ON EXAMPLES OF TWO-DIMENSIONAL PERIODIC CONTINUED FRACTIONS.

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CONTENTS

INTRODUCTION.

This paper is a short survey of the recent results on examples of periodic two-dimensional continued fractions (in Klein's model). These examples were constructed in the last fifteen years by many mathematicians in their works. In the last part of this paper we formulate some questions, problems and conjectures on geometrical properties concerning to this subject.

Klein's model of the multidimensional continued fractions is one of the most well-known generalizations of one-dimensional continued fractions to the multidimensional case. The definition of Klein continued multidimensional fractions was given by F. Klein in his works [13] and [14] (later on we will call them just multidimensional continued fractions). F. Klein considered the integer approximations for zeroes of quadratic forms $f(x, y) = ax^2 + bxy + cy^2$, where $f(x, y)$ is a product of two linear factors. Ordinary continued fractions provide a good algorithm for these approximations. Suppose the set of all zeroes for the form $f(x, y)$ does not contain any integer point other than the origin, and the linear factors of the form $f(x, y)$ are not proportional. Consider the convex hulls of all integer points in each of the four connected components of the complement to the set of zeroes of the form $f(x, y)$. The boundaries of the convex hulls are called sails and the union of them — the one-dimensional continued fraction (the notions of ordinary continued fraction and one-dimensional continued fraction are differs a little, the relation between these notions is described in Section 1). It turns out that the vertices of the

1

sails provide the best in some sense integer approximations of zeroes for the form f (the detailed description see in [9]). The main idea of the algorithm is to construct the vertices of the sail recursively. The description of the algorithm is not in the limelight of the present work, so we omit it (it can be found, for example, in [2]).

The idea of F. Klein was to find some effective generalization of the algorithm to the case of homogeneous cubic forms of three variables that is decomposable in a product of three linear factors. He generalized the one-dimensional continued fractions to the two-dimensional case. F. Klein considered the convex hulls of all integer points in each of the eight connected components of the complement to the set of all zeroes for the cubic form (we again suppose that three planes of zeroes are in general position and there is no integer points in this planes other than the origin). The boundaries of the convex hulls are called the sails and the union of the convex hulls — the two-dimensional continued fraction. This definition generalizes to the case the other dimensions (see Section 1 for the precise definitions) Unfortunately, F. Klein failed to find an effective algorithm that generalizes the one-dimensional algorithm.

The interest for multidimensional continued fractions was revived by V. I. Arnold's work [1]. He realized the reasons of Klein's failure and proposed a new approach to the study of the multidimensional continued fractions. We describe one of in a few words. The one-dimensional sails are infinite polygonal lines and hence the set of all vertices possesses the natural ordering. This ordering is used in the algorithm. The multidimensional sails are infinite polyhedrons without any natural ordering of the vertices. In order to overcome such difficulties V. I.Arnold suggested to investigate the geometry and combinatorics of the sails at first (it contains the study of properties for such notions as affine types of the faces of the sails, their quantities and frequencies, integer angles between the faces, integer distances, volumes and so on, see also [2]).

Klein multidimensional continued fractions have many relations with some other branches of mathematics. For example, J.-O. Moussafir showed the connection between the sails of multidimensional continued fractions and Hilbert bases in his work [23]. In [27] H. Tsuchihashi found the relation between periodic multidimensional continued fractions and multidimensional cusp singularities which generalizes the relation between the ordinary continued fractions and two-dimensional cusp singularities.

We will mostly consider periodic Klein continued fractions (i.e. the fractions that are invariant under some particular group action described in Section 1) in this paper. Some important properties concerning to non-periodic Klein multidimensional continued fractions are described in the works of G. Lachaud [21], J.-O. Moussafir [23] and O. N. German [10]. The other generalizations of continued fractions can be found in the work [22] of H. Minkowski and the work [28] of G. F. Voronoi. Some attempts of comparison of the efficiency for the algorithms for calculation different types of continued fractions are given in [6] and [7] by A. D. Bruno and V. I. Parusnikov.

This paper is organized as follows. In the first section we give all definitions and notions that are necessary to our subject. In the second section we show the simplest three examples of torus decompositions for cubic irrationalities. Then we present the classification of all continued fractions of cubic irrationalities associated to the operator with the

small norm (the sum of absolute numbers of the coefficients is less or equal than six). In the third section we give some examples of the infinite sequences of nonequivalent torus decompositions. Finally, in the fourth section we formulate some interesting problems and pose appropriate conjectures.

Acknowledgement. The author is grateful to professor V. I. Arnold for constant attention to this work and for good remarks and questions, to N. G. Moshevitin and R. Uribe-Vargas for useful discussions and CEREMADE (Paris-Dauphine) for the hospitality.

1. Notions and definitions.

In this section we recall some basic notions and definitions (see also [11]). Consider the space $\mathbb{R}^{n+1}$ ($n \geq 1$) over the field of real numbers. The point of $\mathbb{R}^{n+1}$ is said to be *integer* if all coordinates of this point are integer. The polyhedron is said to be *integer* if all its vertices are integer. Consider a set of $n+1$ hyperplanes of $\mathbb{R}^{n+1}$ passing through the origin in general position. The complement to the union of these hyperplanes consists of 2^{n+1} open orthants. Let us choose an arbitrary orthant.

Definition 1.1. The boundary of the convex hull of all integer points except the origin in the closure of the orthant is called *the sail*. The union of all 2^{n+1} sails of the space $\mathbb{R}^{n+1}$ is called *n-dimensional continued fraction* associated to the given $n+1$ hyperplanes in general position in $(n+1)$-dimensional space.

Two n-dimensional sails are said to be *equivalent* if there exists a linear transformation that preserves the integer lattice of the $(n+1)$-dimensional space and maps one continued fraction to the other.

Two n-dimensional continued fractions are said to be *equivalent* if there exists a linear transformation that preserves the integer lattice of the $(n+1)$-dimensional space and maps one continued fraction to the other.

The definition of one-dimensional continued fraction is a little different to the definition of ordinary continued fraction. We will briefly discuss the relation between these two definitions. The definition of the ordinary continued fraction for some positive real number a is given by the following construction. Consider a ray $\{(x, y)|y = ax, x > 0\}$ on the plane OXY with the vertex in the origin. This ray divides the first quadrant of the plane $Q = \{(x, y)|x > 1, y > 1\}$ into two parts. Let us construct two boundaries of the convex hulls for the integer points in the closure of each part without the origin. These boundaries are polygonal lines. The polygonal line for each boundary consists of a ray and an infinite numbers of segments in the case of irrational number a, and it consists of two rays and some finite number of segments in the case of rational number a. Consider the union of all segments of these two polygonal lines. This set is called *the ordinary continued fraction* for the positive real number a.

Consider some arbitrary one-dimensional continued fraction L constructed by two lines passing through the origin. Let $y = a_1 x$ and $y = a_2 x$ be the equations for these lines (note that we regards now only positive irrational real numbers a_1 and a_2). The following statement holds.

Statement 1.1. *There exist such radius $R > 0$ that the union of ordinary continued fractions for the numbers a_1 and a_2 coincides (as a set) with the one-dimensional continued fraction L in the intersection of the quadrant Q and the complement to the disk with center in the origin and radius equals R.*

The case of negative a_1 or a_2 are similar and are left to the reader as an exercise.

Now let $A \in GL(n + 1, \mathbb{Z})$ be the operator with irreducible characteristic polynomial over the field $\mathbb{Q}$ whose roots are all real and distinct. Let us take the n-dimensional spaces that spans all possible subsets of n linearly independent eigenvectors of the operator A. The spans of every n eigenvectors defines $n + 1$ hyperplanes passing through the origin in general position. These hyperplanes defines the multidimensional continued fraction.

Definition 1.2. The n-dimensional continued fraction associated to the operator $A \in GL(n + 1, \mathbb{Z})$ as above is called *the n-dimensional continued fraction of $(n + 1)$-algebraic irrationality*. The case of $n = 1(2)$ corresponds to *one(two)-dimensional continued fractions of quadratic (cubic) irrationalities*.

The following statement holds (for proof see [11]).

Statement 1.2. *Continued fractions associated to the operators A and B of the group $GL(n + 1, \mathbb{Z})$ with distinct real irrational eigenvalues are equivalent iff there exists such an integer operator X with determinant equals one that the operator $\tilde{A}$, obtained from the operator A by means of the conjugation by the operator X commutes with B.*

Now let us formulate the notion of periodic continued fractions associated to some algebraic irrationalities. Suppose the characteristic polynomial of the operator A with all real roots be irreducible over the field of rational numbers. Consider the group of integer operators with the determinant equals one and positive eigenvalues that commutes with A. These operators form an Abelian group. It follows from Dirichlet unity elements theorem (see. [3]) that this group is isomorphic to $\mathbb{Z}^n$ and its action is free. Each operator of this group preserves the integer lattice and the union of all $n+1$ hyperplanes, and hence it preserves the n-dimensional continued fraction (as far as all eigenvalues are positive the sails maps to themselves also). The factor of a sail under such group action is isomorphic to n-dimensional torus at that. The converse is also true:

Statement 1.3. *For any double-periodic sail of a continued fraction there exist an operator A of the group $GL(n + 1, \mathbb{Z})$ with irreducible over the field of rational numbers characteristic polynomial with three real roots such that one of the sails of the continued fraction associated to A coincides with the given one.*

These two statements generalize Lagrange theorem for the ordinary continued fractions (see [15], [27], [18] and [20]).

Suppose some characteristic (object) is invariant under the action of the group $SL(n + 1, \mathbb{Z})$. Then this characteristic (object) is invariant under the action of the subgroup that consists of all integer operators with the determinant equals one and positive eigenvalues that commutes with A. Hence this characteristic (object) naturally corresponds to the

induced characteristic (object) for the factorization of the sail by the described above group action.

The first example of such object is k-dimensional planes in $\mathbb{R}^{n+1}$ for some fixed $k \leq n$. Under the action of some operator from the group $SL(n+1, \mathbb{Z})$ any k-dimensional plane maps to some k-dimensional plane. The second important example is sublattices of all integer points in the k-dimensional subspaces. Under the action of some operator from the group $SL(n+1, \mathbb{Z})$ any k-dimensional plane maps to some k-dimensional plane and the integer sublattice of the first one maps to the integer sublattice of the second plane in the one-to-one way. To be more precise, the k-dimensional planes with the integer sublattice are invariant under the group of all affine transformations that preserves the integer lattice of the space $\mathbb{R}^{n+1}$. This group contains the group $SL(n+1, \mathbb{Z})$ as a subgroup.

Definition 1.3. We say that two faces in the space $\mathbb{R}^k$ equipped with lattice have the same *integer affine type* if there exist a lattice preserving affine transformation of the space that maps one of the faces to the other.

It follows from this definition that integer affine type of the face is invariant under the action of the group of all affine transformations, that preserves the lattice of integer points. As we have already mentioned this group contains the group $SL(n+1, \mathbb{Z})$ as a subgroup. Hence our tori are equipped with face decomposition. All faces are some polyhedrons of the dimension less than or equal n.

Let us describe one important series of $SL(n+1, \mathbb{Z})$-action invariants.

Definition 1.4. The ratio of the Euclidean distance from an integer k-dimensional plane (containing a k-dimensional integer lattice) to the integer point to the minimal Euclidean distance from this plane to some integer point in the complement of the plane is called the *integer distance* from this plane to the integer point.

Let us rewrite the integer distance from the integer k-dimensional plane to the integer point in the integer affine invariant terms. Consider the lattice of integer points in the $(k+1)$-dimensional plane that spans the given integer k-dimensional plane and given integer point. Consider the sublattice of this lattice generated by the given integer point and all integer points in the given k-dimensional plane. Let us choose some zero and standard addition for the lattice and the sublattice. We have an Abelian group $\mathbb{Z}^{k+1}$ and some its subgroup. The index of this subgroup is equal to the integer distance from the integer k-dimensional plane to the integer point. This index is an integer affine invariant. So this distance is also an integer affine invariant and therefore it is a characteristic of the torus face decomposition.

Let us generalize the notion of distance to the case of two integer planes of dimension k_1 and k_2 (we only need that each of them is not contained in the other one). Consider the lattice of the integer points in the plane that spans the given two planes. Now consider the sublattice of this lattice generated by the all integer points in the given two planes.

Definition 1.5. If the intersection of the given two planes is empty, then the index of this sublattice is called *the integer distance* between the planes. If the intersection of the given

two planes is not empty, then corresponding index is called *the integer angle* between the planes.

All these indices are integer affine invariant, and therefore all integer distances and integer angles are also characteristic of the torus face decomposition.

Remark. The last definition has natural generalization to the case of l different planes of dimensions $k_1, \ldots, k_l$.

Definition 1.6. We call by *torus decomposition corresponding to the sail of the continued n-dimensional continued fraction for the $(n+1)$-algebraic irrationality* (or torus decomposition, for short) the above torus face decomposition equipped with integer affine structure for all faces of all dimensions, and all integer distances and integer angles between the planes containing two faces, for any two faces.

By *a fundamental region* of the sail we call a union of some faces that contains exactly one face from each equivalence class.

2. Three single examples of torus decompositions.

Let us start this section with three single examples of the torus decomposition for two-dimensional continued fractions of cubic irrationalities. One can find a nice and rich collection of beautiful single decompositions, and also a lot of useful references concerning to the examples in the work [4] by K. Briggs. I hope that new examples from the present paper would sooner or later appear in the list of the work of K. Briggs. Further we work in $\mathbb{R}^3$. In this and the next sections we describe only one of the eight possible sails for each continued fraction.

To describe more precisely the torus decompositions for two-dimensional continued fractions we need the following geometric $SL(n+1, \mathbb{Z})$-invariants: an integer length of a segment, an integer area of the face, an integer angle between two rays, and an integer distance from the plane (of a face) to some integer point. The definition of the integer distance was given in the last part of the previous section. Let us give all other definitions. They are all included in the definition of an affine type of the face and hence are $SL(3, \mathbb{Z})$-invariant.

Definition 2.1. The number of inner integer points of an integer segment in $\mathbb{R}^3$ plus one is called the *integer length* of the segment. Integer points divide this segment in the congruent parts. The ratio of the Euclidean area for an integer polygon in $\mathbb{R}^3$ to the Euclidean area for the minimal integer triangle lying in the plane of the polygon is called its *integer area*.

Definition 2.2. By *the integer angle* between two integer rays (i.e. containing more than one integer point) with the vertex at the same integer point we call the value $S(u, v)/(|u| \cdot |v|)$, where u and v are arbitrary integer vectors passing along the rays and $S(u, v)$ is the integer volume of the triangle with edges u and v.

Remark. The smallest possible integer volume of a triangle is equal to one (in standard parallelogram measuring all area will be two times less). The integer area of the triangle

is equal to the index of the lattice subgroup generated by its edges having the common vertex. The integer angle between two integer rays with vertex in the point is equal to the index of the lattice subgroup generated by all integer points lying on edges of the angle.

Example 2.1. Now all is done to describe the simplest example. We consider two-dimensional fraction that generalizes the ordinary continued fraction corresponding to the golden ratio $\frac{1+\sqrt{5}}{2}$ (see [17]). All four one-dimensional sails of the one-dimensional continued fraction corresponding to the golden ratio are equivalent. They consist of the segments of integer length one. The corresponding circle decompositions consist of one vertex and one edge. This fraction is given by the following operator:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

The generalization of this one-dimensional continued fraction to the multidimensional case was given by E. I. Korkina in the work [16]. The multidimensional continued fraction associated to the operator
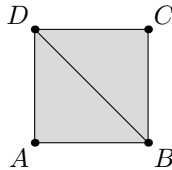
$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

is called *the generalized golden ratio.*

Here we consider the continued fraction associated to the generalized gold ratio operator:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

The torus decomposition corresponding to this operator is homeomorphic to the following one:

Here the segment AB is identified with the segment DC and the segment AD to the segment BC.

Remark. Note that only homeomorphic type of the faces is given on the picture above. This means, that we erase all integer points. We have only the torus, equipped with the graph. Each face of the graph corresponds to some face of the torus decomposition of the considering periodic sail, each segment of the graph corresponds to some segment of the torus decomposition, and each vertex of the graph corresponds to some vertex of the torus decomposition. The integer affine types of the corresponding faces are given below. Let us find all integer invariants mentioned above.

The integer areas of this two triangles are equal to 1.

The integer lengths of all three segments are equal to 1.

All six integer angles are equal to 1.

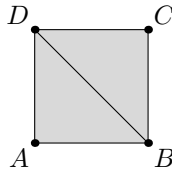All two triangles have integer affine types of the simplest triangle:

The integer distance from the triangle ABD to the origin is equal to 2, and from the triangle BCD is equal to 1.

This torus decomposition was constructed by E. I. Korkina [16], G. Lachaud [19] and A. D. Bryuno and V. I. Parusnikov [5] approximately at the same time.

Example 2.2. The second example was given by A. D. Bryuno and V. I. Parusnikov [5]. They construct the continued fraction that associated to the following operator:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The torus decomposition corresponding to this operator is also homeomorphic to the following one:

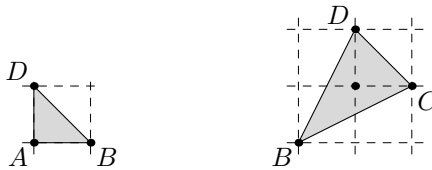Here the segment AB is identified with the segment DC and the segment AD to the segment BC.

The integer area of the triangle ABD is equal to 1 and the integer area of the triangle BCD is equal to 3.

The integer lengths of all three segments are equal to 1.

All three integer angles of the triangle ABD are equal to 1. All three integer angles of the triangle BCD are equal to 3.

The integer distance from the triangle ABD to the origin is equal to 2, and from the triangle BCD is equal to 1.

The triangle ABD has an integer affine type of the simplest triangle, The triangle BCD has an integer affine type of the triangle with the vertices $(-1,-1)$, $(0,1)$, and $(1,0)$.
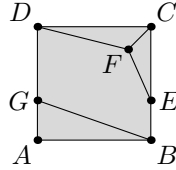
Example 2.3. The third example was given by V. I. Parusnikov [24]. This continued fraction is associated to the following operator:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -3 \end{pmatrix}.$$

The torus decomposition corresponding to this operator is also homeomorphic to the following one:

Here the segment AB is identified with the segment DC and the polygonal line AGD — to the polygonal line BEC (the point G is identified with the point E).
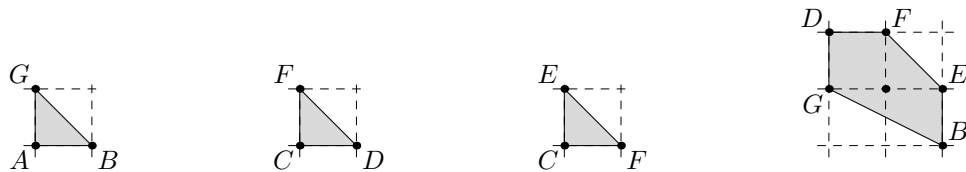
The integer areas of all triangles are equal 1 and the integer area of the pentagon $BEFDG$ is equal to 5.

The integer lengths of all three segments are equal to 1.

All integer angles of all triangle are equal to 1. The angle in the vertex D is equal to 1 and in other four vertices the angles are equal to 2.

The integer distance from the triangle ABG and the pentagon $BEFDG$ to the origin is equal to 1, from the triangle CDF to the origin is equal to 2, and from the triangle CFE is equal to 3.

All triangles have an integer affine type of the simplest triangle, The pentagon $BEFDG$ has an integer affine type of the pentagon with the vertices $(-1,0)$, $(-1,1)$, $(0,1)$ $(1,0)$ and $(1,-1)$:

Some other single torus decomposition for some two-dimensional continued fractions was investigated in the works [25] and [26].

The continued fractions constructed in the examples 2.1, 2.2 and 2.3 are also known as the continued fractions corresponding to the first, the second and the third Davenport form.

The first steps in classification of continued fractions with a small norm based on these three examples were made by the author of this issue in the work [12]. Let us now consider the following norm for the space of matrices: the sum of absolute values of all the coefficients for the matrix.

Theorem 2.1. *There is no hyperbolic matrix among the set of matrices with integer coefficients and irrational eigenvalues, and hence there is no continued fraction associated to the operator with such norm.*

If the norm of the hyperbolic matrix with integer coefficients and irrational eigenvectors is equal to five (their number is 48), then the corresponding continued fraction is equivalent to the generalization of the ordinary fraction for the golden ratio. This fraction is shown in Example 2.1.

The amount of such matrices with norm equals six is 912, and only three different two-dimensional continued fractions associated to them: 480 continued fractions are equivalent to the fraction of Example 2.1, 240 continued fractions are equivalent to the fraction of Example 2.3 and 192 — to the fraction of Example 2.2 (for more information see [12]).

The classification of two-dimensional continued fractions with the norm equals seven or more is unknown.
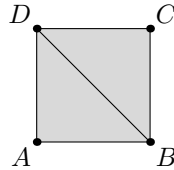
3. Infinite sequences of torus decomposition examples.

Now we continue the description of examples with the infinite sequences of the torus decompositions. The first two infinite sequences of torus decompositions was calculated by E. I. Korkina in the work [16]. One of this sequences is shown below.

Example 3.1. This sequence of continued fractions is associated to the following operators for $m \geq 0$:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -m-5 \\ 0 & 1 & m+6 \end{pmatrix}.$$

The torus decomposition corresponding to this operator is homeomorphic to the following one:

Here the segment AB is identified with the segment DC and the segment AD to the segment BC.

The integer areas of this two triangles are equal to 1.

The integer lengths of all three segments are equal to 1.

All six integer angles are equal to 1.

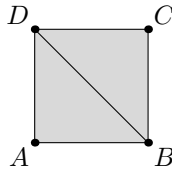All two triangles have integer affine types of the simplest triangle:

The integer distance from the triangle ABD to the origin is equal to $m+2$, and from the triangle BCD is equal to 1.

Many other examples of infinite sequence of continued fractions were done by the author of this issue in the work [11]. Now we show some of them. The following sequence generalize the one from the previous example.

Example 3.2. This sequence depends on two integer parameters $a, b \geq 0$. The continued fractions of the sequence are associated to the following operators:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1+a-b & -(a+2)(b+1) \end{pmatrix}.$$

The torus decomposition corresponding to this operator is homeomorphic to the following one:

Here the segment AB is identified with the segment DC and the segment AD to the segment BC.

The integer areas of this two triangles are equal to $b+1$.

The integer lengths of the segment BD is equal to $b+1$ and the integer of all others is equal to 1.

The integer angles DAB and DCB are equal to $b+1$, and the others — to 1.

All two triangles have the same integer affine type of the triangle with the vertices $(0,0)$, $(0,1)$ and $b+1, 0$ $1, 0$ (b=5 at the picture):

The integer distance from the triangle ABD to the origin is equal to $a+2$, and from the triangle BCD is equal to 1.

Remark. If we pose $b = 0$, then we will have the sequence of operators that have equivalent continued fractions with the sequence of operators of Example 3.1.

Statement 3.1. *The continued fractions are associated to the following operators are equivalent (for integer $t \geq 0$):*

$$A_t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1+t & -t-2 \end{pmatrix} \qquad M_t = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -t-5 \\ 0 & 1 & t+6 \end{pmatrix}.$$

The proof of this statement is straightforward. It bases on the fact that the operators $(I - A_t)^{-1}$ and M_t are conjugate by the operator X in the group $SL(3, \mathbb{Z})$ (here I — is
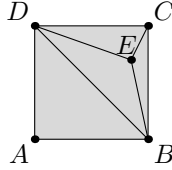
the identity operator):

$$X = \begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}.$$

(i.e. $M_t = X^{-1}(I - A_t)^{-1}X$).

Example 3.3. The sequence of this example depends on an integer parameters $a \geq 1$. The continued fractions of the sequence are associated to the following operators:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a & -2a - 3 \end{pmatrix}.$$

The torus decomposition corresponding to this operator is homeomorphic to the following one:

Here the segment AB is identified with the segment DC and the segment AD to the segment BC.

The integer areas, integer lengths and integer angles of all four triangles are equal to 1 here.

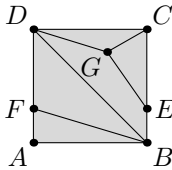All two triangles have integer affine types of the simplest triangle:

The integer distance from the triangle ABD to the origin is equal to $a+2$, from BDE — to $a + 1$ and from BCE and CED — to 1.

Example 3.4. The last sequence of this example that we will show in the issue depends on two integer parameters $a > 0$ and $b \geq 0$. The continued fractions of the sequence are associated to following operators:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & (a+2)(b+2) - 3 & 3 - (a+2)(b+3) \end{pmatrix}.$$

The torus decomposition corresponding to this operator is homeomorphic to the following one:
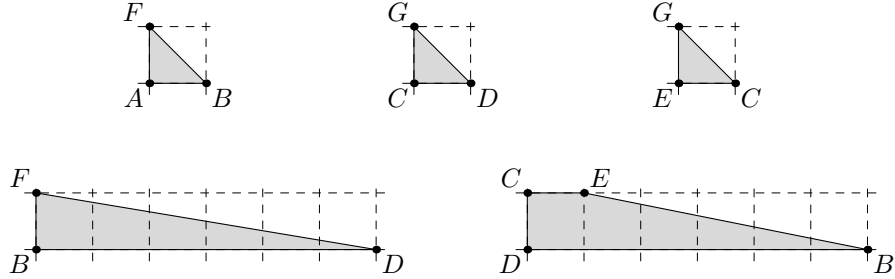
Here the segment AB is identified with the segment DC and the polygonal line AFD —
to the polygonal line BEC (the point F is identified with the point E).

The integer length of the segment DB is equal to $b+2$ and all other integer lengths
to 1.

The integer areas of the quadrangle $BECD$ is equal to $b+3$. The integer areas of the
triangle BDF is equal to $b+2$, and the integer lengths for all other triangles are equal
to 1.

The integer angle DFB is equal to $b+2$, All other integer angles are equal to 1.

The triangles ABF, CGE and CDG have an integer affine type of the simplest triangle;
the triangle BDF has an integer affine type of the triangle with the vertices $(0,0)$, $(b+2,0)$
and $(0,1)$; the quadrangle $DBEC$ has an integer affine type of the quadrangle with the
vertices $(0,0)$, $(b+2,0)$, $(1,1)$ and $(0,1)$:

The integer distance from the triangles ABF and BFD to the origin is equal to 1, from
the triangle CDG — to $2+2a+2b+ab$, from the triangle CEG — to $3+2a+2b+ab$,
from the quadrangle $DBEC$ — to 1.

For more examples of such sequences and some method of finding new such sequences,
see [11].

4. Some problems and conjectures.

In the last section we formulate some of the most actual geometrical problems and
conjectures on Klein multidimensional continued fractions. We start with the problem of
the recognizing the sails by their characteristics.

Problem 1. (Arnold) *Find the collection of integer characteristics of the sail that uniqu-
ely up to the equivalence relation determines the two-dimensional (n-dimensional) contin-
ued fraction.*

The same question is very important also for the periodic two-dimensional sails of cubic
irrationalities (and for dimension $n>2$).

For the one-dimensional case these characteristics are known. Suppose we know the
number of ordered sequences of integer lengths and integer angles for one of the sails
of the one-dimensional continued fraction and their correspondence to each other. This
means that for any two segments with the common vertex we know the integer lengths of
this segments and the corresponding integer angle between these edges. The whole sail

and even the whole continued fraction is constructed by these characteristics (up to the equivalence relation) in the unique way. Moreover if we are given some arbitrary collection of this characteristics then there exist the sail with this characteristics. In the case of the periodic one-dimensional continued fraction it is enough to know only the part of these two corresponding to each other sequences of integer characteristics — the integer lengths and angles for some fundamental region (i.e. the decomposition of the circle).

There is no answer to this question in the two-dimensional case. It is conjectured the following.
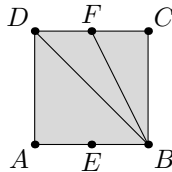
Conjecture 1. *To construct the whole continued fraction up to equivalence relation it is sufficient to know only the following characteristics for one of the sail: the homeomorphic types of all facets for one of the sails of the fraction, their adjacency to each other; all their integer volumes and integer distances to them; integer lengths of all edges; and all integer angles for any two adjacent edges.*

The whole classification of torus decompositions than are obtained by some continued fraction of cubic irrationality is unknown. Here we formulate an important problem on the structure of torus decomposition for the sails.

Problem 2. (Arnold) *Describe all torus decompositions that are possible for some periodic two-dimensional continued fraction.*

Only a few is known in this direction. There is a lot of trivial examples for torus decompositions that do not corresponds to some sails. The following example was constructed by the author in [11]:

Example 4.1. The following torus decomposition does not correspond to any sail for the periodic continued fraction of cubic irrationality (there is one integer point in the interval AD, we denote it by E):
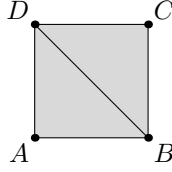

Here the polygonal line AEB maps to the polygonal line DFC under one of the operators of the group $SL(3, \mathbb{Z})$ that preserves the sail. As far AEB is a segment, DFC is also a segment. Therefore the points B, C, F and D lie in the same plane. And hence BF is not a edge of some face.

Now we present the first nontrivial example of torus decomposition that was made by E. I. Korkina in the work [16].

Example 4.2. Consider the simplest torus triangulation. It consists of two triangles with the simplest integer affine type of the triangle $(0, 0)$, $(0, 1)$ and $(1, 0)$. The integer distances to all of these faces is equal to 1. This decomposition does not correspond to

any sail for the periodic continued fraction of cubic irrationality:

This example as far as a large number of sail calculations allow us to formulate the following conjecture.
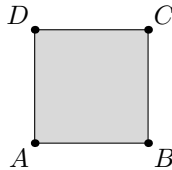
Conjecture 2. *For any torus decomposition for the continued fraction of the cubic irrationality there exists some face with integer distance to the origin more than one.*

On the other hand each of the calculated sails have some face with the integer distance to the origin equals one.

Conjecture 3. *For any torus decomposition for the continued fraction of the cubic irrationality there exists some face with integer distance to the origin equals one.*

The following example is on the simplest torus decomposition, that consist of one parallelogram. The example was constructed by the author of this issue.

Example 4.3. Consider the torus decomposition consisting of the only one face with integer affine type of the simplest parallelogram with the vertices $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$. The integer distances to all of the faces is equal to some natural number n. This decomposition does not correspond to any sail for the periodic continued fraction of cubic irrationality:

It seems that the torus decomposition with one rectangular face is not possible for the decompositions associated to the two-dimensional continued fractions of cubic irrationalities. Moreover we conjecture the following.

Conjecture 4. *For any torus decomposition for the continued fraction of the cubic irrationality there exists the face that has an integer affine type of some triangle.*

Let us describe the relation between two-dimensional continued fractions of cubic irrationalities and cubic extensions of the field of rational numbers. For any such two-dimensional continued fraction there exist an operator in $SL(3, \mathbb{Z})$ with irreducible characteristic polynomial over the field $\mathbb{Q}$ such that the continued fraction is associated to the operator. This polynomial determines the cubic extension of the field of rational numbers. All integer operators that identifies the continued fraction are all commutes with A and hence have the same cubic extension. However the converse is not true (the example is from the work [11]).

Example 4.4. The following two operators having the same characteristic polynomial $x^3 + 11x^2 - 4x - 1$ (and hence the same cubic extension) define nonequivalent continued fractions:

$$\left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -2 \end{array} \right)^3, \qquad \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 4 & -11 \end{array} \right).$$

So, the natural question here is the following one.

Problem 3. (Arnold) *Classify the continued fractions that corresponds to the same cubic extensions of the field of rational numbers.*

It is known almost nothing here. For example, it is even unknown the finiteness of the number of possible continued fractions associated to the same extension. (For properties of cubic extensions of rational numbers see in the work of B. N. Delone and D. K. Faddeev [8].)

In conclusion we show the table that was taken from my work [12] with squares filled with torus decomposition of the sails for continued fractions associated to the family of Sylvester operators
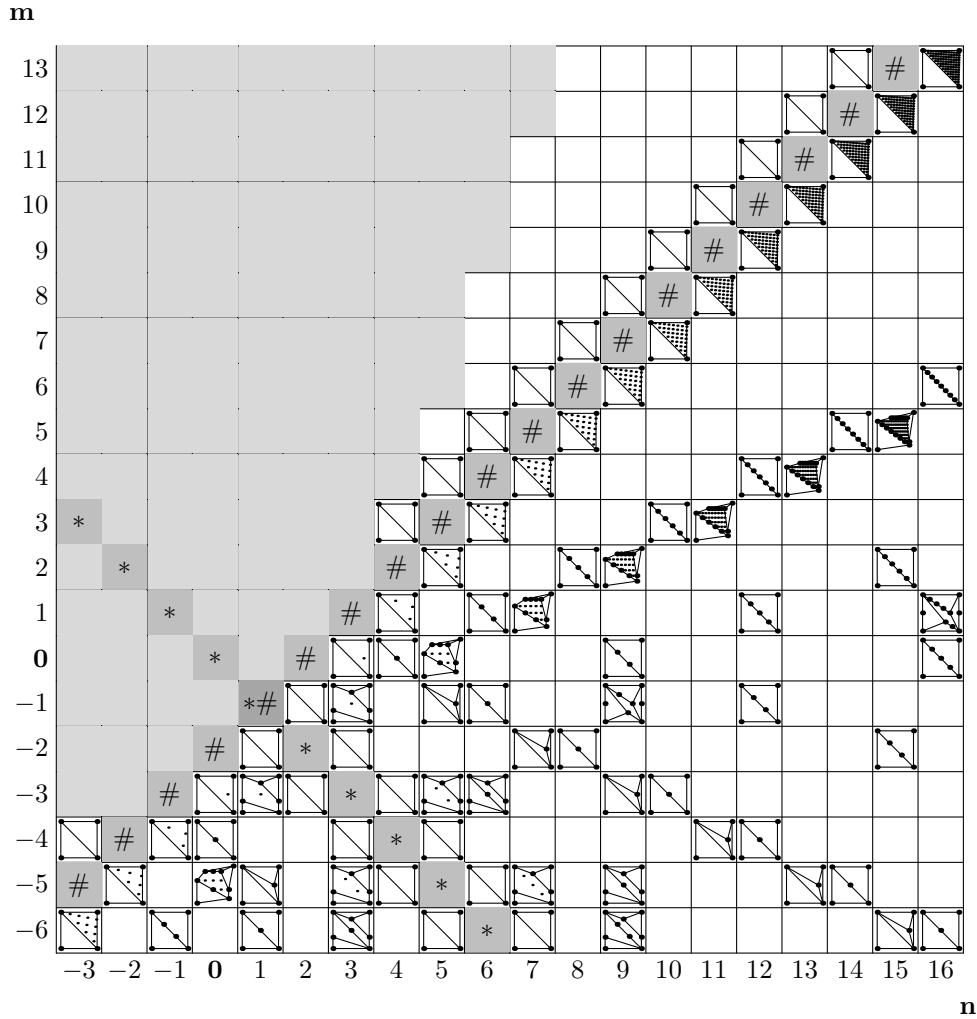
$$A_{m,n} := \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -m & -n \end{array} \right),$$

see fig 1. Some aspects of torus decomposition for the sail of the two-dimensional continued fraction for the cubic irrationality, associated to the operator $A_{m,n}$ is shown in the square sited at the intersection of the string with number n and the column with number m. If one of the roots of characteristic polynomial for the operator is equal to 1 or -1 at that than the square (m, n) is marked with the sign $*$ or $\#$ correspondingly. The squares that correspond to the operators which characteristic polynomial has two complex conjugate roots are painted over with light gray color.

References

[1] V. I. Arnold, *A-graded algebras and continued fractions*, Commun. Pure Appl. Math., 142(1989), pp. 993-1000.

[2] V. I. Arnold, *Continued fractions*, M.: MCCME, (2002).

[3] Z. I. Borevich, I. R. Shafarevich, *Number theory*, 3 ed, M., (1985).

[4] K. Briggs, *Klein polyhedra*, http://www.btexact.com/people/briggsk2/klein-polyhedra.html (2002), http://members.lycos.co.uk/keithmbriggs/klein-polyhedra.html (2002).

[5] A. D. Bryuno, V. I. Parusnikov, *Klein polyhedrals for two cubic Davenport forms*, Mathematical notes, 56(4), (1994), 9-27.

[6] A. D. Bryuno, V. I. Parusnikov, *Comparison of various generalizations of continud fractions*, Mathematical notes, 61, (1997), 278-286.

[7] A. D. Bryuno, V. I. Parusnikov, *To the generalisations of continued fraction*, preprint 137 of Keldysh Institute of the RAS, Moscow, (2004).

[8] B. N. Delone, D. K. Faddeev, *The third power irrationalities theory*, M.-L.: Ac. Sci. USSR, (1940).

[9] A. Ya. Hinchin, *Continued fractions*, M.: FISMATGIS, (1961).

[10] O. N. German, *Sails and Hilbert Bases*, Proc. of Steklov Ins. Math, v. 239(2002), pp. 88-95.

FIGURE 1. Torus decompositions for operators $A_{m,n}$.

[11] O. Karpenkov, *On tori decompositions associated with two-dimensional continued fractions of cubic irrationalities*, to appear in Func. An. Appl., (2004).

[12] O. Karpenkov, *On two-dimensional continued fractions for integer hyperbolic matrices with small norm*, to appear in Uspekhi mat. nauk, (2004).

[13] F. Klein, *Ueber einegeometrische Auffassung der gewöhnliche Kettenbruchentwicklung*, Nachr. Ges. Wiss. Göttingen Math-Phys. Kl., 3, (1895), 357-359.

[14] F. Klein, *Sur une représentation géométrique de développement en fraction continue ordinaire*, Nouv. Ann. Math. 15(3), (1896), pp. 327–331.

[15] E. I.. Korkina, *La périodicité des fractions continues multidimensionelles*, C. R. Ac. Sci. Paris, v. 319(1994), pp. 777–780.

[16] E. I. Korkina, *Two-dimensional continued fractions. The simplest examples*, Proceedings of V. A. Steklov Math. Ins., v. 209(1995), pp. 143–166.

[17] E. I. Korkina, *The simplest 2-dimensional continued fraction.*, J. Math. Sci., 82(5), (1996), pp. 3680–3685.

[18] G. Lachaud, *Le théorème de Lagrange pour less irrationels de degáuelconque*, preprint, Laboratoire de Mathématiques Discrètes du C.N.R.S., Luminy (1992).

[19] G. Lachaud, *Polyèdre d'Arnold et voile d'un cône simplicial: analogues du thèoreme de Lagrange pour less irrationels de degeáuelconque*, preprint, Laboratoire de Mathématiques Discrètes du C.N.R.S., Luminy (1993).

[20] G. Lachaud, *Sailes and Klein Polyhedra*, Contemp. Math., v. 210(1998), pp 373-385.

[21] G. Lachaud, *Polyèdre d'Arnold et voile d'un cône simplicial: analogues du thèoreme de Lagrange*, C. R. Acad. Sci. Paris, v. 317(1993), pp. 711-716.

[22] H. Minkowski, *Généralisation de le théorie des fractions continues*, Ann. Sci. Ec. Norm. Super. ser III, v. 13(1896), pp. 41-60.

[23] J.-O. Moussafir, *Sales and Hilbert bases*, Func. an. and appl., v. 34(2000), n. 2, pp. 43-49.

[24] V.I. Parusnikov, *Klein's polyhedra for the third extremal ternary cubic form*, preprint 137 of Keldysh Institute of the RAS, Moscow, (1995).

[25] V.I. Parusnikov, *Klein's polyhedra for the fourth extremal cubic form*, Mat. Zametki, 67(1), (2000), 110-128.

[26] V.I. Parusnikov, *Klein's polyhedra for the fifth extremal cubic form*, preprint 69 of Keldysh Institute of the RAS, Moscow, (1998).

[27] H. Tsuchihashi, *Higher dimensional analogues of periodic continued fractions and cusp singularities*, Tohoku Math. Journ. v. 35(1983), pp. 176–193.

[28] G. F. Voronoi, *On one generalization of continued fraction algorithm*, USSR Ac. Sci., v.1(1952), pp.197-391.

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