Phenomenological survey of free fermionic heterotic–string models

Alon E. Faraggi
Theoretical Physics Division, University of Liverpool, Liverpool L69 7ZL, UK

June 30th, 2008

Talk at the Sixth Simons Workshop in Mathematics and Physics
Stony Brook University, June 16 - July 12, 2008

Abstract

A glossary overview of the phenomenological studies of quasi–realistic free fermionic heterotic string models is presented. I elaborate on the correspondence of these models with $Z_2 \times Z_2$ orbifolds.

Introduction

The standard model of particle physics passes all experimental observations with flying colors. The gauge charges of the standard model matter states are strongly suggestive of the embedding of the standard model in larger grand unified groups. This is particularly striking in the context of $SO(10)$ grand unification in which each of the standard model matter generation is embedded in a single 16 spinorial representation of $SO(10)$. We recall that the standard model gauge charges were experimentally discovered and therefore are experimental observables. To account for these charges in the framework of the standard model requires $3 \times 3 \times 6 = 54$ distinct parameters, taking into account the three group factors, the three generations and the six multiplets (including the right–handed neutrino) within each generation. The embedding of the standard model reduces this number to one, being the number of spinorial representations needed to accommodate the three generations of the standard model, namely three. The evidence for the realisation of grand unification structures in nature is therefore striking indeed.

The standard model and grand unification in themselves cannot, however, be the end of the story. The first apparent question that pops to mind is how three generations came to be and not two, four or five. Next, there are the various mass and flavor mixing parameters of the standard model. The origin of these parameters is not explained in the context of the standard model, nor in grand unified theories.

It is plausible therefore that to seek answers to these questions one must explore the origins of the Standard Model at a more basic level. In modern parlance this means at an energy scale, which is above the the GUT energy scale, *i.e.* the Planck scale, where the strength of the gravitational interaction is comparable to that of the gauge interactions. We are lead to this conclusion by the structure of the standard model itself.

String theory provides a self consistent framework for the synthesis of quantum mechanics and gravity. It is a natural extension of point quantum field theories. It admits a quantised particle interpretation, which in is a highly non–trivial result. Furthermore, the internal particle attributes, which in point particle gauge theories are ad hoc, arise in string theory from the internal consistency conditions. We can interpret these internal degrees of freedom as extra space time dimensions. The important feature of string theory is that while providing a consistent approach to quantum gravity it gives rise to the gauge and matter structures that are used in contemporary quantum field theories and the standard model. This enables the development of a phenomenological approach to quantum gravity by constructing string models that aim to reproduce the standard model and in turn can be used to explore the dynamics of string theory and its fundamental properties from a phenomenological point of view.

The five ten dimensional string theories, as well as eleven supergravity are believed to be limits of a more fundamental theory. Any one of this limits can be used to construct phenomenological string models. As limits of a more fundamental theory we should not expect any of the limits to provide a complete description of the true vacuum but merely to probe some of its properties. As the standard model data favor its embedding in $SO(10)$, the two pivotal requirements from a phenomenological string vacuum is the existence of three generations and their embedding into $SO(10)$ multiplets. The perturbative string limit that facilitates the embedding in $SO(10)$ is the heterotic–string as it is the limit that produces spinorial representations in the perturbative spectrum. Thus, to preserve these two key properties of the standard model spectrum the perturbative string limit that should be used is the heterotic string. It is likely that to obtain insight into other properties of the true vacuum other limits other perturbative string limits should be used. For example, the dilaton exhibits a run away behaviour in the perturbative heterotic limit and its stabilisation requires moving away from that limit.

The study of phenomenological string vacua proceeds with the compactification of the heterotic–string from ten to four dimensions. A class string compactifications that preserve the $SO(10)$ embedding of the Standard Model spectrum are those that are based on the $Z_2 \times Z_2$ orbifold and have been extensively studied by utilizing the so–called free fermionic formulation [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

There are of course a large number of requirements that a realistic string vacuum should satisfy. Here I list a few of these requirements:

- $\longrightarrow \quad SU(3) \times SU(2) \times U(1)^n \times hidden$
- Three generations
- Proton stable $\qquad (\tau_{\rm P} > 10^{30}$ years)
- Higgs doublets $\oplus$ potentially realistic Yukawa couplings
- N=1 SUSY $\qquad$ (N=0)
- Agreement with $\underline{\sin^2\theta_W}$ and $\underline{\alpha_s}$ at M_Z (+ other observables).
- Light left–handed neutrinos
- $SU(2) \times U(1)$ breaking
  SUSY breaking
  No flavor changing neutral currents
  No strong CP violation
  Exist family mixing and weak CP violation
- + ...
- + **GRAVITY**

The fermionic formulation was developed in the mid–eighties [13]. Just as the point particle time parameter spans a world–line, the string time and internal parameters span the two dimensional string world–sheet. The equivalence of bosons and fermions of a two dimensional conformal field theory entails that a model constructed using the fermionic approach correspond to a model constructed using the bosonic approach in which the target–space is compactified on a six dimensional internal manifold. In this vein the free fermionic formalism correspond to using a free bosonic formalism in which the radii of the internal dimensions are fixed at a special point in the compact space. Deformation from the special point in the moduli space are parametrized in terms of world–sheet Thirring interactions among the world–sheet fermions [14]. This equivalence is merely the simplest illustration of the relation between world–sheet rational conformal field theories and manifolds with $SU(n)$ holonomy [15]. The simplicity of the free fermionic formalism entails that the string consistency constraints are solved in terms of the world–sheet free fermion transformation properties on the string world–sheet, which are encoded in sets of basis vectors and one–loop GSO projection coefficients among the basis vectors. The formalism to extract the physical spectrum and superpotential interaction terms are also straightforward. The simplest free fermionic constructions correspond to a $Z_2 \times Z_2$ orbifold of a six dimensional toroidal manifold, augmented with discrete Wilson lines that are needed to break the $SO(10)$ GUT symmetry. The quasi–realistic free fermionic heterotic–string standard–like models were constructed in the late eighties and early nineties. They provide a concrete framework to study many of the issue that pertain to the phenomenology of the Standard Model and string unification. A few highlights of these studies are listed below:

- Top quark mass $\sim$ 175–180GeV [6, 16]
- Generation mass hierarchy [17]
- CKM mixing [18]
- Stringy seesaw mechanism [19, 20]
- Gauge coupling unification [21, 22]
- Proton stability [23]
- Squark degeneracy [24]
- Minimal Standard Heterotic String Model (MSHSM) [9]
- Moduli fixing [25]
- Classification & spinor–vector duality [26]

Perhaps, the most tantalising achievement is the successful calculation of the top quark mass, which was obtained several years prior to the experimental discovery, and in the correct mass range. This calculation demonstrated how string theory enables the calculation of the fermion–scalar Yukawa couplings in terms of the unified gauge coupling. Furthermore, the string models offered an explanation for the hierarchical mass splitting between the top and bottom quarks. The top quark Yukawa coupling is obtained at the cubic level of the superpotential and is of order one, whereas the Yukawa couplings of the lighter quarks and leptons are obtained from nonrenormalizable operators that are suppressed relative to the leading cubic level term. Thus, only the top quark mass is characterised by the electroweak scale and the masses of the lighter quarks and leptons are naturally suppressed compared to it. As the heavy generation Yukawa couplings are obtained at low orders in the superpotential, the calculation of these Yukawa couplings is robust and is common to a large class of models. The analysis of fermion masses was then further pursued, and quasi–realistic fermion mass textures were shown to arise for reasonable choices of supersymmetric flat directions. Issues like left–handed neutrino masses, gauge coupling unification, proton stability and squark degeneracy were studied in concrete quasi–realistic free fermionic string models and for detailed solutions of the supersymmetric flat direction constraints. While an attempt to find a single solution that satisfies all the variety of phenomenological requirements listed above was not pursued, it was demonstrated that all of the above requirements can find satisfactory solutions in the context of the free fermionic string models. It was also demonstrated in ref. [9] that the free fermionic heterotic string vacua give rise to models that produce in the observable charged sector below the string unification scale solely the matter spectrum of the minimal supersymmetric standard model. Such models are dubbed Minimal Standard Heterotic String Models (MSHSM). The free fermionic models also provide important clues to the problem of moduli fixing in string theory. They highlight the fact that string theory may utilize geometrical structures that do not have a classical correspondence. Primarily, they allow boundary conditions that distinguish between the left– and right–moving coordinates of the six dimensional compactified space. Such boundary conditions necessarily lead to the projection of the moduli fields associated with the extra internal coordinates. The free fermionic models have also been instrumental in recent years to unravel a new duality symmetry under the exchange of spinor and vector representations of the GUT group.

String theory predicts that the number of degrees of freedom giving rise to the gauge symmetries of the standard model should be augmented by a specific number of additional degrees of freedom. An naive interpretation of some of those is as extra space–time dimensions. These additional degrees of freedom may be out of reach of contemporary experiments, and the development of phenomenological string models aims at bridging the gap. String models give rise to additional symmetries and matter sectors that do not arise in grand unified theories. These include: gauge symmetries that are external to the GUT symmetries and may play a pivotal role in explaining proton stability [27]; matter states that arise due to the breaking of non–Abelian gauge symmetries by Wilson lines, which gives rise to matter states that do not obey the GUT charge quantisation, and may lead to stable string relics [28]; specific soft SUSY breaking patterns and consequently specific predictions for the superpartners mass spectrum [24, 29]. While all of these will be parametrised in terms of point quantum field theory parameters, their experimental observations will provide further evidence for the validity of string theory and specific string compactifications with which they are compatible. The final step in this program is to seek the all elusive dynamical mechanism, based on first principles, that singles out the string vacuum. The free fermionic models, and the association of the free fermionic point in the moduli space with the self–dual point under T–duality, suggests that self–duality play a vital role in this selection principle [30].

Correspondence with $Z_2 \times Z_2$ orbifold

I elaborate here on the relation to $Z_2 \times Z_2$ orbifold in which there has been some recent interest [31]. In general, due to the equivalence of two dimensional fermions and bosons we can anticipate that any model constructed by using world–sheet fermions can also be constructed by using world–sheet bosons. The constructions using world–sheet bosons are the toroidal orbifolds, whereas the fermionic construction is formulated at the point in the moduli space at which the fermions are free. Models in this formalism are defined in terms of boundary condition basis vectors and one loop generalized GSO projection coefficients. The correspondence between free fermion models and bosonic constructions can be illustrated by starting with the set of basis vectors

$$\{1, S, \xi_1, \xi_2\}, \tag{1}$$

generates a model with $N = 4$ space-time supersymmetry and $\mathrm{SO}(12) \times \mathrm{E}_8 \times \mathrm{E}_8$ gauge group. The basis vector S is the space–time supersymmetry generator and the two basis vectors ξ_1 and ξ_2 produce the two spinorial 128 representation of $SO(16)$ and enhance $SO(16) \times SO(16)$ to $\mathrm{E}_8 \times \mathrm{E}_8$. The free fermionic realization of the six compactified dimensions gives rise to the maximal $SO(12)$ enhanced symmetry. The same model can be constructed by using the bosonic construction. The action for the D–dimensional compactified string is given by,

$$S = \frac{1}{8\pi}\int d^2\sigma (G_{ij}\partial^\alpha X^i \partial_\alpha X^j + \epsilon^{\alpha\beta} B_{ij}\partial_\alpha X^i \partial_\beta X^j) \ ,$$

where,

$$G_{ij} = \frac{1}{2}\sum_{I=1}^{D} R_i e_i^I R_j e_j^I \ ,$$

is the metric of the six dimensional compactified space and $B_{ij} = -B_{ji}$ is the antisymmetric tensor field. The $e^i = \{e_i^I\}$ are six linear independent vectors normalized to $(e_i)^2 = 2$. The left– and right–moving momenta are given by,

$$P^I_{R,L} = [m_i - \frac{1}{2}(B_{ij} \pm G_{ij})n_j] e_i^{I^*} \ , \tag{2}$$

where the $e_i^{I^*}$ are dual to the e_i, and $e_i^* \cdot e_j = \delta_{ij}$. The left– and right–moving momenta span a Lorentzian even self–dual lattice. The mass formula for the left and right–movers is,

$$M_L^2 = -c + \frac{P_L \cdot P_L}{2} + N_L = -1 + \frac{P_R \cdot P_R}{2} + N_R = M_R^2 \ ,$$

where $N_{L,R}$ are the sum on the left–moving and right–moving oscillators and c is a normal ordering constant equal to $\frac{1}{2}$ and 0 for the antiperiodic (NS) and periodic (R) sectors of the NSR fermions [32]. The background fields that produce the toroidal $SO(12)$ lattice are given by the metric,

$$g_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \ , \tag{3}$$

and the antisymmetric tensor,

$$b_{ij} = \begin{cases} g_{ij} & ; i > j, \\ 0 & ; i = j, \\ -g_{ij} & ; i < j. \end{cases} \tag{4}$$

When all the radii of the six-dimensional compactified manifold are fixed at $R_I = \sqrt{2}$, it is seen that the right–moving momenta given by eqs. (2) produce the root vectors of $SO(12)$ [33]. The next step in the construction of the $Z_2 \times Z_2$ orbifold is to add the two basis vectors b_1 and b_2 that each breaks $N = 4$ space–time supersymmetry to $N = 2$. With a suitable choice of the GSO projection coefficients the model possesses an $\mathrm{SO}(4)^3 \times \mathrm{E}_6 \times \mathrm{U}(1)^2 \times \mathrm{E}_8$ gauge group and $N = 1$ space-time supersymmetry. The matter fields include 24 generations in the 27 representation of E_6, eight from each of the sectors $b_1 \oplus b_1 + \xi_1$, $b_2 \oplus b_2 + \xi_1$ and $b_3 \oplus b_3 + \xi_1$. Three additional 27 and $\overline{27}$ pairs are obtained from the Neveu-Schwarz $\oplus\, \xi_1$ sector. The same spectrum is obtained by acting with the $Z_2 \times Z_2$ orbifold on the $SO(12)$ lattice with standard embedding.

A $Z_2 \times Z_2$ orbifold at a generic point, however, produces forty–eight fixed points, and hence forty–eight generations rather than twenty–four. There is therefore a mismatch by a factor of two between the two models. This mismatch seems puzzling because a priori we do not expect that the number of fixed points does not depend on the moduli.

To investigate this issue further we can start with the $Z_2 \times Z_2$ orbifold on $T_2^1 \times T_2^2 \times T_2^3$, which gives $(h_{11}, h_{21}) = (51, 3)$. I will denote the manifold of this model as X_1. We can then add a freely acting twist or shift [34] to this model, which reduces the number of fixed points. Let us first start with the compactified $T_2^1 \times T_2^2 \times T_2^3$ torus parameterized by three complex coordinates z_1, z_2 and z_3, with the identification

$$z_i = z_i + 1\,, \qquad z_i = z_i + \tau_i\,, \tag{5}$$

where τ is the complex parameter of each T_2 torus. With the identification $z_i \to -z_i$, a single torus has four fixed points at

$$z_i = \{0, \tfrac{1}{2}, \tfrac{1}{2}\tau, \tfrac{1}{2}(1+\tau)\}. \tag{6}$$

With the two Z_2 twists

$$\begin{aligned} \alpha &: (z_1, z_2, z_3) \to (-z_1, -z_2, \;\; z_3)\,, \\ \beta &: (z_1, z_2, z_3) \to (\;\; z_1, -z_2, -z_3)\,, \end{aligned} \tag{7}$$

there are three twisted sectors in this model, α, β and $\alpha\beta = \alpha \cdot \beta$, each producing 16 fixed tori, for a total of 48. Adding to the model generated by the $Z_2 \times Z_2$ twist in (7), the additional shift

$$\gamma : (z_1, z_2, z_3) \to (z_1 + \tfrac{1}{2}, z_2 + \tfrac{1}{2}, z_3 + \tfrac{1}{2}) \tag{8}$$

produces again fixed tori from the three twisted sectors α, β and $\alpha\beta$. The product of the γ shift in (8) with any of the twisted sectors does not produce any additional fixed tori. Therefore, this shift acts freely. Under the action of the γ-shift, the fixed tori from each twisted sector are paired. Therefore, γ reduces the total number of fixed tori from the twisted sectors by a factor of 2, yielding $(h_{11}, h_{21}) = (27, 3)$. This model therefore reproduces the data of the $Z_2 \times Z_2$ orbifold at the free-fermion point in the Narain moduli space.

We note that the freely acting shift (8), added to the $Z_2 \times Z_2$ orbifold at a generic point of $T_2^1 \times T_2^2 \times T_2^3$, reproduces the data of the $Z_2 \times Z_2$ orbifold acting on the SO(12) lattice. This observation does not prove, however, that the vacuum which includes the shift is identical to the free fermionic model. While the massless spectrum of the two models may coincide their massive excitations, in general, may differ. The matching of the massive spectra is examined by constructing the partition function of the $Z_2 \times Z_2$ orbifold of an SO(12) lattice, and subsequently that of the model at a generic point including the shift. In effect since the action of the $Z_2 \times Z_2$ orbifold in the two cases is identical the problem reduces to proving the existence of a freely acting shift that reproduces the partition function of the SO(12) lattice at the free fermionic point. Then since the action of the shift and the orbifold projections are commuting it follows that the two $Z_2 \times Z_2$ orbifolds are identical.

The realization of the $SO(12)$ lattice as an orbifold in achieved by incorporating idenitifications on the internal lattice by shift symmetries. It is instructive for this purpose to study the partition function at a generic point in the moduli space, incorporate the shifts, and fix the internal radii at the self–dual point, which then reproduces the partition function of the $SO(12)$ lattice. The partition function of the $N=4$ supersymmetric $SO(12)\times E_8\times E_8$ heterotic vacuum is given by

$$Z=(V_8-S_8)\left[|O_{12}|^2+|V_{12}|^2+|S_{12}|^2+|C_{12}|^2\right]\left(\bar{O}_{16}+\bar{S}_{16}\right)\left(\bar{O}_{16}+\bar{S}_{16}\right)\,, \tag{9}$$

where Z has been written in terms of level-one SO($2n$) characters (see, for instance, [35])

$$\begin{aligned} O_{2n} &= \tfrac{1}{2}\left(\frac{\vartheta_3^n}{\eta^n}+\frac{\vartheta_4^n}{\eta^n}\right)\,, \\ V_{2n} &= \tfrac{1}{2}\left(\frac{\vartheta_3^n}{\eta^n}-\frac{\vartheta_4^n}{\eta^n}\right)\,, \\ S_{2n} &= \tfrac{1}{2}\left(\frac{\vartheta_2^n}{\eta^n}+i^{-n}\frac{\vartheta_1^n}{\eta^n}\right)\,, \\ C_{2n} &= \tfrac{1}{2}\left(\frac{\vartheta_2^n}{\eta^n}-i^{-n}\frac{\vartheta_1^n}{\eta^n}\right)\,. \end{aligned} \tag{10}$$

On the compact coordinates there are actually three inequivalent ways in which the shifts can act. In the more familiar case, they simply translate a generic point by half the length of the circle. As usual, the presence of windings in string theory allows shifts on the T-dual circle, or even asymmetric ones, that act both on the circle and on its dual. More concretely, for a circle of length $2\pi R$, one can have the following options [36]:

$$\begin{aligned} A_1 &: \quad X_{\rm L,R}\to X_{\rm L,R}+\tfrac{1}{2}\pi R\,, \\ A_2 &: \quad X_{\rm L,R}\to X_{\rm L,R}+\tfrac{1}{2}\left(\pi R\pm\frac{\pi\alpha'}{R}\right)\,, \\ A_3 &: \quad X_{\rm L,R}\to X_{\rm L,R}\pm\tfrac{1}{2}\frac{\pi\alpha'}{R}\,. \end{aligned} \tag{11}$$

There is, however, a crucial difference among these three choices: while A_1 and A_3 shifts can act consistently on any number of coordinates, level-matching requires instead that the A_2-shifts act on (mod) four real coordinates.

Our problem is to find the shift that when acting on the lattice $T_2^1\otimes T_2^2\otimes T_2^3$ at a generic point in the moduli space reproduces the $SO(12)$ lattice when the radii are fixed at the self–dual point $R=\sqrt{\alpha'}$ [37]. Let us consider for simplicity the case of six orthogonal circles or radii R_i. The partition function reads

$$Z_+=(V_8-S_8)\left(\sum_{m,n}\Lambda_{m,n}\right)^{\otimes 6}\left(\bar{O}_{16}+\bar{S}_{16}\right)\left(\bar{O}_{16}+\bar{S}_{16}\right)\,, \tag{12}$$

where as usual, for each circle,

$$p^i_{\rm L,R} = \frac{m_i}{R_i} \pm \frac{n_i R_i}{\alpha'}\,, \tag{13}$$

and

$$\Lambda_{m,n} = \frac{q^{\frac{\alpha'}{4}p^2_{\rm L}}\,\bar q^{\frac{\alpha'}{4}p^2_{\rm R}}}{|\eta|^2}\,. \tag{14}$$

We can now act with the $Z_2 \times Z_2$ shifts generated by

$$\begin{aligned} g &: \quad (A_2, A_2, 0)\,,\\ h &: \quad (0, A_2, A_2)\,, \end{aligned} \tag{15}$$

where each A_2 acts on a complex coordinate. The resulting partition function then reads

$$\begin{aligned} Z_+ \;=\; & \tfrac{1}{4}\,(V_8 - S_8) \sum_{m_i,n_i} \Big\{ \big[1 + (-1)^{m_1+m_2+m_3+m_4+n_1+n_2+n_3+n_4} \\ & + (-1)^{m_1+m_2+m_5+m_6+n_1+n_2+n_5+n_6} + (-1)^{m_3+m_4+m_5+m_6+n_3+n_4+n_5+n_6} \big] \\ & \times \left(\Lambda^{1,\ldots,6}_{m_i,n_i} + \Lambda^{1,\ldots,4}_{m_i+\frac12,n_i+\frac12}\Lambda^{5,6}_{m_i,n_i} + \Lambda^{1,2,5,6}_{m_i+\frac12,n_i+\frac12}\Lambda^{3,4}_{m_i,n_i} + \Lambda^{1,2}_{m_i,n_i}\Lambda^{3,4,5,6}_{m_i+\frac12,n_i+\frac12} \right) \Big\} \\ & \times \left(\bar O_{16} + \bar S_{16}\right)\left(\bar O_{16} + \bar S_{16}\right) \end{aligned} \tag{16}$$

After some tedious algebra, it is then possible to show that, once evaluated at the self-dual radius $R_i = \sqrt{\alpha'}$, the partition function (16) reproduces that at the SO(12) point (9). To this end, it suffices to notice that

$$\begin{aligned} \sum_{m,n} \Lambda_{m,n}(R=\sqrt{\alpha'}) &= |\chi_0|^2 + |\chi_{\frac12}|^2\,,\\ \sum_{m,n} (-1)^{m+n}\Lambda_{m,n}(R=\sqrt{\alpha'}) &= |\chi_0|^2 - |\chi_{\frac12}|^2\,,\\ \sum_{m,n} \Lambda_{m+\frac12,n+\frac12}(R=\sqrt{\alpha'}) &= \chi_0\bar\chi_{\frac12} + \chi_{\frac12}\bar\chi_0\,,\\ \sum_{m,n} (-1)^{m+n}\Lambda_{m+\frac12,n+\frac12}(R=\sqrt{\alpha'}) &= \chi_{\frac12}\bar\chi_0 - \chi_0\bar\chi_{\frac12}\,, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \chi_0 &= \sum_\ell q^{\ell^2}\,,\\ \chi_{\frac12} &= \sum_\ell q^{(\ell+\frac12)^2}\,, \end{aligned} \tag{18}$$

are the two level-one SU(2) characters, while, standard branching rules, decompose the SO(12) characters into products of SU(2) ones. For instance,

$$\begin{aligned} O_{12} \;=\;& \chi_0\chi_0\chi_0\chi_0\chi_0\chi_0 + \chi_0\chi_0\chi_{\frac{1}{2}}\chi_{\frac{1}{2}}\chi_{\frac{1}{2}}\chi_{\frac{1}{2}} + \\ & \chi_{\frac{1}{2}}\chi_{\frac{1}{2}}\chi_0\chi_0\chi_{\frac{1}{2}}\chi_{\frac{1}{2}} + \chi_{\frac{1}{2}}\chi_{\frac{1}{2}}\chi_{\frac{1}{2}}\chi_{\frac{1}{2}}\chi_0\chi_0 \,. \end{aligned} \tag{19}$$

The precise form of the orbifold shifts that produces the $SO(12)$ lattice is given in eq. (15). On the other hand, the shifts given in Eq. (8), and similarly the analogous freely acting shift given by (A_3, A_3, A_3), do not reproduce the partition function of the $SO(12)$ lattice. Therefore, the shift in eq. (8) does reproduce the same massless spectrum and symmetries of the $Z_2 \times Z_2$ of the $SO(12)$ lattice, but the partition functions of the two models differ!

Another method to exhibit the reduction of the number of fixed points of the $Z_2 \times Z_2$ orbifold of the $SO(12)$ lattice was presented in ref. [39], using the canonical orbifold method [38]. The basis vectors of the $SO(12)$ lattice are given by the simple roots

$$\begin{aligned} e_1 &= (1,-1,0,0,0,0)\,, \\ e_2 &= (0,1,-1,0,0,0)\,, \\ e_3 &= (0,0,1,-1,0,0)\,, \\ e_4 &= (0,0,0,1,-1,0)\,, \\ e_5 &= (0,0,0,0,1,-1)\,, \\ e_6 &= (0,0,0,0,1,1)\,. \end{aligned} \tag{20}$$

The $Z_2 \times Z_2$ orbifold action on a set of six Cartesian coordinates $x^1, \ldots, x^6$ of the compact space is specified by:

$$\begin{pmatrix} x^1 \\ \vdots \\ x^6 \end{pmatrix} \to \theta_1 \begin{pmatrix} x^1 \\ \vdots \\ x^6 \end{pmatrix}, \;\; \text{with } \theta_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

and

$$\begin{pmatrix} x^1 \\ \vdots \\ x^6 \end{pmatrix} \to \theta_2 \begin{pmatrix} x^1 \\ \vdots \\ x^6 \end{pmatrix}, \;\; \text{with } \theta_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{22}$$

where θ_1 and θ_2 are the generators of $Z_2 \times Z_2$.

The orbifold action, by e.g. (22), leaves sets of points invariant, i.e. these points differ from their orbifold image by an SO(12) root lattice shift. For our particular choice of the orbifold action these sets appear as two dimensional fixed tori. In the following I will list 16 such two-tori and afterwards argue that some of these 16 tori are identical. That will leave eight distinct fixed two-tori. The trivial fixed torus is given as the set

$$\left\{(x, y, 0, 0, 0, 0) \,\middle|\, x, y \in \mathbb{R}^2/\Lambda^2\right\}. \tag{23}$$

The compactification lattice Λ^2 is generated by the vectors $(1,1)$ and $(1,-1)$. This can be verified by writing

$$(x, y, 0, 0, 0, 0) = xe_1 + (x+y)\left(e_2 + e_3 + e_4 + \frac{1}{2}e_5 + \frac{1}{2}e_6\right)$$

and identifying minimal shifts in (x, y) shifting the coefficients in front of lattice vectors by integers. Now, consider the fixed torus

$$\left\{(x, y, 1, 0, 0, 0) \,\middle|\, x, y \in \mathbb{R}^2/\Lambda^2\right\}. \tag{24}$$

Points on that torus differ from their image point by the lattice vector $(0, 0, 2, 0, 0, 0)$. The position of the 1 entry can be altered within the last four components by adding SO(12) root vectors, e.g. $(0, 0, -1, 1, 0, 0)$. Next there are fixed tori of the form

$$\left\{\left(x, y, \underline{\frac{1}{2}, \frac{1}{2}, 0, 0}\right) \,\middle|\, x, y \in \mathbb{R}^2/\Lambda^2\right\}, \tag{25}$$

where the underlined entries can be permuted. Points on these fixed tori differ from their orbifold image by an SO(12) root, e.g. $(0, 0, 1, 1, 0, 0)$. There are $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$ such fixed two-tori. Very similar fixed tori are

$$\left\{\left(x, y, \underline{\frac{1}{2}, -\frac{1}{2}, 0, 0}\right) \,\middle|\, x, y \in \mathbb{R}^2/\Lambda^2\right\}, \tag{26}$$

where the position of the minus sign can be changed by lattice shifts ($(1/2, -1/2) + (-1, 1) = (-1/2, 1/2)$). This yields another set of six fixed tori. Finally, there are the fixed tori

$$\left\{\left(x, y, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \,\middle|\, x, y \in \mathbb{R}^2/\Lambda^2\right\} \tag{27}$$

and

$$\left\{\left(x, y, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \,\middle|\, x, y \in \mathbb{R}^2/\Lambda^2\right\}. \tag{28}$$

So, altogether there are 16 fixed tori. Some of these are equivalent. Consider the fixed torus (24) and add the SO(12) root vector $(1, 0, -1, 0, 0, 0)$. This yields an equivalent expression for (24)

$$\left\{(x+1, y, 0, 0, 0, 0) \,\middle|\, x, y \in \mathbb{R}^2/\Lambda^2\right\}. \tag{29}$$

But this is the same fixed torus as (23), merely the origin for the x coordinate has been shifted by one. Similar arguments show that the tori in (25) and (26) as well as the tori (27) and (28) are mutually equivalent. So, finally we are left with eight inequivalent fixed tori.

For the $Z_2 \times Z_2$ orbifold we add another Z_2 action θ_1 (21). For this $Z_2 \times Z_2$ action we obtain eight fixed tori under the action of θ_1, eight fixed tori under the action of θ_2 and eight fixed tori under the action of $\theta_1\theta_2$. Hence, the total number of fixed tori is 24. On each fixed tori, labeled by a complex coordinate z_i $i = 1, 2, 3$, there is a non–trivial identification imposed by the second Z_2 orbifold $z_i \to -z_i$. The results of this identification is that the fixed torus degenerates to P_1. This degeneration is the origin of the chirality in this construction [40].

The model discussed above represent an explicit case in which the correspondence between the free fermion construction and the bosonic construction has been explicitly demonstrated at the level of the string partition function, *i.e.* at the massless as well as the massive string spectrum. Of course, there are many more vacua that can be constructed. The classification of symmetric $Z_2 \times Z_2$ orbifolds with standard embedding using bosonic techniques was studied in [40, 31]. In ref. [26] a more general classification using fermionic techniques was presented. In this classification the gauge degrees of freedom are separated into four modular blocks, which is the most general separation compatible with modular invariance. The space of vacua spanned includes models with $(2, 2)$ world–sheet supersymmetry as well as models in which the right–moving $N = 2$ world–sheet supersymmetry is broken. Hence, this analysis also includes vacua with non–standard embedding, and revealed a spinor–vector duality map in the space of these vacua [26]. The two dimensional Fermi–Bose equivalence, however, entails that every model constructed using the fermionic techniques can also be constructed using the bosonic techniques. Deriving this dictionary will provide further insight into the properties of the string vacua.

Acknowledgements

I would like to thank the organisers for the opportunity to speak at the 6^{th} Simons workshop; the University of Oxford for hospitality while this work was written; and my collaborators over the years, with special thanks to Costas Kounnas and John Rizos. This work is supported in part by the STFC under contract PP/D000416/1 and by the EU UniverseNet network under contract MRTN–CT–2006–035863–1.

References

[1] I. Antoniadis, J. Ellis, J. Hagelin and D.V. Nanopoulos *Phys. Lett.* **B231** (1989) 65.

[2] A.E. Faraggi, D.V. Nanopoulos and K. Yuan, *Nucl. Phys.* **B335** (1990) 347;
A.E. Faraggi, *Phys. Rev.* **D46** (1992) 3204.

[3] I. Antoniadis. G.K. Leontaris and J. Rizos, *Phys. Lett.* **B245** (1990) 161.

[4] A.E. Faraggi and D.V. Nanopoulos, *Phys. Rev.* **D48** (1993) 3288.

[5] A.E. Faraggi, *Phys. Lett.* **B278** (1992) 131; *Nucl. Phys.* **B403** (1993) 101.

[6] A.E. Faraggi, *Phys. Lett.* **B274** (1992) 47; *Phys. Lett.* **B339** (1994) 223.

[7] A.E. Faraggi, *Nucl. Phys.* **B387** (1992) 239.

[8] S. Chaudhuri *et al*, *Nucl. Phys.* **B469** (1996) 357;
G.B. Cleaver *et al*, *Nucl. Phys.* **B525** (1998) 3; *Nucl. Phys.* **B545** (1998) 47.

[9] G.B. Cleaver, A.E. Faraggi and D.V. Nanopoulos, *Phys. Lett.* **B455** (1999) 135; *Int. J. Mod. Phys.* **A16** (2001) 425;
G.B. Cleaver, A.E. Faraggi, D.V. Nanopoulos and J.W. Walker, *Mod. Phys. Lett.* **A15** (2000) 1191; *Nucl. Phys.* **B593** (2001) 471; *Nucl. Phys.* **B620** (2002) 259;
G. Cleaver, *Int. J. Mod. Phys.* **A16S1C** (2001) 949.

[10] G.B. Cleaver, A.E. Faraggi and C. Savage, *Phys. Rev.* **D63** (2001) 066001;
G.B. Cleaver, D.J. Clements and A.E. Faraggi, *Phys. Rev.* **D65** (2002) 106003.

[11] G. Cleaver, A.E. Faraggi and S. Nooij, *Nucl. Phys.* **B672** (2003) 64.

[12] A.E. Faraggi, E. Manno and C. Timirgaziu, Eur. Phys. Jour. C **50** (2007) 701;
G. Cleaver, A.E. Faraggi, E. Manno and C. Timirgaziu, *Phys. Rev.* **D78** (2008) 046009.

[13] I. Antoniadis, C. P. Bachas, and C. Kounnas, *Nucl. Phys.* **B289** (1987) 87;
H. Kawai, D.C. Lewellen, and S.H.-H. Tye, *Nucl. Phys.* **B288** (1987) 1;
I. Antoniadis and C. P. Bachas, *Nucl. Phys.* **B289** (1987) 87.

[14] S. Elitzur, E. Gross, E. Rabinovici and N. Seiberg, *Nucl. Phys.* **B283** (1987) 413.

[15] D. Gepner, *Phys. Lett.* **B199** (1987) 380.

[16] A.E. Faraggi, *Phys. Lett.* **B377** (1996) 43; *Nucl. Phys.* **B487** (1997) 55.

[17] A.E. Faraggi, *Nucl. Phys.* **B407** (1993) 57.

[18] I. Antoniadis, J. Rizos and K. Tamvakis, *Phys. Lett.* **B278** (1992) 257;
A.E. Faraggi and E. Halyo, *Phys. Lett.* **B307** (1993) 305; *Nucl. Phys.* **B416** (1994) 63.

[19] I. Antoniadis, J. Rizos and K. Tamvakis, *Phys. Lett.* **B279** (1992) 281;
A.E. Faraggi and E. Halyo, *Phys. Lett.* **B307** (1993) 311.

[20] C. Coriano and A.E. Faraggi, *Phys. Lett.* **B581** (2004) 99.

[21] I. Antoniadis, J. Ellis, R. Lacaze and D.V. Nanopoulos, *Phys. Lett.* **B268** (1991) 188;
A.E. Faraggi, *Phys. Lett.* **B302** (1992) 202.

[22] K.R. Dienes and A.E. Faraggi, *Phys. Rev. Lett.* **75** (1995) 2646; *Nucl. Phys.* **B457** (1995) 409.

[23] A.E. Faraggi, *Nucl. Phys.* **B428** (1994) 111; *Phys. Lett.* **B499** (2001) 147; *Phys. Lett.* **B520** (2001) 337;
J.C. Pati, *Phys. Lett.* **B388** (1996) 532.

[24] I. Antoniadis, J. Ellis, A. Lahanas and D.V. Nanopoulos, *Phys. Lett.* **B241** (1990) 24;
A.E. Faraggi and J.C. Pati, *Nucl. Phys.* **B526** (1998) 21;
A.E. Faraggi and O. Vives, *Nucl. Phys.* **B641** (2002) 93.

[25] A.E. Faraggi, *Nucl. Phys.* **B728** (2005) 83.

[26] A.E. Faraggi, C. Kounnas, S.E.M. Nooij and J. Rizos, *Nucl. Phys.* **B695** (2004) 41;
A.E.Faraggi, C. Kounnas and J. Rizos, *Phys. Lett.* **B648** (2007) 84; *Nucl. Phys.* **B774** (2007) 208; *Nucl. Phys.* **B799** (2008) 19;
T. Catelin-Jullien, A.E. Faraggi, C. Kounnas and J. Rizos, arXiv:0807.4084.

[27] A.E. Faraggi and D.V. Nanopoulos, *Mod. Phys. Lett.* **A6** (1991) 61;
C. Coriano, A.E. Faraggi and M. Guzzi, Eur. Phys. Jour. C **53** (2008) 421; *Phys. Rev.* **D78** (2008) 015012.

[28] S. Chang, C. Coriano and A.E. Faraggi, *Nucl. Phys.* **B477** (1996) 65;
C. Coriano, A.E. Faraggi and M. Plumacher, *Nucl. Phys.* **B614** (2001) 233.

[29] A. Dedes and A.E. Faraggi, *Phys. Rev.* **D62** (2000) 016010.

[30] A.E. Faraggi, *Int. J. Mod. Phys.* **A19** (2004) 5523; hep-th/0411118.

[31] R. Donagi and K. Wendland, arXiv:0808.0330.

[32] K.S. Narain, *Phys. Lett.* **B169** (1986) 41;
K.S. Narain, M.H. Sarmadi and E. Witten, *Nucl. Phys.* **B279** (1987) 369.

[33] A.E. Faraggi, *Phys. Lett.* **B326** (1994) 62.

[34] P. Berglund *et al*, *Phys. Lett.* **B433** (1998) 269; *Int. J. Mod. Phys.* **A15** (2000) 1345.

[35] See *e.g.*: C. Angelantonj and A. Sagnotti, *Phys. Rep.* **371** (2002) 1.

[36] C. Vafa and E. Witten, 1996 *Nucl.Phys.Proc.Suppl.* **46** (1996) 225;
C. Angelantonj, I. Antoniadis and K. Förger, *Nucl. Phys.* **B555** (1999) 116.

[37] A.E. Faraggi, *Phys. Lett.* **B544** (2002) 207.

[38] L. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Nucl. Phys.* **B274** (1986) 285.

[39] A.E. Faraggi, S. Forste and C. Timirgaziu, *JHEP* **0608** (2006) 057.

[40] R. Donagi and A.E. Faraggi, *Nucl. Phys.* **B694** (2004) 187.