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APPROXIMATING REALS BY RATIONALS OF THE FORM a/b^2.

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Abstract. In this note we formulate some questions in the study of approximations of reals by rationals of the form a/b^2 arising in theory of Shrödinger equations. We hope to attract attention of specialists to this natural subject of number theory.

1. Introduction

Some background. In this note we formulate some questions in the study of approximations of reals by rationals of the form a/b^2 arising in theory of Shrödinger equations (see [3] and [5] for further information). We hope to attract attention of specialists to this natural subject of number theory. Good references to theory of approximations by arbitrary algebraic numbers are for instance [11], [8] and [9], especially for approximations by quadratic irrationals, see [4]. A metric approach to the study (in a more general situation) was proposed by [7] and further developed by M. Weber in [10], V. Beresnevich, M. Dodson, S. Kristensen, and J. Levesley in [1] and other works. This approach is a good test of the proposed problems, nevertheless it does not give the answers. Some upper bound estimates were made by A. Zaharescu in [12].

2. Questions

We start with formulation of one of the main results in classical approximation theory of reals by rationals (see [6] for the proofs).

Theorem 1. I. *For any reals α and $c \geq 1/\sqrt{5}$ there exists an infinite number of integer solutions (a,b), $b>0$ for the following inequality*

$$\left|\alpha - \frac{a}{b}\right| < \frac{c}{b^2}.$$

II. *Let α be the Golden Ratio (i. e. $\alpha = (\sqrt{5}+1)/2$). Then for any $c < 1/\sqrt{5}$ the inequality of item I. has only finitely many solutions.* □

Similar results for the approximations by rationals of the form a/b^2 are not known. One of the reasons of that is the following: lattice geometry of continued fractions corresponding to the approximations by rationals can not be naturally expanded to the case of rationals a/b^2.

Problem 1. Find a good generalization for geometry of numbers to the case of a/b^2-approximations.

Let us give some known estimates for the case of a/b^2-approximations. The lower estimate seems to be quite precise.

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Theorem 2. *For any positive ε and for any real α there exist a positive constant $c = c(\varepsilon)$, such that the following inequality does not have integer solutions:*

$$\left|\alpha - \frac{a}{b^2}\right| < \frac{c}{b^3 \ln^{1+\varepsilon}(b)}.$$

The proof of a more general statement is given by I. Borosh and A. S. Fraenkel in [2]. We suppose that the logarithm in the formula can be eliminated.

All known proofs of previous theorem are general, and do not give the examplpes of badly approximable reals, like it was $(\sqrt{5}+1)/2$ for the case of approximations by a/b. So the following problem is actual here.

Problem 2. Find any particular example of α that satisfies the condition of Theorem 2.

The following estimate for the upper bound case is known.

Theorem 3. A. Zaharescu [12]. *For any real α and any positive $\theta < 2/3$ there exists infinitely many solutions of the following inequality:*

$$\left|\alpha - \frac{a}{b^2}\right| < \frac{1}{b^{2+\theta}\xi(b)}.$$

As one can see there is a gap between upper and lover estimates for the baddly approximable reals by a/b^2-rationals. The results [2] for almost all reals and numerical experiments support the following classical conjecture.

Conjecture 3. For any real α there exists a constant $c(\alpha)$ such that the inequality:

$$\left|\alpha - \frac{a}{b^2}\right| < \frac{c(\alpha)}{b^3}$$

has infinitely many integer solutions (a, b).

We conclude this note with the following problem wich is supposed to be close to the subject.

Problem 4. Find the estimates for the upper and lower bounds of the "best" approximations of reals by rationals of the form z/p, where z is integer, and p is prime or unity.

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