Quantum estimation of coupled parameters and the role of entanglement

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The quantum Cramér-Rao bound places a limit on the mean square error of a parameter estimation procedure, and its numerical value is determined by the quantum Fisher information. For single parameters, this leads to the well-known Heisenberg limit that surpasses the classical shot-noise limit. When estimating multiple parameters, the situation is more complicated and the quantum Cramér-Rao bound is generally not attainable. In such cases, the use of entanglement typically still offers an enhancement in precision. Here, we demonstrate that entanglement is detrimental when estimating some nuisance parameters. In general, we find that the estimation of coupled parameters does not benefit from either classical or quantum correlations. We illustrate this effect in a practical application for optical gyroscopes.

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Quantum metrology and quantum parameter estimation offer great potential improvements in precision measurements. Recent experiments have demonstrated quantum improvements in measuring protein concentration [1], tracking lipid granules in yeast cells [2], and searching for gravitational waves [3]. In the longer term, the aim is to improve the precision of measurements from the classical shot noise limit (SNL) to the quantum mechanical Heisenberg limit (HL) [4]. It has been recognised that any practical implementation of quantum metrology requires methods to deal with effects due to environmental noise and dissipation [5]. Quantum error correction has been proposed for combating noise [6–8], and loss-tolerant metrology protocols have been designed and implemented to address some of the negative effects of dissipation [9–12]. It was shown recently that the measurement of d phases in an interferometer can obtain an improvement of a factor $O(d)$ in the precision when multi-mode entanglement is used [13]. This behaviour persists in the presence of photon loss [14], even though multi-mode entanglement is highly susceptible to loss [15]. When loss is also estimated, there is a trade-off between the attainable precision of the phase estimation and the loss estimation [16]. However, loss and noise are not the only causes for imperfect metrology. Other sources of imperfection can include badly characterised responses to non-standard stimuli, or couplings between the parameters of interest. The performance of any larger scale system—i.e., one containing a number of individual sensors—will be limited by the presence of such nuisance parameters. The accuracy of a composite sensor system is only partially determined by the precision of the individual measurements, but this aspect of quantum metrology has been somewhat overlooked.

In this Letter, we address the problem of nuisance parameters arising from unwanted coupling in practical quantum parameter estimation. Such couplings affect the measurement precision—defined by the mean square error (MSE)—and must also be estimated, even if we are ultimately not interested in their numerical value [17]. For a single parameter, the quantum Cramér-Rao bound (QCRB) puts a lower limit on the MSE, determined by the inverse of the quantum Fisher information (QFI) [18, 19]. Multiple parameters lead to a QFI matrix, the inverse of which provides lower bounds for the MSE covariance matrix [20, 21]. Nuisance parameters are part of this multi-parameter estimation. While the QCRB for a single parameter can generally be attained, this is not always true of the QCRB for multiple parameters [22, 23]. Where multiple parameters are being estimated, it matters whether the generators of translation of the parameters commute or not, with implications for the optimal strategies of the parameter estimation procedures [24–29]. Even though multi-mode entanglement can be used to improve the estimation of multiple phase parameters beyond the classical SNL [13], the question is whether entanglement can be used to improve the precision of general multi-parameter estimation including nuisance parameters. Here, we show that this is not the case.

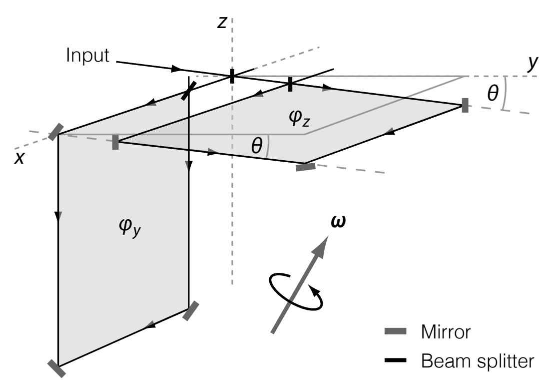

FIG. 1: Two nearly orthogonal (coupled) Sagnac interferometers that are misaligned by an angle θ. The entire system rotates with angular velocity $\boldsymbol{\omega}$, and the resulting phase shifts φ_y and φ_z in the interferometers can be used to estimate $\boldsymbol{\omega}$. For clarity, the third interferometers measuring φ_x and the photodetectors are omitted.

We consider the practical situation of a (simplified) optical gyroscope based on two Sagnac interferometers, shown in Fig. 1, in which the two orthogonal interferometers are misaligned by a small angle θ. We find that both entanglement and classical correlations in the quantum state between the two interferometers do not help the estimation of θ and the phases of interest, φ_y and φ_z. We will argue that this is very general behaviour that does not depend on the commutation relation of the generators of the parameters. Our result is an important waypoint in the development of the general quantum theory of multi-parameter estimation.

The Sagnac interferometer [30] can be described quantum mechanically in a very similar way to the Mach-Zehnder interferometer, but instead of two spatially different paths in the latter, the Sagnac interferometer has a single loop with two counter-propagating modes, a and b. The phase shift induced by a rotation of the interferometer can be written as a unitary transformation

$$U(t) = \exp\left[-i\boldsymbol{\omega}\cdot\mathbf{e}\left(\hat{n}_a - \hat{n}_b\right)t\right] , \tag{1}$$

where $\boldsymbol{\omega}$ is the rotation vector of the interferometer, and $\mathbf{e}$ is the normal vector to the plane of the interferometer. For simplicity we assume that the Sagnac interferometer lies entirely in the xy-plane ($\mathbf{e} = \hat{\mathrm{e}}_z$), and we define $\varphi_j \equiv \omega_j t$ and $\hat{n} \equiv \hat{n}_a - \hat{n}_b$. Then we can write the transformation in Eq. (1) as $U(\varphi_z) = \exp\left[-i\varphi_z\hat{n}\right]$. In the usual notation where $U = \exp(-iHt/\hbar)$, the Hamiltonian becomes $H = \hbar\omega_z\hat{n}$, and we will now set $\hbar = 1$. Clearly, measurements of the phase φ_z can be used to determine the rotation ω_z of the gyroscope. We can construct a second Sagnac interferometer in the xz-plane ($\mathbf{e} = \hat{\mathrm{e}}_y$) to determine the rotation ω_y, and a third to determine ω_x. The use of three such gyroscopes allows a general rotation about an arbitrary axis to be determined [31]. Here, we will limit the discussion to two interferometers, but the extension to three interferometers is straightforward.

In any practical construction, the two Sagnac interferometers will not be perfectly perpendicular (and when the interferometer is constructed from optical fibres it may not lie perfectly in a plane). Let $\hat{n}_y$ be the number difference operator for the counter-propagating modes of the interferometer in the xz-plane, and $\hat{n}_z$ the equivalent operator for the interferometer in the xy-plane. Furthermore, let θ be the angle with which the φ_z interferometer is misaligned: $\hat{\mathrm{e}}_z' = \cos\theta\,\hat{\mathrm{e}}_z + \sin\theta\,\hat{\mathrm{e}}_y$. The transformation of the optical state inside the interferometers then becomes

$$U(\boldsymbol{\varphi}) = \exp\left[-i\left(\varphi_y\hat{n}_y + \cos\theta\,\varphi_z\hat{n}_z + \sin\theta\,\varphi_y\hat{n}_z\right)\right] , \tag{2}$$

leading to a Hamiltonian for the system $H = \omega_y\hat{n}_y + \cos\theta\;\omega_z\hat{n}_z + \sin\theta\;\omega_y\hat{n}_z$. There is now a coupling between the two interferometers given by the term $\sin\theta\,\omega_y\hat{n}_z$. We can also include a tilt ϕ of the interferometer towards the yz-plane for greater generality. This induces a term $\sin\phi\;\omega_x\hat{n}_z$ in the Hamiltonian and a multiplicative factor $\cos\phi$ on $\omega_z\hat{n}_z$.

If we wish to estimate $\boldsymbol{\omega}$, we not only have to measure the individual phases φ_y and φ_z, but also the coupling between the two interferometers, parameterised by θ. To this end, we define the generators of translation in the phases φ_y and φ_z, as well as the generator of translations in the nuisance parameter θ. Generally, the generator of translation G_α of a parameter α can be defined as $G_\alpha \equiv iU^\dagger\partial_\alpha U$ [32]. Applying this to $U(\boldsymbol{\varphi})$ in Eq. (2) for the parameters φ_y, φ_z, and θ, we obtain the generators

$$\begin{aligned} G_y &= \hat{n}_y + \sin\theta\;\hat{n}_z\,, \\ G_z &= \cos\theta\;\hat{n}_z\,, \\ G_\theta &= (\varphi_y\cos\theta - \varphi_z\sin\theta)\hat{n}_z \equiv \beta\hat{n}_z\,. \end{aligned} \tag{3}$$

We will use these generators in the next section, when we derive a Cramér-Rao bound for the estimation of φ_y, φ_z, and θ. Note that, in this case, all generators commute.

To determine the ultimate precision with which we can estimate the Sagnac phases and the coupling between them ($\boldsymbol{\varphi} \equiv (\varphi_y, \varphi_z, \theta)$), we consider the quantum Cramér-Rao bound

$$\textbf{Cov}(\boldsymbol{\varphi}) \geq \frac{1}{N}\,\mathbf{I}_Q^{-1}(\boldsymbol{\varphi})\,, \tag{4}$$

where $\textbf{Cov}(\boldsymbol{\varphi})$ is the covariance matrix of the three variables φ_y, φ_z, and θ, N is the number of independent measurements, and $\mathbf{I}_Q(\boldsymbol{\varphi})$ is the quantum Fisher information (QFI) matrix of the three variables [18, 33] with elements:

$$[\mathbf{I}_Q(\boldsymbol{\varphi})]_{ij} = 2\partial_i\partial_{\tilde{j}}\log\left|\langle\psi(\boldsymbol{\varphi})|\psi(\tilde{\boldsymbol{\varphi}})\rangle\right|^2_{\tilde{\boldsymbol{\varphi}}=\boldsymbol{\varphi}}\,, \tag{5}$$

where ∂_i is the derivative with respect to φ_i, and $\partial_{\tilde{j}}$ the derivative with respect to $\tilde{\varphi}_j$. Evaluating the matrix elements of the QFI matrix for pure states $|\psi\rangle$ then yields

$$\begin{aligned} [\mathbf{I}_Q(\boldsymbol{\varphi})]_{ij} &= 4\left(\frac{1}{2}\left\langle\{G_i, G_j\}\right\rangle_{\boldsymbol{\varphi}} - \langle G_i\rangle_{\boldsymbol{\varphi}}\langle G_j\rangle_{\boldsymbol{\varphi}}\right) \\ &\equiv 4[\mathbf{C}_S(\mathbf{G})]_{ij}\,, \end{aligned} \tag{6}$$

where $\langle O\rangle_{\boldsymbol{\varphi}} \equiv \langle\psi(\boldsymbol{\varphi})|O|\psi(\boldsymbol{\varphi})\rangle$ for some operator O, and $[\mathbf{C}_S(\mathbf{G})]_{ij}$ is the symmetrized covariance matrix element between operators G_i and G_j, originating from the fact that the quantum Fisher information matrix in equation (5) is derived from the symmetric logarithmic derivative [18]. Since all our generators commute with each other, we can ignore this technical requirement and drop the subscript S.

The diagonal elements of $\textbf{Cov}(\boldsymbol{\varphi})$ are the variances of the parameters of interest, namely $(\Delta\varphi_y)^2$, $(\Delta\varphi_z)^2$, and the nuisance parameter $(\Delta\theta)^2$. We can choose to optimise any one of these variances, two of them, or all three. In the latter case, we need to choose a quantum state that minimises

$$\mathrm{Tr}[\textbf{Cov}(\boldsymbol{\varphi})] \geq \frac{1}{N}\mathrm{Tr}\left[\mathbf{I}_Q^{-1}(\boldsymbol{\varphi})\right]. \tag{7}$$

The right-hand side of Eq. (7) provides a bound on the optimal joint estimation of φ_y, φ_z, and θ that may be achieved in the asymptotic limit of large N.

Based on the generators in Eq. (3), the QFI matrix $\mathbf{I}_Q(\boldsymbol{\varphi})$ becomes singular. This means that we cannot evaluate the usual

Cramér-Rao bound in Eq. (4) because the inverse does not exist. In general, when the Fisher information matrix is singular, unbiased estimators do not exist [34, 35] (this is true for the both the standard Fisher information matrix and the QFI matrix, since the QFI is merely the optimal Fisher information over all possible quantum measurements, and leaves the derivation of the Cramér-Rao bound unaffected). However, when we consider constraints on the parameters, we can use the Moore-Penrose generalised inverse of the QFI matrix to obtain a Cramér-Rao-like inequality that yields the minimum variance of φ over all possible (minimum) constraint functions [36]. The new Cramér-Rao-like inequality then becomes $\mathrm{Tr}[\mathbf{Cov}(\boldsymbol{\varphi})] \geq N^{-1}\mathrm{Tr}\left[\mathbf{I}_Q^{-\mathscr{I}}(\boldsymbol{\varphi})\right]$, where $\mathbf{I}_Q^{-\mathscr{I}}$ is the Moore-Penrose generalised inverse of $\mathbf{I}_Q$. The new Cramér-Rao-like inequality is a *constrained* Cramér-Rao bound, given certain constraints on the parameters $\boldsymbol{\varphi}$ [36]. Unfortunately, for the optical gyroscope considered here the constraints take a rather complicated form that is not very instructive.

The QFI based on the generators in Eq. (3) becomes the singular matrix

$$\mathbf{I}_Q(\boldsymbol{\varphi}) = 4\begin{pmatrix} (\Delta\hat{n}_y)^2 + \sin^2\theta(\Delta\hat{n}_z)^2 + 2\sin\theta\,\mathbf{C}(\mathbf{G})_{yz} & \cos\theta[\mathbf{C}(\mathbf{G})_{yz} + \sin\theta(\Delta\hat{n}_z)^2] & \beta[\mathbf{C}(\mathbf{G})_{yz} + \sin\theta(\Delta\hat{n}_z)^2] \\ \cos\theta[\mathbf{C}(\mathbf{G})_{yz} + \sin\theta(\Delta\hat{n}_z)^2] & \cos^2\theta(\Delta\hat{n}_z)^2 & \beta\cos\theta(\Delta\hat{n}_z)^2 \\ \beta[\mathbf{C}(\mathbf{G})_{yz} + \sin\theta(\Delta\hat{n}_z)^2] & \beta\cos\theta(\Delta\hat{n}_z)^2 & \beta^2(\Delta\hat{n}_z)^2 \end{pmatrix}, \quad (8)$$

whose generalised inverse has the diagonal elements

$$\begin{aligned} [\mathbf{I}_Q^{-\mathscr{I}}(\boldsymbol{\varphi})]_{yy} &= \frac{1}{4}\frac{(\Delta\hat{n}_z)^2}{(\Delta\hat{n}_y)^2(\Delta\hat{n}_z)^2 - \mathbf{C}(\mathbf{G})_{yz}^2}, \\ [\mathbf{I}_Q^{-\mathscr{I}}(\boldsymbol{\varphi})]_{zz} &= \frac{1}{2}\frac{2(\Delta\hat{n}_y)^2 + (1-\cos 2\theta)(\Delta\hat{n}_z)^2 + 4\sin\theta\,\mathbf{C}(\mathbf{G})_{yz}}{(\Delta\hat{n}_y)^2(\Delta\hat{n}_z)^2 - \mathbf{C}(\mathbf{G})_{yz}^2}\frac{\cos^2\theta}{(1+2\beta^2+\cos 2\theta)^2}, \\ [\mathbf{I}_Q^{-\mathscr{I}}(\boldsymbol{\varphi})]_{\theta\theta} &= \frac{1}{2}\frac{2(\Delta\hat{n}_y)^2 + (1-\cos 2\theta)(\Delta\hat{n}_z)^2 + 4\sin\theta\,\mathbf{C}(\mathbf{G})_{yz}}{(\Delta\hat{n}_y)^2(\Delta\hat{n}_z)^2 - \mathbf{C}(\mathbf{G})_{yz}^2}\frac{\beta^2}{(1+2\beta^2+\cos 2\theta)^2}. \end{aligned} \quad (9)$$

The total precision over the three parameters is then bounded by

$$\mathrm{Tr}[\mathbf{Cov}(\boldsymbol{\varphi})] \geq \frac{1}{N}\mathrm{Tr}\left[\mathbf{I}_Q^{-\mathscr{I}}(\boldsymbol{\varphi})\right] = \frac{1}{2N}\frac{(\Delta\hat{n}_y)^2 + (\Delta\hat{n}_z)^2(1-\beta^2) + 2\sin\theta\,\mathbf{C}(\mathbf{G})_{yz}}{(\Delta\hat{n}_y)^2(\Delta\hat{n}_z)^2 - \mathbf{C}(\mathbf{G})_{yz}^2}\frac{1}{1+2\beta^2+\cos 2\theta}. \quad (10)$$

From the expressions in Eqs. (9) we can deduce that the minimum errors are obtained when $\mathbf{C}(\mathbf{G})_{yz} = 0$ in the relevant limit $(\Delta\hat{n}_y)^2,\ (\Delta\hat{n}_z)^2 \to \infty$ (see Fig. 2). Since it is well-known that maximising $(\Delta\hat{n})^2$ maximises the phase sensitivity of the individual Sagnac interferometer (for example using NOON states [37]), the optimal estimation of the parameters governing the coupled interferometers can be achieved by estimating the phases φ_y and φ_z in each interferometer separately, leading to maximal $(\Delta\hat{n}_y)^2$ and $(\Delta\hat{n}_z)^2$, and $\mathbf{C}(\mathbf{G})_{yz} = 0$. To find θ, the outcomes of φ_y and φ_z must be correlated classically for a suitably chosen set of rotations $\boldsymbol{\omega}$. In other words, the phase estimation achieves maximum precision when there are *no correlations*, either classical or quantum, in the joint state inside the Sagnac interferometers with respect to the phase generators. Therefore, entanglement cannot help us to obtain a greater precision in determining the coupling between the two Sagnac interferometers.

In the limit $\theta \to 0$, the angle β reduces to φ_y. We are interested in small angles φ_y, so we evaluate the covariances at $\varphi_y = 0$. Then the constrained Cramér-Rao bound reduces to the bounds for individual uncoupled Sagnac interferometers:

$$\mathbf{Var}(\varphi_j) \geq \frac{1}{4N(\Delta\hat{n}_j)^2}, \quad (11)$$

where $j = y, z$, and $\mathbf{Var}(\varphi_j)$ is the MSE in the phase φ_j. For these values the MSE in θ becomes zero, but since there is now no coupling between the interferometers, this quantity has become devoid of meaning.

The above discussion deals with a specific problem, but it reveals a rather general effect in estimating coupled parameters. For example, the form of the variances in Eq. (9) does not depend on the commutation relation between the generators G_y, G_z and G_θ; only the numerical value of $\mathbf{C}(\mathbf{G})_{yz}$ depends on $[G_y, G_z]$. The question arises whether the behaviour we observe persists for other couplings. To this end, consider the general generators G_1, G_2 and $G_\theta = \alpha R$ for the parameters φ_1, φ_2 and θ, respectively. The operators G_1, G_2 and G_θ do not necessarily commute and may themselves be functions of the parameters, and α is a function of φ_1, φ_2 and θ, such that $\alpha \to 0$ when the coupling is turned off (possibly taking limiting values on the parameters as well). The QFI takes a

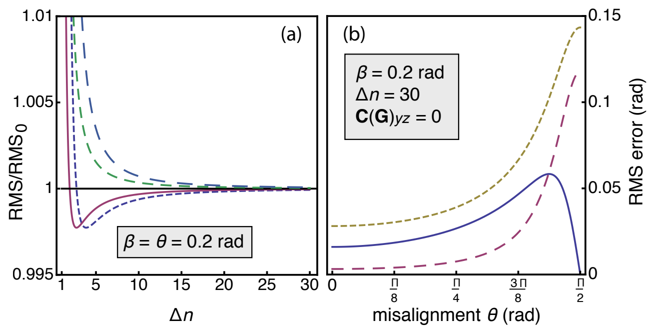

FIG. 2: Color online. (a) Ratio of the total RMS $= \sqrt{\mathrm{Tr}[\mathbf{Cov}(\boldsymbol{\varphi})]}$ for $\mathbf{C}(\mathbf{G})_{yz} = -1$ (lower dashed), -0.5 (lower solid), 0.5, 1 (top) to the value of RMS$_0$ for $\mathbf{C}(\mathbf{G})_{yz} = 0$. Note that for large deviation in the photon number difference $\Delta n = \Delta\hat{n}_y = \Delta\hat{n}_z$, the RMS reduces to RMS$_0$. (b) RMS for φ_z (solid), θ (lower dashed), and the total RMS (top) as a function of the misalignment angle θ.

particularly simple form:

$$\mathbf{I}_Q(\boldsymbol{\varphi}) = 4\begin{pmatrix} (\Delta G_1)^2 & \mathbf{C}(\boldsymbol{G})_{12} & \alpha\,\mathbf{C}(\boldsymbol{G})_{1\theta} \\ \mathbf{C}(\boldsymbol{G})_{12} & (\Delta G_2)^2 & \alpha\,\mathbf{C}(\boldsymbol{G})_{2\theta} \\ \alpha\,\mathbf{C}(\boldsymbol{G})_{1\theta} & \alpha\,\mathbf{C}(\boldsymbol{G})_{2\theta} & \alpha^2(\Delta R)^2 \end{pmatrix}. \qquad (12)$$

Inverting this matrix yields the RMS errors on the parameters:

$$\begin{aligned} \mathbf{Var}(\varphi_1) &\geq \frac{(4\alpha)^2}{N}\,\frac{(\Delta G_2)^2(\Delta R)^2 - \mathbf{C}(\boldsymbol{G})_{2\theta}^2}{\det(\mathbf{I}_Q)}\,, \\ \mathbf{Var}(\varphi_2) &\geq \frac{(4\alpha)^2}{N}\,\frac{(\Delta G_1)^2(\Delta R)^2 - \mathbf{C}(\boldsymbol{G})_{1\theta}^2}{\det(\mathbf{I}_Q)}\,, \\ \mathbf{Var}(\theta) &\geq \frac{(4\alpha)^2}{N}\,\frac{(\Delta G_2)^2(\Delta G_1)^2 - \mathbf{C}(\boldsymbol{G})_{12}^2}{\det(\mathbf{I}_Q)}\,, \end{aligned} \qquad (13)$$

where $\det(\mathbf{I}_Q)$ denotes the determinant of the QFI matrix. When this is zero, as above, the Moore-Penrose generalised inverse must be used. For general forms of G_1, G_2 and R the optimal choice will not be $\mathbf{C}(\boldsymbol{G})_{ij} = 0$ for $i \neq j$. However, R may not be completely independent of G_1 and G_2 (as in the example of the coupled Sagnac interferometers), which can have a profound effect on the parameter estimation procedure.

In conclusion, we have studied a practical application of quantum metrology in optical gyroscopes. When we account for effects that couple the parameters of interest, we find that entanglement and classical correlations provide *no benefits* to estimating the coupling. It has been shown recently that multi-mode entanglement is more beneficial than bi-partite entanglement for certain multi-parameter estimation protocols [13], a fact which suggests that entanglement generally provides an advantage in multi-parameter estimation. Our result suggests a more complicated picture in which entanglement can play a positive or a negative role. The potential for quantum enhanced estimation depends critically on the geometry of the system described by the parameters. For practical applications—which inevitably include nuisance parameters—it is an open question whether and how any correlations, classical or quantum, can be used effectively to improve precision and accuracy of the metrology protocol.

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