Satellites and surgery invariants

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Abstract

A satellite formula relating the quantum invariants of a satellite knot and those of its companion and pattern links will be described briefly. The $SU(2)_q$ invariants of a framed k-component link L, when the variable q is replaced by an rth root of unity yield a natural map

$$J_r(L) : \mathcal{R}_r^{\otimes k} \to \Lambda_r,$$

where $\mathcal{R}_r$ is a finite-dimensional truncation of the representation ring of $SU(2)$, and $\Lambda_r = \mathbf{Z}[e^{\pi i/2r}]$. The effect on J_r of a framing change on L is given by applying a suitable power of an automorphism F_r of $\mathcal{R}_r$ to each factor.

Use of the satellite formula for a simple choice of companion exhibits Reshetikhin and Turaev's invariant of the 3-manifold given by surgery on L as the evaluation of $J_r(L)$ on a fixed element M_0 in each $\mathcal{R}_r$, after slight normalisation. Explicit calculation of M_0 can be made easily because of a beautiful relation between F_r and the invariants $J_r(H)$ of the Hopf link. This relation can be viewed in terms of an action of $\mathrm{PSL}(2,\mathbf{Z})$ on $\mathcal{R}_r$, at least up to scalar multiples by roots of unity, and shows how the invariant of the manifold given by Dehn surgery with coefficients a_i/b_i on a link L can be found by evaluating $J_r(L)$ on suitable elements $M_{a_i/b_i} \in \mathcal{R}_r$. An indication is also given of how these results extend to other quantum groups.

In the final section we give an explicit formula for the invariant when any Dehn surgery is used, confirming its correctness via the Rolfsen moves.

Introduction

This is an account of a 3-manifold invariant for $SU(2)_q$ which was conceived, following Reshetikhin and Turaev's original description, as a direct approach with the emphasis on using the multilinearity and the explicit formula of the satellite calculations in [MS], avoiding specialisation of link invariants to a root of unity until as late as possible. Much of the paper is an expansion of a talk presented in Oberwolfach in September 1989. Its eventual form followed the unexpected discovery, prompted by explicit calculations, that apart from scalar factors, as detailed later, the invariants of the Hopf link, together with the factors associated with a change of framing, can be organised to represent the modular group $\mathrm{PSL}(2,\mathbf{Z})$, once the variable is specialised to a root of unity. This suggests a way to calculate the invariant for manifolds given by general Dehn surgery from a link, in terms of the link invariants. The conformal field theory approaches outlined by Segal as a concrete means of handling Witten's ideas make this very plausible, although suitable normalisations to deal with the scalar factors have been surprisingly elusive.

Independent work, both by Kirby and Melvin [KM], and very elegantly by Lickorish [L], has provided for different aspects of the invariants to be explored. Our approach is somewhere between these two. We use the quantum group $SU(2)_q$ as in our development [MS] of Kirillov and Reshetikhin [KR], and draw on the explicit form of multilinearity of parallels and satellites given there. We do not use the finite-dimensional Hopf algebras, where a root of unity has been introduced. There is then no need to take account of the more complicated representation theory which arises in that case, as is done in [RT], and avoided by the use of clever arguments in [KM].

Roots of unity will appear here only in specialisations of existing invariants, and we develop enough background to ensure that such moves are completely legitimate where we need them.

1. Link invariants

The $SU(2)_q$ invariants of a framed link L with k components are described in [KR]. Assign irreducible $SU(2)_q$ modules W_{i_j} to the j th component of L, and there is an invariant $J(L; W_{i_1}, \ldots, W_{i_k}) \in \Lambda = \mathbf{Z}[q^{\pm 1/4}]$. The definition can be extended multilinearly to allow the use of a Λ-linear combination of modules on each component. The invariants for L can then be viewed as a single Λ-linear map

$$J(L) : \mathcal{R}^{\otimes k} \to \Lambda,$$

where $\mathcal{R}$ is the representation ring of $SU(2)_q$. It can be shown, for example in [MS], that the product in the ring $\mathcal{R}$ has a nice interpretation in terms of invariants of parallel links. The result can be summarised as follows.

Let $m : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$ be the product in the ring, and let L be a framed knot. Then

$$J(L) \circ m = J(L^{(2)}),$$

where $L^{(2)}$ is the framed 2-parallel of L.

This result extends naturally to multiparallels of links with more than one component, as in [MS]. It allows for an alternative description of $J(L)$ to that in terms of the irreducible module assignments.

The ring $\mathcal{R}$ is spanned by the irreducible representations W_i, one of each dimension i, or equally well by the powers $(W_2)^j$, since $\mathcal{R} \cong \Lambda[W_2]$ as a ring. Since the evaluation of $J(L)$ on the element W_2 for each component is essentially the bracket polynomial version of the Jones polynomial of L it is then possible to calculate $J(L)$ in terms of the Jones polynomials for multiparallels of L. This corresponds to the use of powers of W_2 as the Λ-basis for $\mathcal{R}$ rather than the irreducibles. The change of basis information needed to pass from one basis to the other is noted in [MS], and also in a nice form in [KM]. The basis of powers has been used by Lickorish [L] in his approach to the 3-manifold invariants, allowing him to avoid any use of the other irreducibles $W_i, i > 2$, and so present the manifold invariants without having to consider the quantum group $SU(2)_q$ at all.

Satellites. Where a link P is given which has one distinguished unknotted component, the remaining components form a closed tangle relative to the distinguished component as axis. The tangle can then be used as a pattern to form satellites of given companion knots or links, based on the pattern P. The relation between the invariants of the companion, satellite and pattern is summarised in the next theorem, from [MS].

Theorem 1.1. *Let P be a pattern link with k components, with one distinguished unknotted component. There is a Λ-linear map $G : \mathcal{R}^{\otimes(k-1)} \to \mathcal{R}$ such that any satellite K formed from a companion C using P as pattern has invariant*

$$J(K) = J(C) \circ G.$$

Proof: This is given in [MS] by constructing G in terms of the basis of irreducibles for $\mathcal{R}$. If C has more than one component then G is used on that component which is to be embellished by the pattern. □

Remark. Although it is not initially clear that the ring Λ can be used without extension to permit some denominators, this follows by working with the basis of powers of W_2 and using the skein relations for the bracket polynomials on the resulting tangles in the construction of G.

The pattern link P itself can be considered as a satellite of the Hopf link H using P as pattern, so that

$$J(P) = J(H) \circ (G \otimes \mathrm{id}_{\mathcal{R}}).$$

The map G can then be recovered from the invariants $J(H)$ and $J(P)$.

We always assume that we are considering links with a given choice of framing, and that when satellite and parallel constructions are made they respect the framing. It is easy to calculate the change which takes place in the invariant when the same underlying link is used, but the framing on one or more components is altered.

Theorem 1.2. *There is a linear isomorphism $F : \mathcal{R} \to \mathcal{R}$ which can be used on the copy of $\mathcal{R}$ corresponding to one component, L_i say, of L before applying*

$J(L)$, and will then give the invariant J for the link whose framing on L_i has been increased by one.

Proof: It is known [KR] how the framing change on a component affects the invariant when an irreducible is selected in $\mathcal{R}$ for that component. This determines the map F explicitly by $F(W_i) = f_i W_i$ where the 'framing factor' f_i is given by $f_i = (-1)^{i-1} a^{i^2-1}$. □

Remark. We retain the sign in f_i as in [KR] and [MS]. Kirby and Melvin use a variant where the sign does not appear, but this needs a little caution in interpreting the relation with the bracket polynomial. One source of signs can be accounted for by considering $-W_2$ in place of W_2 as the polynomial generator for $\mathcal{R}$.

2. Roots of unity

We now consider the behaviour of J when the variable q in Λ is specialised to be an rth root of unity. We shall suppose that $q^{1/4}$ is a primitive $4r$th root of unity, and we consider the ring Λ_r given by factoring out the cyclotomic polynomial φ_{4r} generated by the $4r$-th root a in $\Lambda = \mathbf{Z}[a^{\pm 1}]$. We then have a specialisation homomorphism $e_r : \Lambda \to \Lambda_r = \Lambda / < \varphi_{4r}(a) >$.

Proposition 2.1. *For any link L the evaluation $e_r(J(L; W_{i_1}, \ldots, W_{i_k})) = 0$ if $W_{i_j} = W_r$ for any j.*

Proof: For each j we can find $\lambda_j \in \Lambda$ such that $J(L; W_{i_1}, \ldots, W_{i_k}) = \lambda_j \delta_{i_j}$, where $\delta_i = (-1)^{i-1} \dfrac{a^{2i} - a^{-2i}}{a^2 - a^{-2}}$, as in [MS]. Now $e_r(\delta_r) = 0$. □

Corollary 2.2. *If $V_j \in \mathcal{R}$ lies in the ideal generated by W_r, for some j, then $e_r(J(L; V_1, \ldots, V_k)) = 0$.*

Proof: Without loss of generality we may consider the case where $k = 1$, and $V = W_r V'$ with $V' \in \mathcal{R}$. Then

$$e_r(J(L; V)) = e_r(J(L^{(2)}; W_r, V')) = 0. \square$$

Notation. Write $W_r = \Delta_r(W_2)$ as a polynomial in W_2. The polynomial Δ_r satisfies $\Delta_r(2\cos\theta) = \sin r\theta/\sin\theta$, and is closely related to the Tchebychev polynomial T_r with $T_r(2\cos\theta) = \cos r\theta$.

Write $\mathcal{R}_r = (\mathcal{R}\otimes\Lambda_r)/< W_r > \cong \Lambda_r[W_2]/< \Delta_r(W_2) >$, and write $p_r : \mathcal{R} \to \mathcal{R}_r$ for the projection. The ring $\mathcal{R}_r$ is known as a *Verlinde algebra*; similar algebras may be defined for representation rings of other simple Lie groups.

We shall extend the definition in $\mathcal{R}$ to allow W_i with $i \le 0$ by setting $W_{-k} = -W_k$ for $k > 0$. Multiplication in $\mathcal{R}$ can then be described simply, for $i, j > 0$ as

$$W_i W_j = \sum_{k=i-j+1}^{i+j-1} W_k = \sum_{k=j-i+1}^{j+i-1} W_k,$$

where the sum is in steps of 2, [MS], since the excess terms in one of these sums will cancel. (Under a suitable convention for sums, the same result holds for all i, j.) It then follows that

$$(W_{j+1} - W_{j-1})W_r = W_{r+j} + W_{r-j}, \text{ for all } r, j.$$

Proposition 2.3. *The invariant $J(L) : \mathcal{R}^{\otimes k} \to \Lambda$ induces a Λ_r-linear map $J_r(L) : \mathcal{R}_r^{\otimes k} \to \Lambda_r$ with $e_r \circ J(L) = J_r(L) \circ p_r^{\otimes k}$.*

Proof: The map $e_r \circ J(L)$ is zero on the kernel of $p_r^{\otimes k}$, by corollary 2.2. □

Proposition 2.4. *The isomorphism $F : \mathcal{R} \to \mathcal{R}$ induces $F_r : \mathcal{R}_r \to \mathcal{R}_r$ such that $F_r \circ p_r = p_r \circ F$.*

Proof: We must show that $p_r \circ F$ is zero on the ideal generated by W_r. This ideal is spanned by the elements W_r and

$$(W_{j+1} - W_{j-1})W_r = W_{r-j} + W_{r+j} \quad \text{for } j \in \mathbf{N}.$$

Now

$$p_r \circ F(W_r) = f_r p_r(W_r) = 0$$

and

$$\begin{aligned} p_r \circ F(W_{r-j} + W_{r+j}) &= p_r\left((-1)^{r-j-1} a^{(r-j)^2-1} W_{r-j} + (-1)^{r+j-1} a^{(r+j)^2-1} W_{r+j}\right) \\ &= (-1)^{r-j-1} a^{r^2+j^2-1}\left(a^{-2rj} p_r(W_{r-j}) + a^{2rj} p_r(W_{r+j})\right) = 0 \end{aligned}$$

as $p_r(W_{r-j}) = -p_r(W_{r+j})$ and $e_r(a^{4r}) = 1$. □

Notation. We abuse notation by writing W_i for $p_r(W_i) \in \mathcal{R}_r$. We shall write $i \in j \otimes k$ for i, j, k between 1 and $r-1$ to mean that W_i has non-zero coefficient in the product $W_j \otimes W_k$ in $\mathcal{R}_r$.

Then we have

Lemma 2.5. $j \in i \otimes k \Leftrightarrow k \in i \otimes j$.

Proof: We have $W_{r+n} = -W_{r-n}$ in $\mathcal{R}_r$. Thus

$$W_i \otimes W_k = \sum_{|i-k|+1}^{m} W_j$$

where $m = \min(i+k-1, 2r-i-k-1)$, and the sum runs in steps of two. If we let $\alpha = (i-1)/2$, $\beta = (j-1)/2$ and $\gamma = (k-1)/2$ then the condition for W_i to be a summand of $W_j \otimes W_k$ is that α, β and γ form the sides of a triangle, with perimeter an integer less that $r-1$. Since this is clearly a symmetric condition the lemma is proved. $\square$

Note: In the case of a deformation of a general simple Lie algebra, the statement would need to be altered to allow for multiplicities, and for the fact that conjugate representations enter in.

We may now use the reduced invariants $J_r(L)$ of framed links L, which can be calculated from their standard $SU(2)_q$ invariant $J(L)$, to determine the Reshetikhin-Turaev invariant of the manifold given by surgery on L. We construct $U \in \mathcal{R}_r$ so that the invariant in Λ_r is a simple multiple of $J_r(L; U, \ldots, U)$.

Write $M^3(L)$ for the manifold constructed from a framed link L by surgery.

Theorem. (Kirby, Fenn-Rourke)

1 *Every closed oriented M^3 arises in this way.*

2 *There is an orientation preserving homeomorphism $M^3(L) \cong M^3(L')$ if and only if L, L' are related by a sequence of Kirby moves.*

Kirby moves are of two types, shown in figure 1.

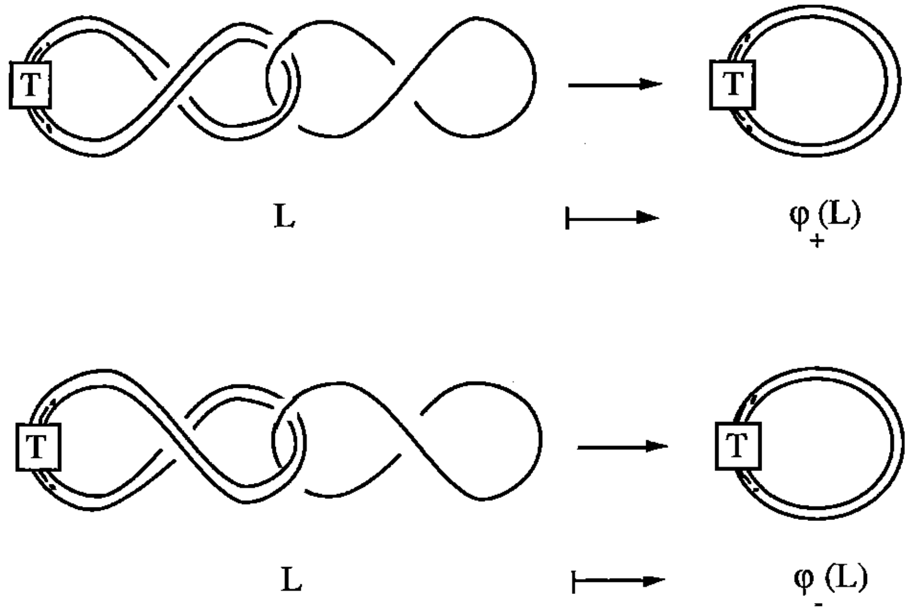

Figure 1

Remark. Fenn and Rourke [FR] showed that Kirby's original moves could be reduced to these.

In all diagrams we shall use the convention that the link is framed with the *planar* framing, i.e. the chosen parallel is given by the edge of a ribbon following one side of the curve in the plane of the diagram.

To a framed *oriented* link $L = L_1 \cup L_2 \cup \ldots \cup L_k$ we can associate a quadratic form with $k \times k$ matrix (ℓ_{ij}) where

$$\ell_{ij} = \mathrm{lk}(L_i, L_j),\ \ i \neq j,$$
$$\ell_{ii} = \text{framing on } L_i.$$

Write $\mathrm{sig}(L)$ for the signature of this form. (This is not generally the signature of the link L in the usual sense.)

Then $\mathrm{sig}(L)$ is independent of the choice of orientation of L, and

$$\mathrm{sig}\varphi_\pm(L) = \mathrm{sig}\, L \mp 1.$$

Invariants of $M^3(L)$

For each root of unity a, find $c(a), T_L(a) \in \Lambda_r$, for each L, such that

$$T_{\varphi_+(L)} = c(a) T_L$$
$$T_{\varphi_-(L)} = (c(a))^{-1} T_L$$

Then $\mathcal{I}(M) = c(a)^{\mathrm{sig}\, L} T_L$ depends only on $M(L)$.

To demonstrate independence of the invariant on the choice of framed link it is sufficient to ensure that it is unaltered by the Kirby moves. We must then compare the invariants of the two links L and $\varphi_+(L)$.

The pattern determined by the tangle T defines $G : \mathcal{R}^{\otimes k-1} \to \mathcal{R}$. For a fixed assignment of elements $V_1, \ldots, V_{k-1}$ of $\mathcal{R}$ to the strings of T write $X = G(V_1, \ldots, V_{k-1})$. The satellite formula then shows that, for a choice of element $Y \in \mathcal{R}$ on the unknotted component of the first link, this link has invariant $J(H; F(X), F(Y))$. This must be compared with the invariant of the second link, which is $J(O; X) = J(H; X, W_1)$, where O is the unknot, as indicated in figure 2.

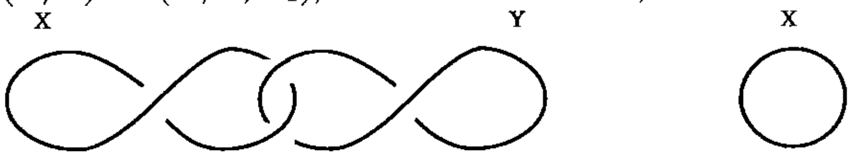

Figure 2

The reduced invariants of the two links will then be $J_r(H; F_r(X), F_r(Y))$ and $J_r(H; X, W_1)$.

Notation. Write $< \;,\; >_r$ for the bilinear form on $\mathcal{R}_r$ determined by $J_r(H)$ as

$$< V, W >_r = J_r(H; V, W).$$

Theorem 2.6. *The element* $U = \sum_{j=1}^{r-1} \delta_j W_j \in \mathcal{R}_r$ *satisfies*

$$< F_r(X), F_r(U) >_r = c_+ < X, W_1 >_r$$

$$< F_r^{-1}(X), F_r^{-1}(U) >_r = c_- < X, W_1 >_r$$

for every $X \in \mathcal{R}_r$, *where* $c_+ = \sum f_j \delta_j^2$, *and* $c_- = \bar{c}_+ \in \Lambda_r$.

Notation. Write $c_+ = \rho(a)c(a)$ with $\rho(a) > 0$ and $|c(a)| = 1$.

Remark. The element $c(a)^2$ turns out to be the power a^{2r^2-r-6} of a.

Corollary 2.7. *The element* $\rho(a)^{-k} c(a)^{\mathrm{sig}(L)} J_r(L; U, \ldots, U)$ *depends only on the manifold given from the k component framed link L by surgery, where* $\mathrm{sig}(L)$ *is the signature of the quadratic form determined by the linking numbers and framings of L.*

Proof: Take $T_L = J_r(L; \rho(a)^{-1}U, \dots, \rho(a)^{-1}U)$ and use theorem 2.6 to compare the invariants arising from L and $\varphi_\pm(L)$. $\square$

Remark. Apart from a factor of $c(a)$ this gives the invariant of Reshetikhin and Turaev. Kirby and Melvin use exactly this normalisation, which ensures that oppositely oriented manifolds have conjugate invariants.

We shall give the proof of theorem 2.6 shortly, in the context of further properties of F_r and $< \ , \ >_r$, leading to a means for finding the invariant for the manifold given by general Dehn surgery on a link in terms of the invariant of the link.

It is helpful to view each copy of $\mathcal{R}_r$ associated to a link component as depending on a choice of parallel and meridian for the peripheral torus. The automorphism F_r corresponds to altering the choice of parallel, by Dehn twists about the meridian, to allow for integer framing change when calculating the invariant of the link exterior. We use the bilinear form $< \ , \ >_r$ to construct another automorphism which will correspond to a Dehn twist about the parallel.

We shall prove

Theorem 2.8. *For any $X \in \mathcal{R}_r$ we have*

$$< X, U >_r = < F_r(X), U >_r.$$

Theorem 2.9. *The symmetric bilinear form $< \ , \ >_r$ is non-degenerate. Its matrix H_r relative to the basis of irreducibles $W_1, \dots, W_{r-1}$ satisfies*

$$H_r^2 = \rho(a)^2 I.$$

Definition. We may then define $\Phi_r : \mathcal{R}_r \to \mathcal{R}_r$ to be the adjoint of F_r, that is $< F_r(X), Y >_r = < X, \Phi_r(Y) >_r$ for all X, Y.

The matrices of F_r and Φ_r in the basis of irreducibles are then related by $\Phi_r = H_r^{-1} F_r H_r$. This necessitates extending the coefficient ring Λ_r to include an inverse for $\det(H)$; making $2r$ invertible will be sufficient, as we shall see in the proof of theorem 2.9. In order to include $\rho(a)$ and $c(a)$ we also need the square root of two, or equivalently of i; one possibility would be to take Λ_r to be the cyclotomic field generated by the $8r$th roots of unity.

Corollary 2.10. *The element U is an eigenvector of Φ_r with eigenvalue* 1.

Proof: We have $< X, U >_r = < F_r(X), U >_r = < X, \Phi_r(U) >_r$ for all X, so $U = \Phi_r(U)$. □

Remark. From theorem 2.6 we have $c_+ < X, W_1 >_r = < F_r(X), F_r(U) >_r = < X, \Phi_r F_r(U) >_r$ so that $\Phi_r F_r(U) = c_+ W_1$. This could be used as a definition of U.

Proof of theorem 2.6: To give a self-contained argument we shall work in coordinates relative to the basis of irreducibles in $\mathcal{R}_r$. Let X have coordinate vector $\mathbf{x} = (x_1, \dots, x_{r-1})$. Now U has coordinate vector $\delta = (\delta_1, \dots, \delta_{r-1})$ which is also the first column of the matrix H_r so we must show that

$$\mathbf{x}^T F_r H_r F_r \delta = c_+ \mathbf{x}^T \delta,$$

for all $\mathbf{x}$.

It is then enough to show that the vector $\delta = (\delta_1, \delta_2, \dots \delta_{r-1})$ is an eigenvector of the matrix $F_r H_r F_r$ with eigenvalue $c_+ = \sum_1^{r-1} f_k \delta_k^2$.

The ijth entry of H_r is given by $\displaystyle\sum_{k \in i \otimes j} f_k f_j^{-1} f_i^{-1} \delta_k$.

So the ith entry of $F_r H_r F_r \delta$ is $\displaystyle\sum_{j=1}^{r-1} \sum_{k \in i \otimes j} f_k \delta_k \delta_j$. By lemma 2.5, we can rearrange this sum as

$$\sum_{k=1}^{r-1} f_k \delta_k \sum_{j \in i \otimes k} \delta_j = \sum_{k=1}^{r-1} f_k \delta_k \delta_k \delta_i = c_+ \delta_i,$$

and the result is proved.

Conjugation gives the other half of the result, since $\bar{\delta} = \delta$ and $\overline{H}_r = H_r$ while $\overline{F}_r = F_r^{-1}$ and $\bar{c}_+ = c_-$. □

We can represent theorem 2.6 diagrammatically by figure 3.

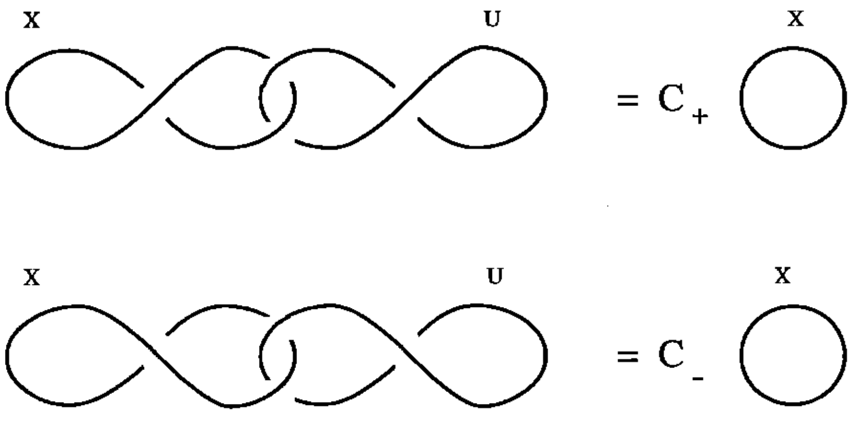

Figure 3

As noted before theorem 2.6, it then follows from the satellite formula that the invariants of the two links L and $\varphi_+(L)$ are the same, up to the factor c_+, when U is used on the unknotted string, with the same assignments made to the strings of the tangle T in each case.

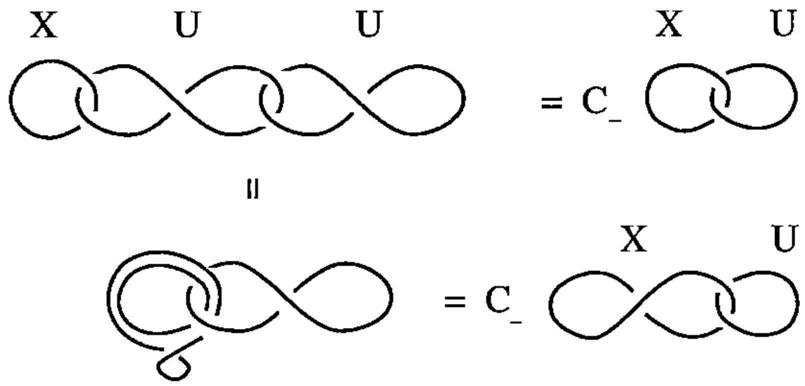

Figure 4

Proof of theorem 2.8: Apply this result to the link shown in figure 4 in which each component labelled with the element U plays the role of the unknotted curve in turn, for a suitable choice of T, gives two links with the same invariant, one being $< X, U >_r$ and the other $< F_r(X), U >_r$. □

The original proof of theorem 2.9, which we give here, relies on explicit knowledge of the entries in H_r, and follows the details in [S]. A similar technique has also been used to show that $(H_rF_r)^3$ is a scalar matrix. In the next section we give a more diagrammatic argument for these results, using connected sums of links, and the other results of this section.

Proof of theorem 2.9: As shown in [MS], the invariant $J(H; W_i, W_j)$ of a Hopf link labelled by irreducibles W_i and W_j is

$$H_{ij} = (-1)^{i+j} \frac{s^{ij} - s^{-ij}}{s - s^{-1}},$$

where $s = a^2$. Let c_{ik} be the general entry in H_r^2, then

$$\begin{aligned}
(-1)^{i+k}(s - s^{-1})^2 c_{ik} &= \sum_{j=1}^{r-1} (s^{ij} - s^{-ij})(s^{jk} - s^{-jk}) \\
&= \sum_{1}^{r-1} (s^{i+k})^j + (s^{-(i+k)})^j - \sum_{1}^{r-1} (s^{i-k})^j + (s^{-(i-k)})^j.
\end{aligned}$$

Writing $z_1 = s^{i+k}$, $z_2 = s^{i-k}$ we have

$$\begin{aligned}
\sum_{j=-(r-1)}^{r-1} (z_1^j - z_2^j) &= \sum_{j=1}^{r-1} (z_1^j + z_1^{-j}) - \sum_{j=1}^{r-1} (z_2^j + z_2^{-j}) + z_1^r - z_2^r + z_1^0 - z_2^0 \\
&= (s - s^{-1})^2 c_{ik} + (-1)^{i+k} - (-1)^{i-k} \\
&= (s - s^{-1})^2 c_{ik}.
\end{aligned}$$

Now the sum of any $2r$ consecutive powers of a $2r$th root of unity, other than 1, is zero; and $z_2 = 1$ exactly when $i = j$, whereas $z_1 \neq 1$. Then

$$(-1)^{i+j}(s - s^{-1})^2 c_{ij} = \begin{cases} 0, & i \neq j \\ -2r, & i = j. \end{cases}$$

This shows that $H_r^2 = \dfrac{-2r}{(s - s^{-1})^2} I$. We can now use the proof of theorem 2.6 to identify the scalar with $\rho(a)^2$. For we have $F_r H_r F_r \delta = c_+ \delta$ and $F_r^{-1} H_r F_r^{-1} \delta = c_- \delta$. It follows that

$$F_r H_r^2 F_r^{-1} \delta = c_+ c_- \delta = \rho(\alpha)^2 \delta.$$

On the other hand, $F_r H_r^2 F_r^{-1} = \dfrac{-2r}{(s - s^{-1})^2} I$, and so $\rho(a)^2 = \dfrac{-2r}{(s - s^{-1})^2}$, completing the proof. □

3. Modular group

In this section we shall show that F_r and Φ_r, as automorphisms of $\mathcal{R}_r$, obey the same relations as the generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ of $\mathrm{SL}(2, \mathbf{Z})$, up to multiplication by powers of the scalar $c(a)^2$, giving us a 'projective' representation for the modular group $\mathrm{PSL}(2, \mathbf{Z})$ on $\mathcal{R}_r$. As John Humphreys has pointed out, one could easily make this a genuine representation of a central extension of PSL(2,**Z**) by an element whose $4r$th power was the identity.

Theorem 3.1. *The automorphisms F_r and Φ_r of $\mathcal{R}_r$ satisfy*

$$F_r \Phi_r F_r = \Phi_r F_r \Phi_r.$$

Theorem 3.2. *The automorphism $(F_r\Phi_rF_r)^2$ is scalar multiplication by $c(a)^2$.*

These two results will follow by establishing

Proposition 3.3. *The matrices H_r and F_r satisfy*

$$(H_rF_r)^3 = c_+\rho(a)^2I = \rho(a)^3c(a)I.$$

Proof of theorem 3.1: Using the matrices in the basis of irreducibles, we have

$$\begin{aligned} F_r\Phi_rF_r &= F_rH_r^{-1}F_rH_rF_r \\ &= \rho(a)^{-2}F_rH_rF_rH_rF_r \text{ by theorem 2.9} \\ &= c_+H_r^{-1} \text{ by theorem 3.3} \end{aligned}$$

and

$$\begin{aligned} \Phi_rF_r\Phi_r &= H_r^{-1}F_rH_rF_rH_r^{-1}F_rH_r \\ &= \rho(a)^{-2}H_r^{-1}F_rH_rF_rH_rF_rH_r \\ &= c_+H_r^{-1}. \end{aligned}$$

□

Proof of theorem 3.2: Again using the matrices we have

$$\begin{aligned} (F_r\Phi_rF_r)^2 &= F_r\Phi_rF_r\Phi_rF_r\Phi_r \\ &= \rho(a)^{-6}(F_rH_r)^6 \\ &= \rho(a)^{-6}(\rho(a)^3c(a))^2I \\ &= c(a)^2I. \end{aligned}$$

□

In order to prove proposition 3.3, we will use the work of the previous section, together with the following result on the invariant of a connected sum of two links;

Lemma 3.4. *Let $K = K_1 \cup \ldots \cup K_m$ and let $L = L_1 \cup \ldots \cup L_n$ be two framed links. Let W be any irreducible representation of $SU(2)_q$, and let $L\#K$ denote the connected sum of L and K along the first components of L and K. Then*

$$J(L; W, X_2, \ldots, X_m)J(K; W, Y_2, \ldots, Y_n) = \delta_W J(L\#K; W, X_2, \ldots, X_m, Y_2, \ldots, Y_n)$$

for any X_i and Y_j in $\mathcal{R}$.

Proof: Present K and L as the closures of 1-1 tangles S and T on the first strings; then the connected sum will be the closure of the tangle ST. By Schur's lemma, the invariants of S and T are scalars σ and τ say times the identity map on W. The invariants of K, L and $K\#L$ are then $\sigma\delta_W$, $\tau\delta_W$ and $\sigma\tau\delta_W$ respectively, proving the lemma. □

Figure 5

Proof of proposition 3.3: The ijth entry, b_{ij} say, of the matrix H_rF_r is $J_r(H; W_i, F_r(W_j))$. This is the $SU(2)_q$ invariant of a Hopf link with a positive twist on the second component, and labelled by representations W_i and W_j, , as shown in figure 5. Let T_{ijk} be the invariant of the link shown in figure 6, with elements W_i, W_j and W_k assigned to the components as shown. Regarding the link as the connected sum of two links shows, by lemma 3.4, that $\delta_j T_{ijk} = b_{ij}b_{jk}$. The ikth entry, c_{ik} say, of $(H_rF_r)^2$ is then $\sum_1^{r-1} \delta_j T_{ijk}$.

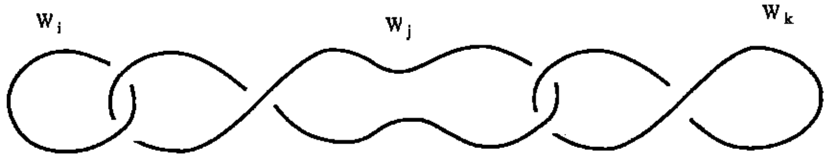

Figure 6

We may rewrite this sum as the invariant of the same zonent link, in which the central component has the element $\sum_{j=1}^{r-1} \delta_j W_j = U$ attached, while the other two strings have W_i and W_k respectively. Make a positive Kirby move on this central string, to get a new framed link L, as in figure 7, and then $c_{ik} = c_+ J_r(L; W_i, W_k)$, by theorem 2.6. Now L is a Hopf link, with altered framing, and its invariant $J_r(L; W_i, W_k) = J_r(H; F_r^{-1}(W_i), W_k)$ is the ikth entry of $F_r^{-1}H_r$.

Then $(H_rF_r)^2 = c_+F_r^{-1}H_r$, giving $(H_rF_r)^3 = c_+\rho(a)^2 I$. □

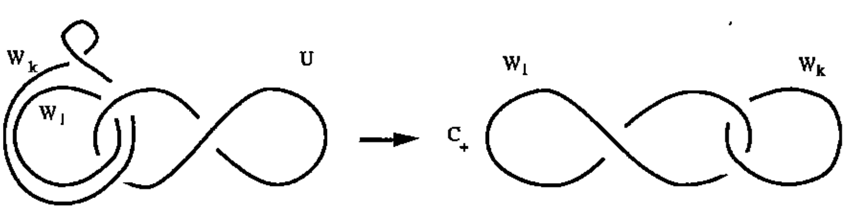

Figure 7

We now give an alternative proof of theorem 2.9, using the same methods.

This proof does not use explicit knowledge of the Hopf link invariants, and can be modified to give a similar result for other quantum groups, with a permutation matrix (of order 2) in place of the identity.

Alternative proof of theorem 2.9: As in the proof of 3.3, we can write the ikth element, c_{ik}, of H_r^2 as the invariant $J_r(L; W_i, U, W_k)$ of a 3-component link, which is the connected sum of two Hopf links, this time with zero framing on all components, as shown in figure 8.

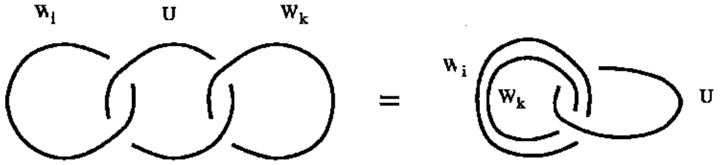

Figure 8

Now this 3-component link is the 2-parallel of a Hopf link, in which one of the components is replaced by two, carrying the elements W_i and W_k. By the result for invariants of parallels, we can then write

$$c_{ik} = J_r(L; W_i, U, W_k) = J_r(H; W_i W_k, U) = \sum_{j \in i \otimes k} J_r(H; W_j, U).$$

Since $1 \in i \otimes k$ if and only if $i = k$, for $1 \leq i, k \leq r-1$, it is enough to show that $J_r(H; W_j, U) = 0$ when $1 < j \leq r-1$, giving $c_{ik} = 0$ for $i \neq k$ and $H_r^2 = J_r(H; W_1, U)I$. Now $J_r(H; W_j, U) = J_r(H; F_r(W_j), U) = e_r(f_j) J_r(H; W_j, U)$, using theorem 2.8, and $e_r(f_j) = a^{j^2-1} = 1$ in Λ_r for $j \leq r-1$ if and only if $j = 1$, at least for r prime. An explicit proof that $J_r(H; W_j, U) = 0$ for $j \neq 1$ is needed to complete the proof in general by this method.

As for the earlier proof of theorem 2.9, it now follows, knowing that H_r^2 is a scalar matrix, that the scalar is $\rho(a)^2$. This gives another calculation for $\rho(a)^2$ since $J_r(H; W_1, U) = \sum_{j=1}^{r-1} \delta_j^2$, which is clearly the product of the first row and column of H_r. □

For general simple Lie algebras, and hence for their quantum groups by [Ro], it can be shown that if U, V and W are three irreducible representations, then U is a summand of $V \otimes W$ if and only if $V^\dagger$ is a summand of $W \otimes U^\dagger$, where $\dagger$ denotes the conjugate representation. Thus the one dimensional representation will be contained as a summand of $V \otimes W$ exactly when V and W are conjugate. In the case of quantum groups having representations which are not self conjugate (so that the invariants are orientation dependent), this has the consequence that the matrices H_r do not square to a scalar, but to a multiple of the permutation matrix P_r which interchanges conjugate modules. On the other hand, it is still true that the universal module is a scalar times the sum of $\delta_W W$ as W runs over the relevant irreducibles; this follows because the δ for W is identical to that for $W^\dagger$, as changing the orientation of an unknotted component does not alter a link.

In particular, following the proof of theorem 3.3 will show that $(H_r F_r)^3$ is now a multiple of P_r. It can be seen that the theorems 3.1 and 3.2 will also go through for other quantum groups, as P_r commutes with H_r and F_r, and hence does not affect the proof in any essential way.

4. General Dehn surgery

In the final section we indicate how the modular group action allows us to handle the manifold invariant for a manifold given from a framed link L by general Dehn surgery. The principle adopted is to regard the invariant $J_r(L)$ of a link as an invariant of a 3-manifold with boundary components which carry a choice of parallel and meridian. In the spirit of Segal's views of Witten's work it is appropriate to think of the map $J_r(L) : \mathcal{R}_r^{\otimes k} \to \Lambda_r$ as determined by the link exterior; each boundary component, with chosen parallel and meridian coordinates, having associated with it a copy of $\mathcal{R}_r$. Evaluation of $J_r(L)$ at $W_1 \in \mathcal{R}_r$ for a given boundary component gives the invariant of the link with that boundary component removed, and can be thought of as the manifold given by attaching a solid torus whose meridian disc spans the meridian of the boundary torus.

To perform any other surgery, say on a (p, q) curve (relative to the meridian and parallel coordinates on L), we apply an automorphism of the modular group, as represented by F_r and Φ_r on $\mathcal{R}_r$, to the appropriate copy of $\mathcal{R}_r$ which will carry the meridian $(1, 0)$ to the (p, q) curve before evaluating $J_r(L)$ at W_1. Thus the invariant of the new manifold, with one fewer boundary component, might be expected to be given by evaluating $J_r(L)$ on an element $M_{p/q} = \theta_{p/q}(W_1)$. In this notation we should write $W_1 = M_\infty$.

While there is some choice of automorphism to carry the meridian to the (p, q) curve there will be no ambiguity, apart from powers of $c(a)$, in the choice of $M_{p/q}$ because the automorphisms will differ on W_1 by an automorphism which carries the meridian to itself. These automorphisms are represented by powers of F_r, and $F_r(W_1) = W_1$. The precise choice will be governed by the fact that the signature of the generalised linking matrix of L, whose diagonal entries are now possibly fractional framings, may be changed under the third Rolfsen move described below; and this will affect the calculation of corollary 2.7.

We take the following to constitute the exact definition of M_α for each α. First, define $M_\infty = W_1$, and then let

$$\begin{aligned} F_r(M_\alpha) &= M_{\alpha+1}, \\ \Phi_r(M_\alpha) &= \begin{cases} M_{\alpha'} & \text{if } \alpha < 1 \\ c(a)^2 M_{\alpha'} & \text{if } \alpha > 1 \end{cases}, \text{ with } \alpha' = \frac{\alpha}{1-\alpha}. \\ \Phi_r(M_1) &= c(a) M_\infty \\ \Phi_r(M_\infty) &= c(a) M_{-1} \end{aligned}$$

To show that this leads to a well-defined choice, we must prove that

$$\Phi_r F_r \Phi_r(M_\alpha) = F_r \Phi_r F_r(M_\alpha) \qquad \text{and}$$
$$(F_r \Phi_r F_r)^2(M_\alpha) = c(a)^2 M_\alpha$$

for all α. To do this we shall temporarily allow F and Φ to stand for their counterparts in PSL$(2, \mathbf{Z})$ acting on $\mathbf{Q}\mathrm{P}^1$, so that we can represent the two products above by

$$\alpha \overset{\Phi}{\longmapsto} \frac{\alpha}{1-\alpha} \overset{F}{\longmapsto} \frac{1}{1-\alpha} \overset{\Phi}{\longmapsto} -\frac{1}{\alpha}$$

$$\alpha \overset{F}{\longmapsto} \alpha+1 \overset{\Phi}{\longmapsto} \frac{\alpha+1}{-\alpha} \overset{F}{\longmapsto} -\frac{1}{\alpha}$$

The second route will give rise to a factor of $c(a)^2$ exactly when $\alpha > 0$ (∞ is neither positive nor negative).The first route will do the same if $\alpha > 1$ (on the first application of Φ) or $1 > \alpha > 0$ (on the second); if $\alpha = 1$, the factor will also arise in two halfs. For $\alpha = \infty$ or 0, either route introduces a factor of $c(a)$.

For the second identity above, we follow the second route throughout. If α is finite and non-zero, then exactly one of α and $-\frac{1}{\alpha}$ is positive, so the factor $c(a)^2$ comes in; for $\alpha = 0$ or ∞ we get two separate factors of $c(a)$.

Definition. A 3-manifold is said to be given Dehn surgery on a *framed* k-component link L, with surgery coefficients $(\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathbf{Q} \cup \{\infty\}$, when it is constructed by gluing a solid torus to each boundary component of the link exterior along a curve of slope α_i relative to the meridian and chosen parallel for the ith component.

Thus slope ∞ will always refer to the meridian, while slope 0 will give the parallel chosen by the framing.

Rolfsen [R] shows that if two oriented manifolds given by Dehn surgeries on links L, L' are homeomorphic then L and L' are related by a sequence of moves of three types. (Rolfsen only used framing zero, but the modifications for arbitrary framing are straightforward.)

I. Change the framing, and the surgery coefficients so that the underlying link and surgery curves are unchanged. This has the effect of adding or subtracting an integer to the surgery coefficient when the framing is changed on a component.

II. Add or remove a component with surgery coefficient ∞.

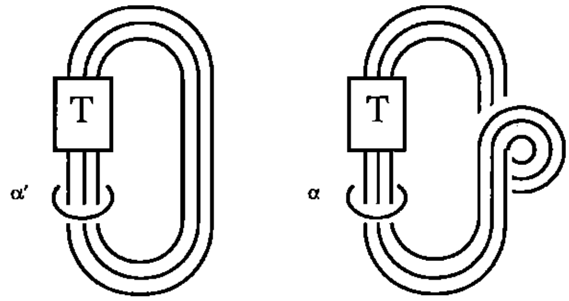

Figure 9

III. Replace a link L which has one distinguished unknotted component of framing 0 by L' as shown in figure 9. The framing in each diagram is planar (so that the framing on components has been changed depending on their linking number with the unknotted curve) and the surgery coefficients are unchanged on all except the unknotted component, where the coefficient α' becomes $\alpha = \dfrac{\alpha'}{1+\alpha'}$.

Theorem 4.1. *Let a closed 3-manifold be given by Dehn surgery on a framed k-component link L with surgery coefficients $\alpha_1, \ldots, \alpha_k$ relative to the framing coordinates on L. Then the invariant of the manifold can be calculated as $c(a)^{\mathrm{sig}(L)} J_r(L; M_{\alpha_1}, \ldots, M_{\alpha_k})$, where $sig(L)$ is calculated as the signature of the linking matrix with the absolute surgery coefficients (i.e. relative to the topological framings) down the diagonal, omitting any components with infinite surgery coefficients.*

In order to prove this result, we will need the following lemma

Lemma 4.2. *Let A and A' be the $k \times k$ matrices*

$$\begin{pmatrix} \alpha & \ell^T \\ \ell & B \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \alpha' & \ell^T \\ \ell & B - \ell\ell^T \end{pmatrix}$$

where $\alpha' = \frac{\alpha}{1-\alpha}$, $\alpha \neq 1$ and ℓ and B are $(k-1)\times 1$ and $(k-1)\times(k-1)$ matrices respectively. Then the signatures of these matrices are related by

$$\mathrm{sig}(A) = \mathrm{sig}(A') + \mathrm{sign}(\alpha) - \mathrm{sign}(\alpha').$$

Proof: Let P be the matrix $\begin{pmatrix} 1 & 0 \\ -\alpha^{-1}\ell & I \end{pmatrix}$; then, for $\alpha \neq 0$,

$$PAP^T = \begin{pmatrix} \alpha & 0 \\ 0 & B - \alpha^{-1}\ell\ell^T \end{pmatrix}.$$

So $\mathrm{sig}(A) = \mathrm{sign}(\alpha) + \mathrm{sig}(B - \alpha^{-1}\ell\ell^T)$. Similarly, $\mathrm{sig}(A') = \mathrm{sign}(\alpha') + \mathrm{sig}(B - \ell\ell^T - \alpha'^{-1}\ell\ell^T)$. But $1 + \alpha'^{-1} = \alpha^{-1}$, so the last term in each of these expressions is the same, proving the lemma for $\alpha \neq 0$.

When $\alpha = 0$, we take $P = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\ell & I \end{pmatrix}$, so that $PAP^T = A'$, completing the proof. □

Proof of theorem 4.1: We use Rolfsen's moves on framed links to pass between surgery descriptions of a manifold.

It is readily seen that the first two moves leave the invariant unaltered, since with the definitions chosen $M_\infty = W_1$ has the effect of ignoring a component, while the reframing works because $F_r(M_\alpha) = M_{\alpha+1}$. Neither of these affect the generalised linking matrix.

When move III is applied the two links L and L' will yield invariants $< X, M_{\alpha'} >_r$ and $< F_r(X), M_\alpha >_r$, for some X, by the satellite theorem. Now

$$< F_r(X), M_\alpha >_r = < X, \Phi_r(M_\alpha) >_r$$

by the adjoint property of F_r and Φ_r, and we have $\Phi_r(M_\alpha) = M_{\alpha'}$ (up to powers of $c(a)$). Any discrepancy in the power of $c(a)$ is compensated for by a change in the signature of the generalised linking matrix, as follows. Let A and A' be the matrices for the two links in figure 9; then these are related as in lemma 4.2. For $\alpha < 1$ we have $\text{sign}(\alpha) = \text{sign}(\alpha')$ and $\Phi_r(M_\alpha) = M_{\alpha'}$. For $\alpha > 1$ we have $\text{sign}(\alpha) = \text{sign}(\alpha') + 2$, compensating for the fact that $\Phi_r(M_\alpha) = c(a)^2 M_{\alpha'}$. Finally, for $\alpha = 1, \alpha' = \infty$ the method of the lemma shows that $\text{sig}(A) = 1 + \text{sig}(B - \ell\ell^T)$; and for $\alpha = \infty, \alpha' = -1$ we have $\text{sig}(A') = -1 + \text{sig(B)}$, which deals with the special cases. □

We have been able to make some calculations for lens spaces given both by framed surgery on torus knots, and also by Dehn surgery on the unknot, which confirm the above result.

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