9 correspond to estimates for  $\sigma$  and  $\tau$  from [63, 65, 66]. Again the constrained Padé estimates are not dissimilar to the central values.

$d$	$\eta$	$\eta_{\mathcal{O}}$	$\gamma$	$\nu$	$\omega$
5	- 0.0578	- 0.3122	1.1788	0.5729	0.7540
4	- 0.1229	- 0.6431	1.4346	0.6758	1.3880
3	- 0.1903	- 0.9800	1.8097	0.8262	1.9937
2	- 0.2588	- 1.3199	2.4057	1.0651	2.5870

Table 6. Critical exponent estimates using Padé-Borel method for three loop expressions.

$d$	$\eta$	$\eta_{\mathcal{O}}$	$\gamma$	$\nu$	$\omega$
5	- 0.0582	- 0.3160	1.1814	0.5740	0.7539
4	- 0.1253	- 0.6631	1.4517	0.6831	1.3966
3	- 0.1968	- 1.0464	1.9096	0.8693	2.0170
2	- 0.2717	- 1.4503	2.7656	1.2174	2.6320

Table 7. Critical exponent estimates using Padé-Borel method for four loop expressions.

We close this section by recording the renormalization group functions for the  $N = 2$  case. While this is not directly related to the percolation problem it does correspond to a specific Potts model and so we give the relevant expressions for completeness here. The main reason for treating this case specially lies in the nature of the group invariants  $T_n$  given in (5.4). From the explicit values most vanish at  $N = 2$ . If one analyses the underlying Feynman graph which each invariant relates to, then the non-zero invariants have the same feature as the  $SU(3) \times SU(3)$  theory analysed earlier. Though they have no other connection aside from the graphical one. This similarity is that when  $T_n$  is non-zero the corresponding Feynman graph has no subgraph with an odd number of Feynman propagators. If any  $T_i = 0$  then there is at least one subgraph with an odd number of propagators. This is clearly the case for  $T_3$  which is the one loop triangle. So in (3.1) a large number of terms are absent. Moreover, the non-zero values of  $T_n$  are

$$T_2 = T_5 = 9 \quad , \quad T_{71} = 729 \quad , \quad T_{94} = 6561 \quad (5.9)$$

which are all powers of 3 in contrast to (3.10) where the non-zero invariants were powers of  $1/2$  and  $T_5$  is non-zero here. With these values the renormalization group functions are

$$\begin{aligned}
\beta(g) &= -\frac{\epsilon}{2}g - \frac{9}{8}g^3 - \frac{171}{32}g^5 + 9[288\zeta_3 + 18883]\frac{g^7}{512} \\
&\quad + 81[-231648\zeta_3 - 288\zeta_4 - 69120\zeta_5 + 187069]\frac{g^9}{2048} + O(g^{11}) \\
\gamma_\phi(g) &= -\frac{3}{4}g^2 - \frac{33}{16}g^4 + 3[288\zeta_3 - 1325]\frac{g^6}{256} \\
&\quad + 9[46176\zeta_3 - 864\zeta_4 - 207360\zeta_5 + 165887]\frac{g^8}{1024} + O(g^{10}) \\
\gamma_{\mathcal{O}}(g) &= -\frac{9}{2}g^2 - \frac{27}{16}g^4 - \frac{783}{4}g^6 \\
&\quad + 9[844128\zeta_3 + 2592\zeta_4 - 1866240\zeta_5 + 1624571]\frac{g^8}{1024} + O(g^{10}) \quad (5.10)
\end{aligned}$$

which can be used for a Wilson-Fisher fixed point analysis.

## 6 $O(N)$ symmetric theory.

We now turn to another version of  $\phi^3$  theory which is the one endowed with an  $O(N)$  symmetry. It has been considered recently in [18, 20, 26, 27, 28] in the context of the conformal bootstrap programme as a way of accessing five dimensional quantum field theories with a conformal symmetry. The basic Lagrangian in this case is

$$L = \frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{g_1}{2} \sigma \phi^i \phi^i + \frac{g_2}{6} \sigma^3 \quad (6.1)$$

where there is an  $O(N)$  multiplet of fields  $\phi^i$  together with a single scalar field,  $\sigma$ . To ensure these fields produce a renormalizable Lagrangian in six dimensions there are two massless coupling constants  $g_1$  and  $g_2$ . For ease of comparison with [18, 20] we use the same notation as those articles. As the main interest in this section is the extension of the three loop analysis to the next loop order and as the computational techniques have already been described we will mention only those features which are new to this calculation. First, with the extra fields we have carried out the full computation rather than identify the group structures of section 3 with those of the  $O(N)$  symmetric Lagrangian as was noted in [20]. While this may seem to be inefficient it is actually a necessary first step in the derivation of the renormalization of the mass operators of the fields in (6.1) which we will discuss later. What is worth noting is the number of 2-point graphs which are required for the renormalization of (6.1). For  $\phi^i$  there are 1, 5, 48 and 637 one, two, three and four loop graphs respectively. The corresponding numbers for the  $\sigma$  2-point function are 2, 7, 60 and 723. The resultant four loop field anomalous dimensions are

$$\begin{aligned} \gamma_\phi(g_1, g_2) = & -\frac{g_1^2}{6} + \left[ -11Ng_1^2 + 26g_1^2 + 48g_1g_2 - 11g_2^2 \right] \frac{g_1^2}{432} \\ & + \left[ 13N^2g_1^4 - 232g_1^4N + 5184\zeta_3g_1^4 - 9064g_1^4 + 2646Ng_1^3g_2 - 3264g_1^3g_2 \right. \\ & \quad \left. - 386Ng_1^2g_2^2 + 5184\zeta_3g_1^2g_2^2 - 11762g_1^2g_2^2 + 942g_1g_2^3 + 327g_2^4 \right] \frac{g_1^2}{31104} \\ & + \left[ 1296\zeta_3N^3g_1^6 + 3N^3g_1^6 + 46656\zeta_3N^2g_1^6 + 21412N^2g_1^6 + 3649536\zeta_3Ng_1^6 \right. \\ & \quad + 1026432\zeta_4Ng_1^6 - 3732480\zeta_5Ng_1^6 - 1600648Ng_1^6 - 1275264\zeta_3g_1^6 \\ & \quad + 1306368\zeta_4g_1^6 - 7464960\zeta_5g_1^6 + 9095944g_1^6 - 15552\zeta_3N^2g_1^5g_2 \\ & \quad + 52452N^2g_1^5g_2 - 2799360\zeta_3Ng_1^5g_2 - 839808\zeta_4Ng_1^5g_2 + 3945432Ng_1^5g_2 \\ & \quad + 995328\zeta_3g_1^5g_2 + 3359232\zeta_4g_1^5g_2 - 2784240g_1^5g_2 + 1296\zeta_3N^2g_1^4g_2^2 \\ & \quad - 3874N^2g_1^4g_2^2 + 1034208\zeta_3Ng_1^4g_2^2 - 116640\zeta_4Ng_1^4g_2^2 - 2864316Ng_1^4g_2^2 \\ & \quad + 3037824\zeta_3g_1^4g_2^2 + 233280\zeta_4g_1^4g_2^2 - 14929920\zeta_5g_1^4g_2^2 + 13929064g_1^4g_2^2 \\ & \quad + 77760\zeta_3Ng_1^3g_2^3 + 35544Ng_1^3g_2^3 - 1772928\zeta_3g_1^3g_2^3 + 2239488\zeta_4g_1^3g_2^3 \\ & \quad + 1910496g_1^3g_2^3 - 1296\zeta_3Ng_1^2g_2^4 + 40951Ng_1^2g_2^4 + 1648512\zeta_3g_1^2g_2^4 \\ & \quad \left. + 886464\zeta_4g_1^2g_2^4 - 3732480\zeta_5g_1^2g_2^4 + 1056620g_1^2g_2^4 - 342144\zeta_3g_1g_2^5 \right] \end{aligned}$$

$d$	$\nu$	$\gamma$	$\eta$	$\beta$	$\omega$	$\sigma$	$\tau$
5	0.5746	1.1817	-0.0565	0.8457	0.7178	0.4933	2.4171
4	0.6920	1.4500	-0.0954	0.6590	1.2198	0.4742	2.3124
3	0.8968	1.8357	-0.0470	0.4273	1.6334	0.4419	2.1888
2	1.3333	2.3888	0.2083	0.1389	2.0000	0.3956	2.0549

Table 8. Critical exponent estimates using constrained Padé approximant of four loop results.

$d$	$\nu$	$\gamma$	$\eta$	$\beta$	$\sigma$	$\tau$
5	0.571(3)	1.185(5)	- 0.075(20)	0.845(5)	-	2.412(4)
4	0.678(50)	1.435(15)	- 0.12(4)	0.639(20)	-	2.313(2)
3	0.872(70)	1.805(20)	- 0.07(5)	0.405(25)	0.445(10)	2.190(2)
2	1.3333	2.3888	0.2083	0.1389	-	-

Table 9. Results for critical exponents from [60, 63, 65, 66].

$$\begin{aligned}
& - 279936\zeta_4 g_1 g_2^5 + 459612 g_1 g_2^5 + 68688\zeta_3 g_2^6 + 23328\zeta_4 g_2^6 \\
& - 204484 g_2^6 \Big] \frac{g_1^2}{6718464} + O(g_i^{10})
\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}
\gamma_\sigma(g_1, g_2) = & - \left[ N g_1^2 + g_2^2 \right] \frac{1}{12} + \left[ 2N g_1^4 + 48N g_1^3 g_2 - 11N g_1^2 g_2^2 + 13g_2^4 \right] \frac{1}{432} \\
& + \left[ - 2762N^2 g_1^6 + 5184\zeta_3 N g_1^6 - 8560N g_1^6 + 1152N^2 g_1^5 g_2 + 1056N g_1^5 g_2 \right. \\
& + 3N^2 g_1^4 g_2^2 + 12960\zeta_3 N g_1^4 g_2^2 - 26646N g_1^4 g_2^2 - 1560N g_1^3 g_2^3 + 952N g_1^2 g_2^4 \\
& \left. + 2592\zeta_3 g_2^6 - 5195g_2^6 \right] \frac{1}{62208} \\
& + \left[ - 41472\zeta_3 N^3 g_1^8 + 54266N^3 g_1^8 + 1897344\zeta_3 N^2 g_1^8 + 513216\zeta_4 N^2 g_1^8 \right. \\
& - 1866240\zeta_5 N^2 g_1^8 - 605816N^2 g_1^8 - 238464\zeta_3 N g_1^8 + 653184\zeta_4 N g_1^8 \\
& - 3732480\zeta_5 N g_1^8 + 3883280N g_1^8 - 8064N^3 g_1^7 g_2 - 2488320\zeta_3 N^2 g_1^7 g_2 \\
& + 4171512N^2 g_1^7 g_2 + 1679616\zeta_4 N g_1^7 g_2 - 1008768N g_1^7 g_2 + 2592\zeta_3 N^3 g_1^6 g_2^2 \\
& - 354N^3 g_1^6 g_2^2 + 1542240\zeta_3 N^2 g_1^6 g_2^2 - 233280\zeta_4 N^2 g_1^6 g_2^2 - 2324552N^2 g_1^6 g_2^2 \\
& + 3535488\zeta_3 N g_1^6 g_2^2 + 746496\zeta_4 N g_1^6 g_2^2 - 14929920\zeta_5 N g_1^6 g_2^2 \\
& + 10883728N g_1^6 g_2^2 - 233280\zeta_3 N^2 g_1^5 g_2^3 + 416016N^2 g_1^5 g_2^3 - 1026432\zeta_3 N g_1^5 g_2^3 \\
& + 2799360\zeta_4 N g_1^5 g_2^3 - 240816N g_1^5 g_2^3 + 12960\zeta_3 N^2 g_1^4 g_2^4 - 19101N^2 g_1^4 g_2^4 \\
& + 1031616\zeta_3 N g_1^4 g_2^4 + 1283040\zeta_4 N g_1^4 g_2^4 - 7464960\zeta_5 N g_1^4 g_2^4 \\
& + 6462626N g_1^4 g_2^4 - 108864\zeta_3 N g_1^3 g_2^5 + 289416N g_1^3 g_2^5 + 196992\zeta_3 N g_1^2 g_2^6 \\
& - 93312\zeta_4 N g_1^2 g_2^6 - 306528N g_1^2 g_2^6 + 272160\zeta_3 g_2^8 + 489888\zeta_4 g_2^8 \\
& \left. - 1866240\zeta_5 g_2^8 + 1443123g_2^8 \right] \frac{1}{6718464} + O(g_i^{10})
\end{aligned} \tag{6.3}$$

where the order symbol refers to any combination of the two couplings. As we do not use the method of subtractions but follow the automatic renormalization algorithm of [41] the cancellation of the double, triple and quadruple poles in  $\epsilon$  act as a useful computational check. Moreover, we correctly reproduced the three loop results given in [20].

For the two  $\beta$ -functions,  $\beta_1(g_1, g_2)$  and  $\beta_2(g_1, g_2)$ , we again do not need to compute any nullified 3-point vertex graphs but instead use the method outlined in section 2 where the propagators of each field had an extra term. In terms of graphs this means we avoid computing 6455 graphs at four loops for  $g_1$  and 6681 for  $g_2$ . However, in expanding the propagator within a graph to reproduce the corresponding nullified 3-point vertices, one has to label the extra term of the  $\phi^i$  propagator with the coupling constant  $g_1$ . This is because this is the only vertex with two  $\phi^i$  fields. Equally the additional term for the  $\sigma$  propagator is labelled with  $g_2$ . Once this step has been achieved the renormalization process outlined for the basic  $\phi^3$  Lagrangian is

followed. The result is the two  $\beta$ -functions

$$\begin{aligned}
\beta_1(g_1, g_2) = & -\frac{\epsilon g_1}{2} + \left[ -N g_1^2 + 8g_1^2 + 12g_1 g_2 - g_2^2 \right] \frac{g_1}{24} \\
& + \left[ -86N g_1^4 - 536g_1^4 + 132N g_1^3 g_2 - 360g_1^3 g_2 - 11N g_1^2 g_2^2 - 628g_1^2 g_2^2 \right. \\
& \quad \left. - 24g_1 g_2^3 + 13g_2^4 \right] \frac{g_1}{864} \\
& + \left[ 3662N^2 g_1^6 + 129600\zeta_3 N g_1^6 - 40688N g_1^6 + 20736\zeta_3 g_1^6 + 251360g_1^6 \right. \\
& \quad - 36N^2 g_1^5 g_2 - 155520\zeta_3 N g_1^5 g_2 + 124704N g_1^5 g_2 + 186624\zeta_3 g_1^5 g_2 + 18000g_1^5 g_2 \\
& \quad + 3N^2 g_1^4 g_2^2 + 12960\zeta_3 N g_1^4 g_2^2 - 53990N g_1^4 g_2^2 - 41472\zeta_3 g_1^4 g_2^2 + 358480g_1^4 g_2^2 \\
& \quad - 4560N g_1^3 g_2^3 + 124416\zeta_3 g_1^3 g_2^3 + 97776g_1^3 g_2^3 + 952N g_1^2 g_2^4 + 62208\zeta_3 g_1^2 g_2^4 \\
& \quad \left. + 9960g_1^2 g_2^4 - 31104\zeta_3 g_1 g_2^5 + 33612g_1 g_2^5 + 2592\zeta_3 g_2^6 - 5195g_2^6 \right] \frac{g_1}{124416} \\
& + \left[ 93312\zeta_3 N^3 g_1^8 - 12310N^3 g_1^8 + 21959424\zeta_3 N^2 g_1^8 - 5365440\zeta_4 N^2 g_1^8 \right. \\
& \quad - 1866240\zeta_5 N^2 g_1^8 - 11535384N^2 g_1^8 - 172969344\zeta_3 N g_1^8 + 14183424\zeta_4 N g_1^8 \\
& \quad + 111974400\zeta_5 N g_1^8 + 12401088N g_1^8 + 31290624\zeta_3 g_1^8 + 1492992\zeta_4 g_1^8 \\
& \quad - 82114560\zeta_5 g_1^8 - 104680384g_1^8 - 31104\zeta_3 N^3 g_1^7 g_2 + 4248N^3 g_1^7 g_2 \\
& \quad - 17262720\zeta_3 N^2 g_1^7 g_2 + 2799360\zeta_4 N^2 g_1^7 g_2 + 11998152N^2 g_1^7 g_2 \\
& \quad - 29673216\zeta_3 N g_1^7 g_2 + 3919104\zeta_4 N g_1^7 g_2 + 89579520\zeta_5 N g_1^7 g_2 \\
& \quad - 93820080N g_1^7 g_2 - 89268480\zeta_3 g_1^7 g_2 + 12877056\zeta_4 g_1^7 g_2 - 78382080\zeta_5 g_1^7 g_2 \\
& \quad + 5902944g_1^7 g_2 + 2592\zeta_3 N^3 g_1^6 g_2^2 - 354N^3 g_1^6 g_2^2 + 3403296\zeta_3 N^2 g_1^6 g_2^2 \\
& \quad - 233280\zeta_4 N^2 g_1^6 g_2^2 - 4985248N^2 g_1^6 g_2^2 + 27454464\zeta_3 N g_1^6 g_2^2 \\
& \quad - 12223872\zeta_4 N g_1^6 g_2^2 - 59719680\zeta_5 N g_1^6 g_2^2 + 25092064N g_1^6 g_2^2 \\
& \quad - 176380416\zeta_3 g_1^6 g_2^2 + 18382464\zeta_4 g_1^6 g_2^2 + 82114560\zeta_5 g_1^6 g_2^2 \\
& \quad - 109678192g_1^6 g_2^2 - 342144\zeta_3 N^2 g_1^5 g_2^3 + 576648N^2 g_1^5 g_2^3 - 17324928\zeta_3 N g_1^5 g_2^3 \\
& \quad + 839808\zeta_4 N g_1^5 g_2^3 + 111974400\zeta_5 N g_1^5 g_2^3 - 41588256N g_1^5 g_2^3 \\
& \quad - 70232832\zeta_3 g_1^5 g_2^3 + 559872\zeta_4 g_1^5 g_2^3 + 44789760\zeta_5 g_1^5 g_2^3 - 146215152g_1^5 g_2^3 \\
& \quad + 12960\zeta_3 N^2 g_1^4 g_2^4 - 19101N^2 g_1^4 g_2^4 + 3478464\zeta_3 N g_1^4 g_2^4 - 3195936\zeta_4 N g_1^4 g_2^4 \\
& \quad - 7464960\zeta_5 N g_1^4 g_2^4 + 7579786N g_1^4 g_2^4 - 85784832\zeta_3 g_1^4 g_2^4 + 5412096\zeta_4 g_1^4 g_2^4 \\
& \quad + 3732480\zeta_5 g_1^4 g_2^4 - 34030688g_1^4 g_2^4 - 2255040\zeta_3 N g_1^3 g_2^5 + 1119744\zeta_4 N g_1^3 g_2^5 \\
& \quad + 1918896N g_1^3 g_2^5 - 24292224\zeta_3 g_1^3 g_2^5 + 8957952\zeta_4 g_1^3 g_2^5 - 55987200\zeta_5 g_1^3 g_2^5 \\
& \quad + 17096616g_1^3 g_2^5 + 196992\zeta_3 N g_1^2 g_2^6 - 93312\zeta_4 N g_1^2 g_2^6 - 306528N g_1^2 g_2^6 \\
& \quad - 6371136\zeta_3 g_1^2 g_2^6 + 4665600\zeta_4 g_1^2 g_2^6 - 17426144g_1^2 g_2^6 - 4494528\zeta_3 g_1 g_2^7 \\
& \quad - 4199040\zeta_4 g_1 g_2^7 + 22394880\zeta_5 g_1 g_2^7 - 9944304g_1 g_2^7 + 272160\zeta_3 g_2^8 \\
& \quad \left. + 489888\zeta_4 g_2^8 - 1866240\zeta_5 g_2^8 + 1443123g_2^8 \right] \frac{g_1}{13436928} + O(g_i^{11}) \tag{6.4}
\end{aligned}$$

and

$$\begin{aligned}
\beta_2(g_1, g_2) = & -\frac{\epsilon g_2}{2} + \left[ 4N g_1^3 - N g_1^2 g_2 + 3g_2^3 \right] \frac{1}{8} \\
& + \left[ -24N g_1^5 - 322N g_1^4 g_2 - 60N g_1^3 g_2^2 + 31N g_1^2 g_2^3 - 125g_2^5 \right] \frac{1}{288} \\
& + \left[ 27696N^2 g_1^7 + 34224N g_1^7 - 38474N^2 g_1^6 g_2 + 5184\zeta_3 N g_1^6 g_2 + 59408N g_1^6 g_2 \right. \\
& \quad \left. + 11304N^2 g_1^5 g_2^2 + 62208\zeta_3 N g_1^5 g_2^2 + 25296N g_1^5 g_2^2 - 789N^2 g_1^4 g_2^3 \right]
\end{aligned}$$

$$\begin{aligned}
& + 44064\zeta_3 N g_1^4 g_2^3 + 127890 N g_1^4 g_2^3 - 20736\zeta_3 N g_1^3 g_2^4 - 8688 N g_1^3 g_2^4 \\
& - 6272 N g_1^2 g_2^5 + 12960\zeta_3 g_2^7 + 33085 g_2^7 \Big] \frac{1}{41472} \\
+ & \left[ 1088640\zeta_3 N^3 g_1^9 - 1031208 N^3 g_1^9 - 8771328\zeta_3 N^2 g_1^9 - 3359232\zeta_4 N^2 g_1^9 \right. \\
& + 6915984 N^2 g_1^9 - 6117120\zeta_3 N g_1^9 - 559872\zeta_4 N g_1^9 + 11197440\zeta_5 N g_1^9 \\
& - 20404128 N g_1^9 - 1021248\zeta_3 N^3 g_1^8 g_2 + 706478 N^3 g_1^8 g_2 + 3856896\zeta_3 N^2 g_1^8 g_2 \\
& + 8071488\zeta_4 N^2 g_1^8 g_2 - 13063680\zeta_5 N^2 g_1^8 g_2 - 26286776 N^2 g_1^8 g_2 \\
& - 34763904\zeta_3 N g_1^8 g_2 - 4385664\zeta_4 N g_1^8 g_2 + 1866240\zeta_5 N g_1^8 g_2 \\
& - 401008 N g_1^8 g_2 + 279936\zeta_3 N^3 g_1^7 g_2^2 - 147384 N^3 g_1^7 g_2^2 \\
& - 12192768\zeta_3 N^2 g_1^7 g_2^2 + 1399680\zeta_4 N^2 g_1^7 g_2^2 + 22394880\zeta_5 N^2 g_1^7 g_2^2 \\
& + 5773632 N^2 g_1^7 g_2^2 - 39688704\zeta_3 N g_1^7 g_2^2 + 8398080\zeta_4 N g_1^7 g_2^2 \\
& + 44789760\zeta_5 N g_1^7 g_2^2 - 67219056 N g_1^7 g_2^2 - 23328\zeta_3 N^3 g_1^6 g_2^3 \\
& + 9906 N^3 g_1^6 g_2^3 + 4388256\zeta_3 N^2 g_1^6 g_2^3 - 3172608\zeta_4 N^2 g_1^6 g_2^3 \\
& + 10267192 N^2 g_1^6 g_2^3 - 56619648\zeta_3 N g_1^6 g_2^3 + 7744896\zeta_4 N g_1^6 g_2^3 \\
& + 7464960\zeta_5 N g_1^6 g_2^3 - 9887792 N g_1^6 g_2^3 - 1477440\zeta_3 N^2 g_1^5 g_2^4 \\
& + 559872\zeta_4 N^2 g_1^5 g_2^4 - 3730536 N^2 g_1^5 g_2^4 - 8398080\zeta_3 N g_1^5 g_2^4 + 559872\zeta_4 N g_1^5 g_2^4 \\
& - 55987200\zeta_5 N g_1^5 g_2^4 - 8554440 N g_1^5 g_2^4 + 44064\zeta_3 N^2 g_1^4 g_2^5 + 224817 N^2 g_1^4 g_2^5 \\
& - 22187520\zeta_3 N g_1^4 g_2^5 - 956448\zeta_4 N g_1^4 g_2^5 + 14929920\zeta_5 N g_1^4 g_2^5 \\
& - 44490442 N g_1^4 g_2^5 - 9782208\zeta_3 N g_1^3 g_2^6 + 1959552\zeta_4 N g_1^3 g_2^6 \\
& + 22394880\zeta_5 N g_1^3 g_2^6 + 3707040 N g_1^3 g_2^6 + 1223424\zeta_3 N g_1^2 g_2^7 \\
& - 513216\zeta_4 N g_1^2 g_2^7 + 1351296 N g_1^2 g_2^7 - 12677472\zeta_3 g_2^9 + 1049760\zeta_4 g_2^9 \\
& \left. + 3732480\zeta_5 g_2^9 \right] \frac{1}{4478976} + O(g_i^{11}) . \tag{6.5}
\end{aligned}$$

As with the two field anomalous dimensions we have reproduced the three loop expressions given in [20] and equally recovered the expressions given earlier for (2.1) when  $N = 0$ . This comparison is made with reference to the comments on our conventions. In particular the leading term of each  $\beta$ -function derives from our choice of  $d = 6 - 2\epsilon$  and the factor of 2 in our definition of the renormalization group functions in comparison to [20, 21].

To complete the four loop renormalization of (6.1) we compute the renormalization group functions associated with the two mass operators which we will denote by

$$\mathcal{O}_1 = \frac{1}{2}\phi^i\phi^i \quad , \quad \mathcal{O}_2 = \frac{1}{2}\sigma^2 . \tag{6.6}$$

These two operators have the same canonical dimension of 2 and therefore mix under renormalization. Here we note that our canonical dimension convention derives from the dimensionality of the associated coupling constant of the operators in a Lagrangian. This is an important distinction and is motivated in part for later discussion in relation to checks with the large  $N$  expansion of critical exponents. In other words denoting the bare operators with a subscript  $o$

$$\mathcal{O}_{i o} = Z_{ij}\mathcal{O}_j \tag{6.7}$$

where  $Z_{ij}$  is the mixing matrix of renormalization constants. These produce a mixing matrix of mass anomalous dimensions denoted by  $\gamma_{ij}(g_1, g_2)$ . To extract the renormalization constants one ordinarily inserts the operators into separate  $\phi^i$  and  $\sigma$  2-point functions of all possible 1-particle irreducible Feynman graphs and follows the normal procedure. However, as the operators do not

involve derivatives there is no complication with mixing into total derivative operators. Moreover, this means that for the determination of the  $\overline{\text{MS}}$  renormalization constants the operators are inserted with no momentum flowing in or out of the vertex itself. This is a standard method and reduces the problem to a simple 2-point function computation. In the context of the underlying  $\phi^3$  interaction this leads to a similar computational simplification which we exploited before. Again one need not generate any more Feynman graphs than those already used for the wave function renormalization. Similar to the coupling constant we expand each  $\phi^i$  and  $\sigma$  propagator as if there was a respective mass present using the mapping

$$\frac{1}{k^2} \mapsto \frac{1}{k^2} + \frac{m^2}{(k^2)^2}. \quad (6.8)$$

The additional complication here is that one has to label the  $O(m^2)$  term to indicate whether that insertion is from a  $\phi^i$  or  $\sigma$  field mass term. By contrast if one instead evaluated the 3-point functions with a nullified operator insertion then there would be 4 one loop, 38 two loop, 722 three loop and 13136 four loop graphs to determine for the mixing matrix in total at each loop order. Following the procedure for renormalizing a mixing matrix we find the elements of  $\gamma_{ij}(g_1, g_2)$  to four loops are

$$\begin{aligned} \gamma_{11}(g_1, g_2) = & \frac{g_1^2}{3} + \left[ -22Ng_1^2 - 134g_1^2 - 30g_1g_2 + 5g_2^2 \right] \frac{g_1^2}{216} \\ & + \left[ 803N^2g_1^4 + 15552\zeta_3Ng_1^4 - 4016g_1^4 + 2592\zeta_3g_1^4 + 31420g_1^4 - 7776\zeta_3Ng_1^3g_2 \right. \\ & \quad + 2259Ng_1^3g_2 + 15552\zeta_3g_1^3g_2 - 2964g_1^3g_2 + 3926Ng_1^2g_2^2 - 5184\zeta_3g_1^2g_2^2 \\ & \quad \left. + 18512g_1^2g_2^2 - 2859g_1g_2^3 - 51g_2^4 \right] \frac{g_1^2}{15552} \\ & + \left[ 16848\zeta_3N^3g_1^6 - 8322N^3g_1^6 + 2507760\zeta_3N^2g_1^6 - 734832\zeta_4N^2g_1^6 \right. \\ & \quad - 1366196N^2g_1^6 - 21591360\zeta_3Ng_1^6 + 1691280\zeta_4Ng_1^6 + 14463360\zeta_5Ng_1^6 \\ & \quad + 1064726Ng_1^6 + 3911328\zeta_3g_1^6 + 186624\zeta_4g_1^6 - 10264320\zeta_5g_1^6 \\ & \quad - 13085048g_1^6 - 909792\zeta_3N^2g_1^5g_2 + 139968\zeta_4N^2g_1^5g_2 + 370827N^2g_1^5g_2 \\ & \quad - 225504\zeta_3Ng_1^5g_2 + 1959552\zeta_4Ng_1^5g_2 + 2799360\zeta_5Ng_1^5g_2 \\ & \quad - 7312056Ng_1^5g_2 - 8040384\zeta_3g_1^5g_2 + 979776\zeta_4g_1^5g_2 - 8398080\zeta_5g_1^5g_2 \\ & \quad + 3962952g_1^5g_2 + 104328\zeta_3N^2g_1^4g_2^2 - 31177N^2g_1^4g_2^2 - 1489104\zeta_3Ng_1^4g_2^2 \\ & \quad - 991440\zeta_4Ng_1^4g_2^2 - 2799360\zeta_5Ng_1^4g_2^2 + 3771432Ng_1^4g_2^2 - 9916992\zeta_3g_1^4g_2^2 \\ & \quad + 1877904\zeta_4g_1^4g_2^2 + 8864640\zeta_5g_1^4g_2^2 - 11767142g_1^4g_2^2 - 524880\zeta_3Ng_1^3g_2^3 \\ & \quad - 69984\zeta_4Ng_1^3g_2^3 + 2799360\zeta_5Ng_1^3g_2^3 - 2267862Ng_1^3g_2^3 + 1741824\zeta_3g_1^3g_2^3 \\ & \quad - 979776\zeta_4g_1^3g_2^3 - 5598720\zeta_5g_1^3g_2^3 + 32664g_1^3g_2^3 + 76464\zeta_3Ng_1^2g_2^4 \\ & \quad - 104976\zeta_4Ng_1^2g_2^4 - 303299Ng_1^2g_2^4 - 3335904\zeta_3g_1^2g_2^4 - 443232\zeta_4g_1^2g_2^4 \\ & \quad + 7464960\zeta_5g_1^2g_2^4 - 2428708g_1^2g_2^4 - 73872\zeta_3g_1g_2^5 + 139968\zeta_4g_1g_2^5 \\ & \quad \left. + 134451g_1g_2^5 - 11016\zeta_3g_2^6 - 11664\zeta_4g_2^6 + 36596g_2^6 \right] \frac{g_1^2}{1679616} + O(g_i^{10}) \\ \gamma_{12}(g_1, g_2) = & \frac{Ng_1^2}{2} + \left[ -2g_1^2 - 18g_1g_2 - 3g_2^2 \right] \frac{Ng_1^2}{24} \\ & + \left[ 1154Ng_1^4 + 1426g_1^4 - 992Ng_1^3g_2 + 1822g_1^3g_2 + 141Ng_1^2g_2^2 + 864\zeta_3g_1^2g_2^2 \right. \\ & \quad \left. + 1430g_1^2g_2^2 + 864\zeta_3g_1g_2^3 + 1420g_1g_2^3 - 21g_2^4 \right] \frac{Ng_1^2}{1728} \\ & + \left[ 45360\zeta_3N^2g_1^6 - 42967N^2g_1^6 - 365472\zeta_3Ng_1^6 - 139968\zeta_4Ng_1^6 + 288166Ng_1^6 \right. \end{aligned}$$



$$\begin{aligned}
& -254880\zeta_3 g_1^6 - 23328\zeta_4 g_1^6 + 466560\zeta_5 g_1^6 - 850172g_1^6 - 27216\zeta_3 N^2 g_1^5 g_2 \\
& + 18117N^2 g_1^5 g_2 + 111456\zeta_3 N g_1^5 g_2 + 233280\zeta_4 N g_1^5 g_2 - 311040\zeta_5 N g_1^5 g_2 \\
& - 856464N g_1^5 g_2 - 1161216\zeta_3 g_1^5 g_2 - 186624\zeta_4 g_1^5 g_2 + 155520\zeta_5 g_1^5 g_2 \\
& + 88128g_1^5 g_2 + 3888\zeta_3 N^2 g_1^4 g_2^2 - 1935N^2 g_1^4 g_2^2 - 260928\zeta_3 N g_1^4 g_2^2 \\
& + 66096\zeta_4 N g_1^4 g_2^2 + 311040\zeta_5 N g_1^4 g_2^2 + 130289N g_1^4 g_2^2 - 1468800\zeta_3 g_1^4 g_2^2 \\
& + 124416\zeta_4 g_1^4 g_2^2 + 2488320\zeta_5 g_1^4 g_2^2 - 2360254g_1^4 g_2^2 + 111888\zeta_3 N g_1^3 g_2^3 \\
& - 42768\zeta_4 N g_1^3 g_2^3 + 135612N g_1^3 g_2^3 - 506304\zeta_3 g_1^3 g_2^3 + 62208\zeta_4 g_1^3 g_2^3 \\
& - 777600\zeta_5 g_1^3 g_2^3 + 107418g_1^3 g_2^3 - 7776\zeta_3 N g_1^2 g_2^4 - 19521N g_1^2 g_2^4 \\
& - 251856\zeta_3 g_1^2 g_2^4 + 31104\zeta_4 g_1^2 g_2^4 - 466560\zeta_5 g_1^2 g_2^4 - 209457g_1^2 g_2^4 \\
& - 247104\zeta_3 g_1 g_2^5 + 50544\zeta_4 g_1 g_2^5 - 363531g_1 g_2^5 - 37584\zeta_3 g_2^6 + 11664\zeta_4 g_2^6 \\
& + 51165g_2^6 \Big] \frac{N g_1^2}{186624} + O(g_i^{10}) \\
\gamma_{21}(g_1, g_2) = & \frac{g_1^2}{2} + [7N g_1^2 - 20g_1^2 - 54g_1 g_2 - 2g_2^2] \frac{g_1^2}{72} \\
& + [-99N^2 g_1^4 - 7776\zeta_3 N g_1^4 + 8798N g_1^4 + 5184\zeta_3 g_1^4 + 3476g_1^4 - 4896N g_1^3 g_2 \\
& + 17532g_1^3 g_2 - 250N g_1^2 g_2^2 + 10368\zeta_3 g_1^2 g_2^2 + 10054g_1^2 g_2^2 + 5184\zeta_3 g_1 g_2^3 \\
& + 864g_1 g_2^3 - 2592\zeta_3 g_2^4 + 2801g_2^4] \frac{g_1^2}{10368} \\
& + [-1296\zeta_3 N^3 g_1^6 + 513N^3 g_1^6 - 312336\zeta_3 N^2 g_1^6 + 69984\zeta_4 N^2 g_1^6 + 202501N^2 g_1^6 \\
& - 1161216\zeta_3 N g_1^6 - 559872\zeta_4 N g_1^6 + 2799360\zeta_5 N g_1^6 - 1429786N g_1^6 \\
& - 1039392\zeta_3 g_1^6 + 209952\zeta_4 g_1^6 - 466560\zeta_5 g_1^6 - 1075028g_1^6 \\
& + 42768\zeta_3 N^2 g_1^5 g_2 - 100470N^2 g_1^5 g_2 + 1492992\zeta_3 N g_1^5 g_2 - 209952\zeta_4 N g_1^5 g_2 \\
& - 933120\zeta_5 N g_1^5 g_2 - 665130N g_1^5 g_2 - 4043520\zeta_3 g_1^5 g_2 + 139968\zeta_4 g_1^5 g_2 \\
& + 466560\zeta_5 g_1^5 g_2 - 647544g_1^5 g_2 - 4536\zeta_3 N^2 g_1^4 g_2^2 + 6693N^2 g_1^4 g_2^2 \\
& - 504144\zeta_3 N g_1^4 g_2^2 - 58320\zeta_4 N g_1^4 g_2^2 + 3732480\zeta_5 N g_1^4 g_2^2 - 966856N g_1^4 g_2^2 \\
& - 3506976\zeta_3 g_1^4 g_2^2 + 349920\zeta_4 g_1^4 g_2^2 + 3732480\zeta_5 g_1^4 g_2^2 - 6103186g_1^4 g_2^2 \\
& + 76464\zeta_3 N g_1^3 g_2^3 - 151632\zeta_4 N g_1^3 g_2^3 + 147648N g_1^3 g_2^3 - 2462400\zeta_3 g_1^3 g_2^3 \\
& + 373248\zeta_4 g_1^3 g_2^3 - 2332800\zeta_5 g_1^3 g_2^3 - 608376g_1^3 g_2^3 - 89424\zeta_3 N g_1^2 g_2^4 \\
& + 46656\zeta_4 N g_1^2 g_2^4 + 67895N g_1^2 g_2^4 - 987552\zeta_3 g_1^2 g_2^4 + 326592\zeta_4 g_1^2 g_2^4 \\
& - 2332800\zeta_5 g_1^2 g_2^4 + 667542g_1^2 g_2^4 - 261792\zeta_3 g_1 g_2^5 + 198288\zeta_4 g_1 g_2^5 \\
& - 738288g_1 g_2^5 - 187272\zeta_3 g_2^6 - 174960\zeta_4 g_2^6 + 933120\zeta_5 g_2^6 \\
& - 414346g_2^6] \frac{g_1^2}{559872} + O(g_i^{10}) \tag{6.9}
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{22}(g_1, g_2) = & [-N g_1^2 + 5g_2^2] \frac{1}{12} + [-80N g_1^4 - 30N g_1^3 g_2 + 26N g_1^2 g_2^2 - 97g_2^4] \frac{1}{216} \\
& + [-20618N^2 g_1^6 + 5184\zeta_3 N g_1^6 + 27800N g_1^6 + 11304N^2 g_1^5 g_2 + 62208\zeta_3 N g_1^5 g_2 \\
& - 14064N g_1^5 g_2 - 1185N^2 g_1^4 g_2^2 + 28512\zeta_3 N g_1^4 g_2^2 + 154038N g_1^4 g_2^2 \\
& - 31104\zeta_3 N g_1^3 g_2^3 - 11496N g_1^3 g_2^3 - 9884N g_1^2 g_2^4 + 18144\zeta_3 g_2^6 \\
& + 52225g_2^6] \frac{1}{62208} \\
& + [-132840\zeta_3 N^3 g_1^8 + 95093N^3 g_1^8 + 206064\zeta_3 N^2 g_1^8 + 863136\zeta_4 N^2 g_1^8
\end{aligned}$$

$$\begin{aligned}
& - 1866240\zeta_5 N^2 g_1^8 - 2073638 N^2 g_1^8 - 2555712\zeta_3 N g_1^8 - 46656\zeta_4 N g_1^8 \\
& - 233280\zeta_5 N g_1^8 - 1428940 N g_1^8 + 69984\zeta_3 N^3 g_1^7 g_2 - 36846 N^3 g_1^7 g_2 \\
& - 1912896\zeta_3 N^2 g_1^7 g_2 - 69984\zeta_4 N^2 g_1^7 g_2 + 5598720\zeta_5 N^2 g_1^7 g_2 + 471072 N^2 g_1^7 g_2 \\
& - 1664064\zeta_3 N g_1^7 g_2 + 1819584\zeta_4 N g_1^7 g_2 - 5598720\zeta_5 N g_1^7 g_2 \\
& - 3838764 N g_1^7 g_2 - 9072\zeta_3 N^3 g_1^6 g_2^2 + 3759 N^3 g_1^6 g_2^2 + 445824\zeta_3 N^2 g_1^6 g_2^2 \\
& - 775656\zeta_4 N^2 g_1^6 g_2^2 + 2920258 N^2 g_1^6 g_2^2 - 17117568\zeta_3 N g_1^6 g_2^2 \\
& + 2251152\zeta_4 N g_1^6 g_2^2 + 11664000\zeta_5 N g_1^6 g_2^2 - 6035150 N g_1^6 g_2^2 \\
& - 454896\zeta_3 N^2 g_1^5 g_2^3 + 209952\zeta_4 N^2 g_1^5 g_2^3 - 1275264 N^2 g_1^5 g_2^3 - 754272\zeta_3 N g_1^5 g_2^3 \\
& - 419904\zeta_4 N g_1^5 g_2^3 - 16796160\zeta_5 N g_1^5 g_2^3 - 1292700 N g_1^5 g_2^3 + 14904\zeta_3 N^2 g_1^4 g_2^4 \\
& + 86694 N^2 g_1^4 g_2^4 - 6225336\zeta_3 N g_1^4 g_2^4 - 973944\zeta_4 N g_1^4 g_2^4 + 6531840\zeta_5 N g_1^4 g_2^4 \\
& - 14219965 N g_1^4 g_2^4 - 3316464\zeta_3 N g_1^3 g_2^5 + 629856\zeta_4 N g_1^3 g_2^5 \\
& + 8398080\zeta_5 N g_1^3 g_2^5 + 893478 N g_1^3 g_2^5 + 434160\zeta_3 N g_1^2 g_2^6 - 180792\zeta_4 N g_1^2 g_2^6 \\
& + 545052 N g_1^2 g_2^6 - 4788072\zeta_3 g_2^8 + 332424\zeta_4 g_2^8 + 1632960\zeta_5 g_2^8 \\
& - 4010301 g_2^8 \Big] \frac{1}{1679616} + O(g_i^{10}) . \tag{6.10}
\end{aligned}$$

The immediate checks on these expressions are the internal ones similar to those alluded to for the wave function and coupling constant renormalizations. Specifically the higher order poles in  $\epsilon$  in the renormalization constants are not independent but depend on the lower order simple poles. That these higher order poles correctly emerge indicate that the procedure is not inconsistent. We note that unlike the expression given in (3.4) we have not included the wave function renormalization constants in the determination of the renormalization constants of the mixing matrix as is the usual procedure for a set of operators. See, for instance, [67] for a similar renormalization in QCD.

A more appropriate check on our results rests in comparing with results from another quantum field theory. This is possible through the presence of the  $O(N)$  symmetry which means that the anomalous dimensions can be extracted using another expansion method which is the large  $N$  expansion. While this was noted in [20] it is worth summarizing the background to the large  $N$  technique because of the overlap with the renormalization group at a critical point. If the  $\beta$ -function has a non-trivial fixed point at the value  $g_c$ , where this could also represent a vector of coupling constants such as we have here, then the renormalization group functions evaluated at  $g_c$  are termed critical exponents which are renormalization group invariants. Moreover at a fixed point the critical exponents from field theories which are invariably different in nature can be the same. This universality is the key to our large  $N$  checks. Here the relevant scalar field theories which lie in the same universality class as the  $O(N)$  nonlinear  $\sigma$  model,  $O(N)$   $\phi^4$  theory and (6.1). The former two ordinarily reside in dimensions less than six and are respectively perturbatively renormalizable in two and four dimensions. In dimensions differing from their canonical dimension they may cease to be perturbatively renormalizable. Instead they are renormalizable above their canonical dimension in the sense that at their Wilson-Fisher fixed point in  $d$ -dimensions the critical exponents can be determined in the large  $N$  expansion. As  $N$  is a dimensionless parameter, which remains dimensionless in  $d$ -dimensions unlike the coupling constant in dimensionally regularized perturbation theory, then the quantity  $1/N$  becomes a valid parameter for a perturbative expansion when  $N$  is large. Since the critical exponents can be computed to several orders in  $1/N$  and in  $d$ -dimensions and because they are related to the critical renormalization group functions, they contain information on the anomalous dimensions of *all* theories in the same universality class at that fixed point. For the case we consider here, (6.1), the critical exponents corresponding to the two wave function renormalization constants and masses are known to  $O(1/N^2)$  in  $d$ -dimensions and for  $\phi^i$  to  $O(1/N^3)$ , [35, 36, 37, 38].

To be more specific and make contact with earlier work, [35, 36], it is worth recalling the situation for  $O(N)$   $\phi^4$  theory and define the relevant critical exponents. The form of the Lagrangian which is most appropriate is

$$L = \frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{1}{2} \sigma \phi^i \phi^i - \frac{1}{2g} \sigma^2 \quad (6.11)$$

where  $\sigma$  is regarded as an auxiliary field which if eliminated produces the usual  $\phi^4$  Lagrangian. The single coupling constant  $g$  here appears with the quadratic term in  $\sigma$  as it is in this particular form that the large  $N$  evaluation of the critical exponents is developed at high order in  $1/N$ , [35, 36, 37]. Moreover, this formulation allows one to observe which theories lie in the same universality class. For instance, the  $O(N)$  nonlinear  $\sigma$  model has a similar formulation but the final term is linear rather than quadratic in  $\sigma$  and has a different coupling constant which is dimensionless in two dimensions. Moreover, there is a similarity to (6.1) from the point of view of the interaction but the six dimensional theory has an additional coupling. However, only one of the critical points of (6.1) is in the same universality class as (6.11). As a point of reference we use similar notation to [35, 36] to define the full scaling dimensions of the fields. In [35, 36] the critical exponents of  $\phi^i$  and  $\sigma$  were  $\tilde{\alpha}$  and  $\tilde{\beta}$  respectively. It should be noted that these are not related to the physical exponents of earlier sections and are distinct to this section. We have modified the notation of [35, 36] here for the full dimension of the fields to avoid confusion with the usual use of  $\alpha$  and  $\beta$  as critical exponents. In terms of the anomalous contributions they are defined by

$$\tilde{\alpha} = \frac{1}{2}d - 1 + \frac{1}{2}\eta \quad , \quad \tilde{\beta} = 2 - \eta - \chi \quad (6.12)$$

where  $\eta$  is the anomalous dimension of  $\phi^i$  and  $\chi$  is the anomalous dimension of the interaction of (6.11). The canonical dimension of  $\tilde{\beta}$  is in keeping with our mass operator canonical dimension convention. In terms of the renormalization group functions of (6.1)

$$\gamma_\phi(g_{1c}, g_{2c}) = \frac{1}{2}\eta \quad , \quad \gamma_\sigma(g_{1c}, g_{2c}) = -\eta - \chi \quad (6.13)$$

Using (6.4) and (6.5) we have determined the location of fixed point in the large  $N$  expansion,  $(g_{1c}, g_{2c})$ , and evaluated the field anomalous dimensions to the orders in  $1/N$  to which they are known. It is satisfying to record that we find total agreement with our expressions.

It is possible to repeat this check for  $\gamma_{ij}(g_1, g_2)$  which requires some care. In the  $\phi^4$  formulation of the universal theory at the fixed point, (6.11), the critical exponents of the two masses are straightforward to deduce. That for  $\mathcal{O}_1$  was discussed in early sections and in fact is equivalent to  $\tilde{\beta}$  above. The critical exponent for  $\mathcal{O}_2$  can be deduced from the final term of (6.11). In relation to extracting information on the  $\beta$ -function of  $O(N)$   $\phi^4$  theory in the neighbourhood of four dimensions the relevant critical exponent is denoted by  $\omega$  and is proportional to the critical slope of the  $\beta$ -function. However, in relation to the equivalent theory defined in six dimensions that exponent would give the mass anomalous dimension of  $\sigma$  as is evident from the form of the final term of (6.11). Therefore, in terms of the exponents the critical point mass dimensions of the two mass operators are

$$\Delta_1 = 2 - \eta - \chi \quad , \quad \Delta_2 = 2\omega \quad (6.14)$$

The remaining matter is to reconcile this argument with  $\gamma_{ij}(g_1, g_2)$  at criticality and check if the exponents computed from it at the large  $N$  fixed point agree with the above known exponents at  $O(1/N^2)$ . The key to this is to compute the eigen-anomalous dimensions of  $\gamma_{ij}(g_1, g_2)$ . These are given by

$$\gamma_\pm(g_1, g_2) = 2 - [\gamma_{11}(g_1, g_2) + \gamma_{22}(g_1, g_2) \pm \Delta_d(g_1, g_2)] \quad (6.15)$$

where the discriminant is given by

$$\Delta_d(g_1, g_2) = \sqrt{[\gamma_{11}(g_1, g_2) - \gamma_{22}(g_1, g_2)]^2 + 4\gamma_{12}(g_1, g_2)\gamma_{21}(g_1, g_2)} \quad (6.16)$$

and we have included the canonical dimension of 2 here. In [20] it appears that the canonical dimension of  $(d - 2)$  was used for the mass dimensions based on the dimensions of the constituent fields. However, here we retain the value of 2 as that convention is essential in getting consistency with this particular check between different theories. Evaluating  $\gamma_{\pm}(g_1, g_2)$  at the point  $(g_{1c}, g_{2c})$  and expanding to  $O(1/N^2)$  we find that  $\gamma_{-}(g_{1c}, g_{2c})$  agrees exactly to four loops with  $\Delta_1$ . Similarly  $\gamma_{+}(g_{1c}, g_{2c})$  is in precise agreement with  $\Delta_2$  to the same accuracy. In particular the canonical dimension of 2 is crucial for ensuring the consistency of the latter as it derives from the exponent  $\omega$  in the  $O(N)$   $\phi^4$  theory and that exponent near four dimensions corresponds to corrections to scaling rather than a mass operator. We regard these large  $N$  comparisons and in particular the second on the mass operators as non-trivial checks on our perturbative computations. It is worth mentioning that a one loop evaluation of a mass mixing matrix was given in [18] as well as a two loop version for the related  $Sp(N)$  version of (6.1) in [21]. The renormalization group functions of the  $Sp(N)$  symmetric version of (6.1) are related to those of the  $O(N)$  theory by replacing  $N$  in each expression by  $(-N)$ . However, as far as we can see both of the mass mixing matrix computations appear to have included 1-particle *reducible* diagrams. Low order in  $\epsilon$  checks of the critical exponents with respect to the large  $N$  mass operator exponents were discussed in [18, 20].

One of the motivations of extending the three loop results of [20] to the next loop is to examine the critical point structure of the  $O(N)$  two coupling theory. In [20] it was noted that there are various critical value of  $N$  for which the fixed point structure has different properties. For instance for  $N > N_{cr}$  there are three distinct critical points at real values of  $g_1$  and  $g_2$  which are perturbatively unitary. In addition for values of  $N$  in the range  $(N'_{cr}, N''_{cr})$  there are non-unitary fixed points. In [20] the values of these critical values of  $N$  were computed in an  $\epsilon$  expansion from the three loop  $\beta$ -functions building on the one loop work of [25, 18]. Therefore, we extend those estimates here using (6.4) and (6.5) and use the same notation and method but in our conventions. First, we introduce the new scaled coupling variables  $x$  and  $y$  by

$$g_1 = i\sqrt{\frac{12\epsilon}{N}}x \quad , \quad g_2 = i\sqrt{\frac{12\epsilon}{N}}y \quad (6.17)$$

where we recall  $d = 6 - 2\epsilon$ . The presence of  $i$  here is to be consistent with [20, 21] given our coupling constant conventions. Then to find the critical couplings and  $N_{cr}$  one solves the set of equations

$$\beta_1(g_1, g_2) = \beta_2(g_1, g_2) = \frac{\partial\beta_1}{\partial g_1} \frac{\partial\beta_2}{\partial g_2} - \frac{\partial\beta_1}{\partial g_2} \frac{\partial\beta_2}{\partial g_1} = 0 \quad (6.18)$$

where the final equation is the condition for at least one zero eigenvalue of the Hessian of the matrix of derivatives of the  $\beta$ -functions. Solving the resultant three equations perturbatively in  $d$ -dimensions there are three solutions which we designate  $A$ ,  $B$  and  $C$ . The critical values in each of the three cases are

$$\begin{aligned} N_{cr}^A &= 1038.26605 - 1219.67959\epsilon - 1456.69332\epsilon^2 + 3621.68482\epsilon^3 + O(\epsilon^4) \\ x_{cr}^A &= 1.018036 - 0.01879\epsilon + 0.027606\epsilon^2 - 0.02587\epsilon^3 + O(\epsilon^4) \\ y_{cr}^A &= 8.90305 - 0.42045\epsilon + 4.06719\epsilon^2 - 2.00941\epsilon^3 + O(\epsilon^4) \end{aligned} \quad (6.19)$$

$$\begin{aligned} N_{cr}^B &= 1.02145 + 0.06506\epsilon - 0.00652\epsilon^2 + 0.20347\epsilon^3 + O(\epsilon^4) \\ x_{cr}^B &= i[0.23185 + 0.17773\epsilon - 0.15822\epsilon^2 + 0.61640\epsilon^3 + O(\epsilon^4)] \\ y_{cr}^B &= i[0.25582 + 0.22746\epsilon - 0.17106\epsilon^2 + 0.77176\epsilon^3 + O(\epsilon^4)] \end{aligned} \quad (6.20)$$

	$O(1)$	$O(\epsilon)$	$O(\epsilon^2)$	$O(\epsilon^3)$
$N_{cr}^A$	1038.2660	654.0820	454.7593	421.7574
$x_{cr}^A$	1.0180	1.0087	1.0126	1.0133
$y_{cr}^A$	8.9031	8.6977	8.8670	9.5082

Table 10. Estimates of critical value of  $N$  and location of fixed points for solution  $A$  in five dimensions using Padé approximants.

	$O(1)$	$O(\epsilon)$	$O(\epsilon^2)$	$O(\epsilon^3)$
$N_{cr}^B$	1.0215	1.0551	1.0524	1.0539
$x_{cr}^B$	0.2318 <i>i</i>	–	0.2933 <i>i</i>	0.3073 <i>i</i>
$y_{cr}^B$	0.2558 <i>i</i>	–	0.3385 <i>i</i>	0.3564 <i>i</i>

Table 11. Estimates of critical value of  $N$  and location of fixed points for solution  $B$  in five dimensions using Padé approximants.

and

$$\begin{aligned}
N_{cr}^C &= -0.08750 + 0.69453\epsilon - 3.53076\epsilon^2 + 22.49021\epsilon^3 + O(\epsilon^4) \\
x_{cr}^C &= 0.13175 - 0.33427\epsilon + 0.48270\epsilon^2 - 3.84349\epsilon^3 + O(\epsilon^4) \\
y_{cr}^C &= -0.03277 + 0.26911\epsilon - 1.43791\epsilon^2 + 10.20700\epsilon^3 + O(\epsilon^4). \quad (6.21)
\end{aligned}$$

We have checked that the results to  $O(\epsilon^3)$  agree exactly with those given in [20]. With the  $O(\epsilon^3)$  terms now present we can revisit the analysis of the location of the boundaries for the unitary versus non-unitary theories for five dimensions. In [20] a value of  $N_{cr} = 64.253$  emerged by setting  $\epsilon = \frac{1}{2}$  in  $N_{cr}^A$ . This is a significant distance from the leading value of 1038.266. As the  $O(\epsilon^3)$  correction is both large and has a positive sign this means that the estimate for  $N_{cr}$  will increase. Using the simple substitution approach then  $N_{cr} = 516.963$  which is larger than expectations from other methods. However, this large increase is perhaps more indicative of a lack of convergence and it seems more appropriate to estimate  $N_{cr}$  by using Padé approximants. Using a  $[0, L - 1]$  Padé approximant for the  $L$ th loop we find the values given in Table 10. It appears that the value of  $N_{cr}$  settles to around 400 which while larger than the three loop estimate given in [20] is significantly smaller than the leading value. The corresponding values for solutions  $B$  and  $C$  are given respectively in Tables 11 and 12. In both instances the values of  $N_{cr}$  appear to converge to 1.05 and 0 respectively. Indeed for  $B$  the  $O(\epsilon^3)$  correction confirms the observation of [20] that  $N'_{cr} > 1$ .

One interesting application of our analysis is to compare estimates for the dimension of  $\phi$  with recent estimates using the conformal bootstrap method of [28]. There the  $N = 500$  theory was considered directly in strictly five dimensions and the estimate of  $\Delta_\phi = 1.500409$  was given.

	$O(1)$	$O(\epsilon)$	$O(\epsilon^2)$	$O(\epsilon^3)$
$N_{cr}^C$	-0.0875	-0.0176	-0.0082	-0.0035
$x_{cr}^C$	0.1317	0.0581	0.0347	-0.0112
$y_{cr}^C$	-0.0328	-0.0064	0.0039	0.0228

Table 12. Estimates of critical value of  $N$  and location of fixed points for solution  $C$  in five dimensions using Padé approximants.

This is remarkably close to the estimate obtained using the exponent evaluated to  $O(1/N^3)$  in the large  $N$  method which was  $\Delta_\phi = 1.500414$ . From the location of the  $\beta$ -function zeroes for  $N = 500$  we have

$$\begin{aligned} x &= 0.805458 + 0.276177\epsilon + 0.939812\epsilon^2 + 8.067242\epsilon^3 + O(\epsilon^4) \\ y &= -9.455850 - 4.724456\epsilon + 3.073550\epsilon^2 + 140.713929\epsilon^3 + O(\epsilon^4). \end{aligned} \quad (6.22)$$

Using the four loop term and a Padé approximant we find the estimate of  $\Delta_\phi = 1.500537$  compared to a value of 1.500976 using the three loop expression. While not precisely on top of the other two methods there is a hint that a five loop computation may bridge the difference.

As our final part of the analysis we briefly consider several low values of  $N$  in order to extend the results of [25, 18, 20] where  $N = 0$  and 1 were considered. The former value has been already considered in section 3 as it is the case of purely one field. For  $N = 1$  there are two non-trivial fixed points one of which is stable at  $g_2 = \frac{6}{5}g_1$  to leading order in  $\epsilon$  and the other is at  $g_1 = g_2$  exactly, [25, 20]. For this unstable one the Lagrangian can be rewritten in such a way as to have a double copy of the basic theory (2.1) and it is believed that the flow is away from the unstable case to the infrared stable fixed point, [20]. Given our four loop analysis it is a simple exercise to extend the expressions for the critical exponents at each fixed point to the next order in  $\epsilon$ . At the  $g_2 = g_1$  fixed point we have

$$\Delta_\phi = \Delta_\sigma = 2 - 1.111111\epsilon - 0.117970\epsilon^2 + 0.174760\epsilon^3 - 0.631636\epsilon^4 + O(\epsilon^5) \quad (6.23)$$

for the dimensions of the fields and

$$\begin{aligned} \omega_+ &= 2.000000\epsilon - 3.086420\epsilon^2 + 12.725343\epsilon^3 - 72.522012\epsilon^4 + O(\epsilon^5) \\ \omega_- &= -0.222222\epsilon - 0.235940\epsilon^2 + 0.349520\epsilon^3 - 1.263272\epsilon^4 + O(\epsilon^5) \end{aligned} \quad (6.24)$$

for the eigen-critical exponents of the matrix

$$\omega_{ij} = \left. \frac{\partial \beta_i}{\partial g_j} \right|_{g_1=g_{1c} \ g_2=g_{2c}} \quad (6.25)$$

at the fixed point. In (6.24) we use the convention that  $d$  is not added to the eigenvalues of  $\omega_{ij}$  and also note that the signs of the leading terms confirms the fixed point is unstable. In addition

$$\begin{aligned} \Delta_+ &= 2 - 1.111111\epsilon - 0.117970\epsilon^2 + 0.174760\epsilon^3 - 0.631636\epsilon^4 + O(\epsilon^5) \\ \Delta_- &= 2 + 0.222222\epsilon + 0.235940\epsilon^2 - 0.349520\epsilon^3 + 1.263272\epsilon^4 + O(\epsilon^5) \end{aligned} \quad (6.26)$$

for the mass matrix eigen-exponents which shows  $\Delta_+ = \Delta_\sigma$ . By contrast at the infrared stable point we have

$$\begin{aligned} \Delta_\phi &= 2 - 1.100200\epsilon - 0.093791\epsilon^2 + 0.160519\epsilon^3 - 0.545803\epsilon^4 + O(\epsilon^5) \\ \Delta_\sigma &= 2 - 1.122244\epsilon - 0.143537\epsilon^2 + 0.188846\epsilon^3 - 0.721707\epsilon^4 + O(\epsilon^5) \\ \omega_+ &= 2.000000\epsilon - 3.092766\epsilon^2 + 12.776556\epsilon^3 - 72.867332\epsilon^4 + O(\epsilon^5) \\ \omega_- &= 0.220441\epsilon + 0.175093\epsilon^2 - 0.316680\epsilon^3 + 1.200781\epsilon^4 + O(\epsilon^5) \\ \Delta_+ &= 2 - 1.122244\epsilon - 0.143537\epsilon^2 + 0.188846\epsilon^3 - 0.721707\epsilon^4 + O(\epsilon^5) \\ \Delta_- &= 2 + 0.100200\epsilon - 0.046403\epsilon^2 - 0.156191\epsilon^3 + 0.336654\epsilon^4 + O(\epsilon^5). \end{aligned} \quad (6.27)$$

We note that our expressions for  $\omega_i$  agree with [20] at three loops when the different conventions are accommodated. Clearly  $\Delta_- = \Delta_\sigma$  as expected, [20], in contrast to the situation at the

unstable fixed point. There the other eigen-exponent of the mass matrix was equivalent to the field dimension for  $N = 1$ . In Appendix B we have recorded the field and mass eigen-exponents for a variety of values of  $N$  for the three fixed points that occur in the solution of  $\beta_i(g_1, g_2) = 0$ . The specific cases we considered are those chosen in [28] for a conformal bootstrap analysis. The same feature which has just been noted is apparent there. In other words at certain fixed points one of the mass eigen-exponents is equivalent to the anomalous dimension of the  $\sigma$  field similar to the large  $N$  results. However, which mass exponent is identified depends on the nature of the underlying fixed point.

## 7 Discussion.

We close with various observations. First, we have carried out the four loop renormalization of  $\phi^3$  theory in six dimensions. Clearly this has been a nontrivial exercise since we had to calculate the full set of master four loop massless 2-point functions to the requisite orders in  $\epsilon$ . Though tedious due to the large amount of integration by parts required, the master evaluation rested on the corresponding known masters in four dimensions, [46]. That one can relate them was possible through the methods of [51, 52] and using general properties such as Weinberg's theorem, [47]. One of the original aims was to refine the  $\epsilon$  expansion estimates of critical exponents for several physical problems. Overall the exponents we computed at the next order of precision are in reasonable accord with numerical approaches. Though the accuracy for low dimensions was driven by a deeper underlying property and that was the use of two dimensional conformal symmetry, [32]. There since critical exponents are known exactly one could constrain the exponent estimates and allow us to extract the behaviour across several dimensions. Such techniques now make the  $\epsilon$ -expansion reasonably competitive, but it is perhaps in the application to the more recent studies of  $O(N)$  symmetric theories that will be beneficial in future. One issue examined in [20] was the range of  $N$  defining the conformal window which was suggested in [20] to drop from the one loop value of  $N_{cr} = 1038$ , [25, 18], to around  $N_{cr} = 64$ . Here we took a more conservative approach in applying summation techniques to suggest that while the bounding value drops, it maybe does not reduce so far as this. While successive three and four loop estimates suggest a value settling to around  $N_{cr} = 400$  it would be premature to regard this problem as having been resolved. It may be that the perturbative approach is not as fully equipped to give a definitive answer in comparison to, say, the conformal bootstrap machinery. Also as noted in [20] the  $N = 1$  theory may be related to the deformed (3,10) minimal conformal field theory in two dimensions. Therefore, if true then one could perform a similar analysis across the gap to six dimensions. It would be interesting then to try and match predictions with physical systems.

One intriguing possibility is the potential application of the universality between 2 and 6 dimensions to non-scalar theories such as those with gauge symmetry or supersymmetry. Indeed the conformal window of QCD is of interest, [14]. Akin to the connections of four dimensional  $\phi^4$  theory and six dimensional  $\phi^3$  theory one question which would be worth considering in future is what if any is the higher dimensional theory which is in the same universality class as the Banks-Zaks fixed point in QCD. To an extent there is already a parallel in QCD with what has been discussed here and in [20] for the large  $N$  connection in the scalar field theories. At the Wilson-Fisher fixed point in  $2 < d < 4$  it is known that QCD is in the same universality class as the non-abelian Thirring model in the large  $N_f$  expansion, [68]. Here  $N_f$  is the number of (massless) quarks and it is important to appreciate that the connection is with respect to this particular parameter being large as opposed to  $N_c$  being large where  $N_c$  is the number of colours in  $SU(N_c)$  QCD. Like  $\phi^4$  theory QCD is perturbatively renormalizable in four dimensions and

the non-abelian Thirring model is renormalizable in two dimensions. Indeed this fact has been used to compute large  $N_f$  critical exponents to varying orders in  $1/N_f$  in the Thirring model and demonstrate that their  $\epsilon$  expansion with respect to four dimensions agrees with the analogous renormalization group functions of QCD. See, for example, [69, 70]. While parallel to the lower end of the chain of nonlinear  $\sigma$  model,  $\phi^4$  and  $\phi^3$  theories which are renormalizable in two, four and six dimensions respectively, what is missing in the QCD instance is the corresponding six dimensional theory. To extend the QCD chain would require an understanding of the critical point properties of a spin-1 field in contrast to a spin-0 field. We have referred to this field not as a gauge field or gluon because in the two dimensional non-abelian Thirring model the spin-1 field plays the role of an auxiliary field rather than a gauge field much in the same way that one replaces the interaction in  $\phi^4$  theory by a 3-point vertex at the expense of introducing an auxiliary field. In (6.1) this appears as  $\sigma$  and is the connecting field for the chain in  $4 < d < 6$  as well as being the Lagrange multiplier field in the two dimensional  $O(N)$  nonlinear  $\sigma$  model. There it imposes the constraint that the fields lie on a multidimensional sphere. So the  $\sigma$  field is effectively the lynchpin field which also underlies the chain of theories at the Wilson-Fisher fixed point across the dimensions.

**Acknowledgements.** This work was carried out with the support of STFC Consolidated Grant ST/L000431/1. We thank R.M. Simms for discussions.

## A Master Integrals.

In this appendix we record the explicit values of the master integrals required for the four loop renormalization to various orders in  $\epsilon$  near six dimensions. Terms beyond the  $O(1/\epsilon)$  are needed due to the presence of spurious poles in  $\epsilon$  from the integration by parts algorithm of Laporta, [42]. We have not included all the basic integrals used but only those which are not products of lower loop integrals nor which contain only simple self-energy subgraphs in order to save space. These are straightforward to construct directly by expanding products and ratios of Euler  $\Gamma$ -functions. We use the same notation used in [46] for the definition of the topology. The explicit graphs of the integrals are given in [46]. We have

$$\begin{aligned}
M_{21} &= \frac{1}{373248} \frac{1}{\epsilon^3} + \frac{4093}{261273600} \frac{1}{\epsilon^2} + \frac{17541299}{329204736000} \frac{1}{\epsilon} + \left[ \frac{12061889939}{138265989120000} + \frac{19}{466560} \zeta_3 \right] \\
&+ \left[ -\frac{26183347978621}{58071715430400000} + \frac{19}{311040} \zeta_4 + \frac{15131}{65318400} \zeta_3 \right] \epsilon \\
&+ \left[ -\frac{151264019628699781}{24390120480768000000} + \frac{341}{155520} \zeta_5 + \frac{15131}{43545600} \zeta_4 + \frac{124760533}{82301184000} \zeta_3 \right] \epsilon^2 \\
&+ \left[ -\frac{490045281662428129541}{10243850601922560000000} + \frac{251}{46656} \zeta_6 + \frac{275209}{21772800} \zeta_5 + \frac{124760533}{54867456000} \zeta_4 \right. \\
&\quad \left. + \frac{388425339013}{34566497280000} \zeta_3 - \frac{493}{233280} \zeta_3^2 \right] \epsilon^3 \\
&+ \left[ -\frac{1331159757988712929353901}{4302417252807475200000000} + \frac{16619}{311040} \zeta_7 + \frac{1583}{51030} \zeta_6 + \frac{1700403287}{27433728000} \zeta_5 \right. \\
&\quad \left. + \frac{388425339013}{23044331520000} \zeta_4 + \frac{1138231977555493}{14517928857600000} \zeta_3 - \frac{493}{77760} \zeta_3 \zeta_4 - \frac{397643}{32659200} \zeta_3^2 \right] \epsilon^4 \\
&+ O(\epsilon^5) \\
M_{22} &= -\frac{1}{544320} \frac{1}{\epsilon^3} - \frac{517}{36578304} \frac{1}{\epsilon^2} - \frac{3058789}{38407219200} \frac{1}{\epsilon} + \left[ \frac{1065260317}{10754021376000} - \frac{1}{90720} \zeta_3 \right]
\end{aligned}$$



$$\begin{aligned}
& + \left[ -\frac{398545304569}{188195374080000} + \frac{1}{9072}\zeta_4 + \frac{1609}{3048192}\zeta_3 \right] \epsilon \\
& + \left[ -\frac{20429737763758291}{1897009370726400000} + \frac{37}{27216}\zeta_5 + \frac{1609}{2032128}\zeta_4 + \frac{27941057}{9601804800}\zeta_3 \right] \epsilon^2 \\
& + \left[ -\frac{65407910058930860539}{1195115903557632000000} + \frac{25}{7776}\zeta_6 + \frac{29203}{3048192}\zeta_5 + \frac{27941057}{6401203200}\zeta_4 \right. \\
& \quad \left. + \frac{13768662841}{896168448000}\zeta_3 - \frac{17}{11340}\zeta_3^2 \right] \epsilon^3 \\
& + \left[ -\frac{279112798944169319679083}{1003897358988410880000000} + \frac{13157}{544320}\zeta_7 + \frac{365}{16128}\zeta_6 + \frac{500709689}{9601804800}\zeta_5 \right. \\
& \quad \left. + \frac{13768662841}{597445632000}\zeta_4 + \frac{22798640898757}{282293061120000}\zeta_3 - \frac{17}{3780}\zeta_3\zeta_4 - \frac{5353}{508032}\zeta_3^2 \right] \epsilon^4 + O(\epsilon^5) \\
M_{27} = & \frac{1}{1555200}\frac{1}{\epsilon^3} + \frac{43}{11664000}\frac{1}{\epsilon^2} + \frac{88957}{9797760000}\frac{1}{\epsilon} + \left[ -\frac{86009717}{2743372800000} + \frac{7}{194400}\zeta_3 \right] \\
& + \left[ -\frac{34572378221}{57610828800000} + \frac{7}{129600}\zeta_4 + \frac{301}{1458000}\zeta_3 \right] \epsilon \\
& + \left[ -\frac{2548180213275329}{483930961920000000} + \frac{221}{129600}\zeta_5 + \frac{301}{972000}\zeta_4 + \frac{2924447}{2449440000}\zeta_3 \right] \epsilon^2 \\
& + \left[ -\frac{1134533046211608059}{30487650600960000000} + \frac{649}{155520}\zeta_6 + \frac{9503}{972000}\zeta_5 + \frac{2924447}{1632960000}\zeta_4 \right. \\
& \quad \left. + \frac{5613786161}{685843200000}\zeta_3 - \frac{29}{19440}\zeta_3^2 \right] \epsilon^3 \\
& + \left[ -\frac{12113190120183510866647}{51219253009612800000000} + \frac{3763}{86400}\zeta_7 + \frac{27907}{1166400}\zeta_6 + \frac{37569353}{816480000}\zeta_5 \right. \\
& \quad \left. + \frac{5613786161}{457228800000}\zeta_4 + \frac{4104046049389}{72013536000000}\zeta_3 - \frac{29}{6480}\zeta_3\zeta_4 - \frac{1247}{145800}\zeta_3^2 \right] \epsilon^4 + O(\epsilon^5) \\
M_{32} = & -\frac{1}{7776}\frac{1}{\epsilon^4} - \frac{13}{31104}\frac{1}{\epsilon^3} - \frac{3281}{5598720}\frac{1}{\epsilon^2} + \left[ \frac{188299}{111974400} - \frac{7}{3888}\zeta_3 \right] \frac{1}{\epsilon} \\
& + \left[ \frac{397898731}{20155392000} - \frac{7}{2592}\zeta_4 - \frac{91}{15552}\zeta_3 \right] \\
& + \left[ \frac{16226423357}{134369280000} - \frac{7}{144}\zeta_5 - \frac{91}{10368}\zeta_4 - \frac{65699}{2799360}\zeta_3 \right] \epsilon \\
& + \left[ \frac{44885504009599}{72559411200000} - \frac{455}{3888}\zeta_6 - \frac{91}{576}\zeta_5 - \frac{65699}{1866240}\zeta_4 - \frac{7076939}{55987200}\zeta_3 + \frac{113}{3888}\zeta_3^2 \right] \epsilon^2 \\
& + O(\epsilon^3) \\
M_{33} = & \frac{1}{25920}\frac{1}{\epsilon^4} + \frac{109}{777600}\frac{1}{\epsilon^3} + \frac{71}{1866240}\frac{1}{\epsilon^2} \\
& + \left[ -\frac{1037519}{311040000} + \frac{31}{12960}\zeta_3 \right] \frac{1}{\epsilon} + \left[ -\frac{1856641247}{55987200000} + \frac{31}{8640}\zeta_4 + \frac{3379}{388800}\zeta_3 \right] \\
& + \left[ -\frac{89611835501}{373248000000} + \frac{449}{4320}\zeta_5 + \frac{3379}{259200}\zeta_4 + \frac{1013903}{23328000}\zeta_3 \right] \epsilon \\
& + \left[ -\frac{924742645942453}{604661760000000} + \frac{329}{1296}\zeta_6 + \frac{48941}{129600}\zeta_5 + \frac{1013903}{15552000}\zeta_4 + \frac{47128153}{155520000}\zeta_3 \right. \\
& \quad \left. - \frac{983}{12960}\zeta_3^2 \right] \epsilon^2 \\
& + \left[ -\frac{65933071345592731}{7255941120000000} + \frac{5669}{2160}\zeta_7 + \frac{35861}{38880}\zeta_6 + \frac{11025007}{7776000}\zeta_5 + \frac{47128153}{103680000}\zeta_4 \right. \\
& \quad \left. + \frac{11647067957}{5598720000}\zeta_3 - \frac{983}{4320}\zeta_3\zeta_4 - \frac{107147}{388800}\zeta_3^2 \right] \epsilon^3 + O(\epsilon^4)
\end{aligned}$$

$$\begin{aligned}
M_{34} &= -\frac{1}{103680} \frac{1}{\epsilon^4} - \frac{253}{6220800} \frac{1}{\epsilon^3} - \frac{6031}{74649600} \frac{1}{\epsilon^2} + \left[ \frac{10551067}{22394880000} - \frac{5}{10368} \zeta_3 \right] \frac{1}{\epsilon} \\
&+ \left[ \frac{3403463359}{447897600000} - \frac{5}{6912} \zeta_4 - \frac{253}{124416} \zeta_3 \right] \\
&+ \left[ \frac{588761983211}{8957952000000} - \frac{31}{1152} \zeta_5 - \frac{253}{82944} \zeta_4 - \frac{2444147}{186624000} \zeta_3 \right] \epsilon \\
&+ \left[ \frac{2243912853260021}{4837294080000000} - \frac{685}{10368} \zeta_6 - \frac{7843}{69120} \zeta_5 - \frac{2444147}{124416000} \zeta_4 - \frac{1133018717}{11197440000} \zeta_3 \right. \\
&\quad \left. + \frac{1247}{51840} \zeta_3^2 \right] \epsilon^2 \\
&+ \left[ \frac{171045558222119119}{58047528960000000} - \frac{12503}{17280} \zeta_7 - \frac{34661}{124416} \zeta_6 - \frac{131641}{256000} \zeta_5 - \frac{1133018717}{7464960000} \zeta_4 \right. \\
&\quad \left. - \frac{161255734073}{223948800000} \zeta_3 + \frac{1247}{17280} \zeta_3 \zeta_4 + \frac{315491}{3110400} \zeta_3^2 \right] \epsilon^3 + O(\epsilon^4) \\
M_{35} &= \frac{1}{41472} \frac{1}{\epsilon^3} + \frac{173}{1382400} \frac{1}{\epsilon^2} + \left[ \frac{245651}{746496000} + \frac{1}{28800} \zeta_3 \right] \frac{1}{\epsilon} \\
&+ \left[ -\frac{46303}{4976640000} + \frac{1}{19200} \zeta_4 + \frac{2989}{5184000} \zeta_3 \right] \\
&+ \left[ -\frac{18489907021}{2687385600000} - \frac{23}{28800} \zeta_5 + \frac{2989}{3456000} \zeta_4 + \frac{13087}{3840000} \zeta_3 \right] \epsilon \\
&+ \left[ -\frac{1087166778487}{17915904000000} - \frac{1}{480} \zeta_6 + \frac{25751}{1728000} \zeta_5 + \frac{13087}{2560000} \zeta_4 + \frac{318793381}{18662400000} \zeta_3 \right. \\
&\quad \left. + \frac{29}{28800} \zeta_3^2 \right] \epsilon^2 \\
&+ \left[ -\frac{3908335481909509}{9674588160000000} - \frac{221}{11520} \zeta_7 + \frac{9283}{259200} \zeta_6 + \frac{404999}{3840000} \zeta_5 + \frac{318793381}{12441600000} \zeta_4 \right. \\
&\quad \left. + \frac{33530592221}{373248000000} \zeta_3 + \frac{29}{9600} \zeta_3 \zeta_4 - \frac{67019}{5184000} \zeta_3^2 \right] \epsilon^3 + O(\epsilon^4) \\
M_{36} &= \frac{1}{69120} \frac{1}{\epsilon^2} + \frac{47}{497664} \frac{1}{\epsilon} + \frac{9241}{29859840} + \left[ \frac{383903}{358318080} - \frac{1}{288} \zeta_5 + \frac{1}{405} \zeta_3 \right] \epsilon \\
&+ \left[ \frac{49253441}{4299816960} - \frac{5}{576} \zeta_6 - \frac{65}{3456} \zeta_5 + \frac{1}{270} \zeta_4 + \frac{107}{38880} \zeta_3 + \frac{7}{1440} \zeta_3^2 \right] \epsilon^2 \\
&+ \left[ \frac{160435019}{1146617856} - \frac{127}{2880} \zeta_7 - \frac{325}{6912} \zeta_6 + \frac{10889}{207360} \zeta_5 + \frac{107}{25920} \zeta_4 - \frac{36259}{233280} \zeta_3 + \frac{7}{480} \zeta_3 \zeta_4 \right. \\
&\quad \left. + \frac{91}{3456} \zeta_3^2 \right] \epsilon^3 + O(\epsilon^4) \\
M_{41} &= -\frac{1}{20736} \frac{1}{\epsilon^3} - \frac{383}{746496} \frac{1}{\epsilon^2} + \left[ -\frac{128657}{44789760} + \frac{1}{2880} \zeta_3 \right] \frac{1}{\epsilon} \\
&+ \left[ -\frac{1111315}{107495424} + \frac{1}{1920} \zeta_4 + \frac{253}{103680} \zeta_3 \right] \\
&+ \left[ -\frac{93732961}{6449725440} - \frac{49}{8640} \zeta_5 + \frac{253}{69120} \zeta_4 - \frac{19}{746496} \zeta_3 \right] \epsilon \\
&+ \left[ \frac{82848167}{573308928} - \frac{13}{864} \zeta_6 - \frac{199}{103680} \zeta_5 - \frac{19}{497664} \zeta_4 - \frac{677957}{8957952} \zeta_3 + \frac{31}{1728} \zeta_3^2 \right] \epsilon^2 + O(\epsilon^3) \\
M_{42} &= \frac{1}{62208} \frac{1}{\epsilon^3} + \frac{79}{746496} \frac{1}{\epsilon^2} + \left[ \frac{5689}{4976640} + \frac{1}{2880} \zeta_3 \right] \frac{1}{\epsilon} + \left[ \frac{1277555}{107495424} + \frac{1}{1920} \zeta_4 - \frac{11}{311040} \zeta_3 \right] \\
&+ \left[ \frac{635773633}{6449725440} - \frac{13}{2880} \zeta_5 - \frac{11}{207360} \zeta_4 - \frac{67793}{3732480} \zeta_3 \right] \epsilon
\end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{17878937029}{25798901760} - \frac{7}{576} \zeta_6 - \frac{6019}{103680} \zeta_5 - \frac{67793}{2488320} \zeta_4 - \frac{903631}{4976640} \zeta_3 + \frac{19}{2880} \zeta_3^2 \right] \epsilon^2 \\
& + O(\epsilon^3) \\
M_{43} = & - \frac{1}{5760} \frac{1}{\epsilon^2} + \left[ -\frac{131}{69120} + \frac{1}{2880} \zeta_3 \right] \frac{1}{\epsilon} + \left[ -\frac{10553}{829440} + \frac{1}{1920} \zeta_4 + \frac{101}{34560} \zeta_3 \right] \\
& + \left[ -\frac{684343}{9953280} - \frac{1}{360} \zeta_5 + \frac{101}{23040} \zeta_4 + \frac{8687}{414720} \zeta_3 \right] \epsilon \\
& + \left[ -\frac{13206019}{39813120} - \frac{1}{128} \zeta_6 - \frac{101}{4320} \zeta_5 + \frac{8687}{276480} \zeta_4 + \frac{642169}{4976640} \zeta_3 - \frac{1}{180} \zeta_3^2 \right] \epsilon^2 \\
& + \left[ -\frac{719292013}{477757440} - \frac{67}{960} \zeta_7 - \frac{101}{1536} \zeta_6 - \frac{4367}{51840} \zeta_5 + \frac{642169}{3317760} \zeta_4 + \frac{13908749}{19906560} \zeta_3 \right. \\
& \quad \left. - \frac{1}{60} \zeta_3 \zeta_4 - \frac{101}{2160} \zeta_3^2 \right] \epsilon^3 + O(\epsilon^4) \\
M_{44} = & \frac{7}{103680} \frac{1}{\epsilon^3} + \frac{61}{77760} \frac{1}{\epsilon^2} + \frac{32939}{7464960} \frac{1}{\epsilon} + \left[ \frac{277411}{17915904} + \frac{1}{1296} \zeta_3 \right] \\
& + \left[ \frac{19619333}{1074954240} + \frac{1}{864} \zeta_4 + \frac{781}{155520} \zeta_3 \right] \epsilon \\
& + \left[ -\frac{4976176237}{12899450880} - \frac{147}{64} \zeta_7 + \frac{1999}{1728} \zeta_5 + \frac{781}{103680} \zeta_4 + \frac{113243}{93312} \zeta_3 \right] \epsilon^2 + O(\epsilon^3) \\
M_{45} = & \frac{7}{20736} \frac{1}{\epsilon^3} + \frac{277}{138240} \frac{1}{\epsilon^2} + \left[ \frac{88751}{14929920} + \frac{1}{2880} \zeta_3 \right] \frac{1}{\epsilon} \\
& + \left[ -\frac{30421}{59719680} + \frac{1}{1920} \zeta_4 + \frac{1183}{103680} \zeta_3 \right] \\
& + \left[ -\frac{333199777}{2149908480} - \frac{23}{2880} \zeta_5 + \frac{1183}{69120} \zeta_4 + \frac{5915}{82944} \zeta_3 \right] \epsilon \\
& + \left[ -\frac{88173683267}{60197437440} - \frac{1}{48} \zeta_6 + \frac{511}{1280} \zeta_5 + \frac{5915}{55296} \zeta_4 + \frac{39770197}{104509440} \zeta_3 + \frac{55}{4032} \zeta_3^2 \right] \epsilon^2 \\
& + O(\epsilon^3) \\
M_{51} = & \frac{1}{864} \frac{1}{\epsilon^3} + \left[ \frac{85}{10368} - \frac{1}{216} \zeta_3 \right] \frac{1}{\epsilon^2} + \left[ \frac{1171}{41472} - \frac{1}{144} \zeta_4 - \frac{7}{324} \zeta_3 \right] \frac{1}{\epsilon} \\
& + \left[ \frac{575}{1492992} - \frac{1}{108} \zeta_5 - \frac{7}{216} \zeta_4 + \frac{43}{7776} \zeta_3 \right] \\
& + \left[ -\frac{15492679}{17915904} - \frac{5}{432} \zeta_6 - \frac{191}{1296} \zeta_5 + \frac{43}{5184} \zeta_4 + \frac{45173}{93312} \zeta_3 - \frac{5}{36} \zeta_3^2 \right] \epsilon + O(\epsilon^2) \\
M_{52} = & \frac{1}{216} \frac{1}{\epsilon^3} + \left[ \frac{37}{1296} - \frac{1}{108} \zeta_3 \right] \frac{1}{\epsilon^2} + \left[ \frac{779}{7776} - \frac{1}{72} \zeta_4 - \frac{11}{324} \zeta_3 \right] \frac{1}{\epsilon} \\
& + \left[ \frac{10289}{46656} + \frac{2}{27} \zeta_5 - \frac{11}{216} \zeta_4 - \frac{25}{486} \zeta_3 \right] \\
& + \left[ \frac{3109}{93312} + \frac{5}{24} \zeta_6 + \frac{22}{81} \zeta_5 - \frac{25}{324} \zeta_4 + \frac{445}{5832} \zeta_3 - \frac{7}{54} \zeta_3^2 \right] \epsilon + O(\epsilon^2) \\
M_{61} = & - \frac{1}{72} \frac{1}{\epsilon^2} + \left[ -\frac{17}{108} + \frac{5}{36} \zeta_5 - \frac{1}{36} \zeta_3 \right] \frac{1}{\epsilon} \\
& + \left[ -\frac{407}{432} + \frac{25}{72} \zeta_6 + \frac{85}{216} \zeta_5 - \frac{1}{24} \zeta_4 + \frac{25}{216} \zeta_3 - \frac{1}{36} \zeta_3^2 \right] + O(\epsilon) \\
M_{62} = & \left[ \frac{1}{216} - \frac{1}{72} \zeta_3 \right] \frac{1}{\epsilon^2} + \left[ \frac{37}{648} - \frac{1}{48} \zeta_4 - \frac{13}{216} \zeta_3 \right] \frac{1}{\epsilon} \\
& + \left[ \frac{505}{1296} - \frac{1}{16} \zeta_5 - \frac{13}{144} \zeta_4 - \frac{41}{144} \zeta_3 \right] + O(\epsilon)
\end{aligned}$$

$$\begin{aligned}
M_{63} = & \left[ \frac{1}{216} - \frac{1}{72}\zeta_3 \right] \frac{1}{\epsilon^2} + \left[ \frac{65}{1296} - \frac{1}{48}\zeta_4 - \frac{13}{216}\zeta_3 \right] \frac{1}{\epsilon} + \left[ \frac{179}{648} - \frac{1}{36}\zeta_5 - \frac{13}{144}\zeta_4 - \frac{2}{9}\zeta_3 \right] \\
& + \left[ \frac{4109}{5832} - \frac{5}{144}\zeta_6 - \frac{97}{432}\zeta_5 - \frac{1}{3}\zeta_4 - \frac{457}{972}\zeta_3 - \frac{5}{12}\zeta_3^2 \right] \epsilon + O(\epsilon^2). \tag{A.1}
\end{aligned}$$

In several cases we have given terms up to the level of  $\zeta_7$  despite the fact that this number does not appear in the final renormalization group functions. It would be present in the finite parts of the various Green's functions. For compactness we have expressed the master integrals in (A.1) in the corresponding  $G$ -scheme in six dimensions which is the standard way to present them, [46]. In other words for *each* loop order the factor

$$G = \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)} \tag{A.2}$$

is included. This circumvents the need to include, for instance, the Euler-Mascheroni constant  $\gamma$  which would otherwise appear throughout and which is ordinarily included in part in the change from the minimal subtraction to modified minimal subtraction scheme, [71].

## B Field dimensions at various fixed points.

In this appendix we give the various critical exponents for (6.1) to  $O(\epsilon^4)$  at the same fixed points value of  $N$  as given in the appendix of [28]. This is partly for completeness but also to note interesting structure. For the various values of  $N$  considered there are three distinct fixed points which are labelled in the same way as [28]. These were termed 'critical', 'theory 2' and 'theory 3' and we retain that nomenclature here but label the exponents with the respective superscripts  $c$ , 1 and 2. The first case considered in [28] was  $N = 600$  and the three sets of exponents are

$$\begin{aligned}
\Delta_\phi^c &= 2 + (0.000036i - 0.996357)\epsilon + (0.000142i - 0.008332)\epsilon^2 \\
&\quad + (-0.000852i + 0.001801)\epsilon^3 + (-0.002328i + 0.014101)\epsilon^4 + O(\epsilon^5) \\
\Delta_\sigma^c &= 2 + (-0.060308i + 0.167456)\epsilon + (0.172529i - 0.263035)\epsilon^2 \\
&\quad + (0.274055i + 0.000784)\epsilon^3 + (-0.846566i + 0.464339)\epsilon^4 + O(\epsilon^5) \\
\Delta_+^c &= 2 + (-1.594272 - 0.673433i)\epsilon + (-0.231855 + 1.280485i)\epsilon^2 \\
&\quad + (0.890531 + 3.353446i)\epsilon^3 + (1.853548 - 1.733472i)\epsilon^4 + O(\epsilon^5) \\
\Delta_-^c &= 2 + (0.167456 - 0.060308i)\epsilon + (-0.263035 + 0.172529i)\epsilon^2 \\
&\quad + (0.000784 + 0.274055i)\epsilon^3 + (0.464339 - 0.846567i)\epsilon^4 + O(\epsilon^5) \tag{B.1}
\end{aligned}$$

$$\begin{aligned}
\Delta_\phi^2 &= 2 + (-0.000036i - 0.996357)\epsilon + (-0.000142i - 0.008332)\epsilon^2 \\
&\quad + (0.000853i + 0.001801)\epsilon^3 + (0.002328i + 0.014101)\epsilon^4 + O(\epsilon^5) \\
\Delta_\sigma^2 &= 2 + (0.060308i + 0.167456)\epsilon + (-0.172529i - 0.263035)\epsilon^2 \\
&\quad + (-0.274055i + 0.000784)\epsilon^3 + (0.846566i + 0.464339)\epsilon^4 + O(\epsilon^5) \\
\Delta_+^2 &= 2 + (-1.594272 + 0.673433i)\epsilon + (-0.231855 - 1.280485i)\epsilon^2 \\
&\quad + (0.890531 - 3.353446i)\epsilon^3 + (1.853548 + 1.733472i)\epsilon^4 + O(\epsilon^5) \\
\Delta_-^2 &= 2 + (0.167457 + 0.060308i)\epsilon + (-0.263035 - 0.172530i)\epsilon^2 \\
&\quad + (0.000784 - 0.274055i)\epsilon^3 + (0.464339 + 0.846566i)\epsilon^4 + O(\epsilon^5) \tag{B.2}
\end{aligned}$$

and

$$\Delta_\phi^3 = 2 - 0.997795\epsilon - 0.00219\epsilon^2 + 0.003555\epsilon^3 + 0.034922\epsilon^4 + O(\epsilon^5)$$

$$\begin{aligned}
\Delta_\sigma^3 &= 2 - 0.159643\epsilon + 0.167857\epsilon^2 + 0.763443\epsilon^3 + 4.639369\epsilon^4 + O(\epsilon^5) \\
\Delta_+^3 &= 2 - 0.159642\epsilon + 0.167857\epsilon^2 + 0.763443\epsilon^3 + 4.639369\epsilon^4 + O(\epsilon^5) \\
\Delta_-^3 &= 2 + 0.632464\epsilon + 0.506339\epsilon^2 + 4.963931\epsilon^3 + 35.781014\epsilon^4 + O(\epsilon^5). \quad (\text{B.3})
\end{aligned}$$

Here and for the expressions we give for the other values of  $N$  we note that

$$\Delta_\pm^i = \gamma_\pm(g_{1c}, g_{2c}) \quad (\text{B.4})$$

includes the canonical dimension and these mass exponents definitions are the same as those used for the large  $N$  comparison with the large  $N$  critical exponents. For this value of  $N$  and others two of the points have critical exponents which are complex conjugates. Next the parallel values for  $N = 1000$  are

$$\begin{aligned}
\Delta_\phi^c &= 2 + (-0.000010i - 0.997921)\epsilon + (0.000167i - 0.004378)\epsilon^2 \\
&\quad + (0.001518i + 0.000406)\epsilon^3 + (0.021951i + 0.007247)\epsilon^4 + O(\epsilon^5) \\
\Delta_\sigma^c &= 2 + (0.010791i + 0.115748)\epsilon + (-0.186373i - 0.172907)\epsilon^2 \\
&\quad + (-1.392417i - 0.116672)\epsilon^3 + (-21.819202i - 0.017897)\epsilon^4 + O(\epsilon^5) \\
\Delta_+^c &= 2 + (-1.424029 + 0.158680i)\epsilon + (-0.198690 - 2.582892i)\epsilon^2 \\
&\quad + (-0.142920 - 22.664631i)\epsilon^3 + (-2.551351 - 349.159818i)\epsilon^4 + O(\epsilon^5) \\
\Delta_-^c &= 2 + (0.115748 + 0.010791i)\epsilon + (-0.172907 - 0.186373i)\epsilon^2 \\
&\quad + (-0.116672 - 1.392417i)\epsilon^3 + (-0.017897 - 21.819201i)\epsilon^4 + O(\epsilon^5) \quad (\text{B.5})
\end{aligned}$$

$$\begin{aligned}
\Delta_\phi^2 &= 2 + (0.000010i - 0.997921)\epsilon + (-0.000167i - 0.0043778)\epsilon^2 \\
&\quad + (-0.001518i + 0.000406)\epsilon^3 + (-0.021951i + 0.007247)\epsilon^4 + O(\epsilon^5) \\
\Delta_\sigma^2 &= 2 + (-0.010791i + 0.115748)\epsilon + (0.186373i - 0.172907)\epsilon^2 \\
&\quad + (1.392417i - 0.116671)\epsilon^3 + (21.819201i - 0.017897)\epsilon^4 + O(\epsilon^5) \\
\Delta_+^2 &= 2 + (-1.424029 - 0.158680i)\epsilon + (-0.198690 + 2.582892i)\epsilon^2 \\
&\quad + (-0.142920 + 22.664631i)\epsilon^3 + (-2.551351 + 349.159818i)\epsilon^4 + O(\epsilon^5) \\
\Delta_-^2 &= 2 + (0.115748 - 0.010791i)\epsilon + (-0.172907 + 0.186373i)\epsilon^2 \\
&\quad + (-0.116671 + 1.392417i)\epsilon^3 + (-0.017897 + 21.819202i)\epsilon^4 + O(\epsilon^5) \quad (\text{B.6})
\end{aligned}$$

and

$$\begin{aligned}
\Delta_\phi^3 &= 2 - 0.998612\epsilon - 0.001535\epsilon^2 + 0.001864\epsilon^3 + 0.018909\epsilon^4 + O(\epsilon^5) \\
\Delta_\sigma^3 &= 2 - 0.127848\epsilon + 0.138289\epsilon^2 + 0.598452\epsilon^3 + 3.499963\epsilon^4 + O(\epsilon^5) \\
\Delta_+^3 &= 2 - 0.127848\epsilon + 0.138289\epsilon^2 + 0.598452\epsilon^3 + 3.499963\epsilon^4 + O(\epsilon^5) \\
\Delta_-^3 &= 2 + 0.525620\epsilon + 0.386259\epsilon^2 + 3.806034\epsilon^3 + 27.088909\epsilon^4 + O(\epsilon^5). \quad (\text{B.7})
\end{aligned}$$

Finally, for  $N = 1400$  we find

$$\begin{aligned}
\Delta_\phi^c &= 2 - 0.998526\epsilon - 0.002924\epsilon^2 - 0.000056\epsilon^3 + 0.004014\epsilon^4 + O(\epsilon^5) \\
\Delta_\sigma^c &= 2 + 0.067715\epsilon - 0.143131\epsilon^2 + 0.005374\epsilon^3 + 0.348439\epsilon^4 + O(\epsilon^5) \\
\Delta_+^c &= 2 - 1.765085\epsilon - 0.767483\epsilon^2 + 0.291437\epsilon^3 + 2.728177\epsilon^4 + O(\epsilon^5) \\
\Delta_-^c &= 2 + 0.067715\epsilon - 0.143131\epsilon^2 + 0.005374\epsilon^3 + 0.348439\epsilon^4 + O(\epsilon^5) \quad (\text{B.8})
\end{aligned}$$

$$\begin{aligned}
\Delta_\phi^2 &= 2 - 0.998575\epsilon - 0.002939\epsilon^2 + 0.000342\epsilon^3 + 0.005877\epsilon^4 + O(\epsilon^5) \\
\Delta_\sigma^2 &= 2 + 0.115801\epsilon - 0.123622\epsilon^2 - 0.2790432\epsilon^3 - 0.776356\epsilon^4 + O(\epsilon^5) \\
\Delta_+^2 &= 2 - 0.924175\epsilon + 0.427891\epsilon^2 - 1.113372\epsilon^3 - 10.101218\epsilon^4 + O(\epsilon^5) \\
\Delta_-^2 &= 2 + 0.115801\epsilon - 0.1236220\epsilon^2 - 0.279043\epsilon^3 - 0.776356\epsilon^4 + O(\epsilon^5) \quad (\text{B.9})
\end{aligned}$$

and

$$\begin{aligned}
\Delta_\phi^3 &= 2 - 0.998982\epsilon - 0.001191\epsilon^2 + 0.001224\epsilon^3 + 0.012696\epsilon^4 + O(\epsilon^5) \\
\Delta_\sigma^3 &= 2 - 0.109951\epsilon + 0.120821\epsilon^2 + 0.507233\epsilon^3 + 2.895587\epsilon^4 + O(\epsilon^5) \\
\Delta_+^3 &= 2 - 0.109951\epsilon + 0.120821\epsilon^2 + 0.507233\epsilon^3 + 2.895587\epsilon^4 + O(\epsilon^5) \\
\Delta_-^3 &= 2 + 0.462254\epsilon + 0.321631\epsilon^2 + 3.184495\epsilon^3 + 22.495823\epsilon^4 + O(\epsilon^5) . \quad (\text{B.10})
\end{aligned}$$

We note that the expression for  $\Delta_\sigma^2$  corrects an obvious typographical error in equation (A.23) of [28]. There the  $O(\epsilon)$  term is not recorded although its actual coefficient appears as the coefficient of the  $O(\epsilon^2)$  term. For the  $N = 1400$  set the three fixed points again produces real critical exponents. Unfortunately in each set improving the series convergence using Padé approximants only applies to  $\Delta_\phi$  along the lines discussed in the main text for  $N = 500$ . In each case we have recorded the mass mixing matrix eigen-critical exponents in our conventions as there is an interesting feature which extends the observation in the large  $N$  comparison in section 6. In each set of exponents and values of  $N$   $\Delta_\sigma$  is equivalent to one of the mass eigen-critical exponents. In other words for finite values of  $N$  the field critical exponent and its mass exponent are equivalent. However, the particular eigen-exponent the field dimension equates to depends on the specific fixed point. In each of the cases presented here  $\Delta_\sigma$  corresponds to the minus exponent for the points designated critical and theory 2 but to the plus exponent for theory 3. Of the three only theory 3 has real exponents and this picture tallies with the large  $N$  checks discussed earlier.

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