A basis for the full Homfly skein of the annulus

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Abstract

A basis, denoted $\{Q_{\lambda,\mu}\}$, for the full Homfly skein of the annulus $\mathcal{C}$ was introduced in [18], where λ and μ are partitions of integers n and p into k and k^* parts respectively. The basis consists of eigenvectors of the two meridian maps on $\mathcal{C}$; these maps are the linear endomorphisms of $\mathcal{C}$ induced by the insertion of a meridian loop with either orientation around a diagram in the annulus.

In this paper we give an explicit expression for each $Q_{\lambda,\mu}$ as the determinant of a $(k^*+k)\times(k^*+k)$ matrix whose entries are simple elements h_n, h_n^* in the skein $\mathcal{C}$. In the case $p=0$ ($\mu=\phi$) the determinant gives the Jacobi-Trudy formula for the Schur function s_λ of N variables as a polynomial in the complete symmetric functions h_n of the variables, [12]. The Jacobi-Trudy determinants have previously been used by Kawagoe [6] and Lukac [10] in discussing the elements in the skein of the annulus represented by closed braids in which all strings are oriented in the same direction. Our results and techniques here form a natural extension of the work of Lukac.

Introduction

The Homfly polynomials of satellites of a knot K have provided an extensive source of invariants of K since the discovery of the Homfly polynomial in 1984. It has proved helpful to organise these invariants by means of the Homfly skein of the annulus, $\mathcal{C}$, which consists of linear combinations of diagrams in the annulus, modulo the Homfly skein relations, and encapsulates the relevant information about the decorating curves used in constructing a satellite.

The skein $\mathcal{C}$, which carries a natural product making it a commutative algebra, has been studied widely following the initial work of Turaev [22],

[1] The first author was supported by EPSRC grant 99801479

who showed it to be a polynomial algebra with an explicit set of generators consisting of simple closed braids oriented in either direction around the annulus. Other related invariants of K emerged subsequently, using the quantum groups $sl(N)_q$. The relation between the invariants is particularly close on the subalgebra $\mathcal{C}^+ \subset \mathcal{C}$ generated by closed braids all oriented in the same direction. In this context Wenzl [23] and Murakami [20] gave details of elements of $\mathcal{C}^+$ derived from the closure of idempotents of Hecke algebras, each determined by a partition λ, which provide a direct translation between Homfly invariants and the $sl(N)_q$ invariants arising from the irreducible representation given by λ. More explicitly skein-based versions of the idempotents and the corresponding elements $Q_\lambda \in \mathcal{C}^+$ were developed by Aiston [1, 16].

It has more recently been observed, [6, 10], that the elements Q_λ can be naturally characterised as eigenvectors of the meridian maps of $\mathcal{C}$, which are linear endomorphisms of $\mathcal{C}$ induced at the level of diagrams in the annulus by inserting a meridian curve around a given diagram. The work in [6, 10] shows how to describe Q_λ as the determinant of a matrix with entries drawn from a sequence of elements h_n in $\mathcal{C}$. In interpretations of $\mathcal{C}^+$ as the representation ring of the quantum groups $sl(N)_q$ the elements h_n correspond to the irreducible representations indexed by partitions with a single part n, and the determinants used give the irreducible representation corresponding to the partition λ in terms of the Jacobi-Trudy formula, [12, I(3.4)].

In the full skein $\mathcal{C}$ the eigenspaces of the meridian maps are all known to be 1-dimensional [18]. We present here in section 3 an explicit determinantal formula for the corresponding eigenvectors $Q_{\lambda,\mu}$, using as matrix entries the elements h_n and the corresponding elements h_n^* where the string orientation is reversed. As in [10] we use an auxiliary skein $\mathcal{A}$, also a commutative algebra, based on diagrams in the annulus which include one arc joining a point on each boundary component.

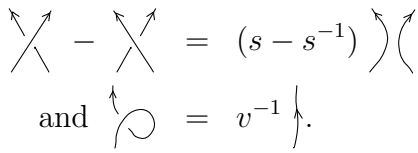
In section 1 we introduce the basic features of Homfly skein theory needed, including properties of the skeins $\mathcal{C}$ and $\mathcal{A}$ and the elements h_n. In section 2 we describe the sort of matrices with entries in $\mathcal{A}$ or $\mathcal{C}$ which are used, and deduce some properties of their determinants from simple skein relations in section 1. In the final section we construct the basis elements $Q_{\lambda,\mu}$ and prove that they are eigenvectors of the meridian maps. We show how they appear naturally as eigenvectors when considering the Homfly satellite invariants of a framed knot K, with eigenvalues that give $(1,1)$-tangle invariants of K. We also oberve that the structure constants in the algebra $\mathcal{C}$ with basis $Q_{\lambda,\mu}$ are non-negative integers.

Much of this work has been developed from a part of the first author's PhD thesis [5].

1 The skein models

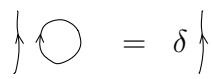
The account here largely follows those of [16], [10] and [14].

A *tangle* in a planar surface F, with some designated input and output boundary points, consists of oriented arcs in F joining input points to output points and possibly some further oriented closed curves, up to Reidemeister moves II and III. The term *diagram* is often used for a tangle when there are no input or output points. The *framed Homfly skein* $\mathcal{S}(F)$ of F is defined to be Λ-linear combinations of oriented tangles in F, modulo the two local relations



The *coefficient ring* can be taken as $\Lambda = \mathbf{Z}[v^{\pm 1}, s^{\pm 1}]$ with the elements $\{k\} = s^k - s^{-k}$ admitted as denominators for $k \geq 1$.

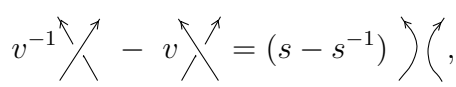
The local relation



is a consequence of the main relations. It allows the removal of a null-homotopic closed curve without crossings, at the expense of multiplication by the scalar $\delta = \frac{v^{-1}-v}{s-s^{-1}}$, except in the case where removal of the curve leaves the empty diagram. The result can be extended to this case too, without inconsistency, provided that the empty diagram is admitted when F has no designated boundary points.

1.1 The plane

When $F = \mathbf{R}^2$ every element can be represented uniquely as a scalar multiple of the empty diagram. For a diagram D the resulting scalar $< D > \in \Lambda$ is the *framed Homfly polynomial* of D. The more traditional Homfly polynomial P is defined as the *ambient* isotopy invariant which satisfies the local relation



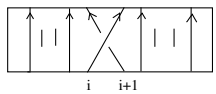
and which takes the value 1 on the unknot. The framed Homfly polynomial of D satisfies

$$< D >= v^{-wr(D)} \delta P(D),$$

where $wr(D)$ is the writhe of the diagram D.

1.2 The Hecke algebras and extended variants

Write H_n for the skein $\mathcal{S}(F)$ of n-tangles, where F is a rectangle with n inputs at the bottom and n outputs at the top. Composing n-tangles by placing one above another induces a product which makes H_n into an algebra. It has a linear basis of $n!$ elements, and is isomorphic to the Hecke algebra $H_n(z)$, with coefficients extended to the ring Λ. This algebra has a presentation generated by the elementary braids

$\sigma_i \;=$ 

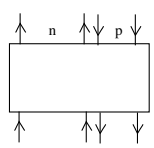
subject to the braid relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i, \quad |i-j|>1, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \end{aligned}$$

and the quadratic relations $\sigma_i^2 = z\sigma_i + 1$, with $z = s - s^{-1}$, giving the alternative form $(\sigma_i - s)(\sigma_i + s^{-1}) = 0$.

A simple adjustment of the skein relations, as in [16], allows for a skein model H_n whose parameters can be readily adapted to match any of the different appearances of the algebra, [14].

Definition. Write $H_{n,p}$ for the skein $\mathcal{S}(F)$ of (n,p)-tangles, where F is the rectangle with n outputs and p inputs at the top, and matching inputs and outputs at the bottom.

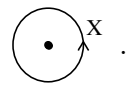
$F \;=$ 

There is again a natural algebra structure on $H_{n,p}$ induced by placing tangles one above the other. When $p = 0$ we have the Hecke algebra $H_n = H_{n,0}$. The resulting algebra $H_{n,p}$ has been studied by Kosuda and Murakami, [9], in the context of $sl(N)_q$ endomorphisms of the module $V^{\otimes n} \otimes \overline{V}^{\otimes p}$, where V is the fundamental N-dimensional module. Hadji gives an explicit skein-theoretic basis for it in [4], although in his account the string orientations are the reverse of those used here.

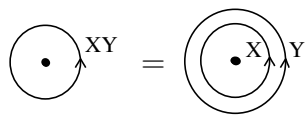
1.3 The annulus

The Homfly skein of the annulus, $\mathcal{C}$, as discussed in [13] and originally in [22], is the framed Homfly skein $\mathcal{S}(S^1 \times I)$. An element $X \in \mathcal{C}$ will be indicated

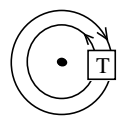
schematically as



The skein $\mathcal{C}$ has a product induced by placing one annulus outside another, under which $\mathcal{C}$ becomes a commutative algebra;

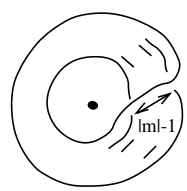


The closure map $H_{n,p} \to \mathcal{C}$, induced by taking an (n,p)-tangle T to its closure $\hat{T}$ in the annulus, is defined by

$\hat{T} \;=$ 

This is a Λ-linear map, whose image we call $\mathcal{C}_{n,p}$. Every diagram in the annulus represents an element in some $\mathcal{C}_{n,p}$.

Turaev [22] showed that $\mathcal{C}$ is freely generated as an algebra by the set $\{A_m : m \in \mathbf{Z}\}$ where $A_m, m \neq 0$, is represented by the closure of the braid $\sigma_{|m|-1} \cdots \sigma_2 \sigma_1$,

$A_m \;=$ 

The orientation of the curve around the annulus is counter-clockwise for positive m and clockwise for negative m. The element A_0 is the identity element and is represented by the empty diagram.

The algebra $\mathcal{C}$ is the product of two subalgebras $\mathcal{C}^+$ and $\mathcal{C}^-$ generated by $\{A_m : m \in \mathbf{Z}, m \geq 0\}$ and $\{A_m : m \in \mathbf{Z}, m \leq 0\}$ respectively.

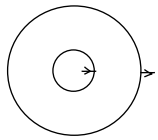
The algebra $\mathcal{C}^+$ is spanned by the subspaces $\mathcal{C}_{n,0}$ which arise from the closure of elements in H_n, and has been studied quite extensively [16, 10, 14, 6]. In particular, there is a good basis of $\mathcal{C}^+$ consisting of closures of certain idempotents of H_n.

The basis elements $\{Q_{\lambda,\mu}\}$ for the whole of $\mathcal{C}$, which we construct here, are identified in section 3 as eigenvectors of the meridian maps defined in section 1.7, following the Schur function methods of Lukac [10] for $\mathcal{C}^+$. This fact suggests that it should be useful and relatively easy to express other elements of

$\mathcal{C}$ in this basis. Lukac also shows that the Gyoja-Aiston idempotents for H_n, [16], close to the basis elements $Q_{\lambda,\phi}$ which span $\mathcal{C}^+$. Our results here complement the work of Kosuda and Murakami [9] on skein-based idempotents in $H_{n,p}$ and their closures in $\mathcal{C}$ along the same lines.

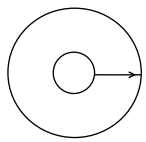
1.4 The annulus with two boundary points

Denote by $\mathcal{A}$ the Homfly skein of the annulus with one input and one output boundary point,



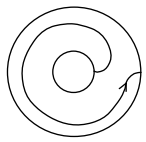
one on each boundary component, as indicated.

In a similar way to $\mathcal{C}$, the skein $\mathcal{A}$ becomes an algebra under the product induced by placing one annulus outside another. The identity element $e \in \mathcal{A}$ is represented by the diagram

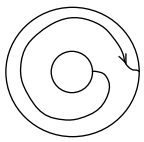
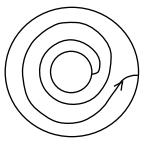
$e \; =$ 

obtained by joining the two boundary points by a single straight arc.

A further element a of $\mathcal{A}$ is represented by

$a \; =$  .

Powers of this element, a^m for $m \in \mathbf{Z}$, are also represented by diagrams with no crossings. For example,

$a^{-1} \; =$  , $a^2 \; =$  .

As an algebra, $\mathcal{A}$ is commutative. A straightforward skein theoretic proof is given in [14], although unlike the case of $\mathcal{C}$ the result is not immediately obvious.

Remark. A skein which is linearly isomorphic to $\mathcal{A}$ is used by Kawagoe [6] and other authors. Their version is based on the annulus with input and

output points on the same component. More recently $\mathcal{A}$ has come into use, as its algebraic properties allow for some satisfyingly clean proofs. See also [14], and work by Lukac [10].

1.5 Skein involutions

We make use of two readily defined involutions on the skeins $\mathcal{C}$ and $\mathcal{A}$ in our subsequent calculations.

- For every surface F we can define the *mirror map*, $\overline{} : \mathcal{S}(F) \to \mathcal{S}(F)$ as follows. For a tangle T in F define $\bar{T}$ to be T with all its crossings switched. In the coefficient ring Λ define conjugation by $\bar{s} = s^{-1}$ and $\bar{v} = v^{-1}$. The operation of switching crossings in tangles and conjugating coefficients respects the skein relations and so induces a conjugate-linear automorphism $\overline{} : \mathcal{S}(F) \to \mathcal{S}(F)$, which we call the mirror map on $\mathcal{S}(F)$.

- Rotation of diagrams in the annulus $S^1 \times I$ by π about the horizontal axis through the distinguished boundary points induces a linear automorphism of each of the two skeins $\mathcal{C}$ and $\mathcal{A}$ of the annulus which we denote by $*$ in each case.

Thus for the element $a \in \mathcal{A}$ we have $a^* = a^{-1}$, while $\bar{a} = a$. We can see that $(A_m)^* = A_{-m}$, so that $(\mathcal{C}^+)^* = \mathcal{C}^-$. It is also readily verified that $(\mathcal{C}_{n,p})^* = \mathcal{C}_{p,n}$, and that $\mathcal{C}_{n,p}$ is invariant under the mirror map, since $H_{n,p}$ is.

1.6 Some basic elements in the annulus

In H_n there is a known set of idempotent elements, E_λ, one for each partition λ of n. These elements were originally given a purely algebraic description by Gyoja [3], and were subsequently given a skein picture based on the Young diagram for λ by the second author and Aiston [16]. The closure of E_λ gives the element $Q_\lambda \in \mathcal{C}^+$, making up the basis elements used in [16], and indirectly those used in [10]. Our construction makes use only of the sequence of elements $\{h_n\} \in \mathcal{C}^+$ which arise from the single row Young diagrams, and the corresponding elements $\{h_n^*\} \in \mathcal{C}^-$.

The two simplest idempotents in H_n correspond to the single row and single column Young diagrams, as described in [13]. In the interpretation of the subalgebra $\mathcal{C}^+$ of $\mathcal{C}$ as the algebra of symmetric functions in infinitely many variables the closures of these idempotents correspond to the complete

symmetric function and the elementary symmetric function of degree n respectively. We give brief details of the definition and properties of these idempotents and their closure.

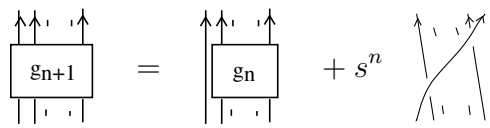
Let w_π be the positive permutation braid ([2]) corresponding to $\pi \in S_n$. Define two quasi-idempotents $a_n, b_n \in H_n$ by

$$a_n = \sum_{\pi \in S_n} s^{l(\pi)} w_\pi \qquad \text{and} \qquad b_n = \sum_{\pi \in S_n} (-s)^{-l(\pi)} w_\pi,$$

where $l(\pi) = \mathrm{wr}(w_\pi)$, the writhe of the braid w_π.

Write $g_n = 1 + s\sigma_{n-1} + s^2\sigma_{n-1}\sigma_{n-2} + \ldots + s^{n-1}\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1$, where $\{\sigma_i\}$ are the usual elementary braid generators of the braid group B_n.

We have $g_{n+1} = 1 + s\sigma_n g_n$, and also the immediate skein relation



(1)

for tangles on $n+1$ strings.

There is an algebra homomorphism $\varphi_s : H_n \to \Lambda$ defined on the elementary braids σ_i by $\varphi_s(\sigma_i) = s$.

The following two lemmas are readily established, as in [13]:

Lemma 1.1 $a_n = a_{n-1}g_n$.

Lemma 1.2 *For any $g \in H_n$ we have $a_n g = \varphi_s(g)a_n = g a_n$.*

The element a_n then satisfies

$$a_n^2 = \varphi_s(a_n)a_n = \varphi_s(a_{n-1})\varphi_s(g_n)a_n.$$

Now since $\varphi_s(g_n) = 1 + s^2 + \ldots + s^{2n-2} = s^{n-1}[n]$ with $[k] = \frac{s^k - s^{-k}}{s - s^{-1}}$, we have the immediate corollary:

Corollary 1.3 *We can write*

$$s^{n-1}[n]h_n = h_{n-1}g_n,$$

where $h_n = a_n/\varphi_s(a_n)$ is the true idempotent.

The element h_n constructed above is the idempotent which corresponds to the single row Young diagram with n cells. The single column idempotent, denoted e_n, is constructed in an analogous way from b_n. It can be obtained from h_n by using $-s^{-1}$ in place of s.

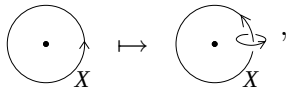
With a slight abuse of the notation we shall write $h_n, e_n \in \mathcal{C}$ for the closures $\hat{h}_n$, $\hat{e}_n$ of these idempotents as elements of $\mathcal{C}$.

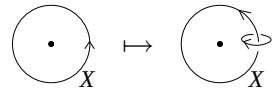
The skein $\mathcal{C}^+$ is spanned by the monomials in $\{h_m : m \geq 0\}$. The skein $\mathcal{C}^-$ is thus spanned by monomials in $\{h_l^* : l \geq 0\}$. The whole skein $\mathcal{C}$ is then spanned by monomials in $\{h_l^*, h_m : l, m \geq 0\}$.

1.7 The meridian maps of the skein $\mathcal{C}$

We introduce here the linear endomorphisms $\varphi, \bar{\varphi}$ of $\mathcal{C}$ which are of central importance in this work.

Take a diagram X in the annulus and link it once with a simple meridian loop, oriented in either direction, to give diagrams $\varphi(X)$ and $\bar{\varphi}(X)$ in the annulus as indicated. This induces linear endomorphisms $\varphi, \bar{\varphi}$ of the skein $\mathcal{C}$, called the *meridian maps*.

$\varphi : \mathcal{C} \to \mathcal{C}$, $\mapsto$ 

, and $\bar{\varphi} : \mathcal{C} \to \mathcal{C}$, $\mapsto$ 

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Each subspace $\mathcal{C}_{n,p}$ is invariant under φ and $\bar{\varphi}$. In [10] the eigenvectors of φ on $\mathcal{C}_{n,0}$ are identified with the closures Q_λ of the idempotents in H_n, for partitions λ of n, and are also expressed as explicit integer polynomials in $\{h_r\}$, as in [6]. Lukac makes use of the skein $\mathcal{A}$ in doing this, and our methods here are an extension of his work.

In [18] we calculated the complete set of eigenvalues $\{t_{\lambda,\mu}\}$ of φ, where (λ, μ) run over all pairs of partitions of integers, and showed that they are all distinct. Our goal in this paper is an explicit expression, as an integer polynomial in $\{h_n\}$ and $\{h_n^*\}$, for an eigenvector $Q_{\lambda,\mu}$ corresponding to each eigenvalue $t_{\mu,\lambda}$. The general result is given in section 3.1, along with an explicit example where λ and μ are the partitions with parts $4, 2, 2$ and $3, 2$ respectively.

Under the mirror map and the involution $*$ it is clear that

$$\overline{\varphi(X)} = \bar{\varphi}(\bar{X}) \tag{2}$$

$$(\varphi(X))^* = \bar{\varphi}(X^*). \tag{3}$$

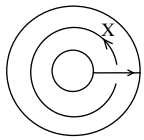
It can be established that h_n is fixed by the mirror map, and is an eigenvector of φ, see for example [10]. Use of the mirror map and $*$ then show at once that h_n and h_n^* are eigenvectors both of φ and of $\bar{\varphi}$. The eigenvalues are listed explicitly in the following lemma.

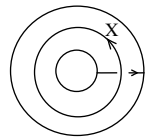
Lemma 1.4

$$\begin{aligned}
\varphi(h_n) &= (v^{-1}(s^{2n-1} - s^{-1}) + \delta)h_n, \\
\varphi(h_n^*) &= (v(s^{-2n+1} - s) + \delta)h_n^*, \\
\bar{\varphi}(h_n) &= (v(s^{-2n+1} - s) + \delta)h_n, \\
\bar{\varphi}(h_n^*) &= (v^{-1}(s^{2n-1} - s^{-1}) + \delta)h_n^*.
\end{aligned}$$

1.8 Relations between $\mathcal{C}$ and $\mathcal{A}$

Two algebra homomorphisms $l, r : \mathcal{C} \to \mathcal{A}$ can be induced by placing an element of $\mathcal{C}$ respectively under or over the simple arc e joining the distinguished boundary points of the annulus. Thus

$l(X) =$ 

and $r(X) =$ 

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The effects of the mirror map and the involution $*$ are readily seen to give

$$\overline{l(X)} = r(\overline{X}) \tag{4}$$

$$(l(X))^* = r(X^*). \tag{5}$$

Then $\overline{l(h_n)} = r(h_n)$, where $h_n \in \mathcal{C}$ is the closure of the idempotent in H_n.

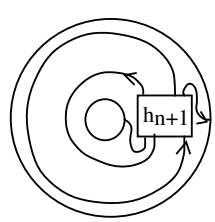
Define a closure map $\diamond : \mathcal{A} \to \mathcal{C}$ on diagrams by joining the distinguished boundary points with a simple arc lying above the annulus. This map interacts with l and r to give

$$\begin{aligned} \diamond(r(X)) &= \bar{\varphi}(X), \\ \diamond(l(X)) &= \delta X, \\ \diamond(Yl(X)) &= \diamond(Y)X, \end{aligned}$$

for any $X \in \mathcal{C}, Y \in \mathcal{A}$.

We shall construct eigenvectors of φ and $\bar{\varphi}$ as determinants of matrices with entries h_n, h_n^* in $\mathcal{C}$. When M is any such a matrix, with $X = \det M$, we have $\bar{\varphi}(X) = \diamond(r(\det M)) = \diamond(\det r(M))$, where $r(M)$ is the matrix with entries in $\mathcal{A}$ given by applying the algebra homomorphism r to all entries in M. Our main calculation in the next section involves determinants of matrices such as $r(M)$ with entries in $\mathcal{A}$. In advance of this we introduce some relations in $\mathcal{A}$ which allow us to perform certain column operations on the matrices.

Define elements $y_n \in \mathcal{A}$, fixed under the mirror map, by

$y_n := [n+1] \times$ 

where h_{n+1} here is the idempotent in H_{n+1}.

Lemma 1.5 *We have the relation*

$$y_n = s^{-1}ay_{n-1} + l(h_n). \tag{6}$$

Proof: Apply Corollary 1.3 to write $[n+1]h_{n+1} = s^{-n}h_n g_{n+1}$. Use the skein relation (1) to write g_{n+1}, and hence y_n, as the sum of two parts. One of these gives $l(h_n)$ at once and the other gives $s^{-1}ay_{n-1}$ on moving the box g_n in (1) round the annulus to combine with h_n from above. □

Apply the mirror map to (6), noting that $\bar{s} = s^{-1}$, to get

$$y_n = say_{n-1} + r(h_n). \tag{7}$$

Since $a^* = a^{-1}$ we can apply $*$ to equations (6) and (7) to obtain

$$y_n^* = s^{-1}a^{-1}y_{n-1}^* + r(h_n^*) \tag{8}$$
$$y_n^* = sa^{-1}y_{n-1}^* + l(h_n^*). \tag{9}$$

We rewrite equations (6) and (9) as

$$y_n = s^{-1}ay_{n-1} + l_n, \tag{10}$$
$$y_{n-1}^* = s^{-1}ay_n^* + L_{n-1}, \tag{11}$$

with $l_n = l(h_n)$ and $L_{n-1} = -s^{-1}al(h_n^*)$, for ease of use in our later matrix work.

Similarly we rewrite (7) and (8) as

$$y_n = say_{n-1} + r_n \tag{12}$$
$$y_{n-1}^* = say_n^* + R_{n-1}. \tag{13}$$

with $r_n = r(h_n)$ and $R_{n-1} = -sar(h_n^*)$.

From equations (10) and (12) we get

$$(s - s^{-1})y_n = sl_n - s^{-1}r_n, \tag{14}$$

and from (11) and (13) we get

$$(s - s^{-1})y_{n-1}^* = sL_{n-1} - s^{-1}R_{n-1}. \tag{15}$$

2 Matrix-based calculations

In this section we work with square matrices whose entries, in $\mathcal{C}$ or $\mathcal{A}$, are arranged in a fairly restricted format. Because of the restrictions we can use a simplified notation for them, which we now describe.

2.1 Notation

The entries in each matrix are drawn from sequences such as $h = \{h_n\}, y^* = \{y_n^*\}, R = \{R_n\}$ with integer subscripts. On any given row the subscripts either increase in steps of 1 along the row, (a *standard* row), or decrease in steps of 1, (a *starred* row). The subscript for every entry of the matrix is then determined by knowing which rows are starred, and the subscripts to be used in the first column.

We shall always use matrices in which the starred rows are the first k^* rows, followed by k standard rows. The column vector $\mathbf{v}$ of subscripts to be used for the first column will be called the *index vector* for the matrix.

The matrices used have the further restriction that in any given column the entries in the starred rows are drawn from some fixed sequence, h, R, for example, while all entries in the standard rows of a given column again come from some fixed sequence.

By way of example the 8×8 matrix M below has entries in such a format.

$$M = \begin{pmatrix} a_3^* & a_2^* & a_1^* & b_0 & b_{-1} & c_{-2}^* & c_{-3}^* & c_{-4}^* \\ a_5^* & a_4^* & a_3^* & b_2 & b_1 & c_0^* & c_{-1}^* & c_{-2}^* \\ a_4^* & a_3^* & a_2^* & b_1 & b_0 & c_{-1}^* & c_{-2}^* & c_{-3}^* \\ a_{-1} & a_0 & a_1 & d_2 & c_3 & c_4 & c_5 & c_6 \\ a_1 & a_2 & a_3 & d_4 & c_5 & c_6 & c_7 & c_8 \\ a_2 & a_3 & a_4 & d_5 & c_6 & c_7 & c_8 & c_9 \\ a_1 & a_2 & a_3 & d_4 & c_5 & c_6 & c_7 & c_8 \\ a_0 & a_1 & a_2 & d_3 & c_4 & c_5 & c_6 & c_7 \end{pmatrix}.$$

In M the first 3 rows are starred and the index vector is

$$\mathbf{v} = \left(\begin{array}{c} 3 \\ 5 \\ 4 \\ \hline -1 \\ 1 \\ 2 \\ 1 \\ 0 \end{array}\right)$$

The complete matrix M can be recovered, given $\mathbf{v}$, once the sequences to be used in the starred and standard rows are known.

This information is given by the *template matrix*, in this case

$$\begin{pmatrix} a^* & a^* & a^* & b & b & c^* & c^* & c^* \\ a & a & a & d & c & c & c & c \end{pmatrix}.$$

In general the template matrix is the $2 \times (k^* + k)$ matrix giving the sequences from which the starred and standard entries in each column are drawn.

As a further notational simplification we use the symbol $\dashv$ in the template matrix to indicate that the same sequence is repeated, and the symbol $\cdots$ where the choice of sequence is unimportant. We label a column or columns in the template as necessary to show where a repetition ends.

With this notation the template matrix for M can be written

$$\begin{array}{c} \begin{array}{cccccc} & & \overset{4}{\downarrow} & & & \end{array} \\ \begin{pmatrix} a^* & \dashv & b & b & c^* & \dashv \\ a & \dashv & d & c & & \dashv \end{pmatrix} \end{array}$$

and the complete matrix M with index vector $\mathbf{v}$ can then denoted simply by

$$M = \begin{array}{c} \begin{array}{cccccc} & & \overset{4}{\downarrow} & & & \end{array} \\ \begin{pmatrix} a^* & \dashv & b & b & c^* & \dashv \\ a & \dashv & d & c & & \dashv \end{pmatrix}_{\mathbf{v}} \end{array}.$$

For most of our calculations the index vector $\mathbf{v}$ will be fixed, but the choice of $\mathbf{v}$ will be unimportant, and we frequently suppress it, giving simply the template.

2.2 Calculations

Throughout this section we use matrices whose entries are drawn from the sequences of elements of $\mathcal{A}$ or $\mathcal{C}$ described in section 1.8. The sequences will be denoted by the letters used there, $h, h^*, y, y^*, l, r, L, R$. The definitions can in all cases be extended to allow for negative index by setting $h_j = h_j^* = y_j = y_j^* = l_j = r_j = L_j = R_j = 0$ when $j < 0$. The equations at the end of section 1.8 relating the sequences continue to hold in the extended range.

Lemma 2.1 *For each $i \geq 1$ and fixed index vector we have the following equations in $\mathcal{A}$.*

$$\det\begin{pmatrix} \cdots & \overset{i}{\downarrow} & & \\ \cdots & y^* & R & \cdots \\ \cdots & y & r & \cdots \end{pmatrix} = \det\begin{pmatrix} \cdots & \overset{i}{\downarrow} & & \\ \cdots & y^* & y^* & \cdots \\ \cdots & y & y & \cdots \end{pmatrix} \tag{16}$$

$$\det\begin{pmatrix} \cdots & \overset{i}{\downarrow} & & \\ \cdots & y^* & L & \cdots \\ \cdots & y & l & \cdots \end{pmatrix} = \det\begin{pmatrix} \cdots & \overset{i}{\downarrow} & & \\ \cdots & y^* & y^* & \cdots \\ \cdots & y & y & \cdots \end{pmatrix} \tag{17}$$

Proof: The column operation $C_{i+1} \mapsto saC_i + C_{i+1}$ applied to the matrix on the lefthand side of equation (16) gives the matrix on the righthand side, using (13) on starred rows and (12) on standard rows.

The column operation $C_{i+1} \mapsto s^{-1}aC_i + C_{i+1}$ works similarly for equation (17) using (11) on starred rows and (10) on standard rows. $\square$

Corollary 2.2 *For each $j \geq 1$ we have*

$$\det\begin{pmatrix} & \overset{j}{\downarrow} & & \\ \cdots & y^* & R & \dashv \\ \cdots & y & r & \dashv \end{pmatrix} = \det\begin{pmatrix} & \overset{j}{\downarrow} & \\ \cdots & y^* & \dashv \\ \cdots & y & \dashv \end{pmatrix} = \det\begin{pmatrix} & \overset{j}{\downarrow} & & \\ \cdots & y^* & L & \dashv \\ \cdots & y & l & \dashv \end{pmatrix}$$

Proof: Apply equation (16) repeatedly, for i from j onwards, to get the first equation, and similarly use (17) repeatedly to get the second equation. $\square$

Lemma 2.3 *For each $j \geq 1$ we have*

$$\begin{aligned}(s - s^{-1}) \det\begin{pmatrix} & \overset{j}{\downarrow} & \\ \cdots & y^* & \cdots \\ \cdots & y & \cdots \end{pmatrix} &= s \det\begin{pmatrix} & \overset{j}{\downarrow} & \\ \cdots & L & \cdots \\ \cdots & l & \cdots \end{pmatrix} \\ &\quad - s^{-1} \det\begin{pmatrix} & \overset{j}{\downarrow} & \\ \cdots & R & \cdots \\ \cdots & r & \cdots \end{pmatrix}.\end{aligned}$$

Proof: Expand the determinants by the jth column, using equations (15) on the starred rows and (14) on the standard rows. $\square$

Notation. Write

$$\begin{aligned}\Delta_j &= \det\begin{pmatrix} & & \overset{j+1}{\downarrow} & \\ L & \dashv & R & \dashv \\ l & \dashv & r & \dashv \end{pmatrix}, \ 0 \leq j < k^* + k, \\ \Delta_{k^*+k} &= \det\begin{pmatrix} L & \dashv \\ l & \dashv \end{pmatrix},\end{aligned}$$

for any choice of index vector.

Lemma 2.4 *We have*

$$(s-s^{-1})\det\begin{pmatrix} L & \dashv & \overset{j}{\underset{\downarrow}{y^*}} & L & \dashv \\ l & \dashv & y & l & \dashv \end{pmatrix} = s\Delta_j - s^{-1}\Delta_{j-1},\ 1 \le j \le k^* + k.$$

Proof:

$$\begin{aligned}(s-s^{-1})\det\begin{pmatrix} L & \dashv & \overset{j}{\underset{\downarrow}{y^*}} & L & \dashv \\ l & \dashv & y & l & \dashv \end{pmatrix} &= (s-s^{-1})\det\begin{pmatrix} L & \dashv & \overset{j}{\underset{\downarrow}{y^*}} & R & \dashv \\ l & \dashv & y & r & \dashv \end{pmatrix}, \\ &\quad \text{by corollary 2.2} \\ &= s\Delta_j - s^{-1}\Delta_{j-1},\ \text{by lemma 2.3}\end{aligned}$$

□

Expanding further by lemma 2.3 gives

$$s\Delta_j - s^{-1}\Delta_{j-1} = s\det\begin{pmatrix} L & \dashv \\ l & \dashv \end{pmatrix} - s^{-1}\det\begin{pmatrix} L & \dashv & \overset{j}{\underset{\downarrow}{R}} & L & \dashv \\ l & \dashv & r & l & \dashv \end{pmatrix}. \tag{18}$$

Corollary 2.5 *We have*

$$\begin{aligned}&\sum_{j=1}^{k^*+k-1} s^{2j}\det\begin{pmatrix} L & \dashv \\ l & \dashv \end{pmatrix} + \det\begin{pmatrix} R & \dashv \\ r & \dashv \end{pmatrix} \\ &= \sum_{j=1}^{k+k^*} s^{2j-2}\det\begin{pmatrix} L & \dashv & \overset{j}{\underset{\downarrow}{R}} & L & \dashv \\ l & \dashv & r & l & \dashv \end{pmatrix}.\end{aligned}$$

Proof: Multiply both sides of (18) by s^{2j-1} and sum to get

$$\begin{aligned}s^{2(k^*+k)}\Delta_{k^*+k} - \Delta_0 &= \sum_{j=1}^{k+k^*} s^{2j}\det\begin{pmatrix} L & \dashv \\ l & \dashv \end{pmatrix} \\ &\quad - \sum_{j=1}^{k+k^*} s^{2j-2}\det\begin{pmatrix} L & \dashv & \overset{j}{\underset{\downarrow}{R}} & L & \dashv \\ l & \dashv & r & l & \dashv \end{pmatrix},\end{aligned}$$

and then rearrange the terms. □

Lemma 2.6 *For any index vector* $\mathbf{w}$ *we have*

$$s^{-k^*}\left(\sum_{j=1}^{k^*+k-1} s^{2j}\right)\det\begin{pmatrix} l(h^*) & \dashv \\ l(h) & \dashv \end{pmatrix}_{\mathbf{w}} + s^{k^*}\det\begin{pmatrix} r(h^*) & \dashv \\ r(h) & \dashv \end{pmatrix}_{\mathbf{w}}$$

$$= s^{-k^*}\sum_{j=1}^{k+k^*}\det\begin{pmatrix} l(h^*) & \dashv & \overset{j}{\downarrow} & & \\ & & s^{2j}r(h^*) & l(h^*) & \dashv \\ l(h) & \dashv & s^{2j-2}r(h) & l(h) & \dashv \end{pmatrix}_{\mathbf{w}}.$$

Proof: Recall that $L_{n-1} = (-a)s^{-1}l(h_n^*)$ and $R_{n-1} = (-a)sr(h_n^*)$. Use corollary 2.5 with index vector $\mathbf{v}$ given by reducing all starred indices of $\mathbf{w}$ by 1, and leaving the standard indices unaltered. Then take out a factor of $-a$ from each starred row to get

$$(-a)^{k^*}\left(\sum_{j=1}^{k^*+k-1} s^{2j}\right)\det\begin{pmatrix} s^{-1}l(h^*) & \dashv \\ l(h) & \dashv \end{pmatrix}_{\mathbf{w}} + (-a)^{k^*}\det\begin{pmatrix} sr(h^*) & \dashv \\ r(h) & \dashv \end{pmatrix}_{\mathbf{w}}$$

$$= (-a)^{k^*}\sum_{j=1}^{k+k^*} s^{2j-2}\det\begin{pmatrix} s^{-1}l(h^*) & \dashv & \overset{j}{\downarrow} & & \\ & & sr(h^*) & s^{-1}l(h^*) & \dashv \\ l(h) & \dashv & r(h) & l(h) & \dashv \end{pmatrix}_{\mathbf{w}}.$$

The factor $(-a)^{k^*}$ can be cancelled. Extract k^* further factors of s or s^{-1} from the starred rows, and insert the factor s^{2j-2} into the jth column of the appropriate matrix to complete the proof. □

3 Eigenvectors of the meridian maps

We now come to the main result, giving potential eigenvectors $A_{\mathbf{w}}$ in $\mathcal{C}$ for the meridian maps. We show later that sufficiently many of these vectors are non-zero, to provide our claimed basis $Q_{\lambda,\mu}$ by suitable choices of index vector $\mathbf{w}$.

Theorem 3.1 *Let* $\mathbf{w}$ *be any index vector. Then*

$$A_{\mathbf{w}} = \det\begin{pmatrix} h^* & \dashv \\ h & \dashv \end{pmatrix}_{\mathbf{w}} \in \mathcal{C}$$

satisfies $\bar{\varphi}(A_{\mathbf{w}}) = c_{\mathbf{w}}A_{\mathbf{w}}$*, and hence also* $\varphi(A_{\mathbf{w}}) = \overline{c_{\mathbf{w}}}A_{\mathbf{w}}$*, for some* $c_{\mathbf{w}} \in \Lambda$*.*

Proof: Write $M_{\mathbf{w}} = \begin{pmatrix} h^* & \dashv \\ h & \dashv \end{pmatrix}_{\mathbf{w}}$, with entries $m_{i\,j}$ and cofactors $M_{i\,j} \in \mathcal{C}$. Write $c_{i\,j}$ for the entries in the distinguished jth column of

$$\begin{array}{c} \qquad\qquad\quad j \\ \qquad\qquad\quad \downarrow \\ \begin{pmatrix} l(h^*) & \dashv & s^{2j}r(h^*) & l(h^*) & \dashv \\ l(h) & \dashv & s^{2j-2}r(h) & l(h) & \dashv \end{pmatrix}_{\mathbf{w}} \end{array}.$$

The determinant of this matrix, expanded by the jth column is

$$\sum_{i=1}^{k^*+k} c_{i\,j} l(M_{i\,j}).$$

Consequently lemma 2.6 gives

$$s^{-2k^*}\left(\sum_{j=1}^{k^*+k-1} s^{2j}\right)\det l(M_{\mathbf{w}}) + \det r(M_{\mathbf{w}}) = s^{-2k^*}\sum_{i,j=1}^{k^*+k} c_{i\,j} l(M_{i\,j}). \tag{19}$$

Now apply the closure map $\diamond$ to the elements of $\mathcal{A}$ on each side of equation (19). Recall that

$$\begin{aligned} \diamond(l(X)) &= \delta X, \\ \diamond(r(X)) &= \bar{\varphi}(X), \\ \diamond(Yl(X)) &= \diamond(Y)X, \text{ for any } X \in \mathcal{C}, Y \in \mathcal{A}, \end{aligned}$$

and that r and l are algebra homomorphisms, so that $\det l(M_{\mathbf{w}}) = l(A_{\mathbf{w}})$ and $\det r(M_{\mathbf{w}}) = r(A_{\mathbf{w}})$. We then have

$$s^{-2k^*}\left(\sum_{j=1}^{k^*+k-1} s^{2j}\right)\delta A_{\mathbf{w}} + \bar{\varphi}A_{\mathbf{w}} = s^{-2k^*}\sum_{i,j=1}^{k^*+k} \diamond(c_{i\,j})M_{i\,j}. \tag{20}$$

We now show that $\diamond(c_{i\,j})$ is a multiple of $m_{i\,j}$, and that the righthand side of equation (20) can be written as a multiple of $A_{\mathbf{w}}$, thus proving the theorem.

We deduce this from the following lemma.

Lemma 3.2 *There exist $\alpha_i, \beta_j \in \mathcal{A}, 1 \le i, j \le k^* + k$, such that*

$$\diamond(c_{i\,j}) = (\alpha_i + \beta_j)m_{i\,j}.$$

Corollary 3.3

$$\begin{aligned}
\sum_{i,j=1}^{k^*+k} \diamond(c_{i\,j})M_{i\,j} &= \sum_{i,j=1}^{k^*+k} (\alpha_i+\beta_j)m_{i\,j}M_{i\,j} \\
&= \sum_{i,j=1}^{k^*+k} \alpha_i m_{i\,j}M_{i\,j} + \sum_{i,j=1}^{k^*+k} \beta_j m_{i\,j}M_{i\,j} \\
&= \sum_{i=1}^{k^*+k} \alpha_i \left(\sum_{j=1}^{k^*+k} m_{i\,j}M_{i\,j}\right) + \sum_{j=1}^{k^*+k} \beta_j \left(\sum_{i=1}^{k^*+k} m_{i\,j}M_{i\,j}\right) \\
&= \sum_{i=1}^{k^*+k} \alpha_i A_{\mathbf{w}} + \sum_{j=1}^{k^*+k} \beta_j A_{\mathbf{w}},
\end{aligned}$$

using the expansion of $A_{\mathbf{w}}$ by the ith row in the first sum and by the jth column in the second.

Proof of lemma 3.2: Since

$$c_{i\,j} = \begin{cases} s^{2j}r(m_{i\,j}) & \text{if } i \le k^*, \\ s^{2j-2}r(m_{i\,j}) & \text{if } i > k^*, \end{cases}$$

we have

$$\diamond(c_{i\,j}) = \begin{cases} s^{2j}\bar{\varphi}(m_{i\,j}) & \text{if } i \le k^*, \\ s^{2j-2}\bar{\varphi}(m_{i\,j}) & \text{if } i > k^*. \end{cases}$$

In every case $m_{i\,j}$ is either h_n^* or h_n for some n depending on the index vector $\mathbf{w}$, and is hence an eigenvector of $\bar{\varphi}$.

For $i \le k^*$ we have $m_{i\,j} = h_n^*$ with $n = w_i - j + 1$ and $\bar{\varphi}(m_{i\,j}) = (v^{-1}s^{2n-1} + \delta - v^{-1}s^{-1})m_{i\,j}$, while for $i > k^*$ we have $m_{i\,j} = h_n$ with $n = w_i + j - 1$ and $\bar{\varphi}(m_{i\,j}) = (v^{-1}s^{-2n+1} + \delta - vs)m_{i\,j}$. In each case $\diamond(c_{i\,j}) = (\alpha_i + \beta_j)m_{i\,j}$, where

$$\begin{aligned}
\alpha_i &= \begin{cases} v^{-1}s^{2w_i+1} & \text{if } i \le k^*, \\ vs^{1-2w_i} & \text{if } i > k^* \end{cases} \\
\beta_j &= s^{2j}(\delta - v^{-1}s^{-1}) = s^{2j-2}(\delta - vs) \\
&= \frac{v^{-1}s^{2j-2} - vs^{2j}}{s - s^{-1}}.
\end{aligned}$$

□

The theorem is now established, with

$$c_{\mathbf{w}} = s^{-2k^*}\left(\sum_{i=1}^{k^*+k} \alpha_i + \sum_{j=1}^{k^*+k} \beta_j - \delta \sum_{j=1}^{k^*+k-1} s^{2j}\right) \tag{21}$$

□

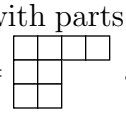
3.1 Construction of $Q_{\lambda,\mu}$

Given two partitions λ and μ with k and k^* parts respectively we define $Q_{\lambda,\mu} = A_{\mathbf{w}}$ for a suitably chosen index vector $\mathbf{w}$ with k standard and k^* starred rows. We choose $\mathbf{w}$ so that the subscripts of the *diagonal* entries in the standard rows are the parts $\lambda_1, \lambda_2, \ldots, \lambda_k$ of λ in order, while the subscripts of the *diagonal* entries in the starred rows are the parts $\mu_1, \mu_2, \ldots, \mu_{k^*}$ of μ in *reverse* order.

The entries in the index vector are then explicitly

$$w_i = \begin{cases} \mu_{k^*-i+1} + i - 1 & i \leq k^*, \\ \lambda_{i-k^*} - i + 1 & i > k^*. \end{cases}$$

Remark. Similar determinants are used by Koike [8] in giving universal formulae for the irreducible characters of rational representations of $GL(N)$, along with interpretations in terms of skew Schur functions.

Example. For the partitions λ and μ with parts $4, 2, 2$ and $3, 2$ respectively, represented by the Young diagrams $\lambda =$  and $\mu =$  this gives

$$Q_{\lambda,\mu} = \det \begin{pmatrix} \mathbf{h_2^*} & h_1^* & 1 & 0 & 0 \\ h_4^* & \mathbf{h_3^*} & h_2^* & h_1^* & 1 \\ h_2 & h_3 & \mathbf{h_4} & h_5 & h_6 \\ 0 & 1 & h_1 & \mathbf{h_2} & h_3 \\ 0 & 0 & 1 & h_1 & \mathbf{h_2} \end{pmatrix}$$

Remark. Unless the matrix $M_{\mathbf{w}}$ has two equal rows or an entirely zero row the elements $A_{\mathbf{w}} = \det M_{\mathbf{w}}$ of $\mathcal{C}$ are $\pm Q_{\lambda,\mu}$ for some partitions λ, μ.

If the index vector $\mathbf{w}$ has a repeated entry in the starred rows, or a repeated entry in the standard rows then $M_{\mathbf{w}}$ has a repeated row. When $w_i < 0$ for any $i \leq k^*$ or $w_i < k^* + k$ for any $i \geq k^*$ the matrix $M_{\mathbf{w}}$ will have a zero row. Otherwise, by permuting the rows we can assume that the starred entries in $\mathbf{w}$ increase with i and the standard entries decrease with i, and that $w_1 \geq 0$ and $w_{k^*+k} \geq -k^* - k$. If $w_1 = 0$ or $w_{k^*+k} = -k^* - k$ then we can find an index vector with one fewer starred or standard entries respectively which determines the same element $A_{\mathbf{w}}$, on expanding the determinant by either the first or the last row. We may then assume that the starred entries increase strictly, with $w_1 > 0$ and that the standard entries decrease strictly, with $w_{k^*+k} > -k^* - k$. The indices on the diagonal then determine the partitions λ and μ for which $A_{\mathbf{w}} = Q_{\lambda,\mu}$.

Definition. Given two partitions λ and μ, write

$$s_{\lambda,\mu} = (s - s^{-1}) \left(v^{-1} \sum_{x \in \lambda} s^{2c(x)} - v \sum_{x \in \mu} s^{-2c(x)} \right) + \delta,$$

where the sum is taken over cells x in the Young diagram of the partition, and $c(x) = j - i$ is the *content* of the cell x in row i and column j of the Young diagram.

Remark. In [18] the notation $t_{\lambda,\mu}$ is used, with $t_{\lambda,\mu} = s_{\mu,\lambda}$. It is shown there that the set $\{t_{\lambda,\mu}\}$ forms a complete set of eigenvalues of φ, each occurring with multiplicity 1.

Theorem 3.4 *The element $Q_{\lambda,\mu} \in \mathcal{C}$ defined above is an eigenvector of the meridian map φ, with eigenvalue $s_{\lambda,\mu}$.*

Proof: We know already from theorem 3.1 that either $Q_{\lambda,\mu} = 0$ or it is an eigenvector of φ. We first identify its eigenvalue as $s_{\lambda,\mu}$ by showing that $\overline{\varphi}(Q_{\lambda,\mu}) = s_{\mu,\lambda} Q_{\lambda,\mu}$. This is sufficient, since then $\varphi(Q_{\lambda,\mu}) = \overline{s_{\mu,\lambda}} Q_{\lambda,\mu} = s_{\lambda,\mu} Q_{\lambda,\mu}$. Finally we establish that $Q_{\lambda,\mu} \neq 0$ in $\mathcal{C}$.

From the explicit formula (21) in theorem 3.1 for $c_{\mathbf{w}}$ in terms of the index vector $\mathbf{w}$ which defines $Q_{\lambda,\mu}$ we have

$$\begin{aligned} c_{\mathbf{w}} &= s^{-2k^*} \left(\sum_{i=1}^{k^*+k} \alpha_i + \sum_{j=1}^{k^*+k} \beta_j - \delta \sum_{j=1}^{k^*+k-1} s^{2j} \right) \\ &= v^{-1} \sum_{i=1}^{k^*} s^{2(w_i - k^*)+1} + v \sum_{i=k^*+1}^{k^*+k} s^{-2(w_i+k^*)+1} \\ &\quad + s^{-2k^*} \left(\sum_{j=1}^{k^*+k} \beta_j - \delta \sum_{j=1}^{k^*+k-1} s^{2j} \right). \end{aligned}$$

Now $\beta_j = \frac{v^{-1}}{s-s^{-1}} s^{2j-2} - \frac{v}{s-s^{-1}} s^{2j}$ and $\delta = \frac{v^{-1}}{s-s^{-1}} - \frac{v}{s-s^{-1}}$. Then

$$\sum_{j=1}^{k^*+k} \beta_j - \delta \sum_{j=1}^{k^*+k-1} s^{2j} = v^{-1} \frac{1}{s - s^{-1}} - v \frac{s^{2k^*+2k}}{s - s^{-1}}.$$

For our choice of $\mathbf{w}$ in terms of λ and μ we have

$$\sum_{i=1}^{k^*} s^{2(w_i - k^*)+1} = \sum_{i=1}^{k^*} s^{2(\mu_{k^*-i+1} - k^* + i - 1)+1} = \sum_{j=1}^{k^*} s^{2(\mu_j - j)+1},$$

and

$$\sum_{i=k^*+1}^{k^*+k} s^{-2(w_i+k^*)+1} = \sum_{i=k^*+1}^{k^*+k} s^{-2(\lambda_{i-k^*}+k^*-i+1)+1} = \sum_{j=1}^{k} s^{-2(\lambda_j-j)-1}.$$

Then

$$c_{\mathbf{w}} = v^{-1}\sum_{j=1}^{k^*} s^{2(\mu_j-j)+1} + v\sum_{j=1}^{k} s^{-2(\lambda_j-j)-1} + \frac{v^{-1}s^{-2k^*} - vs^{2k}}{s-s^{-1}}.$$

Rewrite $s_{\mu,\lambda}$ by summing over cells in the same row for each of λ and μ. Thus

$$\begin{aligned} s_{\mu,\lambda} &= v^{-1}(s-s^{-1})\sum_{x\in\mu} s^{2c(x)} - v(s-s^{-1})\sum_{x\in\lambda} s^{-2c(x)} + \delta \\ &= v^{-1}(s-s^{-1})\sum_{i=1}^{k^*}\sum_{j=1}^{\mu_i} s^{2(j-i)} - v(s-s^{-1})\sum_{i=1}^{k}\sum_{j=1}^{\lambda_i} s^{-2(j-i)} + \delta \\ &= v^{-1}\sum_{i=1}^{k^*}(s^{2(\mu_i-i)+1} - s^{-2i+1}) + v\sum_{i=1}^{k}(s^{-2(\lambda_i-i)-1} - s^{2i-1}) + \delta. \end{aligned}$$

Comparison of the terms in the formulae above then shows that $c_{\mathbf{w}} = s_{\mu,\lambda}$, noting that

$$\delta - v^{-1}\sum_{i=1}^{k^*} s^{-2i+1} - v\sum_{i=1}^{k} s^{2i-1} = v^{-1}\frac{s^{-2k^*}}{s-s^{-1}} - v\frac{s^{2k}}{s-s^{-1}}.$$

We now show that $Q_{\lambda,\mu} \neq 0 \in \mathcal{C}$ by proving that its framed Homfly evaluation $< Q_{\lambda,\mu} >$ in Λ is non-zero. This calculation uses an explicit formula from Macdonald [12] for $< Q_{\lambda,\mu} >$ in the case where the partition μ is empty.

There is an expression for $H(t) = \sum_{n=0}^{\infty} < h_n > t^n$ as an infinite product

$$H(t) = \prod_{i=0}^{\infty} \frac{1-bq^it}{1-aq^it}$$

with $a = vs, b = v^{-1}s$ and $q = s^2$, [1, 17]. For such an infinite product $H(t)$ Macdonald [12] gives the formula

$$s_\lambda = q^{n(\lambda)}\prod_{x\in\lambda}\frac{a-bq^{c(x)}}{1-q^{h(x)}} \tag{22}$$

where $s_\lambda =< Q_{\lambda,\phi} >$ is the Schur function for the partition λ given as a determinant by the Jacobi-Trudy formula from the coefficients $< h_n >$ of $H(t)$.

The product in (22) is taken over cells x of the partition λ with content $c(x)$ and hook length $h(x)$. Then $< Q_{\lambda,\phi} >\neq 0$ so long as $ab^{-1} \neq q^{c(x)}$ for any cell $x \in \lambda$. After the substitution $v = \pm s^N$ this expression is non-zero in the ring Λ if there are no cells $x \in \lambda$ with $c(x) = N$. This is the case so long as $N \geq \lambda_1$.

Given partitions λ, μ we now choose N and a partition ν with $\nu_1 \leq N$ such that $< Q_{\lambda,\mu} >=< Q_{\nu,\phi} >$ after substituting $v = -s^N$. It follows that the resulting value of $< Q_{\lambda,\mu} >$ is non-zero, and hence that $Q_{\lambda,\mu} \neq 0 \in \mathcal{C}$.

It is enough to choose $N \geq \lambda_1 + \mu_1$ and take

$$\nu_i = \begin{cases} N - \mu_{k^*-i+1}, & i \leq k^* \\ \lambda_{i-k^*}, & k^* < i \leq k^* + k. \end{cases}$$

In [17] there is an explicit formula

$$< h_n >= (-1)^n \prod_{i=1}^{n} \frac{vs^{-i+1} - v^{-1}s^{i-1}}{s^i - s^{-i}}.$$

Then $< h_n >=< h_{N-n} >, 0 \leq n \leq N$, when $v = -s^N$, since

$$\begin{aligned} \prod_{i=1}^{N}(s^i - s^{-i}) &= \prod_{i=1}^{n}(s^i - s^{-i}) \prod_{j=1}^{N-n}(s^j - s^{-j}) < h_n > \\ &= \prod_{i=1}^{n}(s^i - s^{-i}) \prod_{j=1}^{N-n}(s^j - s^{-j}) < h_{N-n} > \end{aligned}$$

for this value of v. The result holds also for $n < 0$ and $n > N$.

When we replace every entry h_n^* in the starred rows of a matrix $M_{\mathbf{w}} = \begin{pmatrix} h^* & \dashv \\ h & \dashv \end{pmatrix}_{\mathbf{w}}$ by h_{N-n} we get a matrix $M_{\mathbf{v}} = \begin{pmatrix} h & \dashv \end{pmatrix}_{\mathbf{v}}$ with standard rows only and index vector given by

$$v_i = \begin{cases} N - w_i, & i \leq k^*, \\ w_i, & i > k^*. \end{cases} \tag{23}$$

The matrices $< M_{\mathbf{w}} >$ and $< M_{\mathbf{v}} >$ given by evaluating the framed Homfly polynomial of their entries are then identical after the substitution $v = -s^N$, since $< h_n^* >=< h_n >\in \Lambda$.

When $M_{\mathbf{w}}$ is given by partitions λ and μ, and $N \geq \lambda_1 + \mu_1$, the matrix $M_{\mathbf{v}}$ arises from the partition $\nu := \lambda \cup (N - \mu)$ above with $k^* + k$ parts $\nu_1 = N - \mu_{k^*} \geq N - \mu_{k^*-1} \geq \cdots \geq N - \mu_1 \geq \lambda_1 \geq \cdots \geq \lambda_k > 0$.

Now $< Q_{\lambda,\mu} >=< \det M_{\mathbf{w}} >= \det < M_{\mathbf{w}} >$ and $< Q_{\nu,\phi} >= \det < M_{\mathbf{v}} >$. Then $< Q_{\lambda,\mu} >=< Q_{\nu,\phi} >$ after substituting $v = -s^N$, and $< Q_{\nu,\phi} >\neq 0$ after this substitution, since $\nu_1 \leq N$. It follows that $Q_{\lambda,\mu} \neq 0$, and is hence an eigenvector for φ. $\square$

We thus have identified an eigenvector $Q_{\lambda,\mu}$ for each eigenvalue $s_{\lambda,\mu}$ of φ, giving the explicit basis of $\mathcal{C}$.

3.2 Properties of the basis elements $Q_{\lambda,\mu}$

When an element $X \in \mathcal{C}$ is used to decorate a framed knot K, the Homfly polynomial, $P(K; X) \in \Lambda$, of the decorated knot gives a 2-variable invariant of K determined by X. Each basis element $Q_{\lambda,\mu}$ gives such an invariant of K, and these determine the invariants for all choices of X. When $\mu = \phi$ these invariants, for each fixed λ, give a family of 1-variable invariants after setting $v = s^{-N}$ which coincide, up to a power of s, with the quantum group invariants of K determined by the irreducible $sl(N)_q$-module corresponding to the partition λ.

In general, the 2-variable invariant from $Q_{\lambda,\mu}$, when evaluated at $v = s^{-N}$, gives the $sl(N)_q$ quantum invariant for an irreducible module whose partition depends on λ, μ and N. The simplest example where both λ and μ occur non-trivially is $Q_{\square,\square}$, where $\square$ represents the unique partition of 1. The 2-variable invariant in this case, after setting $v = s^{-N}$, evaluates the quantum invariant when the knot is coloured by the adjoint representation of $sl(N)_q$ for each N. Since $Q_{\square,\square} = h_1 h_1^* - 1$ this invariant is very nearly the Homfly polynomial of the reverse parallel of K with one strand in each direction.

We have already noted that $\overline{Q_{\lambda,\mu}} = Q_{\lambda,\mu}$ and that $Q^*_{\lambda,\mu} = Q_{\mu,\lambda}$. Hence the invariant of K when decorated by $Q_{\lambda,\mu}$ is symmetric in λ and μ, and is conjugated in Λ when K is replaced by its mirror image.

In the skein $\mathcal{C}$ the replacement of s by $-s^{-1}$, while fixing v and leaving diagrams unaltered, has the effect of interchanging rows and columns in the Young diagram of each partition λ to give its conjugate partition $\lambda^\vee$. Then $Q_{\lambda,\mu}$ becomes $Q_{\lambda^\vee,\mu^\vee}$, and the invariant $P(K, Q_{\lambda^\vee,\mu^\vee})$ is given by changing s to $-s^{-1}$ in $P(K : Q_{\lambda,\mu})$. Where both partitions λ and μ are self-conjugate the invariant is symmetric in s and $-s^{-1}$, and can then be written in terms of v and $z = s - s^{-1}$, as is the case for the ordinary Homfly polynomial of K, which comes from the decoration of K by the single string $Q_{\square,\phi}$.

Theorem 3.5 *The basis elements $Q_{\lambda,\mu}$ of $\mathcal{C}$ have the property that the prod-*

uct of any two is an integer linear combination of basis elements with non-negative coefficients.

Proof: In the case of the subalgebra $\mathcal{C}^+$, the basis restricts to the elements $Q_{\lambda,\phi}$ which behave like the basis $\{s_\lambda\}$ of Schur functions for the algebra of symmetric functions in many variables. Products of these are non-negative integer combinations of Schur functions, whose coefficients can be determined combinatorially by the Littlewood-Richardson rules.

The algebra $\mathcal{C}$ is the polynomial algebra generated by $\{h_n\}, \{h_n^*\}, n \geq 1$, with $1 = h_0 = h_0^*$ and $0 = h_n = h_n^*, n < 0$. Define a homomorphism $\varphi_N : \mathcal{C} \to \mathcal{C}^+$ by

$$\varphi_N(h_n^*) = h_{N-n},\ 0 < n, \quad \varphi_N(h_n) = \begin{cases} h_n, & 0 < n < N, \\ 1, & n = N, \\ 0, & n > N. \end{cases}$$

The definition ensures that $\varphi_N(h_n^*) = \varphi_N(h_{N-n})$ for *all* integers n. Then $\varphi_N(Q_{\lambda,\mu}) = \varphi_N(Q_{\nu,\phi})$ when $N \geq \lambda_1 + \mu_1$ and $\nu = \lambda \cup (N - \mu)$ is the partition with parts $\{\lambda_i\}, \{N - \mu_j\}$. The map $\nu \to \varphi_N(Q_{\nu,\phi})$ is injective on partitions ν with $\nu_1 < N$, since the monomial $h_{\nu_1} h_{\nu_2} \ldots h_{\nu_k}$ can be recovered unambiguously from the polynomial $\varphi_N(Q_{\nu,\phi})$. Indeed the set $\{\varphi_N(Q_{\nu,\phi}) : \nu_1 < N\}$ forms a basis for $\varphi_N(\mathcal{C})$.

For any finite set of elements $Q_{\lambda,\mu}$ we can choose N sufficiently large so that φ_N is injective on that set. It is enough to take $N > 2\max(\lambda_1, \mu_1)$ for all λ, μ in the set, as then the pair λ, μ can be recovered from the partition $\lambda \cup (N - \mu)$ by considering the parts $< N/2$ and those $> N/2$.

Expand any product $Q_{\lambda,\mu} Q_{\lambda',\mu'}$ as a linear combination $\sum a_{\lambda'',\mu''} Q_{\lambda'',\mu''}$ of basis elements of $\mathcal{C}$, and choose N so that the supporting basis elements are mapped by φ_N to distinct basis elements of $\varphi_N(\mathcal{C})$. Write $\varphi_N(Q_{\lambda'',\mu''}) = \varphi_N(s_{\nu''})$ with $\nu_1'' < N$ for these basis elements, and also write $\varphi_N(Q_{\lambda,\mu}) = \varphi_N(s_\nu)$ and $\varphi_N(Q_{\lambda',\mu'}) = \varphi_N(s_{\nu'})$. Expand the product $s_\nu s_{\nu'} = \sum b_\rho s_\rho$ with non-negative integer coefficients b_ρ. Then

$$\sum a_{\lambda'',\mu''} \varphi_N(s_{\nu''}) = \sum b_\rho \varphi_N(s_\rho).$$

Now $\varphi_N(s_\rho)$ is a basis element of $\varphi_N(\mathcal{C})$ when $\rho_1 < N$. It is also a basis element if $\rho_1 = N$, and is zero if $\rho_1 > N$. A comparison of the two sides shows that each non-zero coefficient $a_{\lambda'',\mu''}$ is a sum of one or more positive integers b_ρ. □

Remark. This result has a close connection with the behaviour of irreducible mixed tensor representations of $GL(N)$, which are characterised by

pairs of partitions (λ, μ). Work of King [7], Stembridge [21] and Koike [8] could be used to give an explicit combinatorial calculation of the coefficients in a product $Q_{\lambda,\mu}Q_{\lambda',\mu'}$.

3.3 $(1,1)$-tangle invariants

For a framed knot K the invariants $P(K : Q_{\lambda,\mu})$ can be expressed in terms of the eigenvalues of a map $K : \mathcal{C} \to \mathcal{C}$, described as follows. Draw a diagram of K as a closed (1,1)-tangle in the annulus and decorate this diagram by any element $X \in \mathcal{C}$ to give the element $K(X) \in \mathcal{C}$. Thus $P(K : X)$ is the Homfly polynomial $< K(X) >$.

Now the map K commutes with the meridian map φ, and all eigenspaces of φ are 1-dimensional, so every eigenvector of φ is also an eigenvector of K. We can then write $K(Q_{\lambda,\mu}) = K_{\lambda,\mu}Q_{\lambda,\mu}$ for some scalar $K_{\lambda,\mu}$. The eigenvalues of K are

$$K_{\lambda,\mu} = \frac{P(K : Q_{\lambda,\mu})}{< Q_{\lambda,\mu} >},$$

giving a normalised version of $P(K : Q_{\lambda,\mu})$ which takes the value 1 on the unknot. These eigenvalues are sometimes known as (1,1)-tangle invariants of K, and have the advantage that they often remain non-zero under evaluations which send the corresponding invariant P to zero irrespective of K.

When $\mu = \phi$ it can be shown [15], using properties of the Gyoja-Aiston idempotents, that the eigenvalues $K_{\lambda,\phi}$, although conceivably rational functions in v and s, are in fact integral elements of Λ, lying in the subring $\Lambda_0 = \mathbf{Z}[s^{\pm 1}, v^{\pm 1}]$. This is also the case for $K_{\square,\square}$, and we believe that it is true in general.

By way of example we give the invariant $K_{\square,\square}$ for the trefoil and for the figure-eight knot, in terms of v and $z = s - s^{-1}$.

For the trefoil we get $v^2 - 4v^4 + 4v^6 + z^2(1 + 2v^2 - 7v^4 - 4v^6) + z^4(v^2 - 2v^4 + v^6)$ for some choice of framing - there is a factor of v^2 to be used for each change of framing.

For the figure-eight with zero framing the result is symmetric in $v^{\pm 1}$, and is $3 - 2z^2 - 6z^4 - 2z^6 + (v^2 + v^{-2})(-2 - z^2 - 2z^4 + z^6) + (v^4 + v^{-4})(1 + 2z^2 + z^4)$.

Version 1.7, July 2004

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