STATISTICAL LINEARIZATION OF NONLINEAR STRUCTURAL SYSTEMS WITH SINGULAR MATRICES

Vasileios C. Fragkoulis[1], S.M. ASCE, Ioannis A. Kougioumtzoglou*[2], M. ASCE and Athanasios A. Pantelous[3]

ABSTRACT

A generalized statistical linearization technique is developed for determining approximately the stochastic response of nonlinear dynamic systems with singular matrices. This system modeling can arise when a greater than the minimum number of coordinates is utilized, and can be advantageous, for instance, in cases of complex multibody systems where the explicit formulation of the equations of motion can be a nontrivial task. In such cases, the introduction of additional/redundant degrees of freedom can facilitate the formulation of the equations of motion in a less labor intensive manner. Specifically, relying on the generalized matrix inverse theory and on the Moore-Penrose (M-P) matrix inverse, a family of optimal and response dependent equivalent linear matrices is derived. This set of equations in conjunction with a generalized excitation-response relationship for linear systems leads to an iterative determination of the system response mean vector and covariance matrix. Further, it is proved that setting the arbitrary element in the M-P solution for the equivalent linear matrices equal to zero yields a mean square error at least as low as the error corresponding to any non-zero value of the arbitrary element. This proof greatly facilitates the practical implementation of the technique as it promotes the utilization of the intuitively simplest solution among a family of possible solutions. A pertinent numerical example demonstrates the validity of the generalized technique.

[1]PhD Candidate, Department of Mathematical Sciences, University of Liverpool, Peach Str., Liverpool, L69 7ZL, UK. E-mail: v.fragkoulis@liverpool.ac.uk.

[2]Assistant Professor, Department of Civil Engineering and Engineering Mechanics, Columbia University, 610 S.W. Mudd Building, 500 W. 120th Str., New York, NY 10027. (corresponding author). E-mail: ikougioum@columbia.edu.

[3]Reader, Department of Mathematical Sciences, University of Liverpool, Peach Str., Liverpool, L69 7ZL, UK. E-mail: a.pantelous@liverpool.ac.uk.

Keywords: Structural Dynamics; Random Vibration; Statistical Linearization; Singular Matrices; Moore-Penrose Inverse.

INTRODUCTION

The dynamic analysis of nonlinear systems subjected to stochastic excitation has received considerable attention over the last six decades; see Lin (1967), Newland (1993), Grigoriu (2002), and Li and Chen (2009) for some indicative books, as well as Naess and Johnsen (1993), Pirrotta and Santoro (2011), Kougioumtzoglou and Spanos (2014) and Kougioumtzoglou et al. (2015) for some recently developed techniques such as the ones based on path integrals.

Undoubtedly, a critical role in the study of linear and nonlinear structural dynamic systems plays the procedure that is followed for the formulation of the system equations of motion, and in particular, the number of coordinates that are utilized. In general, using the minimum number of coordinates (generalized coordinates) for formulating the system equations of motion yields matrices that are not only non-singular, but also symmetric and positive definite (e.g. Pars 1979, Roberts and Spanos 2003). This feature of "well-behaved" matrices greatly facilitates the analysis of such dynamic systems since a number of techniques exist for determining efficiently the system response statistics (e.g. Roberts and Spanos 2003).

Nevertheless, it can be argued that there are cases, especially for large scale multi-body systems, where utilizing generalized coordinates for the system modeling is not always the most efficient approach. Specifically, the complexity of the equations of motion (and thus, the effort for their formulation) may increase rapidly with increasing the number of constituent bodies (e.g. Pradhan et al. 1997, Nikravesh et al. 1985, Schiehlen 1984, Schutte and Udwadia 2011, Mariti et al. 2011). In fact, in many cases the choice of modeling utilizing the minimum number of degrees-of-freedom (DOFs)/coordinates relates to excessive computational cost (e.g. Featherstone 1987, Bae and Haug 1987, Critchley and Anderson 2003, de Falco et al. 2009). On the other hand, employing additional/redundant, not independent, coordinates in the structural system modeling yields, typically, equations with singular mass, damping and stiffness matrices (e.g. Laulusa and Bauchau 2007, Nikravesh et al. 1985, Udwadia and Wanichanon 2013). Note in passing that utilizing redundant coordinates is not the only reason for the appearance of singular matrices in the system equations of motion. For instance, singularities may arise in certain applications such as in the rotational motion of rigid bodies even if generalized coordinates are employed (Udwadia and Wanichanon 2013, Nikravesh et al. 1985, Udwadia and Schutte 2010). Further, besides the case where theoretically non-singular, but numerically ill-conditioned matrices may also appear (e.g. Kawano et al. 2013), singular matrices are naturally met in the formulation of the equations of motion of a certain class of smart structures. In this class of vibrating systems, the system mechanical equation of motion is coupled with the electrical equation yielding a differential-algebraic system of equations with a singular mass matrix (e.g. Xu and Koko 2004, Kawano et al. 2013, Kamada et al. 1997). Note that systems described by a set of differential-algebraic equations belong to the wider class of the so-called descriptor systems (e.g. Kalogeropoulos et al. 2014, Gashi and Pantelous 2015).

Although it can be argued that in many cases (in particular when relatively complex systems are considered) the latter ”unconventional” modeling can be advantageous from a computational efficiency perspective (e.g. Udwadia and Kalaba 2007, Mariti et al. 2011), standard solution techniques (e.g. a state-variable formulation in conjunction with a complex modal analysis), that inherently assume the existence of non-singular matrices, cannot be applied in a straightforward manner. To address this challenge, the authors recently developed a solution framework for determining the stochastic response of linear dynamic systems with singular matrices (Fragkoulis et al. 2016).

In this paper, the aforementioned solution framework is generalized to account for nonlinear systems. Specifically, the popular and versatile statistical linearization approximate methodology (e.g. Roberts and Spanos 2003) is generalized herein to account for systems with singular matrices. In this regard, relying on the generalized matrix inverse theory and on the Moore-Penrose (M-P) matrix inverse, a family of optimal and response dependent equivalent linear matrices is derived. This set of equations in conjunction with a recently derived (e.g. Fragkoulis et al. 2016) linear system generalized excitation-response relationship leads to an iterative determination of the system response mean vector and covariance matrix. Further, it is proved that setting the arbitrary element in the M-P solution for the equivalent linear matrices equal to zero yields a mean square error at least as low as the error corresponding to any non-zero value of the arbitrary element. A pertinent numerical example demonstrates the validity of the generalized technique.

MOORE-PENROSE THEORY ELEMENTS

In this section, elements of the generalized matrix inverse theory, and in particular of the Moore-Penrose (M-P) inverse, are provided for completeness.

Definition. *If* $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ *then* $\boldsymbol{A}^+$ *is the unique matrix in* $\mathbb{C}^{n \times m}$ *so that*

$$\begin{aligned} \boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A} \quad &, \quad \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+, \\ (\boldsymbol{A}\boldsymbol{A}^+)^* = \boldsymbol{A}\boldsymbol{A}^+ \quad &, \quad (\boldsymbol{A}^+\boldsymbol{A})^* = \boldsymbol{A}^+\boldsymbol{A}. \end{aligned} \tag{1}$$

The matrix $\boldsymbol{A}^+$ is known as the M-P inverse of $\boldsymbol{A}$. The M-P inverse of a square matrix exists for any arbitrary $\boldsymbol{A} \in \mathbb{C}^{n \times n}$, and if $\boldsymbol{A}$ is non-singular, $\boldsymbol{A}^+$ coincides with $\boldsymbol{A}^{-1}$. Eq. (1) represents the so-called M-P equations. Further, the M-P inverse of any $m \times n$ matrix $\boldsymbol{A}$ can be determined by employing various techniques and methodologies, such as a number of recursive formulae (e.g., Campbell and Meyer 1979, Greville 1960), and provides a tool for solving equations of the form

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{2}$$

where $\boldsymbol{A}$ is a rectangular $m \times n$ matrix, $\boldsymbol{x}$ is an n vector and $\boldsymbol{b}$ is an m vector. For a singular square matrix $\boldsymbol{A}$, i.e. $det\boldsymbol{A} \neq 0$, utilizing the M-P inverse, Eq. (2) yields

$$\boldsymbol{x} = \boldsymbol{A}^+\boldsymbol{b} + (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{y}, \tag{3}$$

where $\boldsymbol{y}$ is an arbitrary n vector. A more detailed presentation of the topic can be found in Ben-Israel and Greville (2003) and Campbell and Meyer (1979).

EQUATIONS OF MOTION OF A NONLINEAR MDOF SYSTEM WITH SINGULAR MATRICES

The equations of motion of a general nonlinear n degree-of-freedom ($n-$DOF) system are given by

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} + \boldsymbol{\Phi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \boldsymbol{Q}(t), \tag{4}$$

where $\boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{K}$ are the $n \times n$ mass, damping and stiffness matrices, respectively. Further, $\boldsymbol{q}$ is the n vector containing the n (generalized) displacements of the system and $\boldsymbol{Q}$ is the n vector containing the n (generalized) forces, corresponding to $\boldsymbol{q}$. Finally, $\boldsymbol{\Phi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ is a nonlinear n vector of the (generalized) coordinates vector $\boldsymbol{q}$ and its derivatives. Considering next an alternative formulation of the equations of motion, where more than the minimum number coordinates are employed (e.g. Udwadia and Schutte 2010; Fragkoulis et al. 2016), Eq. (4) can take the form

$$\boldsymbol{M}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} + \boldsymbol{C}_{\boldsymbol{x}}\dot{\boldsymbol{x}} + \boldsymbol{K}_{\boldsymbol{x}}\boldsymbol{x} + \boldsymbol{\Phi}_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \boldsymbol{Q}_{\boldsymbol{x}}(t), \tag{5}$$

where $\boldsymbol{x}$ stands for an $l-$DOF vector of coordinates ($l \geq n$), $\boldsymbol{Q}_{\boldsymbol{x}}$ is the l vector of the external forces and $\boldsymbol{M}_{\boldsymbol{x}}, \boldsymbol{C}_{\boldsymbol{x}}$ and $\boldsymbol{K}_{\boldsymbol{x}}$ are the $l \times l$ mass, damping and stiffness matrices, respectively. The augmented nonlinear vector for the $l-$DOF system is given by $\boldsymbol{\Phi}_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}})$. Further, a number of constraint equations of the form

$$\boldsymbol{A}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\ddot{\boldsymbol{x}} = \boldsymbol{b}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t), \tag{6}$$

may arise that practically enforce the connection relations between the considered subsystems (e.g. Udwadia and Phohomsiri 2006). These arising constraints yield in turn a number of additional forces, and thus, Eq. (5) becomes

$$\boldsymbol{M}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} + \boldsymbol{C}_{\boldsymbol{x}}\dot{\boldsymbol{x}} + \boldsymbol{K}_{\boldsymbol{x}}\boldsymbol{x} + \boldsymbol{\Phi}_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \boldsymbol{Q}_{\boldsymbol{x}}(t) + \boldsymbol{Q}^c_{\boldsymbol{x}}(t), \tag{7}$$

where $\boldsymbol{Q}^c_{\boldsymbol{x}}(t)$ are the additional aforementioned forces. Also, the presence of constraints yields virtual displacements described by the l vector $\boldsymbol{w}$, which is any non-zero vector satisfying the condition

$$\boldsymbol{A}\boldsymbol{w} = \boldsymbol{0}, \tag{8}$$

and at any instant of time t can be expressed as

$$\boldsymbol{w}^T \boldsymbol{Q}_{\boldsymbol{x}}^c = \boldsymbol{w}^T \boldsymbol{N}. \tag{9}$$

The l vector $\boldsymbol{N}$ describes the nature of the non-ideal constraints and can be obtained by experimentation and/or observation. Taking into consideration Eq. (3), the solution to Eq. (8) becomes

$$\boldsymbol{w} = (\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A})\boldsymbol{y}, \tag{10}$$

or, equivalently,

$$\boldsymbol{w} = \tilde{\boldsymbol{A}}\boldsymbol{y}, \tag{11}$$

where

$$\tilde{\boldsymbol{A}} = \boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A}, \tag{12}$$

and $\boldsymbol{y}$ is an arbitrary l vector; therefore, Eq. (9) takes the form

$$\tilde{\boldsymbol{A}}\boldsymbol{Q}_{\boldsymbol{x}}^c = \tilde{\boldsymbol{A}}\boldsymbol{N}. \tag{13}$$

Next, multiplying Eq. (7) by $\tilde{\boldsymbol{A}}$ and considering Eq. (13) yields

$$\tilde{\boldsymbol{A}}\left\{\boldsymbol{M}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} + \boldsymbol{C}_{\boldsymbol{x}}\dot{\boldsymbol{x}} + \boldsymbol{K}_{\boldsymbol{x}}\boldsymbol{x} + \boldsymbol{\Phi}_{\boldsymbol{x}}\right\} = \tilde{\boldsymbol{A}}(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N}). \tag{14}$$

Further, without loss of generality and for facilitating the ensuing analysis, the m vector $\boldsymbol{b}$ in Eq. (6) is assumed to be of the form

$$\boldsymbol{b} = \boldsymbol{F} - \boldsymbol{E}\dot{\boldsymbol{x}} - \boldsymbol{L}\boldsymbol{x}. \tag{15}$$

Considering next Eqs. (6), (14) and (15) yields

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{A}}(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N}) \\ \boldsymbol{F} \end{bmatrix} - \begin{bmatrix} \tilde{\boldsymbol{A}}\boldsymbol{C}_{\boldsymbol{x}}\dot{\boldsymbol{x}} \\ \boldsymbol{E}\dot{\boldsymbol{x}} \end{bmatrix} - \begin{bmatrix} \tilde{\boldsymbol{A}}\boldsymbol{K}_{\boldsymbol{x}}\boldsymbol{x} \\ \boldsymbol{L}\boldsymbol{x} \end{bmatrix} - \begin{bmatrix} \tilde{\boldsymbol{A}}\boldsymbol{\Phi}_{\boldsymbol{x}} \\ \boldsymbol{0} \end{bmatrix}, \tag{16}$$

or, equivalently,

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{A}}(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S}) \\ \boldsymbol{b} \end{bmatrix}. \tag{17}$$

In Eq. (17), the $(m+l)\times l$ matrix $\bar{\boldsymbol{M}}_{\boldsymbol{x}}$ and the l vector $\boldsymbol{S}$ are given by

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{A}}\boldsymbol{M}_{\boldsymbol{x}} \\ \boldsymbol{A} \end{bmatrix}, \tag{18}$$

and

$$\boldsymbol{S} = -\boldsymbol{\Phi}_{\boldsymbol{x}} - \boldsymbol{C}_{\boldsymbol{x}}\dot{\boldsymbol{x}} - \boldsymbol{K}_{\boldsymbol{x}}\boldsymbol{x}, \tag{19}$$

respectively. Considering the M-P inverse, $\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}$, of the $(m+l)\times l$ matrix $\bar{\boldsymbol{M}}_{\boldsymbol{x}}$, the solution to Eq. (17) is given by

$$\ddot{\boldsymbol{x}} = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \begin{bmatrix} \tilde{\boldsymbol{A}}(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S}) \\ \boldsymbol{b} \end{bmatrix} + (\boldsymbol{I} - \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{M}}_{\boldsymbol{x}})\boldsymbol{y}. \tag{20}$$

Further, according to $Lemma\ 4$ in Udwadia and Shutte (2010), the relationship

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \begin{bmatrix} (\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{A}^{+}\boldsymbol{z}) + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix} = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \begin{bmatrix} \boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix}, \tag{21}$$

where $\bar{\boldsymbol{M}}_{\boldsymbol{x}}$ is the matrix defined in Eq. (18), holds true for any l vector $\boldsymbol{z}$. Therefore, by setting $\boldsymbol{z} = -\boldsymbol{A}(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S})$, Eq. (21) yields

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \begin{bmatrix} \tilde{\boldsymbol{A}}(\boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S}) \\ \boldsymbol{b} \end{bmatrix} = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \begin{bmatrix} \boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix}. \tag{22}$$

Taking into consideration Eq. (22), Eq. (20) degenerates to the form

$$\ddot{\boldsymbol{x}} = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \begin{bmatrix} \boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix} + (\boldsymbol{I} - \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \bar{\boldsymbol{M}}_{\boldsymbol{x}}) \boldsymbol{y}; \tag{23}$$

whereas considering ideal constraints, i.e. $\boldsymbol{N} = \boldsymbol{0}$, Eq. (23) becomes

$$\ddot{\boldsymbol{x}} = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \begin{bmatrix} \boldsymbol{Q}_{\boldsymbol{x}} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix} + (\boldsymbol{I} - \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \bar{\boldsymbol{M}}_{\boldsymbol{x}}) \boldsymbol{y}. \tag{24}$$

Taking into account Eqs. (19) and (24), the acceleration vector of the system takes the form

$$\ddot{\boldsymbol{x}} = \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \left\{ -\tilde{\boldsymbol{C}}_{\boldsymbol{x}} \dot{\boldsymbol{x}} - \tilde{\boldsymbol{K}}_{\boldsymbol{x}} \boldsymbol{x} - \tilde{\boldsymbol{\Phi}}_{\boldsymbol{x}} + \tilde{\boldsymbol{Q}}_{\boldsymbol{x}} \right\} + (\boldsymbol{I} - \bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} \bar{\boldsymbol{M}}_{\boldsymbol{x}}) \boldsymbol{y}, \tag{25}$$

where the $(m + l) \times l$ matrices $\tilde{\boldsymbol{C}}_{\boldsymbol{x}}$, $\tilde{\boldsymbol{K}}_{\boldsymbol{x}}$, as well as the $(m + l)$ vector $\tilde{\boldsymbol{Q}}_{\boldsymbol{x}}$ are given by

$$\tilde{\boldsymbol{C}}_{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{C}_{\boldsymbol{x}} \\ \boldsymbol{E} \end{bmatrix}, \tag{26}$$

$$\tilde{\boldsymbol{K}}_{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{K}_{\boldsymbol{x}} \\ \boldsymbol{L} \end{bmatrix}, \tag{27}$$

and

$$\tilde{\boldsymbol{Q}}_{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{Q}_{\boldsymbol{x}} \\ \boldsymbol{F} \end{bmatrix}, \tag{28}$$

respectively. Finally, the $(m + l)$ nonlinear vector $\tilde{\boldsymbol{\Phi}}_{\boldsymbol{x}}$ is given by

$$\tilde{\boldsymbol{\Phi}}_{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{\Phi}_{\boldsymbol{x}} \\ \boldsymbol{0} \end{bmatrix}. \tag{29}$$

It is noted that the simplified expression for the response acceleration, Eq. (25), facilitates significantly (e.g. Fragkoulis et al. 2016) an efficient state variable formulation of the original equations of motion. Overall, the augmented system of equations can be concisely written in the alternative form

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} + \bar{\boldsymbol{C}}_{\boldsymbol{x}}\dot{\boldsymbol{x}} + \bar{\boldsymbol{K}}_{\boldsymbol{x}}\boldsymbol{x} + \bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \bar{\boldsymbol{Q}}_{\boldsymbol{x}}(t) \tag{30}$$

where $\bar{\boldsymbol{M}}_{\boldsymbol{x}}$, $\bar{\boldsymbol{C}}_{\boldsymbol{x}}$ and $\bar{\boldsymbol{K}}_{\boldsymbol{x}}$ denote the $(m+l) \times l$ augmented mass, damping and stiffness matrices, $\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}})$ is the $(m+l)$ nonlinear vector of the system and $\bar{\boldsymbol{Q}}_{\boldsymbol{x}}$ denotes the $(m+l)$ augmented excitation vector. The augmented mass matrix is given by Eq. (18), whereas the augmented damping and stiffness matrices are given by

$$\bar{\boldsymbol{C}}_{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{A}}\boldsymbol{C}_{\boldsymbol{x}} \\ \boldsymbol{E} \end{bmatrix} \tag{31}$$

and

$$\bar{\boldsymbol{K}}_{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{A}}\boldsymbol{K}_{\boldsymbol{x}} \\ \boldsymbol{L} \end{bmatrix}, \tag{32}$$

respectively. Finally, the $(m+l)$ vector $\bar{\boldsymbol{Q}}_{\boldsymbol{x}}$ as well as the $(m+l)$ nonlinear vector $\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}$ are given by

$$\bar{\boldsymbol{Q}}_{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{A}}\boldsymbol{Q}_{\boldsymbol{x}} \\ \boldsymbol{F} \end{bmatrix} \tag{33}$$

and

$$\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{A}}\boldsymbol{\Phi}_{\boldsymbol{x}} \\ \boldsymbol{0} \end{bmatrix}. \tag{34}$$

A GENERALIZED STATISTICAL LINEARIZATION METHODOLOGY

Statistical linearization has been one of the most versatile approximate methodologies for determining the stochastic response of nonlinear systems efficiently (e.g. Roberts and Spanos 2003). The main objective of the methodology relates to the replacement of the original nonlinear system with an ”equivalent linear” one by appropriately minimizing the error vector corresponding to the difference between the two systems. The rationale behind this procedure is that closed form analytical expressions exist for the response statistics of a linear system, and thus, the response statistics of the equivalent linear system can be used as an approximation for the response statistics of the original nonlinear system. According to the standard implementation of the methodology, the minimization criterion relates typically to the mean square error, whereas the Gaussian assumption for the system response probability density functions (PDFs) is commonly adopted (e.g. Roberts and Spanos 2003). Note, that although more sophisticated implementations of the statistical linearization that relax the aforementioned assumptions and/or employ various other minimization criteria (e.g. Socha 2008) exist, these versions typically lack versatility. In this regard, one of the reasons for the wide utilization of the standard statistical linearization methodology has been, undoubtedly, its versatility in addressing a wide range of nonlinear behaviors. In particular, the Gaussian response assumption in conjunction with the mean square error minimization criterion facilitates the derivation of closed form expressions for the equivalent linear elements (e.g. stiffness, damping coefficients, etc) as functions of the response statistics. Further, regarding the stochastic response determination of linear systems, the authors have recently generalized certain concepts and solution techniques of the standard random vibration theory (e.g. Roberts and Spanos 2003, Li and Chen 2009) to account for systems with singular matrices (see Fragkoulis et al. 2016). These generalized techniques are utilized in the ensuing analysis for developing a generalized statistical linearization methodology.

Next, the statistical linearization approximate methodology is generalized to account for the nonlinear system with singular matrices of Eq. (30). To this aim, following closely Roberts and Spanos (2003), an equivalent linear system is sought in the form

$$(\bar{\boldsymbol{M}}_{\boldsymbol{x}} + \bar{\boldsymbol{M}}_e)\ddot{\boldsymbol{x}} + (\bar{\boldsymbol{C}}_{\boldsymbol{x}} + \bar{\boldsymbol{C}}_e)\dot{\boldsymbol{x}} + (\bar{\boldsymbol{K}}_{\boldsymbol{x}} + \bar{\boldsymbol{K}}_e)\boldsymbol{x} = \bar{\boldsymbol{Q}}_{\boldsymbol{x}}(t), \tag{35}$$

where $\bar{\boldsymbol{M}}_e, \bar{\boldsymbol{C}}_e$ and $\bar{\boldsymbol{K}}_e$ denote the equivalent linear $(m + l) \times l$ mass, damping and stiffness matrices, respectively, to account for the nonlinearity of the original system.

The error vector, $\boldsymbol{\varepsilon}$, between the nonlinear and the equivalent linear system is defined as

$$\boldsymbol{\varepsilon} = \bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) - \bar{\boldsymbol{M}}_e \ddot{\boldsymbol{x}} - \bar{\boldsymbol{C}}_e \dot{\boldsymbol{x}} - \bar{\boldsymbol{K}}_e \boldsymbol{x}. \tag{36}$$

Further, the mean square error is chosen to be minimized (e.g. Roberts and Spanos 2003), i.e.,

$$\mathbb{E}[\boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon}] = \text{minimum}, \tag{37}$$

for determining the equivalent linear matrices. This yields the equations

$$\frac{\partial}{\partial m_{ij}} \mathbb{E}[\boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon}] = 0, \tag{38}$$

$$\frac{\partial}{\partial c_{ij}} \mathbb{E}[\boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon}] = 0 \tag{39}$$

and

$$\frac{\partial}{\partial k_{ij}} \mathbb{E}[\boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon}] = 0, \tag{40}$$

where m_{ij}^e, c_{ij}^e and k_{ij}^e are the (i, j) elements of the matrices $\bar{\boldsymbol{M}}_e, \bar{\boldsymbol{C}}_e$ and $\bar{\boldsymbol{K}}_e$, respectively. Furthermore, combining Eqs. (36) with (37), the minimization criterion can be equivalently written as

$$\sum_{i=1}^{m+l} D_i^2 = \text{minimum}, \tag{41}$$

where

$$D_i^2 = \mathbb{E}\left\{\left[\bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}} - \sum_{j=1}^{l}(m_{ij}^e \ddot{x}_j + c_{ij}^e \dot{x}_j + k_{ij}^e x_j)\right]^2\right\}, \quad i = 1, 2, \ldots, (m+l) \tag{42}$$

and

$$\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}} = \left[\bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}})\right]^T, \quad i = 1, 2, \ldots, (m+l). \tag{43}$$

Clearly, due to the form of the expression in Eq. (41), the minimization criterion can be equivalently written as

$$D_i^2 = \text{minimum}, \quad i = 1, 2, \ldots, (m+l). \tag{44}$$

Next, minimizing Eq. (44) yields the equations

$$\mathbb{E}\left[\bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}\hat{\boldsymbol{x}}\right] = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]\begin{bmatrix}\boldsymbol{k}_{i*}^{eT}\\ \boldsymbol{c}_{i*}^{eT}\\ \boldsymbol{m}_{i*}^{eT}\end{bmatrix}, \quad i = 1, 2, \ldots, (m+l). \tag{45}$$

The $3l$ vector $\hat{\boldsymbol{x}}$ is defined as $\hat{\boldsymbol{x}} = (\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}})^T$ and $\boldsymbol{m}_{i*}^{eT}, \boldsymbol{c}_{i*}^{eT}$ and $\boldsymbol{k}_{i*}^{eT}$ correspond to the i^{th} row of $\bar{\boldsymbol{M}}_e, \bar{\boldsymbol{C}}_e$ and $\bar{\boldsymbol{K}}_e$, respectively. Further, adopting the standard Gaussian response assumption, the term on the left hand side of Eq. (45) is given by (Roberts and Spanos 2003)

$$\mathbb{E}[\bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}\hat{\boldsymbol{x}}] = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]\mathbb{E}[\nabla\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\hat{\boldsymbol{x}})]. \tag{46}$$

Combining next Eqs. (45) with (46) yields

$$\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]\begin{bmatrix}\boldsymbol{k}_{i*}^{eT}\\ \boldsymbol{c}_{i*}^{eT}\\ \boldsymbol{m}_{i*}^{eT}\end{bmatrix} = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]\mathbb{E}\begin{bmatrix}\dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}}{\partial\boldsymbol{x}}\\ \dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}}{\partial\dot{\boldsymbol{x}}}\\ \dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}}{\partial\ddot{\boldsymbol{x}}}\end{bmatrix}, \quad i = 1, 2, \ldots, (m+l). \tag{47}$$

Clearly, the determination of the equivalent linear elements in Eq. (47) requires the inversion of $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]$. Thus, the question arises whether this $3l \times 3l$ matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]$, which appears in both sides of Eq. (47), is singular or not. As proved in Spanos and Iwan (1978), a necessary and sufficient condition for $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]$ to be singular is that at least one of the components of the $3l$ vector $\hat{\boldsymbol{x}}$, can be expressed as a linear combination of the remaining components. In this regard, note that in the formulation herein it is assumed a priori that the elements of the coordinates vector $\boldsymbol{x}$ are not independent with each other as more than the minimum coordinates are utilized in forming the equations of motion. Thus, it is anticipated that some of the elements of $\hat{\boldsymbol{x}}$ are linearly dependent.

Therefore, the matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]$ in Eq. (47) is singular and its M-P inverse, denoted as $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]^+$, is employed next for determining an expression for the elements m^e_{ij}, c^e_{ij} and k^e_{ij} of the equivalent linear augmented matrices. Considering Eq. (3), Eq. (47) is written in the form

$$\begin{bmatrix} \boldsymbol{k}^{eT}_{\boldsymbol{i}*} \\ \boldsymbol{c}^{eT}_{\boldsymbol{i}*} \\ \boldsymbol{m}^{eT}_{\boldsymbol{i}*} \end{bmatrix} = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]^+\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]\mathbb{E}\begin{bmatrix} \dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}}{\partial\boldsymbol{x}} \\ \dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}}{\partial\dot{\boldsymbol{x}}} \\ \dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}}{\partial\ddot{\boldsymbol{x}}} \end{bmatrix} + \boldsymbol{g}(\boldsymbol{y}), \quad i = 1, 2, \ldots, (m+l), \tag{48}$$

where the $3l$ vector

$$\boldsymbol{g}(\boldsymbol{y}) = (\boldsymbol{I} - \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]^+\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T])\boldsymbol{y}, \tag{49}$$

is the arbitrary part of the M-P inverse based general solution of Eq. (47). At this point, it is deemed important to note that when the minimum number of coordinates, n, is utilized, $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]$ is a non-singular matrix yielding

$$\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]^+ = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]^{-1}. \tag{50}$$

In that case, $\hat{\boldsymbol{x}} = (\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})^T$ and, therefore, combining Eqs. (49) with (50), Eq. (48) takes the well-established form

$$\begin{bmatrix} \boldsymbol{k}^{eT}_{\boldsymbol{i}*} \\ \boldsymbol{c}^{eT}_{\boldsymbol{i}*} \\ \boldsymbol{m}^{eT}_{\boldsymbol{i}*} \end{bmatrix} = \mathbb{E}\begin{bmatrix} \dfrac{\partial\boldsymbol{\Phi}_{i,\boldsymbol{q}}}{\partial\boldsymbol{q}} \\ \dfrac{\partial\boldsymbol{\Phi}_{i,\boldsymbol{q}}}{\partial\dot{\boldsymbol{q}}} \\ \dfrac{\partial\boldsymbol{\Phi}_{i,\boldsymbol{q}}}{\partial\ddot{\boldsymbol{q}}} \end{bmatrix}, \quad i = 1, 2, \ldots, n. \tag{51}$$

Specifically, Eq. (51) represents the celebrated expressions for determining the elements of the equivalent linear mass, damping and stiffness matrices in the standard formulation of the statistical linearization methodology (e.g. Roberts and Spanos 2003). Nevertheless, when formulating the system equations of motion by employing additional DOFs, $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]$ is singular and the generalized version of Eq. (51) (i.e. Eq. (48)) needs to be considered. Regarding Eq. (48), it can be readily seen that a critical step for the practical implementation of the generalized statistical linearization methodology is the choice of the arbitrary element $\boldsymbol{y}$. It is proved in the following Proposition that the solution obtained by setting the arbitrary element $\boldsymbol{y}$ equal to zero is not only intuitively the simplest but it is also at least as good (in the sense of minimizing the mean square error, where the error $\boldsymbol{\varepsilon}$ is defined in Eq. (36)) as any other solution obtained by selecting an arbitrary non-zero value for $\boldsymbol{y}$. Specifically, setting $\boldsymbol{y} = \boldsymbol{0}$, Eq. (48) becomes

$$\begin{bmatrix} \boldsymbol{k}_{\boldsymbol{i*}}^{eT} \\ \boldsymbol{c}_{\boldsymbol{i*}}^{eT} \\ \boldsymbol{m}_{\boldsymbol{i*}}^{eT} \end{bmatrix} = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]^{+}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]\mathbb{E}\begin{bmatrix} \dfrac{\partial \bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}}{\partial \boldsymbol{x}} \\ \dfrac{\partial \bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}}{\partial \dot{\boldsymbol{x}}} \\ \dfrac{\partial \bar{\boldsymbol{\Phi}}_{i,\boldsymbol{x}}}{\partial \ddot{\boldsymbol{x}}} \end{bmatrix}, \quad i = 1, 2, \ldots, (m+l). \tag{52}$$

Assume next that $(m_{ij}^e, c_{ij}^e, k_{ij}^e)$ is the set of parameters arising from solving Eq. (52) and corresponding to the equivalent matrices $\bar{\boldsymbol{M}}_e, \bar{\boldsymbol{C}}_e$ and $\bar{\boldsymbol{K}}_e$. Also, selecting an arbitrary vector $\boldsymbol{y} = \boldsymbol{y}_0 \neq \boldsymbol{0}$ in Eq. (49), a different set of parameters, $(m_{ij}^{\prime e}, c_{ij}^{\prime e}, k_{ij}^{\prime e})$, corresponding to matrices $\bar{\boldsymbol{M}}_e^{'}, \bar{\boldsymbol{C}}_e^{'}, \bar{\boldsymbol{K}}_e^{'}$, is obtained via Eq. (48); see also Spanos and Iwan (1978).

Proposition. *Let $D_i^2(m_{ij}^e, c_{ij}^e, k_{ij}^e)$ and $D_i^2(m_{ij}^{\prime e}, c_{ij}^{\prime e}, k_{ij}^{\prime e})$ denote the errors as defined in Eq. (42) corresponding to the parameters values $m_{ij}^e, c_{ij}^e, k_{ij}^e$ and $m_{ij}^{\prime e}, c_{ij}^{\prime e}, k_{ij}^{\prime e}$, respectively. Then,*

$$D_i^2(m_{ij}^e, c_{ij}^e, k_{ij}^e) \leq D_i^2(m_{ij}^{\prime e}, c_{ij}^{\prime e}, k_{ij}^{\prime e}), \tag{53}$$

for $i = 1, 2, \ldots, (m+l)$ *and* $j = 1, 2, \ldots, l$.

Proof. From Eq. (42) it is seen that the quantity $\boldsymbol{D}_i^2(m_{ij}^e, c_{ij}^e, k_{ij}^e)$ is a quadratic polynomial with respect to the parameters m_{ij}^e, c_{ij}^e and k_{ij}^e. Therefore, its mixed partial derivatives concerning $m_{ij}^e, c_{ij}^e, k_{ij}^e$ of order higher that two vanish. Taking into account Eq. (48), the two sets of parameters are connected via the expressions

$$m_{ij}^{\prime e} = m_{ij}^e + g_{m,i}(y_0), \tag{54}$$

$$c_{ij}^{\prime e} = c_{ij}^e + g_{c,i}(y_0), \tag{55}$$

$$k_{ij}^{\prime e} = k_{ij}^e + g_{k,i}(y_0), \tag{56}$$

where the terms $g_{m,i}(y_0), g_{c,i}(y_0), g_{k,i}(y_0), i = 1, 2, \ldots, m + l$, represent the arbitrary elements as defined in Eq. (49). Next, considering a Taylor's expansion around $(m_{ij}^e, c_{ij}^e, k_{ij}^e)$, yields

$$\begin{aligned} D_i^2(m_{ij}^{\prime e}, c_{ij}^{\prime e}, k_{ij}^{\prime e}) &= D_i^2(m_{ij}^e, c_{ij}^e, k_{ij}^e) + \sum_{j=1}^{l}(\frac{\partial D_i^2}{\partial m_{ij}^e} g_{m,i}(y_0) + \frac{\partial D_i^2}{\partial c_{ij}^e} g_{c,i}(y_0) + \frac{\partial D_i^2}{\partial k_{ij}^e} g_{k,i}(y_0)) \\ &+ \frac{1}{2}\mathbb{E}\left\{\left[\sum_{j=1}^{l}(g_{m,i}(y_0)\ddot{x}_j + g_{c,i}(y_0)\dot{x}_j + g_{k,i}(y_0)x_j)\right]^2\right\}, \end{aligned} \tag{57}$$

for $i = 1, 2, \ldots, m + l$, where the terms $g_{m,i}(y_0), g_{c,i}(y_0)$ and $g_{k,i}(y_0)$ denote the distance between the two sets of parameters.

Also, taking into account Eqs. (38)-(40), the necessary conditions for minimizing Eq. (44) are

$$\frac{\partial D_i^2}{\partial m_{ij}^e} = 0, \tag{58}$$

$$\frac{\partial D_i^2}{\partial c_{ij}^e} = 0 \tag{59}$$

and

$$\frac{\partial D_i^2}{\partial k_{ij}^e} = 0. \tag{60}$$

Utilizing then Eqs. (58)-(60), the first sum on the right hand side of Eq. (57) is zero and Eq. (57) takes the form

$$D_i^2(m_{ij}^{\prime e}, c_{ij}^{\prime e}, k_{ij}^{\prime e}) = D_i^2(m_{ij}^e, c_{ij}^e, k_{ij}^e) + \frac{1}{2}\mathbb{E}\left\{J_i^2\right\},\ i = 1, 2, \ldots, m + l, \tag{61}$$

where

$$J_i = \sum_{j=1}^{l}(g_{m,i}(y_0)\ddot{x}_j + g_{c,i}(y_0)\dot{x}_j + g_{k,i}(y_0)x_j). \tag{62}$$

Finally, taking into account that $\mathbb{E}\left\{J_i^2\right\} \geq 0$, for all $i = 1, 2, \ldots, m + l$, Eq. (61) proves the argument stated in Eq. (53). □

Clearly, based on Eq. (53), utilizing Eq. (52) yields equivalent linear elements corresponding to an error that is at least as small (in a mean square sense) as any other obtained by utilizing a non-zero $\boldsymbol{y}$ vector in Eq. (48).

At this point, it is noted that comparing the standard Eq. (51) with its generalized counterpart Eq. (52) the equivalent linear matrices in Eq. (52) have typically a more complex structure than their counterparts in Eq. (51). Specifically, due to the fact that in Eq. (52) the product $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]^+\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]$ does not yield a unitary matrix, the equivalent linear matrices are anticipated to have many more non-zero components than in the case of utilizing Eq. (51). This observation is further highlighted in the numerical example section. Additionally, the determination of the equivalent linear matrices in Eq. (52) requires the knowledge of the response covariance matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]$. Obviously, an additional system of equations is needed that relates the two sets of unknowns, i.e. the response covariance matrix and the equivalent linear elements. In this regard, focusing on the linearized system of Eq. (35), generalized excitation-response relationships recently derived by the authors can be employed. Specifically, the standard state-variable formulation and the complex modal analysis were generalized for treating systems with singular matrices and for determining the augmented system response covariance matrix (see Fragkoulis et al. 2016). In the following subsections, the basic elements of these approaches are included for completeness.

State variable formulation and analysis

Considering the M-P inverse of the $\bar{\boldsymbol{M}}_{\boldsymbol{x}} + \bar{\boldsymbol{M}}_{\boldsymbol{e}}$ matrix, the augmented equivalent linear system of Eq. (35) can be cast in the form

$$\dot{\boldsymbol{p}} = \boldsymbol{G}_{\boldsymbol{x}}\boldsymbol{p} + \boldsymbol{f}_{\boldsymbol{x}}, \tag{63}$$

where $\boldsymbol{p} = \begin{bmatrix} \boldsymbol{x} & \dot{\boldsymbol{x}} \end{bmatrix}^T$; and the $2l \times 2l$ matrix $\boldsymbol{G}_{\boldsymbol{x}}$ and the $2l$ vector $\boldsymbol{f}_{\boldsymbol{x}}$, are given by

$$\boldsymbol{G}_{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -(\bar{\boldsymbol{M}}_{\boldsymbol{x}} + \bar{\boldsymbol{M}}_e)^+(\bar{\boldsymbol{K}}_{\boldsymbol{x}} + \bar{\boldsymbol{K}}_e) & -(\bar{\boldsymbol{M}}_{\boldsymbol{x}} + \bar{\boldsymbol{M}}_e)^+(\bar{\boldsymbol{C}}_{\boldsymbol{x}} + \bar{\boldsymbol{C}}_e) \end{bmatrix} \tag{64}$$

and

$$\boldsymbol{f_x} = \begin{bmatrix} \mathbf{0} \\ (\bar{\boldsymbol{M}}_{\boldsymbol{x}} + \bar{\boldsymbol{M}}_e)^+ \bar{\boldsymbol{Q}}_{\boldsymbol{x}} \end{bmatrix}, \tag{65}$$

respectively. Further, for zero initial conditions, i.e. $\boldsymbol{p}(0) = \mathbf{0}$, the solution of Eq. (63) is given by

$$\boldsymbol{p}(t) = \int_0^t \exp(\boldsymbol{G_x}\tau)\boldsymbol{f_x}(t-\tau)d\tau, \tag{66}$$

where the $2l \times 2l$ transition matrix $exp(\boldsymbol{G_x}t)$ has the block matrix form

$$exp(\boldsymbol{G_x}t) = \begin{bmatrix} \boldsymbol{a_x}(t) & \boldsymbol{b_x}(t) \\ \boldsymbol{c_x}(t) & \boldsymbol{d_x}(t) \end{bmatrix}. \tag{67}$$

Combining next Eqs. (66)-(67), the response displacement vector $\boldsymbol{x}$ is given by

$$\boldsymbol{x}(t) = \int_0^t \boldsymbol{h_x}(\tau)\bar{\boldsymbol{Q}}_{\boldsymbol{x}}(t-\tau)d\tau, \tag{68}$$

where

$$\boldsymbol{h_x}(t) = \boldsymbol{b_x}(t)(\bar{\boldsymbol{M}}_{\boldsymbol{x}} + \bar{\boldsymbol{M}}_e)^+ \tag{69}$$

can be construed as the uniquely defined ”*generalized*” impulse response matrix.

Note that in deriving Eq. (68) arguments for neglecting the arbitrary term associated with the M-P inverse of the $\bar{\boldsymbol{M}}_{\boldsymbol{x}} + \bar{\boldsymbol{M}}_e$ matrix have been employed. These relate to uniquely defining a response acceleration vector (see also Eq. (25)) as suggested by experimental observations; see Udwadia and Phohomsiri (2006) and Fragkoulis et al. (2016) for a detailed discussion.

Next, manipulating Eq. (63) and taking expectations yields the equation for the system response covariance matrix in the form

$$\dot{\boldsymbol{V}}_{\boldsymbol{x}} = \boldsymbol{G_x V_x} + \boldsymbol{V_x G_x^T} + \boldsymbol{S_x}, \tag{70}$$

where

$$\boldsymbol{S_x} = \int_0^t exp(\boldsymbol{G_x}(t-\tau)) \left[\boldsymbol{w_{\eta_x}}(t,\tau) + \boldsymbol{w}^T_{\boldsymbol{\eta_x}}(t,\tau)\right] d\tau; \tag{71}$$

and $\boldsymbol{w_{\eta_x}}$ denotes the covariance matrix of the vector

$$\boldsymbol{\eta_x} = \boldsymbol{f_x}(t) - \mathbb{E}[\boldsymbol{f_x}(t)]. \tag{72}$$

Further, for the case where the elements of $\boldsymbol{\eta_x}$ are regarded to be stationary white noises, Eq. (70) degenerates to

$$\dot{\boldsymbol{V}}_{\boldsymbol{x}} = \boldsymbol{G_x V_x} + \boldsymbol{V_x G_x^T} + \boldsymbol{D_x}, \tag{73}$$

where $\boldsymbol{D_x}$ is a real, symmetric, non-negative matrix of constants. Focusing next on the system stationary response, i.e. $\dot{\boldsymbol{V}}_{\boldsymbol{x}} = \boldsymbol{0}$, Eq. (73) becomes a Lyapunov equation of the form

$$\boldsymbol{G_x V_x} + \boldsymbol{V_x G_x^T} + \boldsymbol{D_x} = \boldsymbol{0} \tag{74}$$

that, notably, does not have a unique solution due to the form of the augmented matrix $\boldsymbol{G_x}$ as highlighted in Fragkoulis et al. (2016). Nevertheless, recasting the Lyapunov equation in a form that utilizes the Kronecker product, it has been shown that a solution for the response covariance matrix can be provided.

Complex modal analysis

Focusing next on a complex modal analysis treatment, due to the form of matrix $\boldsymbol{G_x}$, its eigenvectors that correspond to its zero eigenvalues are linearly dependent. Thus, a standard eigendecomposition analysis cannot be performed as is the case for modeling using generalized coordinates. In this regard, the singular value decomposition (SVD) method can be applied for matrix $\boldsymbol{G_x}$ yielding

$$\boldsymbol{G_x} = \boldsymbol{U \eta_x \Psi^*}, \tag{75}$$

where the diagonal $2l \times 2l$ matrix $\boldsymbol{\eta_x}$ is given by

$$\boldsymbol{\eta_x} = diag(\sigma_1, \sigma_2, \dots, \sigma_r, 0, 0, \dots, 0). \tag{76}$$

In Eq. (76), $\sigma_j = \sqrt{\lambda_j}, j = 1, 2, \dots, 2l$ denote the singular values of $\boldsymbol{G_x}$, whereas the $2l \times 2l$ matrices $\boldsymbol{U} = \begin{bmatrix} \boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_{2l} \end{bmatrix}$ and $\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \dots, \boldsymbol{\psi}_{2l} \end{bmatrix}$ are unitary. Further, $\boldsymbol{\psi}_j$ is the eigenvector corresponding to the singular value σ_j $(j = 1, 2, \dots, 2l)$ whereas $\boldsymbol{u}_j$ is equal to $\boldsymbol{u}_j = \dfrac{\boldsymbol{G_x \psi}_j}{\sigma_j}$.

Utilizing next the SVD of Eq. (75), Eq. (63) can be alternatively written as

$$\dot{\boldsymbol{z}}_{\boldsymbol{x}} = \boldsymbol{\Psi}^* \boldsymbol{U} \boldsymbol{\eta_x} \boldsymbol{z_x} + \boldsymbol{g_x}, \tag{77}$$

where

$$\boldsymbol{g_x} = \boldsymbol{\Psi}^* \boldsymbol{f_x} \tag{78}$$

and

$$\boldsymbol{p} = \boldsymbol{\Psi} \boldsymbol{z_x}. \tag{79}$$

Thus, Eq. (66) becomes

$$\boldsymbol{z_x}(t) = \int_0^t \boldsymbol{H_x}(s) \boldsymbol{g_x}(t-s) ds, \tag{80}$$

where

$$\boldsymbol{H_x}(t) = exp(\boldsymbol{\Psi}^* \boldsymbol{U} \boldsymbol{\eta_x} t). \tag{81}$$

As pointed out in Fragkoulis et al. (2016), a complex modal analysis does not result in uncoupling the coupled system of Eq. (77). Specifically, the product $\boldsymbol{\Psi}^* \boldsymbol{U}$ does not yield a unitary matrix as in the case of utilizing the minimum number of coordinates, and thus, $\boldsymbol{H_x}(t)$ is not diagonal. Nevertheless, relying on a SVD of matrix $\boldsymbol{G_x}$ facilitates significantly the numerical computation of the system response statistics. In particular, considering Eq. (80) and manipulating yields the covariance matrix $\boldsymbol{w}_{\boldsymbol{z_x}}$ of the response vector $\boldsymbol{z_x}$ in the form

$$\boldsymbol{w}_{\boldsymbol{z_x}}(\tau) = \int_0^\infty \int_0^\infty \boldsymbol{H_x}(s_1)\boldsymbol{w}_{\boldsymbol{g_x}}(\tau + s_1 - s_2)\boldsymbol{H}^*_{\boldsymbol{x}}(s_2)\, ds_1 ds_2. \tag{82}$$

Of course, the relationship $\boldsymbol{p} = \boldsymbol{\Psi}\boldsymbol{z_x}$ can be used for determining the covariance matrix of the response vector $\boldsymbol{p}$ in the form

$$\boldsymbol{w_p}(\tau) = \boldsymbol{\Psi}\boldsymbol{w}_{\boldsymbol{z_x}}(\tau)\boldsymbol{\Psi}^*. \tag{83}$$

Mechanization of the generalized statistical linearization methodology

Clearly, based on a modeling utilizing more than the minimum number degrees-of-freedom Eqs. (52) and (70) (or alternatively Eqs. (52) and (82)-(83) if a complex modal analysis is employed) constitute a coupled nonlinear system of equations to be solved for determining the system response covariance matrix and the equivalent linear elements. This can be solved by utilizing any appropriate standard numerical optimization scheme (e.g. Nocedal and Wright 2006), or even the following simple iterative procedure. Specifically,

(i) Assume zero initial (starting) values for the equivalent linear matrices $\bar{\boldsymbol{M}}_e$, $\bar{\boldsymbol{C}}_e$, and $\bar{\boldsymbol{K}}_e$.

(ii) Determine the system response covariance matrix via Eq. (70) (or alternatively via Eqs. (82)-(83)).

(iii) Utilize the system response covariance matrix values calculated in (ii) to determine the equivalent linear elements via Eq. (52).

(iv) Repeat steps (ii) and (iii) until convergence.

NUMERICAL EXAMPLE

As a numerical example the system of two rigid masses m_1 and m_2 shown in Figure 1 is considered. It is assumed that the mass m_1 is connected to the ground by a nonlinear spring of the linear-plus-cubic type and by a linear damper with coefficient c_1. Further, a mass m_2 is connected to m_1 by a linear spring and a linear damper with coefficients k_2 and c_2, respectively. The applied random force $Q_2(t)$ is modeled as a white-noise process with a correlation function $w_{Q_2}(t) = 2\pi S_0 \delta(t)$, where S_0 is the (constant) power spectrum value of $Q_2(t)$. Finally, q_1, q_2 are the generalized displacements, as shown in Figure 1. Further, utilizing generalized coordinates the equations of motion governing the system in Figure 1 can be written in the matrix form of Eq. (4), where the matrices $\boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{K}$ are given by (see also Roberts and Spanos 2003)

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \ \boldsymbol{C} = \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \ \boldsymbol{K} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}; \tag{84}$$

the coordinate vector $\boldsymbol{q}$ and the excitation vector $\boldsymbol{Q}(t)$ are given by

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{85}$$

and

$$\boldsymbol{Q} = \begin{bmatrix} 0 \\ Q_2(t) \end{bmatrix}, \tag{86}$$

respectively. Finally, the nonlinear function $\boldsymbol{\Phi}$ takes the form

$$\boldsymbol{\Phi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \begin{bmatrix} \varepsilon_1 k_1 q_1^3 \\ 0 \end{bmatrix}. \tag{87}$$

Next, taking into account Eqs. (51) and (87) yields the equivalent linear stiffness matrix

$$\boldsymbol{K}_e = \begin{bmatrix} 3\varepsilon_1 k_1 \sigma_{q_1}^2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{88}$$

Focusing next on the stationary system response, i.e. $\dot{\boldsymbol{V}} = \boldsymbol{0}$, a standard statistical linearization procedure in conjunction with a complex modal analysis treatment (e.g. Roberts and Spanos 2003) for the values $m_1 = m_2 = m = 1, c_1 = c_2 = c = 0.1, k_1 = k_2 = k = 1$ and $S_0 = 10^{-3}$, is applied. Regarding the numerical implementation, convergence based on the criterion $\left|\dfrac{\boldsymbol{K}_e^{j+1} - \boldsymbol{K}_e^{j}}{\boldsymbol{K}_e^{j}}\right| > 10^{-5}$, where the j index denotes the $j-th$ iteration, is satisfied after eight iterations. The initial value $\boldsymbol{K}_e^0$ has been set equal to zero. Further, by applying a complex modal analysis treatment, the eigenvalues of the system after the last iteration are

$$\begin{aligned} \lambda_1 &= -0.1308 - 1.6389i \quad , \quad \lambda_2 = -0.1308 + 1.6389i, \\ \lambda_3 &= -0.0192 - 0.6422i \quad , \quad \lambda_4 = -0.0192 + 0.6422i, \end{aligned} \tag{89}$$

whereas the corresponding eigenvectors are

$$\begin{aligned} \boldsymbol{v}_1^T &= \begin{bmatrix} -0.0357 - 0.4466i & 0.0188 + 0.2626i & 0.7366 & -0.4328 - 0.0036i \end{bmatrix}, \\ \boldsymbol{v}_2^T &= \begin{bmatrix} -0.0357 + 0.4466i & 0.0188 - 0.2626i & 0.7366 & -0.4328 + 0.0036i \end{bmatrix}, \\ \boldsymbol{v}_3^T &= \begin{bmatrix} -0.4260 - 0.0014i & -0.7255 & 0.0090 - 0.2736i & 0.0139 - 0.4659i \end{bmatrix}, \\ \boldsymbol{v}_4^T &= \begin{bmatrix} -0.4260 + 0.0014i & -0.7255 & 0.0090 + 0.2736i & 0.0139 + 0.4659i \end{bmatrix}. \end{aligned} \tag{90}$$

Finally, the obtained covariance matrix of the system response is given by

$$\boldsymbol{V} = \begin{bmatrix} 0.0386 & 0.0639 & 0 & -0.0010 \\ 0.0639 & 0.1102 & 0.0010 & 0 \\ 0 & 0.0010 & 0.0178 & 0.0252 \\ -0.0010 & 0 & 0.0252 & 0.0462 \end{bmatrix}. \tag{91}$$

Consider next the system of two masses m_1 and m_2 of the above example modeled as a multi-body one, and consisting of two separate subsystems as shown in Figure 2; see also Fragkoulis et al. (2016). In this regard, the two subsystems are related based on the constraint $x_2 = x_1 + d$ (where d is the length of mass m_1). The "*unconstrained*" equations of motion are derived by treating the three coordinates ($\bar{x}_1, x_2$ and $\bar{x}_3$) as independent with each other. Next, the equation of motion of the composite system is derived by including the constraint

$$x_2 = x_1 + d \tag{92}$$

or, equivalently,

$$x_2 = \bar{x}_1 + l_{1,0} + d, \tag{93}$$

where $l_{1,0}$ is the unstretched length of the spring k_1. Further, based on a Lagrangian formulation of the equations of motion, Eq. (5) becomes (Fragkoulis et al. 2016)

$$\boldsymbol{M_x} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & m_2 \\ 0 & m_2 & m_2 \end{bmatrix}, \; \boldsymbol{C_x} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_2 \end{bmatrix}, \; \boldsymbol{K_x} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_2 \end{bmatrix} \tag{94}$$

and

$$\boldsymbol{\Phi_x}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \begin{bmatrix} \varepsilon_1 k_1 \bar{x}_1^3 \\ 0 \\ 0 \end{bmatrix}, \; \boldsymbol{Q_x} = \begin{bmatrix} 0 \\ Q_3 \\ Q_3 \end{bmatrix}, \; \boldsymbol{x} = \begin{bmatrix} \bar{x}_1 \\ x_2 \\ \bar{x}_3 \end{bmatrix}, \tag{95}$$

where the variables $\bar{x}_1$ and $\bar{x}_3$ are defined as

$$\bar{x}_1 = x_1 - l_{1,0} \quad \text{and} \quad \bar{x}_3 = x_3 - l_{2,0}. \tag{96}$$

In Eq. (96), $l_{2,0}$ is the unstretched length of the spring k_2. Further, differentiating the constraint of Eq. (93), the two sub-systems are subject to, yields

$$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\bar{x}}_1 \\ \ddot{x}_2 \\ \ddot{\bar{x}}_3 \end{bmatrix} = 0. \tag{97}$$

Thus, the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{b}$ of Eq. (6) take the form

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad b = 0. \tag{98}$$

Furthermore, utilizing Eqs. (30), (94), (95) and (98), the new augmented equation of motion can be determined. The matrices $\bar{\boldsymbol{M}}_{\boldsymbol{x}}, \bar{\boldsymbol{C}}_{\boldsymbol{x}}, \bar{\boldsymbol{K}}_{\boldsymbol{x}}$, as well as the vectors $\bar{\boldsymbol{Q}}_{\boldsymbol{x}}$ and $\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}$ are given by

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad \bar{\boldsymbol{C}}_{\boldsymbol{x}} = \begin{bmatrix} 0.05 & 0 & 0 \\ 0.05 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\boldsymbol{K}}_{\boldsymbol{x}} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{99}$$

and

$$\bar{\boldsymbol{Q}}_{\boldsymbol{x}} = \begin{bmatrix} 0.5w(t) \\ 0.5w(t) \\ w(t) \\ 0 \end{bmatrix}, \quad \bar{\boldsymbol{\Phi}}_{\boldsymbol{x}} = \begin{bmatrix} 0.5\varepsilon_1 k_1 \bar{x}_1^3 \\ 0.5\varepsilon_1 k_1 \bar{x}_1^3 \\ 0 \\ 0 \end{bmatrix}. \tag{100}$$

Applying next Eq. (52) for determining the equivalent linear stiffness matrix $\bar{\boldsymbol{K}}_e$ yields

$$\boldsymbol{k}^{eT}_{\mathbf{1*}} = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix} \begin{bmatrix} \frac{3}{2}\varepsilon_1 k_1 \sigma^2_{\bar{x}_1} \\ 0 \\ 0 \end{bmatrix} \quad , \quad \boldsymbol{k}^{eT}_{\mathbf{2*}} = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix} \begin{bmatrix} \frac{3}{2}\varepsilon_1 k_1 \sigma^2_{\bar{x}_1} \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{k}^{eT}_{\mathbf{3*}} = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix} \mathbb{E} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \quad , \quad \boldsymbol{k}^{eT}_{\mathbf{4*}} = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix} \mathbb{E} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}, \qquad (101)$$

where $r_{i,j}, i, j = 1, 2, \ldots, 9$ denotes the element of the matrix $\boldsymbol{r} = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]^{+}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]$ in position (i, j). Hence, considering Eq. (101) the equivalent linear matrix $\bar{\boldsymbol{K}}_e$ can be concisely written as

$$\bar{\boldsymbol{K}}_e = \frac{3}{2}\varepsilon_1 k_1 \sigma^2_{\bar{x}_1} \begin{bmatrix} \boldsymbol{r}_{1,1} & \boldsymbol{r}_{2,1} & \boldsymbol{r}_{3,1} \\ \boldsymbol{r}_{1,1} & \boldsymbol{r}_{2,1} & \boldsymbol{r}_{3,1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (102)$$

At this point, comparing Eqs. (88) and (102) it is noted that although the general form of the equivalent linear stiffness matrices is similar, the equivalent linear matrix of Eq. (102) has more non-zero elements. Clearly, this is due to the presence of matrix $\boldsymbol{r}$ which, unlike the generalized coordinates modeling case, is not unitary. Next, employing Eq. (64), the matrix $\boldsymbol{G}_{\boldsymbol{x}}$ takes the form

$$\boldsymbol{G}_{\boldsymbol{x}} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\bar{\boldsymbol{M}}^{+}_{\boldsymbol{x}}(\bar{\boldsymbol{K}}_{\boldsymbol{x}} + \bar{\boldsymbol{K}}_e) & -\bar{\boldsymbol{M}}^{+}_{\boldsymbol{x}}\bar{\boldsymbol{C}}_{\boldsymbol{x}} \end{bmatrix}, \qquad (103)$$

where the M-P inverse of $\bar{\boldsymbol{M}}_{\boldsymbol{x}}$ is found by Eq. (18) to be equal to

$$\bar{\boldsymbol{M}}_{\boldsymbol{x}}^{+} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 2 & 1 \end{bmatrix}. \tag{104}$$

Further, as in the case of the covariance matrix obtained in Eq. (91) for the $2-$DOF system, a complex modal analysis treatment is utilized for deriving the covariance matrix of the system response. Also, to be consistent with the previously obtained result, the convergence criterion and error are the same as those utilized for deriving Eq. (91). In this regard, convergence is reached after eight iterations. Employing Eqs. (75)-(81), the eigenvalues of the matrix $\boldsymbol{\Psi}^{*}\boldsymbol{U}\boldsymbol{\eta}_{\boldsymbol{x}}$, where $\boldsymbol{\Psi}, \boldsymbol{U}, \boldsymbol{\eta}_{\boldsymbol{x}}$ are defined in Eq. (75), after the last iteration are

$$\begin{aligned} &\lambda_1 = -0.1308 - 1.6389i \quad , \quad \lambda_2 = -0.1308 + 1.6389i \ , \ \lambda_3 = -0.0192 - 0.6422i, \\ &\lambda_4 = -0.0192 + 0.6422i \quad , \quad \lambda_5 = 0 \ , \ \lambda_6 = 0, \end{aligned} \tag{105}$$

whereas the corresponding eigenvectors are

$$\boldsymbol{v}_1^T = \begin{bmatrix} -0.0145 - 0.4629i \\ -0.0432 - 0.0020i \\ 0.4009 + 0.0278i \\ 0.7540 \\ 0.0051 - 0.0227i \\ -0.0343 + 0.2281i \end{bmatrix}, \boldsymbol{v}_2^T = \begin{bmatrix} -0.0145 + 0.4629i \\ -0.0432 + 0.0020i \\ 0.4009 - 0.0278i \\ 0.7540 \\ 0.0051 + 0.0227i \\ -0.0343 - 0.2281i \end{bmatrix}, \boldsymbol{v}_3^T = \begin{bmatrix} -0.0308 + 0.0028i \\ 0.0006 - 0.4181i \\ -0.0177 - 0.3418i \\ -0.0060 - 0.0025i \\ 0.6740 \\ 0.5027 - 0.0111i \end{bmatrix},$$

$$\boldsymbol{v}_4^T = \begin{bmatrix} -0.0308 - 0.0028i \\ 0.0006 + 0.4181i \\ -0.0177 + 0.3418i \\ -0.0060 + 0.0025i \\ 0.6740 \\ 0.5027 + 0.0111i \end{bmatrix}, \boldsymbol{v}_5^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{v}_6^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{106}$$

After determining the eigenvalues and eigenvectors of the matrix $\boldsymbol{\Psi}^* \boldsymbol{U} \boldsymbol{\eta}_{\boldsymbol{x}}$, Eq. (82) evaluated at $\tau = 0$ takes the form

$$\boldsymbol{w}_{\boldsymbol{z}_{\boldsymbol{x}}}(0) = -\sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\boldsymbol{p}_i (\boldsymbol{\Psi}^* \boldsymbol{D}_{\boldsymbol{x}} \boldsymbol{\Psi}) \boldsymbol{p}_j^*}{\lambda_i + \bar{\lambda}_j}, \tag{107}$$

where $\lambda_i, i = 1, 2, 3, 4$ are given by Eq. (105) and $\boldsymbol{D}_{\boldsymbol{x}}$ is a real, symmetric, non-negative matrix of constants given by

$$\boldsymbol{D}_{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\pi 10^{-3} \end{bmatrix}. \tag{108}$$

In Eq. (107), the expressions $\boldsymbol{p}_i, i = 1, 2, 3, 4$ denote 6×6 matrices defined in terms of the matrix $\boldsymbol{\Psi}^* \boldsymbol{U} \boldsymbol{\eta_x}$, as well as the eigenvalues calculated in Eq. (105). For example, $\boldsymbol{p}_1$ is defined as (see Fragkoulis et al. 2016 for more details)

$$\boldsymbol{p}_1 = \frac{(\boldsymbol{\Psi}^* \boldsymbol{U} \boldsymbol{\eta_x} - \lambda_2 \boldsymbol{I})(\boldsymbol{\Psi}^* \boldsymbol{U} \boldsymbol{\eta_x} - \lambda_3 \boldsymbol{I})(\boldsymbol{\Psi}^* \boldsymbol{U} \boldsymbol{\eta_x} - \lambda_4 \boldsymbol{I})(\boldsymbol{\Psi}^* \boldsymbol{U} \boldsymbol{\eta_x})^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)\lambda_1^2}. \tag{109}$$

Finally, employing Eq. (83), the covariance matrix of the system response is given by

$$\boldsymbol{w_p}(0) = \begin{bmatrix} 0.0386 & 0.0386 & 0.0253 & 0 & 0 & -0.0010 \\ 0.0386 & 0.0386 & 0.0253 & 0 & 0 & -0.0010 \\ 0.0253 & 0.0253 & 0.0210 & 0.0010 & 0.0010 & 0 \\ 0 & 0 & 0.0010 & 0.0178 & 0.0178 & 0.0074 \\ 0 & 0 & 0.0010 & 0.0178 & 0.0178 & 0.0074 \\ -0.0010 & -0.0010 & 0 & 0.0074 & 0.0074 & 0.0136 \end{bmatrix}. \tag{110}$$

Indicatively, comparing Eqs. (91) and (110), the variance $\mathbb{E}[q_1^2]$ as well as $\mathbb{E}[\dot{q}_1^2]$ obtained in the first example, coincide with the respective ones in the second one, i.e $\mathbb{E}[\bar{x}_1^2]$ and $\mathbb{E}[\dot{\bar{x}}_1^2]$. Further, taking expectations in the equation that connects the two reference systems, that is $\bar{x}_3 = q_2 - q_1$, and utilizing Eq. (91) yields

$$\mathbb{E}[\bar{x}_3^2] = \mathbb{E}[q_2^2] + \mathbb{E}[q_1^2] - 2\mathbb{E}[q_1 q_2] = 0.0210 \tag{111}$$

and

$$\mathbb{E}[\dot{\bar{x}}_3^2] = \mathbb{E}[\dot{q}_2^2] + \mathbb{E}[\dot{q}_1^2] - 2\mathbb{E}[\dot{q}_1 \dot{q}_2] = 0.0136, \tag{112}$$

which are indeed in agreement with the corresponding values in Eq. (110). It can be readily verified that the rest of the elements of the matrix given in Eq. (110) are also in agreement with the respective ones of Eq. (91). It is noted that, alternatively, the response covariance matrix $\boldsymbol{V_x}$ can be obtained by utilizing a state variable formulation in conjunction with the Lyapunov equation of Eq. (74); see Fragkoulis et al. (2016) for more details.

CONCLUSION

In this paper the standard and popular statistical linearization methodology for determining approximately the stochastic response of nonlinear dynamic systems (Roberts and Spanos 2003) has been generalized to account for systems with singular matrices. This kind of modeling can appear for various reasons, among which is the utilization of additional/redundant coordinates. This can be advantageous for cases of complex multi-body systems, for instance, where formulating the equations of motion in terms of the independent/generalized coordinates can be a non-trivial task.

Specifically, relying on the generalized matrix inverse theory and on the M-P inverse of a singular matrix, a family of optimal and response dependent equivalent linear matrices has been derived. Next, this set of equations has been combined with a recently developed by the authors generalized linear system input-output (excitation-response) relationship to yield a coupled system of nonlinear algebraic equations. This can be readily solved via an iterative procedure for determining the system response mean vector and covariance matrix. A significant additional contribution of the paper relates to the proof that the solution obtained by setting the arbitrary element in the M-P expression for the equivalent linear matrices equal to zero is at least as good (in a mean square error minimization sense) as any other solution corresponding to a non-zero value for the arbitrary element. This proof greatly facilitates the practical implementation of the technique as it promotes the utilization of the intuitively simplest solution among a family of possible solutions. Finally, a $2-$DOF nonlinear system modeled by utilizing redundant coordinates is employed in the numerical examples section to demonstrate the validity of the herein developed generalized statistical linearization methodology.

ACKNOWLEDGMENT

The first and the third authors would like to gratefully acknowledge the support of this work through the EPSRC and ESRC Centre for Doctoral Training on Quantification and Management of Risk & Uncertainty in Complex Systems & Environments (EP/L015927/1). Further, the authors gratefully acknowledge the fruitful interaction with Prof. Antonina Pirrotta (University of Palermo and University of Liverpool).

REFERENCES

Bae, D. S., and Haug, E. J. (1987). ”A recursive formulation for constraint mechanical system dynamics: Part II Closed loop systems.” *Mech. Struct. Mach.*, 15(4), 481-506.

Ben-Israel, A., and Greville, T. N. E. (2003). *Generalized Inverses: Theory and applications, 2nd Edition*, Springer-Verlag, New York.

Campbell, S. L., and Meyer, Jr C. D. (1979). *Generalized Inverses of Linear Transformations*, Dover Publications, New York.

Critchley, J. H., and Anderson, K. S. (2003). ”A generalized recursive coordinate reduction method for multi-body dynamic systems.” *Int. J. Multiscale Comp. Eng.*, 1(2-3), 181-200.

de Falco, D., Pennestrì, E., and Vita, L. (2009). ”Investigation of the Influence of Pseudoinverse Matrix Calculations on Multibody Dynamics Simulations by Means of the Udwadia-Kalaba Formulation.” *J. Aerosp. Eng.*, 22(4), 365-372.

Featherstone, R. (1987). *Robot Dynamics Algorithms*, Kluwer, New York.

Fragkoulis, V. C., Kougioumtzoglou, I. A., and Pantelous, A. A. (2016). ”Linear random vibration of structural systems with singular matrices.” ASCE *J. Eng. Mech.*, 142(2), 04015081: 1-11.

Gashi, B., and Pantelous, A. A. (2015). ”Linear backward stochastic differential systems of descriptor type with structure and applications to engineering.” *Prob. Eng. Mech.*, 40, 1-11.

Greville, T. N. E. (1960). ”Some Applications of the Pseudoinverse of a Matrix.” *SIAM Rev.*, 2, 15-22.

Grigoriu, M. (2002). *Stochastic calculus, applications in science and engineering*, Birkhauser, Boston.

Kalogeropoulos, G. I., Karageorgos, A. D., and Pantelous, A. A. (2014). ”On the solution of higher order linear homogeneous complex $\sigma - \alpha$ descriptor matrix differential systems of Apostol-Kolodner type.” *J. Frank. Inst.*, 351(3), 1756-1777.

Kamada, T., Fujita, T., Hatayama, T., Arikabe, T., Murai, N., Aizawa, S., and Tohyama, K. (1997). ”Active vibration control of frame structures with smart structures using piezoelectric actuators (Vibration control by control of bending moments of columns).” *Smart Mater. Struct.* 6, 448-456.

Kawano, D. T., Morzfeld, M., and Ma, F. (2013). ”The decoupling of second order linear systems with a singular mass matrix.” *J. Sound Vibration*, 332(25), 6829-6846.

Kougioumtzoglou, I. A., Di Matteo, A., Spanos, P. D., Pirrotta, A., and Di Paola, M. (2015). ”An Efficient Wiener Path Integral Technique Formulation for Stochastic Response Determination of Nonlinear MDOF Systems.” *J. Appl. Mech.*, 82(10):101005-101005-7. doi:10.1115/1.4030890.

Kougioumtzoglou, I. A., and Spanos, P. D. (2014). ”Nonstationary Stochastic Response Determination of Nonlinear Systems: A Wiener Path Integral Formalism.” *J. Eng. Mech.*, 10.1061/(ASCE)EM.1943-7889.0000780, 04014064.

Laulusa, A., and Bauchau, O. A. (2007). ”Review of classical approaches for constraint enforcement in multibody systems.” ASME. *J. Comput. Nonlinear Dynam.*, 3(1):011004-011004-8. doi:10.1115/1.2803257.

Li, J., and Chen, J. (2009). *Stochastic dynamics of structures*, J. Wiley and Sons, Singapore.

Lin, Y. K. (1967). *Probabilistic theory of structural dynamics*, McGraw-Hill, New York.

Mariti, L., Belfiore, N. P., Pennestrì, E., and Valentini, P. P. (2011), ”Comparison of solution strategies for multibody dynamics equations.” *Int. J. Numer. Meth. Eng.*, 88(7), 637656.

Naess, A., and Johnsen, J. M. (1993). ”Response statistics of nonlinear, compliant offshore structures by the path integral solution method.” *Probab. Eng. Mech.*, 8(2), 91-106.

Newland, D. E. (1993). *An introduction to random vibrations, spectral and wavelet analysis, 3rd Edition*, Dover, New York.

Nikravesh, P. E., Wehage, R. A., and Kwon, O. K. (1985). ”Euler Parameters in Computational Kinematics and Dynamics. Part 1.” ASME *J. Mech., Trans., and Automation.*, 107(3):358-365. doi:10.1115/1.3260722.

Nocedal, J., and Wright, S. (2006). *Numerical Optimization, 2nd Edition*, Springer, New York.

Pars, L. A. (1979). *A treatise on analytical dynamics*, CT: Oxbow Press, Woodridge.

Pirrotta, A., and Santoro R. (2011). ”Probabilistic response of nonlinear systems under combined normal and Poisson white noise via path integral method.” *Probab. Eng. Mech.*, 26(1), 26-32.

Pradhan, S., Modi, V. J., and Misra, A. K. (1997). ”Order N formulation for flexible multi-body systems in tree topology: Lagrangian approach.” *J. Guid. Control Dyn.*, 20(4), 665-672.

Roberts, J. B., and Spanos, P. D. (2003). *Random Vibration and Statistical Linearization*, Dover Publications, New York.

Schiehlen, W. O. (1984). ”Dynamics of complex multibody systems.” *SM Arch.*, 9: 159-195.

Schutte, A. D., and Udwadia, F. E. (2011). ”New approach to the modeling of complex multi-body dynamical systems.” *J. Appl. Mech.*, 78(2), 021018.

Socha, L. (2008). *Linearization Methods for Stochastic Dynamic Systems*, Lect. Notes Phys. 730, Springer-Verlag, Berlin Heidelberg.

Spanos, P-T. D., and Iwan, W. D. (1978). ”On the existence and uniqueness of solutions generated by equivalent linearization.”, *Int. J. Non-Linear Mech.*, 13(2), 71-78.

Udwadia, F. E., and Kalaba, R. E. (2007). *Analytical Dynamics. A New Approach*, Cambridge University Press, New York.

Udwadia, F. E., and Phohomsiri, P. (2006). ”Explicit equations of motion for constrained mechanical systems with singular mass matrices and applications to multi-body dynamics.” *Proc. R. Soc. A*, 462, 2097-2117.

Udwadia, F. E., and Schutte, A. D. (2010). ”Equations of motion for general constrained systems in Lagrangian mechanics.”, *Acta Mech.*, 213(1), 111-129.

Udwadia, F. E., and Wanichanon, T. (2013). ”On general nonlinear constrained mechanical systems.”, *Numerical Algebra, Control and Optimization*, 3(3), 425-443.

Xu, S. X., and Koko, T. S. (2004). ”Finite element analysis and design of actively controlled piezoelectric smart structures.” *Finite Elem. Anal. Des.*, 40(3), 241-262.

List of Figures

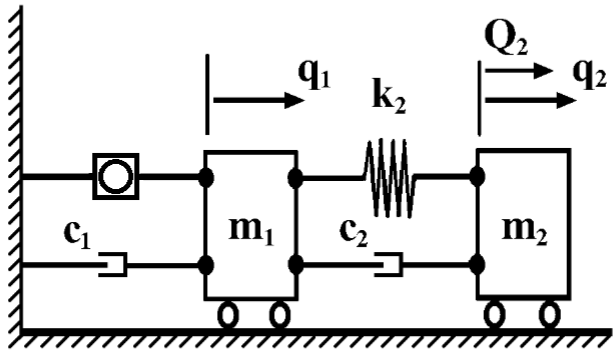

FIG. 1. A two degree-of-freedom nonlinear structural system under stochastic excitation.

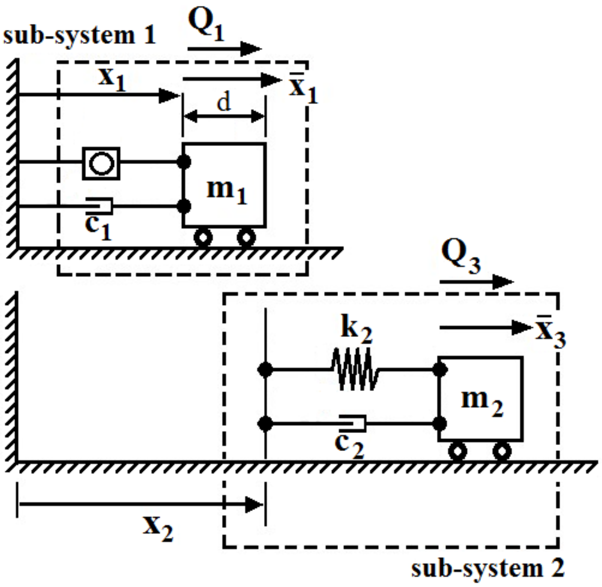

FIG. 2. Modeling of the system shown in Figure 1 using more than two coordinates.