LINEAR RANDOM VIBRATION OF STRUCTURAL SYSTEMS WITH SINGULAR MASS MATRICES

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ABSTRACT

A framework is developed for determining the stochastic response of linear multi-degree-of-freedom (MDOF) structural systems with singular mass matrices, potentially arising when utilizing more than the minimum number of coordinates in modeling the system. Specifically, relying on the generalized matrix inverse theory and on the Moore-Penrose (M-P) matrix inverse, standard concepts, relationships and equations of the linear random vibration theory are extended and generalized herein to account for systems with singular mass matrices. In this regard, adopting a state-variable formulation, equations governing the system response mean vector and covariance matrix are formed and solved. Further, it is shown that a complex modal analysis treatment, unlike the standard system modeling case, does not lead to decoupling of the equations of motion. Nevertheless, relying on a singular value decomposition of the system transition matrix facilitates significantly the efficient computation of the impulse response matrix, and ultimately, of the system response statistics. A linear structural system with a singular mass is considered as a numerical example for demonstrating the applicability of the methodology and for elucidating certain related numerical aspects.

Keywords: Structural dynamics, Random vibration, Singular mass matrix, Moore-Penrose inverse

INTRODUCTION

Inherent randomness in a wide range of complex and time-evolving natural phenomena has motivated the modeling and study of systems with stochastic parameters, input, and initial/boundary conditions (e.g. Grigoriu (2002)). Further, to quantify the uncertain behavior of complex dynamical structural and mechanical systems, several random vibration methodologies have been developed over the past six decades with varying degree of success; see Lin (1967), Naess and Johnsen (1993), Newland (1993), Li and Chen (2009), Pirrotta and Santoro (2011), and Kougioumtzoglou and Spanos (2014) for some indicative references.

Typically, in the field of multibody system dynamics the smallest possible number of coordinates is utilized for various reasons such computational efficiency. Indeed, it can be argued that forming the multibody system equations of motion in terms of the independent degrees of freedom can ideally increase computational performance (e.g. Featherstone (1987); Bae and Haug (1987); Critchley and Anderson (2003)). Further, utilizing the minimum number of coordinates

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(generalized coordinates) for formulating the equations of motion of a multi-degree-of-freedom (MDOF) dynamical structural system yields mass matrices that are not only non-singular, but also symmetric and positive definite (e.g. Pars (1979); Roberts and Spanos (2003)).

However, it can be argued (e.g. Schutte and Udwadia (2011)) that formulating the equations of motion for general large-scale multibody systems can be a non-trivial task where the complexity of the equations of motion grows rapidly with increasing the number of constituent bodies and/or the number of DOFs. In fact, the degree of simplicity and the amount of effort required for deriving the governing equations are among the factors for assessing the performance of a methodology for obtaining the system equations of motion (e.g. Schiehlen (1984)). In this regard, several approaches for generating the equations of motion (e.g. Shabana (1998)), such as the ones relying on the computation of Lagrange multipliers (e.g. Nikravesh (1988); Pradhan et al. (1997)), require the application of constraints that are functionally independent. Clearly, verifying the above requirement is not a straightforward task, especially for large-scale complex systems. Further, employing the minimum number of coordinates can lead to limited flexibility regarding the form and nature of the constraints. Specifically, altering a constraint might require a complete remodeling of the multibody system.

Thus, it can be argued that modeling utilizing more than the minimum number of DOFs circumvents some of the above limitations and provides the modeler with enhanced flexibility (e.g. Schutte and Udwadia (2011)). In this regard, any complex multibody system can be decomposed into its constituent parts for each of which the equations of motion can be obtained readily (e.g. Udwadia and Kalaba (2007)). These equations can then be used to form the composite system equations of motion in a less labor-intensive manner by also incorporating appropriate constraints. Note in passing that some of the structural systems considered herein are related to the so-called descriptor systems described, in general, by a set of differential-algebraic equations (e.g. Gashi and Pantelous (2013), Pantelous et al. (2014), Kalogeropoulos et al. (2014) and Gashi and Pantelous (2015)). However, due to the fact that the coordinates used are not independent with each other, a singular mass matrix can arise in the system equations of motion. Thus, determining the system response by employing standard concepts/techniques, such as recasting the equations of motion in a state-variable formulation, is not possible.

In this paper, some theoretical and practical elements pertaining to linear random vibration theory of MDOF systems with singular mass matrices are developed and discussed. Specifically, based on the generalized matrix inverse theory (e.g. Ben-Israel and Greville (2003)), the Moore-Penrose (M-P) inverse of a singular mass matrix can be determined and arguably uniquely defined for systems of engineering interest; see Udwadia and Phohomsiri (2006). Further, it is shown that relying on the M-P inverse of a matrix, the Lyapunov equation for the system response covariance matrix can be formed and solved for systems with singular mass matrices. Also, it is shown that a complex modal analysis treatment, unlike the standard system modeling case, does not lead to decoupling of the equations of motion. Nevertheless, relying on a singular value decomposition of the system transition matrix greatly facilitates the efficient computation of the impulse response matrix and of the system response statistics. A linear structural system modeled by utilizing more than the minimum number of coordinates (thus, yielding a singular mass matrix) is considered as a numerical example.

MATHEMATICAL FORMULATION

Moore-Penrose theory elements

The topic of generalized matrix inverses has received considerable interest in recent years both from a theoretical and a practical perspective; see Campbell and Meyer (1979) for a detailed presentation. In this subsection, the basic mathematical elements are discussed briefly.

Definition. *If* $\boldsymbol{A} \in C^{m\times n}$ *then* $\boldsymbol{A}^+$ *is the unique matrix in* $C^{m\times n}$ *such that*

$$\begin{aligned} \boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A} \quad &, \quad \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+, \\ (\boldsymbol{A}\boldsymbol{A}^+)^* = \boldsymbol{A}\boldsymbol{A}^+ \quad &, \quad (\boldsymbol{A}^+\boldsymbol{A})^* = \boldsymbol{A}^+\boldsymbol{A}. \end{aligned} \tag{1}$$

The matrix $\boldsymbol{A}^+$ is known in the literature as the Moore-Penrose inverse of $\boldsymbol{A}$. The Moore-Penrose inverse of a square matrix exists for any arbitrary $\boldsymbol{A} \in C^{n\times n}$. Note that if $\boldsymbol{A}$ is non-singular, $\boldsymbol{A}^+$ coincides with $\boldsymbol{A}^{-1}$. Eqs. (1) represent the so-called Moore-Penrose equations. Further, the Moore-Penrose inverse of any $m \times n$ matrix $\boldsymbol{A}$ can be determined by employing various techniques and methodologies, such as a number of recursive formulae (e.g. see Campbell and Meyer (1979), Greville (1960)) and provides a tool for solving equations of the form

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{2}$$

where $\boldsymbol{A}$ is a rectangular $m \times n$ matrix, $\boldsymbol{x}$ is an n vector and $\boldsymbol{b}$ is an m vector. For a square matrix $\boldsymbol{A}$, the Moore-Penrose inverse can be applied when $\boldsymbol{A}$ is singular, i.e. $det\boldsymbol{A} \neq 0$. For such cases, the solution to eq. (2) is given by

$$\boldsymbol{x} = \boldsymbol{A}^+\boldsymbol{b} + (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{y}, \tag{3}$$

where $\boldsymbol{y}$ is an arbitrary m vector. A more detailed presentation of the topic can be found in Ben-Israel and Greville (2003) and Campbell and Meyer (1979).

Stochastic response analysis of multi-degree-of-freedom (MDOF) structural systems

Standard state-variable formulation

Following closely Roberts and Spanos (2003), the general form of the equations of motion of a lumped-parameter $n-$degree-of-freedom ($n-$DOF) system is

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{Q}(t), \tag{4}$$

where $\boldsymbol{M}, \boldsymbol{C}, \boldsymbol{K}$ are symmetric $n \times n$ matrices, representing the mass, the damping and the stiffness of the system, respectively. The symbol $\boldsymbol{q}$ stands for an n vector containing the n (generalized) displacements of the system, and $\boldsymbol{Q}$ is an n vector containing the n (generalized) forces, corresponding to $\boldsymbol{q}$.

Further, the equations of motion for the $n-$DOF system of eq. (4) can be cast into the state variable form by defining a $2n$ vector,

$$\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix}. \tag{5}$$

Next, taking into account eq. (5), eq. (4) can be written, equivalently, in the form

$$\dot{\boldsymbol{z}} = \boldsymbol{G}\boldsymbol{z} + \boldsymbol{f}, \tag{6}$$

where

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix}, \tag{7}$$

and

$$\boldsymbol{f} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{Q} \end{bmatrix}. \tag{8}$$

Note that in deriving eqs. (7) and (8) it is assumed that the mass matrix $\boldsymbol{M}$ is non-singular, since systems with singular mass matrices are not common in a standard formulation of the system equations of motion in classical dynamics. In fact, when the minimum number of independent coordinates is utilized for formulating the equations of motion, the mass matrices are not only non-singular, but also symmetric and positive definite; see also Pars (1979) and Roberts and Spanos (2003).

Further, the response of the system of eq. (6) can be determined by utilizing the convolution integral

$$\boldsymbol{q}(t) = \int_0^t \boldsymbol{h}(\tau)\boldsymbol{Q}(t-\tau)d\tau, \tag{9}$$

where $\boldsymbol{h}(t)$ represents the impulse response matrix of the system, given by

$$\boldsymbol{h}(t) = \boldsymbol{b}(t)\boldsymbol{M}^{-1}. \tag{10}$$

In eq. (10), $\boldsymbol{b}(t)$ is obtained by the relationship

$$exp(\boldsymbol{G}t) = \begin{bmatrix} \boldsymbol{a}(t) & \boldsymbol{b}(t) \\ \boldsymbol{c}(t) & \boldsymbol{d}(t) \end{bmatrix}, \tag{11}$$

where all the submatrices are $n \times n$; see Roberts and Spanos (2003) for a more detailed presentation.

Furthermore, statistical moments of the response of the linear MDOF system of equation (4) can be determined readily by direct manipulation of the equation of motion (6). In this regard, by denoting

$$\boldsymbol{m_z} = \boldsymbol{E}\left\{\boldsymbol{z}(t)\right\}, \tag{12}$$

and taking expectations on eq. (6), yields

$$\dot{\boldsymbol{m}}_{\boldsymbol{z}} = \boldsymbol{G}\boldsymbol{m_z} + \boldsymbol{m_f}. \tag{13}$$

The last equation, can be solved to find $\boldsymbol{m_z}$ as a function of time. For a zero-mean excitation, the solution for $\boldsymbol{m_z}$ is given in the form

$$\boldsymbol{m_z} = exp(\boldsymbol{G}t)\boldsymbol{m_z}(0), \tag{14}$$

where $\boldsymbol{m_z} \to 0$, as $t \to \infty$. Next, considering eq. (6) and (13), the equation

$$\dot{\boldsymbol{\lambda}} = \boldsymbol{G}\boldsymbol{\lambda} + \boldsymbol{\eta}(t), \tag{15}$$

is obtained, where

$$\boldsymbol{\lambda}(t) = \boldsymbol{z}(t) - \boldsymbol{m_z}(t) \ \text{ and } \ \boldsymbol{\eta}(t) = \boldsymbol{f}(t) - \boldsymbol{m_f}(t). \tag{16}$$

Further, taking into account the covariance matrix

$$\boldsymbol{V} = \boldsymbol{E}\left\{[\boldsymbol{z}(t) - \boldsymbol{m}_{\boldsymbol{z}}(t)][\boldsymbol{z}(t) - \boldsymbol{m}_{\boldsymbol{z}}(t)]^T\right\}, \tag{17}$$

and considering eqs. (15) to (17), equation

$$\dot{\boldsymbol{V}} = \boldsymbol{G}\boldsymbol{V}^T + \boldsymbol{V}\boldsymbol{G}^T + \boldsymbol{S} \tag{18}$$

is obtained, where

$$\boldsymbol{S}(t) = \int_0^t exp(\boldsymbol{G}(t-\tau))\left[\boldsymbol{w}_{\boldsymbol{\eta}}(t,\tau) + \boldsymbol{w}_{\boldsymbol{\eta}}^T(t,\tau)\right] d\tau. \tag{19}$$

In eq. (19), $\boldsymbol{w}_{\boldsymbol{\eta}}(t,\tau)$ is the covariance matrix for $\boldsymbol{\eta}(t)$. If the elements of $\boldsymbol{\eta}(t)$ are modeled as stationary white-noises, then

$$\boldsymbol{w}_{\boldsymbol{\eta}}(t,\tau) = \boldsymbol{D}\delta(t-\tau), \tag{20}$$

where $\boldsymbol{D}$ is a real, symmetric, non-negative matrix of constants. Hence, substituting eq. (20) into eq. (19), eq. (18) becomes

$$\dot{\boldsymbol{V}} = \boldsymbol{G}\boldsymbol{V}^T + \boldsymbol{V}\boldsymbol{G}^T + \boldsymbol{D}. \tag{21}$$

State-variable formulation based on the Moore-Penrose theory

An inherent assumption for all the equations obtained in the standard state-variable formulation section, is that the mass matrix $\boldsymbol{M}$ is non-singular. Nevertheless, note that singular mass matrices can arise in the system equations of motion. This is the case when more than the minimum number of generalized coordinates are considered. This kind of modeling can be advantageous in cases of complex, multi-body systems. In this regard, the complex multi-body system can be decomposed into its constituent parts for each of which the equations of motion can be readily obtained. These equations can then be used to form the equations of motion of the overall composite system in a less labor-intensive manner. This is done by also incorporating constraints associated with the fact that the coordinates are not independent any more; thus, yielding mass-matrices which are singular; see also Udwadia and Kalaba (2001), Udwadia and Phohomsiri (2006) and references therein. In this regard, following the standard state-variable formulation section, consider an $l-$DOF system of the form

$$\boldsymbol{M}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} + \boldsymbol{C}_{\boldsymbol{x}}\dot{\boldsymbol{x}} + \boldsymbol{K}_{\boldsymbol{x}}\boldsymbol{x} = \boldsymbol{Q}_{\boldsymbol{x}}(t), \tag{22}$$

where $\boldsymbol{x}$ is the l vector of the coordinates ($l \geqslant n$), $\boldsymbol{Q}_{\boldsymbol{x}}$ is the l vector of external forces, and $\boldsymbol{M}_{\boldsymbol{x}}, \boldsymbol{C}_{\boldsymbol{x}}$ and $\boldsymbol{K}_{\boldsymbol{x}}$ are the mass, damping and stiffness matrices, respectively, corresponding to the system of eq. (22). Next, consider the case where the system of eq. (22) is subjected to m constraints of the form

$$\boldsymbol{A}(\boldsymbol{x},\dot{\boldsymbol{x}},t)\ddot{\boldsymbol{x}} = \boldsymbol{b}(\boldsymbol{x},\dot{\boldsymbol{x}},t), \tag{23}$$

where $\boldsymbol{A}$ is an $m \times l$ matrix and $\boldsymbol{b}$ is an m vector.

In general, while the unconstrained system becomes constrained, additional forces arise to ensure that the constraints are satisfied (see also Udwadia and Phohomsiri (2006)). Therefore, eq. (22) becomes

$$\boldsymbol{M}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} + \boldsymbol{C}_{\boldsymbol{x}}\dot{\boldsymbol{x}} + \boldsymbol{K}_{\boldsymbol{x}}\boldsymbol{x} = \boldsymbol{Q}_{\boldsymbol{x}}(t) + \boldsymbol{Q}_{\boldsymbol{x}}^c(t), \tag{24}$$

where $\boldsymbol{Q}^c_{\boldsymbol{x}}(t)$ are the additional aforementioned forces. The presence of constraints yields virtual displacements, described by the l vector $\boldsymbol{w}$, which is any non-zero vector satisfying the condition

$$\boldsymbol{A}\boldsymbol{w} = \boldsymbol{0}, \tag{25}$$

and at any instant of time t can be expressed as

$$\boldsymbol{w}^T \boldsymbol{Q}^c_{\boldsymbol{x}} = \boldsymbol{w}^T \boldsymbol{N}. \tag{26}$$

The l vector $\boldsymbol{N}$ describes the nature of the non-ideal constraints and can be obtained by experimentation and/or observation. By employing the Moore-Penrose inverse, $\boldsymbol{A}^+$, of the matrix $\boldsymbol{A}$, eq. (25) is rewritten as

$$\boldsymbol{w} = (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{y}, \tag{27}$$

where $\boldsymbol{y}$ is an arbitrary l vector. Substituting eq. (27) in (26), yields

$$(\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{Q}^c_{\boldsymbol{x}} = (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{N}. \tag{28}$$

Next, premultiplying eq. (24) by $\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A}$, and considering eq. (28), the equation

$$(\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{M}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} + (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{C}_{\boldsymbol{x}}\dot{\boldsymbol{x}} + (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{K}_{\boldsymbol{x}}\boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N}), \tag{29}$$

is obtained. Hence, the additional forces that appeared due to the presence of the constraints, are eliminated. Further, for sake of simplicity, the m vector $\boldsymbol{b}$ of the constrained eq. (23), can be assumed to be of the form

$$\boldsymbol{b} = \boldsymbol{F} - \boldsymbol{E}\dot{\boldsymbol{x}} - \boldsymbol{L}\boldsymbol{x}. \tag{30}$$

Subsequently, considering eqs. (23), (30) together with eq. (29), gives

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} = \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N}) \\ \boldsymbol{F} \end{bmatrix} - \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{C}_{\boldsymbol{x}}\dot{\boldsymbol{x}} \\ \boldsymbol{E}\dot{\boldsymbol{x}} \end{bmatrix} - \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{K}_{\boldsymbol{x}}\boldsymbol{x} \\ \boldsymbol{L}\boldsymbol{x} \end{bmatrix}$$

or, equivalently

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} = \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S}) \\ \boldsymbol{b} \end{bmatrix}, \tag{31}$$

where the m vector $\boldsymbol{b}$ and the $(m + l) \times l$ matrix $\bar{\boldsymbol{M}}_{\boldsymbol{x}}$ are given by eq. (30) and

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}} = \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{M}_{\boldsymbol{x}} \\ \boldsymbol{A} \end{bmatrix}, \tag{32}$$

respectively; and the l vector S is given by

$$\boldsymbol{S} = -\boldsymbol{C}_{\boldsymbol{x}}\dot{\boldsymbol{x}} - \boldsymbol{K}_{\boldsymbol{x}}\boldsymbol{x}. \tag{33}$$

The solution to eq. (31) is

$$\ddot{\boldsymbol{x}} = \bar{\boldsymbol{M}}^+_{\boldsymbol{x}} \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S}) \\ \boldsymbol{b} \end{bmatrix} + (\boldsymbol{I} - \bar{\boldsymbol{M}}^+_{\boldsymbol{x}}\bar{\boldsymbol{M}}_{\boldsymbol{x}})\boldsymbol{y}, \tag{34}$$

where the $l \times (l+m)$ matrix $\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}$ is the Moore-Penrose inverse of $\bar{\boldsymbol{M}}_{\boldsymbol{x}}$.

Note that for the specific matrix $\bar{\boldsymbol{M}}_{\boldsymbol{x}}$ of eq. (32), the equation

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}} \begin{bmatrix} (\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{A}^{+}\boldsymbol{z}) + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix} = \bar{\boldsymbol{M}}_{\boldsymbol{x}} \begin{bmatrix} \boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix}, \tag{35}$$

holds true for any m vector $\boldsymbol{z}$ (Udwadia and Schutte (2010)). Thus, by setting $\boldsymbol{z} = -\boldsymbol{A}(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S})$, eq. (35) yields

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{A}^{+}\boldsymbol{A})(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S}) \\ \boldsymbol{b} \end{bmatrix} = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \begin{bmatrix} \boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix}. \tag{36}$$

Thus, considering eqs. (34) and (36) together, the response acceleration vector is given by

$$\ddot{\boldsymbol{x}} = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \begin{bmatrix} \boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix} + (\boldsymbol{I} - \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{M}}_{\boldsymbol{x}})\boldsymbol{y}. \tag{37}$$

The preceding analysis is for the general case, where the constraints are considered to be non-ideal. Nevertheless, assuming in the ensuing analysis that the constraints are ideal, i.e. $\boldsymbol{N} = \boldsymbol{0}$, eq. (37) becomes

$$\ddot{\boldsymbol{x}} = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \begin{bmatrix} \boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix} + (\boldsymbol{I} - \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{M}}_{\boldsymbol{x}})\boldsymbol{y} \tag{38}$$

which, considering eq. (33), can be written as

$$\ddot{\boldsymbol{x}} = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \left[-\bar{\boldsymbol{C}}_{\boldsymbol{x}}\dot{\boldsymbol{x}} - \bar{\boldsymbol{K}}_{\boldsymbol{x}}\boldsymbol{x} + \bar{\boldsymbol{Q}}_{\boldsymbol{x}} \right] + (\boldsymbol{I} - \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{M}}_{\boldsymbol{x}})\boldsymbol{y}; \tag{39}$$

the $(m+l) \times l$ matrices $\bar{\boldsymbol{C}}_{\boldsymbol{x}}$, $\bar{\boldsymbol{K}}_{\boldsymbol{x}}$, as well as the $(l+m)$ vector $\bar{\boldsymbol{Q}}_{\boldsymbol{x}}$ are given by

$$\bar{\boldsymbol{C}}_{\boldsymbol{x}} = \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{C}_{\boldsymbol{x}} \\ \boldsymbol{E} \end{bmatrix}, \tag{40}$$

$$\bar{\boldsymbol{K}}_{\boldsymbol{x}} = \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{K}_{\boldsymbol{x}} \\ \boldsymbol{L} \end{bmatrix} \tag{41}$$

and

$$\bar{\boldsymbol{Q}}_{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{Q}_{\boldsymbol{x}} \\ \boldsymbol{F} \end{bmatrix}, \tag{42}$$

respectively.

In a similar manner as in the standard state-variable formulation section, the augmented system

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} + \bar{\boldsymbol{C}}_{\boldsymbol{x}}\dot{\boldsymbol{x}} + \bar{\boldsymbol{K}}_{\boldsymbol{x}}\boldsymbol{x} = \bar{\boldsymbol{Q}}_{\boldsymbol{x}} \tag{43}$$

can be cast into the state variable form by defining a $2l$ vector, $\boldsymbol{p}(t)$ of the form

$$\boldsymbol{p}(t) = \begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{bmatrix}. \tag{44}$$

Taking into account eqs. (39) and (44), eq. (43) becomes

$$\dot{\boldsymbol{p}} = \boldsymbol{G}_{\boldsymbol{x}}\boldsymbol{p} + \boldsymbol{f}_{\boldsymbol{x}}, \tag{45}$$

where

$$\boldsymbol{G}_{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{K}}_{\boldsymbol{x}} & -\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{C}}_{\boldsymbol{x}} \end{bmatrix}, \tag{46}$$

and

$$\boldsymbol{f}_{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{0} \\ \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{Q}}_{\boldsymbol{x}} + (\boldsymbol{I} - \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{M}}_{\boldsymbol{x}})\boldsymbol{y} \end{bmatrix}. \tag{47}$$

Setting $\boldsymbol{f}_{\boldsymbol{x}} = 0$, eq. (45) becomes homogeneous and its general solution is given by

$$\boldsymbol{p}(t) = exp(\boldsymbol{G}_{\boldsymbol{x}}t)\boldsymbol{p}(0), \tag{48}$$

where the $2l \times 2l$ matrix $exp(\boldsymbol{G}_{\boldsymbol{x}}t)$ represents the transition matrix for the system.

Next, based on the solution of the homogeneous equation, the response to a non-zero forcing, $\boldsymbol{f}_{\boldsymbol{x}}$, is given by

$$\boldsymbol{p(t)} = exp(\boldsymbol{G}_{\boldsymbol{x}}t)\boldsymbol{p}(0) + \int_0^t exp[\boldsymbol{G}_{\boldsymbol{x}}(t-\tau)]\boldsymbol{f}_{\boldsymbol{x}}(\tau)d\tau, \tag{49}$$

which, under the assumption that $\boldsymbol{p}(0) = 0$, becomes

$$\boldsymbol{p}(t) = \int_0^t \exp(\boldsymbol{G}_{\boldsymbol{x}}\tau)\boldsymbol{f}_{\boldsymbol{x}}(t-\tau)d\tau. \tag{50}$$

Clearly, eq. (50) is a convolution integral between the input $\boldsymbol{f}_{\boldsymbol{x}}(t)$ and the output $\boldsymbol{p}(t)$. Further, defining

$$exp(\boldsymbol{G}_{\boldsymbol{x}}t) = \begin{bmatrix} \boldsymbol{a}_{\boldsymbol{x}}(t) & \boldsymbol{b}_{\boldsymbol{x}}(t) \\ \boldsymbol{c}_{\boldsymbol{x}}(t) & \boldsymbol{d}_{\boldsymbol{x}}(t) \end{bmatrix}, \tag{51}$$

eq. (50) yields

$$\begin{bmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{bmatrix} = \int_0^t \begin{bmatrix} \boldsymbol{a}_{\boldsymbol{x}}(\tau) & \boldsymbol{b}_{\boldsymbol{x}}(\tau) \\ \boldsymbol{c}_{\boldsymbol{x}}(\tau) & \boldsymbol{d}_{\boldsymbol{x}}(\tau) \end{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{H}(t-\tau) \end{bmatrix} d\tau, \tag{52}$$

where

$$\boldsymbol{H}(t-\tau) = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{Q}}_{\boldsymbol{x}}(t-\tau) + (\boldsymbol{I} - \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{M}}_{\boldsymbol{x}})\boldsymbol{y}(t-\tau). \tag{53}$$

Taking into account eqs. (52)-(53) yields

$$\boldsymbol{x}(t) = \int_0^t \boldsymbol{b}_{\boldsymbol{x}}(\tau)\boldsymbol{H}(t-\tau)d\tau. \tag{54}$$

At this point it is deemed appropriate to make two remarks. First, as noted also in Udwadia and Phohomsiri (2006) the expression of eq. (39) for the response acceleration vector is not unique. This is due to the arbitrary vector $\boldsymbol{y}$ involved in the definition of the Moore-Penrose inverse matrix. Second, the expression for $\boldsymbol{H}(t)$ in eq. (53), which can be construed as a ”generalized” impulse response matrix is also not unique for the same reason as above. As a result, the response displacement vector $\boldsymbol{x}(t)$ in eq. (54) is not unique. In this regard, due to the fact that, in general, for systems of the form of eq. (3) a unique response displacement/acceleration vector is experimentally observed, and that the respective impulse response matrix is also unique (depending only on the system properties/parameters), it is reasonable to apply conditions so that the system impulse response matrix is uniquely defined. To this end, as shown in Udwadia and Phohomsiri (2006), when the $(m+l)\times l$ matrix $\bar{\boldsymbol{M}}_{\boldsymbol{x}}$ has full rank, i.e. $rank\bar{\boldsymbol{M}}_{\boldsymbol{x}} = l$, yields

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} = (\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{T}\bar{\boldsymbol{M}}_{\boldsymbol{x}})^{-1}\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{T}, \tag{55}$$

so that

$$(\boldsymbol{I} - \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{M}}_{\boldsymbol{x}}) = \boldsymbol{0}. \tag{56}$$

Hence, eq. (54) can be equivalently written as

$$\boldsymbol{x}(t) = \int_0^t \boldsymbol{h}_{\boldsymbol{x}}(\tau)\bar{\boldsymbol{Q}}_{\boldsymbol{x}}(t-\tau)d\tau, \tag{57}$$

where

$$\boldsymbol{h}_{\boldsymbol{x}}(t) = \boldsymbol{b}_{\boldsymbol{x}}(t)\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \tag{58}$$

can be considered as the uniquely defined ”generalized” impulse response matrix. Considering next eq. (56), eq. (39) and (47) become

$$\ddot{\boldsymbol{x}} = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}(-\bar{\boldsymbol{C}}_{\boldsymbol{x}}\dot{\boldsymbol{x}} - \bar{\boldsymbol{K}}_{\boldsymbol{x}}\boldsymbol{x} + \bar{\boldsymbol{Q}}_{\boldsymbol{x}}) \tag{59}$$

and

$$\boldsymbol{f}_{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{0} \\ \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{Q}}_{\boldsymbol{x}} \end{bmatrix}, \tag{60}$$

respectively.

Further, for the determination of the system response statistical moments, in a similar manner as in the previous section, taking expectations on eq. (45) yields an equation for the system response mean vector in the form

$$\dot{\boldsymbol{m}}_{\boldsymbol{x}} = \boldsymbol{G}_{\boldsymbol{x}}\boldsymbol{m}_{\boldsymbol{x}} + \boldsymbol{m}_{\boldsymbol{f}_{\boldsymbol{x}}}. \tag{61}$$

Furthermore, the corresponding equation for the system response covariance matrix becomes

$$\dot{\boldsymbol{V}}_{\boldsymbol{x}} = \boldsymbol{G}_{\boldsymbol{x}}\boldsymbol{V}_{\boldsymbol{x}} + \boldsymbol{V}_{\boldsymbol{x}}\boldsymbol{G}_{\boldsymbol{x}}^{T} + \boldsymbol{S}_{\boldsymbol{x}}, \tag{62}$$

where

$$\boldsymbol{S}_{\boldsymbol{x}} = \int_0^t exp(\boldsymbol{G}_{\boldsymbol{x}}(t-\tau))\left[\boldsymbol{w}_{\boldsymbol{\eta}}(t,\tau) + \boldsymbol{w}_{\boldsymbol{\eta}}^{T}(t,\tau)\right]d\tau. \tag{63}$$

For the case where the elements of $\boldsymbol{\eta}$ are regarded to be stationary white noises, eq. (62) becomes

$$\dot{\boldsymbol{V}}_{\boldsymbol{x}} = \boldsymbol{G}_{\boldsymbol{x}}\boldsymbol{V}_{\boldsymbol{x}} + \boldsymbol{V}_{\boldsymbol{x}}\boldsymbol{G}_{\boldsymbol{x}}^{T} + \boldsymbol{D}_{\boldsymbol{x}}, \tag{64}$$

where $\boldsymbol{D}_{\boldsymbol{x}}$ is a real, symmetric, non-negative matrix of constants and $\boldsymbol{G}_{\boldsymbol{x}}, \boldsymbol{f}_{\boldsymbol{x}}$ are given by eq. (46) and (47), respectively. Focusing on the system stationary response, i.e. $\dot{\boldsymbol{V}}_{\boldsymbol{x}} = \boldsymbol{0}$, eq. (64) becomes

$$\boldsymbol{G}_{\boldsymbol{x}}\boldsymbol{V}_{\boldsymbol{x}} + \boldsymbol{V}_{\boldsymbol{x}}\boldsymbol{G}_{\boldsymbol{x}}^{T} + \boldsymbol{D}_{\boldsymbol{x}} = \boldsymbol{0}. \tag{65}$$

Clearly, eq. (65) is a Lyapunov equation which is a special case of the Sylvester equation of the form

$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{B} + \boldsymbol{Q} = \boldsymbol{0}. \tag{66}$$

Eq. (66) has a unique solution if and only if the matrices $\boldsymbol{A}$ and $-\boldsymbol{B}$ have no common eigenvalues. Thus, eq. (65) has a unique solution if all the eigenvalues of the matrix $\boldsymbol{G_x}$ are not equal to zero or, equivalently, the rows of $\boldsymbol{G_x}$ are linearly independent with each other (for more details see Chen (1999)). However, due to the fact that more than the minimum number of coordinates are used for the system modeling, most likely the rows of $\boldsymbol{G_x}$ will not be linearly independent; thus, a special treatment is needed for solving eq. (65).

In this regard, eq. (65) can be rewritten as

$$(\boldsymbol{I}_{2l} \otimes \boldsymbol{G_x} + \boldsymbol{G_x} \otimes \boldsymbol{I}_{2l}) vec\boldsymbol{V_x} = vec(-\boldsymbol{D_x}), \tag{67}$$

where $vec\boldsymbol{V_x}$ and $vec(-\boldsymbol{D_x})$ are $(2l)^2$ vectors formed by stacking all columns of $\boldsymbol{V_x}$ and $-\boldsymbol{D_x}$ respectively, on top of one another; also, by $\boldsymbol{I}_{2l} \otimes \boldsymbol{G_x}$ and $\boldsymbol{G_x} \otimes \boldsymbol{I}_{2l}$ is denoted the Kronecker product of the pairs of matrices $\boldsymbol{I}_{2l}, \boldsymbol{G_x}$ and $\boldsymbol{G_x}, \boldsymbol{I}_{2l}$. Equivalently, eq. (67) is expressed in the form

$$\boldsymbol{W}\boldsymbol{v} = \boldsymbol{d}, \tag{68}$$

where $\boldsymbol{W}$ is a $(2l)^2 \times (2l)^2$ matrix and $\boldsymbol{v} = vec(\boldsymbol{V_x}), \boldsymbol{d} = vec(-\boldsymbol{D_x})$. Next, involving eq. (3) for the M-P inverse of a matrix, the general solution to eq. (68) is

$$\boldsymbol{v} = \boldsymbol{W}^+\boldsymbol{d} + (\boldsymbol{I}_{(2l)^2} - \boldsymbol{W}^+\boldsymbol{W})\boldsymbol{y}, \tag{69}$$

where $\boldsymbol{y}$ is an arbitrary $(2l)^2$ vector.

Moore-Penrose state-variable formulation: A numerical example

Consider the system of two rigid masses m_1 and m_2 in Figure 1. The masses move horizontally as a result of an applied random force $Q_2(t)$. Let the mass m_1 be connected to the foundation by a linear spring and a linear damper with coefficients k_1 and c_1, respectively. Further, a mass m_2 is connected to m_1 by a linear spring and a linear damper with coefficients k_2 and c_2, respectively. Furthermore, $Q_2(t)$ is a white-noise process with a correlation function $w_{Q_2(t)} = 2\pi S_0 \delta(\tau)$, where S_0 is the (constant) power spectrum value for $Q_2(t)$. Finally, q_1, q_2 are the generalized displacements, shown in Figure 1. The equations of motion governing the system in Figure 1 can be written in the matrix form of eq. (4), where the matrices $\boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{K}$ are given by (see also Roberts and Spanos (2003))

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \ \boldsymbol{C} = \begin{bmatrix} (c_1 + c_2) & -c_2 \\ c_2 & c_2 \end{bmatrix}, \tag{70}$$

$$\boldsymbol{K} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \text{and} \tag{71}$$

$$\boldsymbol{Q} = \begin{bmatrix} 0 \\ Q_2(t) \end{bmatrix}. \tag{72}$$

Next, eq. (21) is formed; thus, obtaining a system of algebraic equations to be solved for the 16 unknowns of matrix $\boldsymbol{V}$. Focusing next on the stationary system response, i.e. $\dot{\boldsymbol{V}} = \boldsymbol{0}$, and considering the parameters values $m_1 = m_2 = m = 1, c_1 = c_2 = c = 0.1, k_1 = k_2 = k = 1$ and $S_0 = 10^{-3}$, numerical solution of the Lyapunov eq. (21) yields

$$\boldsymbol{V} = \begin{bmatrix} 0.0438 & 0.0690 & 0.0000 & -0.0012 \\ 0.0690 & 0.1132 & 0.0012 & 0.0000 \\ 0.0000 & 0.0012 & 0.0188 & 0.0251 \\ -0.0012 & 0.0000 & 0.0251 & 0.0441 \end{bmatrix}. \tag{73}$$

Consider next the system of two masses m_1 and m_2 of the above example modeled as a multi-body one, and consisting of two separate subsystems as shown in Figure 2; see also Udwadia and Phohomsiri (2006). In this regard, the two sub-systems are related based on the constraint $x_2 = x_1 + d$ (where d is the length of mass m_1). The "unconstrained" equations of motion are derived by treating the three coordinates ($\bar{x}_1, x_2$ and $\bar{x}_3$) as independent with each other. Next, the equation of motion of the composite system is derived by including the constraint

$$x_2 = x_1 + d \tag{74}$$

or, equivalently

$$x_2 = \bar{x}_1 + l_{10} + d, \tag{75}$$

where l_{10} is the unstretched length of the spring k_1.

The total kinetic and potential energies for the two sub-systems are

$$T = \frac{1}{2} m_1 \dot{\bar{x}}_1^2 + \frac{1}{2} m_2 (\dot{x}_2 + \dot{\bar{x}}_3)^2 \tag{76}$$

and

$$V = \frac{1}{2} k_1 \bar{x}_1^2 + \frac{1}{2} k_2 \bar{x}_3^2, \tag{77}$$

respectively. Next, by forming the Lagrangian function $\boldsymbol{L}(\bar{x}_1, x_2, \bar{x}_3, \dot{\bar{x}}_1, \dot{x}_2, \dot{\bar{x}}_3) = \boldsymbol{T} - \boldsymbol{V}$, and utilizing the Euler-Lagrange equations (e.g. Mestdag et al. (2011)), yields

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{L}}{\partial \dot{x}_i}\right) - \frac{\partial \boldsymbol{L}}{\partial x_i} - \frac{\partial \boldsymbol{F}}{\partial \dot{x}_i} = \boldsymbol{Q}, \tag{78}$$

where $\boldsymbol{F}(\bar{x}_i, \dot{\bar{x}}_i) = -\frac{1}{2} c_i \dot{\bar{x}}_i^2$ is the damping force ($i = 1, 3$) and $\boldsymbol{Q}$ the external excitation. Manipulating eq. (78) yields

$$
\begin{aligned}
m_1 \ddot{\bar{x}}_1 + c_1 \dot{\bar{x}}_1 + k_1 \bar{x}_1 &= 0 & (79)\\
m_2 \ddot{x}_2 + m_2 \ddot{\bar{x}}_3 &= 0 & (80)\\
m_2 \ddot{x}_2 + m_2 \ddot{\bar{x}}_3 + c_2 \dot{\bar{x}}_3 + k_2 \bar{x}_3 &= Q_3 & (81)
\end{aligned}
$$

where $\bar{x}_1 = x_1 - l_{10}, \bar{x}_3 = x_3 - l_{20}$ and l_{10}, l_{20} are the unstretched lengths of the springs k_1, k_2 respectively.

The matrix form for the equations of motion becomes

$$\boldsymbol{M_x} \ddot{\boldsymbol{x}} + \boldsymbol{C_x} \dot{\boldsymbol{x}} + \boldsymbol{K_x} \boldsymbol{x} = \boldsymbol{Q_x}, \tag{82}$$

where

$$\boldsymbol{M_x} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & m_2 \\ 0 & m_2 & m_2 \end{bmatrix}, \quad \boldsymbol{C_x} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_2 \end{bmatrix}, \quad \boldsymbol{K_x} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_2 \end{bmatrix} \tag{83}$$

and

$$\boldsymbol{Q_x} = \begin{bmatrix} 0 \\ 0 \\ Q_3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} \bar{x}_1 \\ x_2 \\ \bar{x}_3 \end{bmatrix}. \tag{84}$$

Differentiating the constraint of eq. (75) the two sub-systems are subject to, yields

$$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x_1} \\ x_2 \\ \bar{x_3} \end{bmatrix} = 0. \tag{85}$$

Thus, eq. (23) takes the form

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \text{ and } b = 0. \tag{86}$$

As in the previous example, assume that $m_1 = m_2 = m = 1, c_1 = c_2 = c = 0.1, k_1 = k_2 = k = 1$ and Q_3 is a white noise excitation with power spectrum amplitude $S_0 = 10^{-3}$. Next, note that $rank\bar{\boldsymbol{M}}_{\boldsymbol{x}} = 3$, i.e. the 4×3 matrix $\bar{\boldsymbol{M}}_{\boldsymbol{x}}$ has full rank. Hence, eqs. (32), (40) and (42), become

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad \bar{\boldsymbol{C}}_{\boldsymbol{x}} = \begin{bmatrix} 0.05 & 0 & 0 \\ 0.05 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\boldsymbol{K}}_{\boldsymbol{x}} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{87}$$

and

$$\bar{\boldsymbol{Q}}_{\boldsymbol{x}} = \begin{bmatrix} 0 \\ 0 \\ w(t) \\ 0 \end{bmatrix}. \tag{88}$$

Further, the Moore-Penrose inverse of the matrix $\bar{\boldsymbol{M}}_{\boldsymbol{x}}$, yields

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 2 & 1 \end{bmatrix}. \tag{89}$$

Thus, substituting eq. (89) into eq. (46), yields

$$\boldsymbol{G}_{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -0.1 & 0 & 0.1 \\ -1 & 0 & 1 & -0.1 & 0 & 0.1 \\ 1 & 0 & -2 & 0.1 & 0 & -0.2 \end{bmatrix}. \tag{90}$$

Focusing next on the stationary system response, i.e. $\dot{\boldsymbol{V}}_{\boldsymbol{x}} = \boldsymbol{0}$, and considering that Q_3 is a white-noise excitation, the matrix $\boldsymbol{D}_{\boldsymbol{x}}$ in eq. (65), takes the form

$$\boldsymbol{D}_{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\pi 10^{-3} \end{bmatrix}, \tag{91}$$

whereas the Lyapunov eq. (65) becomes

$$\boldsymbol{G_x V_x} + \boldsymbol{V_x G_x^T} = -\boldsymbol{D_x}. \tag{92}$$

Note that due to the fact that not all rows of $\boldsymbol{G_x}$ are linearly independent with each other (compare fourth and fifth row), the solution of (92) is not unique. Hence, following eq. (67), eq. (69) yields

$$\boldsymbol{V_x} = \begin{bmatrix} 0.0438 & 0.0438 & 0.0252 & 0 & 0 & -0.0012 \\ 0.0438 & y_1 & 0.0252 & 0 & 0 & -0.0012 \\ 0.0252 & 0.0252 & 0.0190 & 0.0012 & 0.0012 & 0 \\ 0 & 0 & 0.0012 & 0.0188 & 0.0188 & 0.0063 \\ 0 & 0 & 0.0012 & 0.0188 & 0.0188 & 0.0063 \\ -0.0012 & -0.0012 & 0 & 0.0063 & 0.0063 & 0.0127 \end{bmatrix}. \tag{93}$$

Note also that almost all the elements of the matrix $(\boldsymbol{I}_{(2l)^2} - \boldsymbol{W}^+\boldsymbol{W})$ in eq. (69) are zero. Interestingly, the non-zero one is the element in the diagonal in position corresponding to the additional auxiliary DOFs; in this case it is the element $\boldsymbol{V}_{\boldsymbol{x}(2,2)} = \boldsymbol{E}(\boldsymbol{x}_2^2)$. Hence, the presence of the arbitrary vector $\boldsymbol{y}$ does not affect, in essence, the calculated $\boldsymbol{V_x}$.

Indicatively, comparing eq. (73) and (93), the variance $E[q_1^2]$ as well as $E[\dot{q}_1^2]$ obtained in the first example, coincide with the respective ones in the second one, i.e $E[\bar{x}_1^2]$ and $E[\dot{\bar{x}}_1^2]$. Further, taking expectations in the equation that connects the two reference systems, that is $\bar{x}_3 = q_2 - q_1$, and utilizing eq. (73) yields

$$E[\bar{x}_3^2] = E[q_2^2] + E[q_1^2] - 2E[q_1 q_2] = 0.0190 \tag{94}$$

and

$$E[\dot{\bar{x}}_3^2] = E[\dot{q}_2^2] + E[\dot{q}_1^2] - 2E[\dot{q}_1 \dot{q}_2] = 0.0127, \tag{95}$$

which are indeed in agreement with the corresponding values in eq. (93). It can be readily verified that the rest of the elements of matrix (93) are also in agreement with the respective ones of eq. (73).

Complex modal analysis

In the standard formulation of the linear random vibration theory, computing the "complex modal matrix" whose columns are the eigenvectors, or "complex modes" of matrix $\boldsymbol{G}$ of eq. (7) facilitates not only the efficient evaluation of $exp(\boldsymbol{G}t)$ in eq. (11), and thus, of the system impulse response matrix of eq. (10), but also plays an instrumental role in decoupling the original coupled system of equations (eq. (4)); see for example Roberts and Spanos (2003). In this section it is shown that a similar treatment of the system of eq. (43) does not lead in general in a decoupling of the equations of motion. Further insights are provided regarding the efficient computation of the impulse response matrix $\boldsymbol{h_x}(t)$ of eq. (58).

Let $\lambda_1, \lambda_2, \ldots, \lambda_{2l}$ be the eigenvalues of the $2l \times 2l$ matrix $\boldsymbol{G_x}$ given by eq. (46), so that the first r of them are non zero and the remaining $2l - r$ are equal to zero. Then, the eigen-decomposition of $\boldsymbol{G_x}$ yields

$$\boldsymbol{G_x T} = \boldsymbol{T \eta_x}, \tag{96}$$

where $\boldsymbol{\eta_x}$ is the diagonal matrix given by

$$\boldsymbol{\eta_x} = diag(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, 0, \ldots, 0), \tag{97}$$

and

$$\boldsymbol{T} = [\boldsymbol{v}_1 \, \boldsymbol{v}_2 \ldots \boldsymbol{v}_{2l}] \tag{98}$$

is the $2l$ "complex modal matrix" formed by the eigenvectors of $\boldsymbol{G_x}$. Due to the presence of zero eigenvalues, the eigenvectors are not linearly independent, which means that the matrix $\boldsymbol{T}$ is singular. Next, the singular value decomposition (SVD) of $\boldsymbol{G_x}$, yields

$$\boldsymbol{G_x} = \boldsymbol{U}\boldsymbol{\eta_x}\boldsymbol{\Phi}^*, \tag{99}$$

where the matrix $\boldsymbol{\eta_x}$ is $2l \times 2l$ diagonal of the form

$$\boldsymbol{\eta_x} = diag(\sigma_1, \sigma_2, \ldots, \sigma_r, 0, 0, \ldots, 0). \tag{100}$$

In eq. (100), $\sigma_j = \sqrt{\lambda_j}, j = 1, 2, \ldots, 2l$ are the singular values of $\boldsymbol{G_x}$. The $2l \times 2l$ matrix $\boldsymbol{\Phi} = \left[\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_{2l}\right]$ is a unitary matrix (i.e. $\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \boldsymbol{\Phi}^*\boldsymbol{\Phi} = \boldsymbol{I}$), where $\boldsymbol{\phi}_j$ is an eigenvector corresponding to each singular value σ_j for $j = 1, 2, \ldots, 2l$. Finally, $\boldsymbol{U} = \left[\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{2l}\right]$ is a $2l \times 2l$ unitary matrix (i.e. $\boldsymbol{U}\boldsymbol{U}^* = \boldsymbol{U}^*\boldsymbol{U} = \boldsymbol{I}$) and each one of the $2l-$vectors $\boldsymbol{u}_j$ is equal to

$$\boldsymbol{u}_j = \frac{\boldsymbol{G_x}\boldsymbol{\phi}_j}{\sigma_j}, j = 1, 2, \ldots, 2l. \tag{101}$$

Further, to determine the impulse response matrix, $\boldsymbol{h_x}(t)$, of eq. (58) the matrix $exp(\boldsymbol{G_x})$ of eq. (51) has to be evaluated first.

In this regard, the transformation

$$\boldsymbol{p} = \boldsymbol{\Phi}\boldsymbol{z_x}, \tag{102}$$

is used and the state variable form of eq. (45), becomes

$$\dot{\boldsymbol{z}}_{\boldsymbol{x}} = \boldsymbol{\Phi}^*\boldsymbol{G_x}\boldsymbol{\Phi}\boldsymbol{z_x} + \boldsymbol{\Phi}^*\boldsymbol{f_x}. \tag{103}$$

Next, taking into consideration eq. (99), eq. (103) can be rewritten as

$$\dot{\boldsymbol{z}}_{\boldsymbol{x}} = \boldsymbol{\Phi}^*\boldsymbol{U}\boldsymbol{\eta_x}\boldsymbol{z_x} + \boldsymbol{g_x}, \tag{104}$$

where

$$\boldsymbol{g_x} = \boldsymbol{\Phi}^*\boldsymbol{f_x}. \tag{105}$$

At this point, it is critical to note that due to the form of eq.(104) the equations of motion cannot be decoupled. Unlike a standard complex modal analysis (e.g. Roberts and Spanos (2003)) utilizing the minimum number degrees of freedom, the formulation herein yields a matrix $\boldsymbol{G_x}$ with some of its eigenvalues being zero. As a result, not all the eigenvectors forming the "complex modal matrix" $\boldsymbol{T}$ are linearly independent with each other; thus, leading to inability to perform a standard eigenvalue decomposition of G_x (see eq. (96)). In other words, the matrix $\boldsymbol{\Phi}^*\boldsymbol{U}$ in eq. (104) cannot be the unitary matrix, and thus, rendering the system of coupled equations of eq. (104) an uncoupled one. Overall, in contrast to a standard analysis modeling when a complex modal analysis yields an uncoupled system of equations, this is not possible when utilizing more than the minimum number of degrees of freedom. Nevertheless, it is shown in the ensuing analysis that relying on an SVD of the matrix $\boldsymbol{G_x}$ greatly facilitates the numerical computation of the system response statistics.

Proceeding with the analysis, eq. (45) has been cast into eq. (104), which has the general solution

$$\boldsymbol{z_x}(t) = exp(\boldsymbol{\Phi}^*\boldsymbol{U}\boldsymbol{\eta_x}t)\boldsymbol{z_x}(0) + \int_0^t exp(\boldsymbol{\Phi}^*\boldsymbol{U}\boldsymbol{\eta_x}(t-\tau))\boldsymbol{g_x}(\tau)d\tau. \tag{106}$$

Under the assumption that the system is initially at rest, eq. (106) becomes

$$\boldsymbol{z_x}(t) = \int_0^t exp(\boldsymbol{\Phi}^*\boldsymbol{U}\boldsymbol{\eta_x}s)\boldsymbol{g_x}(t-s)ds. \tag{107}$$

Notably, the impulse response function $\boldsymbol{h_x}(t)$ is given by

$$\boldsymbol{h_x}(t) = exp(\boldsymbol{\Phi}^*\boldsymbol{U}\boldsymbol{\eta_x}t). \tag{108}$$

Further, once $\boldsymbol{z_x}$ is computed, the $2l$ vector $\boldsymbol{p}$ can be determined by using the transformation given in eq. (102).

Next, taking expectation on eq. (104), taking into account eq.(105), and considering the stationary response (i.e. $\dot{\boldsymbol{m}}_{\boldsymbol{z_x}} = \boldsymbol{0}$) and assuming that $\dot{\boldsymbol{m}}_{\boldsymbol{z_x}} = 0$, the equation

$$\boldsymbol{\eta_x}\boldsymbol{m}_{\boldsymbol{z_x}} = -\boldsymbol{U}^*\boldsymbol{m}_{\boldsymbol{f_x}}, \tag{109}$$

arises, which has the general solution

$$\boldsymbol{m}_{\boldsymbol{z_x}} = -\boldsymbol{\eta}_{\boldsymbol{x}}^{+}\boldsymbol{U}^*\boldsymbol{m}_{\boldsymbol{f_x}} + (\boldsymbol{I}_{2l} - \boldsymbol{\eta}_{\boldsymbol{x}}^{+}\boldsymbol{\eta_x})\boldsymbol{y}. \tag{110}$$

In eq. (110), $\boldsymbol{\eta}_{\boldsymbol{x}}^{+}$ is the Moore-Penrose inverse of $\boldsymbol{\eta_x}$ and $\boldsymbol{y}$ is an arbitrary $2l$ vector. Also, using eq. (102), the formula

$$\boldsymbol{m_p} = -\boldsymbol{\Phi}\boldsymbol{\eta}_{\boldsymbol{x}}^{+}\boldsymbol{U}^*\boldsymbol{m}_{\boldsymbol{f_x}} + \boldsymbol{\Phi}(\boldsymbol{I}_{2l} - \boldsymbol{\eta}_{\boldsymbol{x}}^{+}\boldsymbol{\eta_x})\boldsymbol{y}, \tag{111}$$

is obtained, where $\boldsymbol{U}, \boldsymbol{\Phi}$ are the SVD unitary matrices. Regarding the determination of the Moore-Penrose inverse of the $2l \times 2l$ matrix $\boldsymbol{\eta_x}$, this is given by

$$\sigma_j = \begin{cases} \sigma_j^{-1} & \text{, if } \sigma_j \neq 0 \\ 0 & \text{, if } \sigma_j = 0 \end{cases}. \tag{112}$$

Next, the covariance matrices of the transformed state vector $\boldsymbol{z_x}$ and the $2l$ vector $\boldsymbol{g_x}$ given by eqs. (107) and (105), respectively, can be easily related as follows. In this regard, defining the covariance matrix of $\boldsymbol{z_x}$ as

$$\boldsymbol{w}_{\boldsymbol{z_x}}(\tau) = \boldsymbol{E}\left[(\boldsymbol{z_x}(\tau) - \boldsymbol{m}_{\boldsymbol{z_x}})(\boldsymbol{z_x}(t+\tau) - \boldsymbol{m}_{\boldsymbol{z_x}})^*\right] \tag{113}$$

and the covariance matrix of $\boldsymbol{g_x}$ as

$$\boldsymbol{w}_{\boldsymbol{g_x}}(\tau) = E\left[(\boldsymbol{g_x}(\tau) - \boldsymbol{m}_{\boldsymbol{g_x}})(\boldsymbol{g_x}(t+\tau) - \boldsymbol{m}_{\boldsymbol{g_x}})^*\right], \tag{114}$$

and considering eq. (107), the covariance input-output relationship is given by

$$\boldsymbol{w}_{\boldsymbol{z}_{\boldsymbol{x}}}(\tau) = \int_0^\infty \int_0^\infty \boldsymbol{h}_{\boldsymbol{x}}(s_1)\boldsymbol{w}_{\boldsymbol{g}_{\boldsymbol{x}}}(\tau + s_1 - s_2)\boldsymbol{h}^*_{\boldsymbol{x}}(s_2)\, ds_1 ds_2, \tag{115}$$

where $\boldsymbol{h}_{\boldsymbol{x}}(t)$ is the impulse response function determined by eq. (108).

As far as the determination of the elements of the impulse response function is concerned, the Cayley-Hamilton Theorem can be employed yielding

$$\boldsymbol{h}_{\boldsymbol{x}}(t) = exp(\boldsymbol{\Phi}^*\boldsymbol{U}\boldsymbol{\eta}_{\boldsymbol{x}}t) = \sum_{k=0}^{r-1} \alpha_k(\boldsymbol{\Phi}^*\boldsymbol{U}\boldsymbol{\eta}_{\boldsymbol{x}})^k. \tag{116}$$

The coefficients $\alpha_k,\ k = 1, 2, \ldots, r-1$ can be found by solving the following system of linear equations

$$exp(\lambda_i) = \sum_{k=0}^{r-1} \alpha_k \lambda_i^k, \tag{117}$$

where $i = 1, 2, \ldots, r$ and λ_i are the eigenvalues of the matrix $\boldsymbol{\Phi}^*\boldsymbol{U}\boldsymbol{\eta}_{\boldsymbol{x}}$. Next, using the obtained formula for the determination of the elements of the impulse response matrix, the elements of the covariance matrix $\boldsymbol{w}_{\boldsymbol{z}_{\boldsymbol{x}}}$, can be determined.

Thus, after determining $\boldsymbol{w}_{\boldsymbol{z}_{\boldsymbol{x}}}$, by utilizing the transformation given in eq. (102), the covariance matrix $\boldsymbol{w}_{\boldsymbol{p}}$ can be determined as well. In this regard,

$$\boldsymbol{w}_{\boldsymbol{p}}(\tau) = \boldsymbol{\Phi}\boldsymbol{w}_{\boldsymbol{z}_{\boldsymbol{x}}}(\tau)\boldsymbol{\Phi}^*. \tag{118}$$

Similarly, using eq. (105), the matrices of $\boldsymbol{g}_{\boldsymbol{x}}(t)$ and $\boldsymbol{f}(t)$ are related via the formula

$$\boldsymbol{w}_{\boldsymbol{g}_{\boldsymbol{x}}}(\tau) = \boldsymbol{\Phi}^*\boldsymbol{w}_{\boldsymbol{f}_{\boldsymbol{x}}}(\tau)\boldsymbol{\Phi}. \tag{119}$$

Finally, assuming that $\boldsymbol{f}_{\boldsymbol{x}}$ is a white noise vector process with correlation function

$$\boldsymbol{w}_{\boldsymbol{f}_{\boldsymbol{x}}}(t, \tau) = \boldsymbol{D}_{\boldsymbol{x}}\delta(t - \tau), \tag{120}$$

where $\boldsymbol{D}_{\boldsymbol{x}}$ is a real, symmetric, non-negative matrix of constants, the covariance matrix of $\boldsymbol{g}_{\boldsymbol{x}}$ is given by

$$\boldsymbol{w}_{\boldsymbol{g}_{\boldsymbol{x}}}(\tau) = \boldsymbol{\Phi}^*\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi}\delta(t - \tau). \tag{121}$$

Note, however, that the impulse response function $\boldsymbol{h}_{\boldsymbol{x}}(t)$ can be determined by other alternative more elegant methods than by using the Cayley-Hamilton theorem (see Cheng and Yau (1997)). In this regard, setting

$$\boldsymbol{R} = \boldsymbol{\Phi}^*\boldsymbol{U}\boldsymbol{\eta}_{\boldsymbol{x}}, \tag{122}$$

the determination of the impulse response function is equivalent to the determination of the matrix $exp(\boldsymbol{R}t)$, which can be determined as a finite polynomial in $\boldsymbol{R}$, with analytic functions of t as coefficients. Moreover, once the eigenvalues of $\boldsymbol{R}$ are known, i.e. $\mu_1, \mu_2, \ldots, \mu_s$, it might be more convenient to express $exp(\boldsymbol{R}t)$ in terms of polynomials in $(\boldsymbol{R} - \mu_i\boldsymbol{I})$. In the following analysis, the arising systems that have to be solved for determining the coefficient functions, are proven to be triangular, and thus, can be readily solved (for more details see Cheng and Yau (1997)).

As a first step for the determination of $exp(\boldsymbol{R}t)$, assume that the matrix $\boldsymbol{R}$ is in its Jordan form, $\mu_1, \mu_2, \ldots, \mu_s$ are its s distinct eigenvalues and $m_i, i = 1, 2, \ldots, s$ the algebraic multiplicity of each eigenvalue μ_i. Finally, assume that

$$M_{\boldsymbol{R}}(x) = \prod_{j=1}^{s}(x - \mu_j)^{m_j}, \tag{123}$$

is the minimal polynomial of $\boldsymbol{R}$ (i.e. the monic polynomial $\boldsymbol{P}$ of least degree such that $\boldsymbol{P}(\boldsymbol{R}) = \mathbf{0}$). Next, for $r \geq 0, s \geq 1$ and $1 \leq k \leq s$, let

$$H_s(r) = \left\{(a_1, a_2, \ldots, a_s) \in \mathbb{N}^s : a_i \geq 0 \text{ and } \sum_{i=1}^{s} a_i = r\right\} \text{ and} \tag{124}$$

$$H_s^{(k)}(r) = \{(a_1, a_2, \ldots, a_s) \in H_s(r) : a_k = 0\}. \tag{125}$$

Then, the exponential of matrix $\boldsymbol{R}t$ is equal to

$$exp(\boldsymbol{R}t) = \sum_{k=1}^{s}\left[\sum_{r=0}^{m_k-1} f_{k,r}(t)(\boldsymbol{R} - \mu_k I)^r\right] \prod_{j=1, j\neq k}^{s} (\boldsymbol{R} - \mu_k I)^{m_j}, \tag{126}$$

where the coefficient functions $f_{k,0}(t), f_{k,1}(t), \ldots, f_{k,m_k-1}(t)$ satisfy the equation

$$\sum_{r=0}^{i} f_{k,r}(t) \sum_{a \in H_s^{(k)}(i-r)} \prod_{j=1, j\neq k}^{s} \binom{m_j}{a_j} (\mu_k - \mu_j)^{m_j - a_j} = exp(\mu_k t)\frac{t^i}{i!}, \tag{127}$$

for $i = 0, 1, \ldots, m_k - 1$ and each $k = 1, 2, \ldots, s$.

Complex modal analysis: A numerical example

Consider once again the multi-body system presented as an example in the Moore-Penrose state-variable formulation section, where the matrix $\boldsymbol{G_x}$ is given by eq. (90). The system consists of two separate subsystems of masses m_1 and m_2, respectively, related based on the constraint given in eq. (74).

Next, determining its SVD, the unitary matrices $\boldsymbol{U}, \boldsymbol{\Phi}$ as well as the diagonal matrix of the singular values $\boldsymbol{\eta_x}$, are found to be equal to

$$\boldsymbol{U} = \begin{bmatrix} -0.0214 & 0.8204 & 0 & -0.5692 & -0.0507 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.0309 & 0.5685 & 0 & 0.8213 & -0.0351 & 0 \\ 0.4695 & -0.0326 & 0 & -0.0177 & -0.5274 & 0.7071 \\ 0.4695 & -0.0326 & 0 & -0.0177 & -0.5274 & -0.7071 \\ -0.7468 & -0.0410 & 0 & 0.0281 & -0.6632 & 0 \end{bmatrix}, \tag{128}$$

$$\boldsymbol{\Phi} = \begin{bmatrix} -0.5660 & 0.0242 & 0 & 0.0638 & 0.8216 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0.8168 & 0.0168 & 0 & -0.0921 & 0.5693 & 0 \\ -0.0638 & 0.8216 & 0 & -0.5660 & -0.0242 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.0921 & 0.5693 & 0 & 0.8168 & -0.0168 & 0 \end{bmatrix} \tag{129}$$

and

$$\boldsymbol{\eta_x} = \begin{bmatrix} 2.9784 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0015 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9944 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4768 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{130}$$

respectively.

Further, utilizing eq. (122), the matrix $\boldsymbol{R}$ is given by

$$\boldsymbol{R} = \begin{bmatrix} -0.1827 & -0.0017 & 0 & 0.9912 & -0.0131 & 0 \\ -0.1176 & -0.0208 & 0 & 0.0015 & -0.3875 & 0 \\ 1.3984 & -0.0326 & 0 & -0.0176 & -0.2515 & 0 \\ -2.6207 & -0.0150 & 0 & -0.0785 & -0.1159 & 0 \\ 0.0035 & 1.0006 & 0 & 0 & -0.0180 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \tag{131}$$

which has the following six eigenvalues; that is,

$$\begin{aligned} &\lambda_1 = -0.1309 + 1.6127i \quad , \quad \lambda_2 = -0.1309 - 1.6127i \ , \ \lambda_3 = -0.0191 + 0.6177i, \\ &\lambda_4 = -0.0191 - 0.6177i \quad , \quad \lambda_5 = 0 \ , \ \lambda_6 = 0 \end{aligned} \tag{132}$$

and its minimal polynomial has the form

$$M_{\boldsymbol{R}}(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4)x^2. \tag{133}$$

Taking into consideration the analysis provided in eqs. (124) to (127), the exponential matrix $exp(t\boldsymbol{R})$ is decomposed in the form

$$exp(t\boldsymbol{R}) = \boldsymbol{p}_1 exp(\lambda_1 t) + \boldsymbol{p}_2 exp(\lambda_2 t) + \boldsymbol{p}_3 exp(\lambda_3 t) + \boldsymbol{p}_4 exp(\lambda_4 t) + \boldsymbol{p}_5 t + \boldsymbol{p}_6, \tag{134}$$

where the coefficients $\boldsymbol{p}_i$ are given by

$$\begin{aligned} \boldsymbol{p}_1 &= \frac{\boldsymbol{b}_2\boldsymbol{b}_3\boldsymbol{b}_4}{a_{12}a_{13}a_{14}\lambda_1^2}\boldsymbol{R}^2, \ \ \boldsymbol{p}_2 = \frac{\boldsymbol{b}_1\boldsymbol{b}_3\boldsymbol{b}_4}{a_{21}a_{23}a_{24}\lambda_2^2}\boldsymbol{R}^2, \ \ \boldsymbol{p}_3 = \frac{\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_4}{a_{31}a_{32}a_{34}\lambda_3^2}\boldsymbol{R}^2, \\ \boldsymbol{p}_4 &= \frac{\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3}{a_{41}a_{42}a_{43}\lambda_4^2}\boldsymbol{R}^2, \ \ \boldsymbol{p}_5 = \frac{\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3\boldsymbol{b}_4}{a_{51}a_{52}a_{53}a_{54}}\boldsymbol{R} \end{aligned} \tag{135}$$

and

$$\boldsymbol{p}_6 = \left\{ \boldsymbol{I} - \frac{(a_{52}a_{53}a_{54} + a_{51}a_{53}a_{54} + a_{51}a_{52}a_{54} + a_{51}a_{52}a_{53})}{a_{51}a_{52}a_{53}a_{54}}\boldsymbol{R} \right\}. \tag{136}$$

The expressions a_{ij} for $i,j = 1,2,\ldots,5$ and $\boldsymbol{b}_k$ for $k = 1,2,3,4$ are defined, in turn, by

$$a_{ij} = \lambda_i - \lambda_j, \ i,j = 1,2,\ldots,5 \tag{137}$$

and

$$\boldsymbol{b}_k = \boldsymbol{R} - \lambda_k \boldsymbol{I}, \ k = 1,2,3,4. \tag{138}$$

Clearly, the impulse response function $\boldsymbol{h}_{\boldsymbol{x}}(t)$ (see eq. (108)) is expressed in terms of its eigenvalues and, thus, as a result it can be easily determined. Further, following closely the example presented in the Moore-Penrose state-variable formulation section, assuming that the excitation is modeled as white noise, and employing eqs. (115) and (121), the covariance matrix $\boldsymbol{w}_{\boldsymbol{z}_{\boldsymbol{x}}}$, can be determined.

In this regard, using the decomposition of the impulse response function obtained in eq. (134), the double integral of eq. (115), can be decomposed, and simplified in the form

$$\begin{aligned}\boldsymbol{h}_{\boldsymbol{x}}\boldsymbol{w}_{\boldsymbol{g}_{\boldsymbol{x}}}\boldsymbol{h}_{\boldsymbol{x}}^{*} &= \sum_{i=1}^{4} e^{\lambda_i t}\left\{\sum_{j=1}^{4} e^{\bar{\lambda}_j s}\boldsymbol{p}_i\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi}\boldsymbol{p}_j^{*} + s\boldsymbol{p}_i\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi}\boldsymbol{p}_5^{*} + \boldsymbol{p}_i\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi}\boldsymbol{p}_6^{*}\right\}\delta \\ &+ \sum_{i=1}^{2} t^{2-k}\left\{\sum_{j=1}^{4} e^{\bar{\lambda}_j s}\boldsymbol{p}_{4+k}\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi}\boldsymbol{p}_j^{*} + s\boldsymbol{p}_{4+k}\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi}\boldsymbol{p}_5^{*} + \boldsymbol{p}_{4+k}\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi}\boldsymbol{p}_6^{*}\right\}\delta, \end{aligned} \tag{139}$$

where $\boldsymbol{\Phi}$ is the SVD unitary matrix. The matrices $\boldsymbol{p}_r, r = 1, 2, \ldots, 6$ are given by eqs. (135), (136) and the matrix $\boldsymbol{D}_{\boldsymbol{x}}$ by

$$\boldsymbol{D}_{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\pi 10^{-3} \end{bmatrix}, \tag{140}$$

respectively; see also eq. (91).

Next, evaluating the matrices $\boldsymbol{p}_i(\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi})\boldsymbol{p}_j^{*},\ i, j = 1, 2, \ldots, 6$, it is noted that

$$\boldsymbol{p}_i(\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi})\boldsymbol{p}_5^{*} = \boldsymbol{p}_i(\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi})\boldsymbol{p}_6^{*} = 0 \text{ for } i = 1, 2, 3, 4 \tag{141}$$

and

$$\boldsymbol{p}_5(\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi})\boldsymbol{p}_j^{*} = \boldsymbol{p}_6(\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi})\boldsymbol{p}_j^{*} = 0 \text{ for } j = 1, 2, \ldots, 6. \tag{142}$$

Hence, taking into consideration (141) and (142), eq. (139) takes a much simpler form, which being substituted in eq. (115), yields

$$\boldsymbol{w}_{\boldsymbol{z}_{\boldsymbol{x}}} = \sum_{i=1}^{4}\sum_{j=1}^{4}\boldsymbol{I}_{i,j}, \tag{143}$$

where

$$\boldsymbol{I}_{i,j}(\tau) = \int_0^{\infty}\int_0^{\infty} e^{\lambda_i t} e^{\bar{\lambda}_j s}\boldsymbol{p}_i(\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi})\boldsymbol{p}_j^{*}\delta(\tau + t - s)dtds, \tag{144}$$

or, equivalently,

$$\boldsymbol{I}_{i,j}(\tau) = -\boldsymbol{p}_i(\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi})\boldsymbol{p}_j^{*}\frac{e^{\bar{\lambda}_j\tau}}{\lambda_i + \bar{\lambda}_j}, \tag{145}$$

for $i, j = 1, 2, 3, 4$. Therefore, eq. (143), yields

$$\boldsymbol{w}_{\boldsymbol{z}_{\boldsymbol{x}}}(\tau) = -\sum_{i=1}^{4}\sum_{j=1}^{4}\frac{e^{\bar{\lambda}_j\tau}}{\lambda_i + \bar{\lambda}_j}\boldsymbol{p}_i(\boldsymbol{\Phi}^{*}\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{\Phi})\boldsymbol{p}_j^{*} \tag{146}$$

and for $\tau = 0$, eq.(146) becomes

$$\begin{aligned} \boldsymbol{w}_{\boldsymbol{z_x}}(0) &= -\sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\boldsymbol{p}_i(\boldsymbol{\Phi}^*\boldsymbol{D_x}\boldsymbol{\Phi})\boldsymbol{p}_j^*}{\lambda_i+\bar{\lambda}_j} \\ &= \begin{bmatrix} 0.0035 & 0.0011 & 0.0004 & 0.0006 & -0.0029 & 0.0043 \\ 0.0011 & 0.0227 & 0.0190 & -0.0008 & 0.0011 & -0.0008 \\ 0.0004 & 0.0190 & 0.0188 & -0.0056 & 0.0001 & 0 \\ 0.0006 & -0.0008 & -0.0056 & 0.0086 & -0.0009 & 0.0005 \\ -0.0029 & 0.0011 & 0.0001 & -0.0009 & 0.0593 & -0.0504 \\ 0.0043 & -0.0008 & 0 & 0.0005 & -0.0504 & 0.0438 \end{bmatrix}. \end{aligned} \tag{147}$$

Finally, using eq. (118), the covariance matrix of $\boldsymbol{p}$ becomes

$$\boldsymbol{w_p}(0) = \begin{bmatrix} 0.0438 & 0.0438 & 0.0252 & 0 & 0 & -0.0012 \\ 0.0438 & 0.0438 & 0.0252 & 0 & 0 & -0.0012 \\ 0.0252 & 0.0252 & 0.0190 & 0.0012 & 0.0012 & 0 \\ 0 & 0 & 0.0012 & 0.0188 & 0.0188 & 0.0063 \\ 0 & 0 & 0.0012 & 0.0188 & 0.0188 & 0.0063 \\ -0.0012 & -0.0012 & 0 & 0.0063 & 0.0063 & 0.0127 \end{bmatrix}, \tag{148}$$

which is in total agreement with the respective results determined in section 2.2.3 (see eq. (93)) via the solution of the Lyapunov equation. At this point, it is deemed necessary to mention that in contrast to the matrix calculated in eq. (93), the covariance matrix obtained by eq. (148) does not have any arbitrary elements y. This is due to the fact that for the solution of eq. (92), from which $\boldsymbol{V_x}$ is derived, eq. (3) is involved, whereas the M-P inverse notion is not utilized in any part of the procedure followed to determine eq. (148).

CONCLUSION

In this paper, certain concepts and relationships of the linear random vibration theory have been modified and generalized to account for structural systems with singular mass matrices, potentially arising when utilizing more than the minimum number of coordinates in modeling the system.

Specifically, relying on the generalized matrix inverse theory the Moore-Penrose (M-P) inverse of a singular mass matrix can be determined and arguably uniquely defined for engineering dynamical systems. Further, relying on the aforementioned convenient result, and adopting a state-variable formulation, equations governing the system response mean vector and covariance matrix have been formed and solved. Also, it has been shown that a complex modal analysis treatment, unlike the standard system modeling case, does not lead to decoupling of the equations of motion. Nevertheless, relying on a singular value decomposition of the system transition matrix facilitates significantly the efficient computation of the impulse response matrix, and ultimately, of the system response statistics. A 2-DOF linear structural system modeled by utilizing more than the minimum number of coordinates (thus, yielding a singular mass matrix) has been considered as a numerical example for demonstrating the applicability of the methodology as well as for elucidating certain related numerical aspects.

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List of Figures

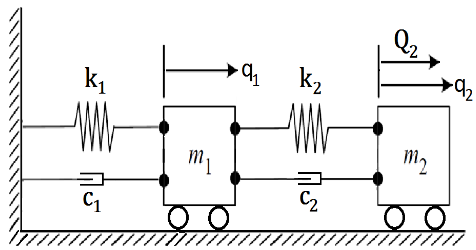

FIG. 1. A two degree-of-freedom linear structural system under stochastic excitation.

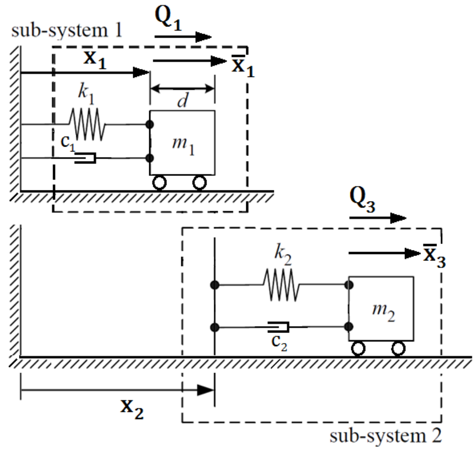

FIG. 2. Modeling of the system shown in Figure 1. using more than two coordinates.