On the Skewed Degree Distribution of Hierarchical Networks

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***Abstract*—This paper proposes a novel dynamical model for linear hierarchical networks. Two evolution processes, dominance-based and prestige-based, are introduced. The former provides a formal framework for the study of hierarchies seen among animal species, while the latter is specific to human-based societies. Due to the deterministic characteristics of the proposed models, we are capable of determining equilibria in closed form. Surprisingly, these stationary points recover the exponential and power-law degree distributions as the shared properties of the resulting hierarchal networks, explaining the prevalence of hierarchies in societies. As another contribution, the closeness centrality of nodes for each model is also computed in closed form. Such results shed light on the evolutionary advantages of hierarchies.**

I. Introduction

Hierarchies, where members are ordered based on their size, weight, or other physical characteristics, can be seen across various species. Examples are wide spread including, but not limited to clownfish size adaptation for conflict prevention [1], chickens' peking order for feeding activities [2], et. cetera. Such hierarchies are not limited to animal species, but can also be seen in human societies. Here, hierarchies are translated to social status, where research has shown that human physique (e.g., stature) and body hormones play a crucial role in enabling dominance [3]. Contrary to animal societies which base hierarchies on dominance, however, human societies replace dominance by "prestige" (i.e., theory of service-for-prestige) to construct reciprocal relationships between leaders and followers [4].

To analyze the behavior of networks, a variety of mathematical models have been proposed [5]. The earliest dates back to the 1900's, where Yule [6] studied the biological evolution of species based on age and population data. Others, e.g., Lotka [7] provided rules required for describing and analyzing scientific publications. Resulting from these and other studies, was the emergence of the power-law degree distribution [8] as a shared common characteristic for a wide-range of networks including but not limited to, the world wide web, protein-protein interaction, and airline, and social networks.

Given such a widely-shared characteristic, Barabási and Albert suggested a linear preferential attachment model for the generation of scale-free graphs exhibiting a power-law degree distribution distribution [9]. As noted by Durret [10], the definition of their process was rather informal. Since then, different precise forms of the Barabási-Albert (BA) model have been studied in literature [11], [12].

Though successful at recovering the power-law degree distribution, these studies impose several restricting assumptions on the underlying graph generating process. For instance, in preferential attachment, newly arriving nodes are assumed to connect to existing nodes with frequency proportional to their highest degree. Furthermore, such techniques typically adopt a binary attachment model, in which two nodes are either connected or not [9], [13], [14], [15], [16].

Evolutionary considerations of real-world networks, however, show the emergence of scale-free behavior (i.e., networks exhibiting a power-law degree distribution) in networks as a resultant of hierarchal attachment processes which are not reflected through current preferential attachment models [3], [4]. Apart from these modeling restrictions, another problem inherent to existing binary models lies in their explanatory capabilities. For instance, they fail to manifest connection strengths between individuals; a property being at the core of behavioral emergence in real networks [17], [18], [19], [20], [21].

To provide more realistic modeling outcomes, in this paper, we contribute by proposing two *deterministic* hierarchal graph attachment processes for both dominance (i.e., animal) and prestige-based (human) societies. Contrary to preferential attachment models, our approach only assumes hierarchal connections between individuals, thus bridging the modeling gap to real-world evolutionary networks. Among many advantages, our deterministic setting enables the derivation of the degree distribution in closed-form. Performing this derivation recovers, surprisingly, the exponential and power-law degree distributions as the main properties of the resultant hierarchal networks, which explains the prevalence of such hierarchies in societies. In short, our contributions can be summarized as:

1) Deterministic modeling of linear (in the sense that if node A is superior to node B, and node B is superior to node C, then node A is also superior to node C) hierarchal networks as complete weighted graphs;
2) Deriving, for the first time, a closed form of the skewed distribution among individuals in networks

having hierarchical access to information or power loads;

3) Explaining the prevalence of hierarchies in societies as a resultant of the characteristics of derived skewed distribution (e.g., high robustness and small average distance [13]); and
4) Understanding the emergence of new behaviors in both animal and human societies as size of groups increases, by amplifying distribution differences between dominance-based hierarchies (i.e., exponential distribution) and prestige-based networks following a power-law distribution.

As another contribution, we have defined and derived the closeness centrality metric in closed form. This can be used to assess the importance of nodes in hierarchical networks. In other words, our centrality measure reflect that in dominance-based networks the shortest path between every two members is their direct link, while in prestige-based hierarchies every shortest path has to pass through the member with highest "prestige".

The remainder of this paper is organized as follows. The notations used in this paper are defined in Section II. Preliminary information on mathematical series and degree distributions are provided in Section III. The dominance-based and prestige-based dynamical models are introduced in Section IV. Each of these models are studied in detail in Sections V and VI; their closeness centrality is further studied in Section VII. Section VIII concludes.

II. NOTATIONS

A network is described via a graph, $\mathbb{G} = (\mathbb{V}, \mathbb{W})$, consisting of a set of N nodes (or vertices) $\mathbb{V} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}$ and an $N \times N$ adjacency matrix $\mathbb{W} = [\boldsymbol{w}_{ij}]$ where non-zero entries $\boldsymbol{w}_{ij}$ indicate the weighted connection from $\boldsymbol{v}_j$ to $\boldsymbol{v}_i$. In this paper we consider the undirected graphs which have symmetrical $\mathbb{W}$. The neighborhood $\mathbb{N}$ of a node $\boldsymbol{v}_i$ is defined as the set formed by its connected vertices, i.e., $\mathbb{N}(\boldsymbol{v}_i) = \cup_j \boldsymbol{v}_j : \boldsymbol{w}_{ij} > 0$. The node's degree, $\deg(\boldsymbol{v}_i)$, is given by the cardinality of its neighborhood.

The strength of a node is of major importance in hierarchical networks. Next, we define three concepts: (1) relative strength (2) strength observation and (3) absolute strength.

Relative Node Strength: The relative strength of j^{th} node with respect to i^{th} node with $i > j$ is denoted by $\boldsymbol{\Psi}_i(\boldsymbol{v}_j)$ and represents the sum over all edge weights between j^{th} node and every k^{th} node where $k < i$. Namely,

$$\boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \sum_{k=1}^{i-1} \boldsymbol{w}_{jk}, \quad i > j. \tag{1}$$

In other words, when node i is monitoring node j with $j < i$, it just observes those connections from other ks to j which $k < i$. The importance of this concept will be shown in Section V.

Strength Observation The Strength observation of the i^{th} node is denoted by the vector

$$\vec{\boldsymbol{\Psi}}_i = [\boldsymbol{\Psi}_i(\boldsymbol{v}_1), \boldsymbol{\Psi}_i(\boldsymbol{v}_2), \ldots, \boldsymbol{\Psi}_i(\boldsymbol{v}_{i-1})]^{\mathsf{T}}$$

with cardinality $i - 1$.

Absolute Node Strength: The absolute strength of j^{th} node is defined as

$$\boldsymbol{\Psi}(\boldsymbol{v}_j) = \sum_{k=1}^{N} \boldsymbol{w}_{jk}. \tag{2}$$

III. BACKGROUND

A. Mathematical Series

The fraction product and harmonic series are two ingredients which play a major role in our analysis for determining closed forms of the strength distributions in hierarchical networks. Here, we provide two lemmas presenting the upper and lower bounds on the values of these summations.

Lemma 1 (Fraction Product Series): Consider the following product of fractions

$$L(i,N) = \prod_{k=i+2}^{N+1} \frac{2k-4}{2k-5},$$

then $\gamma i^{-\frac{1}{2}} < L(i,N) < \gamma(i-1)^{-\frac{1}{2}}$, with $\gamma = \sqrt{N-1}$

Proof: We use the comparison test to compute the lower and upper bounds of $L(i,N)$. Firstly, consider

$$Q(i,N) = \prod_{k=i+2}^{N+1} \frac{2k-5}{2k-6} \tag{3}$$

Clearly, $L(i,N) < Q(i,N)$ and $L(i,N)Q(i,N) = \frac{2N-2}{2i-2}$. Therefore,

$$L(i,N) < \sqrt{\frac{2N-2}{2i-2}} \leq \gamma(i-1)^{-\frac{1}{2}} \tag{4}$$

concluding the upper-bound. To determine the lower bound, define

$$Q'(i,N) = \prod_{k=i+2}^{N+1} \frac{2k-3}{2k-4} \tag{5}$$

It can be shown that $L(i,N) > Q'(i,N)$ and $L(i,N)Q'(i,N) = \frac{2N-1}{2i-1}$. Therefore,

$$L(i,N) > \sqrt{\frac{2N+1}{2i-1}} > \sqrt{\frac{2N-2}{2i}} \geq \gamma i^{-\frac{1}{2}}. \tag{6}$$

■

Lemma 2 (Harmonic Series): Consider the harmonic series

$$L(i,N) = \sum_{k=i}^{N-1} \frac{1}{k},$$

then $\ln\left(\frac{N+1}{i}\right) < L(i,N) < \ln\left(\frac{N}{i-1}\right)$

Proof: The relatively simple proof of the above lemma is based on the integration results of harmonic series, where $L(i,N)$ is lower bounded by $\int_{x=i-1}^{N} \frac{1}{x}\mathrm{d}x$ and upper bounded by $\int_{x=i}^{N+1} \frac{1}{x}\mathrm{d}x$. ■

B. Power-Law & Exponential Degree Distributions

In analysis of weighted networks, typically the Distribution Function (DF) is defined as

$$P(k) = \left|\left\{ v_i | \forall i, k \leq \mathbf{\Psi}(\boldsymbol{v}_i) < k+1 \right\}\right| \tag{7}$$

where $\mathbf{\Psi}(\boldsymbol{v}_i)$ defined in (2) denotes the strength of node v_i and $|\cdot|$ being the cardinality of the set.

To ease the analysis, in this work we make use of the Complementary Cumulative Distribution Function (CCDF) defined as:

$$P_c(k) = \left|\left\{ v_i | \forall i, \mathbf{\Psi}(\boldsymbol{v}_i) \leq k \right\}\right| \tag{8}$$

The following two lemmas clarify the relation between the DF and CCDF for networks with power-law and Exponential Distributions.

Lemma 3 (Power-law Distribution): Consider a power-law distribution in form of $P(k) = ck^{-\alpha}$, where c is the power-law coefficient, and α is the power-law exponent. The CCDF $P_c(k)$ also follows a power-law but with an exponent $\alpha - 1$.

Proof: Can be easily seen by simple integration. ■

Lemma 4 (Exponential Distribution): Consider an exponential distribution in form of $P(k) = ce^{-\alpha k}$. Its CCDF can be written as $P_c(k) = \frac{c}{\alpha}e^{-\alpha k}$.

Proof: Can be easily seen by simple integration. ■

Having laid out our notation and providing the required background knowledge, next we present and analyze two dynamical models that reflect the networks constructed by dominance-based and prestige-based attachment models. Not only we provide such iteration schemes, but also present a set of theorems studying their stationary points, which interestingly relate to the exponential and power-law distributions.

IV. Network Dynamics

Here, we propose a dynamical process which captures the edge dynamics of a complete network. Let $\boldsymbol{\omega} = \{w_{ij} | \forall i, j = 1, 2, ..., N, i > j\}$ denote the state vector of the process, where each state variable w_{ij} corresponds to the weight of the link between the j^{th} and i^{th} node. In the very general case, one considers the rate of changes in w_{ij} as a function of all state variables

$$\dot{w}_{ij} = f(\boldsymbol{\omega}) \tag{9}$$

In this paper, however, the focus is on hierarchical networks in which for any $i > j$, $\dot{w}_{ij}$ is a function of w_{ij} itself and the strength observation of the i^{th} node $\vec{\mathbf{\Psi}}_i$

$$\dot{w}_{ij} = f_\Psi(w_{ij}, \vec{\mathbf{\Psi}}_i), i > j \tag{10}$$

In other words, the dynamics of the linking strength between i and j is independent of any other node l where l is higher than i or j in the hierarchy.

Using f_Ψ from Equation 10, and sorting the state variables w_{ij} increasingly (based on $Ni+j$), the overall dynamic process can be written as

$$\begin{aligned}\dot{\boldsymbol{\omega}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[w_{21}, \ldots, w_{N(N-1)}\right]^\mathsf{T} \\ &= \left[f_\Psi(w_{21}, \vec{\mathbf{\Psi}}_2), \ldots, f_\Psi(w_{N(N-1)}, \vec{\mathbf{\Psi}}_N)\right]^\mathsf{T}\end{aligned}$$

Next, we introduce two possible strategies for $f_\Psi(.)$, namely $f_\Psi^{(\mathbb{D})}(\cdot)$ and $f_\Psi^{(\mathbb{P})}(\cdot)$ which represent the Dominance-based attachment (DA) and Prestige-based attachment (PA) models, respectively.

V. Dominance-based Attachment Model

In the dominance-based attachment model (DA), the weighted links between the i^{th} and every other j^{th} node, where $j < i$, follow a simple dynamical rule defined as

$$\dot{w}_{ij} = f_\Psi^{(\mathbb{D})}\left(w_{ij}, |\vec{\mathbf{\Psi}}_i|\right) \tag{11}$$

where $|\cdot|$ denotes the size of the vector and

$$\begin{aligned} f_\Psi^{(\mathbb{D})}(w_{ij}, |\vec{\mathbf{\Psi}}_i|) &= \frac{1}{|\vec{\mathbf{\Psi}}_i|} - w_{ij} \\ &= \frac{1}{i-1} - w_{ij} \end{aligned} \tag{12}$$

In Equation 12 the difference between $\frac{1}{i-1}$ and w_{ij} determines the direction of changes of w_{ij} (i.e., $\dot{w}_{ij}$).

For computing the equilibrium point of this system, consider an energy function for w_{ij} in form of

$$V_{ij} = \left(\frac{1}{i-1} - w_{ij}\right)^2 \tag{13}$$

The derivative of this energy function, for a fixed i, can be computed as

$$\dot{V}_{ij} = -2\left(\frac{1}{i-1} - w_{ij}\right)\dot{w}_{ij} \tag{14}$$

$$= -2\left(\frac{1}{i-1} - w_{ij}\right)^2 \tag{15}$$

Therefore, based on Invariant Set theorem [22] we can show that the overall dynamical process has a stable equilibrium point, in which the link between the i^{th} and j^{th} node, $j < i$, converges to $w_{i\star}$

$$w_{i\star}^{(\mathbb{D})} = \frac{1}{i-1}. \tag{16}$$

The equilibrium point (16) explains that the links of node i to all nodes with lower order (i.e., $j < i$) depends on i, and the higher this order is the lower the strength of those links are. To illustrate, imagine N agents who are all connected to each other, and continuously each agent shares her available resources with the agents with lower order. The strength of connection between i and j shows the amount of resources which are transmitted from i to j. According to (16) the 2^{nd} individual shares all her resources with 1^{st} individual (i.e., $w_{21} = \frac{1}{2-1} = 1$). The 3^{rd} individual shares half of her resources with the 2^{nd} individual and other half with 1^{st} individual (i.e., $w_{32} = w_{31} = \frac{1}{3-1} = \frac{1}{2}$). With the same respect, i^{th} individual shares $\frac{1}{i-1}$ units of her resources with each of the j individuals where $j < i$. Therefore, one can see that this model directly captures the dominance of individuals in a linear hierarchical network: Every individual is sharing her resources between dominated individuals. Next, we study the amount of resources each individual receives in such dominance-based network (captured by node's strengths), and also compute the distribution of node strengths.

A. Analysis of Node's Strength

Building on $w_{i\star}^{(\mathbb{D})}$'s definition in Equation 16, one can calculate the absolute strength of the i^{th} node, $\mathbf{\Psi}(\boldsymbol{v}_i)$. The node strength $\mathbf{\Psi}(\boldsymbol{v}_i)$ for an arbitrary node $i > 1$ can be calculated as

$$\begin{aligned}\mathbf{\Psi}(\boldsymbol{v}_i) &= \sum_{j=1}^{N} \boldsymbol{w}_{ij}^{(\mathbb{D})} = \sum_{j=1}^{i-1} \boldsymbol{w}_{ij}^{(\mathbb{D})} + \sum_{j=i+1}^{N} \boldsymbol{w}_{ij}^{(\mathbb{D})} \\ &= (i-1)\boldsymbol{w}_{i\star}^{(\mathbb{D})} + \sum_{j=i+1}^{N} \boldsymbol{w}_{j\star}^{(\mathbb{D})} \\ &= 1 + \sum_{j=i+1}^{N} \frac{1}{j-1} = 1 + \sum_{j=i}^{N-1} \frac{1}{j} \end{aligned} \tag{17}$$

By using Lemma 2, it's straightforward to show that

$$1 + \ln\left(\frac{N+1}{i}\right) < \mathbf{\Psi}(\boldsymbol{v}_i) < 1 + \ln\left(\frac{N}{i-1}\right) \tag{18}$$

B. Analysis of Node's Strength Distribution

The distribution of strengths in the DA model can be directly computed from the bounds provided in Equation 18. The following theorem shows how the CCDF and consequently the DF of strengths in this model follow an exponential distribution.

Theorem 1 (Strength Distribution in DA Model): For the weighted network $\mathbb{G}$, generated using the DA model, the DF of the global strength k follows an exponential distribution of the form:

$$P(k) \propto e^{-k}$$

Proof:

Using Equation 18 we have

$$\mathbf{\Psi}(\boldsymbol{v}_i) \geq k, \text{ for } i \in \{1, 2, 3, \ldots, \left\lfloor \frac{N+1}{e^{-(k-1)}} \right\rfloor\}$$

$$P_c(k) = \left| \left\{ 1, 2, 3, \ldots, \left\lfloor \frac{N+1}{e^{-(k-1)}} \right\rfloor \right\} \right| \simeq (N+1)e \cdot e^{-k} \tag{19}$$

Therefore,

$$P_c(k) \propto e^{-k} \tag{20}$$

Using Lemma 4, it's straightforward to see that the DF corresponding to (20) is exponential:

$$P(k) \propto e^{-k}.$$

■

VI. Prestige-based Attachment Model

Here we introduce a more sophisticated form of attachment model which is called prestige-based attachment model.

Firstly, we note that the overall strength of node i in establishing connection with every other j^{th} node is assumed to be limited and sums-up to 1. The prestige-based attachment model can be formally derived as follows. Let

$$\dot{w}_{ij} = f_{\Psi}^{(\mathbb{P})}(w_{ij}, \mathbf{\Psi}_i(\boldsymbol{v}_j), ||\vec{\mathbf{\Psi}}_i||_1)$$

and

$$f_{\Psi}^{(\mathbb{P})}(w_{ij}, \mathbf{\Psi}_i(\boldsymbol{v}_j), ||\vec{\mathbf{\Psi}}_i||_1) = \frac{\mathbf{\Psi}_i(\boldsymbol{v}_j)}{||\vec{\mathbf{\Psi}}_i||_1} - w_{ij}, i > j \tag{21}$$

where $||.||_1$ denotes the first norm.

By studying the dynamic process proposed in Equation 21, it can be easily seen that $\dot{w}_{ij}, i > j$ is a function of every w_{kl} for $k, l < i$. Withough loss of generality we assume $\boldsymbol{w}_{11}^{(\mathbb{P})} = 1$, such that

$$\mathbf{\Psi}_2(\boldsymbol{v}_1) = 1 \tag{22}$$

and $\boldsymbol{w}_{ii}^{(\mathbb{P})} = 0$ for every $i > 1$. Using a similar energy function as in previous section it is straightforward to compute the equilibrium point of this system

$$\boldsymbol{w}_{ij}^{(\mathbb{P})} = \frac{\mathbf{\Psi}_i(\boldsymbol{v}_j)}{||\vec{\mathbf{\Psi}}_i||_1} \tag{23}$$

The equilibrium point (23) explains that the connection strength between node i and node j with lower order (i.e., $j < i$) depends on i and j. To illustrate, imagine N agents who are all connected to each other, and continuously the agents with higher order share their available resources with agents with lower order. The strength of link between i and j shows the amount of resources which are transmitted from i to j. According to (23) the 2^{nd} individual shares all her resources with 1^{st} individual (i.e., $w_{21} = \frac{1}{1} = 1$). The 3^{rd} individual shares one third of her resources with the 2^{nd} individual and two third of it with 1^{st} individual (i.e., $w_{32} = \frac{1}{1+2} = \frac{1}{3}$ and $w_{31} = \frac{2}{1+2} = \frac{2}{3}$). With the same respect, i^{th} individual shares portions of her resources with each of the j individuals where $j < i$, while those with lower order receive more. We call this model prestige-based model as the lower orders reflect a kind of prestige in the group and high prestige agents receive more than agents with lower prestige. Next, we study the amount of resources each individual receives in such prestige-based network (captured by node's strengths), and also compute the distribution of node strengths.

A. Analysis of Node's Strength

Analogous to the DA model, here we can determine a closed form solution for the sum over the strength of every j^{th} node from the perspective of i^{th} node, where $j < i$.

Lemma 5: According to the prestige-based attachment model, the overall strength of every j^{th} node from perspective of i^{th} node, where $j < i$ is derived as following

$$\mathbf{\Psi}^{\dagger}(i) : \sum_{j=1}^{i-1} \mathbf{\Psi}_i(\boldsymbol{v}_j) = 2i - 3.$$

Proof: The above lemma can be proved by using induction:

Initial Step: According to Equation (22), $\boldsymbol{\Psi}_2(\boldsymbol{v}_1) = 1$. This is equivalent to $\sum_{j=1}^{i-1} \boldsymbol{\Psi}_i(\boldsymbol{v}_j) = 2i - 3$ for $i = 2$. Therefore, $\boldsymbol{\Psi}^{\dagger}(2)$ holds for $i = 2$.

Inductive Step: Let

$$\boldsymbol{\Psi}^{\dagger}(i-1) : \sum_{j=1}^{i-2} \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) = 2i - 5,$$

and also consider $\boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j}$. Therefore,

$$\begin{aligned}\sum_{j=1}^{i-1} \boldsymbol{\Psi}_i(\boldsymbol{v}_j) =& \boldsymbol{\Psi}_i(\boldsymbol{v}_{i-1}) + \sum_{j=1}^{i-2} \left(\boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j} \right) \\ =& \boldsymbol{\Psi}_i(\boldsymbol{v}_{i-1}) + \sum_{j=1}^{i-2} \left(\boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) \right) + \sum_{j=1}^{i-2} \left(\boldsymbol{w}^{(\mathbb{P})}_{(i-1)j} \right)\end{aligned}$$

Clearly, it can be seen that the strength of the i^{th} node from the $i+1^{\text{th}}$ perspective equates to 1, since

$$\boldsymbol{\Psi}_{i+1}(\boldsymbol{v}_i) = \sum_{j=1}^{i-1} \boldsymbol{w}^{(\mathbb{P})}_{ij} = 1 \tag{24}$$

By using Equation 24, we'll get

$$\sum_{j=1}^{i-1} \boldsymbol{\Psi}_i(\boldsymbol{v}_j) = 1 + 2i - 5 + 1 = 2i - 3 \tag{25}$$

Therefore, $\boldsymbol{\Psi}^{\dagger}(i)$ holds for every i concluding the proof. ■

B. Analysis of Edge Weights

We can compute the edge weight between i^{th} node and j^{th} node as follows.

Lemma 6 (Edge Weight): For the weighted graph $\mathbb{G}$, evolved with PA model, i^{th} node is connected to j^{th} node with an edge of weight

$$\boldsymbol{\gamma}(i) : \boldsymbol{w}^{(\mathbb{P})}_{ij} = \frac{1}{2i-2} \prod_{k=1}^{i-j} \frac{2i-2k}{2i-2k-1}, \forall j < i. \tag{26}$$

Proof:

The validity of Equation 26 can be proved for each i and for every $j < i$ using induction.

Initial Step: The second node is connected to the first node with $\boldsymbol{w}^{(\mathbb{P})}_{21} = 1$, meaning that $\boldsymbol{\gamma}(2)$ holds.

Inductive Step: Now assume that

$$\boldsymbol{\gamma}(i-1) : \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j} = \frac{1}{2i-4} \prod_{k=1}^{i-j-1} \frac{2i-2k-2}{2i-2k-3}$$

holds for every $j < i-1$. For computing the edge weight between i^{th} and j^{th} node, recall that $\boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j}$. By using Equation 23 and Lemma 5, it can be seen that:

$$\begin{aligned}\boldsymbol{\Psi}_i(\boldsymbol{v}_j) &= \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j} \\ &= (2i-5) w_{(i-1)j} + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j} \\ &= (2i-4) w_{(i-1)j}\end{aligned} \tag{27}$$

Using Equations 23, 27 and Lemma 5, the edge weight between i^{th} and j^{th} nodes can be written as

$$\boldsymbol{w}^{(\mathbb{P})}_{ij} = \frac{\boldsymbol{\Psi}_i(\boldsymbol{v}_j)}{\sum_{k=1}^{i-1} \boldsymbol{\Psi}_i(\boldsymbol{v}_k)} = \frac{1}{2i-2} \prod_{k=1}^{i-j} \frac{2i-2k}{2i-2k-1}$$

for $j < i-1$. Therefore, $\boldsymbol{\gamma}(i)$ holds $\forall i$, concluding the proof. ■

Before, computing the distribution of strengths for the PA model, we present the following proposition providing the relative strength of the j^{th} node from the perspective of the i^{th} for every $i > j$ (i.e., $\boldsymbol{\Psi}_i(\boldsymbol{v}_j)$) in closed form.

Proposition 1 (Relative Node Strength): For the weighted graph $\mathbb{G}$, evolved according to the PA model, the strength of j^{th} node from perspective of the i^{th} is given by

$$\boldsymbol{\Psi}^{\dagger}_{(\mathbb{P})}(i) : \begin{cases} \boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \prod_{k=j+2}^{i} \frac{2k-4}{2k-5} & \text{for } j < i-1 \\ \boldsymbol{\Psi}_i(\boldsymbol{v}_j) = 1 & \text{for } j = i-1. \end{cases} \tag{28}$$

Proof: Again, induction can be used to prove the validity of Equation 28. Starting with the initial step we get

Initial Step: From Equation 22, the strength of the first node from the perspective of the second is $\boldsymbol{\Psi}_2(\boldsymbol{v}_1) = 1$. Besides, using Lemma 6 we can deduce that

$$\boldsymbol{\Psi}_3(\boldsymbol{v}_1) = \frac{\boldsymbol{w}^{(\mathbb{P})}_{11} + \boldsymbol{w}^{(\mathbb{P})}_{21}}{3} = \frac{2}{3}.$$

Therefore, $\boldsymbol{\Psi}^{\dagger}(2)$ and $\boldsymbol{\Psi}^{\dagger}(2)$ hold. For the inductive step we proceed as follows

Inductive Step: Assume that

$$\boldsymbol{\Psi}^{\dagger}(i-1) : \begin{cases} \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) = \prod_{k=j+2}^{i-1} \frac{2k-4}{2k-5} & \text{for } j < i-2 \\ \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) = 1 & \text{for } j = i-2. \end{cases}$$

holds.

For computing $\boldsymbol{\Psi}_i(\boldsymbol{v}_j)$, consider $\boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j}$. Using Equation 23 and Lemma 5, it can show that for every $j < i-1$

$$\boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j} = \prod_{k=j+2}^{i} \frac{2k-4}{2k-5}$$

Besides using Equation (24), $\boldsymbol{\Psi}_j(\boldsymbol{v}_i) = 1$ for $j = i-1$. Therefore, $\boldsymbol{\Psi}^{\dagger}(i)$ holds $\forall i$ and the proof is concluded. ■

Lemma 7 (Global Strength): For the weighted graph $\mathbb{G}$, evolved with the PA model, the global strength of the i^{th} node is

$$\boldsymbol{\Psi}^{\dagger}(i) : \begin{cases} \boldsymbol{\Psi}(\boldsymbol{v}_i) = \prod_{k=i+2}^{N+1} \frac{2k-4}{2k-5} & \text{for } i < N \\ \boldsymbol{\Psi}(\boldsymbol{v}_j) = 1 & \text{for } j = N. \end{cases} \tag{29}$$

Proof: It can be easily seen that

$$\boldsymbol{\Psi}(\boldsymbol{v}_i) = \boldsymbol{\Psi}_N(\boldsymbol{v}_i) + \boldsymbol{w}_{iN}^{(\mathbb{P})}. \tag{30}$$

Using Equation 26 and Proposition 1, we have

$$\begin{aligned} \boldsymbol{\Psi}(\boldsymbol{v}_i) &= \boldsymbol{\Psi}_N(\boldsymbol{v}_i) + \boldsymbol{w}_{iN}^{(\mathbb{P})} \\ &= \boldsymbol{\Psi}_N(\boldsymbol{v}_i) + \frac{\boldsymbol{\Psi}_N(\boldsymbol{v}_i)}{2N-3} = \prod_{k=i+2}^{N+1} \frac{2k-4}{2k-5} \end{aligned}$$

Based on Equation (24), $\boldsymbol{\Psi}(\boldsymbol{v}_N) = 1$, concluding the proof. ■

Finally, we can also compute the strength distribution in a closed form. We define the degree distribution $\mathbb{P}\mathrm{r}(k)$ as the number of nodes with strength k. The following theorem provides the strength distribution of a PA model.

Theorem 2 (Strength Distribution): For the weighted graph $\mathbb{G}$ evolved with the PA model, the distribution of the global strength k follows a power-law of degree -3

$$P(k) \propto k^{-3}.$$

Proof:

Using results of Lemma 1 and Lemma 7, the following lower and upper bounds can be computed for the strength of the i^{th} node

$$\gamma i^{-\frac{1}{2}} < \boldsymbol{\Psi}(\boldsymbol{v}_i) < \gamma(i-1)^{-\frac{1}{2}} \tag{31}$$

where $\gamma = \sqrt{N-1}$.

From Equation (31), we have

$$\boldsymbol{\Psi}(\boldsymbol{v}_i) \geq k, \ \text{for} \ \ i \in \{1, 2, 3, \dots, \left\lfloor \frac{\gamma^2}{k^2} \right\rfloor\} \tag{32}$$

$$P_c(k) = \left| \left\{ 1, 2, 3, \dots, \frac{\gamma^2}{k^2} \right\} \right| \simeq \gamma^2 k^{-2} \tag{33}$$

Therefore,

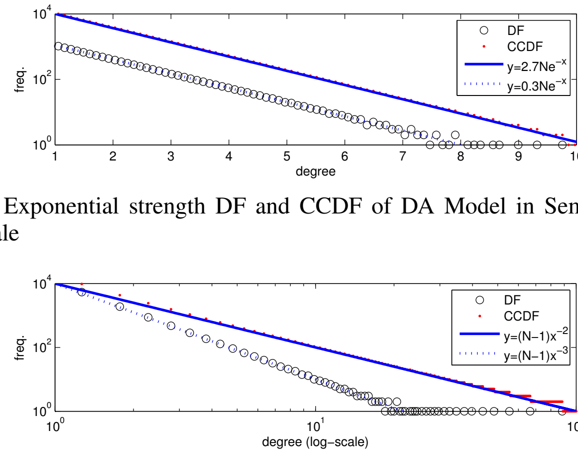
$$P_c(k) \propto k^{-2} \tag{34}$$

Using Lemma 4, the exponent of degree distribution in the proposed model is -3, i.e.,

$$P(k) \propto k^{-3} \tag{35}$$

■

To validate the analytical results on strength distribution of DA and PA models, we initiate a complete graph with 10^4 nodes and random weights. This network is evolved under dynamical processes for the DA and PA models. The strengths of nodes in the equilibrium point of the evolved networks are extracted and their distribution are illustrated in Figure 1. As can be seen, the DA model is generating an exponential strength distribution (i.e., a straight line in semilogarithmic plot) while PA model produces a power-law strength distribution (i.e., a straight line in log-log plot).



(a) Exponential strength DF and CCDF of DA Model in Semi-Log Scale

(b) Power-law strength DF and CCDF of PA Model in Log-Log Scale

Fig. 1: The DF and CCDF of strengths in *DA* model and *PA* models.

VII. Closeness Centrality

The closeness centrality measure is the inverse of sum of distance of an individual to the others (i.e., in turn its farness). For this, a measure of distance needs first to be introduced. We define the distance between two individuals as the inverse of link weight between the corresponding nodes:

$$d_{ij} = 1/w_{ij}$$

if $i \neq j$ and $d_{ii} = 0$ for every i.

The closeness centrality of the i^{th} node c_i is then defined as the inverse of the sum of its distances to other nodes

$$c_i = \left[\sum_{j=1}^{N} d_{ij}^s \right]^{-1} \tag{36}$$

where d_{ij}^s is the shortest distance between i^{th} and j^{th} nodes.

A. Closeness Centrality Measure for DA Model

In order to compute the closeness centrality of the i^{th} individual in DA model $c_i^{(\mathbb{D})}$, we start with the following theorem which computes the shortest distance d_{ij}^s between the i^{th} and j^{th} individuals.

Theorem 3: In the DA model, the shortest distance d_{ij}^s between the i^{th} and j^{th} individual is equal to the distance associated with their link:

$$d_{ij}^s = d_{ij} = \frac{1}{w_{ij}}.$$

Proof: The proof of the above theorem can be attained by contradiction. Without loss of generality, assume $i > j$ (i.e., $d_{ij} = i - 1$). Suppose that the shortest path starts from i^{th} individual and then passes the k^{th} individual where $k \neq i, j$. The distance d_{ij} can be determined as

$$d_{ik} = \begin{cases} i-1 & i > k \\ k-1 & k > i \end{cases} \tag{37}$$

Since $d^s_{ij} > d_{ik}$ and by using Equation 37, it can be easily seen that $d^s_{ij} > d_{ij}$. Hence, the supposition is false and the shortest path can not pass any third individual. The proof is complete. ∎

Therefore, the closeness centrality for i^{th} individual is

$$c_i^{(\mathbb{D})} = \left(\sum_{j=1}^{N} d^s_{ij}\right)^{-1} = \left(\sum_{j=1}^{N} d_{ij}\right)^{-1} = \left(\sum_{j=1}^{i-1} d_{ij} + \sum_{j=i+1}^{N} d_{ij}\right)^{-1} = \frac{2}{i^2 - 3i + (N^2 - N + 2)}$$

B. Closeness Centrality Measure for PA Model

In contrast to the DA model, in which the shortest path between two individuals is the direct link connecting them, the following theorem shows that in the PA model, the shortest path always passes through the first individual.

Theorem 4: In the PA model, the shortest distance d^s_{ij} between the i^{th} and j^{th} individual is

$$d^s_{ij} = d_{i1} + d_{j1} \tag{38}$$

Before giving the proof of this theorem, we use Equation 26 to derive the distance between node i and j

$$d_{ij} = (2i-2)\prod_{k=1}^{i-j}\frac{2i-2k-1}{2i-2k} \tag{39}$$

Proof: The proof follows by contradiction. Without loss of generality we assume that $i > j$. If there exists a $d^s_{ij} = d_{ij} = \frac{1}{w^{\mathbb{P}}_{ij}}$, then

$$d^s_{ij} = d_{ij} = (2i-2)\prod_{k=1}^{i-j}\frac{2i-2k-1}{2i-2k} \tag{40}$$

thus,

$$\begin{aligned} d^s_{ij} &= \prod_{k=i-j+1}^{i-1}\frac{2i-2k}{2i-2k-1}\cdot\prod_{k=1}^{i-1}\frac{2i-2k-1}{2i-2k} \\ &= \prod_{k=i-j+1}^{i-1}\frac{2i-2k}{2i-2k-1}\cdot d_{i1} \\ &= 2\prod_{k=i-j+1}^{i-2}\frac{2i-2k}{2i-2k-1}\cdot d_{i1} \\ &\geq 2d_{i1} \end{aligned} \tag{41}$$

It can be easily seen from Equation 39 that $d_{i1} > d_{j1}, \forall i > j$ and thus

$$d^s_{ij} > d_{i1} + d_{j1}. \tag{42}$$

Therefore, every direct link between two individuals can be replaced via a path that passes through the first individual. Hence, the supposition is false completing the proof. ∎

As a result of this theorem, the shortest distance d^s_{ij} between two individuals is given by Equation 38. The closeness centrality for the i^{th} individual is then

$$\begin{aligned} c_i^{(\mathbb{P})} &= \left(\sum_{\substack{j=1\\ j\neq i}}^{N} d^s_{ij}\right)^{-1} = \left(\sum_{\substack{j=1\\ j\neq i}}^{N}\left(d_{i1} + d_{j1}\right)\right)^{-1} \\ &= \left((N-1)d_{i1} + \sum_{\substack{j=1\\ j\neq i}}^{N} d_{j1}\right)^{-1} \end{aligned} \tag{43}$$

By replacing d_{i1} and d_{j1} from Equation 39 into Equation 43, the closeness centrality for the PA model can be attained in closed form. The closeness centrality of individuals in DA and PA models for a network of 10^4 nodes (as studied in Figure 1) is illustrated in Figure 2. This centrality measure is normalized in a way that the maximum closeness becomes equal to 1. As can be seen, in DA model the individuals centrality decreases much slower compared to the PA model.

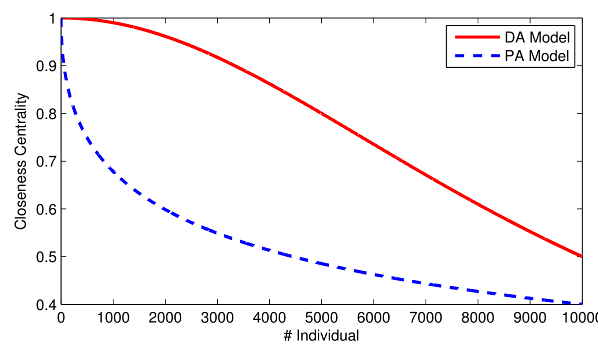


Fig. 2: The normalized closeness centrality of *DA* and *PA* networks.

VIII. Conclusion

In this paper we proposed two dynamical models for dominance-based and prestige-based hierarchical systems. Although, each dynamical system is described using simple hierarchical rules, derived stationary points have been shown to recover the *exponential* (Theorem 1) and *power-law* (Theorem 2) degree distributions. Emergence of such degree distributions despite the simple hierarchical structure explains how hierarchical social structures have survived among different species.

As another contribution, we have defined and derived the closeness centrality metric in closed form. This can be used to assess the importance of nodes in hierarchical networks. In other words, our centrality measure reflects that in dominance-based networks the shortest path between every two members is their direct link (Theorem 3), while in prestige-based hierarchies every shortest path has to pass through the member with highest “prestige” (Theorem 4).

There are various interesting future directions of this work. We plan to validate the attained results through data gathered from real-world networks. Moreover, our model is proposed in form of a dynamical process, which makes it possible for the development of control strategies. The overall idea would be the control of the evolution of the network to arrive at specific network forms.

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