The Burau matrix and Fiedler's invariant for a closed braid

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Abstract

It is shown how Fiedler's 'small state-sum' invariant for a braid β can be calculated from the 2-variable Alexander polynomial of the link which consists of the closed braid $\hat{\beta}$ together with the braid axis A.

AMS Classification numbers Primary: 57M25.

Keywords: small state-sum, Fiedler's invariant, closed braid, Burau matrix, 2-variable Alexander polynomial.

Original version April 1997, revised November 1997.

[1]This work was carried out during the low-dimensional topology program at MSRI, Berkeley in 1996-97. I am grateful to MSRI for their support.

1 Introduction

In a recent paper [1] Fiedler introduced a simple invariant for a knot K in a line bundle over a surface F by means of a 'small' state-sum, which keeps a count of features of the links resulting from smoothing each crossing of the projection of K on F. The invariant takes values in a quotient of the integer group ring of $H_1(F)$. Fiedler gives a number of applications of his general construction. In particular, where K is a closed braid, and can thus be regarded as a knot in a solid torus V, his method gives an invariant of a braid $\beta \in B_n$ in $\mathbf{Z}[H_1(V)] = \mathbf{Z}[x^{\pm 1}]$ modulo the relation $x^n = 1$. This invariant depends only on the closure of the braid in V and hence gives an invariant of β up to conjugacy in B_n. Its behaviour under Birman and Menasco's exchange moves has been used to help in detecting when two braids may be related by such a move.

The purpose of this paper is to show how Fiedler's invariant for a closed braid $\hat{\beta}$ can be found in terms of the Burau representation of β, and hence from the 2-variable Alexander polynomial of the link $\hat{\beta} \cup A$ consisting of the closed braid $\hat{\beta}$ and its axis A. Its construction here from the Alexander polynomial can be compared with methods which yield Vassiliev invariants of degree 1 in other contexts, and suggests possible interpretations of Fiedler's invariants as Vassiliev invariants of degree 1 in the line bundle.

Having seen how the special case of Fiedler's invariant is related to an Alexander polynomial I finish the paper with a suggestion of extracting similar invariants from the 2-variable Alexander polynomial of a more general 2-component link. These might be regarded as degree 1 Vassiliev invariants of one component of the link when considered as a knot in the complement of the other component. It would be interesting to know if there was any similar state sum interpretation of these invariants in the more general setting.

2 Burau matrices

I make use of the fact that the 2-variable Alexander polynomial $\Delta_{\hat{\beta}\cup A}(t,x)$ of a closed braid and its axis can be calculated as the characteristic polynomial, $\det(I - x\overline{B}(t))$, of the reduced $(n-1)\times(n-1)$ Burau matrix $\overline{B}(t)$ of the braid β, [2]. Since the full $n \times n$ Burau matrix $B(t)$ is conjugate to $\begin{pmatrix} \overline{B}(t) & \mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix}$ we can write

$$(1-x)\Delta_{\hat{\beta}\cup A}(t,x) = \det(I - xB(t)).$$

Put $t = e^h$ in $\det(I - xB(t)) = 1 + b_1(t)x + \cdots + b_n(t)x^n$, and expand this as a power series in h to give

$$\det(I - xB(e^h)) = \sum_{i=0}^{\infty} a_i(x)h^i,$$

where each coefficient $a_i(x)$ is a polynomial in x of degree at most n.

When we set $h = 0$, and thus $t = 1$, we must get $\Delta_A(x) \times (1 - x^n)$ by the Torres-Fox formula, since the two components A and $\hat{\beta}$ of the link have linking number n. Hence $a_0(x) = 1 - x^n$. Setting $x = 0$ shows also that $a_1(x) = f_1x + f_2x^2 + \cdots + f_nx^n$ for some integers $f_1, \ldots, f_n$. We know that the determinant of the Burau matrix is $(-t)^{w(\beta)}$, where $w(\beta)$ is the writhe of the braid, and so $b_n(t) = (-1)^n(-t)^{w(\beta)}$. Now $w(\beta) = n - 1 \bmod 2$ since β closes to a single component. Hence $b_n(e^h) = -1 - w(\beta)h + O(h^2)$, giving $f_n = -w(\beta)$. We shall relate the remaining coefficients $f_1, \ldots, f_{n-1}$ directly to Fiedler's invariant.

3 Fiedler's braid invariant.

Fiedler's invariant F_β for an n-braid β which closes to a single curve is a symmetric Laurent polynomial, which is even or odd depending on the parity of n. Suppose that the braid

$$\beta = \prod_{r=1}^{k} \sigma_{i_r}^{\varepsilon_r}$$

has been given in terms of the Artin generators σ_i, where $\varepsilon_r = \pm 1$. Suppose that the product reads from top to bottom in the braid and the strings are oriented downwards. For the rth crossing define a positive integer $m(r)$ by smoothing the crossing and following the 'ascending string' at the smoothed crossing around the closed braid until it closes again after $m(r)$ turns around the axis. Here the ascending string means the string which starts from the end of the overcrossing, and is thus string i_r for a positive crossing and string i_r+1 for a negative crossing. Fiedler's polynomial $F_\beta(X)$ is defined as a sum over the k crossings of β by

$$F_\beta(X) = \sum_{r=1}^{k} \varepsilon_r X^{2m(r)-n}.$$

For a given m we can then write the coefficient of X^{2m-n} as $\sum_{m(r)=m} \varepsilon_r$.

Theorem 1 *Let the n-string braid β have Burau matrix $B(t)$, and write* $\det(I - xB(e^h)) = a_0(x) + a_1(x)h + O(h^2)$. *Fiedler's polynomial for β satisfies*

$$F_\beta(x^{1/2}) = (f_1x + \cdots + f_{n-1}x^{n-1})x^{-\binom{n}{2}},$$

where $a_1(x) = f_1x + \cdots + f_{n-1}x^{n-1} + f_nx^n$.

Proof: Use the classical trace formula for the characteristic polynomial of a matrix B. Suppose that B has eigenvalues $\lambda_1, \ldots, \lambda_n$. Then B^m has eigenvalues

$\lambda_1^m, \ldots, \lambda_n^m$ and $\det(I - xB) = \prod_{i=1}^n (1 - x\lambda_i)$. Hence

$$\begin{aligned}\ln(\det(I - xB)) &= \sum_{i=1}^n \ln(I - x\lambda_i) = -\sum_{m=1}^\infty \sum_{i=1}^n \frac{1}{m} x^m \lambda_i^m \\ &= -\sum_{m=1}^\infty \frac{x^m}{m} \mathrm{tr}\,(B^m),\end{aligned}$$

as power series in x.

Now expand $\ln(a_0(x) + a_1(x)h + \cdots)$ as a power series in h, only as far as the term in h. We get

$$\begin{aligned}\ln(a_0(x) + a_1(x)h + \cdots) &= \ln a_0(x) + \ln(1 + \frac{a_1(x)}{a_0(x)} h + O(h^2)) \\ &= \ln a_0(x) + \frac{a_1(x)}{a_0(x)} h + O(h^2) \\ = -x^n - x^{2n}/2 - \cdots + h(f_1 x &+ f_2 x^2 + \cdots + f_n x^n)(1 + x^n + x^{2n} + \cdots) + O(h^2).\end{aligned}$$

The trace formula above applied to $B(e^h)$ shows at once that $\mathrm{tr}\,((B(e^h))^m) = -m f_m h + O(h^2)$ for $1 \leq m < n$.

The proof will be completed by relating the term in h in the trace of this matrix to the appropriate coefficient of Fiedler's polynomial. It is thus enough to show that $\mathrm{tr}\,((B(e^h))^m) = -m(\sum_{m(r)=m} \varepsilon_r)h + O(h^2)$ for $1 \leq m < n$.

The Burau representation $\rho : B_n \to GL(n, \mathbf{Z}[t^{\pm 1}])$ is the group homomorphism defined on the elementary braid σ_i by

$$\rho(\sigma_i) = B_i = \left(\begin{array}{c|c|c} I_{i-1} & 0 & 0 \\ \hline 0 & \begin{matrix} 1-t & t \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{n-i-1} \end{array}\right).$$

The Burau matrix for the given braid β is then

$$B(t) = \rho(\beta) = \prod_{r=1}^k B_{i_r}^{\varepsilon_r}.$$

Now

$$\begin{aligned}B_i(e^h) &= \left(\begin{array}{c|c|c} I_{i-1} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{n-i-1} \end{array}\right) + h \left(\begin{array}{c|c|c} 0_{i-1} & 0 & 0 \\ \hline 0 & \begin{matrix} -1 & 1 \\ 0 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & 0_{n-i-1} \end{array}\right) + O(h^2) \\ &= T_i + hP_i^+ + O(h^2), \text{ say.}\end{aligned}$$

We can similarly write $B_i^{-1} = T_i + hP_i^- + O(h^2)$ where

$$P_i^- = \left(\begin{array}{c|c|c} 0_{i-1} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & 0 \\ -1 & 1 \end{matrix} & 0 \\ \hline 0 & 0 & 0_{n-i-1} \end{array}\right).$$

Then

$$(B(e^h))^m = (\prod_{r=1}^{k}(T_{i_r} + hP^{\pm}_{i_r}))^m + O(h^2).$$

We can write a matrix of the form $M = \prod_{r=1}^{l}(C_r + hD_r)$ as

$$M = C_1C_2\ldots C_l + h(D_1C_2\ldots C_l + C_1D_2C_3\ldots C_l + \cdots + C_1C_2\ldots C_{l-1}D_l) + O(h^2),$$

and then

$$\operatorname{tr} M = \operatorname{tr}(C_1C_2\ldots C_l) + h(\operatorname{tr}(D_1C_2\ldots C_l) + \operatorname{tr}(C_1D_2C_3\ldots C_l) + \cdots) + O(h^2).$$

The term in h can be rewritten as

$$\operatorname{tr}(C_2\ldots C_lD_1) + \operatorname{tr}(C_3\ldots C_lC_1D_2) + \cdots + \operatorname{tr}(C_1C_2\ldots C_{l-1}D_l)$$

by cycling the matrices so that the rth product ends with the matrix D_r.

Apply this to find the term in h in $\operatorname{tr}((B(e^h))^m$ as the sum of mk terms, each of which is the trace of the product of mk matrices of the form $T_{i_{r+1}}\ldots T_{i_{r-1}}P^{\pm}_{i_r}$ with sign $\pm$ according to the sign of ε_r. For each of the k crossings of the original braid the matrix $T_{i_{r+1}}\ldots T_{i_{r-1}}P^{\pm}_{i_r}$ occurs m times in the sum. Thus

$$f_m = -\sum_{r=1}^{k}\operatorname{tr}(T_{i_{r+1}}\ldots T_{i_{r-1}}P^{\pm}_{i_r}).$$

The proof of theorem 1 will be completed by showing that

$$tr(T_{i_{r+1}}\ldots T_{i_{r-1}}P^{\pm}_{i_r}) = \begin{cases} -\varepsilon_r & \text{if } m(r) = m \\ 0 & \text{otherwise.}\end{cases}$$

The matrix T_i is the permutation matrix for the transposition $(i\ i{+}1)$. Hence a product of these matrices is also a permutation matrix, T say, whose permutation is the product π of the corresponding transpositions. Then the entries in T satisfy

$$T_{ij} = \begin{cases} 1 & \text{if } i = \pi(j), \\ 0 & \text{otherwise.}\end{cases}$$

The matrix $T = T_{i_{r+1}}\ldots T_{i_{r-1}}$ above is thus a permutation matrix with permutation $\pi^{(m)}_r$, say. Notice that the permutation corresponding to the product TT_{i_r} is conjugate to the permutation of the braid β^m. Under the assumption that β closes to a single curve this will be the mth power of an n-cycle, and will hence not fix any number when $1 \leq m < n$. Hence $\pi^{(m)}_r$ can not carry i_r to $i_r + 1$ or vice versa, in this range.

If the rth crossing is smoothed and the strings i_r and $i_r + 1$ are followed upwards around the braid m times, with $m < n$, they will not pass through the smoothed crossing. They then become the strings $\pi^{(m)}_r(i_r)$ and $\pi^{(m)}_r(i_r + 1)$ respectively when they return to the level of the bottom of the rth crossing. Now when $\varepsilon_r = +1$ the ascending string at the rth crossing, which is string i_r, returns to position i_r after the permutation $\pi_r^{(m)}$ if and only if $m = m(r)$. Similarly when $\varepsilon_r = -1$ the ascending string, in this case string $i_r + 1$ returns to position $i_r + 1$ exactly when $m = m(r)$.

The matrices $P_{i_r}^{\pm}$ have only two non-zero entries. Suppose first that $\varepsilon_r = +1$. Then $\mathrm{tr}\,(TP_{i_r}^{+})$ is the sum of two terms. The off-diagonal entry gives a contribution only if the permutation matrix T maps it onto the diagonal. This requires $\pi_r^{(m)}(i_r) = i_r + 1$, which was excluded above. The diagonal entry contributes -1 if and only $\pi_r^{(m)}(i_r) = i_r$, which is the condition that $m = m(r)$. Thus when $\varepsilon_r = +1$ we get a contribution of $-\varepsilon_r$ to the trace if and only if $m = m(r)$, and zero otherwise.

A similar argument holds when $\varepsilon_r = -1$. Again the off-diagonal entry does not contribute to the trace, while the diagonal entry contributes $+1$ if and only if $\pi_r^{(m)}(i_r + 1) = i_r + 1$. This corresponds once more to the condition that $m = m(r)$, and so in each case we have a contribution of $-\varepsilon_r$ if and only if $m = m(r)$. The total coefficient of h in $\mathrm{tr}\,((B(e^h)^m)$ is then $-m \sum\limits_{m=m(r)} \varepsilon_r$, showing that $f_m = \sum\limits_{m=m(r)} \varepsilon_r$ as claimed. This completes the proof of theorem 1.

4 Determination from an Alexander polynomial.

If we are given the Alexander polynomial of the closed braid $\hat{\beta}$ and its axis A as a 2-variable polynomial we can recover Fiedler's invariant for the braid. First multiply by $1-x$, where x is the variable for the axis. This gives the characteristic polynomial of the Burau matrix for β, up to multiplication by a power of x and a power of t, and a sign. Put $t = e^h$ and expand as a power series in h with coefficients depending on x. Then multiply by a power of x and a sign to make the constant term $1-x^n$. The result will be the characteristic polynomial used above, up to a power of $t = e^h$. Extract the coefficient $f_0+f_1x+\cdots+f_{n-1}x^{n-1}+f_nx^n$ of h. This will contain the Fiedler polynomial as before in the terms $f_1x+\cdots+f_{n-1}x^{n-1}$, while the remaining terms will come from a factor of t^{f_0} and will satisfy $f_0 + f_n = -w(\beta)$.

A similar interpretation looks plausible for the coefficients of the linear terms in $h_1, \ldots, h_k$ when the Alexander polynomial of a closed braid with k components and its axis is expanded in terms of the meridian generator x for the axis and meridians $t_i = e^{h_i}$ for the components. This polynomial can again be written in terms of the characteristic polynomial of a suitable 'coloured' Burau matrix. The eventual coefficient of h_i should then have contributions from the overcrossings of the corresponding component of the closed braid, as in the Fiedler polynomial above.

As a possible extension to the case of a general link L with two components X and T say, we might put $t = e^h$ in the Alexander polynomial $\Delta_{X\cup T}(x,t)$ of L

and consider only the terms $a_0(x) + a_1(x)\,h$ up to degree 1 in h. The polynomial $a_0(x)$ is $\Delta_X(x)(1 - x^n)/(1 - x)$, where n is the linking number of X and T, and $\Delta_X(x)$ is the Alexander polynomial of X. Now consider $a_1(x)$ as a polynomial modulo the ideal generated by $a_0(x)$. This is an invariant of L as it is unaffected by any ambiguity of powers of x and t in the Alexander polynomial. This seems to me to be the nearest analogue to Fiedler's invariant for the link component T with meridian t when regarded as a knot in the complement of X; in the case of a closed braid we take X as the braid axis and T as the closed braid. It looks likely to be a Vassiliev invariant of type 1 for knots in the complement of X. There is not, however, any obvious candidate for a state-sum construction of this invariant along Fiedler's lines when the component X is knotted.

References

[1] T. Fiedler. 'A small state-sum invariant for knots'. Topology, 32 (1993), 281-294.

[2] H.R.Morton. 'Infinitely many fibred knots having the same Alexander polynomial'. Topology, 17 (1978), 101-104.