The Gabrielov-Khovanskii problem for polynomials

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We state and consider the Gabrielov-Khovanskii problem of estimating the multiplicity of a common zero for a tuple of polynomials in a subvariety of a given codimension in the space of tuples of polynomials. For a bounded codimension we obtain estimates of the multiplicity of the common zero, which are close to optimal ones. We consider certain generalizations and open questions.

Bibliography: 11 items.

Introduction

0.1. The Gabrielov-Khovanskii problem. Let $\mathbb{C}^N$ be the complex coordinate space and $\mathcal{F}_1,\dots,\mathcal{F}_N$ some linear space of functions, analytic in a neighborhood of the point $o=(0,\dots,0)$ and vanishing at that point. For an arbitrary tuple $(f_1,\dots,f_N)\in\mathcal{F}_1\times\dots\times\mathcal{F}_N$ we define the multiplicity of zero $\mu(f_1,\dots,f_N)\in\mathbb{Z}_+\cup\infty$:

- if the set $\{f_1=\dots=f_N=0\}$ has a component of positive dimension, passing through the point o, we set $\mu(f_1,\dots,f_N)=\infty$,
- otherwise, $\mu(f_1,\dots,f_N)$ is the multiplicity of the isolated common zero o of the functions $f_1,\dots,f_N$, that is, the integer $\dim_{\mathbb{C}}\mathcal{O}/(f_1,\dots,f_N)$.

By the *Gabrielov-Khovanskii problem* we mean the following question: what is the codimension of the closed subset

$$\mathcal{F}(m)=\{(f_1,\dots,f_N)\,|\,\mu(f_1,\dots,f_N)\geqslant m\}$$

in the space $\mathcal{F}=\mathcal{F}_1\times\dots\times\mathcal{F}_N$?

There is an obvious dual form of this problem. Let $B\subset\mathcal{F}$ be an irreducible closed subset of codimension $a\in\mathbb{Z}_+$. Set

$$\mu(B)=\min\{\mu(\underline{f})\,|\,\underline{f}=(f_1,\dots,f_N)\in B\},$$

that is, $\mu(B)$ is the multiplicity at zero of the general tuple of functions $\underline{f}\in B$. Now the Gabrielov-Khovanskii problem takes the form of the question: what is the maximal multiplicity

$$\mu(a)=\max\{\mu(B)\,|\,\operatorname{codim}(B\subset\mathcal{F})\leqslant a\}?$$

Thus we can either fix the multiplicity and look for (or estimate) the codimension, or fix the codimension and estimate the multiplicity. The second form is more natural from the viewpoint of certain geometric applications (see subsection 0.3), and in this paper it is the second form that we consider. (For the original and most general form of the Gabrielov-Khovanskii problem see their original paper [2].) For the spaces $\mathcal{F}_i$ we take the spaces of polynomials of degree $d_i \geqslant 2$, vanishing at the point o; some natural generalizations of the Gabrielov-Khovanskii problem are stated below in subsection 0.2.

Example 0.1. Let us compute $\mu(1)$ for $N = 2$. Let $B \subset \mathcal{F}_1 \times \mathcal{F}_2$ be an irreducible hypersurface. For a general tuple of polynomials $(f_1, f_2) \in B$ the curves $C_1 = \{f_1 = 0\}$ and $C_2 = \{f_2 = 0\}$ are non-singular at the point o (otherwise, $\operatorname{codim}(B \subset \mathcal{F}) \geqslant 2$). If the tangents $L_i = T_oC_i$ are distinct, then $\mu(f_1, f_2) = 1$. Otherwise, the tangents coincide for a general tuple $(f_1, f_2) \in B$, therefore

$$B \subset \{(f_1, f_2) \,|\, df_1(o) \| df_2(o)\}.$$

However, the latter set is closed, irreducible and of codimension 1 in $\mathcal{F}_1 \times \mathcal{F}_2$, so that the inclusion sign can be replaced by the equality. But then for a general tuple $(f_1, f_2) \in B$ the curves C_1 and C_2 have simple tangency at the point o, so that $\mu(B) = 2$ and for that reason $\mu(1) = 2$.

Example 0.2. Let us compute $\mu(2)$ for $N = 2$. In the notations of the previous example set:

$$B_i = \left\{ (f_1, f_2) \,\middle|\, \frac{\partial f_i}{\partial z_1}(o) = \frac{\partial f_i}{\partial z_2}(o) = 0 \right\}.$$

Obviously, B_i is an irreducible closed subset of codimension 2. For a general tuple $(f_1, f_2) \in B_1$ the curve C_1 has multiplicity 2 at the point o, whereas the curve C_2 is non-singular at the point o, and moreover the tangent line L_2 is not tangent to C_1 at the point o, so that $\mu(B_1) = 2$ (and $\mu(B_2) = 2$). Therefore, if $B \subset \mathcal{F}_1 \times \mathcal{F}_2$ is a closed irreducible subset of codimension 2, different from B_1 and B_2, then for a general tuple $(f_1, f_2) \in B$ the curves C_1, C_2 are non-singular at the point o. Set $B_3^\circ \subset \mathcal{F}_1 \times \mathcal{F}_2$ to be the set of such tuples (f_1, f_2), that the curves C_1, C_2 are non-singular at the point o, and moreover

$$\operatorname{ord}_o f_2|_{C_1} \geqslant 3.$$

It is easy to see that the closure $B_3 = \overline{B_3^\circ}$ is irreducible, of codimension 2 in $\mathcal{F}_1 \times \mathcal{F}_2$ and moreover for a general tuple $(f_1, f_2) \in B_3$ the equality $\operatorname{ord}_o f_2|_{C_1} = 3$ holds. This implies that $\mu(B_3) = 3$ and for any irreducible subset $B \subset \mathcal{F}_1 \times \mathcal{F}_2$ of codimension 2, which is not B_1, B_2 or B_3, we have $\mu(B) \leqslant 2$. Therefore, $\mu(2) = 3$.

Example 0.3. The computations of Example 0.1 generalize easily for an arbitrary number of variables N. The closed subset

$$B^* = \{(f_1, \dots, f_N) \,|\, \operatorname{rk}(df_1(o), \dots, df_N(o)) = N - 1\}$$

is of codimension 1 in $\mathcal{F}_1 \times \ldots \times \mathcal{F}_N$. Obviously, $\mu(B^*) = 2$ and for any irreducible hypersurface $B \neq B^*$ we have $\mu(B) = 1$. Therefore, $\mu(1) = 2$.

It is clear that $\mu(a) < \infty$ if and only if

$$a < \operatorname{codim}(\mathcal{F}(\infty) \subset \mathcal{F}).$$

(This, however, does not mean that the Gabrielov-Khovanskii problem makes sense only for those values of a, see subsection 0.2.)

Example 0.4. Let $N = 2$ and $d_1 \leqslant d_2$. The closed set $\mathcal{F}(\infty)$ is reducible: it consists of such tuples (f_1, f_2), that either one of the polynomials f_i is identically zero or (in the notations of Example 0.1) the curves C_1, C_2 have a common component. The degree of that component parametrizes irreducible components of the set $\mathcal{F}(\infty)$. It is easy to check that the least codimension is that of either component consisting of such tuples (f_1, f_2), that the curves C_1, C_2 have a line $L \ni o$ as a common component, or of the component $\{(f_1, f_2) \mid f_1 \equiv 0\}$. Therefore,

$$\operatorname{codim}(\mathcal{F}(\infty) \subset \mathcal{F}) = \min\left(d_1 + d_2 - 2, \frac{(d_1+1)(d_1+2)}{2} - 1\right).$$

Example 0.5. Again let $N = 2$ and $d_1 \leqslant d_2$. Consider the irreducible subvariety $B \subset \mathcal{F}$, given by the condition

$$B = \{(f_1, f_2) \mid \operatorname{mult}_o C_i \geqslant m,\ i = 1, 2\},$$

where $m \leqslant d_1$; we use the notations C_i introduced in Example 0.1 again. Obviously, $\operatorname{codim}(B \subset \mathcal{F}) = m^2 + m - 2$ and $\mu(B) = m^2$. Therefore, the inequality

$$\mu(m^2 + m - 2) \geqslant m^2$$

holds. The following elementary fact is well known (see, for instance, [3, Chapter V, Sec. 3, Ex. 3.2]): if the curves C_1, C_2 have no common irreducible component passing through the point o, then

$$\mu(f_1, f_2) = \sum_{x \geqslant o} (\operatorname{mult}_x C_1)(\operatorname{mult}_x C_2),$$

where the sum is taken over the finite set consisting of the point o and all infinitely near points of intersection of the curves C_1 and C_2, lying over the point o; the multiplicity of a curve at an infinitely near point x is understood as the multiplicity at x of the strict transform of that curve on the surface where x is a point in the usual sense, that is, the surface obtained by a finite sequence of blow ups. (The set of all points of an algebraic surface and all its infinitely near points, equipped with several natural structures, forms a well known classical object, the "bubble space"; for its detailed description see [4].) Experimenting with finite sets of infinitely near points, lying over the point o in the same way as it was done in the beginning of this example, we arrive to the following general conjecture.

Conjecture 0.1. (i) (Stabilization.) *For any fixed N there is a function $\delta\colon \mathbb{Z}_+ \to \mathbb{Z}_+$ such that for $\min(d_i) \geqslant \delta(a)$ the number $\mu(a)$ does not depend on the tuple $\underline{d} = (d_1, \ldots, d_N)$. Denote it by the symbol $\mu^{\mathrm{st}}(a)$.*

(ii) (Asymptotics.) *There is a finite limit*

$$\lim_{a\to\infty}\frac{\mu^{\rm st}(a)}{a}.$$

Now let us consider the behaviour of the numbers $\mu(a)$ for growing values of the number of variables N.

Example 0.6. (See [5, Section 3.5]) Let the closed subset $B\subset\mathcal{F}$ be given by the condition

$$\mathrm{rk}\,\|\partial f_i/\partial z_j(o)\|_{1\leqslant i,j\leqslant N}\leqslant N-b$$

for $b\in\{1,\dots,N\}$. It is easy to see that $\mathrm{codim}(B\subset\mathcal{F})=b^2$. Furthermore, for a general tuple $\underline{f}\in B$ the rank of the Jacobi matrix $\|\partial f_i/\partial z_j(o)\|$ equals $N-b$, so that there is a subset $I\subset\{1,\dots,N\}$, $\sharp I=b$, such that the linear forms $df_i(o)$, $i\notin I$, are linearly independent. Therefore, the subset $X=\{f_i=0\,|\,i\notin I\}$ is a smooth subvariety of dimension b around the point o. The restriction $f_i|_X$ for $i\in I$ has the zero differential at the point o, that is,

$$\mathrm{ord}_o(f_i|_X)\geqslant 2,$$

and moreover, for a general tuple $\underline{f}\in B$ we have the equality $\mathrm{ord}_o(f_i|_X)=2$. Since no other conditions are imposed on $\underline{f}$, for a general tuples $\underline{f}$ we get:

$$\mu(\underline{f})=2^b.$$

Therefore, $\mu(b^2)\geqslant 2^b$. The function $\mu(a)$ is obviously non-decreasing, so that we finally get the inequality $\mu(a)\geqslant 2^{[\sqrt{a}]}$ for $a\leqslant N^2$.

Conjecture 0.2. *For $N\geqslant\sqrt{a}$ there is the limit*

$$\lim_{a\to\infty}\frac{\mu(a)}{2^{\sqrt{a}}}=1$$

In the present paper we will show a weaker statement: for $a\leqslant N$ the function $\mu(a)$ grows as $C^{\sqrt{a}}$, where $C>0$ is some effectively estimated constant. More precisely (see Theorem 3.3 and Remark 3.3), we obtain an upper bound for $\mu(a)$ which asymptotically behaves as

$$\frac{1}{\sqrt{a}}e^{2\sqrt{a}}.$$

0.2. Open questions and generalizations. All the main questions related to the Gabrielov-Khovanskii problem are open. Computing the multiplicities $\mu(a)$ (one should write $\mu(a;\underline{d})$, but the discrete parameter $\underline{d}\in\mathbb{Z}_+^N$ is implicitly meant) seems to be a very difficult problem. It is natural to try to estimate them with various degrees of precision, examples of such estimates are given by Conjectures 0.1 and 0.2.

Here is an example of an open question. As we mentioned in subsection 0.1, the equality $\mu(a) = \infty$ holds for $a \geqslant \operatorname{codim}(\mathcal{F}(\infty) \subset \mathcal{F})$, however this does not mean that the Gabrielov-Khovanskii problem can not be set for such values of a. Set

$$\mu^*(a) = \max\{\mu(B) \,|\, B \not\subset \mathcal{F}(\infty), \operatorname{codim}(B \subset \mathcal{F}) \leqslant a\}.$$

The numbers $\mu^*(a)$ are defined for all $a \leqslant \dim \mathcal{F}$.

Example 0.7. Let $N = 2$, then

$$\dim \mathcal{F} = \frac{1}{2}(d_1^2 + 3d_1 + d_2^2 + 3d_2)$$

and obviously $\mu^*(\dim \mathcal{F}) = d_1 d_2$.

For arbitrary N and $a \leqslant \dim \mathcal{F}$ computing and estimating the numbers $\mu^*(a)$ is a very difficult problem.

Now let us consider a more general setting of the Gabrielov-Khovanskii problem (in the framework of algebraic geometry). Let X be a projective algebraic variety, $o \in X$ some point (not necessarily non-singular!). Set $N = \dim X$. If the point o is an isolated zero of the system of equations $f_1 = \ldots = f_N = 0$, where $f_i \in \mathcal{O}_{o,X}$, $f_i(o) = 0$, then the Samuel multiplicity

$$\mu(\underline{f}) = e_{\mathcal{O}}/(f_1, \ldots, f_N)$$

where $\mathcal{O} = \mathcal{O}_{o,X}$, is well defined (when X is non-singular at the point o, this is just the dimension of the quotient algebra $\mathcal{O}_{o,X}/(f_1, \ldots, f_N)$, see [1, Examples 7.1.2 and 7.1.10]). Now let $L_1, \ldots, L_N \in \operatorname{Pic} X$ be some classes, where $H^0(X, L_i) \neq \{0\}$. Set

$$\mathcal{F} = \{(s_1, \ldots, s_N) \,|\, s_i(o) = 0\} \subset \prod_{i=1}^{N} H^0(X, L_i).$$

Locally the sections s_i are represented by regular functions $f_i \in \mathcal{O}_{o,X}$, so that the multiplicities $\mu(\underline{s}) = \mu(s_1, \ldots, s_N) \in \mathbb{Z}_+ \cup \{\infty\}$ are well defined. This makes it possible to define the numbers

$$\mu(a) = \mu_X(a; L_1, \ldots, L_N)$$

in word for word the same way as it was done in subsection 0.1 and set the generalized Gabrielov-Khovanskii problem.

Example 0.8. Let $N = 2$, $X = \mathbb{P}^1 \times \mathbb{P}^1$ (the surface X can be naturally seen as a quadric in $\mathbb{P}^3$), $L_i = d_i \Delta_i$, where Δ_i are the standard generators of the Picard group, $\operatorname{Pic} X = \mathbb{Z}\Delta_1 \oplus \mathbb{Z}\Delta_2$, that is, Δ_i is the class of a line $\mathbb{P}^1 \times \{\mathrm{pt}\}$ or $\{\mathrm{pt}\} \times \mathbb{P}^1$, respectively. Let $o \in X$ be an arbitrary point. It is easy to see that $\mu_X(a; L_1, L_2)$ is equal to

$$\max\{n_1 n_2 \,|\, n_i \leqslant d_i, n_1 + n_2 \leqslant a + 2\}.$$

For that reason, $\mu_X(a; L_1, L_2) = (\frac{a}{2} + 1)^2$, if $a \in 2\mathbb{Z}$, and $\frac{(a+1)(a+3)}{4}$, if a is odd.

Finally, one more generalization of the Gabrielov-Khovanskii problem for polynomials will be considered in §4.

0.3. One application of the Gabrielov-Khovanskii problem. Let us describe briefly one important application of the Gabrielov-Khovanskii problem for polynomials. In birational geometry of higher-dimensional rationally connected algebraic varieties estimates of the multiplicity of a singular point in terms of the degree of a subvariety are of high importance.Let $X \subset \mathbb{P}^N$ be an irreducible algebraic variety, $o \in X$ a non-singular (for simplicity) point. Consider a subvariety $Y \subset X$. One needs to estimate the ratio of the multiplicity to the degree:

$$\frac{\mathrm{mult}_o Y}{\deg Y} \leqslant c, \tag{1}$$

where the estimate should be true for *every* subvariety Y of a given codimension(for instance, if X is a sufficiently general hypersurface of degree N, where $N \geqslant 4$, then for the codimension $\mathrm{codim}(Y \subset X) = 2$ one can take $c = 3/(N-1)$, see [9, Chapter 3].) Of course, one can always take $c = 1$, but this estimate, as a rule, is insufficient, especially in higher-dimensional problems. The only efficient method of obtaining such estimates, known today, is the *method of hypertangent divisors*, the idea of which can be explained by the following example. Let $(z_1, \ldots, z_N)$ be a system of affine coordinates with the origin at the point o, and

$$f = q_1 + q_2 + \ldots + q_k$$

a polynomial, such that $f|_X \equiv 0$, where $q_i(z)$ are homogeneous of degree i. Now the polynomial

$$f_i = q_1 + \ldots + q_i$$

has degree $\leqslant i$, however

$$f_i|_X = -(q_{i+1} + \ldots + q_K)|_X,$$

so that the multiplicity of the divisor $\{f_i|_X = 0\}$ at the point o is at least $(i+1)$. If $f_i|_Y \not\equiv 0$, then one can form the effective cycle

$$(Y \circ \{f_i|_X = 0\})$$

of codimension $(i+1)$ on X, for which the ratio of the multiplicity at the point o to the degree is at least

$$\frac{\mathrm{mult}_o Y}{\deg Y} \cdot \frac{i+1}{i}.$$

The necessary condition $Y \not\subset \{f_i|_X = 0\}$ is provided by the *regularity conditions* for the equations, defining the variety X. For the details and numerous examples of applications of the method of hypertangent divisors to the problems of higher-dimensional birational geometry see [9, 10, 6]. The procedure described above is iterated and makes it possible to construct, starting from the subvariety Y (the existence of which is assumed), satisfying the estimate

$$\frac{\mathrm{mult}_o Y}{\deg Y} > c,$$

an effective 1-cycle (that is, an integral linear combination of curves) C, such that $\mathrm{mult}_o C > \deg C$. The latter is impossible, whence we conclude that the inequality (1) holds for all subvarieties $Y \subset X$ of the given codimension.

Unfortunately, for certain classes of Fano varieties the procedure described above gives nothing by itself: applying the techniques of hypertangent divisors, one can construct an effective curve C, for which the ratio $(\mathrm{mult}_o C/\deg C)$ is less than 1, although is close to that number. Therefore, no contradiction is obtained and the inequality (1) can not be shown directly. In order to circumvent this obstruction, we use the Gabrielov-Khovanskii problem.

For instance, if X is a Fano complete intersection of quadrics and cubics in $\mathbb{P}^N$ of index 1 (see [8]), then for any irreducible curve $\Gamma \subset X$ of degree $\deg \Gamma \geqslant 2$ the estimate

$$\frac{\mathrm{mult}_o \Gamma}{\deg \Gamma} \leqslant \frac{2}{3}$$

holds, which is sufficient to prove birational rigidity of the variety X, *provided that the lines passing through the point o, form a not too large part of the effective 1-cycle* C, which is the output of the technique of hypertangent divisors. The Gabrielov-Khovanskii problem provides an estimate of the input of the lines. For the details, see [8].

It is this application that initially generated the interest of the author to the Gabrielov-Khovanskii problem, see [7].

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1 Statement of the problem

In this section we give a precise statement of the Gabrielov-Khovanskii problem for polynomials: we introduce the spaces of tuples of polynomials, bi-invariant subvarieties and multiplicities. Then we give an estimate of the codimension of the set of tuples that vanish on a subset of positive dimension. We define the parameter β, characterizing a subvariety of tuples of polynomials.

1.1. The space of tuples of polynomials. Fix the complex coordinate space $\mathbb{A} = \mathbb{C}^N_{(z_1,\dots,z_N)}$ of dimension $N \geqslant 1$ with coordinates $(z_*) = (z_1, \dots, z_N)$. For $d \in \mathbb{Z}_+$ let $\mathcal{P}_{d,N}$ be the linear space of homogeneous polynomials of degree d in z_* (in particular, $\mathcal{P}_{0,N} = \mathbb{C}$), and for $e \leqslant d$ set

$$\mathcal{P}_{[e,d],N} = \bigoplus_{i=e}^{d} \mathcal{P}_{i,N},$$

for instance, $\mathcal{P}_{[1,d],N}$ is the space of (non-homogeneous) polynomials of degree $\leqslant d$ with no free term. On each of these spaces acts the matrix group $G_1 = GL_N(\mathbb{C})$ of linear changes of coordinates. Fix a tuple of integers

$$\underline{d} = (d_1, \dots, d_N),$$

where $2 \leqslant d_1 \leqslant \dots \leqslant d_N$, and set

$$\mathcal{P}(\underline{d}) = \prod_{i=1}^{N} \mathcal{P}_{[1,d_i],N}$$

to be the space of tuples $(f_1, \dots, f_N)$ of polynomials of degree $\leqslant d_1, \dots, d_N$, respectively, with no free term. On the space $\mathcal{P}(\underline{d})$, apart from the above-mentioned group G_1, act two more groups of transformations, which we will now define. The group G_{21} consists of transformations of the form

$$(f_1, \dots, f_N) \mapsto (f_1^+, \dots, f_N^+), \qquad f_i^+ = f_i + \sum_{j=1}^{i-1} s_{i,j}(z) f_j,$$

where $s_{i,j} \in \mathcal{P}_{[0,d_i-d_j],N}$ are polynomials, fixed for the given transformation. Set $\mathcal{D} = \{d_1\} \cup \dots \cup \{d_N\} \subset \mathbb{Z}_+$ and let for $d \in \mathcal{D}$

$$n_d = \sharp\{i \mid d_i = d\},$$

so that $\sum\limits_{d \in \mathcal{D}} n_d = N$. Now the group G_{22} is defined as the matrix group (realized by block-wise diagonal matrices, where the blocks of the size $n_d \times n_d$ are ordered by increasing of the integers d)

$$\prod_{d \in \mathcal{D}} GL_{n_d}(\mathbb{C}),$$

acting on the tuples $(f_*) \in \mathcal{P}(\underline{d})$ by linear transformations of the form

$$(f_1, \dots, f_N) \mapsto (f_1, \dots, f_N)A.$$

Let $G_2 = \langle G_{21}, G_{22} \rangle$ be the group of linear transformations of the space $\mathcal{P}(\underline{d})$, generated by the subgroups G_{21} and G_{22}. The group G_2 is clearly connected, hence irreducible as an algebraic variety. An irreducible subvariety $B \subset \mathcal{P}(\underline{d})$ (respectively, a map from $\mathcal{P}(\underline{d})$ to some set) is said to be *bi-invariant*, if it is invariant with respect to the action of both groups G_1 and G_2.

1.2. Multiplicities. For a tuple $\underline{f} = (f_1, \dots, f_N) \in \mathcal{P}(\underline{d})$ we define the multiplicity $\mu(\underline{f}) = \mu(f_1, \dots, f_N) \in \mathbb{Z}_+ \cup \infty$, setting:

- $\mu(\underline{f}) = \infty$, if the closed algebraic set $\{f_1 = \dots = f_N = 0\}$ has a component of positive dimension, containing the point $o = (0, \dots, 0) \in \mathbb{A}$,
- $\mu(\underline{f}) = \dim \mathcal{O}_{o,\mathbb{A}}/(f_1, \dots, f_N)$, otherwise.

Obviously, the function $\mu\colon \mathcal{P}(\underline{d}) \to \mathbb{Z}_+ \cup \{\infty\}$ is bi-invariant. For an arbitrary irreducible subvariety $B \subset \mathcal{P}(\underline{d})$ set

$$\mu(B) = \min_{\underline{f} \in B} \{\mu(\underline{f})\} \in \mathbb{Z}_+ \cup \infty,$$

so that $\mu(B) = \mu(\underline{f})$ for a general tuple $\underline{f} \in B$. Furthermore, set

$$\mu(a) = \max\{\mu(B) \,|\, \mathrm{codim}(B \subset \mathcal{P}(\underline{d})) \leqslant a\},$$

that is, the maximum is taken over all irreducible subvarieties of codimension a. If $\langle B \rangle$ is the *bi-invariant span* of the subvariety B, that is, the smallest bi-invariant subvariety in $\mathcal{P}(\underline{d})$, containing B, then, obviously, $\mu(\langle B \rangle) = \mu(B)$, and moreover $\mathrm{codim}(\langle B \rangle \subset \mathcal{P}(\underline{d})) \leqslant (\mathrm{codim}(B \subset \mathcal{P}(\underline{d}))$, so that the number $\mu(a)$ can be defined as the maximum of the numbers $\mu(B)$ over all *bi-invariant* irreducible subvarieties $B \subset \mathcal{P}(\underline{d})$ of codimension at most a. This obvious remark will be used in the sequel without special references.

Consider the closed subset

$$X_\infty = \{\underline{f} \in \mathcal{P}(\underline{d}) \,|\, \mu(\underline{f}) = \infty\}$$

and set $\chi_\infty(\underline{d}) = \mathrm{codim}(X_\infty \subset \mathcal{P}(\underline{d}))$. Obviously, $\mu(B) = \infty$, if and only if $B \subset X_\infty$, and $\mu(a) = \infty$ if and only if $a \geqslant \chi_\infty(\underline{d})$. Consider the irreducible subvariety $X_{\mathrm{line}} \subset \mathcal{P}(\underline{d})$, consisting of such tuples $\underline{f}$, that

$$f_1|_L \equiv \ldots \equiv f_N|_L \equiv 0 \qquad (2)$$

for some line $L \ni o$.

Proposition 1.1. *The following equality holds:*

$$\mathrm{codim}(X_{\mathrm{line}} \subset \mathcal{P}(\underline{d})) = 1 - N + \sum_{i=1}^{N} d_i.$$

Proof: a trivial dimension count. When a line $L \ni o$is fixed, the condition (2) defines an irreducible subvariety (in fact, a linear subspace) of codimension $\sum\limits_{i=1}^{N} d_i$. Since a general tuple $\underline{f} \in X_{\mathrm{line}}$ vanishes on exactly one line, considering the direct product $\mathbb{P}^{N-1} \times \mathcal{P}(\underline{d})$ and the second projection, we complete the proof in the standard way. Q.E.D.

Set $X_\infty^+ = \overline{X_\infty \backslash X_{\mathrm{line}}}$.

Proposition 1.2. *The following estimate holds:*

$$\mathrm{codim}(X_\infty^+ \subset \mathcal{P}(\underline{d})) \geqslant d_1 N + 1.$$

Proof. We use the technique developed in [11, Section 3]. The space $\mathbb{A}$ is considered as embedded in the projective space $\mathbb{P} = \mathbb{P}^N_{(x_0:\ldots:x_N)}$ as the affine chart

$(x_0 \neq 0)$, the polynomials $f_1, \dots, f_N$ are represented by polynomials $F_1, \dots, F_N$, where $F_i(o) = 0$. We have to estimate the codimension of the subset of tuples (F_*), for which there exists an irreducible subvariety $Y \ni o$ of positive dimension, which is not a line and such that $F_i|_Y \equiv 0$ for all $i = 1, \dots, N$. For every such tuple there is a uniquely determined integer $k \in \{0, 1, \dots, N-1\}$, satisfying the two conditions:

- $\mathrm{codim}_o(\{F_1 = \ldots = F_k = 0\} \subset \mathbb{P}) = k$ (where codim_o means the codimension in a neighborhood of the point o, and for $k = 0$ the set $\{F_1 = \ldots = F_k = 0\}$ is the whole space $\mathbb{P}$),

- the polynomial F_{k+1} vanishes identically on an irreducible component B of the closed set $\{F_1 = \ldots = F_k = 0\}$, containing the point o, and if $k = N-1$, then B is not a line.

Let

$$\alpha_k = \sum_{i=1}^{k+1} \binom{d_i + N - k}{d_i} - k(N-k)$$

be the codimension of the closed set of such tuples $(F_1, \dots, F_{k+1})$, that $F_i|_\Lambda \equiv 0$ for a certain linear subspace $\Lambda \subset \mathbb{P}$ of codimension k, where $k = 0, 1, \dots, N-2$ (in order to see that the codimension of this closed set is indeed equal to α_k, one argues as in the proof of Proposition 1.1: consider the algebraic set

$$\{(\Lambda, \underline{F}) \mid \underline{F}|_\Lambda \equiv 0\} \subset G(k, N) \times \{(\underline{F})\},$$

where $G(k, N)$ is the projective Grassmanian of k-subspaces in $\mathbb{P}^N$, and two projections on the direct factors $G(k, N)$ and the space $\{(\underline{F})\} = \{(F_1, \dots, F_N)\}$ of tuples of homogeneous polynomials, introduced above; the obvious details are left to the reader). It is easy to check that $\alpha_k \geqslant d_1 N + 1$. Therefore, estimating the codimension of the set of “irregular” tuples (F_*), we may assume that the irreducible component B of the closed set $\{F_1 = \ldots = F_k = 0\}$, on which F_{k+1} vanishes identically, is not a linear subspace. Set

$$\beta_k = \min_{l \in \{1, \dots, k\}} [(d_1 + \ldots + d_{k-l} + d_{k+1} - (k-l))(N - k + l) + 1],$$

$k = 1, \dots, N-1$. Now the technique developed in [11, Section 3] gives the estimate

$$\mathrm{codim}((\overline{X_\infty \backslash X_{\mathrm{line}}}) \subset \mathcal{P}(\underline{d})) \geqslant \min_{k \in \{1, \dots, N-1\}} \beta_k,$$

so that in order to complete the proof of Proposition 1.1 it is sufficient to show that the right-hand side of the last inequality is not smaller than $(d_1 N + 1)$. This is an easy task.

Now replacing in the expression for β_k the numbers $d_1, \dots, d_{k+1}$ by $d = d_1$, we get

$$\beta_k \geqslant \min_{l \in \{1, \dots, k\}} [((k - l + 1)d - (k - l))(N - k + l) + 1].$$

The expression in the square brackets is a quadratic polynomial in l with the senior coefficient $-(d-1)l^2$. Since $d \geqslant 2$, the minimum is attained at one of the endpoints of the interval $[1, k]$. For $l = k$ we get the value $dN + 1$, which is what we want. For $l = 1$ we get

$$(k(d-1)+1)(N-k+1)+1.$$

Here $k \in \{1, \ldots, N-1\}$ and the last expression is again a quadratic polynomial with the senior coefficient $-(d-1)k^2$, that is, the minimum in k is attained at one of the endpoints of the interval $[1, N-1]$. For $k = 1$ we get the required value $dN + 1$. For $k = N-1$ we get $2d(N-1) - 2N + 5 \geqslant dN + 1$. This completes the proof of Proposition 1.2.

Corollary 1.1. *Assume that*

$$a \leqslant \min(d_1 N, \sum_{i=1}^{N} d_i - N).$$

Then the number $\mu(a)$ *is finite.*

Below we obtain estimates from above for $\mu(a)$, which are close to optimal ones, for the values $a \leqslant N$.

1.3. The rank of a system of linear forms. Let us consider the construction of the Example 0.6 more formally.

Example 1.1. (See [5, Section 3.5]) For $b \in \mathbb{Z}_+$ set

$$X(b) = \{\underline{f} \in \mathcal{P}(\underline{d}) \,|\, \mathrm{rk}(df_1(o), \ldots, df_N(o)) \leqslant N - b\}.$$

For $b \leqslant N$ the set $X(b)$ is non-empty, closed and bi-invariant, and of codimension

$$\mathrm{codim}(X(b) \subset \mathcal{P}(\underline{d})) = b^2.$$

For a general tuple $\underline{f}$ in any irreducible component of the set $X(b)$ there is a subset $I \subset \{1, \ldots, N\}$, $\sharp I = N - b$, such that

$$\mathrm{rk}(df_i(o) \,|\, i \in I) = N - b.$$

Therefore, for any polynomials $g_j \in \mathcal{P}_{[1,d_j],N}$, $j \notin I$, such that

$$dg_j(o) \in \langle df_i(o) \,|\, i \in I \rangle,$$

the tuple $(f_1^*, \ldots, f_N^*)$, given by the conditions

- $f_i^* = f_i$ for $i \in I$,
- $f_j^* = g_j$ for $j \notin I$,

belongs to the same irreducible component of the set $X(b)$, as $\underline{f}$. In other words, the closed algebraic set $Z_I(\underline{f}) = \{f_i = 0 \,|\, i \in I\}$ in a neighborhood of the point $o \in \mathbb{A}$ is a non-singular b-dimensional variety, and on the polynomials $f_j, j \notin I$, only one condition is imposed: $df_j|_{T_o Z_I(\underline{f})} \equiv 0$. Therefore, for every irreducible component $B \subset X(b)$ we have

$$\mu(B) = 2^b.$$

Since $\operatorname{codim}(B \subset \mathcal{P}(\underline{d})) \geqslant b^2$, we obtain the following estimate for the function $\mu(a)$ from below:

$$\mu(a) \geqslant 2^{[\sqrt{a}]}.$$

This example motivates introducing a new parameter that characterizes an arbitrary irreducible subvariety $B \subset \mathcal{P}(\underline{d})$ of codimension $a \in \mathbb{Z}_+$: set

$$\beta(B) = N - \max_{\underline{f} \in B} \operatorname{rk}(df_i(o) \,|\, i = 1, \dots, N).$$

Obviously, $\beta(B) = \max\{b \in \mathbb{Z}_+ \,|\, B \subset X(b)\}$. In particular, $\beta(B) \leqslant \sqrt{a}$.

Proposition 1.3. *If $\beta(B) = 0$, then $\mu(b) = 1$.*

Proof. This is obvious: for a general tuple $\underline{f} \in B$ the linear forms $df_i(o)$ are linearly independent. Q.E.D. for the proposition.

Proposition 1.4. *The following equality holds: $\mu(1) = 2$.*

Proof. Let $B \subset \mathcal{P}(\underline{d})$ be an irreducible subvariety of codimension 1. If $\beta(B) = 0$, then $\mu(B) = 1$. Assume that $\beta(B) = 1$. This means that for a general tuple $\underline{f} \in B$ there is an index i, such that the linear forms

$$df_1(o), \dots, df_{i-1}(o),\ df_{i+1}(o), \dots, df_N(o)$$

are linearly independent, so that the set

$$\{f_j = 0 \,|\, j = 1, \dots, i-1, i+1, \dots, N\}$$

in a neighborhood of the point o is a curve $C(\underline{f})$, which is non-singular at the point o, and moreover $df_i|_{T_o C(\underline{f})} \equiv 0$. Since $\operatorname{codim}(B \subset \mathcal{P}(\underline{d})) = 1$, for any polynomial $g \in \mathcal{P}_{[1,d_i],N}$, such that $g|_{T_o C(\underline{f})} \equiv 0$, we have $(f_1, \dots, f_{i-1}, g, f_{i+1}, \dots f_N) \in B$. In particular, this is true for any quadratic form $g \in \mathcal{P}_{2,N}$. Therefore, for a general tuple $\underline{f} \in B$ we have $\mu(\underline{f}) = 2$, as we claimed. Q.E.D. for the proposition.

2 Reduction to a smaller dimension

In this section we construct an inductive procedure of estimating the multiplicity $\mu(B)$ in terms of multiplicities $\mu(B_i)$ for certain subvarieties B_i in the space of tuples of $(N-1)$ polynomials in $(N-1)$ variables. Iterating this procedure, we obtain in §3 estimates for the function $\mu(a)$. In subsection 2.1 we construct the map of bringing a general tuple $(\underline{f}) \in B$ into the standard form, in subsection 2.2 we state the main claim about reduction to a smaller dimension, in subsections 2.3-2.4 we prove it, in subsection 2.5 we consider its generalization.

2.1. Bringing into the standard form. Let $B \subset \mathcal{P}(\underline{d})$ be an irreducible bi-invariant subvariety of codimension a and $\beta(B) = b \geqslant 1$; as we have seen, $a \geqslant b^2$. Consider the subset

$$I \subset \{1, \dots, N\},$$

$\sharp I = b$, such that for a general tuples $\underline{f} \in B$

$$\mathrm{rk}(df_i(o) \,|\, i \notin I) = N - b. \tag{3}$$

Because of the bi-invariance of the set B we may assume that the following conditions are satisfied:

- for $j \in I$ the linear form $df_j(o)$ is a linear combination of the forms $df_i(o)$, where $i \notin I$ and $i < j$;
- for $d_j = d_{j+1}$ if $j \in I$, then $(j+1) \in I$.

By the symbol B° we denote the open subset in B, defined by the condition (3). Set $e = \max\{j \,|\, j \in I\}$. Consider the space of tuples of polynomials

$$\widetilde{\mathcal{P}}(\underline{d}) = \prod_{j \notin I} \mathcal{P}_{[1,d_j],N} \times \prod_{j \in I} \mathcal{P}_{[2,d_j],N}.$$

On the open set B° the map of *bringing into the standard form* $\sigma\colon B^\circ \to \widetilde{\mathcal{P}}(\underline{d})$,

$$\sigma\colon (f_1, \dots, f_N) \mapsto (\tilde{f}_1, \dots \tilde{f}_N),$$

is well defined, where $\tilde{f}_j = f_j$ for $j \notin I$ and $\tilde{f}_j = f_j - \sum\limits_{i \notin I, i<j} \lambda_{j,i} f_i$, the coefficients $\lambda_i \in \mathbb{C}$ are uniquely determined by the relation

$$df_j(o) = \sum_{i \notin I, i<j} \lambda_{j,i} df_i(o).$$

Set $B^{\mathrm{st}} = \overline{\sigma(B^\circ)} \subset \widetilde{\mathcal{P}}(\underline{d})$. Setting for $j = 1, \dots, N$

$$\varepsilon(j) = \sharp\{i \notin I, i < j\},$$

we see that the fibre of general position $\sigma\colon B^\circ \to B^{\mathrm{st}}$ is of dimension

$$\sum_{j \in I} \varepsilon(j) \leqslant b(N-b)$$

(this inequality becomes an equality if $I = \{N-b+1, \dots, N\}$). Taking into account that

$$\dim \widetilde{\mathcal{P}}(\underline{d}) = \dim \mathcal{P}(\underline{d}) - bN,$$

we conclude that

$$\mathrm{codim}(B^{\mathrm{st}} \subset \widetilde{\mathcal{P}}(\underline{d})) = a - bN + \sum_{j \in I} \varepsilon(j) \leqslant a - b^2.$$

Now let us represent the space $\widetilde{\mathcal{P}}(\underline{d})$ as the direct product of the spaces

$$\widetilde{\mathcal{P}}_e(\underline{d}) = \prod_{j \notin I} \mathcal{P}_{[1,d_j],N} \times \prod_{j \in I, j \neq e} \mathcal{P}_{[2,d_j],N}$$

and $\mathcal{P}_{[2,d_e],N}$. The projections onto these direct factors denote by the symbols π and π_e, respectively.

Let us consider the restriction $\pi_B = \pi|_{B^{\mathrm{st}}} \colon B^{\mathrm{st}} \to \widetilde{\mathcal{P}}_e(\underline{d})$ of the projection π onto B^{st}. For a general tuple $\underline{f} \in B^{\mathrm{st}}$ set

$$\gamma = \mathrm{codim}(\pi_B^{-1}(\pi(\underline{f})) \subset \mathcal{P}_{[2,d_e],N}).$$

(Here we identify the fibre $\pi^{-1}(\pi(\underline{f}))$ and the space $\mathcal{P}_{[2,d_e],N}$.) Therefore, the equality

$$\mathrm{codim}(\overline{\pi(B^{\mathrm{st}})} \subset \widetilde{\mathcal{P}}_e(\underline{d})) = a - bN + \sum_{j \in I} \varepsilon(j) - \gamma$$

holds, whereas the right hand side does not exceed $a - b^2 - \gamma$, so that the estimate

$$b^2 + \gamma \leqslant a \tag{4}$$

holds. Before studying the just constructed map π_B in full generality, let us consider some simple examples.

Example 2.1. Assume that $b = 1$ and $\gamma = 0$, that is, every non-empty fibre of the map $\pi_B \colon B^{\mathrm{st}} \to \widetilde{\mathcal{P}}_e(\underline{d})$ is the whole space of polynomials $\mathcal{P}_{[2,d_e],N}$. Since $b = 1$, for a general tuple $\underline{f} \in B^\circ$ the closed set $C = \{f_i = 0 \,|\, i \neq e\}$ in a neighborhood of the point o is a non-singular curve and the polynomial f_e is of the form

$$g(z_*) + \sum_{i \neq e} \lambda_i f_i, \tag{5}$$

where $g \in \mathcal{P}_{[2,d_e],N}$ is an arbitrary polynomial (as bringing into the standard form means subtracting from f_e a linear combination of polynomials f_i, $i \neq e$, and gives $g(z_*)$ as a result).

Therefore, the restriction $f_e|_C$ has at the point o a zero of order exactly 2, that is, the equality $\mu(B) = 2$ holds. This equality does not depend on the codimension a of the subvariety B.

Example 2.2. Now let us assume that $b = \gamma = 1$, and $d_e \geqslant 3$. This case is not much more complicated. Again for a general tuple $\underline{f} \in B^\circ$ the set $C = \{f_i = 0 \,|\, i \neq e\}$ is a curve, non-singular at the point o, and the polynomial f_e is of the form (5), where in this case

$$g \in \pi_B^{-1}(\pi(\underline{f})) \subset \mathcal{P}_{[2,d_e],N}$$

(again, identifying the fibre of the projection π with the space $\mathcal{P}_{[2,d_e],N}$ by means of the projection π_e, which is meant but not written) and $\pi_B^{-1}(\pi(\underline{f}))$ is of codimension 1 in the ambient space $\mathcal{P}_{[2,d_e],N}$. Blowing up the point $o\in\mathbb{A}$, it is easy to see that for $d_e\geqslant 3$ the condition

$$\mathrm{ord}_o\, g|_C\geqslant 3$$

defines an irreducible divisor in $\mathcal{P}_{[2,d_e],N}$, and moreover for a general polynomial g in that divisor the equality $\mathrm{ord}_o\, g|_C=3$ holds. Therefore, in the case under consideration we have $\mu(B)\in\{2,3\}$ for any value a of the codimension of the subvariety B.

2.2. Splitting off a direct factor. Now let us consider the general case. Let $\underline{d}^+=(d_1,\dots,d_{e-1},d_{e+1},\dots,d_N)$ be the truncated tuple of degrees and

$$\mathcal{P}(\underline{d}^+)=\prod_{j\neq e}\mathcal{P}_{[1,d_j],N-1}$$

the corresponding space of tuples of $(N-1)$ polynomials in $(N-1)$ variables. We keep the notations introduced at the beginning os Sec. 2.1: $B\subset\mathcal{P}(\underline{d})$ is an irreducible subvariety of codimension a with $\beta(B)=b$.

Theorem 2.1. *Assume that $\gamma\leqslant N-1$. Then there are irreducible bi-invariant subvarieties $B_i\subset\mathcal{P}(\underline{d}^+)$, $i=1,2$, such that:*

(i) *the inequality $\mu(B)\leqslant\mu(B_1)+\mu(B_2)$ holds,*

(ii) $\beta(B_1)=b-1$ *and the estimate*

$$a_1=\mathrm{codim}(B_1\subset\mathcal{P}(\underline{d}^+))\leqslant a-2b+1 \tag{6}$$

holds,

(iii) *either $\beta(B_2)=b-1$ and the codimension $a_2=\mathrm{codim}(B_2\subset\mathcal{P}(\underline{d}^+))$ satisfies the inequality (6) (in this case we say that this is the case of stable reduction), or $\beta(B_2)=b$ and the inequality*

$$a_2\leqslant a-b \tag{7}$$

holds (in that case we say that this is the case of non-stable reduction).

Before starting to show the theorem, let us explain briefly the strategy of the proof. Bringing the tuples of the subvariety B into the standard form, we obtained a new subvariety B^{st}, where the e-th polynomial of every tuple has no linear term. Now we intersect B^{st} with a special subvariety of tuples, the e-th polynomial in which is a reducible quadratic form (a product of two linear forms), whereas the other polynomials are arbitrary. Calculating dimensions and taking into account the bi-invariance of B, we show that the intersection is non-empty and estimate its (co)dimension. Now we can use the following obvious observation: the multiplicity at zero of a tuple $(f_1,\dots,f_e,\dots,f_N)$ with f_e just a product of two linear forms, say $h_1(z)h_2(z)$, is equal to the sum of multiplicities at zero of the tuples

$$(f_1|_{\{h_i=0\}},\dots,f_{e-1}|_{\{h_i=0\}},f_{e+1}|_{\{h_i=0\}},\dots,f_N|_{\{h_i=0\}}),$$

$i = 1, 2$. The subvarieties $\{h_i = 0\}$ are hyperplanes and can be identified with $\mathbb{C}^{N-1}$. Therefore, we can estimate the original multiplicity $\mu(B)$ in terms of the multiplicities $\mu(B_i)$, $i = 1, 2$, where the subvarieties B_i consist of $(N-1)$-uples of polynomials in $(N-1)$ variables. (It turns out that one of the linear forms, say $h_1(z)$, can be pre-selected.) The main work in the proof is to estimate the parameters of the new subvarieties B_i.

Now we proceed to the rigorous argument.

Proof of Theorem 2.1. Let $h_1(z_*) \in \mathcal{P}_{1,N}$ be a linear form of general position with respect to the subset $\pi(B^{\rm st}) \subset \widetilde{\mathcal{P}_e}(\underline{d})$, in the sense that the hyperplane $\{h_1 = 0\}$ does not contain the linear space

$$\{df_j(o) = 0 \,|\, j \notin I\}$$

for a general tuple $\underline{f} \in B$. Furthermore, let

$$\Pi = \{h_1(z_*)h(z_*) \,|\, h \in \mathcal{P}_{1,N}\} \subset \mathcal{P}_{2,N}$$

be the linear space of reducible homogeneous quadratic polynomials divisible by h_1. Note that $\mathcal{P}_{2,N} \subset \mathcal{P}_{[2,d_e],N}$, so that we may (and will) consider Π as a linear subspace in $\mathcal{P}_{[2,d_e],N}$. Obviously, $\dim \Pi = N$. Set

$$\mathcal{P}_\Pi = \pi_e^{-1}(\Pi) = \widetilde{\mathcal{P}}(\underline{d}) \times \Pi.$$

This is closed irreducible subset of the space $\widetilde{\mathcal{P}}(\underline{d})$. The following claim is true.

Proposition 2.1. *The intersection $B^{\rm st} \cap \mathcal{P}_\Pi$ is non-empty and is of codimension at most $a - bN + \sum\limits_{j \in I} \varepsilon(j) \leqslant a - b^2$ in $\mathcal{P}_\Pi$. Moreover, the equality*

$$\overline{\pi(B^{\rm st})} = \overline{\pi(B^{\rm st} \cap \mathcal{P}_\Pi)}$$

holds, and for a general tuple $\underline{f} \in B^{\rm st}$ the intersection $\pi_B^{-1}(\pi(\underline{f})) \cap \mathcal{P}_\Pi$ has positive dimension.

Proof. As the set B is bi-invariant, for a general tuple $\underline{f} \in B^{\rm st}$ the fibre $\pi_B^{-1}(\pi(\underline{f}))$ is a closed subset of the space $\mathcal{P}_{[2,d_e],N}$ of codimension $\gamma \leqslant N-1$, containing the zero polynomial. Therefore, the intersection $\pi_B^{-1}(\pi(\underline{f})) \cap \mathcal{P}_\Pi$ is a non-empty closed subset in Π of codimension at most γ, so that its dimension is positive. The other claims are now obvious. The proof is complete.

By the symbol π_Π we denote the projection $\mathcal{P}_\Pi \to \Pi$, the fibre of which is the space $\widetilde{\mathcal{P}}_e(\underline{d})$. By the bi-invariance (more precisely, the invariance with respect to the group G_1 of linear changes of variables) either

$$\overline{\pi_\Pi(B^{\rm st} \cap \mathcal{P}_\Pi)} = \Pi,$$

or $\pi_\Pi(B^{\rm st} \cap \mathcal{P}_\Pi)$ is the line $\{\lambda h_1^2(z_*) \,|\, \lambda \in \mathbb{C}\}$ in Π, and then $\gamma = N-1$ (as Π is N-dimensional). The second case is simpler, let us start with the first one.

Let $h_1h_2 \in \pi_\Pi(B^{\rm st} \cap \mathcal{P}_\Pi)$ be a general polynomial, $h_2 \notin \{\lambda h_1 \,|\, \lambda \in \mathbb{C}\}$. The intersection

$$B^{\rm st} \cap \pi_\Pi^{-1}(h_1h_2) \subset \widetilde{\mathcal{P}}_e(\underline{d})$$

is a closed subset of codimension at most

$$a - bN + \sum_{j\in I} \varepsilon(j) - \gamma.$$

Let us fix isomorphisms of the hyperplanes $\{h_i = 0\}$ and $\mathbb{C}^{N-1}$ and let

$$\rho_i \colon \mathcal{P}_{[k,l],N} \to \mathcal{P}_{[k,l],N-1}$$

be the restriction map $\rho_i(f) = f|_{\{h_i=0\}}$, $i = 1, 2$. The same symbols ρ_i will be used for the corresponding maps of the spaces of tuples of polynomials:

$$\rho_i(f_1, \dots, f_k) = (\rho_i(f_1), \dots, \rho_i(f_k)).$$

Omitting the polynomial f_e, we obtain two projections

$$\rho_i \colon \widetilde{\mathcal{P}}_e(\underline{d}) \to \mathcal{P}(\underline{d}^+),$$

corresponding to restrictions onto the hyperplanes $H_i = \{h_i = 0\}$. Finally, set

$$B_i = \overline{\langle \rho_i \circ \pi(B^{\rm st} \cap \pi_\Pi^{-1}(h_1h_2)) \rangle} \subset \mathcal{P}(\underline{d}^+),$$

where the brackets $\langle\cdot\rangle$ mean the bi-invariant span, and the line above means the closure. The sets B_i without loss of generality can be assumed to be irreducible (if this is not the case, take any irreducible component).

Let us show the claim (i) of Theorem 2.1. For a general tuple of polynomials

$$\underline{f} = (f_1, \dots, f_{e-1}, h_1h_2, f_{e+1}, \dots, f_N) \in B^{\rm st} \cap \pi_\Pi^{-1}(h_1h_2)$$

the inequality $\mu(\underline{f}) \geqslant \mu(B)$ holds, because $\underline{f} \in B$. Let

$$\underline{f}^+ = (f_1, \dots, f_{e-1}, f_{e+1}, \dots, f_N)$$

be a truncated tuple. Obviously,

$$\mu(\underline{f}) = \mu(\rho_1(\underline{f}^+)) + \mu(\rho_2(\underline{f}^+)),$$

where $\rho_i(\underline{f}^+)$ are tuples of general position in the algebraic sets

$$\rho_i \circ \pi(B^{\rm st} \cap \pi_\Pi^{-1}(h_1h_2)),$$

so that $\mu(\rho_i(\underline{f}^+)) = \mu(B_i)$. This proves the claim (i).

2.3. Restriction onto the hyperplane H_1. Let us show the claim (ii). By the generality of the linear form h_1, the linear forms $df_i(o)$, $i\notin I$, remain linearly independent after being restricted on the hyperplane $H_1=\{h_1=0\}$:

$$\mathrm{rk}(df_i(o)|_{H_1}\,|\,i\notin I)=N-b,$$

so that $\beta(B_1)=b-1$. Since the projection ρ_1 is a surjective linear map, the codimension of the set $\rho_i\circ\pi(B^{\mathrm{st}}\cap\pi_\Pi^{-1}(h_1h_2))$ in the space

$$\prod_{j\notin I}\mathcal{P}_{[1,d_j],N-1}\times\prod_{j\in I,j\neq e}\mathcal{P}_{[2,d_j],N-1}$$

does not exceed the codimension of the set $\pi(B^{\mathrm{st}}\cap\pi_\Pi^{-1}(h_1h_2))$ in the space $\widetilde{\mathcal{P}}_e(\underline{d})$. Now let us apply the procedure, inverse to the procedure of bringing into the standard form: the variety B_1 with every tuple

$$\underline{f}^+=(f_1,\dots,f_{e-1},f_{e+1},\dots,f_N)\in\rho_i\circ\pi(B^{\mathrm{st}}\cap\pi_\Pi^{-1}(h_1h_2))$$

contains also all tuples $(\widetilde{f}_1,\dots,\widetilde{f}_{e-1},\widetilde{f}_{e+1},\dots,\widetilde{f}_N)$, where $\widetilde{f}_j=f_j$ for $j\notin I$ and

$$\widetilde{f}_j=f_j+\sum_{i\notin I,i<j}\lambda_{j,i}f_i$$

for $j\in I$, $j\neq e$, for all possible tuples of coefficients $(\lambda_{*,*})$, and different tuples of coefficients determine different tuples of polynomials $(\underline{\widetilde{f}})$. Therefore, the codimension $\mathrm{codim}(B_1\subset\mathcal{P}(\underline{d}^+))$ is bounded from above by the number

$$a-bN+\sum_{j\in I}\varepsilon(j)-\gamma+(b-1)(N-1)-\sum_{j\in I,j\neq e}\varepsilon(j)=a-N-b-\gamma+\varepsilon(e)+1.$$

Taking into account that $\varepsilon(e)\leqslant\sharp\{j\notin I\}=N-b$, we obtain the estimate (6). This completes the proof of the claim (ii).

Remark 2.1. If $\gamma=0$, then $\pi_B^{-1}(\pi(\underline{f}))$ is the whole fibre of the projection π, that is, the linear space $\mathcal{P}_{[2,d_e],N}$. In that case $h_2\in\mathcal{P}_{1,N}$ is any form of general position (in fact, in this case we could take $h_2=h_1$ and make no assumption that $h_2\neq\lambda h_1$). Therefore, $\mu(B)\leqslant 2\mu(B_1)$, where $B_1\subset\mathcal{P}(\underline{d}^+)$ is a subvariety with $\beta(B_1)=b-1$, the codimension of which satisfies the estimate (6). The claim (iii) in this case is not needed (obviously, this is the case of stable reduction).

2.4. Restriction onto the hyperplane H_2. Let us show the claim (iii). For a general tuple of polynomials

$$(f_1,\dots,f_{e-1},f_{e+1},\dots,f_N)\in\pi(B^{\mathrm{st}}\cap\pi_\Pi^{-1}(h_1h_2))$$

there are two options:

- the subspace $\{df_i(o)=0\,|\,i\notin I\}$ is not contained in the hyperplane $H_2=\{h_2=0\}$,

- the subspace $\{df_i(o) = 0 \,|\, i \notin I\}$ is contained in H_2.

In the first case we have the stable reduction: $\beta(B_2) = b - 1$ and, arguing in word for word the same way as in subsection 2.3, we get that the codimension of the subvariety B_2 satisfies the inequality (6).

Therefore we assume that the second case takes place, so that

$$\mathrm{rk}(df_i(o)|_{H_2}, i \notin I) = N - b - 1$$

and $\beta(B_2) = b$. Between the linear forms $df_i(o)|_{H_2}, i \notin I$, there is exactly one linear dependence, so that there is a unique index $m \notin I$, satisfying the relation

$$df_m(o)|_{H_2} = \sum_{i<m, i \notin I} \lambda_i df_i(o)|_{H_2}$$

with uniquely determined coefficients λ_i. Therefore, on the Zariski open subset

$$(\rho_2 \circ \pi(B^{\mathrm{st}} \cap \pi_{\Pi}(h_1 h_2)))^{\circ}$$

(see the condition (3) at the beginning of Sec. 2.1) we have a well defined map σ_m of bringing into the standard form in the m-th factor:

$$\sigma_m \colon (g_i \,|\, i \neq e) \mapsto (\widetilde{g}_i \,|\, i \neq e).$$

The natural ambient space for the right-hand side is

$$\prod_{j \notin I, j \neq m} \mathcal{P}_{[1,d_j],N-1} \times \mathcal{P}_{[2,d_m],N-1} \times \prod_{j \in I, j \neq e} \mathcal{P}_{[2,d_j],N-1}$$

(up to a permutation of the direct factors), and the codimension of the closed subset

$$\overline{\sigma_m \circ (\rho_2 \circ \pi(B^{\mathrm{st}} \cap \pi_{\Pi}^{-1}(h_1 h_2)))^{\circ}}$$

with respect to that ambient space does not exceed

$$a - bN + \sum_{j \in I} \varepsilon(j) - \gamma - (N-1) + \varepsilon(m).$$

Now we argue in a word for word the same way as when restricting onto the hyperplane H_1: we apply the procedure, inverse to the procedure of bringing into the standard form in the factors with numbers

$$j \in (I \backslash \{e\}) \cup \{m\}.$$

The variety B_2 with every tuple

$$\underline{g}^{+} = (g_i \,|\, i \neq e) \in \overline{\sigma_m \circ (\rho_2 \circ \pi(B^{\mathrm{st}} \cap \pi_{\Pi}^{-1}(h_1 h_2)))^{\circ}}$$

contains all tuples $(\widetilde{g}_i \,|\, i \neq e)$, where $\widetilde{g}_j = g_j$ for $j \notin I$, $j \neq m$ and

$$\widetilde{g}_j = g_j + \sum_{i \notin I, i \neq m, i < j} \lambda_{j,i} g_i$$

for $j \in (I \backslash \{e\}) \cup \{m\}$. Therefore, the codimension $\mathrm{codim}(B_2 \subset \mathcal{P}(\underline{d}^+))$ is bounded from above by the number

$$a - bN + \sum_{j \in I} \varepsilon(j) - \gamma - (N-1) + \varepsilon(m) + b(N-1) -$$

$$- \sum_{j \in I, j \neq e, j < m} \varepsilon(j) - \sum_{j \in I, j \neq e, j > m} (\varepsilon(j) - 1) - \varepsilon(m) =$$

$$= a - b - \gamma - (N-1) + \varepsilon(e) + \sharp\{j \in I \backslash \{e\} \,|\, j > m\}.$$

Since obviously $\sharp\{j \in I \backslash \{e\} \,|\, j > m\} \leqslant b - 1$ and $\varepsilon(e) \leqslant N - b$, this implies the inequality (7). Proof of the claim (iii) is complete.

In the beginning of the proof of Theorem 2.1 we put off the case when $\pi_{\Pi}(B^{\mathrm{st}} \cap \mathcal{P}_{\Pi})$ is the line $\langle h_1^2(z_*) \rangle$, so that $\gamma = N - 1$. In that case $H_1 = H_2$, so that we have stable reduction (see Remark 2.1).

Proof of Theorem 2.1 is complete. Q.E.D.

2.5. The case of high codimension. Now let us assume that $\gamma \geqslant N$. In that case it is easy to state and prove an analog of Theorem 2.1; however, with γ growing the resulting estimates get less and less useful.

Set $k = \left[\frac{\gamma}{N}\right] + 1 \geqslant 2$.

Proposition 2.2. *Assume that $d_e \geqslant k$. Then there are irreducible bi-invariant subvarieties $B_i \subset \mathcal{P}(\underline{d}^+)$, $i = 1, \dots, k$, such that*

(i) *the inequality $\mu(B) \leqslant \mu(B_1) + \ldots + \mu(B_k)$ holds ,*

(ii) *for every $i \in \{1, \dots, k\}$ we have: either $\beta(B_i) = b - 1$, and then the codimension $a_i = \mathrm{codim}(B_i \subset \mathcal{P}(\underline{d}^+))$ satisfies the inequality (6), or $\beta(B_i) = b$ and the codimension a_i satisfies the inequality (7).*

Proof repeats the proof of Theorem 2.1 word for word, with only one difference: for Π we have to take the irreducible subvariety of decomposable forms of degree k,

$$\Pi = \left\{ \prod_{j=1}^{k} h_j(z_*) \,|\, h_j \in \mathcal{P}_{1,N} \right\} \subset \mathcal{P}_{k,N} \subset \mathcal{P}_{[2,d_e],N}$$

(the last inclusion is provided by our assumption that $d_e \geqslant k$). Obviously, $\dim \Pi > \gamma$, so that the arguments used in the proof of Theorem 2.1 work in this case. Setting

$$B_j = \overline{\langle \rho_j \circ \pi(B^{\mathrm{st}} \cap \pi_{\Pi}^{-1}(h_1 \ldots h_k)) \rangle} \subset \mathcal{P}(\underline{d}^+),$$

where ρ_j is the restriction onto the hyperplane $H_j = \{h_j = 0\}$, and all the symbols that we use have the same meaning as in subsections 2.2-2.4, we obtain the inequality

$$\mu(B) \leqslant \mu(B_1) + \ldots + \mu(B_k).$$

Repeating the arguments of subsections 2.3, 2.4, we obtain the claim (ii). This completes the proof of Proposition 2.2.

The proposition that we have just shown is far from being as useful as Theorem 2.1, because k can be high and, the main point, all subvarieties B_i can have $\beta(B_i) = b$, which essentially weakens the estimates obtained by iterating Proposition 2.2.

3 Explicit estimates for multiplicities

In this section we obtain estimates for $\mu(a)$ for $a \leqslant N$, which are close to optimal ones. First, we consider the case of a subvariety $B \subset \mathcal{P}(\underline{d})$ with $\beta(B) = 1$ as an example, when it is easy to obtain a precise estimate from above for $\mu(B)$. Then using Theorem 2.1, we construct a recurrent procedure of estimating the multiplicity, based on controlling two parameters, the codimension a and $b = \beta(B)$. (Recall that $a \geqslant b^2$.) At first this procedure is applied to obtain the estimates for small values of the codimension $a \leqslant 49$. After that, we consider the general case: in subsections 3.3-3.5 we prove estimates from above for $\mu(a)$, where the estimating function grows as $C^{\sqrt{a}}$, here $C > 0$ is some effectively estimated constant.

3.1. Estimating the multiplicity for $b = 1$**.** Let $\mathcal{P}(\underline{d})$ be an irreducible bi-invariant subvariety of codimension $a \leqslant N$.

Proposition 3.1. *Assume that* $\beta(B) = 1$*. Then the inequality* $\mu(B) \leqslant a + 1$ *holds.*

Proof. As we saw above (Proposition 1.4), for $a = 1$ we have $\mu(B) \leqslant 2$. Therefore we may assume that $N \geqslant a \geqslant 2$ and prove the proposition by induction on N. In the notations of §2, we have $\gamma \leqslant a - 1 \leqslant N - 1$, so that we can apply Theorem 2.1: there are irreducible bi-invariant subvarieties $B_i \subset \mathcal{P}(\underline{d}^+)$, such that $\mu(B) \leqslant \mu(B_1) + \mu(B_2)$. Here $\beta(B_1) = 0$, so that $\mu(B_1) = 1$. On the other hand, $a_2 = \mathrm{codim}(B_2 \subset \mathcal{P}(\underline{d}^+)) \leqslant a - 1$, so that by the inductive assumption

$$\mu(B_2) \leqslant a_2 + 1 \leqslant a.$$

Q.E.D. for the proposition.

Remark 3.1. The estimate in Proposition 3.1 is sharp: for any $a \leqslant N$ there is an irreducible bi-invariant subvariety $B \subset \mathcal{P}(\underline{d})$ of codimension a with $\beta(B) = 1$ and $\mu(B) = a + 1$. Indeed, let $B^\circ \subset \mathcal{P}(\underline{d})$ be defined by the conditions

- the equality $\mathrm{rk}(df_1(o), \ldots, df_{N-1}(o)) = N - 1$ holds, so that the set $C = \{f_1 = \ldots = f_{N-1} = 0\}$ in a neighborhood of the point o is a curve, non-singular at that point,
- the inequality $\mathrm{ord}_o\,(f_N|_C) \geqslant a + 1$ holds.

It is easy to see that $\mathrm{codim}(B^\circ \subset \mathcal{P}(\underline{d})) \leqslant a$, so that for the closure B of the bi-invariant span $\langle B^\circ \rangle$ the more so $\mathrm{codim}(B \subset \mathcal{P}(\underline{d})) \leqslant a$, and $\mu(B) \geqslant a + 1$.

Therefore, the last two inequalities we have the equality (the strict inequalities are impossible by Proposition 3.1).

Let us show now that if the degrees d_i are high enough, then for $\beta(B)=1$ the restriction $a \leqslant N$ for the codimension is not needed.

Proposition 3.2. *Assume that $d_1 \geqslant a+1$ and $\beta(B)=1$. Then $\mu(B) \leqslant a+1$.*

Proof. Let $\underline{f} \in B$ be a general tuple. For some $e \in \{1,\dots,N\}$ we have:

$$\mathrm{rk}(df_i(o) \,|\, i \neq e) = N-1,$$

so that $C=\{f_i=0 \,|\, i \neq e\}$ in a neighborhood of the point o is a curve, non-singular at that point. It is sufficient to show that the condition $\mathrm{ord}_o\,(f_e|_C) \geqslant a+1$ defines a closed subset in $\mathcal{P}_{[1,d_e],N}$ of codimension at least a (and for that reason precisely a). But this is obvious: let $l(z_*)$ be a general linear form, then the polynomials $l^i(z_*)$, $i=1,\dots,a+1$, satisfy the condition $\mathrm{ord}_o\,(l^i(z_*)|_C)=i$. Therefore, the linear subspaces

$$\Delta_i=\{f_e \in \mathcal{P}_{[1,d_e],N} \mid \mathrm{ord}_o\,(f_e|_C) \geqslant i\}$$

are distinct and $\Delta_1=\mathcal{P}_{[1,d_e],N} \supset \Delta_2 \supset \ldots \supset \Delta_{a+1}$ (recall that $d_e \geqslant d_1 \geqslant a+1$). Therefore indeed $\mathrm{codim}(\Delta_{a+1} \subset \Delta_1) \geqslant a$, as we need. Q.E.D. for the proposition.

Remark 3.2. It seems that the assumption $d_1 \geqslant a+1$ can be considerably relaxed.

3.2. Estimating multiplicities for small codimensions. Theorem 2.1 shows that in order to estimate the multiplicity $\mu(B)$ one needs to take into account the value of the parameter $\beta(B)=b$. Let $U \subset \mathbb{Z}_+ \times \mathbb{Z}_+$ be the set $\{(a,b) \,|\, a \geqslant b^2\}$. Let us define inductively the function

$$\overline{\mu}\colon U \to \mathbb{Z}_+,$$

setting $\overline{\mu}(a,0) \equiv 1$, $\overline{\mu}(a,1) \equiv a+1$, for $a < b(b+1)$

$$\overline{\mu}(a,b) = 2\overline{\mu}(a-(2b-1),b-1),$$

for $a \geqslant b(b+1)$

$$\overline{\mu}(a,b) = \overline{\mu}(a-(2b-1),b-1) + \max\{\overline{\mu}(a-(2b-1),b-1), \overline{\mu}(a-b,b)\}.$$

Theorem 2.1 immediately implies

Theorem 3.1. *Let $B \subset \mathcal{P}(\underline{d})$ be an irreducible bi-invariant subvariety of codimension $a \leqslant N$ and $\beta(B)=b$. Then the inequality*

$$\mu(B) \leqslant \bar{\mu}(a,b)$$

holds. In particular, $\mu(a) \leqslant \max\limits_{0 \leqslant b \leqslant \sqrt{a}} \bar{\mu}(a,b)$.

For small values of a the function $\overline{\mu}$ is easy to compute by hand; it is not hard to write a computer program, computing $\overline{\mu}$, either. Below we give the table of values

$\overline{\mu}(a,b)$ for $a \leqslant 49$, $b \leqslant 7$. The symbol $*$ means that the pair $(a,b) \notin U$ and the value of the function $\overline{\mu}$ is not defined. Already for these small values of the codimension the growth of the values $\overline{\mu}(a,b)$ can be clearly seen. In boldface we give the maximal value $\overline{\mu}(a,b)$ for a given a.

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$b=0$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$b=1$	**2**	**3**	**4**	**5**	**6**	7	8	9	10	11	12	13	14	15	16	17
$b=2$	*	*	*	4	6	**8**	**11**	**14**	**18**	**22**	**27**	**32**	**38**	**44**	**51**	**58**
$b=3$	*	*	*	*	*	*	*	*	8	12	16	22	28	36	44	55
$b=4$	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	16
$b=5$	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*

a	17	18	19	20	21	22	23	24	25	26	27	28
$b=0$	1	1	1	1	1	1	1	1	1	1	1	1
$b=1$	18	19	20	21	22	23	24	25	26	27	28	29
$b=2$	66	74	83	92	102	112	123	134	146	158	171	184
$b=3$	**68**	**82**	**99**	**119**	**140**	**165**	**193**	**223**	**257**	**295**	**335**	**380**
$b=4$	24	32	44	56	72	88	110	136	164	198	238	280
$b=5$	*	*	*	*	*	*	*	*	32	48	64	88
$b=6$	*	*	*	*	*	*	*	*	*	*	*	*

a	29	30	31	32	33	34	35	36	37	38	39
$b=0$	1	1	1	1	1	1	1	1	1	1	1
$b=1$	30	31	32	33	34	35	36	37	38	39	40
$b=2$	198	212	227	242	258	274	291	308	326	344	363
$b=3$	**429**	**481**	**538**	**600**	**665**	**736**	812	892	978	1070	1166
$b=4$	330	391	461	537	625	726	**841**	**966**	**1106**	**1264**	**1441**
$b=5$	112	144	176	220	272	328	396	476	560	660	782
$b=6$	*	*	*	*	*	*	*	64	96	128	176
$b=7$	*	*	*	*	*	*	*	*	*	*	*

a	40	41	42	43	44	45	46	47	48	49
$b=0$	1	1	1	1	1	1	1	1	1	1
$b=1$	41	42	43	44	45	46	47	48	49	50
$b=2$	382	402	422	443	464	486	508	531	554	578
$b=3$	1269	1378	1492	1613	1741	1874	2015	2163	2317	2479
$b=4$	**1631**	**1842**	**2076**	**2333**	**2609**	**2912**	**3242**	**3602**	**3987**	**4404**
$b=5$	922	1074	1250	1452	1682	1932	2212	2528	2893	3313
$b=6$	224	288	352	440	544	656	792	952	1120	1320
$b=7$	*	*	*	*	*	*	*	*	*	128
$b=8$	*	*	*	*	*	*	*	*	*	*

Now let us consider the problem of obtaining a simple effective upper bound for the multiplicities $\mu(B)$. From the technical viewpoint, one needs to find a simple and visual formalization of the procedure of estimating these numbers in terms of the numbers $\mu(B')$ for subvarieties $B' \subset \mathcal{P}(\underline{d}')$ in the spaces of truncated tuples $(d'_1, \dots, d'_{N'})$ with $N' < N$.

3.3. The general method of estimating the multiplicity. The symbol B stands for an irreducible bi-invariant subvariety of codimension a with $\beta(B) = b$. Let us consider the three-letter alphabet $\{A, C_0, C_1\}$. Let $\mathcal{W}$ be the set of all words in that alphabet, including the empty word $\emptyset$. The length of the word w is denoted by the symbol $|w| \in \mathbb{Z}_+$. The length of the empty word is equal to zero.

Let us describe a procedure of constructing a sequence of subsets $W_l \subset \mathcal{W}$, $l = 0, 1, \dots$. The length of every word

$$w \in \bigcup_{l \in \mathbb{Z}_+} W_l$$

does not exceed N. Set $N(w) = N - |w| \in \mathbb{Z}_+$. This sequences stabilizes, that is, $W_l = W_{l+1}$, starting from some $l = L$. For every word $w \in \cup W_l$ we assign a multi-index $\underline{d}(w) \in \mathbb{Z}_+^{N(w)}$ and an irreducible bi-invariant subvariety

$$B[w] \subset \mathcal{P}(\underline{d}(w))$$

of codimension $a(w)$ with $\beta(B[w]) = b(w)$.

We start the construction with $W_0 = \{\emptyset\}$. Set $B[\emptyset] = B \subset \mathcal{P}(\underline{d})$, where $\underline{d}(\emptyset) = \underline{d}$, so that $a(\emptyset) = a$ and $b(\emptyset) = b$. If $b(\emptyset) = 0$, then set $W_1 = W_2 = \dots = W_0$: the procedure terminates. Assume that the subsets $W_0, \dots, W_l$ are already constructed. If for every $w \in W_l$ the equality $b(w) = 0$ holds, then we set $W_{l+1} = W_{l+2} = \dots = W_l$, terminating the procedure. Otherwise, take any word $w \in W_l$ with $b(w) \geqslant 1$. Now apply Theorem 2.1 to the subvariety $B[w] \subset \mathcal{P}(\underline{d}(w))$ (constructed at a previous step). Consider the words $w_1 = wA$ and $w_2 = wC_i$, $i \in \{0, 1\}$, where $i = 1$ in the case of stable reduction and $i = 0$, otherwise. Furthermore,

$$B[w_j] = (B[w])_j \subset \mathcal{P}(\underline{d}^+(w)),$$

$j = 1, 2$, in the sense of notations of Theorem 2.1, so that $\underline{d}(w_j) = (\underline{d}(w))^+$ and

$$a(w_j) = (a(w))_j = \mathrm{codim}(B[w_j] \subset \mathcal{P}(\underline{d}(w_j))),$$

$b(w_1) = b(w) - 1$ and $b(w_2) = b(w) - i$. The inequality

$$a(w_1) \leqslant a(w) - (2b(w) - 1)$$

holds, in the case of stable reduction the same inequality ifs satisfied by the second codimension,

$$a(w_2) \leqslant a(w) - (2b(w) - 1),$$

whereas in the case of non-stable reduction the estimate

$$a(w_2) \leqslant a(w) - b(w)$$

holds. In any case, however, $a(w_j) < a(w)$.

The set of words W_{l+1} is obtained from W_l by removing the word w and adding the words w_1, w_2:

$$W_{l+1} = (W_l \setminus \{w\}) \cup \{w_1, w_2\}.$$

In particular, $\sharp W_{l+1} = \sharp W_l + 1$. Theorem 2.1 implies that

$$\sum_{w \in W_l} \mu(B[w]) \leqslant \sum_{w \in W_{l+1}} \mu(B[w]).$$

Therefore, for every l we have the estimate

$$\mu(B) \leqslant \sum_{w \in W_l} \mu(B[w]) \tag{8}$$

Proposition 3.3. *The procedure of constructing the sets $W_l \subset \mathcal{W}$ terminates: for some $l = L$ we have $b(w) = 0$ for all words $w \in W_L$.*

Proof. This is obvious, if, in order to construct W_{l+1} we take a word $w \in W_l$ with the maximal value of the codimension $a(w)$, since $a(w_j) < a(w)$ for both words w_j, $j = 1, 2$. However, this implies the finiteness of the procedure for any choice of the word $w \in W_l$: it is easy to see that the set

$$\bigcup_{l \in \mathbb{Z}_+} W_l \subset \mathcal{W}$$

does not depend on which word $w \in W_l$ with $b(w) \geqslant 1$ is chosen at every step, and for that reason this set is finite.

One can argue in a simpler way: as we mentioned above, the length of every word does not exceed N. Q.E.D. for the proposition.

Set $W = W_L$. For any $w \in W$ we have $b(w) = 0$, so that $\mu(B[w]) = 1$. Therefore,

$$\mu(B) \leqslant \sharp W. \tag{9}$$

So in order to estimate from above the multiplicity $\mu(B)$, we need to estimate the cardinality of the set W.

3.4. An estimate for the cardinality of the set of words. We will write the words in the following way:

$$w = \tau_1 \ldots \tau_K,$$

where $\tau_i \in \{A, C_0, C_1\}$. Now let

$$\nu\colon \{A, C_0, C_1\} \to \{A, C\}$$

be the map from the three-letter alphabet to the two-letter one, given by the equalities $\nu(A) = A$, $\nu(C_i) = C$, and

$$\nu\colon w = \tau_1 \dots \tau_K \mapsto \bar{w} = \nu(\tau_1) \dots \nu(\tau_K)$$

the corresponding map of the set of words. Now we have

Lemma 3.1. *For every $i = 0, 1, \dots$ the map $\nu|_{W_i}$ is injective. In particular, $\nu|_W$ is injective.*

Proof. A stronger claim is true: among all words $\bar{w} = \nu(w)$, $w \in W_i$, no one is a left segment of another one. (In particular, no two words are equal, which means the injectivity of the map $\nu|_{W_i}$.) The last claim is easy to show by induction. The set W_0 consists of one word, and for it the claim is trivial. Assume that we have shown it for W_i, where $i = 0, \dots, e$. If $W_{e+1} = W_e$, then there is nothing to prove. If $W_{e+1} \neq W_e$, then W_{e+1} is obtained from W_e by removing some word $w \in W_e$ and adding two words $w_1 = wA$ and $w_2 = wC_\alpha$, where $\alpha \in \{0, 1\}$. For these words we have $\bar{w}_1 = \bar{w}A$ and $\bar{w}_2 = \bar{w}C$. Obviously, $\bar{w}_1$ and $\bar{w}_2$ are not left segments of each other and no word $\bar{w}'$ for $w' \in W_e \backslash \{w\}$ is not a left segment of $\bar{w}_1$ or $\bar{w}_2$, because otherwise $\bar{w}' = \bar{w}_1$ or $\bar{w}_2$ (since $\bar{w}'$ is not a left segment of the word $\bar{w}$ by the inductive assumption), but then $\bar{w}$ would be a left segment of the word $\bar{w}'$, contrary to the inductive assumption. In a trivial way $\bar{w}_1$ and $\bar{w}_2$ are not left segments of the word $\bar{w}'$, since otherwise this would have been true for $\bar{w}$ as well, contrary to the inductive assumption. Q.E.D. for the lemma.

Let $w \in W$ be a word, w' its left segment (by the construction of the set W we have

$$w' \in \bigcup_{l \in \mathbb{Z}_+} W_l,$$

since w is obtained from the empty word $\emptyset$ by adding letters at the right-hand end when changing from W_k to W_{k+1} for certain values k), and moreover, $w' \neq w$ and $w'\tau$ is the left segment of the word w of length $|w'| + 1$.

Lemma 3.2. (i) *If $\tau = A$ or C_1, then the inequality*

$$a(w'\tau) \leqslant a(w') - (2b(w') - 1)$$

holds and $b(w'\tau) = b(w') - 1$.

(ii) *If $\tau = C_0$, then the inequality*

$$a(w'\tau) \leqslant a(w') - b(w')$$

holds and $b(w'\tau) = b(w')$.

Proof: it follows immediately from Theorem 2.1. Q.E.D. for the lemma.

Besides, the inequality (4) implies that for every word $w \in \bigcup_{l \in \mathbb{Z}_+} W_l$ we have the estimate

$$a(w) \geqslant b^2(w). \tag{10}$$

Example 3.1. Let us prove Proposition 3.1 in terms of the formalism developed above. Let $b = b(\emptyset) = 1$. Now for every word $w \in W_i$ we have the alternative: either $b(w) = 0$ (and then $w \in W$), or $b(w) = 1$ (and then $a(w\tau) \leqslant a(w) - 1$ for any letter τ), so that the set W is of the form

$$A,\ C_0A,\ C_0C_0A,\ \dots,\ \underbrace{C_0C_0\dots C_0}_{k}A,\ \underbrace{C_0\dots C_0}_{k}C_1,$$

where $k + 1 \leqslant a$. Therefore, $\sharp W \leqslant a + 1$, as we claimed above in subsection 3.1.

Let us come back to the general case. Recall that $a \leqslant N$.

Theorem 3.2. *The following inequality holds:*

$$\sharp W \leqslant 2^b \frac{(a - \frac{b(b-1)}{2})^b}{(b!)^2}. \tag{11}$$

Proof. For every word $w \in W$ by construction $b(w) = 0$. Since the letter C_0 does not change the value of the parameter b, and the letters A and C_1 bring it down by 1, we may conclude that in the word w there are precisely b positions, occupied by the letters A and C_1. Let them be the positions with numbers

$$m_1 + 1,\ m_1 + m_2 + 2,\ \dots,\ m_1 + m_2 + \dots + m_b + b,$$

$m_i \in \mathbb{Z}_+$. By Lemma 3.2, the inequality

$$\begin{array}{lllll} 0 \leqslant a(w) \leqslant a & - & m_1 b & - & (2b-1) - \\ & - & m_2(b-1) & - & (2(b-1)-1) - \\ & & \dots & & \\ & - & m_i(b-(i-1)) & - & (2(b-(i-1))-1) - \\ & & \dots & & \\ & - & m_b & - & 1 = \\ & & & = & a - b^2 - \sum_{i=1}^{b} m_i(b-(i-1)) \end{array}$$

holds, so that $(m_1, \dots, m_b)$ is an arbitrary integral point in the polytope

$$\Delta = \{x_1 \geqslant 0, \dots, x_b \geqslant 0, bx_1 + (b-1)x_2 + \dots + x_b \leqslant a - b^2\} \subset \mathbb{R}^b.$$

Therefore, even if we assume that all possible distributions of the letters A and C_1 on the chosen positions are realized by words $w \in W$ (in reality this is not the case: we have a lot less words in W, see Remark 3.3), then the inequality

$$\sharp W \leqslant 2^b \cdot \sharp(\Delta \cap \mathbb{Z}^b)$$

holds. Now let us estimate the number of integral points in Δ. For that purpose, consider a larger polytope

$$\Delta^+ = \{x_1 \geqslant 0, \dots, x_b \geqslant 0, bx_1 + \dots + x_b \leqslant a - \frac{b(b-1)}{2}\} \subset \mathbb{R}^b.$$

Obviously, $\Delta \subset \Delta^+$.

Lemma 3.3. *The following inequality holds:*

$$\sharp(\Delta \cap \mathbb{Z}^b) \leqslant \mathrm{vol}(\Delta^+).$$

Proof. To every point $x = (x_1, \ldots, x_b) \in \mathbb{R}^b$ we correspond the unit cube

$$\Gamma(x) = [x_1, x_1 + 1] \times [x_2, x_2 + 1] \times \ldots \times [x_b, x_b + 1] \subset \mathbb{R}^b,$$

the vertex with the minimum value of the sum of coordinates $x_1 + \ldots + x_b$ of which is the point x. If $x \in \Delta$, then $\Gamma(x) \subset \Delta^+$, since

$$b + (b-1) + \ldots + 1 + a - b^2 = a - \frac{b(b-1)}{2}.$$

Therefore

$$\sharp(\Delta \cap \mathbb{Z}^b) = \sum_{x \in \Delta \cap \mathbb{Z}^b} \mathrm{vol}(\Gamma(x)) = \mathrm{vol}\left(\bigcup_{x \in \Delta \cap \mathbb{Z}^b} \Gamma(x) \right) \leqslant \mathrm{vol}(\Delta^+),$$

as we claimed. Q.E.D. for the lemma.

Computing the volume of the polytope Δ^+, we complete the proof of Theorem 3.2.

4.5. Some calculus. The inequality (11) immediately implies the estimate

$$\mu(a) \leqslant \max_{1 \leqslant b \leqslant [\sqrt{a}]} v_b,$$

where

$$v_b = 2^b \frac{(a - \frac{b(b-1)}{2})^b}{(b!)^2}.$$

We have to estimate the maximum of the sequence v_b on the set $\{1, \ldots, [\sqrt{a}]\}$ by a function that depends on the argument a only. We do it in a few steps. Set

$$u_b = \frac{1}{2\pi b} \left(\frac{2a - b(b-1)}{b^2} e^2 \right)^b.$$

Lemma 3.4. *The inequality $v_b \leqslant u_b$ holds.*

Proof. Applying the Stirling formula, we write

$$v_b = \frac{1}{2\pi b e^{\theta/6b}} \left(\frac{2a - b(b-1)}{b^2} e^2 \right)^b,$$

where $0 < \theta < 1$. Q.E.D. for the lemma.

Lemma 3.5. *The sequence u_b is increasing if the following inequality holds:*

$$2a - b(b-1) \geqslant \frac{5}{2}b^2. \tag{12}$$

Proof. Write

$$\frac{u_{b+1}}{u_b} = \frac{1}{1+\frac{1}{b}} \frac{e^2}{\left(1+\frac{1}{b}\right)^{2b}} \frac{1}{\left(1+\frac{2b}{2a-b(b+1)}\right)^b} \frac{2a-b(b+1)}{(b+1)^2}. \tag{13}$$

Assume first that $b \geqslant 9$. If the numbers a and b satisfy the inequality $2a - b(b+1) \geqslant \frac{5}{2}(b+1)^2$ (that is, the inequality (12) for $b+1$), then the denominator of the third factor in the right hand side can estimated from above in the following way:

$$\left(1 + \frac{2b}{2a - b(b+1)}\right)^b \leqslant \left(1 + \frac{4}{5}\frac{1}{b}\right)^b < e^{\frac{4}{5}}.$$

The second factor in the right hand side of the inequality (13) is strictly higher than one, whereas the fourth is at least $\frac{5}{2}$. As a result, we get:

$$\frac{u_{b+1}}{u_b} > \frac{9}{10} \cdot \frac{5}{2} \cdot e^{-\frac{4}{5}} > 1,$$

which is what we need. For smaller values $b \leqslant 8$ the second and third factors in the right hand side of the inequality (13) can be estimated more precisely, and elementary calculations with a computer complete the proof of the lemma.

Corollary 3.1. *For $a \geqslant 17$ the value $b_{\max} \in \{1, \ldots, [\sqrt{a}]\}$, at which the maximum of the sequence u_b is attained, satisfies the inequality*

$$2a - b_{\max}(b_{\max} - 1) \leqslant \frac{5}{3}a.$$

Proof. By the previous lemma, the value $b_{\max}$ satisfies the inequality

$$2a - b_{\max}(b_{\max} + 1) \leqslant \frac{5}{2}b_{\max}^2$$

(otherwise, the next element of the sequence u_b would be higher). Now elementary computations complete the proof of the corollary.

Corollary 3.2. (i) *For $a \geqslant 17$ the following estimate holds:*

$$\sharp W \leqslant q_b = \frac{1}{2\pi b}\left(\frac{5a}{3b^2}e^2\right)^b.$$

(ii) *For any a the following estimate holds:*

$$\sharp W \leqslant w_b = \frac{1}{2\pi b}\left(\frac{2a}{b^2}e^2\right)^b .$$

Proof. Both claims follow immediately from the inequality (11), taking into account Lemma 3.4 and the previous corollary.

Theorem 3.3. (i) *For $a \geq 17$ the following estimate holds:*

$$\mu(a) \leqslant \frac{e^2}{2\pi[\sqrt{a}]}\left(\frac{5}{3}e^2\right)^{[\sqrt{a}]} .$$

(ii) *For any a the following estimate holds:*

$$\mu(a) \leqslant \frac{e^2}{2\pi[\sqrt{a}]}\left(2e^2\right)^{[\sqrt{a}]} .$$

Proof. The arguments are identical in both cases, the only difference is which of the two claims of Corollary 3.2 is used.

Let us show part (i). Arguing as in the proof of Lemma 3.5, we conclude that the sequence q_b is increasing. Therefore, its maximum is attained for $b = [\sqrt{a}]$. Since

$$a < (b+1)^2 = b^2 + 2b + 1,$$

the inequality

$$\left(\frac{a}{b^2}\right)^b \leqslant \left(1 + \frac{2}{b}\right)^b < e^2,$$

holds, which immediately implies the claim (i). The second part is shown in word for word the same way. Q.E.D.

Remark 3.3. As we can see from the given proof, the estimate we obtained is not optimal and can be essentially improved. For $b \approx \sqrt{a}$ we have $2a - b(b-1) \approx a$, so that in the inequality of Theorem 3.3 the expression $(2e^2)$ can be replaced by e^2. Furthermore, in the proof of Theorem 3.2 we took into account all possible tuples of positions $(m_1, \ldots, m_b)$ and all possible distributions of the letters A and C_1 into b positions. However, since in the set of words $\overline{W} = \nu(W)$ of the two-letter alphabet $\{A, C\}$ no word is a left segment of another word and the map $\nu\colon W \to \overline{W}$ is one-to-one, for a fixed distribution of the letters A and C_1 into b positions, such that at least two letters C_1 follow one another, not all tuples $(m_1, \ldots, m_b) \in \Delta \cap \mathbb{Z}^b$ are realized, since two distinct words $w_1 \neq w_2$, $\{w_1, w_2\} \subset W$ can not differ only on a segment consisting of the letters C_0, C_1. The question of finding a precise upper estimate for the numbers $\mu(a)$, even in the asymptotic sense, remains an open problem.

4 A generalization of the Gabrielov-Khovanskii problem

In this section we consider a generalization of the Gabrielov-Khovanskii problem: the polynomials $f_1, \dots, f_N$ are restricted onto an *arbitrary* (not a fixed) subvariety $R \ni o$ of codimension l, and for the multiplicity we take the multiplicity of zero at the point o of a tuple of generic $N - l$ polynomials in the *polynomial span* of the tuple $\underline{f}$. In subsection 4.1 we give a formal statement of the problem, in subsection 4.2 its alternative setting using the Chow varieties, parameterizing effective cycles of a given degree and codimension in $\mathbb{P}^N$. In subsection 4.3 we show that the estimates of the multiplicity of zero in the generalized problem can be obtained parallel to the estimates for the original problem (§§2-3).

4.1. Polynomial spans and multiplicities. We consider the affine space $\mathbb{A}^N = \mathbb{C}^N_{(z_1,\dots,z_N)}$ as open set embedded in the projective space $\mathbb{P}^N_{(x_0:x_1:\dots:x_N)}$ as the standard affine chart $\{x_0 \neq 0\}$, that is, $z_i = x_i/x_0$. For an ordered multi-index $\underline{d}$ consider the space of tuples of polynomials $\mathcal{P}(\underline{d})$. For an arbitrary tuple $\underline{f} \in \mathcal{P}(\underline{d})$ we define its *polynomial span*

$$[\underline{f}] = [f_1, \dots, f_N] \subset \mathcal{P}(\underline{d})$$

as the smallest bi-invariant set, containing the tuple $\underline{f}$. In particular, the polynomial span contains all tuples of the form $(f_1^+, \dots, f_N^+)$, where

$$f_i^+ = f_i + \sum_{j<i} s_{i,j}(z_*) f_j,$$

the polynomials $s_{i,j}(z_*)$ run through the entire space $\mathcal{P}_{[0,d_i-d_j],N}$ (independently of each other), so that $f_i^+ \in \mathcal{P}_{[1,d_i],N}$ and $\underline{f}^+ \in \mathcal{P}(\underline{d})$.

Now for an irreducible subvariety $R \subset \mathbb{P}^N$ of codimension $l \in \{0, 1, \dots, N\}$ set:

- if $o \notin R$, then $\mu(\underline{f}; R) = 0$,
- if the closed set
$$R \cap \{f_1 = \ldots = f_N = 0\}$$
has an irreducible component of a positive dimension, passing through the point o, then $\mu(\underline{f}; R) = \infty$,
- if none of the two cases described above takes place, then
$$\mu(\underline{f}; R) = e_{\mathcal{O}}(f_{l+1}^+, \dots, f_N^+)$$
where $\mathcal{O} = \mathcal{O}_{o,R}$ is the local ring at the point o and $e_{\mathcal{O}}$ is the Samuel multiplicity, see [1, Chapter 7]; $(f_1^+, \dots, f_N^+) \in [f_1, \dots, f_N]$ is a general tuple.

For an arbitrary effective cycle $R = \Sigma_{j\in J} r_j R_j$ of pure codimension l define $\mu(\underline{f};R)$ by linearity, setting

$$\mu(\underline{f};R) = \sum_{j\in J} r_j \mu(\underline{f};R_j),$$

where the sum in the right hand side is ∞, if at least one value $\mu(\underline{f};R_j)$ is ∞ (and $r_j \geqslant 1$). It is easy to see that if $\mu(\underline{f};R)$ is a finite non-zero number, then it is equal to the multiplicity of the point o in the 0-cycle

$$(\{f^+_{l+1} = 0\} \circ \ldots \circ \{f^+_N = 0\} \circ R),$$

where the scheme-theoretic intersection is taken in a neighborhood of the point o.

Furthermore, set for any $\delta \geqslant 1$

$$\mu(\underline{f},\delta) = \sup_{\deg R=\delta} \{\mu(\underline{f};R)\},$$

where the supremum is taken over all effective cycles R on $\mathbb{P}^N$ of pure codimension l and degree $\deg R = \delta$.

Now let us consider an irreducible bi-invariant subvariety $B \subset \mathcal{P}(\underline{d})$. For an effective cycle R of pure codimension $l \in \{0,\ldots,N\}$ set

$$\mu(B;R) = \mu(\underline{f};R),$$

where $\underline{f} \in B$ is a general tuple of polynomials. For any $\delta \geqslant 1$ set

$$\mu(B,\delta) = \mu(\underline{f},\delta),$$

where $\underline{f} \in B$ is a general tuple of polynomials; obviously,

$$\mu(B,\delta) = \inf\{\mu(\underline{f},\delta) \mid \underline{f} \in B\}.$$

Finally,

$$\mu_l(a,\delta) = \sup_{\alpha(B)\leqslant a} \{\mu(B,\delta)\},$$

where $\alpha(B) = \mathrm{codim}(B \subset \mathcal{P}(\underline{d}))$ and the supremum is taken over all irreducible bi-invariant subvarieties of codimension at most a in $\mathcal{P}(\underline{d})$.

We emphasize that $\mu(B,\delta)$ *is not*

$$\sup_{\deg R=\delta} \{\mu(B;R)\},$$

because the equality $\mu(B;R) = \mu(\underline{f};R)$ holds for any tuple $\underline{f} \in U_R$ from a non-empty Zariski open subset $U_R \subset B$, which *depends on* R.

The generalized Gabrielov-Khovanskii problem, considered in this section, is to compute (or estimate) the function $\mu_l(a,\delta)$. We will show that for $a \leqslant N$ the inductive procedure of estimating this function is totally similar to the absolute problem, considered in §§1-3. The resulting estimates are linear in the degree δ.

4.2. The Chow varieties. By the symbol $\overline{\mathcal{C}}_{l,N}(\delta)$ we denote the Chow variety, parameterizing effective cycles of pure codimension l and degree δ on $\mathbb{P}^N$, so that the definition of the number $\mu(\underline{f},\delta)$, given above, can be written in the following way:

$$\mu(\underline{f},\delta) = \sup_{\overline{\mathcal{C}}_{l,N}(\delta)\ni R} \{\mu(\underline{f};R)\}.$$

Now let us describe an alternative definition of the numbers $\mu(B,\delta)$. Consider the sets

$$\mathcal{X}_{l,N}(m,\delta) \subset \mathcal{P}(\underline{d}) \times \overline{\mathcal{C}}_{l,N}(\delta),$$

consisting of such tuples $(\underline{f},R)$, that

$$\mu(\underline{f};R) \geqslant m \in \mathbb{Z}_+ \cup \{\infty\}.$$

It is easy to see that $\mathcal{X}_{l,N}(m,\delta)$ are closed algebraic sets. Denote by the symbol $\pi_{\mathcal{P}}$ the projection

$$(\underline{f},R) \mapsto \underline{f}.$$

By projectivity of Chow varieties we get that

$$X_{l,N}(m,\delta) = \pi_{\mathcal{P}}(\mathcal{X}_{l,N}(m,\delta)) \subset \mathcal{P}(\underline{d})$$

is a closed algebraic set. Explicitly, it consists of such tuples $\underline{f}$, for which there exists and effective cycle $R \in \overline{\mathcal{C}}_{l,N}(\delta)$, satisfying the inequality $\mu(\underline{f};R) \geqslant m$.

Let $B \subset \mathcal{P}(\underline{d})$ be an irreducible subvariety. We define the multiplicity $\mu(B,\delta) = \mu_{l,N}(B,\delta)$ (in order to simplify the notations, we sometimes omit arguments or indices, the value of which is fixed at the moment), setting

$$\mu(B,\delta) = \max_{m\in\mathbb{Z}_+\cup\{\infty\}} \{m \,|\, B \subset X_{l,N}(m,\delta)\}.$$

Explicitly: $\mu(B,\delta) = m$, if for a general tuple $\underline{f} \in B$ and any effective cycle $R \in \overline{\mathcal{C}}_{l,N}(\delta)$ the inequality

$$\mu(\underline{f};R) \leqslant m$$

holds, and for at least one cycle $R \in \overline{\mathcal{C}}_{l,N}(\delta)$ this inequality turns into the equality. It is clear that if $\mu_{l,N}(B,\delta) = \infty$, then for a general (and therefore, every) tuple $(\underline{f})$ the set of its zeros $Z(f_1,\dots,f_N)$ has a component of positive dimension, passing through the point o. The converse is also true: if there is such a component, for R one can take such a subvariety that $\dim(R \cap Z(f_*)) \geqslant 1$. By the previous remark, in the notations of subsection 1.2 the equality $X_{l,N}(\infty,\delta) = X_\infty$ holds and for that reason Propositions 1.1 and 1.2 give an estimate of the codimension of that set.

By construction, the sets $X_{l,N}(m,\delta)$ are bi-invariant, so that in order to define and estimate the numbers $\mu_l(a,\delta)$ it is sufficient to consider bi-invariant subsets

$B \subset \mathcal{P}(\underline{d})$, as we did it in §§1-2. (If we replace an arbitrary subvariety $B \subset \mathcal{P}(\underline{d})$ by its bi-invariant span $[B] \subset \mathcal{P}(\underline{d})$, then the codimension does not decrease and all the numbers $\mu(B;R)$, $\mu(B,\delta)$ do not change.) The further study of the numbers $\mu(B;R)$, $\mu(B,\delta)$ $\mu_l(a,\delta)$ goes parallel to the constructions of §§1-3, and we will only outline its main steps, paying attention to the additional arguments and modifications.

An analog of Proposition 1.3 is the following

Proposition 4.1. *If $\beta(B)=0$, then $\mu(B;R)=\mathrm{mult}_o R$ for every $R \in \overline{\mathcal{C}}_{l,N}(\delta)$.*

Proof. We may assume that the subvariety B is bi-invariant. Let us fix an effective cycle R. If $o \notin \mathrm{Supp}\, R$, then in a trivial way $\mu(B;R)=\mathrm{mult}_o R=0$. So we assume that $R \subset \mathbb{P}^N$ is an irreducible subvariety containing the point o.

For a general tuple $\underline{f} \in B$ by assumption we have

$$\{df_1(o)=\ldots=df_N(o)=0\}=\{0\},$$

so that

$$\mathrm{codim}\left(\{df_1(o)=\ldots=df_{l+1}(o)=0\} \subset \mathbb{C}^N\right)=l+1.$$

Since the variety B is bi-invariant, with every tuple $\underline{g} \in B$ it contains also the tuple $\underline{g}^+=(g_1^+,\ldots,g_N^+)$, where

$$g_i^+=g_i+\sum_{j=1}^{i-1}\lambda_{i,j}g_j$$

for any $\lambda_{i,j} \in \mathbb{C}$. It follows that for a general tuple $\underline{f} \in B$ the linear form $df_{l+1}(o)$ does not vanish identically on every component of the tangent cone T_oR (they all have codimension l in $\mathbb{C}^N$). Therefore,

$$\mathrm{mult}_o(R \circ \{f_{l+1}=0\})=\mathrm{mult}_o R.$$

This equality holds for every algebraic cycle R of codimension l on $\mathbb{P}^N$ and a general tuple $\underline{f}$. The cycle $(R \circ \{f_{l+1}=0\})$ has codimension $l+1$. Continuing in this way for the codimension $l+1,\ldots,N$, we complete the proof of the proposition.

Corollary 4.1. *If $\beta(B)=0$, then $\mu(B,\delta)=\delta$ for every $\delta \geqslant 1$.*

Proof. Indeed, for every effective cycle R of pure codimension l the inequality $\mathrm{mult}_o R \leqslant \deg R$ holds, and moreover, for the cones we have the equality. Q.E.D. for the corollary.

Assume now that $a=\mathrm{codim}(B \subset \mathcal{P}(\underline{d})) \leqslant N$ and consider the procedure of reducing to the smaller dimensions, constructed in §2, and the resulting explicit estimates for the numbers $\mu_l(a,\delta)$, similar to those obtained for the numbers $\mu(a)$ in §3.

4.3. Reduction to the smaller dimensions and explicit estimates in the generalized problem. The procedure of bringing into the standard form and subsequent splitting off a direct factor yields the following generalization of Theorem 2.1.

Theorem 4.1. *Assume that* $a = \mathrm{codim}(B \subset \mathcal{P}(\underline{d})) \leqslant N$ *and* $b = \beta(B) \geqslant 1$. *Then there are irreducible bi-invariant subvarieties* $B_i \subset \mathcal{P}(\underline{d}^+)$, $i = 1, 2$, *and integers* $\delta_1, \delta_{21}, \delta_{22} \in \mathbb{Z}_+$, *such that* $\delta = \delta_1 + \delta_{21} + \delta_{22}$ *and the inequality*

$$\begin{array}{rcl} \mu_{l,N}(B,\delta) & \leqslant & \mu_{l,N-1}(B_1,\delta_1) + \mu_{l,N-1}(B_2,\delta_1) + \\ & & + \mu_{l-1,N-1}(B_1,\delta_{21}) + \mu_{l-1,N-1}(B_2,\delta_{22}), \end{array} \tag{14}$$

holds, whereas the claims (ii), (iii) of Theorem 2.1 remain true.

Proof is almost word for word the same as the proof of Theorem 2.1. We dwell only on the necessary changes. We use the notations of subsections 2.2-2.4.

Let $\mu_{l,N}(B,\delta) = m$. Consider a general polynomial $h_1h_2 \in \pi_\Pi(B^{\mathrm{st}} \cap \mathcal{P}_\Pi)$ and a general tuple

$$\underline{f} = (f_1, \ldots, f_{e-1}, h_1h_2, f_{e+1}, \ldots, f_N) \in B^{\mathrm{st}} \cap \pi_\Pi^{-1}(h_1h_2).$$

By the definition of multiplicity, there is an effective cycle $R \in \overline{\mathcal{C}}_{l,N}(\delta)$, such that $\mu(\underline{f}; R) \geqslant m$. We define effective cycles R_1, R_{21} and R_{22} by the following conditions:

- $R = R_1 + R_{21} + R_{22}$,
- the component Q of the cycle R is contained in R_1 if and only if $Q \not\subset H_1$ and $Q \not\subset H_2$,
- the component Q of the cycle R is contained in R_{21} if and only if $Q \subset H_1$,
- the component Q of the cycle R is contained in R_{22} if and only if $Q \not\subset H_1$, but $Q \subset H_2$,

Set $\delta_1 = \deg R_1$, $\delta_{2i} = \deg R_{2i}$, $i = 1, 2$. Obviously, $\delta = \delta_1 + \delta_{21} + \delta_{22}$. Consider an arbitrary irreducible component Q of the cycle R. It is an irreducible component of precisely one of the cycles R_1, R_{21}, R_{22}.

If $Q \subset \mathrm{Supp}\, R_1$, then

$$\mu(\underline{f}; Q) \leqslant \mu(\rho_1(\underline{f}^+); (Q \circ H_1)) + \mu(\rho_2(\underline{f}^+); (Q \circ H_2)).$$

Here $(Q \circ H_i)$ are effective cycles of pure codimension l on $H_i \cong \mathbb{P}^{N-1}$.

If $Q \subset \mathrm{Supp}\, R_{21}$, then

$$\mu(\underline{f}; Q) \leqslant \mu(\rho_1(\underline{f}^+); Q),$$

where Q is considered as a subvariety of codimension $(l-1)$ on $H_1 \cong \mathbb{P}^{N-1}$; in a similar way, if $Q \subset \mathrm{Supp}\, R_{22}$, then

$$\mu(\underline{f}; Q) \leqslant \mu(\rho_2(\underline{f}^+); Q),$$

where $Q \subset H_2 \cong \mathbb{P}^{N-1}$ is a subvariety of codimension $(l-1)$. This proves the inequality (14). The remaining part of the proof of Theorem 2.1 works as it is.

Q.E.D. for Theorem 4.1.

Remark 4.1. If we fix an effective cycle $R \in \overline{\mathcal{C}}_{l,N}(\delta)$, then the following claim is an analog of Theorem 2.1: if $a \leqslant N$ and $\beta(B) \geqslant 1$, then there are irreducible bi-invariant subvarieties $B_i \subset \mathcal{P}(\underline{d}^+)$, $i = 1, 2$, and effective cycles $R_1 \in \overline{\mathcal{C}}_{l,N-1}(\delta_1)$ and $R_2 \in \overline{\mathcal{C}}_{l-1,N-1}(\delta_2)$ of degrees $\delta_1, \delta_2 \in \mathbb{Z}_+$, where $\delta = \delta_1 + \delta_2$, such that the inequality

$$\mu(B;R) \leqslant \mu(B_1;R_1) + \mu(B_2;R_1) + \mu(B_2;R_2) \tag{15}$$

holds, and moreover, the claims (ii), (iii) of Theorem 2.1 hold. The proof is word for word the same as the proof of the previous claim, with the only difference: for a *fixed* cycle R the general hyperplane H_1 does not contain any of its irreducible components, so that in the notations of the proof of Theorem 4.1 one can set $R_{21} = 0$.

Now similar to Proposition 3.1 we get

Corollary 4.2. *For $a \leqslant N$, $\beta(B) = 1$ the inequality $\mu_{l,N}(B,\delta) \leqslant (a+1)\delta$ holds.*

Proof is given by induction on N. By the inequality (14) and Corollary 4.1 we have the estimate

$$\mu_{l,N}(B,\delta) \leqslant \delta_1 + \delta_{21} + \mu_{l,N-1}(B_2,\delta_1) + \mu_{l-1,N-1}(B_2,\delta_{22}).$$

If $\beta(B_2) = 0$, then $\mu_{l,N}(B,\delta) \leqslant 2\delta_1 + \delta_{21} + \delta_{22} \leqslant 2\delta$, which what we need. If $\beta(B_2) = 1$, then by the inductive assumption

$$\mu_{l,N}(B,\delta) \leqslant \delta_1 + \delta_{21} + a\delta_1 + a\delta_{22} \leqslant (a+1)\delta.$$

Q.E.D. for the corollary.

In subsection 3.2 we introduced the function $\bar{\mu}(a,b)$.

Corollary 4.3. *The inequality*

$$\mu_{l,N}(B,\delta) \leqslant \bar{\mu}(a,b)\delta,$$

holds, where $a = \mathrm{codim}(B \subset \mathcal{P}(\underline{d})) \leqslant N \quad b = \beta(B)$.

Proof. This follows immediately from Theorem 4.1. Q.E.D. for the corollary.

As an analog on the inequality (9), we have the estimate

$$\mu_{l,N}(B,\delta) \leqslant (\sharp W)\delta, \tag{16}$$

which is obtained by repeating the arguments of subsection 3.3 word for word, taking into account the equality $\delta = \delta_1 + \delta_{21} + \delta_{22}$ at every step. The corresponding formal procedure is constructing irreducible bi-invariant subvarieties $B[w]$, parameterized by the words of three-letter alphabet $\{A, C_0, C_1\}$, and non-negative integers $\delta_j(w)$, $j = 0, \ldots, \min\{l, |w|\}$, satisfying the equality

$$\delta = \sum_{j=0}^{\min\{l,|w|\}} \delta_j(w).$$

As an analog of the estimate (8), we have the estimate

$$\mu_{l,N}(B,\delta) \leqslant \sum_{w\in W_l} \sum_{j=0}^{\min\{l,|w|\}} \mu_{l-j,N-|w|}(B[w],\delta_j(w)),$$

shown by induction on $l = 0,1,\ldots$. If for a word w we have $\beta(B[w]) = 0$, then Corollary 4.1 allows us to replace the summand $\mu_{l-j,N-|w|}(B[w],\delta_j(w))$ by the number $\delta_j(w)$. The details are left to the reader. Proof of the inequality (16) is complete.

Theorem 3.2 now implies

Theorem 4.2. *Assume that* $a = \mathrm{codim}(B \subset \mathcal{P}(\underline{d})) \leqslant N$ *and* $b = \beta(B) \geqslant 1$. *Then for every* $\delta \in \mathbb{Z}_+$ *the following inequality holds:*

$$\mu_{l,N}(B,\delta) \leqslant 2^b \frac{(a - \frac{b(b-1)}{2})^b}{(b!)^2}\delta.$$

More convenient estimates follow from Theorem 3.3.

Theorem 4.3. *For* $a \geq 17$ *the estimate*

$$\mu_{l,N}(B,\delta) \leqslant \frac{e^2}{2\pi[\sqrt{a}]}\left(\frac{5}{3}e^2\right)^{[\sqrt{a}]}\delta$$

holds, and for every $a \geqslant 1$ *the following estimate holds:*

$$\mu_{l,N}(B,\delta) \leqslant \frac{e^2}{2\pi[\sqrt{a}]}\left(2e^2\right)^{[\sqrt{a}]}\delta.$$

Corresponding estimates are true for the suprema $\mu_l(a,\delta)$ as well.

References

[1] Fulton W., Intersection Theory, Springer-Verlag, 1984.

[2] Gabrielov A. and Khovanskii A. G., Multiplicity of a Noetherian intersection. In: Geometry of Differential Equations. AMS, 1998, p.119-131.

[3] Hartshorne R. Algebraic Geometry. Graduate Texts in Mathematics, **52**. Springer-Verlag, New York-Heidelberg, 1977.

[4] Manin Yu. I., Cubic forms. Algebra, geometry, arithmetic. Russian original — Moscow, Nauka, 1972. English translation, second edition. North-Holland Mathematical Library, **4.** North-Holland Publishing Co., Amsterdam, 1986.

[5] Pukhlikov A. V., Birationally rigid varieties with a pencil of Fano double covers. II. Sbornik: Mathematics. **195** (2004), No. 11, 1665-1702.

[6] Pukhlikov A. V., Singularities of algebraic subvarieties and problems of birational geometry. Proceedings of the Steklov Institute of Mathematics, **267** (2009), No. 1, 235-247.

[7] Pukhlikov A. V., On the multiplicity of solutions of a system of algebraic equations. Proceedings of the Steklov Institute of Mathematics. **276** (2012), 234-249.

[8] Pukhlikov A. V., Birationally rigid complete intersections of quadrics and cubics. Izvestiya: Mathematics **77** (2013), No. 4, 795-845.

[9] Pukhlikov Aleksandr, Birationally Rigid Varieties. Mathematical Surveys and Monographs **190**, AMS, 2013.

[10] Pukhlikov A. V., Birational Geometry of Higher-Dimensional Fano Varieties. Proc. Steklov Inst. Math. **288**, suppl. 2 (2015), 1-150.

[11] Pukhlikov A. V., Birationally rigid Fano complete intersections, Crelle J. für die reine und angewandte Mathematik. **541** (2001), 55-79.

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