# Optimal and Robust Control for a Class of Nonlinear Stochastic Systems

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# Abstract

This thesis focuses on theoretical research of optimal and robust control theory for a class of nonlinear stochastic systems. The nonlinearities that appear in the diffusion terms are of a square-root type. Under such systems the following problems are investigated: optimal stochastic control in both finite and infinite horizon; robust stabilization and robust $H_\infty$ control; $H_2/H_\infty$ control in both finite and infinite horizon; and risk-sensitive control. The importance of this work is that explicit optimal linear controls are obtained, which is a very rare case in the nonlinear system. This is regarded as an advantage because with explicit solutions, our work becomes easier to be applied into the real problems. Apart from the mathematical results obtained, we have also introduced some applications to finance.

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This thesis is a result of my own original work and no collaboration outcome is included in it.

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# Contents

# Notation

$\mathbf{0}$: zero matrix, with appropriate dimensions;

$\mathbf{I}$: identity matrix, with appropriate dimensions;

w.r.t.: with respect to;

$M'$: the transpose of any matrix or vector $M$;

$M^\dagger$: the Moore-Penrose pseudo-inverse of a matrix $M$;

$M > 0$: the symmetric matrix $M$ is positive definite;

$M \geq 0$: the symmetric matrix $M$ is positive semidefinite;

$\mathbb{R}^n$: the $n$-dimensional Euclidean space;

$\mathbb{R}^{n\times m}$: the set of all $n \times m$ matrices;

$\mathcal{S}^n$: the set of all $n \times n$ symmetric matrices;

$\mathcal{S}^n_+$: the subset of all nonnegative definite matrices of $\mathcal{S}^n$;

$\widehat{\mathcal{S}}^n_+$: the subset of all positive definite matrices of $\mathcal{S}^n$;

$(\mathcal{S}^n)^l$: $= \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{l}$;

$(\mathcal{S}^n_+)^l$: $= \underbrace{\mathcal{S}^n_+ \times \cdots \times \mathcal{S}^n_+}_{l}$;

$(\widehat{\mathcal{S}}^n_+)^l$: $= \underbrace{\widehat{\mathcal{S}}^n_+ \times \cdots \times \widehat{\mathcal{S}}^n_+}_{l}$;

$C(0,T;\mathbb{R}^{n\times m})$: the set of continuous functions $\phi : [0,T] \to \mathbb{R}^{n\times m}$;

$C^1(0,T;(\mathcal{S}^n)^l)$: the set of continuously differential functions $\phi : [0,T] \to (\mathcal{S}^n)^l$);

$L^p(0,T;\mathbb{R}^{n\times m})$:the set of continuous functions $\phi : [0,T] \to \mathbb{R}^{n\times m}$ such that $\int_0^T |\phi(t)|^p dt < \infty$ where $p \in [1,\infty)$;

$L^\infty(0,T;\mathbb{R}^{n\times m})$:the set of essentially bounded measurable functions $\phi : [0,T] \to \mathbb{R}^{n\times m}$;

$\mathcal{L}^2_{\mathcal{F}}(\mathcal{R}_+, \mathcal{R}^l)$: space of nonanticipative stochastic processes $y(t) \in \mathcal{R}^l$ with respect to an increasing $\sigma$-algebras $\mathcal{F}_t$ $(t \geq 0)$ satisfying $\mathbb{E}\int_0^{\infty} ||y(t)||^2 dt < \infty$;

$\mathcal{L}^2_{\mathcal{F}}([0,T], \mathcal{R}^l)$: space of nonanticipative stochastic processes $y(t) \in \mathcal{R}^l$ with respect to an increasing $\sigma$-algebras $\mathcal{F}_t$ $(t \geq 0)$ satisfying $\mathbb{E}\int_0^{T} ||y(t)||^2 dt < \infty$;

$L_2[0,\infty)$: the space of square-integrable vector functions over $[0,\infty)$;

$L^{\infty}(0,T;\mathbb{R}^{n\times m})$:the set of essentially bounded measurable functions $\phi : [0,T] \to \mathbb{R}^{n\times m}$;

$|\cdot|$: the Euclidean norm for vectors or the trace norm for matrices;

$\|\cdot\|_2$: the usual $L_2[0,\infty)$ norm;

$\mathrm{tr}(M)$: the trace of any square matrix $M$;

$|M|$ : $\sqrt{\mathrm{tr}(MM')}$;

$\chi_A$: the indicator function of a set $A$;

$\mathrm{diag}(a_1, a_2, \ldots, a_m)$: $m \times m$ diagonal matrix, in which the diagonal elements are $a_1, a_2, \ldots, a_m$.

# Chapter 1

# Introduction

## 1.1 Introduction

In this chapter a short literature review of the problem of optimal and robust control is presented. The main contributions of the thesis are outlined. A short introduction for each chapter is given.

## 1.2 The Problem of Stochastic Optimal and Robust Control

We live in an era in which science and technology are developing rapidly, and new technology has introduced higher requirements for automation, which appears in space aircraft, artificial intelligence machinery, automobile making etc. Therefore, the theory of system and control faces more challenges under such circumstances.

The so-called control system includes a controlled plant and a controller. If someone is given the mathematical model of the system, a corresponding controller can be designed according to the properties and the cost functional of the system, and this is a control problem. Uncertainty appears in the real world almost everywhere, and it brings some disadvantages to human beings in their activities.

The key to control theory is feedback. In modern control theory, feedback is treated as a tool to handle uncertainty. In engineering, admissible control input can be adjusted according to the difference between measured output and reference quantity. By doing this, we can make sure that the systems have correct response and dynamic activities without knowing the accurate dynamic response of some systems or faulty response caused by external disturbance. This is a fundamental characteristic of engineering systems, which are required to operate reliably and efficiently. Feedback control is used to ensure the system robustness under uncertain circumstances. Therefore, feedback control systems become widely used in human beings' daily life, for example, automobile, manufacturing factories, communicating systems, military equipments and space systems.

The terminology optimal control theory was proposed about half a century ago. In optimal control theory, if a given system is required to achieve a certain optimal criterion, mathematical optimization method is used to derive certain control laws. Among various classes of optimal control problems, Kalman [62] has made great contributions in investigating the optimal linear quadratic (LQ) regulator problem. Optimal LQ control is one of the fundamental problems in the fields including mathematics, engineering, finance etc. There is a famous book on the topic of optimal LQ control, see [7].

In finance, solutions to stochastic differential equations (SDEs) can be used to model foreign currency exchange rates, interest rates, and stock prices. Stochastic control systems, which are governed by Itô differential equation, appear in many applications. For example, in real financial situation, the state variable in SDEs is usually wealth, and the control is trading strategy.

Here we illustrate some works from literatures on practical applications of stochastic optimal control problems as follows. The stochastic production planning problem was investigated by [12]. The continuous time portfolio consumption model was formulated and solved in [86] and [87]. When stochastic optimal control is applied to the field of insurance, the problems of dividend management were studied in [101]. SDEs are also appropriate to model technology diffusion problems, see for example [71], [97], [98], and [34]. In addition, queueing systems can be modelled by stochastic control problems, and this approach is named as diffusion approximation, which has been explored since 1950s with relevant research outputs, the readers can consult [35], [55], [56], [66] and the references therein. Some significant examples of stochastic control problems are linear quadratic Gaussian

(LQG) problems, which are mostly applied in engineering.

There are two mathematical formulations of stochastic optimal control problems: strong formulation and weak formulation. Note that in this thesis, we consider strong formulation of the problems. Here we present one case of stochastic LQ optimal control problem with some brief explanations and comprehension. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given filtered complete probability space, where there exists a standard one-dimensional Brownian motion $(W(t), 0 \leq t \leq T)$. Consider the following stochastic LQ optimal control problem in a finite time horizon $[s, T]$ :

$$\begin{aligned} &\min \quad J = \mathbb{E}\left\{ \int_s^T [x(t)'Q(t)x(t) + u(t)'R(t)u(t)]dt + x(T)'Hx(T) \right\}, \\ &\text{s.t.} \quad \begin{cases} dx(t) = [A(t)x(t) + B(t)u(t)]dt + [C(t)x(t) + D(t)u(t)]dW(t), \\ x(s) = y \in \mathbb{R}^n. \end{cases} \end{aligned} \tag{1.1}$$

Given that $y$, $s$, and $T$ are fixed data, $J$ is the cost functional, $x(\cdot)$ is the state process, $u(\cdot)$ is the control process. Both $Q(t)$ and $H$ are symmetric positive semi-definite $n \times n$ matrices. $R(t)$ is a symmetric positive definite $m \times m$ matrix. In engineering, $Q(t)$ stands for accuracy and $x(t)'Q(t)x(t)$ penalizes the transient state deviation; $R(t)$ stands for energy and $u(t)'R(t)u(t)$ penalizes the control effort; and $x(T)'Hx(T)$ penalizes the finite state. The selection of $Q(t)$ and $R(t)$ can be seen in [7] Section 6.3. Note that the precise way of denoting the cost functional is $J(x(s), u(\cdot), s)$, and intuitively, this value depends on the initial time, initial state, and the control during time $s$ to $T$.

In general, the cost functional $J$ in (1.1) can be written as follows:

$$J = \Upsilon[X(T), T] + \int_s^T \Xi[X(t), U(t), t]dt. \tag{1.2}$$

There are two parts in the above cost functional: the terminal condition $\Upsilon[X(T), T]$ and the integral part $\int_s^T \Xi[X(t), U(t), t]dt$. When a missile intercepts a target, the circle of the impact point is required to be minimized. Mathematically we use the terminal condition to model the circle of the impact point. In some other control problems, the time for a system to transit from one state to another is required to be minimized, mathematically, $\int_s^T \Xi[X(t), U(t), t]dt \to \min$.

In the case of infinite time horizon, i.e., $T \to +\infty$, the system is time invariant, which means $Q$, $R$, $A$, $B$, $C$, and $D$ in (1.1)are constant matrices. In addition, the terminal cost condition $x(T)'Hx(T)$ in (1.1) is neglected. Then the LQ problem (1.1) becomes:

$$\begin{aligned} \min \quad & J = \mathbb{E}\left\{ \int_s^{+\infty} [x(t)'Qx(t) + u(t)'Ru(t)]dt \right\}, \\ \text{s.t.} \quad & \begin{cases} dx(t) = [Ax(t) + Bu(t)]dt + [Cx(t) + Du(t)]dW(t), \\ x(s) = y \in \mathbb{R}^n. \end{cases} \end{aligned}$$

It is known that the optimal $J$ is always finite in the finite horizon case. However, in the infinite case the optimal $J$ may not be finite, which brings us difficulty. The concept of stability is important in infinite time horizon optimal control problems.

There are several approaches to solving stochastic LQ problems, such as the stochastic maximum principle, dynamic programming, and completion of squares. All these methods involve Riccati equations. If there exist solutions to the Riccati equations, then the stochastic LQ problems can be solved. In addition, we can derive an optimal control via the solutions to the Riccati equations. The method of solving Riccati equations in stochastic LQ problems was first introduced in [110] and [111]. It is notable that the disadvantage of this approach is that the existence and uniqueness of the solution to Riccati equation is difficult to be obtained in some cases.

Due to changes of working conditions, external disturbances, modelling errors, and various faults from the system, there exist unavoidable uncertainties in the mathematical model of the object, and it is difficult to find a precise mathematical model for this actual control plant. Under such circumstances, robust control was introduced in the 1950s and has become popular over the last 20 years. When various uncertainties exist in the system, the system can maintain its proper attributes and still functions well. This is called robustness. If the system is stable when it has uncertainties, then the system is said to have robust stability. If except robust stability, the system can still keep its performance index, then we say the system has its corresponding robust performance. When there are parameter uncertainties and modelling errors in the system, a controller can be designed so that the closed loop system is stable and maintains its own property.

This is named as robust control, see [9]. Robust control focuses on designing controllers that particularly deal with uncertainty. The modern theory of robust control started in the late 1970s, see [14] and [138]. Controllers are designed explicitly to achieve system stability and robust performance. Recently, the topic of robust stability and stabilization of stochastic systems with uncertainties in coefficients is very popular and has been widely studied, see, e.g., [84], [118], [102], [53] [108] and [120].

In order to eliminate the effect of disturbance efficiently, $H_\infty$ control is designed to deal with robust control problems with uncertainties. It was Zames [129] who first formulated the $H_\infty$ control in the frequency domain. Here, $H$ stands for Hardy space, and $\infty$ stands for infinity norm. The development of robust $H_\infty$ control can be divided into two periods. In the first period it is named as classic robust $H_\infty$ control theory. In deterministic $H_\infty$ theory, $H_\infty$ norm is defined by a norm of the rational transfer matrix, and cannot be applied to either nonlinear or stochastic systems, see [104] and [51]. In the second period it is called state space robust $H_\infty$ control theory. In 1988 Glover and Doyle [49] established state space formulae for all stabilizing controllers satisfying an $H_\infty$ norm bound. At roughly the same time, they further developed this research together with Khargonekar and Francis, and this is the famous work usually denoted as DGKF [36]. It is emphasized in [26] that a norm of the transfer function equals to $L_2$-induced norm of the input-output operator from the view of the time domain. Due to this feature, it is possible to develop our research of robust $H_\infty$ control theory based on a class of stochastic nonlinear systems. Since the publication of the work DGKF, more researches have been made, extending previous results from time invariant system to time varying system, see [96]; from linear system to nonlinear system, see [103] and [104]; from continuous time system to discrete time system, see [10] and [79]; from certain system to uncertain system, see [63] and [116]; from system without time delay to system with time delay, see [48], [72], and [50].

In engineering, we design a control $u(t)$ in order to eliminate the effect of disturbance. In addition, when the worst case of disturbance is involved, we require the control $u(t)$ to minimize the desired performance. Since both $H_2$ and $H_\infty$ performances are popular in engineering, the mixed $H_2/H_\infty$ control problem is considered. It is defined in [26] that a controller is designed not only to attenuate the external disturbance efficiently, but also to minimize the $H_2$ performance. Mixed $H_2/H_\infty$ control problem for both deterministic and stochastic systems have attracted many researchers' attention over the past two decades, see [100], [78], [113], [36], [13], [26], [135], [54], [131], [141] and the references therein.

When we are modelling a system it is common that there exist abrupt changes in the system parameters, which are caused by sudden environmental disturbances or component failures, and we use Markov chains to model these abrupt changes. The studies of jumping linear systems started from [65]. There is a classic book on continuous time Markov chains, see [6]. Here we illustrate some other related works as follows. For stability problems, [60], [85] [37], and [82] have been studied. The problem of robust stability and stabilization was studied in [91]. Some other literatures concerning related topics are [92], [99], [17], [28], [33], [43]. The textbook [83] introduces SDEs with Markovian switching. The LQ control with Markovian switching has been widely studied in the last two decades, see [1], [60], [61], and [134].

Most of the literatures mentioned above focus on linear systems. However, in our daily life many systems we see actually are nonlinear, and the control problems for such systems are very complicated. Here we illustrate some recent practical examples in which nonlinaer system is involved, see [106], [125], [41], [73], [95], and [57]. Based on the previous research results, this thesis investigates the stochastic optimal and robust control problems in a class of nonlinear systems.

## 1.3 The Main Contributions and Outlines of the Thesis

In this section we emphasize the main contributions and outlines of the thesis. A class of nonlinear systems are developed for the following problems: optimal stochastic control in both finite and infinite horizon; robust stabilization and robust $H_\infty$ control problem; stochastic $H_2/H_\infty$ control in both finite and infinite horizon and stochastic risk-sensitive control. The nonlinearity is formed by a class of square root processes. Apart from the chapter of risk-sensitive control, Markovian switching is applied to system parameters in this thesis. It is highlighted that although the system is nonlinear, explicit solutions like the optimal controls can still be obtained under such system, which is a very rare case. Some existing works (e.g. [107], [130], [80], [81], [127], [24], [22], [132], [133], and [136]) only focus on the general cases of nonlinear systems without fixing the structure of the nonlinear terms and the results are not obtained explicitly. The advantage of our system is that we can obtain the solutions explicitly, with which the research outputs are easier to be applied to real problems. It is known from the literatures on this subject of study that only some of the nonlinear SDEs can have solutions. Even if the solution exists, it may not be unique. Note that Section 4.4 in the book [64] provides some explicitly solvable SDEs. If the nonlinear system does not permit a solution, the problem will be meaningless. Even if the solution to the nonlinear SDE exists, there may be more than one solution, and then the problem is still impractical. Therefore, in our system, it is significant to ensure the existence and uniqueness of the solution, which is discussed in the thesis.

Next, we outline the main contents and highlight the contributions in each chapter.

## Chapter 2

In this preliminary chapter we review some literatures of optimal and robust control theory, including linear quadratic control in both finite and infinite horizon, robust stabilization and robust $H_\infty$ control, $H_2/H_\infty$ control in both finite and infinite horizon, and risk sensitive control. We recall some definitions, lemmas, and theorems, some of which will be used in our thesis. Attention is focused on the most recent research outputs relating to our thesis. It is worth mentioning that in this chapter we are not only doing literature reviews, but some new results are also obtained. We improve some previous results without changing the spirit of the original work,. In this case the previous results are extended into a more general case.

## Chapter 3

Chapter 3 extends optimal LQ control in [2] and [74] to a more general case, which deals with the optimal control problems of indefinite stochastic nonlinear system with Markovian switching in system coefficients. Two motivating examples are introduced first. The nonlinearity in our system is formulated by a combination of two different diffusion terms. The existence and uniqueness of the solution is discussed. A new type of coupled generalized Riccati equations (CGREs) is introduced when the problem is formulated. The solvability of CGREs is sufficient for the well-posedness of the nonlinear optimal control problem and the existence of optimal controls. Moreover, all the optimal control laws constructed by the solution to the CGREs are obtained explicitly. We assume that the new CGREs have solutions. It is shown that our new CGREs can be transformed into the ones in [74], where the assumption of the solvability of Riccati equation is made. Then we conclude that our assumption is feasible. An application to finance is introduced. An illustrative example is given.

## Chapter 4

After we have discussed the problem of optimal stochastic nonlinear control of systems with Markovian switching in finite time horizon in Chapter 3, the case in infinite time horizon is investigated with its system formulated similarly to the one in the finite time horizon, especially the nonlinear terms. Then the systems considered in [3] and [75] can be regarded as one of the special cases in this chapter. The mean-square stability is considered. The new coupled generalized algebraic Riccati equations (CGAREs) are introduced. We assume that there exists a unique solution to the CGAREs. Explicit optimal control laws can be obtained, then our stochastic nonlinear problem is well-posed. Note that the optimal control laws are linear in state. Furthermore, the value function is obtained.

## Chapter 5

In Chapter 5 we consider the problem of robust stabilization and robust $H_\infty$ control for a class of nonlinear stochastic systems with Markovian switching in coefficients. This chapter generalizes [46], which discusses the linear case in the following aspects. A class of nonlinear term, different from the ones used in Chapter 3 and Chapter 4, is included in the diffusion term. The existence and uniqueness of solution is discussed. Compared with [46], this chapter includes time delay in the system, which is used in [120]. We include element-wise uncertainties in switching probabilities, which is used in [46] . In [120], time delay is only permitted in state, whereas here delay appears in disturbance as well. In [46] and [120], the norm-bounded parameter uncertainty only appear in $x(t)$, $x(t-\tau(t))$, and $u(t)$ in the state equation $dx(t)$, but not in disturbance $v(t)$ or controlled output $z(t)$. Here we extend them by including norm-bounded parameter uncertainty into $v(t)$ and $v(t-\tau(t))$ as well. The function of controlled output is also constructed differently, where disturbance appears in it, which is possible in reality. In summary, the system considered in [115], [114], [20], [120], [121], and [46] are all special cases of the one in this chapter. In the first section, in order to achieve linear state feedback controllers such that the system is robustly stochastically stable, we derive sufficient conditions in forms of matrix inequalities. In the second section, we define and formulate a new generalized robust $H_\infty$ control problem, and we derive a sufficient condition to solve it, also in forms of matrix inequalities. It is noted that, in comparison with the existing literatures, where disturbance attenuation is a constant $\gamma$, here we propose a new type of disturbance attenuation denoted as $R(r_t)$, which are symmetric matrices with Markovian switching. In this case the disturbance attenuation itself is extended to a jumping stochastic process. In general, all of these different kinds of uncertainties are put together into one single system, under which the problems are still solvable.

## Chapter 6

In Chapter 6, $H_2/H_\infty$ control of stochastic nonlinear systems with Markovian switching in both finite and infinite time horizon is considered. Compared with the previous works of nonlinear $H_2/H_\infty$ control that are dealt with in [127], [81], [80], [132], [24], [22], [133], and [136], our research output has its advantage that $u^*(\cdot)$ and $v^*(\cdot)$ are obtained not only explicitly, but also linearly with $x(t)$, which is very similar to the result in the problem of linear $H_2/H_\infty$ control. We extend [141] into a nonlinear case, by involving a square root process in the diffusion term. The nonlinearity term is similar to the one in Chapter 3 and Chapter 4. In the main results we show that the solvability of the coupled differential Riccati equations is sufficient to solve our finite horizon nonlinear stochastic $H_2/H_\infty$ control problem;

and the solvability of coupled algebraic Riccati equations is sufficient to solve our infinite horizon nonlinear stochastic $H_2/H_\infty$ control problem. In this case, some results in [26], [135], [54], and [141] are all special cases of our work.

### Chapter 7

In Chapter 7, the problem of risk-sensitive control of stochastic nonlinear systems in finite time horizon is investigated. Based on [30], we proposed a new nonlinear system, in which the new nonlinear term is similar to the one used in Chapter 6. When a series of assumptions are satisfied, we prove that there exists a unique solution to our optimal control problem, and the optimal cost functional is obtained. We highlight the importance of this chapter by introducing two applications. When it is applied to finance, we introduce a new interest rate model, and based on the result of our risk-sensitive control problem, we find the price of the zero-coupon bond. Moreover, we show that the optimal investment problem for the power utility is an example of our risk-sensitive control problem.

## 1.4 Notation

The list of notation for all chapters is provided at the beginning of the thesis. If some notation is not included in the notation list, then its definition is given in the chapter where it appears. Note that such definition is valid for that particular chapter only. Throughout the thesis, some symbols have the same definition. For example, $x$ is always regarded as state, $u$ is denoted as control, and $W$ is defined to be Brownian motion. However, some symbols have multiple definitions. For example, the letter $H$ has three different definitions given in (1.1) in Section 2.2, (3.13) in Section 3.3.1, and (5.1) in Section 5.2.

## 1.5 Summary

In conclusion, we say that this thesis is of interest in optimal and robust control theory. Aiming to extend the existing works with either linear or nonlinear systems into more general cases, we propose several different nonlinear cases, presented by a class of square root processes. Under such nonlinear systems we are still able to find explicit solutions, which is rare. In addition, our nonlinear systems are easy to be applied to practical usage.

# Chapter 2

# Preliminaries

## 2.1 Introduction

In this chapter we review some basic results of optimal and robust control theory, including LQ control in both finite and infinite horizon, robust stabilization and robust $H_\infty$ control, $H_2/H_\infty$ control in both finite and infinite horizon and risk-sensitive control. Previous results obtained under both deterministic and stochastic systems are concerned. In addition, some works including Markovian switching are discussed. We also review some previous works investigated under some kinds of nonlinear systems, see, e.g. [30], [107], [130], [80], [81], [127], [24], [22], [132], [133], and [136]). Advantages and disadvantages of all these previous nonlinear systems are discussed. Finally some lemmas from the previous works are provided, which will be used throughout this thesis. When we review the literatures of previous works, it is discovered that some results can be further improved to a more general case without changing the spirit of the original work, and this is taken into account by some of the remarks in this chapter.

In addition, it is notable that the recent work [46] is one of author's research results. In Section 2.6, the problem formulation and its main results are provided identically to [46], and these should be regarded as part of this thesis. The reason why we choose to outline [46] in the preliminary chapter, rather than putting the whole work of [46] identically in the later chapter, is that [46] is based on linear systems, and Chapter 5 extends it into a nonlinear case. In this case, all the main content of this thesis is based on nonlinear systems.

## 2.2 Stochastic Linear Quadratic (LQ) Control and Differential Riccati Equation in Finite Time Horizon

In optimal LQ control theory, in order to present an optimal control, some works use Riccati equations to achieve this target, see for example: [62], [94], [7], [110], [8] and [32]. In the past literatures, the control weighting matrix $R$ is usually assumed to be positive definite, and state weighting matrix $Q$ is usually assumed to be positive semidefinite. Under such circumstances the solvability of the optimal LQ control problem equals to the solvability of its corresponding Riccati equations. Later [23], [4] and [2] generalize the above results by allowing $Q$ and $R$ to be indefinite. In this case, the diffusion term $dW(t)$ is required to depend on the control $u(\cdot)$, then the stochastic LQ problem is well-posed. Further researches on indefinite stochastic LQ control in finite time horizon were studied in [27], [76] and [25]. Indefinite stochastic LQ control has various applications, see for example [137], [139], [77] and [67].

One of the recent works focusing on indefinite stochastic LQ control is [23], which proves that problem (1.1) is well posed, if the following Riccati equation has a solution $P(\cdot)$,

$$\begin{cases} \dot{P} + PA + A'P + C'PC - (PB + C'PD)(R + D'PD)^{-1}(B'P + D'PC) \\ \quad + Q = 0, \\ P(T) = H, \\ R + D'PD > 0, \quad \text{a.e.} \quad t \in [0, T]. \end{cases} \tag{2.1}$$

Note that $t$ is omitted in (2.1) for convenience. In addition, the optimal control $u^*(\cdot)$ is achieved by the solution $P(\cdot)$. We emphasized that $R + D'PD > 0$ in (2.1) is restrictive. This is improved in [2], where a generalized Riccati equation

(GRE) is given as follows ($t$ is omitted),

$$\begin{cases} \dot{P} + PA + A'P + C'PC - (PB + C'PD)(R + D'PD)^{\dagger}(B'P + D'PC) \\ \quad + Q = 0, \\ P(T) = H, \\ (R + D'PD)(R + D'PD)^{\dagger}(B'P + D'PC) - (B'P + D'PC) = 0, \\ R + D'PD \geq 0, \quad \text{a.e.} \quad t \in [0, T]. \end{cases} \tag{2.2}$$

Here, the case of $R + D'PD = 0$ is allowed. The difference between (2.1) and (2.2) is that pseudo inverse and one more algebraic constraint are introduced in (2.2).

## 2.3 Optimal Stochastic LQ Control of Systems with Markovian Switching in Finite Time Horizon

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given filtered complete probability space, where there exist a standard one-dimensional Brownian motion $(W(t), 0 \leq t \leq T)$ and a Markov chain $(r_t, 0 \leq t \leq T)$, taking values in $\{1, \ldots, \delta\}$, with transition probabilities given by

$$\mathbb{P}\{r_{t+\Delta t} = j \mid r_t = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t): & \text{if} \quad i \neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t): & \text{if} \quad i = j, \end{cases} \tag{2.3}$$

where $\pi_{ij} \geq 0$ for $i \neq j$ while $\pi_{ii} = -\sum_{j\neq i} \pi_{ij}$. Note that the above setting of Markovian switching will be used throughout this thesis, and we assume that Markov chain $r_t$ is independent of all the Brownian Motions in this thesis.

Based on the results in [2], Markovian switching was included in the indefinite stochastic LQ system by [74], in which the optimal control problem becomes:

$$\min \quad J(s, y, i; u(\cdot))$$

$$
\begin{aligned}
&= \mathbb{E}\left\{ \int_s^T \left( \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q(t,r_t) & L(t,r_t) \\ L(t,r_t)' & R(t,r_t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \right) dt \right. \\
&\qquad \left. + x(T)' H(r_T) x(T) | r_s = i \right\},
\end{aligned}
$$

$$
\text{s.t.} \quad \begin{cases} dx(t) = [A(t,r_t)x(t) + B(t,r_t)u(t)]dt + [C(t,r_t)x(t) \\ \quad + D(t,r_t)u(t)]dW(t), \\ x(s) = y \in \mathbb{R}^n, \end{cases} \tag{2.4}
$$

when $r_t = i$, $i = \{1, \dots, \delta\}$, we denote $A(t, r_t) = A_i(t)$. Here the matrix functions $A_i(\cdot)$, etc., are given with appropriate dimensions.

Similar to the optimal LQ control problem without Markovian switching, a new type of coupled generalized Riccati equations (CGREs) is introduced in [74], ($t$ is omitted for convenience)

$$
\begin{cases} \dot{P}_i + P_iA_i + A_i'P_i + C_i'P_iC_i - (P_iB_i + C_i'P_iD_i + L_i)(R_i + D_i'P_iD_i)^{\dagger}(B_i'P_i \\ \quad + D_i'P_iC_i + L_i') + Q_i + \sum_{j=1}^{\delta} \pi_{ij}P_j = 0, \\ P_i(T) = H_i, \\ (R_i + D_i'P_iD_i)(R_i + D_i'P_iD_i)^{\dagger}(B_i'P_i + D_i'P_iC_i + L_i') - (B_i'P_i + D_i'P_iC_i \\ \quad + L_i') = 0, \\ R_i + D_i'P_iD_i \geq 0, \quad \text{a.e.} \quad t \in [0,T], \quad i = 1, \cdots, \delta. \end{cases} \tag{2.5}
$$

Two more special cases are introduced in [74] as follows. If $D_i'P_iD_i + R_i \neq 0$, for every $i$, then (2.5) becomes the following,

$$
\begin{cases} \dot{P}_i + P_iA_i + A_i'P_i + C_i'P_iC_i - (P_iB_i + C_i'P_iD_i + L_i)(R_i + D_i'P_iD_i)^{-1}(B_i'P_i \\ \quad + D_i'P_iC_i + L_i') + Q_i + \sum_{j=1}^{\delta} \pi_{ij}P_j = 0, \\ P_i(T) = H_i, \\ R_i + D_i'P_iD_i > 0, \quad \text{a.e.} \quad t \in [0,T], \quad i = 1, \cdots, \delta. \end{cases}
$$

When $D_i'P_iD_i + R_i \equiv 0$, for every $i$, (2.5) is reduced to the following,

$$\begin{cases} \dot{P}_i + P_iA_i + A_i'P_i + C_i'P_iC_i + \sum_{j=1}^{\delta} \pi_{ij}P_j + Q_i = 0, \\ P_i(T) = H_i, \\ D_i'P_iC_i + B_i'P_i + L_i' = 0, \\ D_i'P_iD_i + R_i = 0, \quad \text{a.e.} \quad t \in [0,T], \quad i = 1, \cdots, \delta. \end{cases}$$

It is proved in [74] that the solvability of the CGREs (2.5) is not only sufficient, but also necessary for the well-posedness of the LQ problem (2.4). Optimal controls are found explicitly via the solution to CGREs (2.5). Here we provide one of the main results in [74].

**Theorem 2.3.1.** *[74] If the CGREs (2.5) have a solution, then the stochastic LQ problem (2.4) is well-posed. Moreover, the set of all optimal controls w.r.t. the initial $(s,y) \in [0,T) \times \mathbb{R}^n$ is presented as follows:*

$$\begin{aligned} & u^*(t) \\ = & -\sum_{i=1}^{\delta} \Big\{ \Big[ [D_i(t)'P_i(t)D_i(t) + R_i(t)]^{\dagger}[D_i(t)'P_i(t)C_i(t) + B_i(t)'P_i(t) \\ & + L_i(t)'] + Y_i(t) - [D_i(t)'P_i(t)D_i(t) + R_i(t)]^{\dagger}[D_i(t)'P_i(t)D_i(t) \\ & + R_i(t)]Y_i(t) \Big] x(t) + z_i(t) - [D_i(t)'P_i(t)D_i(t) + R_i(t)]^{\dagger}[D_i(t)'P_i(t)D_i(t) \\ & + R_i(t)]z_i(t) \} \chi_{\{r_t = i\}}(t), \end{aligned}$$

*where $Y_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u \times n})$, $z_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u})$. In addition, the value function is obtained as follows:*

$$\begin{aligned} V(s,y,i) & \equiv \inf\nolimits_{u(\cdot)\in\mathcal{U}} J(s,y,i;u(\cdot)) \\ & = y'P_i(s)y, \quad i = 1, \cdots, l. \end{aligned}$$

The proof is omitted here. The idea of involving $Y_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u \times n})$, and $z_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u})$ results in that we are able to obtain infinitely many optimal control laws. This technique is going to be used in both Chapter 3 and Chapter 4. Note that the above theorem can be viewed as a special case of the main result in Chapter 3.

## 2.4 Indefinite Stochastic LQ Control of Systems in Infinite Time Horizon

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given filtered complete probability space, where there exists a standard one-dimensional Brownian motion $(W(t), t \geq 0)$. We introduce the notation of $L_2^{loc}(\mathbb{R}^k)$ which is defined the same way as the one in [3], as follows

$$L_2^{loc}(\mathbb{R}^k) \triangleq \phi(\cdot) : [0, +\infty) \times \Omega \to \mathbb{R}^k, \tag{2.6}$$

if $\phi(\cdot)$ is $\mathcal{F}_t$-adapted, measurable, and $\mathbb{E}\int_0^T |\phi(t,\omega)|^2 dt < +\infty$, $\forall T \geq 0$. In [3] the infinite time horizon stochastic LQ optimal control problem is considered as follows:

$$\begin{aligned} \min \quad & J = \mathbb{E}\left\{\int_0^{+\infty} [x(t)'Qx(t) + 2x(t)'Lu(t) + u(t)'Ru(t)]dt\right\}, \\ \text{s.t.} \quad & \begin{cases} dx(t) = [Ax(t) + Bu(t)]dt + [Cx(t) + Du(t)]dW(t), \\ x(s) = x_0 \in \mathbb{R}^n. \end{cases} \end{aligned} \tag{2.7}$$

The following definitions proposed in [3] will be used in the later chapters.

**Definition 2.4.1.** *[3] A control $u(\cdot)$ is called mean-square stabilizing w.r.t. $x_0$ if*

$$\lim_{t\to+\infty} \mathbb{E}[x(t)'x(t)] = 0. \tag{2.8}$$

*A feedback control $u(t) = Kx(t)$, where $K$ is a constant matrix, is called stabilizing if for $x_0$ the corresponding state $x(\cdot)$ of system (2.7) satisfies (2.8).*

The concept of mean-square stability is very important in infinite horizon optimal control problems.

**Definition 2.4.2.** *[3] Given $x_0 \in \mathbb{R}^n$ in (2.7), we present the definition of admissible controls as follows:*

$$\mathbb{U}_{ad}(x_0) \triangleq \{u(\cdot) \in L_2^{loc}(\mathbb{R}^{n_u})\}, \tag{2.9}$$

*provided that $u(\cdot)$ is mean-square stabilizing w.r.t. $x_0$. The notation of $L_2^{loc}(\cdot)$ is given in (2.6).*

**Definition 2.4.3.** *[3] The value function is defined as*

$$V(x_0) \triangleq \inf_{u(\cdot)\in\mathbb{U}_{ad}(x_0)} J(x_0, u(\cdot)). \tag{2.10}$$

*The LQ problem (2.7) is called well-posed if*

$$V(x_0) > -\infty, \quad \forall x_0 \in \mathbb{R}^n.$$

*Any control $u^*(\cdot)$ that achieves the infimum in (2.10) is called optimal, w.r.t. $x_0$.*

We do not have differential Riccati equations in infinite time horizon optimal control problems. Instead, generalized algebraic Riccati equation (GARE) is introduced in [3] as follows,

$$\begin{cases} A'P + PA + C'PC + Q - (PB + C'PD + L)(R + D'PD)^{\dagger}(B'P \\ \quad + D'PC + L') = 0, \\ [I - (R + D'PD)(R + D'PD)^{\dagger}](B'P + D'PC + L') = 0, \\ R + D'PD \geq 0. \end{cases} \tag{2.11}$$

Similar to the two special cases of (2.5) in the previous section, [3] also provides two special cases as follows,

$$\begin{cases} A'P + PA + C'PC + Q - (PB + C'PD + L)(R + D'PD)^{-1}(B'P \\ \quad + D'PC + L') = 0, \\ R + D'PD > 0, \end{cases} \tag{2.12}$$

and

$$\begin{cases} A'P + PA + C'PC + Q = 0, \\ B'P + D'PC + L' = 0, \\ R + D'PD = 0. \end{cases} \tag{2.13}$$

Before we introduce the main results of [3], for notation convenience, we denote

$$\begin{aligned}\mathcal{M}(P) &\triangleq A'P + PA + C'PC + Q,\\ \mathcal{L}(P) &\triangleq PB + C'PD + L,\\ \mathcal{N}(P) &\triangleq R + D'PD.\end{aligned}$$

**Theorem 2.4.1.** *[3] Assume that GARE (2.11) has a solution and there exist* $Y(\cdot) \in L_2^{loc}(\mathbb{R}^{n_u \times n})$ *and* $z(\cdot) \in L_2^{loc}(\mathbb{R}^{n_u})$ *such that the following control:*

$$\begin{aligned}u_{Y,z}(t) &= -[\mathcal{N}(P)^{\dagger}\mathcal{L}(P)' + (I - \mathcal{N}(P)^{\dagger}\mathcal{N}(P))Y(t)]x(t)\\ &\quad -[I - \mathcal{N}(P)^{\dagger}\mathcal{N}(P)]z(t)\end{aligned}$$

*is admissible w.r.t. any initial* $x_0$*. Then the stochastic LQ problem (2.7) is well-posed and* $u_{Y,z}(\cdot)$ *is an optimal control. Moreover, the value function is*

$$V(x_0) = x_0'Px_0.$$

As we mentioned in the Introduction: the difficulty of the stochastic LQ problem in the infinite time horizon is that the optimal $J$ may not be finite, this is considered in Definition 2.4.3 and solved by the above theorem.

## 2.5 Indefinite Stochastic LQ Control of Systems with Markovian switching in Infinite Time Horizon

Based on [3], Markovian jumps were included in the parameters of system (2.7) by [75]. In this section, we mention the problem formulation and some of the main contributions in [75]. Later in Chapter 4, we extend [75] to a nonlinear case. Define the Markovian switching in the same way as (2.3), the infinite time horizon stochastic LQ optimal control problem with Markovian switching is considered in [75] as follows:

$$\min \quad J(x_0, i; u(\cdot))$$

$$= \mathbb{E}\left\{ \int_s^{+\infty} \left( \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q(r_t) & L(r_t) \\ L(r_t)' & R(r_t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \right) dt \middle| r_s = i \right\},$$

$$\text{s.t.} \quad \begin{cases} dx(t) = [A(r_t)x(t) + B(r_t)u(t)]dt + [C(r_t)x(t) \\ \quad + D(r_t)u(t)]dW(t), \\ x(s) = x_0 \in \mathbb{R}^n, \end{cases} \tag{2.14}$$

where $(r_t, t \geq 0)$ takes values in $\{1, \cdots, \delta\}$.

Similar to Definition 2.4.1, Definition 2.4.2, and Definition 2.4.3, the definition of mean-square stabilizing, admissible control and value function in the case of the system with Markovian switching is stated in [75]. Here we omit the duplicated statements.

The coupled generalized algebraic Riccati equations (CGAREs) are introduced in [75] as follows

$$\begin{cases} A_i'P_i + P_iA_i + C_i'P_iC_i + Q_i + \sum_{j=1}^{\delta} \pi_{ij}P_j - (P_iB_i + C_i'P_iD_i + L_i)(R_i \\ \quad + D_i'P_iD_i)^{-1}(B_i'P_i + D_i'P_iC_i + L_i') = 0, \\ R_i + D_i'P_iD_i > 0, \quad i = 1, \cdots, \delta \end{cases} \tag{2.15}$$

with the unknown $P_1, \cdots, P_\delta$.

The stability condition of system (2.14) is given in [75]. Next, we provide the solution to optimal control problem in [75].

**Theorem 2.5.1.** *[75] Assume that there exists solution to the CGAREs (2.15), and $P_i > 0$, then the LQ problem (2.14) is well-posed and there exists an optimal state feedback control,*

$$u(t) = -\sum_{i=1}^{\delta} (R_i + D_i'P_iD_i)^{-1}(B_i'P_i + D_i'P_iC_i + L_i')x(t)\chi_{r_t=i}(t), \tag{2.16}$$

*and the value function is given by $V(x_0, i) = x_0'P_ix_0$, $\forall x_0 \in \mathbb{R}^n$, $\forall i = 1, 2, \cdots, \delta$. $P_1, \cdots, P_\delta$ is the solution to the CGAREs (2.15).*

**Remark 2.5.1.** *In the section of problem formulation and preliminaries of [75], the concept of pseudo inverse is introduced, which is also used in [2], [74] and [3], but not used in the context of CGAREs (2.15) or the main results, like finding the optimal control. Here we apply the pseudo inverse to CGAREs (2.15), without changing the spirit of [75]. We provide some new results below. The CGAREs become the following:*

$$\begin{cases} P_iA_i + A_i'P_i + C_i'P_iC_i - (P_iB_i + C_i'P_iD_i + L_i)(R_i + D_i'P_iD_i)^{\dagger}(B_i'P_i \\ \qquad + D_i'P_iC_i + L_i') + Q_i + \displaystyle\sum_{j=1}^{\delta} \pi_{ij}P_j = 0, \\ (R_i + D_i'P_iD_i)(R_i + D_i'P_iD_i)^{\dagger}(B_i'P_i + D_i'P_iC_i + L_i') - (B_i'P_i + D_i'P_iC_i \\ \qquad + L_i') = 0, \\ R_i + D_i'P_iD_i \geq 0, \quad i = 1, \cdots, \delta. \end{cases} \tag{2.17}$$

*Note that in (2.17) we allow $R_i + D_i'P_iD_i = 0$, which generalize the CGAREs in (2.15). The importance of the new CGAREs in (2.17) can be seen from the following example. Assume someone can get a solution $P$ to the first equation of (2.15). However, when substituting this value $P$ into the second constraint of (2.15), unfortunately we have $R_i + D_i'P_iD_i = 0$, which is not in agreement with $R_i + D_i'P_iD_i > 0$. In this case, (2.15) has no solution. The involving of pseudo inverse and one more algebraic constraint in (2.17) makes the CGAREs more generalized. In addition, if we compare (2.17) with (2.11) in Section 2.4 without Markovian switching, we see that (2.17) is quite similar to (2.11). The difference is that in (2.17) we have each system coefficient relating to Markovian switching, and accordingly we have one more term $\sum_{j=1}^{\delta} \pi_{ij}P_j$.*

*Following (2.12) and (2.13), we have two more similar results. If the term $D_i'P_iD_i + R_i \neq 0$, for every $i$, then the CGAREs (2.17) become the following,*

$$\begin{cases} P_iA_i + A_i'P_i + C_i'P_iC_i - (P_iB_i + C_i'P_iD_i + L_i)(R_i + D_i'P_iD_i)^{-1}(B_i'P_i \\ \qquad + D_i'P_iC_i + L_i') + Q_i + \displaystyle\sum_{j=1}^{\delta} \pi_{ij}P_j = 0, \\ P_i(T) = H_i, \\ R_i + D_i'P_iD_i > 0, \quad \text{a.e.} \quad t \in [0, T], \quad i = 1, \cdots, \delta. \end{cases}$$

*When $D_i'P_iD_i + R_i \equiv 0$ for every $i$, the CGAREs (2.17) become the following:*

$$\begin{cases} P_iA_i + A_i'P_i + C_i'P_iC_i + \sum_{j=1}^{\delta} \pi_{ij}P_j + Q_i = 0, \\ P_i(T) = H_i, \\ D_i'P_iC_i + B_i'P_i + L_i' = 0, \\ D_i'P_iD_i + R_i = 0, \quad \text{a.e.} \quad t \in [0,T], \quad i = 1, \cdots, \delta. \end{cases}$$

**Remark 2.5.2.** *Similar to including $Y_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u \times n})$, $z_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u})$ in Theorem 2.3.1, and including $Y(\cdot) \in L_2^{loc}(\mathbb{R}^{n_u \times n})$, $z(\cdot) \in L_2^{loc}(\mathbb{R}^{n_u})$ in Theorem 2.4.1, we can also have infinitely many controls $u(\cdot)$, by including $Y(\cdot) \in L_2^{loc}(\mathbb{R}^{n_u \times n})$ and $z(\cdot) \in L_2^{loc}(\mathbb{R}^{n_u})$ into (2.16).*

*If we consider Remark 2.5.1, then we can have a new result below. For notation convenience, we denote*

$$\begin{aligned} \mathcal{M}_i &\triangleq P_iA_i + A_i'P_i + C_i'P_iC_i + Q_i + \sum_{j=1}^{\delta} \pi_{ij}P_j, \\ \mathcal{L}_i &\triangleq P_iB_i + C_i'P_iD_i + L_i, \\ \mathcal{N}_i &\triangleq R_i + D_i'P_iD_i. \end{aligned}$$

*We assume that there exists a solution to the CGAREs (2.17), denoted as $P_1, \cdots, P_\delta$, then the LQ problem (2.14) is well-posed. In addition, there exist optimal controls as follows,*

$$u(t) = -[\mathcal{N}_i^\dagger \mathcal{L}_i' + (I - \mathcal{N}_i^\dagger \mathcal{N}_i)Y_i]x(t) - [I - \mathcal{N}_i^\dagger \mathcal{N}_i]z(t),$$

*with value function $V(x_0, i) = x_0'P_ix_0$, $\forall x_0 \in \mathbb{R}^n$, $\forall i = 1, 2, \cdots, \delta$.*

*The proof is straight forward, according to the previous results. Hence it is omitted here.*

## 2.6 Robust Stabilization and Robust $H_\infty$ Control for Uncertain Stochastic Systems with State Delay

Over the past two decades, $H_\infty$ control theory has been developed rapidly. Some typical works can be found for example in [45], [36], [140] and the references therein. When robustness is considered in $H_\infty$ control theory, [63] studies the problem of robust stabilization. The robust $H_\infty$ control problem was studied by [117], [124], [128] and etc. In addition, [51] studies the stochastic $H_\infty$ control problem. Recently [46] generalizes [120] in the following aspects. One of the advantages over [120] is that in [46] the control appears in the diffusion term as well. In addition, Markovian switching is included in system coefficients. The switching probabilities are assumed to be known precisely in most systems with Markovian switching. However, uncertainties may also appear in the mode transition rate matrix because of modelling errors. There are mainly two different types of uncertain switching probabilities, namely the poly-topic ones, see for example [29] and [40], and the element-wise ones, see for example [11] and [18]. In [46], the switching probabilities is assumed to have element-wise uncertainties.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given filtered complete probability space, where there exists a standard one-dimensional Brownian motion $(W(t), 0 \leq t \leq T)$, and a Markov chain $(r_t, 0 \leq t \leq T)$ taking values in a finite state-space $S = 1, 2, \ldots, N$, with generator $\widehat{\Pi} = (\widehat{\pi}_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\delta) = j \mid r(t) = i\} = \begin{cases} \widehat{\pi}_{ij}\delta + o(\delta): & \text{if} \quad i \neq j, \\ 1 + \widehat{\pi}_{ij}\delta + o(\delta): & \text{if} \quad i = j, \end{cases}$$

for $\delta > 0$, and $\lim_{\delta\to 0}(o(\delta)/\delta) = 0$. Here, $\widehat{\pi}_{ij} \geq 0$ is the transition rate from $i$ to $j$ if $i \neq j$ while $\widehat{\pi}_{ii} = -\sum_{j=1, j\neq i}^{N} \widehat{\pi}_{ij}$. We assume that Markov chain $r(\cdot)$ is independent of Brownian Motion $W(\cdot)$. Additionally, similar to the settings in [119], the mode transition rate matrix $\widehat{\Pi}$ is also assumed to be not exactly known and has the element-wise uncertainties

$$\widehat{\Pi} = \Pi + \Delta\Pi,$$

with $\Pi \triangleq (\pi_{ij})_{N\times N}$ satisfying $\pi_{ij} \geq 0$, $(i, j \in S, j \neq i)$ and $\pi_{ii} \triangleq -\sum_{j=1,j\neq i}^{N} \pi_{ij}$ for all $i \in S$, where $\pi_{ij}$ denotes the estimated value of $\widehat{\pi}_{ij}$, and $\Delta\Pi \triangleq (\Delta\pi_{ij}) = (\widehat{\pi}_{ij} - \pi_{ij})$ where $|\Delta\pi_{ij}| \leq \varepsilon_{ij}$, $\varepsilon_{ij} \geq 0$. $\Delta\pi_{ij}$ denotes the error between $\widehat{\pi}_{ij}$ and $\pi_{ij}$ for all $i, j \in S, j \neq i$ and $\Delta\pi_{ii} \triangleq -\sum_{j=1,j\neq i}^{N} \Delta\pi_{ij}$, $\forall i \in S$.

The following stochastic system with Markovian switching and parameter uncertainties is considered in [46]:

$$\begin{aligned} dx(t) &= [(A(r(t)) + \Delta A(r(t)))x(t) + (B(r(t)) + \Delta B(r(t)))u(t) \\ &\quad +G(r(t))v(t)]dt \\ &\quad +[(E(r(t)) + \Delta E(r(t)))x(t) + (F(r(t)) + \Delta F(r(t)))u(t) \\ &\quad +H(r(t))v(t)]dW(t), \end{aligned} \tag{2.18}$$

and

$$z(t) = C(r(t))x(t) + D(r(t))u(t), \tag{2.19}$$

for $t \geq 0$ with initial data $x(0) = x_0$ and $r(0) = i_0 \in S$. Here $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $v(t) \in \mathbb{R}^p$ is the disturbance input, and $z(t) \in \mathbb{R}^q$ is the controlled output. For each mode $r(t) = i \in S$, $A(i) = A_i$, etc. is denoted for simplicity.

In the above system, $A_i, B_i, C_i, D_i, E_i, F_i, G_i, H_i$ are known real constant matrices. $\Delta A_i, \Delta B_i, \Delta E_i, \Delta F_i$ are unknown matrices representing parameter uncertainties. It is assumed that

$$\begin{bmatrix} \Delta A_i & \Delta B_i & \Delta E_i & \Delta F_i \end{bmatrix} = M_i U_i \begin{bmatrix} N_{ai} & N_{bi} & N_{ei} & N_{f_i} \end{bmatrix},$$

where $M_i, N_{ai}, N_{bi}, N_{ei}, N_{f_i}$ are known real constant matrices and $U_i$'s are unknown matrices such that ${U_i}^T U_i \leq I$, $\forall i \in S$.

Next we present some fundamental definitions that will be used later.

**Definition 2.6.1.** *[82] The SDE with Markovian switching (2.18) is said to be almost surely exponentially stable if for all $x_0 \in \mathbb{R}^n$ and $i_0 \in S$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t; x_0, i_0)| < 0.$$

The definition of mean-square stabilizing can be found in Definition 2.4.1.

**Definition 2.6.2.** *[120] [46] The uncertain stochastic system in (2.18) is said to be robustly stochastically stable if the system (2.18) with $u(t) = 0$ and $v(t) = 0$ is mean-square asymptotically stable for all admissible uncertainties $\Delta A_i$, and $\Delta E_i$.*

Next, we provide sufficient conditions such that system (2.18) is robustly stochastically stable. The following matrices are introduced:

$$\begin{aligned}
\overline{A}(r(t)) &\triangleq A(r(t)) + B(r(t))K(r(t)),\\
\Delta\overline{A}(r(t)) &\triangleq M(r(t))U(r(t))\overline{N_a}(r(t)),\\
\overline{N_a}(r(t)) &\triangleq N_a(r(t)) + N_b(r(t))K(r(t)),\\
\overline{E}(r(t)) &\triangleq E(r(t)) + F(r(t))K(r(t)),\\
\Delta\overline{E}(r(t)) &\triangleq M(r(t))U(r(t))\overline{N_e}(r(t)),\\
\overline{N_e}(r(t)) &\triangleq N_e(r(t)) + N_f(r(t))K(r(t)).
\end{aligned}$$

**Theorem 2.6.1.** *[46] Let $v(t) = 0$, $\forall t \geq 0$. Then the system (2.18) is robustly stochastically stabilizable if there exist scalars $\{\epsilon_{1i} > 0, i \in S\}$, $\{\epsilon_{2i} > 0, i \in S\}$, $\{\lambda_{ij} > 0, i, j \in S, i \neq j\}$, and matrices $\{P_i \in \mathcal{S}^n, i \in S\}$, $\{K_i \in \mathbb{R}^{m\times n}, i \in S\}$, such that the following matrix inequalities hold,*

$$\begin{bmatrix}
\overline{\Lambda}_i & * & * & * & * \\
M_i'P_i & -\epsilon_{1i}^{-1}I & * & * & * \\
\overline{N_{ai}} & 0 & -\epsilon_{1i}I & * & * \\
\overline{N_{ei}} & 0 & 0 & -\epsilon_{2i}I & * \\
\overline{E}_i & 0 & 0 & 0 & \epsilon_{2i}M_iM_i' - P_i^{-1}
\end{bmatrix} < 0, \quad i \in S,$$

*where*

$$\overline{\Lambda}_i = \overline{A}_i'P_i + P_i\overline{A}_i + \sum_{j=1, j\neq i}^{N}\left[\frac{\lambda_{ij}}{4}\varepsilon_{ij}^2 I + \frac{1}{\lambda_{ij}}(P_j - P_i)^2\right].$$

*In this case, the state feedback controller is*

$$u(t) = K(r(t))x(t).$$

The proof is omitted, because we will deal with a more general version of the above problem later in Chapter 5, in which general results are obtained. The idea of proving Theorem 2.6.1 is the same as proving Theorem 5.3.1.

Next, we present a sufficient condition to solve the robust $H_\infty$ control problem for system (2.18) to (2.19). Here, apart from the requirement of robust stabilization, the $H_\infty$ performance must be satisfied. The definition of the $H_\infty$ performance is given below.

**Definition 2.6.3.** *[46] [120] Given a scalar $\gamma > 0$, the stochastic system with $u(t) = 0$ is said to be robustly stochastically stable with disturbance attenuation $\gamma$ if it is robustly stochastically stable and under zero initial conditions, $\|z(t)\| < \gamma\|v(t)\|$ for all non-zero $v(t) \in L_2[0,\infty)$ and all admissible uncertainties $\Delta A_i, \Delta B_i, \Delta E_i, \Delta F_i$, where*

$$\|z(t)\| = \left(\mathbb{E}\left\{\int_0^\infty |z(t)|^2 dt\right\}\right)^{\frac{1}{2}}.$$

The robust $H_\infty$ control problem for system (2.18) to (2.19) is solved by the following theorem.

**Theorem 2.6.2.** *[46] Given a scalar $\gamma > 0$, then this system is robustly stochastically stabilizable with disturbance attenuation $\gamma$ if there exist scalars $\{\epsilon_{1i} > 0, i \in S\}$, $\{\epsilon_{2i} > 0, i \in S\}$, $\{\lambda_{ij} > 0, i, j \in S, i \neq j\}$, and matrices $\{P_i \in \mathcal{S}^n, i \in S\}$, $\{K_i \in \mathbb{R}^{m\times n}, i \in S\}$, such that the following matrix inequalities hold,*

$$\begin{bmatrix} \Gamma_i & * & * & * & * & * \\ G_i' P_i & -\gamma^2 I & * & * & * & * \\ \overline{N}_{ai} & 0 & -\epsilon_{1i} I & * & * & * \\ \overline{N}_{ei} & 0 & 0 & -\epsilon_{2i} I & * & * \\ \overline{E}_i & H_i & 0 & 0 & \epsilon_{2i} M_i M_i' - P_i^{-1} & * \\ \overline{C}_i & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0, \quad i \in S,$$

*where*

$$\Gamma_i \triangleq P_i \overline{A}_i + \overline{A}_i' P_i + \epsilon_{1i} P_i M_i M_i' P_i + \sum_{j=1, j\neq i}^{N} \left[\frac{\lambda_{ij}}{4}\varepsilon_{ij}^2 I + \frac{1}{\lambda_{ij}}(P_j - P_i)^2\right],$$

*and*

$$\overline{C}(r(t)) \triangleq C(r(t)) + D(r(t))K(r(t))$$

*In this case, the state feedback controller is*

$$u(t) = K(r(t))x(t).$$

The proof is omitted for the same reason stated in the previous theorem.

## 2.7 Stochastic $H_2/H_\infty$ Control

Recently [127] studies robust $H_2/H_\infty$ control for a class of nonlinear stochastic systems in discrete time case. [81] focuses on a game theory approach to mixed $H_2/H_\infty$ control for a class of stochastic time-varying systems with randomly occurring nonlinearities, also in discrete time cases. Note that there are some other works such as [80], [132], [24], [22], [133], and [136] focusing on nonlinear $H_2/H_\infty$ control problems. Although some general results are obtained under such general kind of nonlinear system, readers are not provided with any feasible cases that can have explicit solutions.

In this section, we illustrate some basic definitions and results obtained from previous works of stochastic $H_2/H_\infty$ control problem with Markovian switching in both finite and infinite time horizon.

### 2.7.1 Finite Horizon with Markovian switching

Define the Markovian switching in the same way as (2.3), and define the state space $M = \{1, 2, \cdots, l\}$. [141] considers the following linear SDEs with Markovian switching,

$$\begin{cases} dx(t) = [A(r_t)x(t) + B_2(r_t)u(t) + B_1(r_t)v(t)]dt \\ \qquad\qquad + [G(r_t)x(t) + H_2(r_t)u(t) + H_1(r_t)v(t)]dW(t), \\ z(t) = \begin{bmatrix} C(r_t)x(t) \\ D(r_t)u(t) \end{bmatrix}, \end{cases} \tag{2.20}$$

where $x(0) = x_0$ and $D(r_t)'D(r_t) \triangleq \mathbf{I}$. Here, $(W(t), 0 \le t \le T)$ is a one-dimensional standard $\mathcal{F}_t$-Brownian motion. We denote $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^{n_z}$, $u(t) \in \mathbb{R}^{n_u}$ and $v(t) \in \mathbb{R}^{n_v}$ as system state, controlled output, control input, and external disturbance of the system (2.20) respectively. The finite horizon stochastic $H_2/H_\infty$ control problem is stated as follows.

**Definition 2.7.1.** *[141] For given disturbance attenuation level $\gamma > 0$, $0 < T < \infty$, the finite horizon mixed $H_2/H_\infty$ control is to find a state feedback control $u_T^*(t,x) = K_{2i}x(t) \in \mathcal{L}_{\mathcal{F}}^2([0,T], \mathbb{R}^{n_u})$ such that*
*(i) The trajectory of the closed-loop system (2.20) starting from $x(0) = x_0 = 0$ satisfies*

$$\begin{aligned}&\sum_{i=1}^{l} \mathbb{E}\left[\int_0^T \left(|C(r_t)x(t)|^2 + |u_T^*(t)|^2\right) dt | r_0 = i\right] \\ \le\ & \gamma^2 \sum_{i=1}^{l} \mathbb{E}\left[\int_0^T |v(t)|^2 dt | r_0 = i\right] \qquad (2.21)\end{aligned}$$

*for $\forall v \neq 0$, $v \in \mathcal{L}_{\mathcal{F}}^2([0,T], \mathbb{R}^{n_v})$.*
*(ii) When the worst case disturbance $v_T^*(t,x) \in \mathcal{L}_{\mathcal{F}}^2([0,T], \mathbb{R}^{n_v})$ is implemented to (2.20), $u_T^*(t,x)$ minimizes the output*

$$J_2^T(u, v_T^*, x_0, i) = \mathbb{E}\left[\int_0^T |z(t)|^2 dt | r_0 = i\right], \quad i \in M. \qquad (2.22)$$

Game theory approach is used in [141] to solve the stochastic $H_2/H_\infty$ control problem.

**Definition 2.7.2.** *[141] If we define*

$$J_1^T(u, v, x_0, i) \triangleq \mathbb{E}\left[\int_0^T (\gamma^2|v(t)|^2 - |z(t)|^2) dt | r_0 = i\right], \quad i \in M,$$

*and*

$$J_2^T(u, v, x_0, i) \triangleq \mathbb{E}\left[\int_0^T (|z(t)|^2) dt | r_0 = i\right], \quad i \in M,$$

*then the finite horizon stochastic $H_2/H_\infty$ control problem is equivalent to finding the Nash equilibrium $(u_T^*, v_T^*)$ defined as*

$$J_1^T(u_T^*, v_T^*, x_0, i) \leq J_1^T(u_T^*, v, x_0, i), \quad \forall v \in \mathcal{L}_{\mathcal{F}}^2([0,T], \mathbb{R}^{n_v}), \quad i \in M,$$

*and*

$$J_2^T(u_T^*, v_T^*, x_0, i) \leq J_2^T(u, v_T^*, x_0, i), \quad \forall u \in \mathcal{L}_{\mathcal{F}}^2([0,T], \mathbb{R}^{n_u}), \quad i \in M.$$

**Remark 2.7.1.** *[141] The first Nash inequality relates to the $H_\infty$ performance, because $J_1^T(u_T^*, v_T^*, x_0, i) \geq 0$ implies (2.21). The second one deals with the $H_2$ performance. If the Nash equilibrium $(u_T^*, v_T^*)$ exists, $u_T^*$ is our desired controller, and $v_T^*$ is the worst case disturbance. In other words, $(u_T^*, v_T^*)$ is a pair of solutions to the stochastic $H_2/H_\infty$ control problem.*

Next, we consider the following stochastic system in [141]:

$$\begin{cases} dx(t) = [A(r_t)x(t) + B_1(r_t)v(t)]dt \\ \qquad\qquad + [G(r_t)x(t) + H_1(r_t)v(t)]dW(t) \\ \quad z(t) = C(r_t)x(t), \quad x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2.23}$$

The perturbation operator $\mathcal{L}_{[0,T]}$ is defined in [141] as follows,

$$\mathcal{L}_{[0,T]}(v(t)) \triangleq C(r_t)x(t; 0, v), \quad \forall v(t) \in \mathcal{L}_{\mathcal{F}}^2([0,T], \mathbb{R}^{n_v}).$$

Its norm is defined in [141] as follows,

$$\begin{aligned} |\mathcal{L}_{[0,T]}| &\triangleq \sup_{v \in \mathcal{L}_{\mathcal{F}}^2([0,T],\mathbb{R}^{n_v}), v \neq 0, x_0 = 0} \frac{|z(t)|_{[0,T]}}{|v(t)|_{[0,T]}} \\ &\triangleq \sup_{v \in \mathcal{L}_{\mathcal{F}}^2([0,T],\mathbb{R}^{n_v}), v \neq 0, x_0 = 0} \frac{\{\sum_{i=1}^l \mathbb{E}[\int_0^T z(t)'z(t)dt | r_0 = i]\}^{\frac{1}{2}}}{\{\sum_{i=1}^l \mathbb{E}[\int_0^T v(t)'v(t)dt | r_0 = i]\}^{\frac{1}{2}}}. \end{aligned}$$

**Definition 2.7.3.** *[141] Let $\gamma > 0$, system (2.23) is said to have $\mathcal{L}_2$-gain less than or equal to $\gamma$ if for any nonzero $v \in \mathcal{L}_{\mathcal{F}}^2([0,T], \mathbb{R}^{n_v})$, $|\mathcal{L}_{[0,T]}| \leq \gamma$.*

The following lemma indicates the relation between the $\mathcal{L}_2$-gain and the differential Riccati equation (DRE).

**Lemma 2.7.1.** *[141] For system (2.23) and given disturbance attenuation $\gamma > 0$, $|\mathcal{L}_{[0,T]}| \leq \gamma$ iff there exists a solution $P = (P_1, P_2, \cdots, P_l)$ with $P_i \geq 0$, $i \in M$, satisfying the following DRE*

$$\begin{cases} \dot{P}_i + A_i'P_i + P_iA_i + G_i'P_iG_i - C_i'C_i + \displaystyle\sum_{j=1}^{l} \pi_{ij}P_j - (P_iB_{1i} + G_i'P_iH_{1i}) \times \\ \quad (\gamma^2 I + H_{1i}'P_iH_{1i})^{-1}(B_{1i}'P_i + H_{1i}'P_iG_i) = 0, \\ \\ \gamma^2 I + H_{1i}'P_iH_{1i} > 0, \quad i \in M. \end{cases}$$

The above lemma originates from [130], which deals with a class of nonlinear stochastic systems. It is applied in [141] as a special case for a linear system. Later, this lemma will also be applied in our thesis in a nonlinear case.

The following theorem presents the main result of the finite horizon stochastic $H_2/H_\infty$ control. First, some notations are introduced.

$$\begin{aligned} \bar{A}(r_t) &\triangleq A(r_t) + B_2(r_t)K_2(r_t), \\ \bar{G}(r_t) &\triangleq G(r_t) + H_2(r_t)K_2(r_t), \\ \bar{Q}_k(r_t) &\triangleq Q_k(r_t) + K_2(r_t)'R_k(r_t)K_2(r_t), \\ \tilde{A}(r_t) &\triangleq A(r_t) + B_1(r_t)K_1(r_t), \\ \tilde{G}(r_t) &\triangleq G(r_t) + H_1(r_t)K_1(r_t), \\ \tilde{Q}_k(r_t) &\triangleq Q_k(r_t) + K_1(r_t)'S_k(r_t)K_1(r_t). \end{aligned}$$

**Theorem 2.7.1.** *[141] For given disturbance attenuation level $\gamma > 0$, the finite horizon $H_2/H_\infty$ control for system (2.20) has a pair of solutions $(u_T^*, v_T^*)$ with*

$$\begin{aligned} u_T^*(t,x) &= -\sum_{i=1}^{l} K_{2i}\chi_{r_t=i}(t)x(t), \\ v_T^*(t,x) &= -\sum_{i=1}^{l} K_{1i}\chi_{r_t=i}(t)x(t), \end{aligned}$$

*if the following four coupled DREs admit solutions* $(P_1, P_2; K_1, K_2)$ *with* $P_1 = (P_{11}, P_{12}, \cdots, P_{1l}) \geq 0$, *and* $P_2 = (P_{21}, P_{22}, \cdots, P_{2l}) \geq 0$.

$$\begin{cases} \dot{P}_{1i} + P_{1i}\bar{A}_i + \bar{A}_i' P_{1i} + \bar{G}_i' P_{1i} \bar{G}_i + \sum_{i=1}^{l} \pi_{ij} P_j - C_i' C_i - K_{2i}' K_{2i} \\ \quad - (B_{1i}' P_{1i} + H_{1i}' P_{1i} \bar{G}_i)' (\gamma^2 I + H_{1i}' P_{1i} H_{1i})^{-1} (B_{1i}' P_{1i} + H_{1i}' P_{1i} \bar{G}_i) = 0, \\ \gamma^2 I + H_{1i}' P_{1i} H_{1i} > 0, \quad i \in M, \end{cases}$$
$$K_{1i}(t) = (\gamma^2 I + H_{1i}' P_{1i} H_{1i})^{-1} (B_{1i}' P_{1i} + H_{1i}' P_{1i} \bar{G}_i),$$

$$\begin{cases} \dot{P}_{2j} + P_{2j}\tilde{A}_j + \tilde{A}_j' P_{2j} + \tilde{G}_j' P_{2j} \tilde{G}_j + \sum_{k=1}^{l} \pi_{jk} P_{2k} + C_j' C_j \\ \quad - (B_{2j}' P_{2j} + H_{2j}' P_{2j} \tilde{G}_j)' (I + H_{2j}' P_{2j} H_{2j})^{-1} (B_{2j}' P_{2j} + H_{2j}' P_{2j} \tilde{G}_j) = 0, \\ I + H_{2j}' P_{2j} H_{2j} > 0, \quad j \in M, \end{cases}$$
$$K_{2j}(t) = (I + H_{2j}' P_{2j} H_{2j})^{-1} (B_{2j}' P_{2j} + H_{2j}' P_{2j} \tilde{G}_j).$$

The proof can be found in [141] and it is omitted here. Note that this theorem can be regarded as a special case of Theorem 6.2.1 in Chapter 6.

### 2.7.2 Infinite Horizon with Markovian switching

The stochastic $H_2/H_\infty$ control problem in infinite time horizon with Markovian switching is also considered in [141], in which the system is similar to (2.20). The concept of stability is considered in [141], similar to the relevant statements in Section 2.4. It is shown that the solvability of four coupled algebraic Riccati equations (AREs) is sufficient to solve the infinite horizon stochastic $H_2/H_\infty$ control problem with Markovian switching. The statements of the main results are omitted here.

## 2.8 Risk-sensitive Control

Risk-sensitive stochastic control was first considered by [52] and [58]. [44], [58] and [59] study the problems of risk-sensitive control connecting with differential games. Risk-sensitive control can be applied to financial mathematics, see for example [89] and [30]. In addition, [39] and [49] investigates the relationship between risk-sensitive control and robust control. Recently, generalized risk-sensitive control with full and partial state observation was investigated by [31]. By the same author, stochastic risk-sensitive control for a class of nonlinear system was concerned in [30], where the nonlinear term is designed in the drift part. Based on [30], risk-sensitive control for a class of nonlinear square-root processes was studied by [42], which includes nonlinearity in the diffusion term. In [31], [30] and [42] the optimal control is given in an explicit form by using completion of square method. Because the problem formulation of Chapter 7 is based on [30], here we omit introducing the preliminary results in [30].

## 2.9 Some Useful Lemmas

We introduce some lemmas that are useful in this thesis.

**Lemma 2.9.1.** *[112] Let $x(t)$ satisfy*

$$dx(t) = b(t, x(t), r_t)dt + \sigma(t, x(t), r_t)dW(t),$$

*and $\varphi(\cdot, \cdot, i) \in C^2([0, \infty) \times \mathbb{R}^n), i = 1, \cdots, \delta$, be given. Then,*

$$\begin{aligned} & \mathbb{E}\{\varphi(T, x(T), r_T) - \varphi(s, x(s), r_s)|r_s = i\} \\ = \; & \mathbb{E}\left\{\int_s^T [\varphi_t(t, x(t), r_t) + \Gamma_\varphi(t, x(t), r_t)]dt|r_s = i\right\}, \end{aligned}$$

*where*

$$\begin{aligned} \Gamma_\varphi(t, x, i) \; = \; & \frac{1}{2}\mathrm{tr}[\sigma(t, x, i)'\varphi_{xx}(t, x, i)\sigma(t, x, i)] \\ & + b(t, x, i)'\varphi_x(t, x, i) + \sum_{j=1}^{\delta} \pi_{ij}\varphi(t, x, j). \end{aligned}$$

**Lemma 2.9.2.** *[93] Let a matrix $M \in \mathbb{R}^{m\times n}$ be given. Then there exists a unique matrix $M^\dagger \in \mathbb{R}^{n\times m}$ such that*

$$\begin{cases} MM^\dagger M = M, & M^\dagger M M^\dagger = M^\dagger, \\ (MM^\dagger)' = MM^\dagger, & (M^\dagger M)' = M^\dagger M, \end{cases} \tag{2.24}$$

*where the matrix $M^\dagger$ is called the Moore-Penrose pseudo inverse of $M$.*

**Lemma 2.9.3.** *[2] [4] For a symmetric matrix $\mathsf{S}$, we have*

(i)$\mathsf{S}^\dagger = (\mathsf{S}^\dagger)'$,

(ii)$\mathsf{S}\mathsf{S}^\dagger = \mathsf{S}^\dagger\mathsf{S}$,

(iii) $\mathsf{S} \geq 0$ *if and only if* $\mathsf{S}^\dagger \geq 0$.

**Lemma 2.9.4.** *(Extended Schur's Lemma [5]). Let matrices $\mathsf{M} = \mathsf{M}'$, $\mathsf{N}$, and $\mathsf{R} = \mathsf{R}'$ be given with appropriate dimensions. Then the following conditions are equivalent:*

(i) $\mathsf{M} - \mathsf{N}\mathsf{R}^\dagger\mathsf{N}' \geq 0$ *and* $\mathsf{N}(I - \mathsf{R}\mathsf{R}^\dagger) = 0$, $\mathsf{R} \geq 0$*;*

(ii) $\begin{bmatrix} \mathsf{M} & \mathsf{N} \\ \mathsf{N}' & \mathsf{R} \end{bmatrix} \geq 0$*;*

(iii) $\begin{bmatrix} \mathsf{R} & \mathsf{N}' \\ \mathsf{N} & \mathsf{M} \end{bmatrix} \geq 0$.

**Lemma 2.9.5.** *[2] Let matrices $\mathsf{L}$, $\mathsf{M}$ and $\mathsf{N}$ be given with appropriate sizes. Then the following matrix equation*

$$\mathsf{L}\mathsf{X}\mathsf{M} = \mathsf{N}, \tag{2.25}$$

*has a solution $\mathsf{X}$ if and only if*

$$\mathsf{L}\mathsf{L}^\dagger\mathsf{N}\mathsf{M}^\dagger\mathsf{M} = \mathsf{N}. \tag{2.26}$$

*Moreover, any solution to (2.25) is represented by*

$$\mathsf{X} = \mathsf{L}^\dagger\mathsf{N}\mathsf{M}^\dagger + \mathsf{S} - \mathsf{L}^\dagger\mathsf{L}\mathsf{S}\mathsf{M}\mathsf{M}^\dagger, \tag{2.27}$$

*where $\mathsf{S}$ is a matrix with an appropriate size.*

**Lemma 2.9.6.** *(See, e.g., [109]) Let $\mathcal{A}, \mathcal{D}, \mathcal{S}, \mathcal{W}$ and $F$ be real matrices of appropriate dimensions such that $\mathcal{W} > 0$ and $F'F \leq I$. Then we have the following.*

*1) For scalar $\epsilon > 0$ and vectors $x, y \in \mathbb{R}^n$*

$$2x'\mathcal{D}F\mathcal{S}y \leq \epsilon^{-1}x'\mathcal{D}\mathcal{D}'x + \epsilon y'\mathcal{S}'\mathcal{S}y.$$

*2)For any scalar $\epsilon > 0$ such that $\mathcal{W} - \epsilon\mathcal{D}\mathcal{D}' > 0$*

$$(\mathcal{A} + \mathcal{D}F\mathcal{S})'\mathcal{W}^{-1}(\mathcal{A} + \mathcal{D}F\mathcal{S}) \leq \mathcal{A}'(\mathcal{W} - \epsilon\mathcal{D}\mathcal{D}')^{-1}\mathcal{A} + \epsilon^{-1}\mathcal{S}'\mathcal{S}.$$

## 2.10 Summary

In this preliminary chapter we have reviewed some basic results of optimal and robust control theory, including LQ control in both finite and infinite horizon, robust stabilization and robust $H_\infty$ control, $H_2/H_\infty$ control in both finite and infinite horizon and risk-sensitive control. Some previous definitions, lemmas, and theorems that are useful to this thesis are introduced in this chapter. We particularly focus on the most recent research results that are related to our thesis. Note that in this chapter some new results are obtained. Some previous results are improved without changing the spirit of the original work. In this case, they are extended into more general cases. Most of the literatures reviewed in this chapter investigate the linear systems. If the system is nonlinear, which is very common in real situations, it is unknown whether the previous elegant results can still be obtained or not. The following four chapters are going to investigate this issue for the system with a class of nonlinearities.

# Chapter 3

# Nonlinear Optimal Stochastic Control of Systems with Markovian Switching in Finite Time Horizon

## 3.1 Introduction

This chapter extends optimal LQ control in [74] to a more general case, which deals with the optimal control of indefinite stochastic nonlinear system with Markovian switching appearing in system coefficients. Two motivating examples are given first. Then our nonlinear optimal control problem is formulated in the next section, where the nonlinearity is formulated by a combination of two different diffusion terms. It is known that only some of the nonlinear SDEs have solution. Thus in our system the existence and uniqueness of the solution is discussed. In the problem formulation, a new type of coupled generalized Riccati equations (CGREs) is introduced, and it is proved that if there exists solution to CGREs, then our nonlinear optimal control problem is well-posed. Optimal control laws constructed by the solution to the CGREs are obtained. When we solve the optimal control problem, completion of square method is used, and there are difficulties in dealing with the nonlinearity terms. Here it is highlighted that within this nonlinear system an explicit solution is found, which is a very rare case. In addition, the optimal control laws obtained are linear with state, which is very similar to the characteristics of the results in optimal LQ control problems. The feasibility of the assumption of the solvability of the new CGREs is discussed. An application to finance is introduced. An illustrative example is given.

## 3.2 Two motivating examples

As is emphasized in the previous chapter, the important properties of the linear systems are that they have an explicit solution, they appear in various different applications, and several optimal control problems for such systems have explicit closed form solutions. While nonlinear systems appear in many applications, they in general do not have these desirable features of linear systems. Indeed, nonlinear SDEs with an explicit solution are very rare. One such example is the following (see equation (4.29) of [64]):

$$\begin{cases} dx(t) = \frac{1}{2}x(t)dt + \sqrt{x^2(t)+1}dW(t) \\ x(0) = x_0, \end{cases} \tag{3.1}$$

the solution of which is $x(t) = \sinh(W(t) + \operatorname{arcsinh} x_0)$. This equation has a square-root type of nonlinearity, and despite the fact that it admits an explicit solution, no control problems for such an equation have been formulated until now.

Another square-root nonlinearity appears in an optimal investment problem. Consider a market of two assets: the bank account $B(t)$, and a stock $S(t)$, the equations of which are

$$\begin{cases} dB(t) = B(t)r(t)dt, \\ \\ dS(t) = S(t)[\mu(t)dt + \sigma(t)dW_1(t)], \\ \\ B(0) = B_0 \quad \text{and} \quad S(0) = S_0 \quad \text{are given.} \end{cases} \tag{3.2}$$

Here $r(t)$ is the interest rate of the bank account $B(t)$, whereas $\mu(t)$ and $\sigma(t)$ are the appreciation rate and the volatility of the stock $S(t)$, respectively. In this market we consider an investor endowed with the initial wealth $y_0$. Let $v_B(t)$ and $v_S(t)$ denote the number of shares that the investor holds in $B(t)$ and $S(t)$, respectively. Then the investors wealth at time $t$ is $y(t) \triangleq v_B(t)B(t) + v_S(t)S(t)$. If $u(t) \triangleq v_S(t)S(t)$ denotes the amount of the investor's wealth invested in the stock, then the equation of the self-financing portfolio is (see, e.g., [69]):

$$\begin{cases} dy(t) = [r(t)y(t) + (\mu(t) - r(t))u(t)]dt + \sigma(t)u(t)W_1(t), \\ \\ y(0) = y_0 > 0. \end{cases}$$

Let $\mu(t)$ be a given process and $\sigma(t)$ a deterministic function, whereas for the interest rate $r(t)$ we assume that it follows the Cox-Ingersoll-Ross (CIR) process,

$$\begin{cases} dr(t) = [ar(t) + b]dt + \sqrt{r(t)}dW_2(t), \\ \\ r(0) = r_0, \end{cases} \tag{3.3}$$

for some constants $a$ and $b$. Moreover, we assume that $\mu(t) - r(t)$ is a deterministic function (note that this is typical assumption in a market with stochastic interest rate, see, e.g. [15]). We are interested only in the controls $u(t)$ that ensure $y(t) > 0$ $a.s.$ $\forall t \in [0, T]$. For such controls, the differential of the $x(t) \triangleq \log y(t)$ is

$$\begin{cases} dx(t) = [r(t) + (\mu(t) - r(t))v(t) - \sigma^2 v^2(t)/2]dt + \sigma(t)v(t)dW_1(t), \\ \\ x(0) = x_0 = \log y_0 > 0, \end{cases} \tag{3.4}$$

where $v(t) \triangleq u(t)/y(t)$. If for some $\hat{x}_0 \in \mathbb{R}$ we define the process

$$\hat{x}(t) \triangleq \hat{x}_0 + x(t) - x_0 + \int_0^t \frac{1}{2}\sigma^2 v^2(s)ds, \tag{3.5}$$

then its differential is

$$\begin{cases} d\hat{x}(t) = [r(t) + (\mu(t) - r(t))v(t)]dt + \sigma(t)v(t)dW_1(t), \\ \\ \hat{x}(0) = \hat{x}_0. \end{cases}$$

The problem of optimal investment for the logarithmic utility is the optimal control problem of maximizing $\mathbb{E}[x(T)]$ subject to (3.3) and (3.4). From the definition (3.5), it is clear that this is equivalent to the problem of minimizing

$$\mathbb{E}\left[\int_0^T \frac{1}{2}\sigma^2 v^2(s)ds - \hat{x}(T)\right],$$

subject to (3.3) and (3.5). Thus, this is a nonlinear optimal control problem with a quadratic cost.

Motivated by these two examples of nonlinear stochastic systems that either have an explicit solution, or lead to optimal control problems with a quadratic cost, in the next section we introduce a class of nonlinear stochastic control systems with a square-root type of nonlinearity that contains these two examples as special cases. Moreover, we permit for Markovian switching in system coefficients.

## 3.3 Problem Formulation and CGREs

### 3.3.1 Problem Formulation.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given filtered complete probability space, where there exist a $m\times 1$ -dimensional Brownian motion $(W_1(t), 0 \leq t \leq T)$, a one-dimensional standard Brownian motion $(W(t), 0 \leq t \leq T)$, a $\eta \times 1$ -dimensional Brownian motion $(W_2(t), 0 \leq t \leq T)$, and a Markov chain $(r_t \in \{1, 2, \cdots, \delta\}, 0 \leq t \leq T)$ with generator $\Pi = (\pi_{ij})$ specified in (2.3). We assume that $W_1(t)$, $W(t)$, $W_2(t)$ and the process $r_t$ are mutually independent.

**Assumption 3.3.1.** *The data that appear in the nonlinear optimal control prob-*

*lem (3.6)-(3.20) satisfy, for every i,*

$$\begin{cases} H_{1i}(\cdot), L_{ki}(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{m}), \\ A_{1i}(\cdot), C_{1i}(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{n\times m}), \\ A_{2i}(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{n\times n}), \\ C_{2i}(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{m\times m}), \\ B_{1i}(\cdot), D_{1i}(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{n\times n_u}), \\ E_{i}(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{n\times \eta}), \\ Q_{ki}(\cdot) \in L^{\infty}(0,T;\mathcal{S}^{n}), \\ R_{ki}(\cdot), R_i \in L^{\infty}(0,T;\mathcal{S}^{n_u}), \\ Q_{i}(\cdot) \in L^{\infty}(0,T;\mathcal{S}^{m+n}), \\ L_{i}(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{(m+n)\times n_u}), \\ L_{di}(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{m+n}), \\ L_{ei}(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{n_u}), \\ \bar{H}_{i} \in \mathcal{S}^{m+n}, \\ \bar{L}_{ci} \in \mathbb{R}^{m+n}. \end{cases}$$

Considering the financial system introduced in Section 3.2, we set two separate $x_1$ and $x_2$ when we formulate our nonlinear optimal control problem, where the equation for $x_1$ is a special case of the CIR model, whereas equation for $x_2$ is a generalized version of (3.1). Consider the following nonlinear SDEs with Markovian switching:

$$\begin{cases} dx_1(t) = [G_1(t,r_t)x_1(t) + H_1(t,r_t)]dt + \Gamma_1(x_1(t),t,r_t)dW_1(t) \\ dx_2(t) = [A_1(t,r_t)x_1(t) + A_2(t,r_t)x_2(t) + B_1(t,r_t)u(t)]dt \\ \qquad + [C_1(t,r_t)x_1(t) + C_2(t,r_t)x_2(t) + D_1(t,r_t)u(t)]dW(t) \\ \qquad + \Gamma_2(x_1(t),x_2(t),t,r_t)dW_2(t) \\ x_1(0) = x_{10} > 0, \quad x_2(0) = x_{20} > 0, \end{cases} \tag{3.6}$$

where

$$G_1(t, r_t) \triangleq \text{diag}[g_1(t, r_t), g_2(t, r_t), \ldots, g_m(t, r_t)],$$

i.e., a $m \times m$ diagonal matrix, in which the diagonal elements are

$$g_1(t, r_t), g_2(t, r_t), \ldots, g_m(t, r_t).$$

Here, $g_1(t, r_t), g_2(t, r_t), \ldots, g_m(t, r_t)$ are all coefficients in terms of scalars. In addition,

$$\Gamma_1(x_1(t), t, r_t) \triangleq \text{diag}[\sqrt{x_{11}(t)}, \sqrt{x_{12}(t)}, \ldots, \sqrt{x_{1m}(t)}], \tag{3.7}$$

$$\Gamma_2(x_1(t), x_2(t), u(t), t, r_t) \triangleq E(t, r_t) F(x_1(t), x_2(t), u(t), t, r_t), \tag{3.8}$$

and

$$F(x_1(t), x_2(t), u(t), t, r_t) \triangleq \text{diag}(\sqrt{\phi_1}, \sqrt{\phi_2}, \ldots, \sqrt{\phi_\eta}). \tag{3.9}$$

Among $\phi_1, \phi_2, \ldots, \phi_\eta$, we denote each of them as $\phi_k$, where $k = 1, 2, \ldots, \eta$. We define

$$\phi_k \triangleq x_2(t)' Q_k(t, r_t) x_2(t) + u(t)' R_k(t, r_t) u(t) + x_1(t)' L_k(t, r_t) + Z_k(t, r_t). \tag{3.10}$$

We assume that $Q_k(t, r_t) \geq 0$, $R_k(t, r_t) \geq 0$, $Z_k(t, r_t) > 0$, and the components of $L_k(t, r_t)$ are non-negative, for all $k$.

The state $x_1$ is independent of $x_2$, and it is a special case of the CIR model of equation (4.1) in [38] (page 387), where the existence and uniqueness of solution for such an equation is proved.

The equation for $x_2$ may not have a solution for all controls $u$. For our purposes the controls that are affine in $x_2$ are important. Under such controls, the term $\Gamma_2$ in (3.8) has a bounded first derivative with respect to $x_2$. Thus it satisfies Theorem 3.13 in [83], page 89, where existence and uniqueness of general SDEs is proved. In summary, under controls that are affine in $x_2$, the system of equations

(3.6) has a unique solution.

Here we discuss and explain the existence and uniqueness of $x_2$ in a bit detail. We are interested in investigating the property of the nonlinear term in the SDEs of $dx_2$, which is $\Gamma_2(x_1(t), x_2(t), t, r_t)dW_2(t)$. First our aim is to check whether the term $\Gamma_2(x_1(t), x_2(t), t, r_t)$ satisfies the Lipschitz condition. In order to provide an intuitive derivation, the example illustrated here is a scalar case, which is a special case of our original problem. Also, for simplicity we neglect Markovian switching from $\Gamma_2(x_1(t), x_2(t), t, r_t)$, because the condition for the existence and uniqueness of solution to SDEs with and without Markovian switching is quite similar, see [83].

Then we simplify the diagonal matrix $F$ into a scalar, namely, define $F \triangleq \sqrt{\phi}$. Then we rewrite $\Gamma_2 \triangleq EF$, where $E$ is also a scalar. In addition, we define our special case of $\phi$ as

$$\phi \triangleq qx_2^2 + ru^2 + lx_1 + z.$$

Substituting the affine control $u = ax + b$ into the above equation, we have

$$\phi = (q + ra^2)x_2^2 + 2abrx_2 + rb^2 + lx_1 + z.$$

Next, we take the first derivative of $\Gamma_2$ with respect to $x_2$,

$$\frac{d\Gamma_2}{dx_2} = \frac{d(E\sqrt{\phi})}{dx_2} = E\frac{d(\sqrt{\phi})}{dx_2} = E\frac{d(\sqrt{\phi})}{d\phi}\frac{d\phi}{dx_2}.$$

Then

$$\frac{d\Gamma_2}{dx_2} = \frac{E(q + ra^2)x_2 + Eabr}{\sqrt{(q + ra^2)x_2^2 + 2abrx_2 + rb^2 + lx_1 + z}}. \tag{3.11}$$

Note that with our assumption of $Q_k(t, r_t) \geq 0$, $R_k(t, r_t) \geq 0$, $Z_k(t, r_t) > 0$, and the components of $L_k(t, r_t)$ are non-negative, for all $k$, thus here we have $q \geq 0$, $r \geq 0$, $l \geq 0$ and $z > 0$, accordingly. Then we have

$$\phi = qx_2^2 + ru^2 + lx_1 + z > 0,$$

and equivalently,

$$\phi = (q + ra^2)x_2^2 + 2abrx_2 + rb^2 + lx_1 + z > 0.$$

Hence, the denominator of right side of (3.11), i.e., $\sqrt{\phi}$ is well defined. Also, it is straight forward that (3.11) is bounded for all $x_2$. Then our scalar case $\Gamma_2$ is Lipschitz, since its first derivative w.r.t. $x_2$ is bounded.

It is notable that function $\Gamma_2 = E\sqrt{\phi}$ is different from one type of square root function, such as $\sqrt{x}$ because the first derivative of $\sqrt{x}$ w.r.t. $x$ becomes infinitely large when $x$ goes to 0. When the parameters in (3.11) are fixed, the plotted function of $\Gamma_2 = E\sqrt{\phi}$ never becomes infinitely steep whatever value $x_2$ takes, which means the first derivative of $\Gamma_2$ w.r.t. $x_2$ never becomes infinitely large. This can be seen when (3.11) is plotted in software such as Matlab, the value of (3.11) is bounded for all $x_2$.

Additionally, it is easy to find a constant $K$ such that the following holds:

$$|(E\sqrt{\phi})|^2 \leq K(1 + |x_2|^2),$$

which is the linear growth condition in [83]. Up to here, we say our special scalar case satisfies both Lipschitz and linear growth conditions, so the existence and uniqueness of $x_2$ is obtained. Since the scalar case is a special case of our original problem, we can say this result can be extended to our original system (3.6).

If we denote

$$x(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{3.12}$$

then we can rewrite equation (3.6) into the following

$$\begin{cases} dx(t) = [A(t,r_t)x(t) + B(t,r_t)u(t) + H(t,r_t)]dt \\ \qquad + [C(t,r_t)x(t) + D(t,r_t)u(t)]dW(t) \\ \qquad + \theta_1(x(t),u(t),t,r_t)dW_1(t) + \theta_2(x(t),u(t),t,r_t)dW_2(t), \\ x(s) = y, \end{cases} \tag{3.13}$$

where

$$A(t,r_t) \triangleq \begin{bmatrix} G_1(t,r_t) & \mathbf{0} \\ A_1(t,r_t) & A_2(t,r_t) \end{bmatrix}, \quad B(t,r_t) \triangleq \begin{bmatrix} \mathbf{0} \\ B_1(t,r_t) \end{bmatrix},$$

$$C(t,r_t) \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ C_1(t,r_t) & C_2(t,r_t) \end{bmatrix}, \quad D(t,r_t) \triangleq \begin{bmatrix} \mathbf{0} \\ D_1(t,r_t) \end{bmatrix},$$

$$H(t,r_t) \triangleq \begin{bmatrix} H_1(t,r_t) \\ \mathbf{0} \end{bmatrix}, \quad \theta_1(x_1(t),t,r_t) \triangleq \begin{bmatrix} \Gamma_1(x_1(t),t,r_t) \\ \mathbf{0} \end{bmatrix},$$

$$\theta_2((x_1(t),x_2(t),u(t),t,r_t) \triangleq \begin{bmatrix} \mathbf{0} \\ \Gamma_2(x_1(t),x_2(t),u(t),t,r_t) \end{bmatrix}. \tag{3.14}$$

Here, $s \in [0,T)$ is the initial time, and $y \in \mathbb{R}^{m+n}$ is the initial state.

**Definition 3.3.1.** *[74] An admissible control $u(\cdot)$ is any $\mathcal{F}_t$-adapted process under which the equation (3.6) has a unique solution. The set of all admissible controls is denoted by $\mathcal{U}$.*

We give the following notations, which will be used throughout this chapter. We define

$$M \triangleq \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}, \tag{3.15}$$

where $\mathbf{0}$ is a $m \times n$ zero matrix, and $\mathbf{I}$ is a $n \times n$ identity matrix. We define $e_k$ as an $\eta \times 1$ elementary vector, whose $k$-th element is 1, while other elements are 0. For simplicity, we define

$$N_{ki}(t) \triangleq ME_i(t)e_k. \tag{3.16}$$

We define $\epsilon_a$ as an $m \times 1$ elementary vector, whose $a$-th element is 1, while other elements are 0. Then each element of vector $x_1$ can be expressed as

$$x_{1a} = \epsilon'_a x_1. \tag{3.17}$$

Define

$$b_a \triangleq \begin{bmatrix} \epsilon'_a & \mathbf{0} \end{bmatrix}, \tag{3.18}$$

where $\mathbf{0}$ is a $1 \times n$ zero matrix. Define

$$\tilde{M} \triangleq \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}, \tag{3.19}$$

where $\mathbf{0}$ is a $n \times m$ zero matrix, and $\mathbf{I}$ is a $m \times m$ identity matrix. For each $(s, y)$ and $u(\cdot) \in \mathcal{U}$ the cost functional is

$$\begin{aligned} & J(s, y, i; u(\cdot)) \\ = \ & \mathbb{E}\Bigg\{ \int_s^T \Bigg( \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q(t, r_t) & L(t, r_t) \\ L(t, r_t)' & R(t, r_t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + x(t)' L_d(t, r_t) \\ & + u(t)' L_e(t, r_t) \Bigg) dt + x(T)' \bar{H}(r_T) x(T) + \bar{L}_c(r_T)' x(T) \Bigg| r_s = i \Bigg\}. \end{aligned} \tag{3.20}$$

As is emphasized in [74] that since we allow symmetric matrices

$$\begin{bmatrix} Q_i & L_i \\ L_i' & R_i \end{bmatrix}, \quad i = 1, \cdots, \delta,$$

to be indefinite, we say our stochastic nonlinear optimal control problem is an indefinite control problem.

The aim of our optimal control problem is to minimize the cost functional $J(s, y, i; u(\cdot))$ subject to (3.13). Similar to [74] the value function is defined as

$$V(s, y, i) \triangleq \inf_{u(\cdot) \in \mathcal{U}} J(s, y, i; u(\cdot)). \tag{3.21}$$

We provide the following definition that originates from [74].

**Definition 3.3.2.** *[74] The optimal control problem (3.6)-(3.21) is called well-posed if*

$$V(s, y, i) > -\infty, \quad \forall (s, y) \in [0, T) \times \mathbb{R}^n, \quad \forall i = 1, \ldots, \delta.$$

*An admissible pair* $(x^*(\cdot), u^*(\cdot))$ *is called optimal (w.r.t. the initial condition* $(s, y, i)$*) if* $u^*(\cdot)$ *achieves the infimum of* $J(s, y, i; u(\cdot))$.

### 3.3.2 Coupled Generalized Differential Riccati Equations.

First, we assume the following holds, ($t$ is omitted)

$$\begin{cases} L_{ci}(T) = \bar{L}_{ci}, \\ 2P_iH_i + \sum_{a=1}^{m} b'_a P_{aa} + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}\tilde{M}L_{ki} + L_{di} + \dot{L}_{ci} + A'_iL_{ci} - (C'_iP_iD_i \\ \quad + P_iB_i + L_i)(D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} + R_i)^{\dagger}(L_{ei} + B'_iL_{ci}) = 0, \\ (D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} + R_i)(D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} \\ \quad + R_i)^{\dagger}(L_{ei} + B'_iL_{ci}) - (L_{ei} + B'_iL_{ci}) = 0, \qquad \text{a.e.} \quad t \in [0,T], \\ \quad i = 1, \cdots, \delta \end{cases} \tag{3.22}$$

Now we introduce a new type of coupled generalized Riccati equations (CGREs) as follows, ($t$ is omitted)

$$\begin{cases} \dot{P}_i + P_iA_i + A'_iP_i + C'_iP_iC_i + \sum_{j=1}^{\delta} \pi_{ij}P_j + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}MQ_{ki}M' + Q_i \\ \quad - (C'_iP_iD_i + P_iB_i + L_i)(D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} + R_i)^{\dagger}(D'_iP_iC_i \\ \quad + B'_iP_i + L'_i) = 0, \\ P_i(T) = \bar{H}_i, \\ (D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} + R_i)(D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} \\ \quad + R_i)^{\dagger}(D'_iP_iC_i + B'_iP_i + L'_i) - (D'_iP_iC_i + B'_iP_i + L'_i) = 0, \\ D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} + R_i \geq 0, \qquad \text{a.e.} \quad t \in [0,T], \quad i = 1, \cdots, \delta. \end{cases} \tag{3.23}$$

The solvability of CGREs (3.23) is discussed in Section 3.5. We assume the CGREs have a solution. Compare our new CGREs (3.23) with (2.5) introduced in [74], and we see the difference is that we have two additional terms,

$\sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}MQ_{ki}M'$ and $\sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki}$. This is due to the nonlinearity terms $\phi_k$ in (3.10). The detailed derivation can be found in Section 3.4. Note that (2.5) is a special case of (3.23). If $D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} + R_i \neq 0$, for every $i$, then the CGREs (3.23) become:

$$\begin{cases} \dot{P}_i + P_iA_i + A'_iP_i + C'_iP_iC_i + \sum_{j=1}^{\delta} \pi_{ij}P_j + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}MQ_{ki}M' + Q_i \\ \quad - (C'_iP_iD_i + P_iB_i + L_i)(D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} + R_i)^{-1}(D'_iP_iC_i \\ \quad + B'_iP_i + L'_i) = 0, \\ P_i(T) = \bar{H}_i, \\ D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} + R_i > 0, \quad \text{a.e.} \quad t \in [0,T], \quad i = 1, \cdots, \delta. \end{cases} \tag{3.24}$$

When $D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} + R_i = 0$ for every $i$, the CGREs (3.23) become:

$$\begin{cases} \dot{P}_i + P_iA_i + A'_iP_i + C'_iP_iC_i + \sum_{j=1}^{\delta} \pi_{ij}P_j + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}MQ_{ki}M' + Q_i = 0, \\ P_i(T) = \bar{H}_i, \\ D'_iP_iC_i + B'_iP_i + L'_i = 0, \\ D'_iP_iD_i + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} + R_i = 0, \quad \text{a.e.} \quad t \in [0,T], \quad i = 1, \cdots, \delta. \end{cases}$$

## 3.4 Solution to Optimal Control Problem

In this section, we show that the solvability of the CGREs (3.23) is sufficient for the well-posedness of our nonlinear optimal control problem (3.6)-(3.21). Optimal linear state feedback control laws are obtained explicitly, constructed by the solution to the CGREs (3.23).

**Theorem 3.4.1.** *Denote $P_{abi}(t)$ as each element of matrix $P_i(t)$. If the CGREs (3.23) have a solution, then the stochastic nonlinear optimal control problem (3.6)-*

*(3.21) is well-posed. Moreover, all optimal controls are obtained explicitly as follows:*

$$
\begin{aligned}
& u^*(t) \\
= \; & -\sum_{i=1}^{\delta}\Big\{\Big[[D_i(t)'P_i(t)D_i(t)+\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)R_{ki}(t) \\
& +R_i(t)]^{\dagger}[D_i(t)'P_i(t)C_i(t)+B_i(t)'P_i(t)+L_i(t)']+Y_i(t)-[D_i(t)'P_i(t)D_i(t) \\
& +\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)R_{ki}(t)+R_i(t)]^{\dagger}[D_i(t)'P_i(t)D_i(t) \\
& +\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)R_{ki}(t)+R_i(t)]Y_i(t)\Bigg]x(t)+z_i(t) \\
& -[D_i(t)'P_i(t)D_i(t)+\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)R_{ki}(t)+R_i(t)]^{\dagger}[D_i(t)'P_i(t)D_i(t) \\
& +\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)R_{ki}(t)+R_i(t)]z_i(t)+\frac{1}{2}[D_i(t)'P_i(t)D_i(t) \\
& +\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)R_{ki}(t)+R_i(t)]^{\dagger}[L_{ei}(t) \\
& +B_i(t)'L_{ci}(t)]\Big\}\chi_{\{r_t=i\}}(t), \qquad (3.25)
\end{aligned}
$$

*where* $Y_i(\cdot)\in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u\times(m+n)})$, $z_i(\cdot)\in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u})$. *Furthermore, the value function is obtained as follows:*

$$
\begin{aligned}
V(s,y,i) &\equiv \inf\nolimits_{u(\cdot)\in\mathcal{U}}J(s,y,i;u(\cdot)) \\
&= y'P_i(s)y+L_{ci}(s)'y+\mathbb{E}\left[\int_s^T\zeta(t,r_t)dt|r_s=i\right], \qquad (3.26)
\end{aligned}
$$

*where*

$$
\begin{aligned}
\zeta_i(t) &= \sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)Z_{ki}(t)+L'_{ci}(t)H_i(t)-\frac{1}{4}[L_{ei}(t) \\
&\quad +B_i(t)'L_{ci}(t)]'[D_i(t)'P_i(t)D_i(t)+\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)R_{ki}(t) \\
&\quad +R_i(t)]^{\dagger}[L_{ei}(t)+B_i(t)'L_{ci}(t)], \quad i=1,\ldots,\delta. \qquad (3.27)
\end{aligned}
$$

*Proof.* Denote the solution to the CGREs (3.23) by $(P_1(\cdot), \ldots, P_\delta(\cdot)) \in C^1(0, T; (\mathcal{S}^{m+n})^\delta)$. According to the system (3.13), by Lemma 2.9.1, we have

$$\begin{aligned}
& \mathbb{E}[x(T)'P(r_T)(T)x(T)] \\
= \quad & y'P_i(s)y + \mathbb{E}\bigg[\int_s^T \{x'\dot{P}_i(t)x + \sum_{j=1}^{\delta} \pi_{ij} x'P_j(t)x + 2x'P_i(t)[A_i(t)x + B_i(t)u \\
& +H_i(t)] + [C_i(t)x + D_i(t)u]'P_i(t)[C_i(t)x + D_i(t)u] \\
& +\mathrm{tr}[\theta_{1i}(x_1, t)'P_i(t)\theta_{1i}(x_1, t)] \\
& +\mathrm{tr}[\theta_{2i}(x_1, x_2, u, t)'P_i(t)\theta_{2i}(x_1, x_2, u, t)\bigg| r_s = i]\}dt\bigg]. \qquad (3.28)
\end{aligned}$$

We work on $\mathrm{tr}[\theta_{1i}(x_1, t)'P_i(t)\theta_{1i}(x_1, t)]$ first. Note that

$$\mathrm{tr}[\theta_{1i}(x_1, t)'P_i(t)\theta_{1i}(x_1, t)] = \mathrm{tr}[P_i(t)\theta_{1i}(x_1, t)\theta_{1i}(x_1, t)'], \qquad (3.29)$$

in which

$$\theta_{1i}(x_1, t)\theta_{1i}(x_1, t)' = \begin{bmatrix} \Gamma_{1i}(x_1, t)\Gamma_{1i}(x_1, t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad (3.30)$$

and according to (3.7), we have

$$\Gamma_{1i}(x_1, t)\Gamma_{1i}(x_1, t) = \mathrm{diag}(x_{11}, x_{12}, \ldots, x_{1m}). \qquad (3.31)$$

As $P_i(\cdot) \in L^\infty(0, T; \mathcal{S}^{m+n})$, from (3.29) to (3.31), we have

$$\mathrm{tr}[\theta_{1i}(x_1, t)'P_i(t)\theta_{1i}(x_1, t)] = \sum_{a=1}^{m} P_{aai}(t)x_{1a}. \qquad (3.32)$$

According to (3.17), we rewrite (3.32) as follows,

$$\mathrm{tr}[\theta_{1i}(x_1, t)'P_i(t)\theta_{1i}(x_1, t)] = \sum_{a=1}^{m} P_{aai}(t)\epsilon_a' x_1.$$

According to (3.12) and (3.18), we transform $x_1$ in the above equation into forms of $x$ only, then

$$\mathrm{tr}[\theta_{1i}(x_1, t)'P_i(t)\theta_{1i}(x_1, t)]$$

$$\begin{aligned} &= \left[\sum_{a=1}^{m} P_{aai}(t)\epsilon_a' \quad \mathbf{0}\right] x \\ &= \sum_{a=1}^{m} P_{aai}(t) \left[\epsilon_a' \quad \mathbf{0}\right] x \\ &= \sum_{a=1}^{m} x' b_a' P_{aai}. \end{aligned} \tag{3.33}$$

Next, we work on $\mathrm{tr}[\theta_{2i}(x_1, x_2, u, t)' P_i(t) \theta_{2i}(x_1, x_2, u, t)]$. Note that

$$\begin{aligned} &\mathrm{tr}[\theta_{2i}(x_1, x_2, u, t)' P_i(t) \theta_{2i}(x_1, x_2, u, t)] \\ &= \mathrm{tr}[P_i(t) \theta_{2i}(x_1, x_2, u, t) \theta_{2i}(x_1, x_2, u, t)']. \end{aligned} \tag{3.34}$$

According to (3.8) and (3.14), we have

$$\begin{aligned} &\theta_{2i}(x_1, x_2, u, t)\theta_{2i}(x_1, x_2, u, t)' \\ = &\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Gamma_{2i}(x_1, x_2, u, t)\Gamma_{2i}(x_1, x_2, u, t)' \end{bmatrix} \\ = &\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & E_i(t) F_i(x_1, x_2, u, t) F_i(x_1, x_2, u, t)' E_i(t)' \end{bmatrix}. \end{aligned} \tag{3.35}$$

We rewrite (3.34) as

$$\begin{aligned} &\mathrm{tr}[P_i(t) \theta_{2i}(x_1, x_2, u, t) \theta_{2i}(x_1, x_2, u, t)'] \\ = &\mathrm{tr}[M' P_i(t) M E_i(t) F_i(x_1, x_2, u, t) F_i(x_1, x_2, u, t)' E_i(t)'] \\ = &\mathrm{tr}[E_i(t)' M' P_i(t) M E_i(t) F_i(x_1, x_2, u, t) F_i(x_1, x_2, u, t)']. \end{aligned} \tag{3.36}$$

From (3.9) and (3.10), we have

$$\begin{aligned} &F_i(x_1, x_2, u, t) F_i(x_1, x_2, u, t)' \\ = &F_i(x_1, x_2, u, t) F_i(x_1, x_2, u, t) \\ = &\mathrm{diag}(x_2' Q_{ki} x_2 + u' R_{ki} u + x_1' L_{ki} + Z_{ki}), \end{aligned} \tag{3.37}$$

in which $k = 1, 2, \ldots, \eta$. Substituting (3.37) into (3.36), using the notation introduced in (3.15) and (3.16), we have

$$\mathrm{tr}[P_i(t) \theta_{2i}(x_1, x_2, u, t) \theta_{2i}(x_1, x_2, u, t)']$$

$$
\begin{aligned}
&= x_2'(\sum_{k=1}^{\eta} e_k' E_i(t)' M' P_i(t) M_i E_i(t) e_k Q_{ki}(t)) x_2 \\
&\quad + u'(\sum_{k=1}^{\eta} e_k' E_i(t)' M' P_i(t) M_i E_i(t) e_k R_{ki}(t)) u \\
&\quad + x_1' \sum_{k=1}^{\eta} e_k' E_i(t)' M' P_i(t) M_i E_i(t) e_k L_{ki}(t) \\
&\quad + \sum_{k=1}^{\eta} e_k' E_i(t)' M' P_i(t) M_i E_i(t) e_k Z_{ki}(t) \\
&= x_2'(\sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) Q_{ki}(t)) x_2 + u'(\sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) R_{ki}(t)) u \\
&\quad + x_1' \sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) L_{ki}(t) + \sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) Z_{ki}(t). \qquad (3.38)
\end{aligned}
$$

Again, we transform $x_1$ and $x_2$ in the above equation into forms of $x$ only, using (3.12),

$$
\begin{aligned}
&\operatorname{tr}[\theta_{2i}(x_1, x_2, u, t)' P_i(t) \theta_{2i}(x_1, x_2, u, t)] \\
&= x'(\sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) M Q_{ki}(t) M') x + u'(\sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) R_{ki}(t)) u \\
&\quad + x' \sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) \tilde{M} L_{ki}(t) + \sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) Z_{ki}(t). \qquad (3.39)
\end{aligned}
$$

Substituting (3.33) and (3.39) into $\mathbb{E}[x(T)' P(T, r_T) x(T)]$ in (3.28), we have

$$
\begin{aligned}
&\mathbb{E}[x(T)' P(T, r_T) x(T)] \\
&= y' P_i(s) y + \mathbb{E}\Bigg[ \int_s^T \Big\{ x' \Big[ \dot{P}_i(t) + P_i(t) A_i(t) + A_i(t)' P_i(t) + C_i(t)' P_i(t) C_i(t) \\
&\quad + \sum_{j=1}^{\delta} \pi_{ij} P_j(t) + \sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) M Q_{ki}(t) M' \Big] x \\
&\quad + 2u' [D_i(t)' P_i(t) C_i(t) + B_i(t)' P_i(t)] x \\
&\quad + u' \left[ D_i(t)' P_i(t) D_i(t) + \sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) R_{ki}(t) \right] u
\end{aligned}
$$

$$
\begin{aligned}
&+x'\left(2P_i(t)H_i(t)+\sum_{a=1}^{m}b_a'P_{aai}(t)+\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)\tilde{M}L_{ki}(t)\right)\\
&+\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)Z_{ki}(t)\Biggr\}dt\Biggm|r_s=i\Biggr]. \qquad (3.40)
\end{aligned}
$$

Applying Lemma 2.9.1 to $L_c(T,r_T)'x(T)$, we have

$$
\begin{aligned}
&\mathbb{E}[L_{ci}(T)'x(T)]\\
=\ &L_{ci}(s)'y+\mathbb{E}\Biggl[\int_s^T\Biggl[\Bigl(\dot{L}_{ci}(t)'+L_{ci}(t)'A_i(t)\Bigr)x+L_{ci}(t)'B_i(t)u\\
&+L_{ci}(t)'H_i(t)\Biggr]dt\Biggm|r_s=i\Biggr]. \qquad (3.41)
\end{aligned}
$$

Combining (3.40) and (3.41), we have

$$
\begin{aligned}
&\mathbb{E}[x(T)'\bar{H}_{r_T}(T)x(T)]+\mathbb{E}[\bar{L}_{c_{r_T}}(T)'x(T)]-y'P_i(s)y-L_{ci}(s)'y\\
=\ &\mathbb{E}[x(T)'P_{r_T}(T)x(T)+L_{c_{r_T}}(T)'x(T)-x(s)'P_{r_s}x(s)-L_{c_{r_s}}(s)'x(s)|r_s=i]\\
=\ &\mathbb{E}\left\{\int_s^T\Theta_\varphi(t,x(t),r_t)dt\middle|r_s=i\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
&\Theta_\varphi(t,x(t),r_t)\\
=\ &x'\{[\dot{P}_i(t)+P_i(t)A_i(t)+A_i(t)'P_i(t)+C_i(t)'P_i(t)C_i+\sum_{j=1}^{\delta}\pi_{ij}P_j(t)\\
&+\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)MQ_{ki}(t)M']\}x\\
&+2u'[D_i(t)'P_i(t)C_i(t)+B_i(t)'P_i(t)]x+u'[D_i(t)'P_i(t)D_i(t)\\
&+\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)R_{ki}(t)]u+u'B_i(t)'L_{ci}(t)\\
&+x'[2P_i(t)H_i(t)+\sum_{a=1}^{m}b_a'P_{aa}(t)+\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)\tilde{M}L_{ki}(t)\\
&+\dot{L}_{ci}(t)+A_i(t)'L_{ci}(t)]+\sum_{k=1}^{\eta}N_{ki}(t)'P_i(t)N_{ki}(t)Z_{ki}(t)+L_{ci}'(t)H_i(t).
\end{aligned}
$$

Then, the cost functional (3.20) is rewritten as follows,

$$\begin{aligned}
& J(s, y, i; u(\cdot)) \\
= \;& y' P_i(s) y + L_{ci}(s)' y + \mathbb{E}\bigg\{ \int_s^T \Big[ \Theta_\varphi(t, x(t), r_t) + x(t)' Q(t, r_t) x(t) \\
& + 2u(t)' L(t, r_t)' x(t) + u(t)' R(t, r_t) u(t) + x(t)' L_d(t, r_t) \\
& + u(t)' L_e(t, r_t) \Big] dt \bigg| r_s = i \bigg\}. \qquad (3.42)
\end{aligned}$$

For simplicity, we denote

$$\begin{aligned}
\bar{R}_i &\triangleq D_i(t)' P_i(t) D_i(t) + \sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) R_{ki}(t) + R_i(t), \\
\bar{X}_i &\triangleq 2(D_i(t)' P_i(t) C_i(t) + B_i(t)' P_i(t) + L_i(t)'), \\
\bar{Y}_i &\triangleq L_{ei}(t) + B_i(t)' L_{ci}(t), \\
\bar{S}_i &\triangleq \dot{P}_i(t) + P_i(t) A_i(t) + A_i(t)' P_i(t) + C_i(t)' P_i(t) C_i + \sum_{j=1}^{\delta} \pi_{ij} P_j(t) \\
& \quad + \sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) M Q_{ki}(t) M' + Q_i(t) \\
\bar{T}_i &\triangleq 2P_i(t) H_i(t) + \sum_{a=1}^{m} b_a' P_{aa}(t) + \sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) \tilde{M} L_{ki}(t) \\
& \quad + \dot{L}_{ci}(t) + A_i(t)' L_{ci}(t) + L_{di}(t), \\
\bar{Z}_i &\triangleq \sum_{k=1}^{\eta} N_{ki}(t)' P_i(t) N_{ki}(t) Z_{ki}(t) + L_{ci}'(t) H_i(t). \qquad (3.43)
\end{aligned}$$

Then, we rewrite the terms inside the integral of equation (3.42). Applying completion of square method to $u(t)$, we have

$$\begin{aligned}
& \Theta_\varphi(t, x, i) + x' Q_i(t) x + 2u' L_i(t)' x + u' R_i(t) u + x' L_{di}(t) + u' L_{ei}(t) \\
= \;& u' \bar{R}_i u + u' \bar{X}_i x + u' \bar{Y}_i + x' \bar{S} x + x' \bar{T} + \bar{Z} \\
= \;& \left[ u + \frac{1}{2} \bar{R}_i^\dagger \bar{X} x + \frac{1}{2} \bar{R}_i^\dagger \bar{Y} \right]' \bar{R}_i \left[ u + \frac{1}{2} \bar{R}_i^\dagger \bar{X} x + \frac{1}{2} \bar{R}_i^\dagger \bar{Y} \right] - \frac{1}{4} x' \bar{X}_i' \bar{R}_i^\dagger \bar{X}_i x \\
& - \frac{1}{2} x' \bar{X}_i' \bar{R}_i^\dagger \bar{Y}_i - \frac{1}{4} \bar{Y}_i \bar{R}_i^\dagger \bar{Y}_i + x' \bar{S} x + x' \bar{T} + \bar{Z}.
\end{aligned}$$

As we are given $Y_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u\times(m+n)})$ and $z_i(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n_u})$ for every $i$, we define

$$\Psi_i^1(t) \ \triangleq \ Y_i(t) - \bar{R}_i^\dagger \bar{R}_i Y_i(t),$$

and

$$\Psi_i^2(t) \ \triangleq \ z_i(t) - \bar{R}_i^\dagger \bar{R}_i z_i(t).$$

Applying Lemma 2.9.2, Lemma 2.9.3-(ii), and CGREs in (3.23) we have for $\gamma = 1, 2$,

$$\bar{R}_i \Psi_i^\gamma(t) = \bar{R}_i^\dagger \Psi_i^\gamma(t) = 0,$$

and,

$$\bar{X}' \Psi_i^\gamma(t) = 0.$$

Hence, we rewrite

$$\begin{aligned}
&\Theta_\varphi(t,x,i) + x'Q_i(t)x + 2u'L_i(t)'x + u'R_i(t)u + x'L_{di}(t) + u'L_{ei}(t)\\
= \ &\left[u + \left(\frac{1}{2}\bar{R}_i^\dagger \bar{X} + \Psi_i^1(t)\right)x + \Psi_i^2(t) + \frac{1}{2}\bar{R}_i^\dagger \bar{Y}\right]' \bar{R}_i \left[u + \left(\frac{1}{2}\bar{R}_i^\dagger \bar{X} + \Psi_i^1(t)\right)x \right.\\
&\left. + \Psi_i^2(t) + \frac{1}{2}\bar{R}_i^\dagger \bar{Y}\right] + x'\left(\bar{S}_i - \frac{1}{4}\bar{X}_i'\bar{R}_i^\dagger \bar{X}_i\right)x + x'\left(\bar{T}_i - \frac{1}{2}\bar{X}_i'\bar{R}_i^\dagger \bar{Y}_i\right)\\
&+\bar{Z}_i - \frac{1}{4}\bar{Y}_i'\bar{R}_i^\dagger \bar{Y}_i.
\end{aligned}$$

According to the CGREs (3.23), we have $\bar{S}_i - \frac{1}{4}\bar{X}_i'\bar{R}_i^\dagger \bar{X}_i = 0$, and $\bar{T}_i - \frac{1}{2}\bar{X}_i'\bar{R}_i^\dagger \bar{Y}_i = 0$. With $\zeta_i(t)$ provided in (3.27), we rewrite (3.42) as

$$\begin{aligned}
&J(s,y,i;u(\cdot))\\
= \ &y'P_i(s)y + L_{ci}(s)'y + \mathbb{E}\left\{\int_s^T \{\left[u + \left(\frac{1}{2}\bar{R}_i^\dagger \bar{X}_i + \Psi_i^1(t)\right)x + \Psi_i^2(t) + \frac{1}{2}\bar{R}_i^\dagger \bar{Y}_i\right]'\right.\\
&\left.\times \bar{R}_i\left[u + \left(\frac{1}{2}\bar{R}_i^\dagger \bar{X}_i + \Psi_i^1(t)\right)x + \Psi_i^2(t) + \frac{1}{2}\bar{R}_i^\dagger \bar{Y}_i\right] + \zeta(t,r_t)\}dt\right| r_s = i\right\}\\
\geq \ &y'P_i(s)y + L_{ci}(s)'y + \mathbb{E}\left[\int_s^T \zeta(t,r_t)dt\right| r_s = i\right],
\end{aligned}$$

Thus, $J(s,y,i;u(\cdot))$ is minimized by the control given in (3.25). The optimal value is $y'P_i(s)y + L_{ci}(s)'y + \mathbb{E}[\int_s^T \zeta(t,r_t)dt | r_s = i]$. □

We highlight that the importance of this work is that explicit optimal linear controls are obtained, which is a very rare case in the nonlinear system.

Similar to the results in [74], we have the following statements. Any admissible control is optimal if $D_i(t)'P_i(t)D_i(t)+\sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki}+R_i(t) \equiv 0$, a.e. $t \in [s,T]$ for every $i$. In addition, when $D_i(t)'P_i(t)D_i(t) + \sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki} + R_i(t) > 0$, a.e. $t \in [s,T]$ for every $i$, a unique optimal control is given as follows:

$$u(t) = -\sum_{i=1}^{\delta} \left[ \left( \frac{1}{2}\bar{R}_i^{\dagger}\bar{X}_i + \Psi_i^1(t) \right) x + \Psi_i^2(t) + \frac{1}{2}\bar{R}_i^{\dagger}\bar{Y}_i \right] \chi_{\{r_t=i\}}(t),$$

where $\bar{R}_i$, $\bar{X}_i$, and $\bar{Y}_i$ are defined in (3.43). Both of these two statements can be derived from Theorem 3.4.1.

## 3.5 Discussion of Riccati Equation

In this section we focus on discussing the solvability of Riccati equation of (3.24), which is a special case of (3.23). Here $t$ is omitted for convenience.

First, let us rewrite $\sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}MQ_{ki}M'$, where $N_{ki}$ is defined in (3.16). Define $\bar{Q}_{ki} \triangleq MQ_{ki}M'$ and $E'_iM'P_iME_i \triangleq \Lambda_i$. As any matrix can be written as the product of its square root matrix, we rewrite $\bar{Q}_{ki} = \bar{Q}_{ki}^{\frac{1}{2}} \times \bar{Q}_{ki}^{\frac{1}{2}}$. We denote scalar $\Lambda_{kki}$ as each element of matrix $\Lambda_i$, then we have $e'_k\Lambda_ie_k = \Lambda_{kki}$ and

$$\sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}MQ_{ki}M' = \sum_{k=1}^{\eta} \bar{Q}_{ki}^{\frac{1}{2}} \times \Lambda_{kki} \times \mathbf{I} \times \bar{Q}_{ki}^{\frac{1}{2}}. \tag{3.44}$$

Define a $\eta \times (m+n)$ dimensional matrix $\xi_{k\tau}$, in which the element in the $k$th row and the $\tau$th column is 1, whereas other elements are all 0. Then we have

$$\Lambda_{kki} \times \mathbf{I} = \sum_{\tau=1}^{m+n} \xi'_{k\tau}\Lambda_i\xi_{k\tau}. \tag{3.45}$$

Substituting (3.45) into (3.44), we have

$$\sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}MQ_{ki}M' \;\;=\;\; \sum_{k=1}^{\eta} \bar{Q}_{ki}^{\frac{1}{2}} \left( \sum_{\tau=1}^{m+n} \xi'_{k\tau}\Lambda_i\xi_{k\tau} \right) \bar{Q}_{ki}^{\frac{1}{2}}$$

$$= \sum_{k=1}^{\eta} \sum_{\tau=1}^{m+n} \bar{Q}_{ki}^{\frac{1}{2}} \xi'_{k\tau} E'_i M' P_i M E_i \xi_{k\tau} \bar{Q}_{ki}^{\frac{1}{2}}.$$

We denote $\bar{G}_{k\tau i} \triangleq M E_i \xi_{k\tau} \bar{Q}_{ki}^{\frac{1}{2}}$, then

$$\sum_{k=1}^{\eta} N'_{ki} P_i N_{ki} M Q_{ki} M' = \sum_{k=1}^{\eta} \sum_{\tau=1}^{m+n} \bar{G}'_{k\tau i} P \bar{G}_{k\tau i}. \tag{3.46}$$

Similarly, we can transform $\sum_{k=1}^{\eta} N'_{ki} P_i N_{ki} R_{ki}$ into forms of (3.46). In this case, (3.24) can be transformed into a form similar to the Riccati equation in [74], which is the one for the linear case, and its solvability is assumed to be held.

**Remark 3.5.1.** *The solvability of the following type of Riccati equation, (t is omitted),*

$$\begin{cases} \dot{P} + PA + A'P + C'PC - (PB + C'PD)(R + D'PD)^{-1}(B'P + D'PC) \\ \quad + Q = 0, \\ P(T) = H, \\ R + D'PD > 0, \quad \text{a.e.} \quad t \in [0, T], \end{cases} \tag{3.47}$$

*is proved in Lemma 4.1 and Theorem 4.1 in [27]. However, the solvability of the corresponding Riccati equation with Markovian switching in [74], which is (t is omitted)*

$$\begin{cases} \dot{P}_i + P_i A_i + A'_i P_i + C'_i P_i C_i - (P_i B_i + C'_i P_i D_i + L_i)(R_i + D'_i P_i D_i)^{-1}(B'_i P_i \\ \quad + D'_i P_i C_i + L'_i) + Q_i + \sum_{j=1}^{\delta} \pi_{ij} P_j = 0, \\ P_i(T) = H_i, \\ R_i + D'_i P_i D_i > 0, \quad \text{a.e.} \quad t \in [0, T], \quad i = 1, \cdots, \delta, \end{cases} \tag{3.48}$$

*is not proved. The assumption that (3.48) is solvable is made. Since the new type of CGREs (3.23) can be transformed into the similar form as (3.48), then its assumption of solvability is reasonable.*

## 3.6 Application to Finance

In this section we use the results obtained in Section 3.4 to solve the second motivating example provided in Section 3.2, which is the problem of optimal investment for the logarithmic utility with CIR model involved. Using the same notation in Section 3.4, we formulate the problem mathematically again as follows, (where $t$ in coefficients is omitted for convenience)

$$\begin{cases} dr(t) = [ar(t) + b]dt + \sqrt{r(t)}dW_2(t), \\ \\ d\hat{x}(t) = [r(t) + (\mu - r(t))v(t)]dt + \sigma v(t)dW_1(t), \\ \\ r(0) = r_0, \quad \hat{x}(0) = \hat{x}_0, \end{cases} \tag{3.49}$$

with cost functional $J$ to be minimised, where

$$J \triangleq \mathbb{E}\left[\int_0^T \frac{1}{2}\sigma^2 v^2(s)ds - \hat{x}(T)\right]. \tag{3.50}$$

By comparing our optimal control problem (3.6) with the financial problem here, it is easy to see that the $r(t)$ in (3.49) corresponds to state $x_1$ in (3.6), whereas $\hat{x}(t)$ corresponds to state $x_2$ in (3.6), and we regard $v(t)$ as control. We thus see that the problem of minimizing (3.50) subject to (3.49) is just an example of the nonlinear optimal control problem of this chapter, and this can be solved by applying Theorem 3.4.1.

## 3.7 An Example

In this section, we give an example that originates from [74]. Similarly, we assume that the Markov chain has two states, $i = 1, 2$. We also assume $D_i' P_i D_i + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} R_{ki} + R_i = 0$. Moreover, we show that the stochastic nonlinear optimal control problem can be well-posed when $R_1(t) < 0$ and $R_2(t) < 0$. We assume $x_1 = 0$. For simplicity, we consider one-dimensional nonlinear optimal control problem as follows,

$$\min \quad J \ = \ \mathbb{E}\bigg\{ \int_0^T [Q(t, r_t)x(t)^2 + 2L(t, r_t)x(t)u(t) + R(t, r_t)u(t)^2$$

$$
\begin{aligned}
&+L_d(t,r_t)x(t)+L_e(t,r_t)u(t)]dt+\bar{H}x(T)^2 \\
&+\bar{L}_c x(T)\bigg| r_0=i\bigg\},
\end{aligned}
\tag{3.51}
$$

$$
\text{s.t.}\begin{cases}
dx(t)=[A(t,r_t)x(t)+B(t,r_t)u(t)+H(t,r_t)]dt \\
\qquad\quad +[C(t,r_t)x(t)+D(t,r_t)u(t)]dW(t) \\
\qquad\quad +E(t,r_t)[Q_1(t,r_t)x(t)^2+R_1(t,r_t)u(t)^2+Z_1(t,r_t)]dW_2(t), \\
x(0)=x_0,
\end{cases}
\tag{3.52}
$$

where $A(t,r_t)=A_i$, $B(t,r_t)=B_i$, $H(t,r_t)=H_i$, $C(t,r_t)=C_i$, $D(t,r_t)=D_i$, $E(t,r_t)=E_i$, $Q_1(t,r_t)=Q_{1i}$, $R_1(t,r_t)=R_{1i}$, $Z_1(t,r_t)=Z_{1i}$, $Q(t,r_t)=Q_i$, $L(t,r_t)=L_i$, $L_d(t,r_t)=L_{di}$, and $L_e(t,r_t)=L_{ei}$ are all constants, and $R(t,r_t)=R_i(t)$ when $r_t=i$. We assume $D_i\neq 0$, $B_i+D_iC_i=0$, $L_i=0$, $Q_i=0$, $\pi_{ii}<0$ for $i=1,2$, and $\pi_{11}\neq\pi_{22}$. In addition, $R_i(t)=-D_i^2P_i(t)-E_i^2R_{1i}P_i(t)$, $i=1,2$. According to CGREs (3.23) we have

$$
\begin{cases}
\dot{P}_1(t)=-[2A_1+C_1^2+Q_1E_1^2+\pi_{11}]P_1(t)+\pi_{11}P_2(t), \\
\dot{P}_2(t)=\pi_{22}P_1(t)-[2A_2+C_2^2+Q_2E_2^2+\pi_{22}]P_2(t), \\
P_1(T)=\bar{H}, \\
P_2(T)=\bar{H}.
\end{cases}
\tag{3.53}
$$

We denote

$$\alpha\triangleq-(2A_1+C_1^2+Q_1E_1^2+\pi_{11}),$$

and

$$\beta\triangleq-(2A_2+C_2^2+Q_2E_2^2+\pi_{22}).$$

Then (3.53) is rewritten as,

$$
\begin{cases}
\dot{P}_1(t)=\alpha P_1(t)+\pi_{11}P_2(t), \\
\dot{P}_2(t)=\pi_{22}P_1(t)+\beta P_2(t), \\
P_1(T)=\bar{H}, \\
P_2(T)=\bar{H}.
\end{cases}
\tag{3.54}
$$

Now we have transformed this nonlinear example to the linear case stated in [74]. Similar to Section 6 in [74], (3.54) can be solved by

$$
\begin{aligned}
\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} &= \bar{H}e^{\lambda_1(t-T)} \cdot \frac{\lambda_2 - (\alpha + \pi_{11})}{\sqrt{\Xi}} \cdot \begin{bmatrix} 1 \\ \frac{\lambda_1 - \alpha}{\pi_{11}} \end{bmatrix} \\
&\quad + \bar{H}e^{\lambda_2(t-T)} \cdot \frac{(\alpha + \pi_{11}) - \lambda_1}{\sqrt{\Xi}} \cdot \begin{bmatrix} 1 \\ \frac{\lambda_2 - \alpha}{\pi_{11}} \end{bmatrix}.
\end{aligned}
$$

or

$$
\begin{aligned}
\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} &= \bar{H}e^{\lambda_1(t-T)} \cdot \frac{\lambda_2 - (\pi_{22} + \beta)}{\sqrt{\Xi}} \cdot \begin{bmatrix} \frac{\lambda_1 - \beta}{\pi_{22}} \\ 1 \end{bmatrix} \\
&\quad + \bar{H}e^{\lambda_2(t-T)} \cdot \frac{(\pi_{22} + \beta) - \lambda_1}{\sqrt{\Xi}} \cdot \begin{bmatrix} \frac{\lambda_2 - \beta}{\pi_{22}} \\ 1 \end{bmatrix},
\end{aligned}
$$

where

$$
\begin{cases}
\lambda_1 = \frac{1}{2}[(\alpha + \beta) - \sqrt{\Xi}], \\
\lambda_2 = \frac{1}{2}[(\alpha + \beta) + \sqrt{\Xi}], \\
\Xi = (\alpha - \beta)^2 + 4\pi_{11}\pi_{22}.
\end{cases}
$$

Here, $\lambda_1$ and $\lambda_2$ are solutions to $\lambda^2 - (\alpha + \beta)\lambda + \alpha\beta - \pi_{11}\pi_{22} = 0$. In addition, we have

$$
\begin{cases}
\lambda_2 - (\alpha + \pi_{11}) \cdot \frac{\lambda_1 - \alpha}{\pi_{11}} = \lambda_2 - (\pi_{22} + \beta), \\
(\alpha + \pi_{11}) - \lambda_1 \cdot \frac{\lambda_2 - \alpha}{\pi_{11}} = (\pi_{22} + \beta) - \lambda_1, \\
\lambda_2 - (\pi_{22} + \beta) \cdot \frac{\lambda_1 - \beta}{\pi_{22}} = \lambda_2 - (\alpha + \pi_{11}), \\
(\pi_{22} + \beta) - \lambda_1 \cdot \frac{\lambda_2 - \beta}{\pi_{22}} = (\alpha + \pi_{11}) - \lambda_1,
\end{cases}
$$

By Theorem 3.4.1, our nonlinear optimal control problem (3.51)-(3.52) is well-posed. Additionally, any admissible control is optimal. The optimal cost is

$$
P_i(0)x_0^2 + L_{ci}(0)x_0 + \mathbb{E}\left[\int_0^T (E_i^2 Z_{1i} P_i(t) + L_{ci} H_i)dt \middle| r_0 = i\right]. \tag{3.55}
$$

If we choose $\pi_{11}$ and $\pi_{22}$ when the following holds,

$$\begin{cases} -\sqrt{\Xi} \leq (\alpha - \beta) + 2\pi_{11} \leq \sqrt{\Xi}, \\ -\sqrt{\Xi} \leq (\beta - \alpha) + 2\pi_{22} \leq \sqrt{\Xi}, \end{cases}$$

then we have

$$\begin{cases} \lambda_2 - (\alpha + \pi_{11}) \geq 0, \\ (\alpha + \pi_{11}) - \lambda_1 \geq 0, \\ \lambda_2 - (\pi_{22} + \beta) \geq 0, \\ (\pi_{22} + \beta) - \lambda_1 \geq 0. \end{cases}$$

So, if we choose $\bar{H} \geq 0$, then we have $P_i(t) \geq 0$, $i = 1, 2$. Alternatively, if we choose $\bar{H} < 0$, then we have $P_i(t) < 0$, $i = 1, 2$. In conclusion, our nonlinear optimal control problem (3.51)-(3.52) is well-posed even if $R_i(t) = -D_i^2 P_i(t) - E_i^2 R_{1i} P_i(t) \leq 0$ when $P_i(t) \geq 0$.

## 3.8 Summary

This chapter studies the indefinite stochastic nonlinear optimal control problem with Markovian switching in finite time horizon. Due to the nature of nonlinearity, the existence and uniqueness of solution to SDEs of our system is discussed. A new type of Riccati equations is introduced with its solvability discussed. Explicit optimal linear controls are obtained, which is a very rare case when the system is nonlinear. Moreover, the optimal cost value is obtained. An application to finance is introduced. An illustrative example is given. Under such circumstances, some results obtained in [74] are special cases of this chapter.

# Chapter 4

# Nonlinear Optimal Stochastic Control of Systems with Markovian Switching in Infinite Time Horizon

## 4.1 Introduction

In the previous chapter, the problem of optimal nonlinear stochastic control of systems with Markovian switching in finite time horizon is investigated, with value function obtained. Based on that, someone might ask, what will happen if the time $T$ in Chapter 3 goes to infinity? How can we formulate the new problem properly? Can we still obtain the same results? Is there any new topics that need to be concerned? Motivated by these questions, here we investigate the case in infinite time horizon. The system of the problem is formulated similarly to the one in the finite time horizon, especially the nonlinear terms. Note that one of the differences is that in the finite case the Markov jumping parameters are time variant, whereas in infinite case, all the Markov jumping parameters are time invariant. Due to the nature of infinite horizon, the cost functional is constructed differently from the one in Chapter 3. Here we no longer have the terminal coefficient $\bar{H}(r_T)$ or

$\bar{L}_c(r_T)$, which appears in the previous chapter, equation (3.20). As we mentioned in Chapter 1 that there are several difficulties in dealing with infinity horizon problems, one of them is that we may not have a finite optimal performance index, see [7]. Therefore, we have to consider the mean-square stability, which is a standard assumption in the infinite horizon control problems. We propose the stability condition of the system. The coupled generalized algebraic Riccati equations (CGAREs) are introduced and we assume the CGAREs have solutions. By using some similar calculation steps that originate from Chapter 3, we derive the solution to our optimal control problem by completion of square method, and the difficulty appears in dealing with the nonlinearity terms. Here it is highlighted that within this nonlinear system an explicit solution is found, which is a very rare case. In addition, the optimal control laws obtained are linear with state, which is very similar to the characteristics of the results in optimal LQ control problems. Moreover, the existence and uniqueness of solution is discussed, similar to the finite horizon case.

## 4.2 Problem Formulation and CGAREs

### 4.2.1 Problem Formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given filtered complete probability space, where there exist a $m \times 1$ -dimensional Brownian motion $W_1(t)$ on $[0, +\infty)$, a one-dimensional standard Brownian motion $W(t)$ on $[0, +\infty)$, a $\eta \times 1$ -dimensional Brownian motion $W_2(t)$, $t \geq 0$ on $[0, +\infty)$, and a Markov chain $(r_t \in \{1, 2, \cdots, \delta\}, t \geq 0)$ with generator $\Pi = (\pi_{ij})$ specified in (2.3). We assume that $W_1(t)$, $W(t)$, $W_2(t)$ and the process $r_t$ are mutually independent.

**Assumption 4.2.1.** *The data that appear in the nonlinear optimal control problem (4.1)-(4.18) satisfy, for every i,*

$$\begin{aligned}
&H_{1i}, L_{ki} \in \mathbb{R}^{m}, \quad A_{1i}, C_{1i} \in \mathbb{R}^{n\times m}, \quad A_{2i} \in \mathbb{R}^{n\times n}, \quad C_{2i} \in \mathbb{R}^{m\times m},\\
&B_{1i}, D_{1i} \in \mathbb{R}^{n\times n_u}, \quad E_i \in \mathbb{R}^{n\times \eta}, \quad Q_{ki} \in \mathcal{S}^{n}, \quad R_{ki}, R_i \in \mathcal{S}^{n_u},\\
&Q_i \in \mathcal{S}^{m+n}, \quad L_i \in \mathbb{R}^{(m+n)\times n_u}, \quad L_{di} \in \mathbb{R}^{m+n}, \quad L_{ei} \in \mathbb{R}^{n_u}.
\end{aligned}$$

Consider the nonlinear SDEs with Markovian switching as follows,

$$\begin{cases} dx_1(t) = [G_1(r_t)x_1(t) + H_1(r_t)]dt + \Gamma_1(x_1(t), r_t)dW_1(t) \\ dx_2(t) = [A_1(r_t)x_1(t) + A_2(r_t)x_2(t) + B_1(r_t)u(t)]dt \\ \qquad\qquad + [C_1(r_t)x_1(t) + C_2(r_t)x_2(t) + D_1(r_t)u(t)]dW(t) \\ \qquad\qquad + \Gamma_2(x_1(t), x_2(t), r_t)dW_2(t) \\ x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \end{cases} \tag{4.1}$$

where $A_1(r_t) = A_{1i}$, etc., $i = 1, 2, \cdots, \delta$. Define

$$G_1(r_t) \triangleq \text{diag}[g_1(r_t), g_2(r_t), \ldots, g_m(r_t)],$$

i.e., a $m \times m$ diagonal matrix, in which the diagonal elements are $g_1(r_t), g_2(r_t), \ldots, g_m(r_t)$. In addition, we denote

$$\Gamma_1(x_1(t), r_t) \triangleq \text{diag}[\sqrt{x_{11}(t)}, \sqrt{x_{12}(t)}, \ldots, \sqrt{x_{1m}(t)}], \tag{4.2}$$

$$\Gamma_2(x_1(t), x_2(t), u(t), r_t) \triangleq E(r_t)F(x_1(t), x_2(t), u(t), r_t), \tag{4.3}$$

and

$$F(x_1(t), x_2(t), u(t), r_t) \triangleq \text{diag}(\sqrt{\phi_1}, \sqrt{\phi_2}, \ldots, \sqrt{\phi_\eta}). \tag{4.4}$$

Among $\phi_1, \phi_2, \ldots, \phi_\eta$, we denote each of them as $\phi_k$, where $k = 1, 2, \ldots, \eta$. We define

$$\phi_k \triangleq x_2(t)'Q_k(r_t)x_2(t) + u(t)'R_k(r_t)u(t) + x_1(t)'L_k(r_t). \tag{4.5}$$

We assume that $Q_k(r_t) \geq 0$, $R_k(r_t) \geq 0$, $Z_k(r_t) > 0$, and the components of $L_k(r_t)$ are non-negative, for all $k$. If we denote

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{4.6}$$

then we can rewrite equation (4.1) into the following

$$\begin{cases} dx(t) = [A(r_t)x(t) + B(r_t)u(t) + H(r_t)]dt \\ \qquad\qquad + [C(r_t)x(t) + D(r_t)u(t)]dW(t) \\ \qquad\qquad + \theta_1(x(t), u(t), r_t)dW_1(t) + \theta_2(x(t), u(t), r_t)dW_2(t), \\ x(0) = x_0 \in \mathbb{R}^{m+n}, \end{cases} \tag{4.7}$$

where

$$
\begin{aligned}
&A(r_t) \triangleq \begin{bmatrix} G_1(r_t) & \mathbf{0} \\ A_1(r_t) & A_2(r_t) \end{bmatrix}, B(r_t) \triangleq \begin{bmatrix} \mathbf{0} \\ B_1(r_t) \end{bmatrix}, H(r_t) \triangleq \begin{bmatrix} H_1(r_t) \\ \mathbf{0} \end{bmatrix}, \\
&C(r_t) \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ C_1(r_t) & C_2(r_t) \end{bmatrix}, D(r_t) \triangleq \begin{bmatrix} \mathbf{0} \\ D_1(r_t) \end{bmatrix}, \\
&\theta_1(x_1(t), r_t) \triangleq \begin{bmatrix} \Gamma_1(x_1(t), r_t) \\ \mathbf{0} \end{bmatrix}, \\
&\theta_2((x_1(t), x_2(t), u(t), r_t) \triangleq \begin{bmatrix} \mathbf{0} \\ \Gamma_2(x_1(t), x_2(t), u(t), r_t) \end{bmatrix}.
\end{aligned}
\tag{4.8}
$$

We give the following notations, which will be used throughout this chapter. Define

$$
M \triangleq \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}, \tag{4.9}
$$

where $\mathbf{0}$ is a $m \times n$ zero matrix, and $\mathbf{I}$ is a $n \times n$ identity matrix. Define $e_k$ as an $\eta \times 1$ elementary vector, whose $k$-th element is 1, while other elements are 0. For simplicity, we define

$$
N_{ki} \triangleq M E_i e_k. \tag{4.10}
$$

Define $\epsilon_a$ as an $m \times 1$ elementary vector, whose $a$-th element is 1, while other elements are 0. Then each element of vector $x_1$ can be expressed as

$$
x_{1a} = \epsilon_a' x_1. \tag{4.11}
$$

Define

$$
b_a \triangleq \begin{bmatrix} \epsilon_a' & \mathbf{0} \end{bmatrix}, \tag{4.12}
$$

where $\mathbf{0}$ is a $1 \times n$ zero matrix. Define

$$
\tilde{M} \triangleq \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}, \tag{4.13}
$$

where **0** is a $n \times m$ zero matrix, and **I** is a $m \times m$ identity matrix.

The discussion of existence and uniqueness of solution to the system (4.1) is similar to the one discussed in Chapter 3. Here we omit the details.

Next, we provide two definitions which originates from [75] as follows.

**Definition 4.2.1.** *[75] A control $u(\cdot)$ is called mean-square stabilizing w.r.t. a given initial state $(x_0, i)$ if the corresponding state $x(\cdot)$ of (4.7) with $x(0) = x_0$ and $r_0 = i$ satisfies $\lim_{t\to+\infty} \mathbb{E}[x(t)'x(t)] = 0$ .*

**Definition 4.2.2.** *[75] The system (4.7) is called mean-square stabilizable if there exists a linear control $u(t) = \sum_{i=1}^{\delta}\{K_i x(t)\}\chi_{\{r_t=i\}}$, where $K_1, \cdots, K_\delta$ are given matrices, which is mean-square stabilizing w.r.t. any initial state $(x_0, i)$.*

According to Definition 4.2.1 and Definition 4.2.2, we derive sufficient conditions such that our system (4.7) is mean-square stable. First, we define the following notations for convenience,

$$\begin{aligned}
\mathcal{A}_i &\triangleq P_iA_i + A_i'P_i + C_i'P_iC_i + \sum_{j=1}^{\delta}\pi_{ij}P_j + \sum_{k=1}^{\eta} N_{ki}'P_iN_{ki}MQ_{ki}M' + K_i'D_i'P_iC_i \\
&\quad + K_i'B_i'P_i + C_i'P_iD_iK_i + P_iB_iK_i + K_i'D_i'P_iD_iK_i + \\
&\quad \sum_{k=1}^{\eta} K_i'N_{ki}'P_iN_{ki}R_{ki}K_i, \\
\mathcal{B}_i &\triangleq 2P_iH_i + \sum_{a=1}^{m} b_a'P_{aai} + \sum_{k=1}^{\eta} N_{ki}'P_iN_{ki}\tilde{M}L_{ki}. \qquad (4.14)
\end{aligned}$$

**Lemma 4.2.1.** *Substituting the linear control $u^* = K_i x$ into (4.7), if the following matrix inequality*

$$\begin{cases} \mathcal{A}_i < 0, \\ \frac{1}{4}\mathcal{B}_i'\mathcal{A}_i^{-1}\mathcal{B}_i < 0, \quad i = 1, \cdots, \delta, \end{cases} \qquad (4.15)$$

*is satisfied, then our system (4.7) is mean-square stable.*

*Proof.* Similar to the steps from (3.28) to (3.39) in Chapter 3, applying Lemma 2.9.1 to $x(T)'P_{r_T}x(T)$, we have

$$\mathbb{E}[x(T)'P_{r_T}x(T)]$$

$$
\begin{aligned}
= \quad & x_0' P_i x_0 + \mathbb{E}\Bigg[\int_0^T \Bigg\{ x' \Bigg[ P_i A_i + A_i' P_i + C_i' P_i C_i + \sum_{j=1}^{\delta} \pi_{ij} P_j \\
& + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} M Q_{ki} M' \Bigg] x + 2x' K_i' [D_i' P_i C_i + B_i' P_i] x \\
& + x' K_i' \Bigg[ D_i' P_i D_i + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} R_{ki} \Bigg] K_i x \\
& + x' \Bigg( 2 P_i H_i + \sum_{a=1}^{m} b_a' P_{aai} + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} \tilde{M} L_{ki} \Bigg) \Bigg\} dt \Bigg| r_s = i \Bigg]. \qquad (4.16)
\end{aligned}
$$

We rewrite the integrand in the right side of (4.16) by applying completion of square method as follows,

$$
x' \mathcal{A}_i x + x' \mathcal{B}_i = (x + \frac{1}{2} \mathcal{A}_i^{-1} \mathcal{B}_i)' \mathcal{A}_i (x + \frac{1}{2} \mathcal{A}_i^{-1} \mathcal{B}_i) - \frac{1}{4} \mathcal{B}_i' \mathcal{A}_i^{-1} \mathcal{B}_i. \qquad (4.17)
$$

By Definition 4.2.1, Definition 4.2.2, and [68], [83], the matrix inequality (4.15) ensures that our nonlinear system (4.7) is mean-square stable. □

**Definition 4.2.3.** *[75] For a given $(x_0, i) \in \mathbb{R}^{m+n} \times \{1, 2, \cdots \delta\}$, we define the corresponding set of admissible controls $\mathcal{U}(x_0, i)$ where $u(\cdot) \in \mathbb{R}^{n_u}$ such that solution to system (4.7) exists and is unique; the cost $J(x_0, i; u(\cdot))$ is finite; and $u(\cdot)$ is mean-square stabilizing w.r.t. $(x_0, i)$.*

The cost functional is given as follows,

$$
\begin{aligned}
& J(x_0, i; u(\cdot)) \\
= \quad & \mathbb{E}\Bigg\{ \int_0^{+\infty} \Bigg( \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q(r_t) & L(r_t) \\ L(r_t)' & R(r_t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
& + x(t)' L_d(r_t) \Bigg) dt \Bigg| r_0 = i \Bigg\}. \qquad (4.18)
\end{aligned}
$$

The value function is defined as

$$
V(x_0, i) \triangleq \inf_{u(\cdot) \in \mathcal{U}(x_0, i)} J(x_0, i; u(\cdot)). \qquad (4.19)
$$

As is emphasized in [75] that since we allow the symmetric matrices

$$\begin{bmatrix} Q_i & L_i \\ L_i' & R_i \end{bmatrix}, \quad i = 1, \cdots, \delta,$$

to be indefinite, we say our stochastic nonlinear optimal control problem is an indefinite control problem.

**Definition 4.2.4.** *[75] The nonlinear optimal control problem is called well-posed if*

$$-\infty < V(x_0, i) < +\infty, \quad , \forall x_0 \in \mathbb{R}^{m+n}, \quad i = 1, \cdots, \delta.$$

*If there is a control $u^*(\cdot) \in \mathcal{U}(x_0, i)$ that achieves $V(x_0, i)$, then in this case the control $u^*(\cdot)$ is called optimal (w.r.t.$(x_0, i)$).*

### 4.2.2 Coupled Generalized Algebraic Riccati Equations.

Denote $P_{aai}$ as each diagonal element of matrix $P_i$, where $a = 1, \cdots, m$. Now we introduce a new type of coupled generalized algebraic Riccati equations (CGAREs) as follows,

$$\left\{\begin{aligned}
&2P_iH_i + \sum_{a=1}^{m} b_a' P_{aai} + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} \tilde{M} L_{ki} + L_{di} = 0, \\
&P_iA_i + A_i'P_i + C_i'P_iC_i + \sum_{j=1}^{\delta} \pi_{ij} P_j + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} M Q_{ki} M' + Q_i \\
&\quad - (C_i'P_iD_i + P_iB_i + L_i)(D_i'P_iD_i + \sum_{k=1}^{\eta} N_{ki}'P_iN_{ki}R_{ki} + R_i)^{\dagger}(D_i'P_iC_i \\
&\quad + B_i'P_i + L_i') = 0, \\
&(D_i'P_iD_i + \sum_{k=1}^{\eta} N_{ki}'P_iN_{ki}R_{ki} + R_i)(D_i'P_iD_i + \sum_{k=1}^{\eta} N_{ki}'P_iN_{ki}R_{ki} \\
&\quad + R_i)^{\dagger}(D_i'P_iC_i + B_i'P_i + L_i') - (D_i'P_iC_i + B_i'P_i + L_i') = 0, \\
&D_i'P_iD_i + \sum_{k=1}^{\eta} N_{ki}'P_iN_{ki}R_{ki} + R_i \geq 0, \quad i = 1, \cdots, \delta.
\end{aligned}\right. \tag{4.20}$$

If we require $D_i' P_i D_i + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} R_{ki} + R_i \neq 0$, for every $i$, then the CGAREs (4.20) becomes

$$\begin{cases} 2P_iH_i + \sum_{a=1}^{m} b_a' P_{aai} + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} \tilde{M} L_{ki} + L_{di} = 0, \\ P_iA_i + A_i'P_i + C_i'P_iC_i + \sum_{j=1}^{\delta} \pi_{ij} P_j + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} M Q_{ki} M' + Q_i \\ \quad - (C_i'P_iD_i + P_iB_i + L_i)(D_i'P_iD_i + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} R_{ki} + R_i)^{-1}(D_i'P_iC_i \\ \quad + B_i'P_i + L_i') = 0, \\ D_i'P_iD_i + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} R_{ki} + R_i > 0, \quad i = 1, \cdots, \delta. \end{cases} \tag{4.21}$$

When $D_i' P_i D_i + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} R_{ki} + R_i \equiv 0$ for every $i$, the CGAREs (4.20) becomes

$$\begin{cases} 2P_iH_i + \sum_{a=1}^{m} b_a' P_{aai} + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} \tilde{M} L_{ki} + L_{di} = 0, \\ P_iA_i + A_i'P_i + C_i'P_iC_i + \sum_{j=1}^{\delta} \pi_{ij} P_j + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} M Q_{ki} M' + Q_i = 0, \\ D_i'P_iC_i + B_i'P_i + L_i' = 0, \\ D_i'P_iD_i + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} R_{ki} + R_i = 0, \quad i = 1, \cdots, \delta. \end{cases}$$

## 4.3 Solution to Optimal Control Problem

Before we introduce the main theorem, let us provide some notations first, for simplicity purposes. We denote

$$\begin{aligned} \bar{R}_i &\triangleq D_i'P_iD_i + \sum_{k=1}^{\eta} N_{ki}' P_i N_{ki} R_{ki} + R_i, \\ \bar{X}_i &\triangleq 2(D_i'P_iC_i + B_i'P_i + L_i'), \end{aligned}$$

$$
\begin{aligned}
\bar{S}_i &\triangleq P_iA_i + A_i'P_i + C_i'P_iC_i + \sum_{j=1}^{\delta}\pi_{ij}P_j \\
&\quad + \sum_{k=1}^{\eta} N_{ki}'P_iN_{ki}MQ_{ki}M' + Q_i, \\
\bar{T}_i &\triangleq 2P_iH_i + \sum_{a=1}^{m} b_a'P_{aa} + \sum_{k=1}^{\eta} N_{ki}'P_iN_{ki}\tilde{M}L_{ki} + L_{di}, \quad i = 1, \ldots, \delta. \qquad (4.22)
\end{aligned}
$$

In addition, let $Y_i \in \mathbb{R}^{n_u \times (m+n)}$, and $z_i \in \mathbb{R}^{n_u}$ be given for every $i$. Set

$$
\Psi_i^1 \triangleq Y_i - \bar{R}_i^{\dagger}\bar{R}_iY_i, \qquad (4.23)
$$

and

$$
\Psi_i^2 \triangleq z_i - \bar{R}_i^{\dagger}\bar{R}_iz_i. \qquad (4.24)
$$

**Theorem 4.3.1.** *Assume that there exists a unique solution to the CGAREs (4.20) such that the following control*

$$
u^*(t) = -\sum_{i=1}^{\delta}\left\{ \left(\frac{1}{2}\bar{R}_i^{\dagger}\bar{X} + \Psi_i^1\right)x + \Psi_i^2 \right\}\chi_{\{r_t=i\}}, \qquad (4.25)
$$

*is admissible w.r.t. to any initial $x_0$, then the stochastic nonlinear optimal control problem (4.7)-(4.19) is well-posed, and $u^*(t)$ in (4.25) is the optimal control. Furthermore, the value function is*

$$
V(0, x_0) = x_0'P_ix_0. \qquad (4.26)
$$

*Proof.* Similar to the steps from (3.28) to (3.39) in Chapter 3, applying Lemma 2.9.1 to $x(T)'P(r_T)x(T)$, we have

$$
\begin{aligned}
&\mathbb{E}[x(T)'P(r_T)x(T)] \\
= \;& x_0'P_ix_0 + \mathbb{E}\Bigg[\int_0^T \Bigg\{ x' \Bigg[ P_iA_i + A_i'P_i + C_i'P_iC_i + \sum_{j=1}^{\delta}\pi_{ij}P_j \\
& + \sum_{k=1}^{\eta} N_{ki}'P_iN_{ki}MQ_{ki}M' \Bigg] x + 2u'[D_i'P_iC_i + B_i'P_i]x
\end{aligned}
$$

$$
\begin{aligned}
&+u'\left[D_i'P_iD_i+\sum_{k=1}^{\eta}N_{ki}'P_iN_{ki}R_{ki}\right]u \\
&+x'\left(2P_iH_i+\sum_{a=1}^{m}b_a'P_{aai}+\sum_{k=1}^{\eta}N_{ki}'P_iN_{ki}\tilde{M}L_{ki}\right)\Bigg\}dt\Bigg|r_0=i\Bigg].
\end{aligned}
\tag{4.27}
$$

We rewrite the above equation (4.27) as follows,

$$
\mathbb{E}[x(T)'P_{r_T}x(T)]=x_0'P_ix_0+\mathbb{E}\left[\int_0^T\Theta_i(x(t),u(t))dt\Bigg|r_0=i\right],
$$

where

$$
\begin{aligned}
&\Theta_i(x(t),u(t)) \\
= \;& x'\Bigg\{\Bigg[P_iA_i+A_i'P_i+C_i'P_iC_i+\sum_{j=1}^{\delta}\pi_{ij}P_j+\sum_{k=1}^{\eta}N_{ki}'P_iN_{ki}MQ_{ki}M'\Bigg]\Bigg\}x \\
&+2u'[D_i'P_iC_i+B_i'P_i]x+u'\left[D_i'P_iD_i+\sum_{k=1}^{\eta}N_{ki}'P_iN_{ki}R_{ki}\right]u \\
&+x'\left[2P_iH_i+\sum_{a=1}^{m}b_a'P_{aa}+\sum_{k=1}^{\eta}N_{ki}'P_iN_{ki}\tilde{M}L_{ki}\right].
\end{aligned}
$$

As we have the mean-square stability,

$$
\lim_{T\to+\infty}\mathbb{E}[x(T)'P(r_T)x(T)]=0
$$

Then the cost functional (4.18) can be rewritten as the following

$$
\begin{aligned}
&J(x_0,i;u(\cdot)) \\
= \;& x_0'P_ix_0+\mathbb{E}\Bigg\{\int_0^{+\infty}[\Theta(x(t),u(t),r_t)+x(t)'Q(r_t)x(t)+2u(t)'L(r_t)'x(t) \\
&+u(t)'R(r_t)u(t)+x(t)'L_d(r_t)+u(t)'L_e(r_t)]dt|r_0=i\Bigg\}.
\end{aligned}
\tag{4.28}
$$

Using the notation in (4.22), we rewrite the terms inside the integral of equation (4.28) and apply completion of square method to $u$,

$$
\Theta_i(x,u)+x'Q_ix+2u'L_i'x+u'R_iu+x'L_{di}
$$

$$\begin{aligned} &= u'\bar{R}_i u + u'\bar{X}_i x + x'\bar{S}_i x + x'\bar{T}_i \\ &= \left[u + \frac{1}{2}\bar{R}_i^\dagger \bar{X} x\right]' \bar{R}_i \left[u + \frac{1}{2}\bar{R}_i^\dagger \bar{X} x\right] - \frac{1}{4} x'\bar{X}_i'\bar{R}_i^\dagger \bar{X}_i x + x'\bar{S}_i x + x'\bar{T}_i. \end{aligned}$$

Applying Lemma 2.9.2, Lemma 2.9.3-(ii), and according to CGAREs in (4.20) we have for $\gamma = 1, 2$,

$$\bar{R}_i \Psi_i^\gamma = \bar{R}_i^\dagger \Psi_i^\gamma = 0,$$

and,

$$\bar{X}'\Psi_i^\gamma = 0.$$

Hence, we rewrite

$$\begin{aligned} &\Theta(t, x, i) + x'Q_i x + 2u'L_i' x + u'R_i u + x'L_{di} \\ = &\left[u + \left(\frac{1}{2}\bar{R}_i^\dagger \bar{X} + \Psi_i^1\right) x + \Psi_i^2\right]' \bar{R}_i \left[u + \left(\frac{1}{2}\bar{R}_i^\dagger \bar{X} + \Psi_i^1\right) x\right. \\ &\left.\Psi_i^2\right] + x'\left(\bar{S}_i - \frac{1}{4}\bar{X}_i'\bar{R}_i^\dagger \bar{X}_i\right) x + x'\bar{T}_i. \end{aligned}$$

According to the CGAREs in (4.20), we have $\bar{S}_i - \frac{1}{4}\bar{X}_i'\bar{R}_i^\dagger \bar{X}_i = 0$, and $\bar{T}_i = 0$. Then the equation (4.28) can be expressed as

$$\begin{aligned} &J(0, x_0, i; u(\cdot)) \\ = \; &x_0'P_i x_0 + \mathbb{E}\left\{ \int_0^{+\infty} \left\{ \left[u + \left(\frac{1}{2}\bar{R}_i^\dagger \bar{X}_i + \Psi_i^1\right) x + \Psi_i^2\right]' \bar{R}_i \left[u\right.\right.\right. \\ &\left.\left.\left. + \left(\frac{1}{2}\bar{R}_i^\dagger \bar{X}_i + \Psi_i^1\right) x + \Psi_i^2\right]\right\} dt \middle| r_s = i \right\} \\ \geq \; &x_0'P_i x_0, \end{aligned}$$

Thus, $J(s, y, i; u(\cdot))$ is minimized by the control law given by (4.25). The optimal value is $x_0'P_i x_0$. □

We highlight that the importance of this work is that explicit optimal linear controls are obtained, which is a very rare case in the nonlinear system.

Similar to the results in [74], we have the following statements. Any admissible control is optimal, if $[D_i(t)'P_i(t)D_i(t) + \sum_{k=1}^{\eta} N_{ki}'P_i N_{ki} R_{ki} + R_i(t)] \equiv 0$, a.e. $t \in$

$[s, T]$ for every $i$. In addition, when $[D_i(t)'P_i(t)D_i(t)+\sum_{k=1}^{\eta} N'_{ki}P_iN_{ki}R_{ki}+R_i(t)] > 0$, a.e. $t \in [s, T]$ for every $i$, a unique optimal control is given as follows:

$$u(t) = -\sum_{i=1}^{\delta} \frac{1}{2}\bar{R}_i^{\dagger}\bar{X}_i x(t)\chi_{\{r_t=i\}}(t),$$

where $\bar{R}_i$, and $\bar{X}_i$ are defined in (4.22). Both of these two statements can be derived from Theorem 4.3.1.

## 4.4 Summary

This chapter studies the indefinite stochastic nonlinear optimal control with Markovian switching in infinite time horizon. The mean-square stability for our infinite horizon problem is considered. A new type of CGAREs is introduced, and we assume that it is solvable. Linear optimal controls are found explicitly and we also obtain the optimal cost value.

# Chapter 5

# Robust Stabilization and Robust $H_\infty$ Control of Uncertain Nonlinear Markovian Switching Stochastic Systems with Time-Varying Delays

## 5.1 Introduction

The problem of robust stabilization and robust $H_\infty$ control of uncertain linear Markovian switching stochastic systems is introduced in Section 2.6. When time delay is included in the system, the problems of robust control and robust $H_\infty$ control has been widely studied. For example, [114] and [20] focus on systems with Markovian switching for deterministic systems. For stochastic systems, [115] and [120] investigates the problems of uncertain robust $H_\infty$ control with time delays. [121] studies the problem of $H_\infty$ output feedback control for uncertain stochastic systems with time-varying delays. Robust $H_\infty$ control for uncertain discrete stochastic time-delay systems is studied in [122]. In [123], problems of robust stochastic stabilization and $H_\infty$ control are studied for uncertain neutral stochastic time-delay systems. Note that all the literatures mentioned above work on linear systems. For nonlinear system, [107] investigates uncertain stochastic systems with sector nonlinearities and missing measurements in a discrete time case. The sector nonlinearity involved in [107] is a general type of nonlinearity that is typically seen in control analysis and problems of model reduction. In addition, [130] considers state feedback $H_\infty$ control for a class of nonlinear stochastic systems, in which the nonlinearity term is given in a general form.

In this chapter we consider the problem of robust stabilization and robust $H_\infty$ control for a class of nonlinear stochastic systems with time delays, which are more general than the ones considered in [115], [114], [20], [120], [121], and [46]. First, the nonlinear problems are formulated. We discuss the existence and uniqueness of solution to our nonlinear SDEs. Some basic definitions and lemmas are introduced. There are two theorems obtained in our main results. In the section of robust stabilization, we provide sufficient conditions such that the linear state feedback stabilizing controllers exist. The sufficient conditions are presented in forms of matrix inequalities. In the section of robust $H_\infty$ control, in addition to the requirement of robust stabilization, a more generalized type of $H_\infty$ performance is proposed, and it is required to be satisfied. Sufficient conditions for solving this generalized robust $H_\infty$ control problem is proposed. The sufficient conditions are presented in forms of matrix inequalities. The difficulty of this chapter is that we allow parameter uncertainty, interval uncertainty, time delay, uncertain Markovian switching and nonlinearities all included in our system. These uncertainties appear in state, disturbance and output. Under such circumstances, our problems are still solvable. The two theorems derived in this chapter are very advanced, with various applications in complicated situations.

## 5.2 Problem Formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given filtered complete probability space, where there exist a one-dimensional standard Brownian motion $(W(t), 0 \leq t \leq T)$ and $(\tilde{W}_\zeta(t), 0 \leq t \leq T)$ for all $\zeta = 1, 2, \cdots, m$, and a Markov chain $(r_t, 0 \leq t \leq T)$. We assume that $W(t)$, $\tilde{W}_\zeta(t)$ and the process $r_t$ are mutually independent.

Let $r_t$, where $t \geq 0$, be a right-continuous Markov chain, taking values in a finite state-space $\Lambda = 1, 2, \ldots, N$, with generator $\widehat{\Pi} = (\widehat{\pi}_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r_{t+\delta} = j \mid r_t = i\} = \begin{cases} \widehat{\pi}_{ij}\delta + o(\delta): & \text{if} \quad i \neq j, \\ 1 + \widehat{\pi}_{ij}\delta + o(\delta): & \text{if} \quad i = j, \end{cases}$$

for $\delta > 0$, and $\lim_{\delta\to 0}(o(\delta)/\delta) = 0$. Here, $\widehat{\pi}_{ij} \geq 0$ is the transition rate from $i$ to $j$, if $i \neq j$, while $\widehat{\pi}_{ii} = -\sum_{j=1, j\neq i}^{N} \widehat{\pi}_{ij}$.

Consider the following stochastic system with Markovian switching and parameter uncertainties:

$$\begin{aligned} dx(t) \;=\;& [(A(r_t) + \Delta A(t, r_t))x(t) + (A_d(r_t) + \Delta A_d(t, r_t))x(t - \tau(t)) \\ &+(B(r_t) + \Delta B(t, r_t))u(t) \\ &+(G(r_t) + \Delta G(t, r_t))v(t) + (G_d(r_t) + \Delta G_d(t, r_t))v(t - \tau(t))]dt \\ &+[(E(r_t) + \Delta E(t, r_t))x(t) + (E_d(r_t) + \Delta E_d(t, r_t))x(t - \tau(t)) \\ &+(F(r_t) + \Delta F(t, r_t))u(t) \\ &+(H(r_t) + \Delta H(t, r_t)v(t) + (H_d(r_t) + \Delta H_d(t, r_t)v(t - \tau(t))]dW(t) \\ &+\sum_{\zeta=1}^{m} \Gamma_\zeta(x(t), u(t), t, r_t)d\tilde{W}_\zeta(t), \end{aligned} \tag{5.1}$$

$$\begin{aligned} z(t) \;=\;& (C(r_t) + \Delta C(t, r_t))x(t) + (C_d(r_t) + \Delta C_d(t, r_t))x(t - \tau(t)) \\ &+(S(r_t) + \Delta S(t, r_t))u(t) \\ &+(L(r_t) + \Delta L(t, r_t)v(t) + (L_d(r_t) + \Delta L_d(t, r_t)v(t - \tau(t)), \end{aligned} \tag{5.2}$$

$$x(t) = \phi(t) \quad \forall t \in [-\mu, 0], \tag{5.3}$$

where each matrix $\Gamma_\zeta(x(t), u(t), t, r_t)$ is defined as

$$\Gamma_\zeta(x(t), u(t), t, r_t) \triangleq \begin{bmatrix} [x(t)'(Q_{1\zeta}(r_t) + \Delta Q_{1\zeta}(t, r_t))x(t) + u(t)'(R_{1\zeta}(r_t) + \Delta R_{1\zeta}(t, r_t))u(t) \\ +Z_{1\zeta}(r_t) + \Delta Z_{1\zeta}(r_t)]^{\frac{1}{2}} \\ [x(t)'(Q_{2\zeta}(r_t) + \Delta Q_{2\zeta}(t, r_t))x(t) + u(t)'(R_{2\zeta}(r_t) + \Delta R_{2\zeta}(t, r_t))u(t) \\ +Z_{2\zeta}(r_t) + \Delta Z_{2\zeta}(r_t)]^{\frac{1}{2}} \\ \vdots \\ [x(t)'(Q_{n\zeta}(r_t) + \Delta Q_{n\zeta}(t, r_t))x(t) + u(t)'(R_{n\zeta}(r_t) + \Delta R_{n\zeta}(t, r_t))u(t) \\ +Z_{n\zeta}(r_t) + \Delta Z_{n\zeta}(r_t)]^{\frac{1}{2}} \end{bmatrix}, \tag{5.4}$$

for $t \geq 0$ with initial data $x(0) = x_0$ and $r(0) = i_0 \in \Lambda$. We assume that

$$\begin{aligned} Q_{k\zeta}(r_t) + \Delta Q_{k\zeta}(t, r_t) &\geq 0, \\ R_{k\zeta}(r_t) + \Delta R_{k\zeta}(t, r_t) &\geq 0, \\ Z_{k\zeta}(r_t) + \Delta Z_{k\zeta}(r_t) &\geq 0, \end{aligned}$$

for all $k = 1, 2, \cdots, n$. Matrices are assumed to have appropriate dimensions. Here $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $v(t) \in \mathbb{R}^p$ is the disturbance input, and $z(t) \in \mathbb{R}^q$ is the controlled output. Similar to settings of time delay introduced in [120], we assume that $\tau(t)$ is the time-varying delay that satisfies

$$0 < \tau(t) \leq \mu < \infty, \qquad \dot{\tau} \leq h < 1,$$

where $\mu$ and $h$ are real constant scalars. When $r_t = i$, we denote $A(r_t) = A_i$, $\Delta A(t, r_t) = \Delta A_i(t)$, etc. for simplicity.

In the above system, $A_i$, $A_{di}$, $B_i$, $C_i$, $C_{di}$, $E_i$, $E_{di}$, $F_i$, $G_i$, $G_{di}$, $H_i$, $H_{di}$ $L_i$, $L_{di}$, $S_i$, $R_{\sigma\zeta i}$, $H_{ui}$, $Q_{\sigma\zeta i}$, $H_{xi}$, $Z_{\sigma\zeta i}$, $H_{zi}$ are known real constant matrices. $\Delta A_i(t)$, $\Delta A_{di}(t)$, $\Delta B_i(t)$, $\Delta C_i(t)$, $\Delta C_{di}(t)$, $\Delta E_i(t)$, $\Delta E_{di}(t)$, $\Delta F_i(t)$, $\Delta G_i(t)$, $\Delta G_{di}(t)$, $\Delta H_i(t)$, $\Delta H_{di}(t)$, $\Delta L_i(t)$, $\Delta L_{di}(t)$, $\Delta S_i(t)$, $\Delta R_{\sigma\zeta i}(t)$, $\Delta H_{ui}(t)$, $\Delta Q_{\sigma\zeta i}(t)$, $\Delta H_{xi}(t)$, $\Delta Z_{\sigma\zeta i}(t)$, $\Delta H_{zi}(t)$ are unknown matrices and are denoted as parameter uncertainties. The parameter uncertainties are assumed to have the following structures:

$$\begin{aligned} &\begin{bmatrix} \Delta A_i(t) & \Delta A_{di}(t) & \Delta B_i(t) & \Delta C_i(t) & \Delta C_{di}(t) & \Delta E_i(t) & \Delta E_{di}(t) \end{bmatrix} \\ &\qquad = M_i U_i(t) \begin{bmatrix} N_{ai} & N_{adi} & N_{bi} & N_{ci} & N_{cdi} & N_{ei} & N_{edi} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} &\begin{bmatrix} \Delta F_i(t) & \Delta G_i(t) & \Delta G_{di}(t) & \Delta H_i(t) & \Delta H_{di}(t) & \Delta L_i & \Delta L_{di} & \Delta S_i \end{bmatrix} \\ &\qquad = M_i U_i(t) \begin{bmatrix} N_{fi} & N_{gi} & N_{gdi} & N_{hi} & N_{hdi} & N_{li} & N_{ldi} & N_{si} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} &\begin{bmatrix} \Delta R_{\sigma\zeta i}(t) & \Delta H_{ui}(t) & \Delta Q_{\sigma\zeta i}(t) & \Delta H_{xi}(t) \end{bmatrix} \\ &\qquad = M_i U_i(t) \begin{bmatrix} N_{r\sigma\zeta i} & N_{ui} & N_{q\sigma\zeta i} & N_{xi} \end{bmatrix}, \end{aligned} \tag{5.5}$$

where $M_i$, $N_{ai}$, $N_{adi}$, $N_{bi}$, $N_{ci}$, $N_{cdi}$, $N_{ei}$, $N_{edi}$, $N_{fi}$, $N_{gi}$, $N_{gdi}$, $N_{hi}$, $N_{hdi}$, $N_{li}$, $N_{ldi}$, $N_{si}$, $N_{r\sigma\zeta i}$, $N_{ui}$, $N_{q\sigma\zeta i}$, $N_{xi}$, $N_{z\sigma\zeta i}$, $N_{zi}$ are known real constant matrices and $U_i(t)$'s are unknown matrices satisfying $U_i(t)'U_i(t) \leq I$, $\forall i \in \Lambda$. The elements of $U_i(t)$ are assumed to be Lebesgue measurable. Such uncertainty structure has been used by many authors, e.g. [115], [53], [120], [108], [20], [114], [122], [121], [123], [107] and [46].

Additionally, similar to the settings in [119], the mode transition rate matrix $\widehat{\Pi} \triangleq (\widehat{\pi}_{ij})_{N\times N}$ is also assumed to be uncertain and has the element-wise uncertainties

$$\widehat{\Pi} = \Pi + \Delta\Pi,$$

with $\Pi \triangleq (\pi_{ij})_{N\times N}$ satisfying $\pi_{ij} \geq 0$, $(i, j \in \Lambda, j \neq i)$ and $\pi_{ii} \triangleq -\sum_{j=1,j\neq i}^{N} \pi_{ij}$ for all $i \in \Lambda$, where $\pi_{ij}$ denotes the estimated value of $\widehat{\pi}_{ij}$, and $\Delta\Pi \triangleq (\Delta\pi_{ij}) = (\widehat{\pi}_{ij} - \pi_{ij})$ where $|\Delta\pi_{ij}| \leq \varepsilon_{ij}$, $\varepsilon_{ij} \geq 0$. $\Delta\pi_{ij}$ denotes the error between $\widehat{\pi}_{ij}$ and $\pi_{ij}$ for all $i, j \in \Lambda, j \neq i$ and $\Delta\pi_{ii} \triangleq -\sum_{j=1,j\neq i}^{N} \Delta\pi_{ij}$ for all $i \in \Lambda$.

Similar to several definitions stated in Section 2.6 and the references therein, here we have some similar definitions and lemmas.

**Definition 5.2.1.** *[120] The system in (5.1) and (5.3) with $u(t) = 0$ and $v(t) = 0$ is said to be mean-square asymptotically stable if*

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^2 = 0$$

*for any initial conditions.*

**Definition 5.2.2.** *[120] The uncertain stochastic system in (5.1) and (5.3) is said to be robustly stochastically stable if the system associated to (5.1) and (5.3) with $u(t) = 0$ and $v(t) = 0$ is mean-square asymptotically stable for all admissible uncertainties $\Delta A_i$, $\Delta A_{di}$, $\Delta E_i$, and $\Delta E_{di}$.*

Before we provide the definition of the generalized robust $H_\infty$ control, let us recall the classic definition first.

**Definition 5.2.3.** *[120] Given a scalar $\gamma > 0$, the stochastic system from (5.1) to (5.3) with $u(t) = 0$ is said to be robustly stochastically stable with disturbance attenuation $\gamma$ if it is robustly stochastically stable and under zero initial conditions,*

$\|z(t)\| < \gamma\|v(t)\|$ *for all non-zero* $v(t)$ *and all admissible uncertainties* $\Delta A_i$, $\Delta A_{di}$, $\Delta B_i$, $\Delta C_i$, $\Delta C_{di}$, $\Delta E_i$, $\Delta E_{di}$, $\Delta F_i$, $\Delta G_i$, $\Delta G_{di}$, $\Delta H_i$, $\Delta H_{di}$, $\Delta L_i$, $\Delta L_{di}$, $\Delta S_i$, *where*

$$\|z(t)\| = \left(\mathbb{E}\left\{\int_0^\infty |z(t)|^2 dt\right\}\right)^{\frac{1}{2}}.$$

Now we briefly introduce the main idea of solving the classic robust $H_\infty$ control problem in [20], [114], [120] and [46], where $J$ is defined as:

$$J \triangleq \mathbb{E}\bigg\{\int_0^t \Big[z(s)'z(s) - \gamma^2 v(s)'v(s)\Big] ds\bigg\}. \tag{5.6}$$

According to Definition 5.2.3, in order to achieve $\|z(t)\| < \gamma\|v(t)\|$, it is equivalent to make $J < 0$. The following way of calculation is used in [20], [114], [120] and [46].

$$\begin{aligned} J &= \mathbb{E}\bigg\{\int_0^t \Big[z(s)'z(s) - \gamma^2 v(s)'v(s) + LV(x(s), i)\Big] ds\bigg\} - \mathbb{E}\{V(x(t), r_t)\} \\ &\leq \mathbb{E}\bigg\{\int_0^t \Big[z(s)'z(s) - \gamma^2 v(s)'v(s) + LV(x(s), i)\Big] ds\bigg\} \\ &= \mathbb{E}\bigg\{\int_0^t \bigg(\Big[x(s)' \quad x(s-\tau(s))' \quad v(s)'\Big]\Upsilon_i \\ &\qquad \times \Big[x(s)' \quad x(s-\tau(s))' \quad v(s)'\Big]'\bigg) ds\bigg\}. \end{aligned} \tag{5.7}$$

In the above case, if we can find conditions such that $\Upsilon_i < 0$ is achieved, then we have $J < 0$. Motivated by the above idea, we generalize the term $z(s)'z(s) - \gamma^2 v(s)'v(s)$ by changing the constant $\gamma^2$ into a matrix $R_i$, and rewriting $z(s)'z(s) - \gamma^2 v(s)'v(s)$ in (5.6) as follows:

$$\Big[x(s)' \quad x(s-\tau(s))' \quad v(s)' \quad v(s-\tau(s))' \quad z(s)'\Big] R_i \begin{bmatrix} x(s) \\ x(s-\tau(s)) \\ v(s) \\ v(s-\tau(s)) \\ z(s) \end{bmatrix}.$$

Now we provide the definition of generalized robust $H_\infty$ control problem, which is new compared with the ones in the past literatures, see for example [115], [114], [121], [20], [120], [122], [123], [107], and [130].

**Definition 5.2.4.** *Given a matrix with Markovian switching $R(r_t) > 0$, we define*

$$\begin{aligned}\bar{J} \;\triangleq\; & \mathbb{E}\bigg[\int_0^t \bigg\{ \Big[x(s)' \quad x(s-\tau(s))' \quad v(s)' \quad v(s-\tau(s))' \quad z(s)'\Big] R_i \\ & \times \Big[x(s)' \quad x(s-\tau(s))' \quad v(s)' \quad v(s-\tau(s))' \quad z(s)'\Big]' \bigg\} ds\bigg]. \end{aligned} \tag{5.8}$$

*The stochastic system from (5.1) to (5.3) with $u(t) = 0$ is said to be robustly stochastically stable with disturbance attenuation $R(r_t)$, if it is robustly stochastically stable and $\bar{J} < 0$ for all non-zero $v(t)$ and all admissible uncertainties $\Delta A_i$, $\Delta A_{di}$, $\Delta B_i$, $\Delta C_i$, $\Delta C_{di}$, $\Delta E_i$, $\Delta E_{di}$, $\Delta F_i$, $\Delta G_i$, $\Delta G_{di}$, $\Delta H_i$, $\Delta H_{di}$, $\Delta L_i$, $\Delta L_{di}$, and $\Delta S_i$.*

**Remark 5.2.1.** *The above definition involves state $x(s)$, state with delay $x(s-\tau(s))$, and disturbance with delay $v(s-\tau(s))$ in the $H_\infty$ performance. This kind of problem formulation is new, compared with the existing works. The importance of designing such a structure lies in the possible practical requirements. In order to explain intuitively, here we take the motor for example. When we model a motor mathematically, the electric current to the motor is regarded as state $x(s)$. Heat, magnetic field or any other interference are regarded as disturbance $v(s)$. The speed of rotation of the motor is regarded as output $z(s)$. The real situation in practice is that not only the disturbance $v(s)$ has effect on the output $z(s)$, but also the state $x(s)$ affects the output $z(s)$ in some cases. For example, when the electric current is too large for the motor, it will cause overheating to the coil, and finally the motor will break down. Therefore, the magnitude of the electric current must satisfy some criteria. Hence, we do need to consider state in the $H_\infty$ performance. In addition, in many situations, it is assumed that the future states of the system only depend on the present states, and are independent of the past states. Note that in some cases past dependence is important, and cannot be simply neglected. In order to achieve a more precise model, system with time delay has to be considered. Thus we allow time delay to be included in formulating the new $H_\infty$ performance. The selection of $R(r_t)$, corresponding to the $\gamma$ in the classic* *definition, depends on the requirements of the real problem. It is emphasized that Definition 5.2.4 contains Definition 5.2.3 as a special case.*

From the past literatures regarding robust control problems with time delays, see [115], [114], [120], [88], [121], [107], [20], [123], [122] and the references therein, the control law is usually designed to be linear in state only, for example, in forms of $u(t) = kx(t)$. Here in this chapter, we allow time delay to be included in the control law, which is new compared with the existing works. As we have mentioned in Remark 5.2.1, involving time delay in the system is crucial for accurate estimation. Practically, it is necessary to consider a controller that contains state with time delay. Mathematically, for the problems of robust stochastic stabilization and robust $H_\infty$ control, we design a robust controller of the following form:

$$u(t) = K_1(r_t)x(t) + K_2(r_t)x(t - \tau(t)). \tag{5.9}$$

Note that if we choose $K_2(r_t) = 0$, then our control law is exactly the same as the usual ones.

Similar to the previous two chapters, in the nonlinear stochastic systems, here we have to verify the existence and uniqueness of solution. Substituting the control (5.9) to the system (5.1), we rewrite our SDE into (5.17), in which the term $\Gamma_\zeta(x(t), t, r_t)$ has a bounded first derivative with respect to $x(t)$. This satisfies Theorem 7.10 in [83], where existence and uniqueness of SDEs with Markovian switching and time delay is proved.

## 5.3 Robust Stochastic Stabilization

In this section we provide a theorem in which sufficient conditions are derived such that a robust controller of the form (5.9) exists. First we provide a lemma that will be useful in the calculation of the proof for Theorem 5.3.1.

For notation simplicity, we define

$$\Gamma_\zeta(x(t), u(t), t, r_t) \triangleq$$

$$\begin{bmatrix} [x(t)'(Q_{1\zeta}(r_t)+\Delta Q_{1\zeta}(t,r_t))x(t)+u(t)'(R_{1\zeta}(r_t)+\Delta R_{1\zeta}(t,r_t))u(t) \\ +Z_{1\zeta}(r_t)+\Delta Z_{1\zeta}(r_t)]^{\frac{1}{2}} \\ [x(t)'(Q_{2\zeta}(r_t)+\Delta Q_{2\zeta}(t,r_t))x(t)+u(t)'(R_{2\zeta}(r_t)+\Delta R_{2\zeta}(t,r_t))u(t) \\ +Z_{2\zeta}(r_t)+\Delta Z_{2\zeta}(r_t)]^{\frac{1}{2}} \\ \vdots \\ [x(t)'(Q_{n\zeta}(r_t)+\Delta Q_{n\zeta}(t,r_t))x(t)+u(t)'(R_{n\zeta}(r_t)+\Delta R_{n\zeta}(t,r_t))u(t) \\ +Z_{n\zeta}(r_t)+\Delta Z_{n\zeta}(r_t)]^{\frac{1}{2}} \end{bmatrix}$$

$$\triangleq \begin{bmatrix} a_{1\zeta i}(t) \\ a_{2\zeta i}(t) \\ \vdots \\ a_{n\zeta i}(t) \end{bmatrix}. \tag{5.10}$$

Rewrite the $n \times n$ matrix $P_i(t)$ into the following:

$$P_i(t) = \begin{bmatrix} P_{11i}(t) & P_{12i}(t) & \cdots & P_{1ni}(t) \\ P_{21i}(t) & P_{22i}(t) & \cdots & P_{2ni}(t) \\ \vdots & \vdots & & \vdots \\ P_{n1i}(t) & P_{n2i}(t) & \cdots & P_{nni}(t) \end{bmatrix}. \tag{5.11}$$

**Assumption 5.3.1.** *We assume that the following holds:*

$$\begin{aligned} &\sum_{\zeta=1}^{m} \Bigg[ \sum_{k=2}^{n} a_{1\zeta i}(t)a_{k\zeta i}(t)P_{1ki}(t) + \sum_{k=1,k\neq 2}^{n} a_{2\zeta i}(t)a_{k\zeta i}(t)P_{2ki}(t) \\ &\quad + \cdots + \sum_{k=1}^{n-1} a_{n\zeta i}(t)a_{k\zeta i}(t)P_{nki}(t) \Bigg] \\ \triangleq\ & x(t)'(H_{xi}+\Delta H_{xi}(t))x(t) + u(t)'(H_{ui}+\Delta H_{ui}(t))u(t) + H_{zi} \\ & +\Delta H_{zi}. \end{aligned} \tag{5.12}$$

Note that $a_{1\zeta i}(t)$, $a_{2\zeta i}(t)$, $\cdots$, and $a_{n\zeta i}(t)$ are all terms with square roots. The above assumption allow us to cancel the square roots after multiplication. Next, we discuss the feasibility of the above assumption.

**Remark 5.3.1.** *In order to verify that the above assumption is not too strong or too conservative, we provide several cases when it holds.*

*(1) When we have* $a_{1\zeta i}(t) = a_{2\zeta i}(t) = \cdots = a_{n\zeta i}(t)$*, i.e., all the the* $a_{k\zeta i}(t)$ *are the same, the above assumption holds.*

*(2) When we have for example* $a_{1\zeta i}(t) = a_{2\zeta i}(t)$*,* $a_{3\zeta i}(t) = a_{4\zeta i}(t)$ *and* $a_{1\zeta i}(t) \neq a_{3\zeta i}(t)$*, in this case, the square roots are cancelled after multiplication when* $a_{k\zeta i}(t) = a_{(k+1)\zeta i}(t)$*. In the cases when the square roots cannot be cancelled, we choose the corresponding* $P_{\sigma ki}(t) = 0$*, where* $\sigma = 1, 2, \cdots, n$*. Then the above assumption holds.*

*(3) Similar to Case (2), when we have some pairs of the same* $a_{k\zeta i}(t)$*, those square roots can be cancelled after multiplication. As for the other pairs that the square roots cannot be cancelled after multiplication, we can choose the corresponding* $P_{\sigma ki}(t)$ *that have the same absolute value but with different signs, i.e. one positive and the other negative, where* $\sigma = 1, 2, \cdots, n$*. In this case, those pairs are added up with value* $0$*. Then the above assumption holds.*

*Hence, with the above cases illustrated, we say our Assumption 5.3.1 is feasible in many situations.*

**Lemma 5.3.1.** *According to the notation introduced in (5.10) and (5.11), with Assumption 5.3.1, after a series of calculation, we have*

$$\begin{aligned}
&\sum_{\zeta=1}^{m} \mathrm{tr}[P_i(t)\Gamma_{\zeta i}(x,u,t)\Gamma_{\zeta i}(x,u,t)'] \\
= \;& x(t)'[\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}(Q_{\sigma\zeta i} + \Delta Q_{\sigma\zeta i}(t)) + H_{xi} + \Delta H_{xi}(t)]x(t) \\
&+u(t)'[\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}(R_{\sigma\zeta i} + \Delta R_{\sigma\zeta i}(t)) + H_{ui} + \Delta H_{ui}(t)]u(t) \\
&+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}(Z_{\sigma\zeta i} + \Delta Z_{\sigma\zeta i}) + H_{zi} + \Delta H_{zi}. \qquad (5.13)
\end{aligned}$$

*Proof.* According to (5.10) and (5.11), we have

$$\mathrm{tr}[P_i(t)\Gamma_{\zeta i}(x,u,t)\Gamma_{\zeta i}(x,u,t)']$$

$$\begin{aligned}
&= \left[\sum_{k=1}^{n} a_{k\zeta i}(t)P_{1ki}(t)\right]a_{1\zeta i}(t) + \left[\sum_{k=1}^{n} a_{k\zeta i}(t)P_{2ki}(t)\right]a_{2\zeta i}(t) \\
&\quad + \cdots + \left[\sum_{k=1}^{n} a_{k\zeta i}(t)P_{nki}(t)\right]a_{n\zeta i}(t) \\
&= \Bigg[a_{1\zeta i}(t)P_{11i}(t) + \sum_{k=2}^{n} a_{k\zeta i}(t)P_{1ki}(t)\Bigg]a_{1\zeta i}(t) + \Bigg[a_{2\zeta i}(t)P_{22i}(t) \\
&\quad + \sum_{k=1,k\neq 2}^{n} a_{k\zeta i}(t)P_{2ki}(t)\Bigg]a_{2\zeta i}(t) + \cdots + \Bigg[a_{n\zeta i}(t)P_{nni}(t) \\
&\quad + \sum_{k=1}^{n-1} a_{k\zeta i}(t)P_{nki}(t)\Bigg]a_{n\zeta i}(t) \\
&= a_{1\zeta i}(t)^2P_{11i}(t) + a_{2\zeta i}(t)^2P_{22i}(t) + \cdots + a_{n\zeta i}(t)^2P_{nni}(t) \\
&\quad + \sum_{k=2}^{n} a_{k\zeta i}(t)P_{1ki}(t)a_{1\zeta i}(t) + \sum_{k=1,k\neq 2}^{n} a_{k\zeta i}(t)P_{2ki}(t)a_{2\zeta i}(t) \\
&\quad + \cdots + \sum_{k=1}^{n-1} a_{k\zeta i}(t)P_{nki}(t)a_{n\zeta i}(t).
\end{aligned}$$

Then we take the sum of the above terms, and rewrite it as follows,

$$\begin{aligned}
&\sum_{\zeta=1}^{m} \mathrm{tr}\Bigg[P_i(t)\Gamma_{\zeta i}(x,u,t)\Gamma_{\zeta i}(x,u,t)'\Bigg] \\
&= \sum_{\zeta=1}^{m} \Bigg[a_{1\zeta i}(t)^2P_{11i}(t) + a_{2\zeta i}(t)^2P_{22i}(t) + \cdots + a_{n\zeta i}(t)^2P_{nni}(t)\Bigg] \\
&\quad + \sum_{\zeta=1}^{m} \Bigg[\sum_{k=2}^{n} a_{1\zeta i}(t)a_{k\zeta i}(t)P_{1ki}(t) + \sum_{k=1,k\neq 2}^{n} a_{2\zeta i}(t)a_{k\zeta i}(t)P_{2ki}(t) + \cdots \\
&\qquad + \sum_{k=1}^{n-1} a_{n\zeta i}(t)a_{k\zeta i}(t)P_{nki}(t)\Bigg].
\end{aligned}$$

Now it is clear that with terms like $a_{1\zeta i}(t)^2$, $a_{2\zeta i}(t)^2$, $\cdots$, and $a_{n\zeta i}(t)^2$, the square roots are cancelled. In addition, we use Assumption 5.3.1 to rewrite the remaining terms. Then we have

$$\sum_{\zeta=1}^{m} \mathrm{tr}[P_i(t)\Gamma_{\zeta i}(x,u,t)\Gamma_{\zeta i}(x,u,t)']$$

$$
\begin{aligned}
&= x'\Big[\sum_{\zeta=1}^{m}\Big(P_{11i}(t)(Q_{1\zeta i}+\Delta Q_{1\zeta i}(t))+P_{22i}(t)(Q_{2\zeta i}+\Delta Q_{2\zeta i}(t))+\cdots \\
&\quad +P_{nni}(t)(Q_{n\zeta i}+\Delta Q_{n\zeta i}(t))\Big)+H_{xi}+\Delta H_{xi}(t)\Big]x \\
&\quad +u'\Big[\sum_{\zeta=1}^{m}\Big(P_{11i}(t)(R_{1\zeta i}+\Delta R_{1\zeta i}(t))+P_{22i}(t)(R_{2\zeta i}+\Delta R_{2\zeta i}(t))+\cdots \\
&\quad +P_{nni}(t)(R_{n\zeta i}+\Delta R_{n\zeta i}(t))\Big)+H_{ui}+\Delta H_{ui}(t)\Big]u \\
&\quad +\sum_{\zeta=1}^{m}\Big(P_{11i}(t)(Z_{1\zeta i}+\Delta Z_{1\zeta i})+P_{22i}(t)(Z_{2\zeta i}+\Delta Z_{2\zeta i})+\cdots \\
&\quad +P_{nni}(t)(Z_{n\zeta i}+\Delta Z_{n\zeta i})\Big)+H_{zi}+\Delta H_{zi} \\
&= x(t)'[\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}(Q_{\sigma\zeta i}+\Delta Q_{\sigma\zeta i}(t))+H_{xi}+\Delta H_{xi}(t)]x(t) \\
&\quad +u(t)'[\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}(R_{\sigma\zeta i}+\Delta R_{\sigma\zeta i}(t))+H_{ui}+\Delta H_{ui}(t)]u(t) \\
&\quad +\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}(Z_{\sigma\zeta i}+\Delta Z_{\sigma\zeta i})+H_{zi}+\Delta H_{zi}. \qquad (5.14)
\end{aligned}
$$

□

**Remark 5.3.2.** *It is highlighted that the importance of the Lemma 5.3.1 is that when calculating the term* $\sum_{\zeta=1}^{m}\operatorname{tr}[P_i(t)\Gamma_{\zeta i}(x,u,t)\Gamma_{\zeta i}(x,u,t)']$ *we are able to eliminate the square roots that originally appear in* $\Gamma_{\zeta i}(x,u,t)$*.*

Note that the structure of the parameter uncertainty $\Delta Q_{\sigma\zeta i}(t)$, $\Delta R_{\sigma\zeta i}(t)$, $\Delta H_{xi}(t)$, and $\Delta H_{ui}(t)$ in (5.14) is introduced in (5.5). Next, we introduce one more kind of uncertainty in the following assumption, called interval uncertainty, which is used to model $\Delta Z_{\sigma\zeta i}$ and $\Delta H_{zi}$.

**Assumption 5.3.2.** *We assume that the scalars* $\Delta Z_{\sigma\zeta i}$ *and* $\Delta H_{zi}$ *have interval uncertainty as follows,* $\Delta Z_{\sigma\zeta i}\le\alpha_{\sigma\zeta i}$ *and* $\Delta H_{zi}\le\beta_i$*.*

We introduce the following matrices:

$$\bar{A}(r_t) \triangleq A(r_t)+B(r_t)K_1(r_t),$$

$$
\begin{aligned}
\bar{A}_d(r_t) &\triangleq A_d(r_t) + B(r_t)K_2(r_t), \\
\Delta\bar{A}(t, r_t) &\triangleq M(r_t)U(t, r_t)\bar{N}_a(r_t), \\
\Delta\bar{A}_d(t, r_t) &\triangleq M(r_t)U(t, r_t)\bar{N}_{ad}(r_t), \\
\bar{N}_a(r_t) &\triangleq N_a(r_t) + N_b(r_t)K_1(r_t), \\
\bar{N}_{ad}(r_t) &\triangleq N_{ad}(r_t) + N_b(r_t)K_2(r_t), \\
\bar{E}(r_t) &\triangleq E(r_t) + F(r_t)K_1(r_t), \\
\bar{E}_d(r_t) &\triangleq E_d(r_t) + F(r_t)K_2(r_t), \\
\Delta\bar{E}(t, r_t) &\triangleq M(r_t)U(t, r_t)\bar{N}_e(r_t), \\
\Delta\bar{E}_d(t, r_t) &\triangleq M(r_t)U(t, r_t)\bar{N}_{ed}(r_t), \\
\bar{N}_e(r_t) &\triangleq N_e(r_t) + N_f(r_t)K_1(r_t), \\
\bar{N}_{ed}(r_t) &\triangleq N_{ed}(r_t) + N_f(r_t)K_2(r_t).
\end{aligned}
$$

**Theorem 5.3.1.** *Let $v(t) = 0$, $\forall t \geq 0$. Let Assumption 5.3.1 and Assumption 5.3.2 hold, with Lemma 5.3.1, the system (5.1) is robustly stochastically stabilizable if there exist scalars $\{\epsilon_{1i} > 0, i \in \Lambda\}$, $\{\epsilon_{2i} > 0, i \in \Lambda\}$, $\{\lambda_{ij} > 0, i, j \in \Lambda, i \neq j\}$, and matrices $\{P_i, i \in \Lambda\}$, $\{K_i, i \in \Lambda\}$ with appropriate dimensions, such that both of the following two matrix inequalities (5.15) and (5.16) hold,*

$$
\begin{bmatrix}
\mathcal{M}_i & \mathcal{L}_i & \bar{N}_{ai}' & \bar{N}_{ei}' & \bar{E}_i' \\
\mathcal{L}_i' & \mathcal{N}_i & 0 & 0 & 0 \\
\bar{N}_{ai} & 0 & -\epsilon_{1i}I & 0 & 0 \\
\bar{N}_{ei} & 0 & 0 & -\epsilon_{2i}I & 0 \\
\bar{E}_i & 0 & 0 & 0 & \epsilon_{2i}M_iM_i' - P_i^{-1}
\end{bmatrix} < 0, \quad i \in \Lambda, \tag{5.15}
$$

*and*

$$
\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}(Z_{\sigma\zeta i} + \alpha_{\sigma\zeta i}) + H_{zi} + \beta_i < 0, \quad i \in \Lambda, \tag{5.16}
$$

*where*

$$
\begin{aligned}
\mathcal{M}_i \triangleq\ & P_i\bar{A}_i + \bar{A}'P_i + \epsilon_{1i}P_iM_iM_i'P_i + Q_i + \frac{1}{2}\phi_{1i}m\sum_{\sigma=1}^{n} P_{\sigma\sigma i}M_iM_i' \\
&+ \frac{1}{2}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}\phi_{1i}^{-1}N_{q\sigma\zeta i}'N_{q\sigma\zeta i} + \frac{1}{2}\phi_{2i}M_iM_i' + \frac{1}{2}\phi_{2i}^{-1}N_{xi}'N_{xi}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2}\phi_{3i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K_{1i}'M_iM_i'K_{1i}+\frac{1}{2}\phi_{3i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'N_{r\sigma\zeta i}'N_{r\sigma\zeta i}K_{1i} \\
& +\frac{1}{2}\phi_{4i}K_{1i}'M_iM_i'K_{1i}+\frac{1}{2}\phi_{4i}^{-1}K_{1i}'N_{ui}'N_{ui}K_{1i} \\
& +\phi_{5i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K_{1i}'M_iM_i'K_{1i}+\phi_{6i}K_{1i}'M_iM_i'K_{1i} \\
& +\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}Q_{\sigma\zeta i}+H_{xi}+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'R_{\sigma\zeta i}K_{1i}+K_{1i}'H_{ui}K_{1i} \\
& +\sum_{j=1,j\neq i}^{N}\left[\frac{\lambda_{ij}}{4}\varepsilon_{ij}^{2}I+\frac{1}{\lambda_{ij}}(P_j-P_i)^2\right]+\sum_{j=1}^{N}\pi_{ij}P_j,
\end{aligned}
$$

$$
\mathcal{L}_i \triangleq P_i\bar{A}_{di}+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'R_{\sigma\zeta i}K_{2i}+K_{1i}'H_{ui}K_{2i},
$$

$$
\begin{aligned}
\mathcal{N}_i \triangleq\ & \phi_{5i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{2i}'N_{r\sigma\zeta i}'N_{r\sigma\zeta i}K_{2i}+\phi_{6i}^{-1}K_{2i}'N_{ui}'N_{ui}K_{2i} \\
& +\frac{1}{2}\phi_{7i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K_{2i}'M_iM_i'K_{2i}+\frac{1}{2}\phi_{7i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{2i}'N_{r\sigma\zeta i}'N_{r\sigma\zeta i}K_{2i} \\
& +\frac{1}{2}\phi_{8i}K_{2i}'M_iM_i'K_{2i}+\frac{1}{2}\phi_{8i}^{-1}K_{2i}'N_{ui}'N_{ui}K_{2i} \\
& +\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{2i}'R_{\sigma\zeta i}K_{2i}+K_{2i}'H_{ui}K_{2i}-(1-h)Q_i.
\end{aligned}
$$

*In this case the controller can be chosen by (5.9).*

*Proof.* Let us assume that there exist scalars $\{\epsilon_{1i}>0,i\in\Lambda\}$, $\{\epsilon_{2i}>0,i\in\Lambda\}$, $\{\phi_{1i}>0,i\in\Lambda\}$, $\{\phi_{2i}>0,i\in\Lambda\}$, $\{\phi_{3i}>0,i\in\Lambda\}$, $\{\phi_{4i}>0,i\in\Lambda\}$, $\{\phi_{5i}>0,i\in\Lambda\}$, $\{\phi_{6i}>0,i\in\Lambda\}$, $\{\phi_{7i}>0,i\in\Lambda\}$, $\{\phi_{8i}>0,i\in\Lambda\}$, $\{\lambda_{ij}>0,i,j\in\Lambda,i\neq j\}$, and matrices $\{P_i\in\mathcal{S}^n,i\in\Lambda\}$, $\{K_i\in\mathbb{R}^{m\times n},i\in\Lambda\}$, such that (5.15) and (5.16) hold. Also let $v(t)=0$, $\forall t\geq 0$. Substituting the control (5.9) to the system (5.1), we obtain the system

$$
\begin{aligned}
dx(t) =\ & \{[\bar{A}(r_t)+\Delta\bar{A}(t,r_t)]x(t)+[\bar{A}_d(r_t)+\Delta\bar{A}_d(t,r_t)]x(t-\tau(t))\}dt \\
& +\{[\bar{E}(r_t)+\Delta\bar{E}(t,r_t)]x(t)+[\bar{E}_d(r_t)+\Delta\bar{E}_d(t,r_t)]x(t-\tau(t))\}dW(t)
\end{aligned}
$$

$$+\sum_{\zeta=1}^{m}\Gamma_{\zeta}(x(t),t,r_t)d\tilde{W}_{\zeta}(t). \tag{5.17}$$

We consider $V(x(t),r_t)$ as a Lyapunov candidate for (5.17), where

$$V(x(t),r_t) \triangleq x(t)'P(r_t)x(t) + \int_{t-\tau(t)}^{t} x(s)'Qx(s)ds. \tag{5.18}$$

Denote the operator $LV(x(t),i)$ as the drift term after applying Itô's formula to $V(x(t),i)$, according to Lemma 2.9.1, we obtain

$$\mathbb{E}[dV(x(t),i)] = \mathbb{E}[LV(x(t),i)]dt,$$

where the operator

$$\begin{aligned} LV(x(t),i) \;\triangleq\; & 2x(t)'P_i[(\bar{A}_i+\Delta\bar{A}_i(t))x(t)+(\bar{A}_{di}+\Delta\bar{A}_{di}(t))x(t-\tau(t))] \\ & +[(\bar{E}_i+\Delta\bar{E}_i(t))x(t)+(\bar{E}_{di}+\Delta\bar{E}_{di}(t))x(t-\tau(t))]'P_i[(\bar{E}_i \\ & +\Delta\bar{E}_i(t))x(t)+(\bar{E}_{di}+\Delta\bar{E}_{di}(t))x(t-\tau(t))]+x(t)'Q_ix(t) \\ & -(1-\dot{\tau}(t))x(t-\tau(t))'Q_ix(t-\tau(t))+\sum_{j=1}^{N}\widehat{\pi}_{ij}x(t)'P_jx(t) \\ & +\sum_{\zeta=1}^{m}\mathrm{tr}[P_i\Gamma_{\zeta i}(x,u,t)\Gamma_{\zeta i}(x,u,t)']. \end{aligned} \tag{5.19}$$

Substituting the control (5.9) into (5.13) of Lemma 5.3.1, then we have

$$\begin{aligned} & \sum_{\zeta=1}^{m}\mathrm{tr}[P_i\Gamma_{\zeta i}(x,u,t)\Gamma_{\zeta i}(x,u,t)'] \\ = \;& x(t)'(\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}Q_{\sigma\zeta i}+H_{xi}+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'R_{\sigma\zeta i}K_{1i}+K_{1i}'H_{ui}K_{1i})x(t) \\ & +2x(t)'(\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'R_{\sigma\zeta i}K_{2i}+K_{1i}'H_{ui}K_{2i})x(t-\tau(t)) \\ & +x(t-\tau(t))'(\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{2i}'R_{\sigma\zeta i}K_{2i}+K_{2i}'H_{ui}K_{2i})x(t-\tau(t)) \\ & +x(t)'\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}M_iU_i(t)N_{q\sigma\zeta}x(t)+x(t)'M_iU_i(t)N_{xi}x(t) \end{aligned}$$

$$\begin{aligned}
&+x(t)'\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'M_iU_i(t)N_{r\sigma\zeta}K_{1i}x(t)+x(t)'K_{1i}'M_iU_i(t)N_{ui}K_{1i}x(t)\\
&+2x(t)'\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'M_iU_i(t)N_{r\sigma\zeta}K_{2i}x(t-\tau(t))\\
&+2x(t)'K_{1i}'M_iU_i(t)N_{ui}K_{2i}x(t-\tau(t))\\
&+x(t-\tau(t))'\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{2i}'M_iU_i(t)N_{r\sigma\zeta}K_{2i}x(t-\tau(t))\\
&+x(t-\tau(t))'K_{2i}'M_iU_i(t)N_{ui}K_{2i}x(t-\tau(t))\\
&+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}(Z_{\sigma\zeta i}+\Delta Z_{\sigma\zeta i})+H_{zi}+\Delta H_{zi}.
\end{aligned}\tag{5.20}$$

According to Lemma 2.9.6, which is used to eliminate the parameter uncertainties, we have the following inequalities:

$$\begin{aligned}
&\quad 2x(t)'P_i[(\bar{A}_i+\Delta\bar{A}_i(t))x(t)+(\bar{A}_{di}+\Delta\bar{A}_{di}(t))x(t-\tau(t))]\\
&= 2x(t)'P_i\bar{A}_ix(t)+2x(t)'P_i\bar{A}_{di}x(t-\tau(t))+2x(t)'P_iM_iU_i(t)[\bar{N}_{ai}x\\
&\quad +\bar{N}_{adi}x(t-\tau(t))]\\
&\leq 2x(t)'P_i\bar{A}_ix(t)+2x(t)'P_i\bar{A}_{di}x(t-\tau(t))+\epsilon_{1i}x(t)'P_iM_iM_i'P_ix(t)\\
&\quad +\epsilon_{1i}^{-1}[\bar{N}_{ai}x(t)+\bar{N}_{ad}x(t-\tau(t))]'[\bar{N}_{ai}x(t)+\bar{N}_{adi}x(t-\tau(t))],
\end{aligned}$$

$$\begin{aligned}
&\quad [(\bar{E}_i+\Delta\bar{E}_i(t))x(t)+(\bar{E}_{di}+\Delta\bar{E}_{di}(t))x(t-\tau(t))]'P_i[(\bar{E}_i+\Delta\bar{E}_i(t))x(t)\\
&\quad +(\bar{E}_{di}+\Delta\bar{E}_{di}(t))x(t-\tau(t))]\\
&= [\bar{E}_ix(t)+\bar{E}_{di}x(t-\tau(t))+M_iU_i(t)(\bar{N}_{ei}x(t)+\bar{N}_{edi}x(t-\tau(t)))]'P_i[\bar{E}_ix(t)\\
&\quad +\bar{E}_{di}x(t-\tau(t))+M_iU_i(t)(\bar{N}_{ei}x(t)+\bar{N}_{edi}x(t-\tau(t)))]\\
&\leq (\bar{E}_ix(t)+\bar{E}_{di}x(t-\tau(t)))'(P_i^{-1}-\epsilon_{2i}M_iM_i')^{-1}(\bar{E}_ix(t)+\bar{E}_{di}x(t-\tau(t)))\\
&\quad +\epsilon_{2i}^{-1}(\bar{N}_{ei}x(t)+\bar{N}_{edi}x(t-\tau(t))'(\bar{N}_{ei}x(t)+\bar{N}_{edi}x(t-\tau(t)),
\end{aligned}$$

$$\begin{aligned}
&\quad x(t)'\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}M_iU_i(t)N_{q\sigma\zeta}x(t)\\
&\leq x(t)'(\frac{1}{2}\phi_{1i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}M_iM_i'+\frac{1}{2}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}\phi_{1i}^{-1}N_{q\sigma\zeta i}'N_{q\sigma\zeta i})x(t),
\end{aligned}$$

$$x(t)'M_iU_i(t)N_{xi}x(t) \leq x(t)'(\frac{1}{2}\phi_{2i}M_iM_i' + \frac{1}{2}\phi_{2i}^{-1}N_{xi}'N_{xi})x(t),$$

$$\begin{aligned}
& x(t)'\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'M_iU_i(t)N_{r\sigma\zeta}K_{1i}x(t) \\
\leq \; & x(t)'(\frac{1}{2}\phi_{3i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K_{1i}'M_iM_i'K_{1i} \\
& +\frac{1}{2}\phi_{3i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'N_{r\sigma\zeta i}'N_{r\sigma\zeta i}K_{1i})x(t),
\end{aligned}$$

$$\begin{aligned}
& x(t)'K_{1i}'M_iU_i(t)N_{ui}K_{1i}x(t) \\
\leq \; & x(t)'(\frac{1}{2}\phi_{4i}K_{1i}'M_iM_i'K_{1i} + \frac{1}{2}\phi_{4i}^{-1}K_{1i}'N_{ui}'N_{ui}K_{1i})x(t),
\end{aligned}$$

$$\begin{aligned}
& 2x(t)'\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'M_iU_i(t)N_{r\sigma\zeta}K_{2i}x(t-\tau(t)) \\
\leq \; & x(t)'(\phi_{5i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K_{1i}'M_iM_i'K_{1i})x(t) \\
& +x(t-\tau(t))'(\phi_{5i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{2i}'N_{r\sigma\zeta i}'N_{r\sigma\zeta i}K_{2i})x(t-\tau(t)),
\end{aligned}$$

$$\begin{aligned}
& 2x(t)'K_{1i}'M_iU_i(t)N_{ui}K_{2i}x(t-\tau(t)) \\
\leq \; & \phi_{6i}x(t)'K_{1i}'M_iM_i'K_{1i}x(t) + \phi_{6i}^{-1}x(t-\tau(t))'K_{2i}'N_{ui}'N_{ui}K_{2i}x(t-\tau(t)),
\end{aligned}$$

$$\begin{aligned}
& x(t-\tau(t))'\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{2i}'M_iU_i(t)N_{r\sigma\zeta}K_{2i}x(t-\tau(t)) \\
\leq \; & x(t-\tau(t))'(\frac{1}{2}\phi_{7i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K_{2i}'M_iM_i'K_{2i} \\
& +\frac{1}{2}\phi_{7i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{2i}'N_{r\sigma\zeta i}'N_{r\sigma\zeta i}K_{2i})x(t-\tau(t)),
\end{aligned}$$

$$x(t-\tau(t))'K_{2i}'M_iU_i(t)N_{ui}K_{2i}x(t-\tau(t))$$

$$\leq \quad x(t-\tau(t))'(\frac{1}{2}\phi_{8i}K_{2i}'M_iM_i'K_{2i}+\frac{1}{2}\phi_{8i}^{-1}K_{2i}'N_{ui}'N_{ui}K_{2i})x(t-\tau(t)). \qquad (5.21)$$

Following (5.19) with a series of inequalities to (5.21), we have

$$\begin{aligned}
& LV(x(t),i) \\
\leq\ & 2x(t)'P_i\bar{A}_ix(t)+2x(t)'P_i\bar{A}_{di}x(t-\tau(t))+\epsilon_{1i}x(t)'P_iM_iM_i'P_ix(t) \\
& +\epsilon_{1i}^{-1}[\bar{N}_{ai}x(t)+\bar{N}_{ad}x(t-\tau(t))]'[\bar{N}_{ai}x(t)+\bar{N}_{adi}x(t-\tau(t))] \\
& +(\bar{E}_ix(t)+\bar{E}_{di}x(t-\tau(t)))'(P_i^{-1}-\epsilon_{2i}M_iM_i')^{-1}(\bar{E}_ix(t)+\bar{E}_{di}x(t-\tau(t))) \\
& +\epsilon_{2i}^{-1}(\bar{N}_{ei}x(t)+\bar{N}_{edi}x(t-\tau(t))'(\bar{N}_{ei}x(t)+\bar{N}_{edi}x(t-\tau(t)) \\
& +x(t)'(\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}Q_{\sigma\zeta i}+H_{xi}+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'R_{\sigma\zeta i}K_{1i}+K_{1i}'H_{ui}K_{1i})x(t) \\
& +2x(t)'(\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'R_{\sigma\zeta i}K_{2i}+K_{1i}'H_{ui}K_{2i})x(t-\tau(t)) \\
& +x(t-\tau(t))'(\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{2i}'R_{\sigma\zeta i}K_{2i}+K_{2i}'H_{ui}K_{2i})x(t-\tau(t)) \\
& +x(t)'(\frac{1}{2}\phi_{1i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}M_iM_i'+\frac{1}{2}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}\phi_{1i}^{-1}N_{q\sigma\zeta i}'N_{q\sigma\zeta i})x(t), \\
& +x(t)'(\frac{1}{2}\phi_{2i}M_iM_i'+\frac{1}{2}\phi_{2i}^{-1}N_{xi}'N_{xi})x(t) \\
& +x(t)'(\frac{1}{2}\phi_{3i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K_{1i}'M_iM_i'K_{1i} \\
& \quad +\frac{1}{2}\phi_{3i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'N_{r\sigma\zeta i}'N_{r\sigma\zeta i}K_{1i})x(t) \\
& +x(t)'(\frac{1}{2}\phi_{4i}K_{1i}'M_iM_i'K_{1i}+\frac{1}{2}\phi_{4i}^{-1}K_{1i}'N_{ui}'N_{ui}K_{1i})x(t) \\
& +x(t)'(\phi_{5i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K_{1i}'M_iM_i'K_{1i})x(t) \\
& +x(t-\tau(t))'(\phi_{5i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{2i}'N_{r\sigma\zeta i}'N_{r\sigma\zeta i}K_{2i})x(t-\tau(t)) \\
& +\phi_{6i}x(t)'K_{1i}'M_iM_i'K_{1i}x(t)+\phi_{6i}^{-1}x(t-\tau(t))'K_{2i}'N_{ui}'N_{ui}K_{2i}x(t-\tau(t))
\end{aligned}$$

$$
\begin{aligned}
&+x(t-\tau(t))'(\frac{1}{2}\phi_{7i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K_{2i}'M_iM_i'K_{2i}\\
&+\frac{1}{2}\phi_{7i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{2i}'N_{r\sigma\zeta i}'N_{r\sigma\zeta i}K_{2i})x(t-\tau(t))\\
&+x(t-\tau(t))'(\frac{1}{2}\phi_{8i}K_{2i}'M_iM_i'K_{2i}+\frac{1}{2}\phi_{8i}^{-1}K_{2i}'N_{ui}'N_{ui}K_{2i})x(t-\tau(t))\\
&+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}(Z_{\sigma\zeta i}+\Delta Z_{\sigma\zeta i})+H_{zi}+\Delta H_{zi}\\
&+x(t)'Q_ix(t)-(1-\dot{\tau}(t))x(t-\tau(t))'Q_ix(t-\tau(t))\\
&+\sum_{j=1}^{N}\widehat{\pi}_{ij}x(t)'P_jx(t).
\end{aligned}
$$

According to Assumption 5.3.2 and the definition stated in the previous section, and [119], we have the following

$$
\begin{aligned}
\sum_{j=1}^{N}\Delta\pi_{ij}P_j &= \sum_{j=1,j\neq i}^{N}\left[\frac{1}{2}\Delta\pi_{ij}(P_j-P_i)+\frac{1}{2}\Delta\pi_{ij}(P_j-P_i)\right]\\
&\le \sum_{j=1,j\neq i}^{N}\left[\frac{\lambda_{ij}}{4}\varepsilon_{ij}^2I+\frac{1}{\lambda_{ij}}(P_j-P_i)^2\right], \qquad (5.22)
\end{aligned}
$$

where $\lambda_{ij}\in\mathbb{R}^+$. Then we have

$$
\begin{aligned}
&LV(x(t),i)\\
\le\ & x(t)'(P_i\bar{A}_i+\bar{A}'P_i+\epsilon_{1i}P_iM_iM_i'P_i+Q_i+\frac{1}{2}\phi_{1i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}M_iM_i'\\
&+\frac{1}{2}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}\phi_{1i}^{-1}N_{q\sigma\zeta i}'N_{q\sigma\zeta i}+\frac{1}{2}\phi_{2i}M_iM_i'+\frac{1}{2}\phi_{2i}^{-1}N_{xi}'N_{xi}\\
&+\frac{1}{2}\phi_{3i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K_{1i}'M_iM_i'K_{1i}+\frac{1}{2}\phi_{3i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K_{1i}'N_{r\sigma\zeta i}'N_{r\sigma\zeta i}K_{1i}\\
&+\frac{1}{2}\phi_{4i}K_{1i}'M_iM_i'K_{1i}+\frac{1}{2}\phi_{4i}^{-1}K_{1i}'N_{ui}'N_{ui}K_{1i}\\
&+\phi_{5i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K_{1i}'M_iM_i'K_{1i}+\phi_{6i}K_{1i}'M_iM_i'K_{1i}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}Q_{\sigma\zeta i}+H_{xi}+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K'_{1i}R_{\sigma\zeta i}K_{1i}+K'_{1i}H_{ui}K_{1i}\\
&+\sum_{j=1,j\neq i}^{N}\left[\frac{\lambda_{ij}}{4}\varepsilon_{ij}^{2}I+\frac{1}{\lambda_{ij}}(P_j-P_i)^2\right]+\sum_{j=1}^{N}\pi_{ij}P_j)x(t)\\
&+2x(t)'(P_i\bar{A}_{di}+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K'_{1i}R_{\sigma\zeta i}K_{2i}+K'_{1i}H_{ui}K_{2i})x(t-\tau(t))\\
&+x(t-\tau(t))'(\phi_{5i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K'_{2i}N'_{r\sigma\zeta i}N_{r\sigma\zeta i}K_{2i}+\phi_{6i}^{-1}K'_{2i}N'_{ui}N_{ui}K_{2i}\\
&+\frac{1}{2}\phi_{7i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}K'_{2i}M_iM'_iK_{2i}+\frac{1}{2}\phi_{7i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K'_{2i}N'_{r\sigma\zeta i}N_{r\sigma\zeta i}K_{2i}\\
&+\frac{1}{2}\phi_{8i}K'_{2i}M_iM'_iK_{2i}+\frac{1}{2}\phi_{8i}^{-1}K'_{2i}N'_{ui}N_{ui}K_{2i}\\
&+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}K'_{2i}R_{\sigma\zeta i}K_{2i}+K'_{2i}H_{ui}K_{2i}-(1-h)Q_i)x(t-\tau(t))\\
&+\epsilon_{1i}^{-1}[\bar{N}_{ai}x(t)+\bar{N}_{ad}x(t-\tau(t))]'[\bar{N}_{ai}x(t)+\bar{N}_{adi}x(t-\tau(t))]\\
&+(\bar{E}_ix(t)+\bar{E}_{di}x(t-\tau(t)))'(P_i^{-1}-\epsilon_{2i}M_iM'_i)^{-1}(\bar{E}_ix(t)+\bar{E}_{di}x(t-\tau(t)))\\
&+\epsilon_{2i}^{-1}(\bar{N}_{ei}x(t)+\bar{N}_{edi}x(t-\tau(t))'(\bar{N}_{ei}x(t)+\bar{N}_{edi}x(t-\tau(t)),\\
&+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}(Z_{\sigma\zeta i}+\alpha_{\sigma\zeta i})+H_{zi}+\beta_i.
\end{aligned}
$$

We rewrite the above inequality as follows,

$$
\begin{aligned}
& LV(x(t),i)\\
\leq\;& x(t)'\mathcal{M}_ix(t)+2x(t)'\mathcal{L}_ix(t-\tau(t))+x(t-\tau(t))'\mathcal{N}_ix(t-\tau(t))\\
&+\epsilon_{1i}^{-1}[\bar{N}_{ai}x(t)+\bar{N}_{ad}x(t-\tau(t))]'[\bar{N}_{ai}x(t)+\bar{N}_{adi}x(t-\tau(t))]\\
&+(\bar{E}_ix(t)+\bar{E}_{di}x(t-\tau(t)))'(P_i^{-1}-\epsilon_{2i}M_iM'_i)^{-1}(\bar{E}_ix(t)+\bar{E}_{di}x(t-\tau(t)))\\
&+\epsilon_{2i}^{-1}(\bar{N}_{ei}x(t)+\bar{N}_{edi}x(t-\tau(t))'(\bar{N}_{ei}x(t)+\bar{N}_{edi}x(t-\tau(t)),\\
&+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}(Z_{\sigma\zeta i}+\alpha_{\sigma\zeta i})+H_{zi}+\beta_i.
\end{aligned}
$$

The above inequality can be rewritten into forms of matrices as follows,

$$LV(x(t),i)$$

$$
\begin{aligned}
\leq\ & \begin{bmatrix} x(t)' & x(t-\tau(t))' \end{bmatrix} \begin{bmatrix} \mathcal{M}_i & \mathcal{L}_i \\ \mathcal{L}_i' & \mathcal{N}_i \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix} \\
& +\epsilon_{1i}^{-1} \begin{bmatrix} x(t)' & x(t-\tau(t))' \end{bmatrix} \begin{bmatrix} \bar{N}_{ai}' \\ \bar{N}_{adi}' \end{bmatrix} \begin{bmatrix} \bar{N}_{ai} & \bar{N}_{adi} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix} \\
& + \begin{bmatrix} x(t)' & x(t-\tau(t))' \end{bmatrix} \begin{bmatrix} \bar{E}_i' \\ \bar{E}_{di}' \end{bmatrix} (P_i^{-1} - \epsilon_{2i} M_i M_i')^{-1} \begin{bmatrix} \bar{E}_i & \bar{E}_{di} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix} \\
& +\epsilon_{2i}^{-1} \begin{bmatrix} x(t)' & x(t-\tau(t))' \end{bmatrix} \begin{bmatrix} \bar{N}_{ei}' \\ \bar{N}_{edi}' \end{bmatrix} \begin{bmatrix} \bar{N}_{ei} & \bar{N}_{edi} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix} \\
& + \sum_{\sigma=1}^{n} \sum_{\zeta=1}^{m} P_{\sigma\sigma i}(Z_{\sigma\zeta i} + \alpha_{\sigma\zeta i}) + H_{zi} + \beta_i \\
=\ & \begin{bmatrix} x(t)' & x(t-\tau(t))' \end{bmatrix} \Psi_i \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix} \\
& + \sum_{\sigma=1}^{n} \sum_{\zeta=1}^{m} P_{\sigma\sigma i}(Z_{\sigma\zeta i} + \alpha_{\sigma\zeta i}) + H_{zi} + \beta_i, \qquad (5.23)
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_i \ =\ & \begin{bmatrix} \mathcal{M}_i & \mathcal{L}_i \\ \mathcal{L}_i' & \mathcal{N}_i \end{bmatrix} + \epsilon_{1i}^{-1} \begin{bmatrix} \bar{N}_{ai}' \\ \bar{N}_{adi}' \end{bmatrix} \begin{bmatrix} \bar{N}_{ai} & \bar{N}_{adi} \end{bmatrix} \\
& + \begin{bmatrix} \bar{E}_i' \\ \bar{E}_{di}' \end{bmatrix} (P_i^{-1} - \epsilon_{2i} M_i M_i')^{-1} \begin{bmatrix} \bar{E}_i & \bar{E}_{di} \end{bmatrix} \\
& +\epsilon_{2i}^{-1} \begin{bmatrix} \bar{N}_{ei}' \\ \bar{N}_{edi}' \end{bmatrix} \begin{bmatrix} \bar{N}_{ei} & \bar{N}_{edi} \end{bmatrix}.
\end{aligned}
$$

From (5.15), and Lemma 2.9.4, $\Psi_i < 0$ is achieved. Note that there is a constant term

$$\sum_{\sigma=1}^{n} \sum_{\zeta=1}^{m} P_{\sigma\sigma i}(Z_{\sigma\zeta i} + \alpha_{\sigma\zeta i}) + H_{zi} + \beta_i$$

in (5.23). According to inequality (5.16), we have

$$LV(x(t), r_t) < 0. \qquad (5.24)$$

By Definition 5.2.1, Definition 5.2.2, and [83], [68], (5.24) is a sufficient condition such that the system (5.1) is robustly stochastically stable. □

## 5.4 Robust $H_\infty$ Control

In this section we derive a sufficient condition that solves the robust $H_\infty$ control problem for nonlinear uncertain stochastic systems with Markovian switching and time delay. We first introduce the following several matrices:

$$\begin{aligned}
\bar{C}(r_t) &\triangleq C(r_t) + S(r_t)K_1(r_t), \\
\bar{C}_d(r_t) &\triangleq C_d(r_t) + S(r_t)K_2(r_t), \\
\bar{N}_c(r_t) &\triangleq N_c(r_t) + N_s(r_t)K_1(r_t), \\
\bar{N}_{cd}(r_t) &\triangleq N_{cd}(r_t) + N_s(r_t)K_2(r_t), \\
\Delta\bar{C}(r_t) &\triangleq M(r_t)U(t, r_t)\bar{N}_c(r_t), \\
\Delta\bar{C}_d(r_t) &\triangleq M(r_t)U(t, r_t)\bar{N}_{cd}(r_t).
\end{aligned}$$

**Theorem 5.4.1.** *Let Assumption 5.3.1 and Assumption 5.3.2 hold, with Lemma 5.3.1, the system (5.1), (5.2) is robustly stochastically stabilizable with disturbance attenuation $R_i$, where the symmetric matrix $R_i$ is split into*

$$R_i = \begin{bmatrix} R_{11i} & * & * & * & * \\ R_{21i} & R_{22i} & * & * & * \\ R_{31i} & R_{32i} & R_{33i} & * & * \\ R_{41i} & R_{42i} & R_{43i} & R_{44i} & * \\ R_{51i} & R_{52i} & R_{53i} & R_{54i} & R_{55i} \end{bmatrix}, \tag{5.25}$$

*if there exist scalars $\{\epsilon_{1i} > 0, i \in \Lambda\}$, $\{\epsilon_{2i} > 0, i \in \Lambda\}$, $\{\epsilon_{3i} > 0, i \in \Lambda\}$, $\{\epsilon_{4i} > 0, i \in \Lambda\}$, $\{\epsilon_{5i} > 0, i \in \Lambda\}$, $\{\phi_{1i} > 0, i \in \Lambda\}$, $\{\phi_{2i} > 0, i \in \Lambda\}$, $\{\phi_{3i} > 0, i \in \Lambda\}$, $\{\phi_{4i} > 0, i \in \Lambda\}$, $\{\phi_{5i} > 0, i \in \Lambda\}$, $\{\phi_{6i} > 0, i \in \Lambda\}$, $\{\phi_{7i} > 0, i \in \Lambda\}$, $\{\phi_{8i} > 0, i \in \Lambda\}$, $\{\lambda_{ij} > 0, i, j \in \Lambda, i \neq j\}$, and matrices $\{P_i, i \in \Lambda\}$, $\{K_i, i \in \Lambda\}$ with appropriate dimensions, such that the following two matrix inequalities (5.26) and*

*(5.27) hold,*

$$\begin{bmatrix} Y_{11i} & * & * & * & * & * & * & * & * \\ Y_{21i} & Y_{22i} & * & * & * & * & * & * & * \\ Y_{31i} & Y_{32i} & Y_{33i} & * & * & * & * & * & * \\ Y_{41i} & Y_{42i} & Y_{43i} & Y_{44i} & * & * & * & * & * \\ \bar{N}_{ci} & \bar{N}_{cdi} & N_{li} & N_{ldi} & \gamma_{1i} & * & * & * & * \\ \bar{C}_i & \bar{C}_{di} & L_i & L_{di} & 0 & \gamma_{2i} & * & * & * \\ \bar{N}_{ai} & \bar{N}_{adi} & N_{gi} & N_{gdi} & 0 & 0 & \gamma_{3i} & * & * \\ \bar{E}_i & \bar{E}_{di} & H_i & H_{di} & 0 & 0 & 0 & \gamma_{4i} & * \\ \bar{N}_{ei} & \bar{N}_{edi} & N_{hi} & N_{hdi} & 0 & 0 & 0 & 0 & \gamma_{5i} \end{bmatrix} < 0, \quad i \in \Lambda, \tag{5.26}$$

*and*

$$\sum_{\sigma=1}^{n} \sum_{\zeta=1}^{m} P_{\sigma\sigma i}(Z_{\sigma\zeta i} + \alpha_{\sigma\zeta i}) + H_{zi} + \beta_i < 0, \quad i \in \Lambda, \tag{5.27}$$

*where*

$$\begin{aligned} Y_{11i} \;=\;& R_{11i} + 2R_{15i} + 2P_i\bar{A}_i + \epsilon_{1i}P_iM_iM_i'P_i + Q_i \\ & + \sum_{j=1, j\neq i}^{N} \left[\frac{\lambda_{ij}}{4}\varepsilon_{ij}^2 I + \frac{1}{\lambda_{ij}}(P_j - P_i)^2\right] + \sum_{j=1}^{N} \pi_{ij}P_j \\ & + \frac{1}{2}\phi_{1i} m \sum_{\sigma=1}^{n} P_{\sigma\sigma i}M_iM_i' + \frac{1}{2}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}\phi_{1i}^{-1}N_{q\sigma\zeta i}'N_{q\sigma\zeta i} \\ & + \frac{1}{2}\phi_{2i}M_iM_i' + \frac{1}{2}\phi_{2i}^{-1}N_{xi}'N_{xi} \\ & + \frac{1}{2}\phi_{3i} m \sum_{\sigma=1}^{n} P_{\sigma\sigma i}K_{1i}'M_iM_i'K_{1i} + \frac{1}{2}\phi_{3i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}K_{1i}'N_{r\sigma\zeta i}'N_{r\sigma\zeta i}K_{1i} \\ & + \frac{1}{2}\phi_{4i}K_{1i}'M_iM_i'K_{1i} + \frac{1}{2}\phi_{4i}^{-1}K_{1i}'N_{ui}'N_{ui}K_{1i} \\ & + \phi_{5i} m \sum_{\sigma=1}^{n} P_{\sigma\sigma i}K_{1i}'M_iM_i'K_{1i} + \phi_{6i}K_{1i}'M_iM_i'K_{1i} \\ & + \sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}Q_{\sigma\zeta i} + H_{xi} + \sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}K_{1i}'R_{\sigma\zeta i}K_{1i} + K_{1i}'H_{ui}K_{1i}, \end{aligned}$$

$$
\begin{aligned}
Y_{21i} &= R_{21i} + \bar{C}'_{di}R'_{15i} + R_{25i}\bar{C}_i + \bar{A}'_{di}P_i + \sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}K'_{1i}R_{\sigma\zeta i}K_{2i} \\
&\quad + K'_{1i}H_{ui}K_{2i},
\end{aligned}
$$

$$
Y_{31i} = R_{31i} + L'_iR'_{15i} + R_{35i}\bar{C}_i + G'_iP_i,
$$

$$
Y_{41i} = R_{41i} + L'_{di}R'_{15i} + R_{45i}\bar{C}_i + G'_{di}P_i,
$$

$$
\begin{aligned}
Y_{22i} &= R_{22i} + 2R_{25i}\bar{C}_{di} + 2\epsilon_{5i}R_{25i}M_iM'_iR_{52i} + (h-1)Q_i \\
&\quad + \phi_{5i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}K'_{2i}N'_{r\sigma\zeta i}N_{r\sigma\zeta i}K_{2i} + \phi_{6i}^{-1}K'_{2i}N'_{ui}N_{ui}K_{2i} \\
&\quad + \frac{1}{2}\phi_{7i}m\sum_{\sigma=1}^{n} P_{\sigma\sigma i}K'_{2i}M_iM'_iK_{2i} + \frac{1}{2}\phi_{7i}^{-1}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}K'_{2i}N'_{r\sigma\zeta i}N_{r\sigma\zeta i}K_{2i} \\
&\quad + \frac{1}{2}\phi_{8i}K'_{2i}M_iM'_iK_{2i} + \frac{1}{2}\phi_{8i}^{-1}K'_{2i}N'_{ui}N_{ui}K_{2i} \\
&\quad + \sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i}K'_{2i}R_{\sigma\zeta i}K_{2i} + K'_{2i}H_{ui}K_{2i},
\end{aligned}
$$

$$
Y_{32i} = R_{32i} + L'_iR'_{25i} + R_{35i}\bar{C}_{di},
$$

$$
Y_{42i} = R_{42i} + L'_{di}R'_{25i} + R_{45i}\bar{C}_{di},
$$

$$
Y_{33i} = R_{33i} + 2R_{35i}L_i + 2\epsilon_{6i}R_{35i}M_iM'_iR_{53i},
$$

$$
Y_{43i} = R_{43i} + L'_{di}R'_{35i} + R_{45i}L_i,
$$

$$
Y_{44i} = R_{44i} + 2R_{45i}L_{di} + 2\epsilon_{7i}R_{45i}M_iM'_iR_{54i},
$$

$$
\gamma_{1i} = -[\epsilon_{3i}^{-1} + \frac{1}{2}(\epsilon_{4i}^{-1} + \epsilon_{5i}^{-1} + \epsilon_{6i}^{-1} + \epsilon_{7i}^{-1})]I,
$$

$$\begin{aligned}
\gamma_{2i} &= (\epsilon_{3i} M_i M_i' - R_{55i}^{-1})^{-1}, \\
\gamma_{3i} &= -\epsilon_{1i}^{-1} I, \\
\gamma_{4i} &= (\epsilon_{2i} M_i M_i' - P_i^{-1})^{-1}, \\
\gamma_{5i} &= -\epsilon_{2i}^{-1} I.
\end{aligned}$$

*In this case the controller can be chosen by (5.9).*

*Proof.* Let us assume that there exist scalars $\{\epsilon_{1i} > 0, i \in \Lambda\}$, $\{\epsilon_{2i} > 0, i \in \Lambda\}$, $\{\lambda_{ij} > 0, i, j \in S, i \neq j\}$, and matrices $\{P_i \in \mathcal{S}^n, i \in \Lambda\}$, $\{K_i \in \mathbb{R}^{m \times n}, i \in \Lambda\}$, such that (5.26) holds. By (5.26), matrix inequalities (5.15) hold. In addition, the requirement of (5.27) also appears in Theorem 5.3.1. Therefore, from Theorem 5.3.1, the system is robustly stochastically stable. We consider (5.18) as a Lyapunov candidate for (5.17). Denote the operator $LV(x(t), i)$ as the drift term after applying Itô's formula to $V(x(t), i)$.

By Lemma 2.9.1, we have

$$\mathbb{E}[V(x(t)), r_t] = \mathbb{E}\left[\int_0^t LV(x(s), r(s))ds\right]. \tag{5.28}$$

Let us recall the definition of $\bar{J}$ in (5.8) from Definition 5.2.4. According to (5.28) and a series of derivation steps shown in (5.7), we have

$$\begin{aligned}
\bar{J}(t) &= \mathbb{E}\Bigg\{\int_0^t \Bigg[\Big[x(s)' \;\; x(s-\tau(s))' \;\; v(s)' \;\; v(s-\tau(s))' \;\; z(s)'\Big] R_i \\
&\quad \times \Big[x(s)' \;\; x(s-\tau(s))' \;\; v(s)' \;\; v(s-\tau(s))' \;\; z(s)'\Big]' \\
&\quad + LV(x(s), r(s))\Bigg] ds\Bigg\} - \mathbb{E}\left\{V(x(t), r_t)\right\} \\
&\leq \mathbb{E}\Bigg\{\int_0^t \Bigg[\Big[x(s)' \;\; x(s-\tau(s))' \;\; v(s)' \;\; v(s-\tau(s))' \;\; z(s)'\Big] R_i \\
&\quad \times \Big[x(s)' \;\; x(s-\tau(s))' \;\; v(s)' \;\; v(s-\tau(s))' \;\; z(s)'\Big]'
\end{aligned}$$

$$+LV(x(s),r(s))\Big]ds\Big\}.$$

Hence, we are looking for a condition such that the following inequalities hold,

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t \Big\{ \Big[x(s)' \quad x(s-\tau(s))' \quad v(s)' \quad v(s-\tau(s))' \quad z(s)'\Big] R_i \\
&\times \Big[x(s)' \quad x(s-\tau(s))' \quad v(s)' \quad v(s-\tau(s))' \quad z(s)'\Big]' \\
&+LV(x(t),i)\Big\}ds\Big] \leq 0. \qquad (5.29)
\end{aligned}$$

Note that (5.29) is sufficient for $\bar{J} < 0$. Substituting (5.9) to (5.1) and (5.2), we have

$$\begin{aligned}
& LV(x(t),i) \\
= \; & 2x(t)'P_i[(\bar{A}_i + \Delta\bar{A}_i)x(t) + (\bar{A}_{di} + \Delta\bar{A}_{di})x(t-\tau(t)) + (G_i + \Delta G_i)v(t) \\
& +(G_{di} + \Delta G_{di})v(t-\tau(t))] + [(\bar{E}_i + \Delta\bar{E}_i)x(t) + (\bar{E}_{di} + \Delta\bar{E}_{di})x(t-\tau(t)) \\
& +(H_i + \Delta H_i)v(t) + (H_{di} + \Delta H_{di})v(t-\tau(t))]'P_i[(\bar{E}_i + \Delta\bar{E}_i)x(t) \\
& +(\bar{E}_{di} + \Delta\bar{E}_{di})x(t-\tau(t)) + (H_i + \Delta H_i)v(t) + (H_{di} + \Delta H_{di})v(t-\tau(t))] \\
& +x(t)'Q_ix(t) - (1-\dot{\tau}(t))x(t-\tau(t))'Q_ix(t-\tau(t)) + \sum_{j=1}^{N}\widehat{\pi}_{ij}x(t)'P_jx(t) \\
& +\sum_{\zeta=1}^{m}\mathrm{tr}[P_i\Gamma_{\zeta i}(x,u,t)\Gamma_{\zeta i}(x,u,t)'],
\end{aligned}$$

and

$$\begin{aligned}
z(t) &= (\bar{C}_i + \Delta\bar{C}_i)x(t) + (\bar{C}_{di} + \Delta\bar{C}_{di})x(t-\tau(t)) + (L_i + \Delta L_i)v(t) \\
&\quad +(L_{di} + \Delta L_{di})v(t-\tau(t)) \\
&= \bar{C}_ix(t) + \bar{C}_{di}x(t-\tau(t)) + L_iv(t) + L_{di}v(t-\tau(t)) \\
&\quad +M_iU_i(t)[\bar{N}_{ci}x(t) + \bar{N}_{cd}x(t-\tau(t)) + N_{li}v(t) \\
&\quad +N_{ldi}v(t-\tau(t))]. \qquad (5.30)
\end{aligned}$$

Substituting $R_i$ with forms of (5.25), then we have

$$
\begin{aligned}
&\begin{bmatrix} x(t)' & x(t-\tau(t))' & v(t)' & v(t-\tau(t))' & z(t)' \end{bmatrix} R_i \begin{bmatrix} x(t) \\ x(t-\tau(t)) \\ v(t) \\ v(t-\tau(t)) \\ z(t) \end{bmatrix} \\
= \quad & x(t)'R_{11i}x(t) + x(t-\tau(t))'2R_{21i}x(t) + x(t)'2R_{13i}v(t) \\
& + x(t-\tau(t))'2R_{23i}v(t) + x(t-\tau(t))'R_{22i}x(t-\tau(t)) + x(t)'2R_{14i}v(t-\tau(t)) \\
& + v(t)'R_{33i}v(t) + v(t)'2R_{34i}v(t-\tau(t)) + v(t-\tau(t))'R_{44i}v(t-\tau(t)) \\
& + [x(t)'2R_{15i} + x(t-\tau(t))'2R_{25i} + v(t)'2R_{35i} + v(t-\tau(t))'2R_{45i}]z(t) \\
& + z(t)'R_{55i}z(t) + x(t-\tau(t))'2R_{24i}v(t-\tau(t)). \qquad (5.31)
\end{aligned}
$$

Substituting $z(t)$ from (5.30) to (5.31), we have

$$
\begin{aligned}
& [x(t)'2R_{15i} + x(t-\tau(t))'2R_{25i} + v(t)'2R_{35i} + v(t-\tau(t))'2R_{45i}]z(t) \\
= \quad & x(t)'2R_{15i}\bar{C}_i x(t) + x(t)'2R_{15i}\bar{C}_{di}x(t-\tau(t)) + x(t)'2R_{15i}L_i v(t) \\
& + x(t)'2R_{15i}L_{di}v(t-\tau(t)) + x(t-\tau(t))'2R_{25i}\bar{C}_i x(t) \\
& + x(t-\tau(t))'2R_{25i}\bar{C}_{di}x(t-\tau(t)) + x(t-\tau(t))'2R_{25i}L_i v(t) \\
& x(t-\tau(t))'2R_{25i}L_{di}v(t-\tau(t)) + v(t)'2R_{35i}\bar{C}_i x(t) \\
& + v(t)'2R_{35i}L_i v(t) + v(t)'2R_{35i}L_{di}v(t-\tau(t)) + v(t-\tau(t))'2R_{45i}\bar{C}_i x(t) \\
& + v(t-\tau(t))'2R_{45i}\bar{C}_{di}x(t-\tau(t)) + v(t-\tau(t))'2R_{45i}L_i v(t) \\
& + v(t-\tau(t))'2R_{45i}L_{di}v(t-\tau(t)) \\
& + x(t)'2R_{15i}M_iU_i(t)[\bar{N}_{ci}x(t) + \bar{N}_{cd}x(t-\tau(t)) + N_{li}v(t) + N_{ldi}v(t-\tau(t))] \\
& + x(t-\tau(t))'2R_{25i}M_iU_i(t)[\bar{N}_{ci}x(t) + \bar{N}_{cd}x(t-\tau(t)) + N_{li}v(t) \\
& \quad + N_{ldi}v(t-\tau(t))] + v(t)'2R_{35i}\bar{C}_{di}x(t-\tau(t)) \\
& + v(t)'2R_{35i}M_iU_i(t)[\bar{N}_{ci}x(t) + \bar{N}_{cd}x(t-\tau(t)) + N_{li}v(t) + N_{ldi}v(t-\tau(t))] \\
& + v(t-\tau(t))'2R_{45i}M_iU_i(t)[\bar{N}_{ci}x(t) + \bar{N}_{cd}x(t-\tau(t)) + N_{li}v(t) \\
& + N_{ldi}v(t-\tau(t))].
\end{aligned}
$$

According to Lemma 2.9.6, which is used to eliminate the parameter uncertainties, we have the following inequalities:

$$\begin{aligned}
& 2x(t)'P_i[(\bar{A}_i + \Delta\bar{A}_i)x(t) + (\bar{A}_{di} + \Delta\bar{A}_{di})x(t-\tau(t)) + (G_i + \Delta G_i)v(t) \\
& +(G_{di} + \Delta G_{di})v(t-\tau(t))] \\
= \;& 2x(t)'P_i[\bar{A}_i x(t) + \bar{A}_{di}x(t-\tau(t)) + G_i v(t) + G_{di}v(t-\tau(t))] \\
& +2x(t)'P_i M_i U_i(t)[\bar{N}_{ai}x(t) + \bar{N}_{adi}x(t-\tau(t)) + N_{gi}v(t) + N_{gdi}v(t-\tau(t))] \\
\leq \;& 2x(t)'P_i[\bar{A}_i x(t) + \bar{A}_{di}x(t-\tau(t)) + G_i v(t) + G_{di}v(t-\tau(t))] \\
& +\epsilon_{1i}x(t)'P_i M_i M_i' P_i x(t) + \epsilon_{1i}^{-1}[\bar{N}_{ai}x(t) + \bar{N}_{adi}x(t-\tau(t)) + N_{gi}v(t) \\
& +N_{gdi}v(t-\tau(t))]'[\bar{N}_{ai}x(t) + \bar{N}_{adi}x(t-\tau(t)) + N_{gi}v(t) \\
& +N_{gdi}v(t-\tau(t))],
\end{aligned}$$

$$\begin{aligned}
& [(\bar{E}_i + \Delta\bar{E}_i)x(t) + (\bar{E}_{di} + \Delta\bar{E}_{di})x(t-\tau(t)) \\
& +(H_i + \Delta H_i)v(t) + (H_{di} + \Delta H_{di})v(t-\tau(t))]'P_i[(\bar{E}_i + \Delta\bar{E}_i)x(t) \\
& +(\bar{E}_{di} + \Delta\bar{E}_{di})x(t-\tau(t)) + (H_i + \Delta H_i)v(t) + (H_{di} + \Delta H_{di})v(t-\tau(t))] \\
= \;& [\bar{E}_i x(t) + \bar{E}_{di}x(t-\tau(t)) + H_i v(t) + H_{di}v(t-\tau(t)) + M_i U_i(t)(\bar{N}_{ei}x(t) \\
& +\bar{N}_{edi}x(t-\tau(t)) + N_{hi}v(t) + \bar{N}_{hdi}v(t-\tau(t)))]'P_i[\bar{E}_i x(t) \\
& +\bar{E}_{di}x(x-\tau(t)) + H_i v(t) + H_{di}v(t-\tau(t)) + M_i U_i(t)(\bar{N}_{ei}x(t) \\
& +\bar{N}_{edi}x(t-\tau(t)) + N_{hi}v(t) + \bar{N}_{hdi}v(t-\tau(t)))] \\
\leq \;& [\bar{E}_i x(t) + \bar{E}_{di}x(t-\tau(t)) + H_i v(t) + H_{di}v(t-\tau(t))](P_i^{-1} - \epsilon_{2i}M_i M_i')^{-1} \\
& \times[\bar{E}_i x(t) + \bar{E}_{di}x(t-\tau(t)) + H_i v(t) + H_{di}v(t-\tau(t))] + \epsilon_{2i}^{-1}[\bar{N}_{ei}x(t) \\
& +\bar{N}_{edi}x(t-\tau(t)) + N_{hi}v(t) + \bar{N}_{hdi}v(t-\tau(t))]'[\bar{N}_{ei}x(t) + \bar{N}_{edi}x(t-\tau(t)) \\
& +N_{hi}v(t) + \bar{N}_{hdi}v(t-\tau(t))],
\end{aligned}$$

$$\begin{aligned}
& z(t)'R_{55i}z(t) \\
\leq \;& [\bar{C}_i x(t) + \bar{C}_{di}x(t-\tau(t)) + L_i v(t) + L_{di}v(t-\tau(t))]'(R_{55i}^{-1} - \epsilon_{3i}M_i M_i')^{-1} \\
& \times[\bar{C}_i x(t) + \bar{C}_{di}x(t-\tau(t)) + L_i v(t) + L_{di}v(t-\tau(t))] + \epsilon_{3i}^{-1}[\bar{N}_{ci}x(t) \\
& +\bar{N}_{cd}x(t-\tau(t)) + N_{li}v(t) + N_{ldi}v(t-\tau(t))]'[\bar{N}_{ci}x(t) + \bar{N}_{cd}x(t-\tau(t)) \\
& +N_{li}v(t) + N_{ldi}v(t-\tau(t))],
\end{aligned}$$

$$\begin{aligned}
& x(t)'2R_{15i}M_iU_i(t)[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t))+N_{li}v(t)+N_{ldi}v(t-\tau(t))] \\
\leq\ & 2\epsilon_{4i}x(t)'R_{15i}M_iM_i'R_{15i}x(t)+\frac{1}{2}\epsilon_{4i}^{-1}[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t))+N_{li}v(t) \\
& +N_{ldi}v(t-\tau(t))]'[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t))+N_{li}v(t) \\
& +N_{ldi}v(t-\tau(t))],
\end{aligned}$$

$$\begin{aligned}
& x(t-\tau(t))'2R_{25i}M_iU_i(t)[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t))+N_{li}v(t) \\
& +N_{ldi}v(t-\tau(t))] \\
\leq\ & 2\epsilon_{5i}x(t-\tau(t))'R_{25i}M_iM_i'R_{25i}x(t-\tau(t))+\frac{1}{2}\epsilon_{5i}^{-1}[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t)) \\
& +N_{li}v(t)+N_{ldi}v(t-\tau(t))]'[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t))+N_{li}v(t) \\
& +N_{ldi}v(t-\tau(t))],
\end{aligned}$$

$$\begin{aligned}
& v(t)'2R_{35i}M_iU_i(t)[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t))+N_{li}v(t)+N_{ldi}v(t-\tau(t))] \\
\leq\ & 2\epsilon_{6i}v(t)'R_{35i}M_iM_i'R_{35i}v(t)+\frac{1}{2}\epsilon_{6i}^{-1}[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t))+N_{li}v(t) \\
& +N_{ldi}v(t-\tau(t))]'[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t))+N_{li}v(t) \\
& +N_{ldi}v(t-\tau(t))],
\end{aligned}$$

$$\begin{aligned}
& v(t-\tau(t))'2R_{45i}M_iU_i(t)[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t))+N_{li}v(t) \\
& \quad +N_{ldi}v(t-\tau(t))] \\
\leq\ & 2\epsilon_{7i}v(t-\tau(t))'R_{45i}M_iM_i'R_{45i}v(t-\tau(t))+\frac{1}{2}\epsilon_{7i}^{-1}[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t)) \\
& +N_{li}v(t)+N_{ldi}v(t-\tau(t))]'[\bar{N}_{ci}x(t)+\bar{N}_{cd}x(t-\tau(t))+N_{li}v(t) \\
& +N_{ldi}v(t-\tau(t))].
\end{aligned}$$

Based on (5.20), by Lemma 2.9.6, we have

$$\begin{aligned}
& \sum_{\zeta=1}^{m}\mathrm{tr}[P_i\Gamma_{\zeta i}(x,u,t)\Gamma_{\zeta i}(x,u,t)'] \\
\leq\ & x(t)'(\frac{1}{2}\phi_{1i}m\sum_{\sigma=1}^{n}P_{\sigma\sigma i}M_iM_i'+\frac{1}{2}\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m}P_{\sigma\sigma i}\phi_{1i}^{-1}N_{q\sigma\zeta i}'N_{q\sigma\zeta i} \\
& +\frac{1}{2}\phi_{2i}M_iM_i'+\frac{1}{2}\phi_{2i}^{-1}N_{xi}'N_{xi}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\phi_{3i} m \sum_{\sigma=1}^{n} P_{\sigma\sigma i} K'_{1i} M_i M'_i K_{1i} + \frac{1}{2}\phi_{3i}^{-1} \sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i} K'_{1i} N'_{r\sigma\zeta i} N_{r\sigma\zeta i} K_{1i} \\
&+\frac{1}{2}\phi_{4i} K'_{1i} M_i M'_i K_{1i} + \frac{1}{2}\phi_{4i}^{-1} K'_{1i} N'_{ui} N_{ui} K_{1i} \\
&+\phi_{5i} m \sum_{\sigma=1}^{n} P_{\sigma\sigma i} K'_{1i} M_i M'_i K_{1i} + \phi_{6i} K'_{1i} M_i M'_i K_{1i} \\
&+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i} Q_{\sigma\zeta i} + H_{xi} + \sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i} K'_{1i} R_{\sigma\zeta i} K_{1i} + K'_{1i} H_{ui} K_{1i}) x(t) \\
&+2x(t)'(\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i} K'_{1i} R_{\sigma\zeta i} K_{2i} + K'_{1i} H_{ui} K_{2i}) x(t-\tau(t)) \\
&+x(t-\tau(t))'(\phi_{5i}^{-1} \sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i} K'_{2i} N'_{r\sigma\zeta i} N_{r\sigma\zeta i} K_{2i} + \phi_{6i}^{-1} K'_{2i} N'_{ui} N_{ui} K_{2i} \\
&+\frac{1}{2}\phi_{7i} m \sum_{\sigma=1}^{n} P_{\sigma\sigma i} K'_{2i} M_i M'_i K_{2i} + \frac{1}{2}\phi_{7i}^{-1} \sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i} K'_{2i} N'_{r\sigma\zeta i} N_{r\sigma\zeta i} K_{2i} \\
&+\frac{1}{2}\phi_{8i} K'_{2i} M_i M'_i K_{2i} + \frac{1}{2}\phi_{8i}^{-1} K'_{2i} N'_{ui} N_{ui} K_{2i} \\
&+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i} K'_{2i} R_{\sigma\zeta i} K_{2i} + K'_{2i} H_{ui} K_{2i}) x(t-\tau(t)) \\
&+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i} (Z_{\sigma\zeta i} + \alpha_{\sigma\zeta i}) + H_{zi} + \beta_i .
\end{aligned}
$$

Next, we rewrite

$$
\begin{aligned}
&\begin{bmatrix} x(s)' & x(s-\tau(s))' & v(s)' & v(s-\tau(s))' & z(s)' \end{bmatrix} R_i \\
&\times \begin{bmatrix} x(s)' & x(s-\tau(s))' & v(s)' & v(s-\tau(s))' & z(s)' \end{bmatrix}' + LV(x(s), r(s)) \\
= \; &\begin{bmatrix} x(s)' & x(s-\tau(s))' & v(s)' & v(s-\tau(s))' \end{bmatrix} \Upsilon_i \\
&\times \begin{bmatrix} x(s)' & x(s-\tau(s))' & v(s)' & v(s-\tau(s))' \end{bmatrix}' \\
&+\sum_{\sigma=1}^{n}\sum_{\zeta=1}^{m} P_{\sigma\sigma i} (Z_{\sigma\zeta i} + \alpha_{\sigma\zeta i}) + H_{zi} + \beta_i ,
\end{aligned}
\tag{5.32}
$$

where

$$\Upsilon_i \triangleq Y_i + \Theta_i.$$

Here, $Y_i$ can be split into $4 \times 4$ matrix as follows:

$$Y_i = \begin{bmatrix} Y_{11i} & * & * & * \\ Y_{21i} & Y_{22i} & * & * \\ Y_{31i} & Y_{32i} & Y_{33i} & * \\ Y_{41i} & Y_{42i} & Y_{43i} & Y_{44i} \end{bmatrix},$$

and

$$\begin{aligned} \Theta_i \;=\; & \begin{bmatrix} \bar{N}_{ci} & \bar{N}_{cdi} & N_{li} & N_{ldi} \end{bmatrix}' [\epsilon_{3i}^{-1} + \frac{1}{2}(\epsilon_{4i}^{-1} + \epsilon_{5i}^{-1} + \epsilon_{6i}^{-1} + \epsilon_{7i}^{-1})] \\ & \times \begin{bmatrix} \bar{N}_{ci} & \bar{N}_{cdi} & N_{li} & N_{ldi} \end{bmatrix} \\ & + \begin{bmatrix} \bar{C}_i & \bar{C}_{di} & L_i & L_{di} \end{bmatrix}' (R_{55i}^{-1} - \epsilon_{3i} M_i M_i')^{-1} \begin{bmatrix} \bar{C}_i & \bar{C}_{di} & L_i & L_{di} \end{bmatrix} \\ & + \begin{bmatrix} \bar{N}_{ai} & \bar{N}_{adi} & N_{gi} & N_{gdi} \end{bmatrix}' \epsilon_{1i}^{-1} \begin{bmatrix} \bar{N}_{ai} & \bar{N}_{adi} & N_{gi} & N_{gdi} \end{bmatrix} \\ & + \begin{bmatrix} \bar{E}_i & \bar{E}_{di} & H_i & H_{di} \end{bmatrix}' (P_i^{-1} - \epsilon_{2i} M_i M_i')^{-1} \begin{bmatrix} \bar{E}_i & \bar{E}_{di} & H_i & H_{di} \end{bmatrix} \\ & + \begin{bmatrix} \bar{N}_{ei} & \bar{N}_{edi} & N_{hi} & N_{hdi} \end{bmatrix}' \epsilon_{2i}^{-1} \begin{bmatrix} \bar{N}_{ei} & \bar{N}_{edi} & N_{hi} & N_{hdi} \end{bmatrix}. \end{aligned}$$

Following (5.26) and Lemma 2.9.4, we have $\Upsilon_i < 0$, $i \in \Lambda$. We can now rewrite the inequality for $J(t)$ as

$$\begin{aligned} \bar{J}(t) \;\leq\; & \mathbb{E}\left\{ \int_0^t \begin{bmatrix} x(s)' & x(s-\tau(s))' & v(s)' & v(s-\tau(s))' \end{bmatrix} \Upsilon_i \right. \\ & \left. \times \begin{bmatrix} x(s)' & x(s-\tau(s))' & v(s)' & v(s-\tau(s))' \end{bmatrix}' ds \right\} \\ & + \sum_{\sigma=1}^{n} \sum_{\zeta=1}^{m} P_{\sigma\sigma i}(Z_{\sigma\zeta i} + \alpha_{\sigma\zeta i}) + H_{zi} + \beta_i. \end{aligned} \tag{5.33}$$

Together with (5.27), we have $\bar{J}(t) < 0$, $\forall t > 0$. In this case, the $H_\infty$ performance defined in Definition 5.2.4 is achieved. □

## 5.5 Summary

The problems of robust stochastic stabilization and robust $H_\infty$ control for uncertain nonlinear stochastic systems with time delay and Markovian switching have been studied in this chapter. Sufficient conditions for the solvability of these two problems have been proposed, presented by matrix inequalities. In this case, the systems considered in [115], [114], [20], [120], [121], and [46] are all special cases of the system we discuss in this chapter.

# Chapter 6

# Nonlinear $H_2/H_\infty$ Control of Stochastic Systems with Markovian Switching in Finite and Infinite Time Horizon

## 6.1 Introduction

After introducing the nonlinear $H_2$ problems in Chapter 3 and Chapter 4, and the nonlinear $H_\infty$ problem in Chapter 5, naturally we will consider the situation of the mixed nonlinear $H_2/H_\infty$ control problems, which is investigated in this chapter. We solve our problems under the stochastic nonlinear systems with Markovian switching in both finite and infinite time horizon. The nonlinearity part is similar to the one in Chapter 3 and Chapter 4, different from the one in Chapter 5. Based on Nash game approach, we formulate our problem similarly to the linear case with Markovian switching [141]. Following the two Nash inequalities introduced in Section 2.7.1, one associated with the $H_\infty$ performance, and the other related with the $H_2$ performance, we are seeking a pair of solutions $(u_T^*, v_T^*)$, which is a Nash equilibrium. Finally a sufficient condition for solving our nonlinear stochastic $H_2/H_\infty$ control problem is presented by using the completion of square method.

The difficulty appears in dealing with the nonlinearity terms. Here it is highlighted that within this nonlinear system, explicit solutions are found, which is a very rare case. In addition, the optimal control laws obtained are linear with state, which is very similar to the characteristics of the results in linear $H_2/H_\infty$ control problems. We demonstrate our work within two main sections: finite time horizon and infinite time horizon respectively. Note that in the infinite time horizon, when we consider the admissible control, we have to take the concept of mean-square stability into account.

## 6.2 Finite Time Horizon

### 6.2.1 Introduction

In Section 6.2, we formulate the problem of nonlinear stochastic $H_2/H_\infty$ control in finite horizon. Then we present a sufficient condition to solve our problem in Section 6.2.3.

### 6.2.2 Problem Formulation.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given filtered complete probability space, where there exist a one-dimensional standard Brownian motion $(W(t), 0 \leq t \leq T)$, a $\eta \times 1$ -dimensional Brownian motion $(\tilde{W}(t), 0 \leq t \leq T)$, and a Markov chain $(r_t \in M, 0 \leq t \leq T)$ with generator $\Pi = (\pi_{ij})$ specified in (2.3) and state space defined as $M \triangleq \{1, 2, \cdots, l\}$. We assume that $W(t)$, $\tilde{W}(t)$ and the process $r_t$ are mutually independent. The following basic assumption will be used throughout the section of finite time horizon.

**Assumption 6.2.1.** *The data that appear in system (6.1)-(6.3) satisfy, for every*

$i$,

$$\begin{cases} A_i(\cdot), G_i(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}), \\ B_{2i}(\cdot), H_{2i}(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n_u}), \\ B_{1i}(\cdot), H_{1i}(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n_v}), \\ E_i(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times \eta}), \\ Q_{1i}(\cdot),\dots,Q_{\eta i}(\cdot) \in L^\infty(0,T;\mathcal{S}^n), \\ R_{1i}(\cdot),\dots,R_{\eta i}(\cdot) \in L^\infty(0,T;\mathcal{S}^{n_u}), \\ S_{1i}(\cdot),\dots,R_{\eta i}(\cdot) \in L^\infty(0,T;\mathcal{S}^{n_v}). \end{cases}$$

Consider the following nonlinear SDEs with Markovian switching,

$$\begin{cases} dx(t) = [A(t,r_t)x(t) + B_2(t,r_t)u(t) + B_1(t,r_t)v(t)]dt \\ \qquad + [G(t,r_t)x(t) + H_2(t,r_t)u(t) + H_1(t,r_t)v(t)]dW(t) \\ \qquad + E(t,r_t)F(x(t),u(t),t,r_t)d\tilde{W}(t), \\ z(t) = \begin{bmatrix} C(t,r_t)x(t) \\ D(t,r_t)u(t) \end{bmatrix} \end{cases} \tag{6.1}$$

where $x(0) = x_0$ and $D(t,r_t)'D(t,r_t) \triangleq \mathbf{I}$, and

$$F(x(t),u(t),t,r_t) \triangleq \mathrm{diag}(\sqrt{\phi_1},\sqrt{\phi_2},\dots,\sqrt{\phi_\eta}). \tag{6.2}$$

Among $\phi_1,\phi_2,\dots,\phi_\eta$, we denote each of them as $\phi_k$, where $k = 1,2,\dots,\eta$. We define

$$\phi_k \triangleq x(t)'Q_k(t,r_t)x(t) + u(t)'R_k(t,r_t)u(t) + v(t)'S_k(t,r_t)v(t). \tag{6.3}$$

We assume that $Q_k(t,r_t) \geq 0$, $R_k(t,r_t) \geq 0$, $S_k(t,r_t) \geq 0$, for all $k$.

Define $e_k \in \mathbb{R}^\eta$ as an elementary vector, whose $k$-th element is 1, while other elements are 0.

Here, in system (6.1) $x(t) \in \mathbb{R}^n$ is state, $z(t) \in \mathbb{R}^{n_z}$ is controlled output, $u(t) \in \mathbb{R}^{n_u}$ is control input and $v(t) \in \mathbb{R}^{n_v}$ is external disturbance, respectively. The discussion of existence and uniqueness of solution to the system (6.1) is the similar to the one discussed in Chapter 3. Here we omit the details. Similar to

Definition 2.7.1 that originates from [141], the finite horizon stochastic $H_2/H_\infty$ control problem can be stated as follows.

**Definition 6.2.1.** *[141] For given disturbance attenuation level $\gamma > 0$, $0 < T < \infty$, the finite horizon mixed $H_2/H_\infty$ control is to find a state feedback control $u_T^*(t,x) = K_{2i}(t)x(t) \in \mathcal{L}_{\mathcal{F}}^2([0,T],\mathbb{R}^{n_u})$ such that*
*(i) The trajectory of the closed-loop system (6.1) starting from $x(0) = x_0 = 0$ satisfies*

$$\begin{aligned} & \sum_{i=1}^{l} \mathbb{E}\left[\int_0^T \left(|C(t,r_t)x(t)|^2 + |u_T^*(t)|^2\right) dt | r_0 = i\right] \\ \leq\ & \gamma^2 \sum_{i=1}^{l} \mathbb{E}\left[\int_0^T |v(t)|^2 dt | r_0 = i\right] \end{aligned} \tag{6.4}$$

*for $\forall v \neq 0$, $v \in \mathcal{L}_{\mathcal{F}}^2([0,T],\mathbb{R}^{n_v})$.*
*(ii) When the worst case disturbance $v_T^*(t,x) \in \mathcal{L}_{\mathcal{F}}^2([0,T],\mathbb{R}^{n_v})$, if existing, is implemented to (6.1), $u_T^*(t,x)$ minimizes the output energy*

$$J_2^T(u, v_T^*, x_0, i) = \mathbb{E}\left[\int_0^T |z(t)|^2 dt | r_0 = i\right], \quad i \in M.$$

We are going to solve our finite horizon stochastic $H_2/H_\infty$ control problem based on Nash game approach. This requires us to find the Nash equilibria $(u_T^*, v_T^*)$, which is defined in Section 2.7.1.

### 6.2.3 Main Result

Consider the stochastic nonlinear system as follows:

$$\begin{cases} dx(t) = [A(t,r_t)x(t) + B_1(t,r_t)v(t)]dt \\ \qquad + [G(t,r_t)x(t) + H_1(t,r_t)v(t)]dW(t) \\ \qquad + E(t,r_t)F(x(t),u(t),t,r_t)d\tilde{W}(t), \\ z(t) = C(t,r_t)x(t). \end{cases} \tag{6.5}$$

Following (6.2), we have

$$\phi_k = x(t)'Q_k(t, r_t)x(t) + v(t)'S_k(t, r_t)v(t).$$

Similar to Lemma 2.7.1 in Chapter 2, Lemma 2.1 in [141], and Lemma 4.1 in [130], we provide the following lemma, which is useful in deriving Theorem 6.2.1.

**Lemma 6.2.1.** *[130] [141] For system (6.5) and given disturbance attenuation* $\gamma > 0$, $|\mathcal{L}_{[0,T]}| \leq \gamma$ *iff there exists a solution* $P = (P_1, P_2, \cdots, P_l)$ *with* $P_i \geq 0$, $i \in M$, *satisfying the following differential Riccati equations (DREs):*

$$\begin{cases} \dot{P}_i(t) + A_i(t)'P_i(t) + P_i(t)A_i(t) + G_i(t)'P_i(t)G_i(t) - C_i(t)'C_i(t) \\ \quad + \displaystyle\sum_{k=1}^{\eta} e_k' E_i(t)'P_i(t)E_i(t)e_k Q_{ki}(t) + \sum_{j=1}^{l} \pi_{ij} P_j(t) - [P_i(t)B_{1i}(t) \\ \quad + G_i(t)'P_i(t)H_{1i}(t)] \times [\gamma^2 I + H_{1i}(t)'P_i(t)H_{1i}(t) \\ \quad + \displaystyle\sum_{k=1}^{\eta} e_k' E_i(t)'P_i(t)E_i(t)e_k S_{ki}(t)]^{-1}[B_{1i}(t)'P_i(t) + H_{1i}(t)'P_i(t)G_i(t)] = 0, \\ \\ \gamma^2 I + H_{1i}'P_i(t)H_{1i}(t) + \displaystyle\sum_{k=1}^{\eta} e_k' E_i(t)'P_i(t)E_i(t)e_k S_{ki}(t) > 0, \quad i \in M. \end{cases}$$

The proof is similar to Lemma 2.7.1 in Chapter 2, Lemma 2.1 in [141], and Lemma 4.1 in [130], so the details are omitted here.

The following theorem presents the main result of our finite horizon stochastic nonlinear $H_2/H_\infty$ control problem. First, some notations are introduced.

$$\begin{aligned} \bar{A}(t, r_t) &\triangleq A(t, r_t) + B_2(t, r_t)K_2(t, r_t), \\ \bar{G}(t, r_t) &\triangleq G(t, r_t) + H_2(t, r_t)K_2(t, r_t), \\ \bar{Q}_k(t, r_t) &\triangleq Q_k(t, r_t) + K_2(t, r_t)'R_k(t, r_t)K_2(t, r_t), \\ \tilde{A}(t, r_t) &\triangleq A(t, r_t) + B_1(t, r_t)K_1(t, r_t), \\ \tilde{G}(t, r_t) &\triangleq G(t, r_t) + H_1(t, r_t)K_1(t, r_t), \\ \tilde{Q}_k(t, r_t) &\triangleq Q_k(t, r_t) + K_1(t, r_t)'S_k(t, r_t)K_1(t, r_t). \end{aligned} \tag{6.6}$$

**Theorem 6.2.1.** *For given disturbance attenuation level $\gamma > 0$, the finite horizon stochastic nonlinear $H_2/H_\infty$ control for system (6.1) has a pair of solutions $(u_T^*, v_T^*)$ with*

$$\begin{aligned} u_T^*(t,x) &= -\sum_{i=1}^{l} K_{2i}(t)\chi_{r_t=i}(t)x(t) \\ v_T^*(t,x) &= -\sum_{i=1}^{l} K_{1i}(t)\chi_{r_t=i}(t)x(t) \end{aligned}$$

*if the following four coupled DREs have solutions $(P_1(t), P_2(t); K_1(t), K_2(t))$ with $P_1(t) = (P_{11}(t), P_{12}(t), \cdots, P_{1l}(t)) \geq 0$, $P_2(t) = (P_{21}(t), P_{22}(t), \cdots, P_{2l}(t)) \geq 0$.*

$$\begin{aligned} &\begin{cases} L_i(t) - \beta_i(t)'\alpha_i(t)^{-1}\beta_i(t) = 0, \\ \alpha_i(t) > 0, \quad i \in M, \end{cases} \\ &K_{1i}(t) = \alpha_i(t)^{-1}\beta_i(t), \end{aligned} \tag{6.7}$$

$$\begin{aligned} &\begin{cases} T_j(t) - N_j(t)'Z_j(t)^{-1}N_j(t) = 0, \\ Z_j(t) > 0, \quad j \in M, \end{cases} \\ &K_{2j}(t) = Z_j(t)^{-1}N_j(t), \end{aligned} \tag{6.8}$$

*where*

$$\begin{aligned} L_i(t) &\triangleq \dot{P}_{1i}(t) + P_{1i}(t)\bar{A}_i(t) + \bar{A}_i(t)'P_{1i}(t) + \bar{G}_i(t)'P_{1i}(t)\bar{G}_i(t) \\ &\quad + \sum_{k=1}^{\eta} e_k' E_i(t)'P_{1i}(t)E_i(t)e_k\bar{Q}_{ki}(t) + \sum_{i=1}^{l} \pi_{ij}P_j(t) - C_i(t)'C_i(t) \\ &\quad - K_{2i}(t)'K_{2i}(t), \\ \beta_i(t) &\triangleq B_{1i}(t)'P_{1i}(t) + H_{1i}(t)'P_{1i}(t)\bar{G}_i(t), \\ \alpha_i(t) &\triangleq \gamma^2 I + H_{1i}(t)'P_{1i}(t)H_{1i}(t) + \sum_{k=1}^{\eta} e_k' E_i(t)'P_{1i}(t)E_i(t)e_k S_{ki}(t), \\ T_j(t) &\triangleq \dot{P}_{2j}(t) + P_{2j}(t)\tilde{A}_j(t) + \tilde{A}_j(t)'P_{2j}(t) + \tilde{G}_j(t)'P_{2j}(t)\tilde{G}_j(t) \\ &\quad + \sum_{k=1}^{\eta} e_k' E_j(t)'P_{2j}(t)E_j(t)e_k\tilde{Q}_{kj}(t) + \sum_{j=1}^{l} \pi_{ij}P_{2j}(t) + C_j(t)'C_j(t), \end{aligned}$$

$$
\begin{aligned}
N_j(t) &\triangleq B_{2j}(t)'P_{2j}(t) + H_{2j}(t)'P_{2j}(t)\tilde{G}_j(t), \\
Z_j(t) &\triangleq I + H_{2j}(t)'P_{2j}(t)H_{2j}(t) + \sum_{k=1}^{\eta} e_k' E_j(t)'P_{1j}(t)E_j(t)e_k R_{kj}(t).
\end{aligned}
$$

*Proof.* Substituting $u = u_T^*(t,x) = -\sum_{i=1}^{l} K_{2i}(t)\chi_{r_t=i}(t)x(t)$ into (6.1), we have

$$
\begin{cases}
dx(t) = [\bar{A}(t,r_t)x(t) + B_1(t,r_t)v(t)]dt \\
\qquad\quad + [\bar{G}(t,r_t)x(t) + H_1(t,r_t)v(t)]dW(t) \\
\qquad\quad + E(t,r_t)F(x(t),u(t),t,r_t)d\tilde{W}(t), \\
z(t) = \begin{bmatrix} C(t,r_t)x(t) \\ D(t,r_t)K_2(t,r_t)x(t) \end{bmatrix},
\end{cases}
\tag{6.9}
$$

where $x(0) = x_0$.

Applying Lemma 6.2.1 to our system (6.1) to (6.3), with (6.7), $|\mathcal{L}_{[0,T]}| \leq \gamma$ can be achieved immediately, which makes the first condition in Definition 6.2.1 satisfied. Next, we show that $v = v_T^*(t,x)$ is the worst case disturbance. Following (6.2) and (6.3), we have

$$
\phi_k = x(t)'\bar{Q}_k(t,r_t)x(t) + v(t)'S_k(t,r_t)v(t). \tag{6.10}
$$

In addition,

$$
\begin{aligned}
& J_1^T(u_T^*, v, x_0, i) \\
=\ & \mathbb{E}\left[\int_o^T (\gamma^2 v'v - z'z)dt \middle| r_0 = i\right] \\
=\ & x_0'P_{1i}(t)x_0 + \mathbb{E}\left[\int_o^T (\gamma^2 v'v - z'z + d[x'P_{1i}(t)x])dt \middle| r_0 = i\right].
\end{aligned}
$$

Applying Lemma 2.9.1 into $x'P_{1r_T}(t)x$, we have

$$
\begin{aligned}
& \mathbb{E}[x(T)'P_{1r_T}(T)x(T)|r_0 = i] \\
=\ & x_0'P_{1r_0}(0)x_0 + \mathbb{E}\Bigg[\int_0^T \{x'\dot{P}_{1i}(t)x + 2x'P_{1i}(t)[\bar{A}_i(t)x + B_{1i}(t)v] \\
& [\bar{G}_i(t)x + H_{1i}(t)v]'P_{1i}(t)[\bar{G}_i(t)x + H_{1i}(t)v] \\
& + \mathrm{tr}[F_i(x,t)'E_i(t)'P_{1i}(t)E_i(t)F_i(x,t)] + \sum_{i=1}^{l} \pi_{ij}x'P_{1j}(t)x\}dt \Bigg| r_0 = i\Bigg].
\end{aligned}
$$

Similar to the steps from (3.34) to (3.38) in Chapter 3, we have

$$\begin{aligned} & \operatorname{tr}[F_i(x,t)'E_i(t)'P_{1i}(t)E_i(t)F_i(x,t)] \\ = \; & x'\left[\sum_{k=1}^{\eta} e_k' E_i(t)'P_{1i}(t)E_i(t)e_k\bar{Q}_{ki}(t)\right]x \\ & +v'\left[\sum_{k=1}^{\eta} e_k' E_i(t)'P_{1i}(t)E_i(t)e_k S_{ki}(t)\right]v. \end{aligned}$$

Then we have

$$\begin{aligned} & \gamma^2 v'v - z'z + d(x'P_{1i}(t)x) \\ = \; & x'[\dot{P}_{1i}(t) + P_{1i}(t)\bar{A}_i(t) + \bar{A}_i(t)'P_{1i}(t) + \bar{G}_i(t)'P_{1i}(t)\bar{G}_i(t) \\ & + \sum_{k=1}^{\eta} e_k' E_i(t)'P_{1i}(t)E_i(t)e_k\bar{Q}_{ki}(t) + \sum_{i=1}^{l}\pi_{ij}P_{1j}(t) - C_i(t)'C_i(t) \\ & -K_{2i}(t)'K_{2i}(t)]x + 2v'[B_{1i}(t)'P_{1i}(t) + H_{1i}(t)'P_{1i}(t)\bar{G}_i(t)]x \\ & +v'[\gamma^2 I + H_{1i}(t)'P_{1i}(t)H_{1i}(t) + \sum_{k=1}^{\eta} e_k' E_i(t)'P_{1i}(t)E_i(t)e_k S_{ki}(t)]v. \end{aligned}$$

Following the notation in (6.6) and using completion of square method, we have

$$\begin{aligned} & \gamma^2 v'v - z'z + d(x'P_{1i}(t)x) \\ = \; & v'\alpha_i(t)v + 2v'\beta_i(t)x + x'L_i(t)x \\ = \; & [v + \alpha_i(t)^{-1}\beta_i(t)x]'\alpha_i(t)[v + \alpha_i(t)^{-1}\beta_i(t)x] \\ & +x'[L_i(t) - \beta_i(t)'\alpha_i(t)^{-1}\beta_i(t)]x \end{aligned}$$

When RDE $L_i(t) - \beta_i(t)'\alpha_i(t)^{-1}\beta_i(t) = 0$ is satisfied, we can see $v = v_T^*(t,x) = -\sum_{i=1}^{l} K_{1i}(t)\chi_{r_t=i}(t)x(t) = -\alpha_i(t)^{-1}\beta_i(t)x$ is the worst case disturbance. In this case,

$$J_1^T(u_T^*, v; x_0, i) \geq J_1^T(u_T^*, v_T^*; x_0, i) = x_0'P_{1i}(t)x_0.$$

Next, substituting $v = v_T^*(t,x) = -\sum_{i=1}^{l} K_{1i}(t)\chi_{r_t=i}(t)x(t)$ into (6.1), we have

$$\begin{cases} dx = [\tilde{A}(t,r_t)x + B_2(t,r_t)u]dt \\ \quad + [\tilde{G}(t,r_t)x + H_2(t,r_t)u]dW \\ \quad + E(t,r_t)F(x(t),u(t),t,r_t)d\tilde{W}(t), \\ z = \begin{bmatrix} C(t,r_t)x(t) \\ D(t,r_t)u(t) \end{bmatrix}, \end{cases} \tag{6.11}$$

where $x(0) = x_0$. Following (6.2) and (6.3), we have

$$\phi_k = x(t)'\tilde{Q}_k(t,r_t)x(t) + u(t)'R_k(t,r_t)u(t). \tag{6.12}$$

Now minimizing $J_2^T(u, v_T^*; x_0, i)$ is similar to the optimal control problem that we study in Chapter 3. We rewrite $J_2^T(u, v_T^*; x_0, i)$ as follows,

$$\begin{aligned} & J_2^T(u, v_T^*; x_0, i) \\ = \ & \mathbb{E}\left[\int_o^T (z'z)dt | r_0 = i\right] \\ = \ & x_0'P_{2i}(t)x_0 + \mathbb{E}\left[\int_o^T (z'z + d[x'P_{2i}(t)x])dt | r_0 = i\right]. \end{aligned}$$

Applying Lemma 2.9.1 into $x'P_{2r_T}(t)x$, we have

$$\begin{aligned} & \mathbb{E}[x(T)'P_{2r_T}(T)x(T)|r_0 = i] \\ = \ & x_0'P_{2r_0}(0)x_0 + \mathbb{E}\left[\int_0^T \{x'\dot{P}_{2i}(t)x + 2x'P_{2i}(t)[\tilde{A}_i(t)x + B_{2i}(t)u]\right. \\ & [\tilde{G}_i(t)x + H_{2i}(t)u]'P_{2i}(t)[\tilde{G}_i(t)x + H_{2i}(t)u] \\ & \left. + \mathrm{tr}[F_i(x,t)'E_i(t)'P_{2i}(t)E_i(t)F_i(x,t)] + \sum_{i=1}^{l} \pi_{ij}x'P_{2j}(t)x\}dt \middle| r_0 = i\right]. \end{aligned}$$

Similar to the steps from (3.34) to (3.38) in Chapter 3, we have

$$\begin{aligned} & \mathrm{tr}[F_i(x,t)'E_i(t)'P_{2i}(t)E_i(t)F_i(x,t)] \\ = \ & x'\left[\sum_{k=1}^{\eta} e_k'E_i(t)'P_{2i}(t)E_i(t)e_k\tilde{Q}_{ki}(t)\right]x \end{aligned}$$

$$+u'\left[\sum_{k=1}^{\eta} e_k' E_i(t)' P_{2i}(t) E_i(t) e_k R_{ki}(t)\right] u.$$

Then we have

$$\begin{aligned}
& z'z + d[x'P_{2i}(t)x] \\
= \; & x'[\dot{P}_{2i}(t) + P_{2i}(t)\tilde{A}_i(t) + \tilde{A}_i(t)'P_{2i}(t) + \tilde{G}_i(t)'P_{2i}(t)\tilde{G}_i(t) \\
& + \sum_{k=1}^{\eta} e_k' E_i(t)' P_{2i}(t) E_i(t) e_k \tilde{Q}_{ki}(t) + \sum_{i=1}^{l} \pi_{ij} P_{2j}(t) + C_i(t)'C_i(t)]x \\
& +2u'[B_{2i}(t)'P_{2i}(t) + H_{2i}(t)'P_{2i}(t)\tilde{G}_i(t)]x \\
& +u'[I + H_{2i}(t)'P_{2i}(t)H_{2i}(t) + \sum_{k=1}^{\eta} e_k' E_i(t)' P_{1i}(t) E_i(t) e_k R_{ki}(t)]u.
\end{aligned}$$

Following the notation in (6.6), and using completion of square method, we have

$$\begin{aligned}
& z'z + d[x'P_{2i}(t)x] \\
= \; & u'Z_i(t)u + 2u'N_i(t)x + x'T_i(t)x \\
= \; & [u + Z_i(t)^{-1}N_i(t)x]'Z_i(t)[u + Z_i(t)^{-1}N_i(t)x] \\
& +x'[T_i(t) - N_i(t)'Z_i(t)^{-1}N_i(t)]x.
\end{aligned}$$

When RDE $T_i(t) - N_i(t)'Z_i(t)^{-1}N_i(t) = 0$ is satisfied, we can see $u = u_T^*(t,x) = -\sum_{i=1}^{l} K_{2i}(t)\chi_{r_t=i}(t)x(t) = -Z_i(t)^{-1}N_i(t)x$ is the optimal control. In this case,

$$J_2^T(u, v_T^*; x_0, i) \geq J_2^T(u_T^*, v_T^*; x_0, i) = x_0'P_{2i}(t)x_0.$$

□

## 6.3 Infinite Time Horizon

### 6.3.1 Introduction

In Section 6.3.2, we formulate the problem of nonlinear stochastic $H_2/H_\infty$ control in infinite horizon. The mean-square stability condition for our infinite horizon problem is obtained in Section 6.3.3. Then we present a sufficient condition to solve our problem in Section 6.3.4.

### 6.3.2 Problem Formulation.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given complete probability space, where there exist a one-dimensional standard Brownian motion $W(t)$ on $[0, +\infty)$, a $\eta \times 1$ -dimensional Brownian motion $\tilde{W}(t)$ on $[0, +\infty)$, and a Markov chain $(r_t \in M, t \geq 0)$ with generator $\Pi = (\pi_{ij})$ specified in (2.3) and state space defined as $M \triangleq \{1, 2, \cdots, l\}$. We assume that $W(t)$, $\tilde{W}(t)$ and the process $r_t$ are mutually independent. The following basic assumption will be used throughout the section of infinite time horizon.

**Assumption 6.3.1.** *The data appearing in the nonlinear $H_2/H_\infty$ control problem (6.13)-(6.15) satisfy, for every $i$,*

$$\begin{aligned} &A_i, G_i \in \mathbb{R}^{n\times n}, \quad B_{2i}, H_{2i} \in \mathbb{R}^{n\times n_u}, \quad B_{1i}, H_{1i} \in \mathbb{R}^{n\times n_v}, \quad E_i \in \mathbb{R}^{n\times m}, \\ &Q_{1i}, \dots, Q_{\eta i} \in \mathcal{S}^n, \quad R_{1i}, \dots, R_{\eta i} \in \mathcal{S}^{n_u}, \quad S_{1i}, \dots, R_{\eta i} \in \mathcal{S}^{n_v}, \end{aligned}$$

Consider the following nonlinear SDEs with Markovian switching

$$\begin{cases} dx(t) = [A(r_t)x(t) + B_2(r_t)u(t) + B_1(r_t)v(t)]dt \\ \qquad\quad + [G(r_t)x(t) + H_2(r_t)u(t) + H_1(r_t)v(t)]dW(t) \\ \qquad\quad + E(r_t)F(x(t), u(t), r_t)d\tilde{W}(t), \\ z(t) = \begin{bmatrix} C(r_t)x(t) \\ D(r_t)u(t) \end{bmatrix}. \end{cases} \tag{6.13}$$

where $x(0) = x_0$ and $D(t, r_t)'D(t, r_t) \triangleq \mathbf{I}$, and

$$F(x_1(t), x_2(t), u(t), r_t) \triangleq \operatorname{diag}(\sqrt{\phi_1}, \sqrt{\phi_2}, \dots, \sqrt{\phi_\eta}). \tag{6.14}$$

Among $\phi_1, \phi_2, \dots, \phi_\eta$, we denote each of them as $\phi_k$, where $k = 1, 2, \dots, \eta$. We define

$$\phi_k \triangleq x(t)'Q_k(r_t)x(t) + u(t)'R_k(r_t)u(t) + v(t)'S_k(r_t)v(t). \tag{6.15}$$

We assume that $Q_k(r_t) \geq 0$, $R_k(r_t) \geq 0$, $S_k(r_t) \geq 0$, for all $k$.

Define $e_k \in \mathbb{R}^\eta$ as an elementary vector, whose $k$-th element is 1, while other elements are 0.

We denote $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^{n_z}$, $u(t) \in \mathbb{R}^{n_u}$ and $v(t) \in \mathbb{R}^{n_v}$ as state, controlled output, control input, and external disturbance of our system (6.13), respectively. The discussion of existence and uniqueness of solution to the system (6.13) is the similar to the one discussed in Chapter 3. Here we omit the details. Define

$$J_1^{\infty}(u, v; x_0, i) = \mathbb{E}\left[\int_0^{\infty} [\gamma^2 v(t)'v(t) - z(t)'z(t)]dt | r_0 = i\right], \quad i \in M.$$

and

$$J_2^{\infty}(u, v; x_0, i) = \mathbb{E}\left[\int_0^{\infty} z(t)'z(t)dt | r_0 = i\right], \quad i \in M.$$

The infinite horizon stochastic nonlinear $H_2/H_\infty$ control problem can be stated similarly to the one in [141] as follows.

**Definition 6.3.1.** *[141] For given disturbance attenuation level $\gamma > 0$, if we can find $u^*(t) \times v^*(t) \in \mathcal{L}^2_{\mathcal{F}}([0,\infty), \mathbb{R}^{n_u}) \times \mathcal{L}^2_{\mathcal{F}}([0,\infty), \mathbb{R}^{n_v})$, such that*
*(i) When $v(t) = 0$, $u = u^*$, the state trajectory of (6.13) with any initial value $(x_0, i) \in \mathbb{R}^n \times M$ satisfies the mean-square stability*

$$\lim_{t\to\infty} \mathbb{E}[x(t)'x(t)|r_0 = i] = 0.$$

*(ii) $|L_{u^*}|_\infty < \gamma$, where*

$$|L_{u^*}|_\infty = \sup_{v \in \mathcal{L}^2_{\mathcal{F}}\left([0,\infty),\mathbb{R}^{n_v}\right), v\neq 0, u=u^*, x_0=0} \frac{\left\{ \sum_{i=1}^{l} \mathbb{E}\left[ \int_0^\infty z(t)'z(t)dt | r_0 = i\right]\right\}^{\frac{1}{2}}}{\left\{ \sum_{i=1}^{l} \mathbb{E}\left[ \int_0^\infty v(t)'v(t)dt | r_0 = i\right]\right\}^{\frac{1}{2}}}.$$

*(iii) When the worst case disturbance $v^*(t) \in \mathcal{L}^2_{\mathcal{F}}([0,\infty), \mathbb{R}^{n_v})$, if existing, is implemented to (6.13), $u^*(t)$ minimizes the output:*

$$J_2^{\infty}(u, v^*; x_0, i) = \mathbb{E}\left[\int_0^{\infty} z(t)'z(t)dt \Big| r_0 = i\right], \quad i \in M. \tag{6.16}$$

*Then we say that the infinite horizon stochastic nonlinear $H_2/H_\infty$ control problem has a pair of solutions $(u^*, v^*)$.*

We are going to solve our infinite horizon stochastic nonlinear $H_2/H_\infty$ control problem based on Nash game approach. This requires us to find the Nash equilibria $(u_T^*, v_T^*)$, which is defined in Section 2.7.1.

### 6.3.3 Stability Condition

Mean-square stability is a standard assumption in an infinite time horizon nonlinear $H_2/H_\infty$ control problem. In this section we derive conditions such that Definition 6.3.1-(i) is satisfied. We denote

$$\begin{aligned} A_{1i} &\triangleq A_i + B_{2i}K_i, \\ G_{1i} &\triangleq G_i + H_{2i}K_i, \\ T_{ki} &\triangleq Q_{ki} + K_i' R_{ki} K_i. \end{aligned}$$

**Lemma 6.3.1.** *Substituting* $v(t) = 0$ *and* $u(t) = \sum_{i=1}^{l} K_i \chi_{r_t=i} x(t)$ *into system (6.13), if the following matrix inequality holds,*

$$P_i A_{1i} + A_{1i}' P_i + G_{1i}' P_i G_{1i} + \sum_{k=1}^{\eta} e_k' E_i' P_i E_i e_k \tilde{Q}_{ki} + \sum_{i=1}^{l} \pi_{ij} P_j < 0, \qquad (6.17)$$

*then the system (6.13) is mean-square stable.*

*Proof.* Substituting $v(t) = 0$ and $u(t) = \sum_{i=1}^{l} K_i \chi_{r_t=i} x(t)$ into system (6.13), then we rewrite system (6.13) as follows:

$$\begin{cases} dx(t) = A_1(r_t)x(t)dt + G_1(r_t)x(t)dW(t) + E(r_t)F(x(t), r_t)d\tilde{W}(t), \\ z(t) = \begin{bmatrix} C(r_t)x(t) \\ D(r_t)K(r_t)x(t) \end{bmatrix}, \quad x(0) = x_0, \end{cases}$$

where

$$F(x(t), r_t) \triangleq \operatorname{diag}(\sqrt{\phi_1}, \sqrt{\phi_2}, \ldots, \sqrt{\phi_\eta}).$$

Following (6.2) and (6.3), we have

$$\phi_k = x(t)' T_k(r_t)_k(t, r_t) x(t).$$

Applying Lemma 2.9.1 to $x(T)' P_i x(T)$, we have

$$\mathbb{E}[x(T)' P_i x(T)] = \Bigg( P_i A_{1i} + A_{1i}' P_i + G_{1i}' P_i G_{1i} + \sum_{k=1}^{\eta} e_k' E_i' P_i E_i e_k \tilde{Q}_{ki}$$

$$+\sum_{i=1}^{l}\pi_{ij}P_j\Bigg)dt. \tag{6.18}$$

If (6.17) is satisfied, then by Definition 4.2.1, Definition 4.2.2, and [68], [83], our nonlinear system (6.13) is mean-square stable. $\square$

Note that the above matrix inequality (6.17) holds for both $P_1(r_t)$ and $P_2(r_t)$.

### 6.3.4 Main Result

Consider the stochastic nonlinear system as follows,

$$\begin{cases} dx(t) = [A(r_t)x(t) + B_1(r_t)v(t)]dt \\ \qquad + [G(r_t)x(t) + H_1(r_t)v(t)]dW(t) \\ \qquad + E(r_t)F(x(t), u(t), r_t)d\tilde{W}(t), \\ z(t) = C(r_t)x(t). \end{cases} \tag{6.19}$$

Following (6.14), we have

$$\phi_k = x(t)'Q_k(t, r_t)x(t) + v(t)'S_k(t, r_t)v(t).$$

Similar to Lemma 2.7.1 in Chapter 2, Lemma 2.1 in [141], and Lemma 4.1 in [130], we provide the following lemma, which is useful in deriving Theorem 6.3.1.

**Lemma 6.3.2.** *[130] [141] Given disturbance attenuation level $\gamma > 0$, we have $|\mathcal{L}_{[0,\infty]}| \leq \gamma$ iff there exists a solution $P = (P_1, P_2, \cdots, P_l)$ with $P_i \geq 0$, $i \in M$ that satisfies the following algebraic Riccati equations (AREs):*

$$\begin{cases} A_i'P_i + P_iA_i + G_i'P_iG_i - C_i'C_i + \sum_{k=1}^{\eta} e_k'E_i'P_iE_ie_kQ_{ki} + \sum_{j=1}^{l}\pi_{ij}P_j - [P_iB_{1i} \\ \quad + G_i'P_iH_{1i}] \times [\gamma^2 I + H_{1i}'P_iH_{1i} + \sum_{k=1}^{\eta} e_k'E_i'P_iE_ie_kS_{ki}]^{-1}[B_{1i}'P_i + H_{1i}'P_iG_i] = 0, \\ \\ \gamma^2 I + H_{1i}'P_iH_{1i} + \sum_{k=1}^{\eta} e_k'E_i'P_iE_ie_kS_{ki} > 0, \quad i \in M. \end{cases}$$

The proof is similar to Lemma 2.7.1 in Chapter 2, Lemma 2.1 in [141], and Lemma 4.1 in [130], so the details are omitted here.

The following theorem presents the main result of the infinite horizon stochastic nonlinear $H_2/H_\infty$ control problem. First, some notations are introduced.

$$\begin{aligned}
\bar{A}(r_t) &\triangleq A(r_t) + B_2(r_t)K_2(r_t), \\
\bar{G}(r_t) &\triangleq G(r_t) + H_2(r_t)K_2(r_t), \\
\bar{Q}_k(r_t) &\triangleq Q_k(r_t) + K_2(r_t)'R_k(r_t)K_2(r_t), \\
\tilde{A}(r_t) &\triangleq A(r_t) + B_1(r_t)K_1(r_t), \\
\tilde{G}(r_t) &\triangleq G(r_t) + H_1(r_t)K_1(r_t), \\
\tilde{Q}_k(r_t) &\triangleq Q_k(r_t) + K_1(r_t)'S_k(r_t)K_1(r_t).
\end{aligned}$$

**Theorem 6.3.1.** *We assume the mean-square stability condition (6.17) holds for both $P_1(r_t)$ and $P_2(r_t)$. For given disturbance attenuation level $\gamma > 0$, the infinite horizon $H_2/H_\infty$ control for system (6.1) has a pair of solutions $(u_T^*, v_T^*)$ with*

$$\begin{aligned}
u_T^*(t,x) &= -\sum_{i=1}^{l} K_{2i}\chi_{r_t=i}x(t), \\
v_T^*(t,x) &= -\sum_{i=1}^{l} K_{1i}\chi_{r_t=i}x(t),
\end{aligned}$$

*if the following four coupled AREs have solutions*

$$(P_1, P_2; K_1, K_2)$$

*with $P_1 = (P_{11}, P_{12}, \cdots, P_{1l}) \geq 0$, $P_2 = (P_{21}, P_{22}, \cdots, P_{2l}) \geq 0$.*

$$\begin{aligned}
&\begin{cases} L_i - \beta_i'\alpha_i^{-1}\beta_i = 0, \\ \alpha_i > 0, \quad i \in M, \end{cases} \\
&K_{1i} = \alpha_i^{-1}\beta_i, \\
&\begin{cases} T_j - N_j'Z_j^{-1}N_j = 0, \\ Z_j > 0, \quad j \in M, \end{cases} \\
&K_{2j} = Z_j^{-1}N_j,
\end{aligned}$$

*where*

$$\begin{aligned}
L_i &\triangleq P_{1i}\bar{A}_i + \bar{A}_i' P_{1i} + \bar{G}_i' P_{1i}\bar{G}_i + \sum_{k=1}^{\eta} e_k' E_i' P_{1i} E_i e_k \bar{Q}_{ki} \\
&\quad + \sum_{i=1}^{l} \pi_{ij} P_j - C_i' C_i - K_{2i}' K_{2i}, \\
\beta_i(t) &\triangleq B_{1i}' P_{1i} + H_{1i}' P_{1i}\bar{G}_i, \\
\alpha_i &\triangleq \gamma^2 I + H_{1i}' P_{1i} H_{1i} + \sum_{k=1}^{\eta} e_k' E_i' P_{1i} E_i e_k S_{ki}, \\
T_j &\triangleq P_{2j}\tilde{A}_j + \tilde{A}_j' P_{2j} + \tilde{G}_j' P_{2j}\tilde{G}_i + \sum_{k=1}^{\eta} e_k' E_j' P_{2j} E_j e_k \tilde{Q}_{kj} \\
&\quad + \sum_{j=1}^{l} \pi_{j\sigma} P_{2\sigma} + C_j' C_j, \\
N_j &\triangleq B_{2j}' P_{2j} + H_{2j}' P_{2j}\tilde{G}_j, \\
Z_j &\triangleq I + H_{2j}' P_{2j} H_{2j} + \sum_{k=1}^{\eta} e_k' E_j' P_{1j} E_j e_k R_{kj}.
\end{aligned}$$

The proof is similar to the previous case in finite time horizon. Here we omit the details.

## 6.4 Summary

This chapter investigates the problem of stochastic $H_2/H_\infty$ control for a class of nonlinear systems with Markovian switching in both finite and infinite time horizon. In the main results we show that the solvability of the coupled DREs is sufficient to solve the finite horizon stochastic nonlinear $H_2/H_\infty$ control problem. In addition, we show that the solvability of the coupled AREs is sufficient to solve the infinite horizon stochastic nonlinear $H_2/H_\infty$ control problem.

# Chapter 7

# Risk-sensitive Control of Stochastic Nonlinear Systems in Finite Time Horizon

## 7.1 Introduction

Based on [30], in which the nonlinearity is in the drift term, we formulate the problem of risk-sensitive control of stochastic nonlinear systems in finite time horizon with one additional nonlinearity in the diffusion term. In this case, our new system includes two different types of nonlinearity in both drift and diffusion terms. In this case, the system considered in this chapter is more generalized then the one in [30]. The problem is formulated in Section 7.2. When the assumptions in Section 7.3 are satisfied, we present our main results in Section 7.4, where we proved that there exists a unique solution to our optimal control problem. We also obtain the optimal cost functional. When we solve the risk-sensitive control problem, completion of square method is used, and the difficulty arises in dealing with the nonlinearity terms. The other difficulty is that we have to make sure that the control law is admissible, because in risk-sensitive control problems the criterion is given in an exponential form, which is different from the cases of LQ optimal control problems, and this is paid attention to in Section 7.3. Here it is highlighted that within this nonlinear system an explicit solution is found, which is a very rare case. In addition, the optimal control law obtained is linear with state, which is very similar to the characteristics of the results in [31] dealing with linear stochastic risk-sensitive control problems. We highlight the importance of this chapter by introducing its applications to finance, which are discussed in Section 7.5.

## 7.2 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered complete probability space, where there exist a one-dimensional standard $\mathcal{F}_t$-Brownian motion $(W_j(t), 0 \le t \le T)$, and a $\eta$-dimensional Brownian motion $(\tilde{W}(t), 0 \le t \le T)$, which is independent of $W(t)$. Following the structure of the nonlinear stochastic system in [30], we define our new nonlinear stochastic system by including one more nonlinear term similar to the one used in (6.1) as follows:

$$\begin{cases} dx_1(t) = [Ax_1(t) + Bu(t)]dt + \sum_{j=1}^{n} C_j dW_j(t), \\[2em] dx_2(t) = [A_1x_1(t) + A_2x_2(t) + D(x_1(t), u(t)) + B_1u(t)]dt \\ \qquad + \sum_{j=1}^{n} [A_{3j}x_1(t) + B_{2j}u(t) + C_{1j}]dW_j(t) \\ \qquad + EF(x_1(t), u(t))d\tilde{W}(t), \\[2em] x_1(0) = x_{10}, \quad x_2(0) = x_{20}. \end{cases} \tag{7.1}$$

where

$$\begin{aligned} A(\cdot) &\in L^\infty(0, T; \mathbb{R}^{n_1 \times n_1}), \\ B(\cdot) &\in L^\infty(0, T; \mathbb{R}^{n_1 \times m}), \\ C_1(\cdot) \cdots, C_n(\cdot) &\in L^\infty(0, T; \mathbb{R}^{n_1}), \\ A_1(\cdot), A_{31}(\cdot), \cdots, A_{3n}(\cdot) &\in L^\infty(0, T; \mathbb{R}^{n_2 \times n_1}), \end{aligned}$$

$$\begin{array}{rcl} A_2(\cdot) & \in & L^\infty(0,T;\mathbb{R}^{n_2\times n_2}), \\ B_1(\cdot),B_{21}(\cdot),\cdots,B_{2n}(\cdot) & \in & L^\infty(0,T;\mathbb{R}^{n_2\times m}), \\ C_{11}(\cdot),\cdots,C_{1n}(\cdot) & \in & L^\infty(0,T;\mathbb{R}^{n_2}), \\ E(\cdot) & \in & L^\infty(0,T;\mathbb{R}^{n_2\times\eta}). \end{array}$$

The vector $D(x_1(t),u(t))$ is defined the same with [30], where

$$D(x_1(t),u(t)) = \begin{bmatrix} x_1(t)'Q_1x_1(t)+u(t)'X_1x_1(t)+u(t)'R_1u(t) \\ \cdots \\ x_1(t)'Q_{n_2}x_1(t)+u(t)'X_{n_2}x_1(t)+u(t)'R_{n_2}u(t) \end{bmatrix},$$

where

$$\begin{array}{rcl} Q_1(\cdot),\cdots,Q_{n_2}(\cdot) & \in & L^\infty(0,T;\mathbb{S}^{n_1}), \\ X_1(\cdot),\cdots,X_{n_2}(\cdot) & \in & L^\infty(0,T;\mathbb{R}^{m\times n_1}), \\ R_1(\cdot),\cdots,R_{n_2}(\cdot) & \in & L^\infty(0,T;\mathbb{S}^{m}). \end{array}$$

We define $F(x(t),u(t))$ as

$$F(x_1(t),u(t)) \triangleq \mathrm{diag}(\sqrt{\phi_1},\sqrt{\phi_2},\ldots,\sqrt{\phi_\eta}). \tag{7.2}$$

Among $\phi_1,\phi_2,\ldots,\phi_\eta$, we denote each of them as $\phi_k$, where $k=1,2,\ldots,\eta$. We define

$$\phi_k \triangleq x_1(t)'\tilde{Q}_k(t)x_1(t)+u(t)'\tilde{R}_k(t)u(t), \tag{7.3}$$

where

$$\begin{array}{rcl} \tilde{Q}_1(\cdot),\cdots,\tilde{Q}_\eta(\cdot) & \in & L^\infty(0,T;\mathbb{S}^{n_1}), \\ \tilde{R}_1(\cdot),\cdots,\tilde{R}_\eta(\cdot) & \in & L^\infty(0,T;\mathbb{S}^{m}). \end{array}$$

We assume that matrices $\tilde{Q}_k$, $\tilde{R}_k$, $k=1,\cdots,\eta$ satisfy $\tilde{Q}_k(t)\geq 0$, $\tilde{R}_k(t)\geq 0$.

Define $e_k\in\mathbb{R}^m$ as an elementary vector, whose $k$-th element is 1, while other elements are 0.

In summary, compare the system (7.1) with the one in [30], the difference is that we involve one more nonlinear term $EF(x_1(t),u(t))d\tilde{W}(t)$ in the system with the other settings unchanged.

The discussion of existence and uniqueness of solution to the system (7.1) is similar to the one discussed in Chapter 3. Here we omit the details.

The criterion is given the same with [30] as follows:

$$\begin{aligned} J(u(\cdot)) &= \gamma\mathbb{E}\Big\{exp\Big[\frac{\gamma}{2}x_1(T)'Sx_1(T) + \frac{\gamma}{2}\int_0^T [x_1(t)'Qx_1(t) + u(t)'Ru(t)]dt \\ &+ \frac{\gamma}{2}S_1'x_1(T) + \frac{\gamma}{2}S_2'x_2(T) + \frac{\gamma}{2}\int_0^T [L_1'x_1(t) + L_2'x_2(t) + L_u'u(t) \\ &+ u(t)'Xx_1(t)]dt\Big]\Big\}, \end{aligned}$$

where

$$\begin{aligned} L_1(\cdot) &\in L^\infty(0,T;\mathbb{R}^{n_1}), \quad L_2(\cdot) \in L^\infty(0,T;\mathbb{R}^{n_2}), \\ L_u(\cdot) &\in L^\infty(0,T;\mathbb{R}^{m}), \quad X(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times n_1}), \\ S_1 &\in \mathbb{R}^{n_1}, \quad S_2 \in \mathbb{R}^{n_2}. \end{aligned}$$

and $\gamma \in \mathbb{R}$, $\gamma \neq 0$, is a given constant. Our aim is to minimize the cost functional $J(u(\cdot))$ subject to our system (7.1).

## 7.3 Problem Assumptions

For the notation convenience, we define

$$\begin{aligned} \bar{R} &\triangleq R + \sum_{i=1}^{n_2} p_{2i}(t)R_i + \frac{\gamma}{4}\sum_{j=1}^{n} B_{2j}'p_2(t)p_2(t)'B_{2j} \\ &\quad +\frac{\gamma}{4}\sum_{k=1}^{\eta} e_k'E'p_2(t)p_2(t)'Ee_k\tilde{R}_k, \\ \bar{X} &\triangleq X + 2B'P(t) + \sum_{i=1}^{n_2} p_{2i}(t)X_i + \frac{\gamma}{4}\sum_{j=1}^{n} 2B_{2j}'p_2(t)[2C_j'P(t) \\ &\quad +p_2(t)'A_{3j}], \\ \bar{Y} &\triangleq L_u + B'p_1(t) + B_1'p_2(t) + \frac{\gamma}{4}\sum_{j=1}^{n} 2B_{2j}'p_2(t)[C_j'p_1(t) + C_{1j}'p_2(t)], \\ \bar{Z} &\triangleq \sum_{j=1}^{n}[2P(t)C_j + A_{3j}'p_2(t)][C_j'p_1(t) + C_{1j}'p_2(t)]. \end{aligned} \tag{7.4}$$

**Assumption 7.3.1.** *$\bar{R}(t) > 0$, a.e. $t \in [0, T]$.*

**Assumption 7.3.2.** *The following Riccati equation has a unique solution.*

$$\begin{cases} Q + A'P(t) + P(t)A + \dot{P} + \sum_{i=1}^{n_2} p_{2i}(t)Q_i + \frac{\gamma}{4}\sum_{j=1}^{n}[2P(t)C_j \\ \quad + A'_{3j}p_2(t)][2C'_jP(t) + p_2(t)'A_{3j}] + \frac{\gamma}{4}\sum_{k=1}^{\eta} e'_kE'p_2(t)p_2(t)'Ee_k\tilde{Q}_k \\ \quad - \frac{1}{4}\bar{X}'\bar{R}^{-1}\bar{X} = 0. \\ P(T) = S. \end{cases} \tag{7.5}$$

**Remark 7.3.1.** *The conditions for Assumption 7.3.2 to be held can be found similarly to the way discussed in [30], because our equation (7.5) is very similar to equation (2.4) in (7.5). The only difference is that in equation (7.5) we have one more term $\frac{\gamma}{4}\sum_{k=1}^{\eta} e'_kE'p_2(t)p_2(t)'Ee_k\tilde{Q}_k$, which has nothing to do with $P(t)$, thus will not affect the later steps in [30]. Here we do not duplicate the derivation.*

Under Assumption 7.3.2 we introduce the following linear differential equation

$$\begin{cases} \dot{p}_2(t) + A'_2p_2(t) + L_2 = 0, \\ p_2(T) = S_2, \end{cases} \tag{7.6}$$

$$\begin{cases} \dot{p}_1(t) + L_1 + A'p_1(t) + A'_1p_2(t) - \frac{1}{2}\bar{X}'\bar{R}^{-1}\bar{Y} + \frac{\gamma}{2}\bar{Z} = 0, \\ p_1(T) = S_1, \end{cases} \tag{7.7}$$

and

$$\begin{cases} \dot{p} + \sum_{j=1}^{n} C'_jP(t)C_j + \frac{\gamma}{4}\sum_{j=1}^{n}[p_1(t)'C_j + p_2(t)'C_{1j}][C'_jp_1(t) + C'_{1j}p_2(t)] = 0, \\ p(T) = 0. \end{cases} \tag{7.8}$$

We denote the solution to (7.6) by

$$p_2(t) = \begin{bmatrix} p_{21}(t), & \cdots, & p_{2n_2}(t) \end{bmatrix}'.$$

**Assumption 7.3.3.** *We assume that the control process $u(t)$ is such that the term*

$$G(t)\Big[x_1(t)'\Big(2P(t)C_j + A'_{3j}p_2(t)\Big) + p_2(t)'B_{2j}u(t) + p_1(t)'C_j + p_2(t)'C_{1j}\Big] \quad (7.9)$$

*is a square integral process, i.e.,*

$$\int_0^T \mathbb{E}\Big\{G(t)^2\Big[x_1(t)'\Big(2P(t)C_j + A'_{3j}p_2(t)\Big) + p_2(t)'B_{2j}u(t) + p_1(t)'C_j + p_2(t)'C_{1j}\Big]^2\Big\}dt < \infty.$$

**Assumption 7.3.4.** *We assume that the control process $u(t)$ is such that the term*

$$G(t)p_2(t)'EF(x_1(t), u(t)) \quad (7.10)$$

*is a square integral process, i.e.,*

$$\int_0^T \mathbb{E}[G(t)^2 p_2(t)'EF(x_1(t), u(t))F(x_1(t), u(t))'E'p_2(t)]dt < \infty.$$

**Definition 7.3.1.** *When Assumption 7.3.3 and Assumption 7.3.4 are satisfied, the control $u(\cdot)$ is defined to be admissible.*

**Remark 7.3.2.** *In [30] the control is proved to be admissible in details under some conditions. Here, we do not prove the control is admissible due to some technical difficulties. In this chapter, we suppose that Assumption 7.3.3 and Assumption 7.3.4 hold. Such similar assumptions have been made by many researchers so far, see for example [90], [47] and [21].*

## 7.4 Main Result

Define

$$dv(t) \triangleq [x_1(t)'Qx_1(t) + u(t)'Ru(t) + u(t)'Xx_1(t) + L'_1x_1(t) + L'_2x_2(t)$$

$$+L_u'u(t)]dt,$$

where $v(0) = 0$.

Define

$$H(t) \triangleq v(t) + x_1(t)'P(t)x_1(t) + p(t) + p_1(t)'x_1(t) + p_2(t)'x_2(t).$$

**Theorem 7.4.1.** *Let the Assumption 7.3.1, Assumption 7.3.2, Assumption 7.3.3, and Assumption 7.3.4 hold. There exists a unique solution to our optimal control problem. The optimal control is given by*

$$u^*(t) = -\frac{1}{2}\bar{R}^{-1}(\bar{X}x_1(t) + \bar{Y}).$$

*The optimal cost functional is obtained as follows,*

$$J^* = \gamma\mathbb{E}\left[exp\left\{\frac{\gamma}{2}[x_{10}'P(0)x_{10} + p(0) + p_1(0)'x_{10} + p_2(0)'x_{20}]\right\}\right],$$

*where $P$, $p_1$, $p_2$ and $p$ are solutions to (7.5), (7.7), (7.6), and (7.8), respectively.*

*Proof.* The proof is similar to the one in [30]. Here we outline the main idea. The differential of $H(t)$ is

$$\begin{aligned} dH(t) &= [x_1(t)'Qx_1(t) + u(t)'Ru(t) + u(t)'Xx_1(t) + L_1'x_1(t) + L_2'x_2(t) \\ &\quad +L_u'u(t)]dt + d[x_1(t)'Px_1(t)] + d[p_1'x_1(t)] + d[p_2'x_2(t)], \end{aligned}$$

where $d[x_1(t)'Px_1(t)]$ and $d[p_1'x_1(t)]$ have been obtained by [30], here we focus on $d[p_2'x_2(t)]$,

$$\begin{aligned} d[p_2'x_2(t)] &= \dot{p_2}'x_2dt + p_2'[A_1x_1(t) + A_2x_2 + D(x_1(t), u(t)) + B_1u(t)]dt \\ &\quad + \sum_{j=1}^{n}[p_2'A_{3j}x_1 + p_2'B_{2j}u(t) + p_2'C_{1j}]dW_j(t) \\ &\quad +p_2'EF(x_1(t), u(t))d\tilde{W}(t). \end{aligned}$$

We rewrite $dH(t)$,

$$dH(t)$$

$$
\begin{aligned}
= \quad & \Big[x_1(t)'Qx_1(t) + u(t)'Ru(t) + u(t)'Xx_1(t) + L_1'x_1(t) + L_2'x_2(t) \\
& +L_u'u(t) + x_1(t)'\dot{P}(t)x_1(t) + x_1(t)'\Big(A'P(t) + P(t)A\Big)x_1(t) \\
& +2u(t)'B'P(t)x_1(t) + \sum_{j=1}^{n} C_j'P(t)C_j + \dot{p}(t) + \dot{p}_1'(t)x_1(t) + p_1'(t)\Big(Ax_1(t) \\
& +Bu(t)\Big) + \dot{p}_2'(t)x_2(t) + p_2'(t)\Big(A_1x_1(t) + A_2x_2(t) + D(x_1(t), u(t) \\
& +B_1u(t)\Big)\Big]dt + \sum_{j=1}^{n}\Big[x_1(t)'\Big(2P(t)C_j + A_{3j}'p_2(t)\Big) + p_2(t)'B_{2j}u(t) \\
& +p_1(t)'C_j + p_2(t)'C_{1j}\Big]dW_j(t) + p_2(t)'EF(x_1(t), u(t))d\tilde{W}(t).
\end{aligned}
$$

We define

$$G(t) \triangleq exp[\frac{\gamma}{2}H(t)].$$

According to the definition of $J(u(\cdot))$, we have

$$J(u(\cdot)) = \gamma\mathbb{E}[G(T)].$$

We apply Itô's formula to $dG(t)$. With

$$\frac{\partial}{\partial H(t)}G(t) = exp(\frac{\gamma}{2}H(t))\frac{\gamma}{2} = \frac{\gamma}{2}G(t),$$

and

$$\frac{\partial^2}{\partial H^2(t)}G(t) = (\frac{\gamma}{2})^2G(t),$$

we have

$$
\begin{aligned}
& dG(t) \\
= \quad & \frac{\gamma}{2}G(t)\Big[x_1(t)'Qx_1(t) + u(t)'Ru(t) + u(t)'Xx_1(t) + L_1'x_1(t) + L_2'x_2(t) \\
& +L_u'u(t) + x_1(t)'\dot{P}(t)x_1(t) + x_1(t)'\Big(A'P(t) + P(t)A\Big)x_1(t)
\end{aligned}
$$

$$
\begin{aligned}
&+2u(t)'B'P(t)x_1(t)+\sum_{j=1}^{n}C_j'P(t)C_j+\dot{p}(t)+\dot{p}_1'(t)x_1(t)+p_1'(t)\Big(Ax_1(t)\\
&+Bu(t)\Big)+\dot{p}_2'(t)x_2(t)+p_2'(t)\Big(A_1x_1(t)+A_2x_2(t)+D(x_1(t),u(t)\\
&+B_1u(t)\Big)\Big]dt+\frac{G(t)}{2}\Big(\frac{\gamma}{2}\Big)^2\sum_{j=1}^{n}\Big[x_1(t)'\Big(2P(t)C_j+A_{3j}'p_2(t)\Big)\Big(2C_j'P(t)\\
&+p_2(t)'A_{3j}\Big)x_1(t)+2u(t)'B_{2j}'p_2(t)\Big(2C_j'P(t)+p_2'A_{3j}\Big)x_1(t)\\
&+2x_1(t)'\Big(2P(t)C_j+A_{3j}'p_2(t)\Big)\Big(C_j'p_1(t)+C_{1j}'p_2(t)\Big)\\
&+u(t)'B_{2j}'p_2(t)p_2(t)'B_{2j}u(t)+2u(t)'B_{2j}'p_2(t)\Big(C_j'p_1(t)+C_{1j}'p_2(t)\Big)\\
&+\Big(p_1(t)'C_j+p_2(t)'C_{1j}\Big)\Big(C_j'p_1(t)+C_{1j}'p_2(t)\Big)\Big]dt\\
&+\frac{G(t)}{2}\Big(\frac{\gamma}{2}\Big)^2 tr\Big[F(x_1(t),u(t))'E'p_2(t)p_2(t)'EF(x_1(t),u(t))\Big]dt\\
&+\sum_{j=1}^{n}\frac{\gamma}{2}G(t)\Big[x_1(t)'\Big(2P(t)C_j+A_{3j}'p_2(t)\Big)+p_2(t)'B_{2j}u(t)+p_1(t)'C_j\\
&+p_2(t)'C_{1j}\Big]dW_j(t)+\frac{\gamma}{2}G(t)p_2(t)'EF(x_1(t),u(t))d\tilde{W}(t).
\end{aligned}
\tag{7.11}
$$

The calculation of $p_2(t)'D(x_1(t),u(t))$ can be found in [30], here we omit the details. Similar to the derivation steps from (3.34) to (3.38) in Chapter 3, we have

$$
\begin{aligned}
& tr\Big[F(x_1(t),u(t))'E'p_2(t)p_2(t)'EF(x_1(t),u(t))\Big]\\
= \;& x_1(t)'\left[\sum_{k=1}^{\eta}e_k'E'p_2(t)p_2(t)'Ee_k\tilde{Q}_k\right]x_1(t)\\
& +u(t)'\left[\sum_{k=1}^{\eta}e_k'E'p_2(t)p_2(t)'Ee_k\tilde{R}_k\right]u(t).
\end{aligned}
$$

After rewriting $dt$ terms in (7.11), we focus on the terms containing $u(t)$, which is,

$$
u(t)'\bar{R}u(t)+u(t)'\bar{X}x_1(t)+u(t)'\bar{Y}
$$

$$\begin{aligned}
&= \Big[u(t)+\frac{1}{2}\bar{R}^{-1}(\bar{X}x_1(t)+\bar{Y})\Big]'\bar{R}\Big[u(t)+\frac{1}{2}\bar{R}^{-1}(\bar{X}x_1(t)+\bar{Y})\Big] \\
&\quad -\frac{1}{4}[\bar{X}x_1(t)+\bar{Y}]'\bar{R}^{-1}[\bar{X}x_1(t)+\bar{Y}],
\end{aligned}$$

where $\bar{R}$, $\bar{X}$, $\bar{Y}$ is given in (7.4). In the drift terms of (7.11), due to Assumption 7.3.2, we have

$$\begin{aligned}
& x_1(t)'Qx_1(t)+x_1(t)'(A'P(t)+P(t)A)x_1(t)+x_1(t)'\dot{P}x_1(t) \\
+ & \sum_{i=1}^{n_2} p_{2i}(t)x_1(t)'Q_ix_1(t)+x_1(t)'\Big\{\frac{\gamma}{4}\sum_{j=1}^{n}[2P(t)C_j+A_{3j}'p_2(t)][2C_j'P(t) \\
+ & p_2(t)'A_{3j}]\Big\}x_1(t)+x_1(t)'\frac{\gamma}{4}\sum_{k=1}^{\eta}e_k'E'p_2(t)p_2(t)'Ee_k\tilde{Q}_kx_1(t) \\
- & \frac{1}{4}x_1(t)'\bar{X}'\bar{R}^{-1}\bar{X}x_1(t)=0.
\end{aligned}$$

Due to (7.7), we have

$$\begin{aligned}
& L_1'x_1(t)+\dot{p}_1'x_1(t)+p_1(t)'Ax_1(t)+p_2(t)'A_1x_1(t)-\frac{1}{2}x_1(t)'\bar{X}'\bar{R}^{-1}\bar{Y} \\
+ & x_1(t)\frac{\gamma}{4}\sum_{j=1}^{n}2[2P(t)C_j+A_{3j}'p_2(t)][C_j'p_1(t)+C_{1j}'p_2(t)]=0.
\end{aligned}$$

Due to (7.6), we have

$$L_2'x_2(t)+\dot{p}_2'(t)x_2(t)+p_2(t)'A_2x_2(t)=0.$$

Due to (7.8), the remaining terms independent of both $x_1(t)$ and $u(t)$ equal to zero. Finally, we rewrite the cost functional for all the admissible controls as follows,

$$\begin{aligned}
J(u(\cdot)) &= \gamma\mathbb{E}[G(0)]+\frac{\gamma^2}{2}\mathbb{E}\int_0^T G(t)\Big[u(t)+\frac{1}{2}\bar{R}^{-1}(\bar{X}x_1(t)+\bar{Y})\Big]'\bar{R}\Big[u(t) \\
&\quad +\frac{1}{2}\bar{R}^{-1}(\bar{X}x_1(t)+\bar{Y})\Big] \\
&\geq \gamma\mathbb{E}[G(0)].
\end{aligned}$$

We obtain the lower bound $\gamma\mathbb{E}[G(0)]$ iff

$$u(t)=u^*(t)=-\frac{1}{2}\bar{R}^{-1}(\bar{X}x_1(t)+\bar{Y}).$$

□

## 7.5 Applications

Recall the financial market of Section 3.2, which consists of a bank account and a stock, the prices of which we repeat here for convenience:

$$\begin{cases} dB(t) = B(t)r(t)dt, \\ \\ dS(t) = S(t)[\mu(t)dt + \sigma(t)dW_1(t)], \\ \\ B(0) = B_0 \quad \text{and} \quad S(0) = S_0 \quad \text{are given.} \end{cases}$$

The zero-coupon bond is a contract with the terminal payoff of 1, i.e. such a contract guarantees a payment of 1 unit at maturity date $T$ to the holder. This is one of the simplest contracts, and yet it is fundamental since the prices of other contracts, such as swaps, caps, floors, or swaptions, are expressed in terms of the zero-coupon bond price (see, e.g., [16]). Thus, by pricing such a contract we solve the pricing problem for various other contracts. In [30] the price of a zero-coupon bond for a particular interest rate model was found as a special case of the risk-sensitive control problem considered there. In this section we introduce a new interest rate model, and based on our result on the risk-sensitive control, we find the price of the zero-coupon bond. Moreover, we show that the optimal investment problem for the power utility is an example of our risk-sensitive control problem.

Let $f_1$ and $f_2$ be the factor processes with equations the same as the states $x_1$ and $x_2$ with $u(t) = 0$, i.e.

$$\begin{cases} df_1(t) = Af_1(t)dt + \sum_{j=1}^{n} C_j dW_j(t), \\ df_2(t) = [A_1 f_1(t) + A_2 f_2(t) + D(f_1(t))]dt \\ \qquad\qquad + \sum_{j=1}^{n} [A_{3j} f_1(t) + C_{1j}]dW_j(t) + EF(f_1(t))d\tilde{W}(t), \\ f_1(0) = f_{10}, \quad f_2(0) = f_{20}. \end{cases} \tag{7.12}$$

We introduce the following interest rate model:

$$r(t) \triangleq f_1(t)'Qf_1(t) + L_1'f_1(t) + L_2'f_2(t).$$

This model appears to be new due to the square-root nonlinearity of our model. The motivation for introducing this model is that it admits an explicit closed form formula for the price of a zero-coupon bond. Indeed, the price of such a contract is (see, e.g., [16]):

$$p(0,T) \triangleq \mathbb{E}[e^{\int_0^T r(\tau)d\tau}].$$

However, this is just our cost functional with $u(t) = 0$ and without any terminal cost, i.e. $S = 0$, $S_1' = 0$, $S_2' = 0$. Let the assumptions of Theorem 7.4.1 hold, and $\gamma = 2$. Then from that theorem we immediately have

$$p(0,T) \quad = \quad J^*/2 = \mathbb{E}\left[exp\left\{[f_{10}'P(0)f_{10} + p(0) + p_1(0)'f_{10} + p_2(0)'f_{20}]\right\}\right].$$

Another application is in the problem of optimal investment. Recall from Section 3.2 that the equation of the self-financing portfolio is

$$\begin{cases} dy(t) = [r(t)y(t) + (\mu(t) - r(t))u(t)]dt + \sigma(t)u(t)dW_1(t), \\ \\ y(0) = y_0 > 0. \end{cases} \tag{7.13}$$

The problem of optimal investment with the power utility is the problem of maximizing

$$\mathbb{E}[y^\beta(T)], \tag{7.14}$$

with $\beta \in (0,1)$. The solution to (7.13) is

$$\begin{aligned} y(T) = y_0 \qquad & \exp\left[\int_0^T [r(s) + (\mu(s) - r(s))v(t) - \sigma^2(s)v^2(s)/2] \right. \\ & \left. + \int_0^T \sigma(s)v(s)dW_1(s)\right], \end{aligned} \tag{7.15}$$

where $v(t) = u(t)/y(t)$. We assume that $h(t) \triangleq \mu(t) - r(t)$ is a deterministic function (note that this is typical assumption in a market with stochastic interest rate, see, e.g. [15]). The expected power utility can thus be written as

$$\begin{aligned}\mathbb{E} \quad & \left\{ y_0^\beta \exp\left[\beta \int_0^T [f_1(s)'Qf_1(s) + L_1' f_1(s) + L_2' f_2(s) + h(t)v(t) - \sigma^2(s)v^2(s)/2]\right.\right. \\ & \left.\left. \beta + \int_0^T \sigma(s)v(s)dW_1(s)\right]\right\}, \qquad (7.16)\end{aligned}$$

The equation for $x(t) = \log y(t)$ is

$$\begin{cases} dx(t) = [f_1(t)'Qf_1(t) + L_1' f_1(t) + L_2' f_2(t) + h(t)v(t) - \sigma^2 v^2(t)]dt \\ \quad +\sigma(t)v(t)dW_1(t), \\ \\ x(0) = x_0 = \log y_0 > 0, \end{cases} \qquad (7.17)$$

Note that the cost (7.16) contains a noise dependent penalty. By introducing a new state variable $\tilde{x}(t)$ with equation

$$\begin{cases} d\tilde{x}(t) = \sigma(t)v(t)dW_1(t), \\ \\ \tilde{x}(0) = 0, \end{cases} \qquad (7.18)$$

we see that such a noise dependent penalty in (7.16) is just a linear penalty in $\tilde{x}(t)$. We thus see that the problem of maximizing (7.16) subject to (7.12), (7.17), (7.18), is just an example of the risk-sensitive control problem of this chapter, and can be solved by applying Theorem 7.4.1.

## 7.6 Summary

This chapter investigates the problem of risk-sensitive control of stochastic nonlinear systems in finite time horizon. When a series of assumptions are satisfied, a unique solution to our optimal control problem is found, and the optimal cost functional is obtained. We emphasize the importance of this chapter by introducing its applications to finance.

# Chapter 8

# Conclusion

## 8.1 Introduction

We summarize the main contributions of this thesis in this last chapter . We also point out some interesting open problems for future research.

## 8.2 Chapter 3

Chapter 3 deals with the finite horizon optimal control of stochastic nonlinear system with indefinite state and control cost weighting matrices with Markovian switching appearing in system coefficients. A new type of CGREs is introduced. The solvability of CGREs is proved to be sufficient to solve our nonlinear optimal control problem. Moreover, under such a nonlinear system, all the optimal controls are obtained explicitly and linearly with state, constructed by the solution to the CGREs. The existence and uniqueness of the solution is discussed. The feasibility of the assumption of the solvability of the new CGREs is discussed. An application to finance is introduced. An illustrative example is given.

Here we list some open problems related to this chapter as follows:

- The solvability of CGREs with Markovian switching in [74] is not proved. If this can be proved in the future work, then the solvability to our new CGREs (3.23) can also be proved without difficulty.

- The necessity of the solvability to our new CGREs for the well-posedness of our nonlinear optimal control problem is still unsolved.
- More applications to finance can be introduced.

## 8.3 Chapter 4

Chapter 4 deals with the infinite horizon optimal control of stochastic nonlinear system with indefinite state and control cost weighting matrices and with Markovian switching appearing in system coefficients. The mean-square stability is considered. The new CGAREs are introduced. We assume that there exists a unique solution to the CGAREs such that the linear optimal control is admissible w.r.t. to any initial state, then our stochastic nonlinear optimal control problem is well-posed. Furthermore, the value function is obtained. Here we list some open problems related to this chapter as follows:

- The proof of the solvability of our new CGAREs is still an open problem.
- We have not reformulated our new CGAREs to LMIs (linear matrix inequalities) yet. After this reformulation, the problem can be solved in polynomial time based on solving a SDP (semidefinite programming) [19] [105].
- We have not reformulated the mean-square stability condition to LMI , which is easier for numerical computation.
- Due to the difficulty in computation caused by the complexity of problem formulation, numerical example is not given.

## 8.4 Chapter 5

In Chapter 5 we consider the problem of robust stabilization and robust $H_\infty$ control for a class of nonlinear stochastic systems with Markovian switching in coefficients. The new system that we investigate in Chapter 5 generalizes the system considered in [115], [114], [20], [120], [121], and [46] in many aspects. Sufficient conditions in forms of matrix inequalities are obtained such that the linear stabilizing controllers exist. In addition, a new type of disturbance attenuation formed by symmetric matrices with Markovian switching is involved. Then we formulate our generalized robust $H_\infty$ control problem. A sufficient condition for the solvability of our generalised robust $H_\infty$ control problem is proposed.

Some open problems are listed as follows.

- In both sections of robust stabilization and robust $H_\infty$ control, we obtained sufficient conditions for the solvability of our problems. Someone may ask whether these sufficient conditions are too strong or not, and whether there are any better conditions that are not so strong, but can still solve our problems.
- In both two theorems the reformulation of our matrix inequalities into LMIs is not dealt with. After this formulation, the controller can be constructed via a convex optimization problem, which can be checked numerically, see [19].
- Due to the difficulty in computation caused by the complexity of problem formulation, numerical example is not given.
- Although we focus on theoretical research, it is better to find some applications in which our new system can be used.

## 8.5 Chapter 6

In Chapter 6, stochastic $H_2/H_\infty$ control for a class of nonlinear systems with Markovian switching in both finite and infinite time horizon is considered. In the main results we show that the solvability of the coupled DREs is sufficient to solve the finite horizon stochastic nonlinear $H_2/H_\infty$ control problem. In addition, we show that the solvability of the coupled AREs is sufficient to solve the infinite horizon stochastic nonlinear $H_2/H_\infty$ control problem.

There are still some unsolved problems based on this work. In addition, there are also some ideas for future research. Here they are listed as follows:

- In [135], it is pointed out that although sufficient conditions are presented for the solvability of stochastic nonlinear $H_2/H_\infty$ control problems in both finite and infinite horizons, how to solve those four cross coupled DREs and AREs is still an open problem. This problem also appears in Chapter 6. When nonlinear term and Markovian switching is included, this problem becomes even more difficult. This can be viewed as a pure mathematical calculus problem.

- We can apply our work into descriptor systems, which is used in [126], focusing on infinite horizon $H_2/H_\infty$ control for descriptor systems. In this case, we might be able to extend [126] into a nonlinear $H_2/H_\infty$ control for descriptor systems with Markovian switchings in both finite and infinite horizons.

## 8.6 Chapter 7

In Chapter 7, the problem of risk-sensitive control for a class of stochastic nonlinear systems in finite time horizon is investigated. Following a series of assumptions, it is proved that there exists a unique solution to our optimal control problem. The optimal cost functional is obtained. We highlight the importance of this chapter by providing applications to finance.

However, although we extend the system in [30] into a more general type, this chapter has some restrictions. Additionally, some future works can be investigated. Here these problems are listed as follows.

- As we mentioned in Remark 7.3.2, future work can be focused on proving the control to be admissible.

- This is the only chapter that Markovian switching is not applied, due to some technical difficulties. This can be left for future study.

- A generalized risk-sensitive control was studied in [30], where noise dependent penalties on the control $u(t)$ and $x_1(t)$ is introduced. In [30] the cost functional becomes:

$$\tilde{J}(u(\cdot)) \;\; = \;\; \gamma\mathbb{E}\Big\{exp\Big[\frac{\gamma}{2}x_1(T)'Sx_1(T) + \frac{\gamma}{2}\int_0^T [x_1(t)'Qx_1(t) + u(t)'Ru(t)]dt$$

$$
\begin{aligned}
&+ \quad \frac{\gamma}{2}S_1'x_1(T)+\frac{\gamma}{2}S_2'x_2(T)+\frac{\gamma}{2}\int_0^T[L_1'x_1(t)+L_2'x_2(t)+L_u'u(t)\\
&+ \quad u(t)'Xx_1(t)]dt+\frac{\gamma}{2}\int_0^T[x_1(t)'Q_x+u(t)'R_u]dW(t)\Big]\Big\},
\end{aligned}
$$

where

$$
Q_x(\cdot) \quad \in \quad L^\infty(0,T;\mathbb{R}^{n_1}), \quad R_u(\cdot)\in L^\infty(0,T;\mathbb{R}^{n_2}).
$$

This cost functional can also be adopted in our new system for future research.

- Recently risk-sensitive control for infinite horizon was investigated by [47], which is a motivation for us to extend this chapter into the case of infinite time horizon in the future.

## 8.7 Summary

In this chapter we conclude the main contributions of this thesis. We also point out some restrictions and open problems. In addition, we give some possible ideas to future work.

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