Linear stochastic degenerate Sobolev equations: well posedness and exact controllability.*

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Abstract

In this paper, we revisit our recently proposed results for a general class of linear stochastic degenerate Sobolev systems with additive noise by using a different approach keeping, however, the main assumptions unchanged for the purpose of comparison. In particular, the mild and strong well posedness for the initial and final value problems are presented and studied by applying a suitable transformation which formulates the degenerate stochastic system as a pseudoparabolic one. Based on the derived results for the forward and backward cases, under this new framework, the conditions for the exact controllability are revisited for a particular class of degenerate Sobolev systems.

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1 Introduction

The Maxwell equations in complex media [12], the one dimensional Dirac equation in the non-relativistic limit [13, 16] and the Leontief input-output model in economics [8, 10] are only a few of the models that can be formulated using equations or systems involving mixed spatial and time partial derivatives. Under parameter or/and environment uncertainties and when singularities appear, such models can often be reduced to linear stochastic degenerate Sobolev equations with additive noise in suitable infinite dimensional spaces, see [9]. H being the Hilbert space, their general form is given by

$$d(Lx(t)) = [Mx(t) + f(t)] \;\; dt + B \; dW(t), \quad t > 0, \tag{1}$$

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where $L : D(L) \to H$ and $M : D(M) \to H$ are linear and possibly unbounded (closed and densely defined) operators and L is *non–invertible*. The Wiener process W is considered to model the uncertainties in an additive way. Equations of type (1) are the infinite dimensional analogue of stochastic *singular* or *descriptor* systems in $\mathbb{R}^n$; see for instance [1], [5], [7]. For a treatment of the initial value problem of Eq. (1), we refer to [11] (where inclusions of multi–valued linear operators are involved) and [17] (concerning controllability with impulsive components) and references therein.

In [9], recently, the initial and final value problems and the exact controllability for Eq. (1) have been studied within a stochastic framework. The analysis on the operators L, M and the operator pencil $\lambda L - M$ are based on [15] where two cases (and respectively, two methods) for the solvability of the deterministic analogue of Eq. (1) are presented. Obviously, the distinction and usefulness of the two approaches studied by Thaller and Thaller (1996) [15] depend on the properties of L, M and $\lambda L - M$. In particular, whenever L has a closed domain, the deterministic analogue of Eq. (1) can be reduced, with some extra assumptions on M and $\lambda L - M$, to a pseudoparabolic problem in the Hilbert space $(\overline{Ran(L^\star)}, \|.\|_H)$, where $\overline{Ran(L^\star)} = (KerL)^\perp$. This is equivalent with the assumption that the operator $L^\perp = L|_{(KerL)^\perp \cap D(L)}$ is bounded and defined on the whole $(KerL)^\perp$. A multiplication with the boundedly invertible operator $(L^\perp)^{-1}$ provides the pseudoparabolicity of the upcoming equation on the space $(KerL)^\perp$. Alternatively, if L has a closed range, that is $\overline{Ran(L)} = Ran(L)$, the deterministic problem can be reduced to a pseudoparabolic one in the Hilbert space $(Ran(L), \|.\|_H)$. This is equivalent with the assumption that $(L^\perp)^{-1}$ is bounded and defined on the whole $Ran(L)$. In this case a transformation $\psi(t) = L^\perp y(t)$ provides the pseudoparabolicity of the upcoming equation on the space $Ran(L)$. The equivalence of the two cases is provided whenever $L^\perp : (KerL)^\perp \to Ran(L)$ is bounded and boundedly invertible. In [9], the pseudoparabolicity of the stochastic equation is provided by a multiplication with the bounded operator $(L^\perp)^{-1}$, in view of the first methodology. In the present paper, the assumptions on the deterministic coefficients L, M and $\lambda L - M$ are made in such way that the two methodologies be equivalent. In particular, L is assumed to be bounded with a closed range and $L^\perp$ to be boundedly invertible.

The aim of this paper is to study the forward, backward and exact controllability problems under the same framework with [9], but applying the second methodology of [15], i.e. using the transformation $\psi(t) = L^\perp y(t)$ (which is briefly presented in Section 2). It is shown that the equivalence of the two methods is valid not only for the deterministic case, but also for the stochastic problems. Indeed, possible relaxations on the assumptions of the coefficients can affect the stochastic problems as well. Thus, the study of the second methodology is also necessary, however further details on this directions are omitted and they will be studied in a sequel paper.

We proceed as follows. In Section 2, some preliminary results are provided. Section 3 is devoted to the main results, where the study of the forward, backward and exact controllability problem is provided. Some concluding remarks are presented in Section 4.

2 Preliminaries

Let us consider for the deterministic and stochastic coefficients of problem (1) the same functional spaces, notation and main assumptions (A_1-A_6) as in Section 2 of [9]. In particular, the deterministic analogue is derived as follows:

$$\left.\begin{aligned} \frac{d}{dt}(L^\perp y(t)) &= M_0 y(t) + f(t), \quad t > 0, \\ y(0) &= P^\perp x_0, \end{aligned}\right\} \tag{2}$$

where $M_0 = MX$. It is assumed that $D_{M_0} = P^\perp D$ (see $(A_2)(ii)$) is dense in $P^\perp H$, M_0 and $(\lambda L - M)X = \lambda L^\perp - M_0$ are closed operators (see [14] and [15]) in $P^\perp D$. Here, we intent to treat problem (2) by applying the transformation

$$\psi(t) = L^{\perp}y(t) \Leftrightarrow y(t) = (L^{\perp})^{-1}\psi(t) \quad \text{for every} \quad y \in P^{\perp}H, \quad t \in [0,T].$$

Now, system (2) is re-written as follows in $Q^{\perp}H$:

$$\left.\begin{aligned} \frac{d}{dt}\psi(t) &= M_0(L^{\perp})^{-1}\psi(t) + f(t), \quad t > 0, \\ \psi(0) &= L^{\perp}P^{\perp}x_0 = Lx_0, \end{aligned}\right\} \tag{3}$$

with $A_2 = M_0(L^{\perp})^{-1}$ and $D_{A_2} = \{\psi \in Q^{\perp}H : (L^{\perp})^{-1}\psi \in D_{M_0}\} = \{\psi \in Q^{\perp}H : (L^{\perp})^{-1}\psi \in P^{\perp}D\} = L(D)$, which is dense on $Q^{\perp}H$. Furthermore, A_2 is closed on D_{A_2} (see [15] for more details) and since

$$(\lambda I - A_2)^{-1} = L(\lambda L - M)^{-1}Q^{\perp} \quad \text{on} \quad Q^{\perp}H,$$

by assumption $(A3)(ii)$ we obtain that $A_2 : D_{A_2} \to Q^{\perp}H$ fulfils also the Hille–Yoshida conditions. From [15], the following result is derived.

Theorem 1 *Let the assumptions of Theorem 1 in [9] hold. Then the deterministic analogue of the linear forward problem has a unique strict solution $x(t)$, $t \in [0,T]$, with $x(t) = Xy(t)$, where $y(t) = (L^{\perp})^{-1}\psi(t)$ and $\psi(t)$ is the unique strict solution of problem* (3)*, i.e.*

$$\psi(t) = S_2(t)Lx_0 + \int_0^t S_2(t-s)f(s)\ ds, \quad t \in [0,T]$$

and

$$x(t) = X(L^{\perp})^{-1}S_2(t)Lx_0 + X\int_0^t (L^{\perp})^{-1}S_2(t-s)f(s)\ ds, \quad t \in [0,T].$$

3 Main results

The main results for the two corresponding problems of the linear degenerate Sobolev systems are presented. Additionally, the study of the exact controllability is also examined for a special class of equations.

3.1 The linear forward problem

Let us formulate here the forward problem for the stochastic equation (1) in a Hilbert space H:

$$\left.\begin{aligned} d(Lx(t)) &= [Mx(t) + f(t)]\ dt + B\ dW(t), \quad t \in [0,T], \\ x(0) &= \xi. \end{aligned}\right\} \tag{4}$$

The mild solution for Eq. (4) is given by the following definition.

Definition 1 *Mild solution for the problem* (4) *is called an $H-$valued, $\mathscr{F}_t-$progressively measurable, stochastic process $x(t), t \in [0,T]$ which has the following expression:*

$$x(t) = X(L^{\perp})^{-1}S_2(t)L\xi + \ X\int_0^t (L^{\perp})^{-1}S_2(t-s)f(s)\ ds + X\int_0^t (L^{\perp})^{-1}S_2(t-s)B\ dW(s),\ t \in [0,T],$$

where $S_2(t),\ t \geq 0$ is the C_0-semigroup generated by the operator $A_2 = M_0(L^{\perp})^{-1}$.

For the definition of the strong solution, we refer to [9].

Theorem 2 *Assume that $(A_1) - (A_6)$ of [9] hold. Then, problem* (4) *is mildly well–posed, with $x(t) \in D$, $\mathcal{P} - a.s.$, a.e. in $[0,T]$, and $Lx \in L^1(0,T;H)$, $\mathcal{P}$-a.s.*

Proof. First, let us apply the transformation $\psi(t) = L^{\perp}y(t) \Leftrightarrow y(t) = (L^{\perp})^{-1}\psi(t)$, then Eq. (4) takes the form:

$$\left.\begin{aligned} d\psi(t) &= [M_0(L^{\perp})^{-1}\psi(t) + f(t)] \;\; dt + B\; dW(t),\; t \in [0,T], \\ \psi(0) &= L^{\perp}P^{\perp}\xi = L\xi. \end{aligned}\right\} \tag{5}$$

From [4] and [6], the system (5) has a unique mild solution, which also belongs to $L(D)$, $\mathcal{P} - a.s.,\;\; a.e.$ in $[0,T]$, if the following two results can be proved

1. $\displaystyle\int_0^T \|S_2(t)B\|^2_{L_2^0}\; dt < \infty,$
2. $\psi(t) \in D(M_0(L^{\perp})^{-1}) = L(D),\;\; \mathcal{P}$-a.s., a.e. in $[0,T]$.

These results can be provided by similar arguments as in the proof of Theorem 2 in [8]. For problem (4), we can prove that the process $x(t) = Xy(t) = X(L^{\perp})^{-1}\psi(t),\;\; t \in [0,T]$, which is defined uniquely, satisfies Definition 1 and belongs in $D,\; \mathcal{P} - a.s.,\; a.e.\; in\; [0,T]$. Additionally, X is densely defined on $P^{\perp}H$ and the process $y(t) = (L^{\perp})^{-1}\psi(t),\;\; t \in [0,T]$ is $\mathscr{F}_t-$progressively measurable, so we derive that $x(t),\;\; t \in [0,T]$ is also $\mathscr{F}_t-$progressively measurable. Furthermore, by considering the boundedness of $L^{\perp}$ on $P^{\perp}H$, we can prove that $Lx \in L^1(0,T;H),\;\; \mathcal{P}$-a.s.. Under assumption $A_3(ii)$, $\overline{X}$ is considered with its domain $D_{\overline{X}}$ being a dense subspace of $\overline{P^{\perp}D} = P^{\perp}H$ (see [14]) and when equipped with the graph norm $\|z\|_G = \|\overline{X}z\|_H$ is a Hilbert space isometrically isomorphic to $\overline{D}$ under the norm of H. By considering the Remark 2.24 in [15], we conclude that the family of operators $\overline{X}(L^{\perp})^{-1}S_2(t)L^{\perp}P^{\perp} = \overline{X}(L^{\perp})^{-1}S_2(t)L,\;\; t \in [0,T]$, is a C_0-semigroup on the space $\overline{D}$. Consequently, for every $\xi \in D$, $\mathcal{P}$-a.s., we have that

$$\|X(L^{\perp})^{-1}S_2(t)L\xi\|_H = \|\overline{X}(L^{\perp})^{-1}S_2(t)L\xi\|_H \leq \|\overline{X}(L^{\perp})^{-1}S_2(t)L\|_{\mathcal{L}(\overline{D})}\|\xi\|_H,$$

and the proof of Theorem 2 is completed. □

In what follows, we investigate the determination of some extra conditions for the unique mild solution of problem (4) to become a strong solution as well.

Theorem 3 *Let the assumptions of Theorem 2 hold. Assume also that for the solution* $\psi(t),\;\; t \in [0,T]$, *of problem* (5) *it holds that* $\displaystyle\int_0^T \|M_0(L^{\perp})^{-1}\psi(t)\|_H\; dt < \infty,\;\; \mathcal{P}$*-a.s. Then the problem* (4) *is strongly well–posed and the unique solution satisfies that* $Lx \in L^2_{\mathscr{F}_t}(0,T;\Omega;H)$ *and the following expression is true:*

$$x(t) = \overline{X}(L^{\perp})^{-1}S_2(t)L\xi \;+\; \int_0^t \overline{X}(L^{\perp})^{-1}S_2(t-s)f(s)\; ds \;+ \int_0^t \overline{X}(L^{\perp})^{-1}S_2(t-s)B\; dW(s), \quad t \in [0,T].$$

Proof. In order to be able to show that the solution $x(t),\;\; t \in [0,T]$ is a strong one for the forward problem (4), it remains to prove that $Mx \in L^1(0,T;H),\;\; \mathcal{P}$-a.s. Indeed, since

$$Mx(t) = MXy(t) = M_0(L^{\perp})^{-1}\psi(t), \quad \text{for every } t \in [0,T],$$

and $\displaystyle\int_0^T \|M_0(L^{\perp})^{-1}\psi(t)\|_H\; dt < \infty,\;\; \mathcal{P}$-a.s, we obtain that $Mx \in L^1(0,T;H),\;\; \mathcal{P}$-a.s. Furthermore, the boundedness of M^{-1} on $Q^{\perp}H$ provides also that $x \in L^1(0,T;H),\;\; \mathcal{P}$-a.s. By replacing X with its closed extension $\overline{X}$ in the solution, the integral representation of the theorem is obtained. The continuous dependence has already been proved in Theorem 2. Consequently, by applying the assumptions on ξ, f, B, it is derived that

$$E\left[\int_0^T \|Lx(t)\|^2_H\right] = E\left[\int_0^T \|L^{\perp}y(t)\|^2_H\right] = E\left[\int_0^T \|\psi(t)\|^2_H\right] < \infty. \qquad \square$$

We note that the integral form of the solution $x(t), t \in [0,T]$ in Theorem 3 can be defined for every $\xi \in \overline{D}$, $\mathcal{P}$-a.s. and $f(t) \in L(\overline{D})$, $\mathcal{P}$-a.s., a.e. in $[0,T]$, as long as $Ran(B) \subseteq L(\overline{D})$, see also Remark 3 in [9].

3.2 The linear backward problem

In this subsection, the following linear degenerate backward problem is investigated in the Hilbert space H:

$$\left.\begin{aligned} -d(Lx(t)) &= [Mx(t)+Nz(t)+f(t)]\ dt - z(t)\ dW(t), \quad t\in[0,T], \\ x(T) &= \xi_T. \end{aligned}\right\} \tag{6}$$

Thus, we seek for the existence of a unique solution pair $(x,z)\in L^2_{\mathscr{F}_t}(0,T;\Omega;H)\times L^2_{\mathscr{F}_t}(0,T;\Omega;L_2^0)$ for the backward problem (6), where L_2^0 denotes the space of all Hilbert-Schmidt operators from U_0 to $Q^\perp H$ (i.e. $L_2(U_0,Q^\perp H)$). Additionally to $(A_1)-(A_4)$, the assumptions $(B_1)-(B_2)$ of [9] also hold. For the notion of a strong solution of problem (6), we refer to [9].

Using similar steps with the provided analysis of the previous section, under the assumptions $(A_1)-(A_4)$ and $(B_1)-(B_2)$, problem (6) takes the following form in $Q^\perp H$:

$$\left.\begin{aligned} -d(L^\perp y(t)) &= [M_0y(t)+Nz(t)+f(t)]\ dt - z(t)\ dW(t), \quad t\in[0,T], \\ y(T) &= P^\perp\xi_T \end{aligned}\right\} \tag{7}$$

Performing the transformation $\psi(t)=L^\perp y(t)\Leftrightarrow y(t)=(L^\perp)^{-1}\psi(t),\ t\in[0,T]$, in problem (7), this takes the form

$$\left.\begin{aligned} -d\psi(t) &= [M_0(L^\perp)^{-1}\psi(t)+Nz(t)+f(t)]dt - z(t)dW(t), \quad t\in[0,T], \\ \psi(T) &= L^\perp P^\perp\xi_T = L\xi_T. \end{aligned}\right\} \tag{8}$$

Definition 2 *Mild solution for the problem (8) is called a pair $(\psi,z)\in L^2_{\mathscr{F}_t}(0,T;\Omega;H)\times L^2_{\mathscr{F}_t}(0,T;\Omega;L_2^0)$ if for every $t\in[0,T]$, $\mathcal{P}$-a.s. the following expression holds:*

$$\psi(t)=S_2(T-t)L\xi_T + \int_t^T S_2(s-t)Nz(s)\ ds + \int_t^T S_2(s-t)f(s)\ ds - \int_t^T S_2(s-t)z(s)\ dW(s), \quad t\in[0,T],$$

where $S_2(t),\ t\geq 0$, is the C_0-semigroup of operators in the closed subspace $Q^\perp H$ generated by the operator $A_2=M_0(L^\perp)^{-1}$.

Definition 3 *Strong solution for the problem (8) is called a pair $(\psi,z)\in L^2_{\mathscr{F}_t}(0,T;\Omega;H)\times L^2_{\mathscr{F}_t}(0,T;\Omega;L_2^0)$ if the following conditions are satisfied:*

1. *$\psi(t)\in L(D)$, $\mathcal{P}$-a.s., a.e. in $[0,T]$ and $\psi\in L^1(0,T;H)$.*
2. *$M_0(L^\perp)^{-1}\psi\in L^1(0,T;H)$, $\mathcal{P}$-a.s.*
3. *$\psi(t)=L\xi_T+\int_t^T[M_0(L^\perp)^{-1}\psi(s)+Nz(s)+f(s)]\ ds-\int_t^T z(s)\ dW(s)$, $\mathcal{P}$-a.s., a.e in $[0,T]$.*

Modifying accordingly the assumptions of Theorem 4 in [9], the next theorem is derived:

Theorem 4 *Under assumptions $(A_1)-(A_4)$ and $(B_1)-(B_2)$, a unique mild solution $(\psi,z)\in L^2_{\mathscr{F}_t}(0,T;\Omega;H)\times L^2_{\mathscr{F}_t}(0,T;\Omega;L_2^0)$ for the problem (8) is derived. Moreover, let us assume that:*

(B_3) *$S_2(s-t)z(s)h\in L(D)$, (equivalently $z(s)h\in L(D)$), $\mathcal{P}$-a.s., for every $s\in[t,T]$ and $h\in U$.*

(B_4) $E\left[\int_0^T\int_t^T\|M_0(L^\perp)^{-1}S_2(s-t)(Nz(s)+f(s))\|_H\ ds\ dt\right]<\infty.$

(B_5) $E\left[\int_0^T\int_t^T\|M_0(L^\perp)^{-1}S_2(s-t)z(s)\|^2_{L_2^0}\ ds\ dt\right]<\infty.$

Then, the continuous version of the mild solution (ψ,z) is the unique strong solution for the problem (8).

Proof. A direct application of Theorem 4.2 in [3] renders the proof of our new theorem, so unnecessary details are omitted. Based on the assumptions on ξ_T, f and (B_1) on N, it is provided that $S_2(T-t)L\xi_T \in L(D)$, $S_2(s-t)f(s) \in L(D)$ and $S_2(s-t)Nz(s) \in L(D)$, $\mathcal{P}$-a.s., for every $s \in [t,T]$. $\square$

Returning now again back to Eq. (7), we find that the solution (y,z) has the form

$$y(t) = (L^{\perp})^{-1}S_2(T-t)L\xi_T + \int_t^T (L^{\perp})^{-1}S_2(s-t)Nz(s)\, ds + \int_t^T (L^{\perp})^{-1}S_2(s-t)f(s)\, ds - \int_t^T (L^{\perp})^{-1}S_2(s-t)z(s)\, dW(s), \quad t \in [0,T],$$

where following similar arguments as for the proof of Theorem 5 in [9], we can prove that $x(t), t \in [0,T]$, satisfies the properties 1-3 of Definition 2 of [9] and hence the following theorem is obtained:

Theorem 5 *Under assumptions $(A_1)-(A_4)$, $(B_1)-(B_5)$, the pair (x,z) is the unique strong solution for problem (6), where:*

$$x(t) = \overline{X}(L^{\perp})^{-1}S_2(T-t)L\xi_T + \int_t^T \overline{X}(L^{\perp})^{-1}S_2(s-t)Nz(s)\, ds + \int_t^T \overline{X}(L^{\perp})^{-1}S_2(s-t)f(s)\, ds - \int_t^T \overline{X}(L^{\perp})^{-1}S_2(s-t)z(s)\, dW(s), \quad t \in [0,T].$$

The solution process $x(t)$, $t \in [0,T]$, is continuously dependent on the $\mathcal{P}$-a.s., $D-$valued final data $\xi_T \in L^2_{\mathscr{F}_T}(\Omega; H)$.

Remark 1 *By the uniqueness of the solutions of the forward and the backward problem provided by the two methodologies and the equivalence between them (since $L^{\perp}$ is bounded and boundedly invertible), the following operator identities hold:*

$$S_2(t) = LS_1(t)(L^{\perp})^{-1} \text{ on } Q^{\perp}H \quad \text{and} \quad S_1(t) = (L^{\perp})^{-1}S_2(t)L \text{ on } P^{\perp}H, \quad t \in [0,T].$$

If L is not a bounded operator, these identities become (see also Remark 2.24 in [15]):

$$S_2(t) = L^{\perp}S_1(t)(L^{\perp})^{-1} \text{ on } Q^{\perp}H \quad \text{and} \quad S_1(t) = (L^{\perp})^{-1}S_2(t)L^{\perp} \text{ on } P^{\perp}H, \quad t \in [0,T].$$

A possible relaxation on the boundedness and the bounded invertibility of $L^{\perp}$ (for example $L^{\perp}$ is not bounded or $(L^{\perp})^{-1}$ is not bounded) lead to the loss of the equivalence between the two methodologies. As an example, in the case that $(L^{\perp})^{-1}$ is bounded but $L^{\perp}$ is not, we have to proceed with the solvability of the problem by applying the transformation techniques, as long as the first methodology (multiplication by left with the unboundedly invertible operator $(L^{\perp})^{-1}$) fails. Similarly, in the case that $L^{\perp}$ is bounded but $(L^{\perp})^{-1}$ is not, we have to proceed with the solvability of the problem by using the first methodology, as long as the transformation may not be well defined.

3.3 Exact controllability for a special class of degenerate Sobolev systems

Finally, in this subsection, a special class of degenerate Sobolev systems is presented which allows us the study of the forward and the backward problem simultaneously. In the previous subsection, we treated the backward problem (6), which can be written also in the form

$$\left.\begin{aligned} d(Lx(t)) &= -[Mx(t) + Nz(t) + f(t)]\, dt + z(t)\, dW(t), \quad t \in [0,T], \\ x(T) &= \xi_T, \end{aligned}\right\}$$

and we proved that the solution is provided in terms of the C_0-semigroup $S_2(t), t \geq 0$, generated by the operator $A_2 = M_0(L^{\perp})^{-1}$. An interesting case of $S_2(t)$ being a C_0-group of operators is considered. In this case, solvability results can be obtained for the following linear backward problem in H:

$$\left.\begin{aligned} d(Lx(t)) &= [Mx(t) + Nz(t) + f(t)] \ dt + z(t) \ dW(t), \quad t \in [0,T], \\ x(T) &= \xi_T, \end{aligned}\right\} \tag{9}$$

Applying the results of Section 3.2, the following theorem is formulated:

Theorem 6 *Under assumptions $(A_1) - (A_4)$, $(B_1) - (B_5)$, with the extra hypothesis that $S_2(t), t \in \mathbb{R}$, is a C_0-group of operators, the pair (x,z) is the unique strong solution for problem (9), where*

$$x(t) = \overline{X}(L^{\perp})^{-1}S_2(t-T)L\xi_T - \int_t^T \overline{X}(L^{\perp})^{-1}S_2(t-s)Nz(s) \ ds - \int_t^T \overline{X}(L^{\perp})^{-1}S_2(t-s)f(s) \ ds$$
$$- \int_t^T \overline{X}(L^{\perp})^{-1}S_2(t-s)z(s) \ dW(s), \quad t \in [0,T].$$

The solution process $x(t)$, $t \in [0,T]$, is continuously dependent on the $\mathcal{P}$-a.s., $D-$valued final data $\xi_T \in L^2_{\mathscr{F}_T}(\Omega; H)$.

Now, let us define the following controlled stochastic system on a separable Hilbert space H:

$$\left.\begin{aligned} d(Lx(t)) &= [Mx(t) + Cu(t) + f(t)] \ dt + Bu(t) \ dW(t), \quad t \in [0,T], \\ x(0) &= \xi, \qquad x(T) = \xi_T, \end{aligned}\right\} \tag{10}$$

where the control parameter which takes values in a separable Hilbert space K is represented by u. Moreover, all progressively measurable, square–integrable processes, i.e. $\mathcal{U} = L^2_{\mathscr{F}_t}(0,T;\Omega;K)$ consists the set of all admissible controls. Additionally, let the assumptions of section 3.1 (see Theorem 3) hold on the coefficients $L,\ M,\ f,\ B$ for the existence of a strong solution and (F_1) assumption of [9] also holds, where the stochastic process $Cu(t), t \in [0,T]$ satisfies now that:

$$E\left[\int_0^T \int_0^t \|M_0(L^{\perp})^{-1}S_2(s-t)(Cu(s) + f(s))\|_H \ ds \ dt\right] < \infty.$$

We refer to [9] for the definition of the exact controllability of the problem (10). Moreover, in order to prove that system (10) is exactly controllable, we have to demonstrate that at least one admissible control, $u \in \mathcal{U}$, exists with $Cu(t) \in L(D)$, $\mathcal{P}$-a.s., $t \in [0,T]$, such that

$$\xi_T = \overline{X}(L^{\perp})^{-1}S_2(T)L\xi + \int_0^T \overline{X}(L^{\perp})^{-1}S_2(T-s)Cu(s) \ ds + \int_0^T \overline{X}(L^{\perp})^{-1}S_2(T-s)f(s) \ ds +$$
$$\int_0^T \overline{X}(L^{\perp})^{-1}S_2(T-s)B \ dW(s), \ \mathcal{P}\text{-a.s.},$$

and removing the operator $\overline{X}$ outside the integrals and replacing it by X, that

$$S_2(-T)L^{\perp}P^{\perp}\xi_T - L\xi - \int_0^T S_2(-s)B \ dW(s) = \int_0^T S_2(-s)[Cu(s) + f(s)]ds, \ \mathcal{P}-\text{a.s.} \tag{11}$$

For the case that $u \in \mathcal{U}$ is given by

$$Cu(t) = S_2(t)v(t) - f(t), \quad t \in [0,T], \tag{12}$$

and for a $\mathscr{F}_t-$progressively measurable stochastic process $v(t) \in L(D)$, $\mathcal{P}$-a.s., a.e. in $[0,T]$, by substituting Eq. (12) in (11), we show that v should satisfy

$$\int_0^T v(s)ds = S_2(-T)L\xi_T - L\xi - \int_0^T S_2(-s)B \ dW(s), \ \mathcal{P}-\text{a.s.} \tag{13}$$

Considering now the backward problem in the space $P^{\perp}H$

$$\left.\begin{aligned} d(\psi(t)) &= [M_0(L^{\perp})^{-1}\psi(t)] \; dt + z(t) \; dW(t), \quad t \in [0,T], \\ \psi(T) &= L\xi_T, \end{aligned}\right\} \tag{14}$$

we see that for $N = 0$ and $f = 0$, this is an analogue of problem (8) of section 3.2, assuming also that $S_2(t), \ t \in \mathbb{R}$, is a C_0-group. For problem (14) there is a unique pair $(\psi_1, z_1) \in L^2_{\mathscr{F}_t}(0,T;\Omega;P^{\perp}H) \times L^2_{\mathscr{F}_t}(0,T;\Omega;L_2^0)$ such that

$$\psi_1(t) = S_2(t-T)L\xi_T - \int_t^T S_2(t-s)z_1(s) \; dW(s),$$

is a strong solution of (14), with $\psi_1(t) \in L(D)$, $\mathcal{P}$-a.s., a.e. in $[0,T]$.
Obviously, the random variable $\psi_1(0) = S_2(-T)L\xi_T - \int_0^T S_2(-s)z_1(s) \; dW(s)$, is $\mathscr{F}_0-$ measurable. Thus, Eq. (13) becomes

$$\int_0^T v(s)ds = \psi_1(0) - L\xi - \int_0^T S_2(-s)(B - z_1(s)) \; dW(s),$$

and following similar steps with the subsection 3.3 in [9], a possible choice for $Cu(t), \ t \in [0,T]$ is given by

$$Cu(t) = S_2(t)\frac{1}{T}(\psi_1(0) - L\xi) - \int_0^t \frac{1}{T-s} S_2(t-s)(B - z_1(s)) \; dW(s) - f(t), \tag{15}$$

for $t \in [0,T]$. Consequently, Theorem 7 of [9] is fully recovered, having now as right hand side this of Eq. (15). Similarly, the condition $Ran(C) = L(D)$ provides the exact controllability of (10).

4 Conclusion

Systems with mixed spatial and time partial derivatives often appear in science and engineering. When singularities and uncertainties are also involved, such systems can be reduced to linear stochastic degenerate Sobolev equations with additive noise in an infinite dimensional Hilbert space.
In this paper, the study of the mild and strong well posedness for the initial and final value problems is provided as a continuation of the research work proposed by Liaskos et al. (2015) [9], which now uses an alternative methodology based on a deterministic framework by Thaller and Thaller (1996) [15]. It is interesting that similarly with the deterministic methodologies, using appropriate assumptions on the operators and the spectral properties of the operator pencil, the equivalence for the stochastic methods is derived. Furthermore, it should be mentioned that in the present paper, an infinite dimensional generalization of the notion of a regular matrix pencil is also considered. Finally, using the results of the forward and backward problems, the conditions for the exact controllability of a particular class of these equations are provided.
Last but not least, it should be emphasised that the proposed stochastic methodology can be very distinct to the one presented in [9] if some of the assumptions are relaxed. This direction will be investigated in a forthcoming paper.

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