Block-decoupling vibration control using eigenstructure assignment

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Abstract

A theoretical study is presented on the feasibility of applying active control for the purpose of vibration isolation in lightweight structures by block diagonalisation of the system matrices and at the same time assigning eigenvalues (natural frequencies and damping) to the chosen substructures separately. The methodology, based on eigenstructure assignment using the method of receptances, is found to work successfully when the eigenvalues of the open-loop system are controllable and the open- and closed-loop eigenvalues are distinct. In the first part of the paper results are obtained under the restriction that the mass matrix is diagonal (lumped). This is certainly applicable in the case of numerous engineering systems consisting of discrete masses with flexible interconnections of negligible mass. Later in the paper this restriction is lifted to allow bandedness of the mass matrix. Several numerical examples are used to illustrate the working of the proposed algorithm.

Keywords: Structural vibration isolation; block decoupling; eigenstructure assignment; method of receptances.

1. Introduction

The classical vibration isolation method is well understood and its application is ubiquitous. It is well suited to industrial problems where a relatively massive piece of engineering hardware, such as an engine-block or a heavy machine tool is to be isolated from its surroundings. Spacecraft structures, such as deployable antennae or solar arrays, and light-weight multi degree of freedom structures generally, are much more difficult in terms of isolating one substructure from the vibration of another. In this paper we consider from a purely theoretical point of view, the feasibility of decoupling multi degree of freedom systems to form substructures that are completely isolated from one another by active vibration control. This problem appears to be one that has received very little attention in the vibrations control literature to date.

A related, but different problem is that of input-output decoupling in linear time-invariant (LTI) multivariable control. The purpose here is simplification to form a number of single variable systems by the elimination of cross-couplings between the variables of the system. There is an extensive literature on this topic spanning several decades using state feedback (Morgan [1], Falb and Wolovich [2], Gilbert [3] and Descusse et al. [4]) and output feedback (Paraskevopoulos and Koumboulis [5], Howze [6], Denham [7] and Descusse [8]). The combined problem of simultaneous decoupling and pole placement in LTI multivariable systems was addressed by several authors [2, 9-11] and the block decoupling problem was investigated [12-18]. A transfer function matrix approach with block-decoupling was proposed by Hautus and Heymann [19] and Commault and Dion [19, 20] and unity-output feedback systems with decoupling and stability was investigated using a transfer function matrix approach [21-24]. Q.-G. Wang provides a detailed account of input-output decoupling control in the research monograph [25].

Although there are considerable volumes of literature devoted to the development of theoretical input-output decoupling methods, far less attention has been paid to the application of decoupling to structural vibration control. Zacharenakis [26, 27] investigated the decoupling problems of civil engineering structures via state/output feedback with the assumption that the number of inputs is equal to the number of outputs. Li et al [28] proposed decoupling control law for vibration control of multi-story building using a diagonal mass matrix and tri-diagonal damping and stiffness matrices. The control laws were based on the second-order matrix differential equations directly.

With the state space formulation, pole placement is an intrinsically ill-conditioned problem and becomes increasingly ill conditioned with the dimension of the system [29] . Another obvious drawback of using a first-order realisation is that the system matrices become $2n \times 2n$, which is computationally expensive if the order of the system n is large [30]. Furthermore, in terms of structural vibration control, converting the equations of motion into a first-order state-space formulation, the bandedness, definiteness and symmetry, of the mass damping and stiffness matrices are lost [31]. The transfer function matrix approach, which requires much algebraic manipulation of rational functions, becomes increasingly complicated as the dimension of the system increases, especially for vibration control of industrial-scale structures.

The research reported in this article is a preliminary study, which might be deemed timely in view of contemporary interest in lightweight and deployable structures, piezo-based actuators and sensors with proven capability and a related literature on active input-output decoupling. In this research, a new block decoupling control algorithm based on eigenstructure assignment using measured receptances is proposed for structural vibration control. Modal degree of freedom constraints are imposed such that the matrix of closed-loop right eigenvectors is block-diagonalised, leading to block diagonal matrices of the second-order system in physical coordinates.

For the purpose of simplicity, we limit the investigation in this article to the problem of block decoupling to form two independent substructures from a linear multi degree of freedom system. It is straightforward to show that the approach can be extended to the case of multiple independent substructures and also diagonal decoupling in physical coordinates. Since the main objective is the introduction of a new conceptual idea eigenvalue assignment is limited to the case of distinct eigenvalues in both open and closed loops. The block-diagonal receptance matrix is introduced in Section 2 and eigenstructure assignment by the method of receptances is briefy reviewed in Section 3. In Sections 4 and 5, block decoupling vibration control algorithm for undamped and damped systems with lumped masses is explained. The number of actuators and sensors required in the case of banded damping and stiffness matrices is considered in Section 6, and in Section 7 the methodology is extended to cope with damped systems with inertia coupling using a hybrid block-decoupling vibration control law by the application of acceleration, velocity and displacement feedback control. Several numerical examples are used to show how the block decoupling control method works.

2. The closed-loop block-diagonal receptance matrix

The equation of motion of the n degree of freedom linear system may be cast in second-order form as,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ $\in \mathbb{R}^{n\times n}$ are symmetric matrices, $\mathbf{M}$ is positive definite and $\mathbf{C}$ and $\mathbf{K}$ are positive semi-definite.

Now, velocity and displacement feedback is applied to decouple and control the system so that the closed-loop system may be written as,

$$\mathbf{M}\ddot{\mathbf{x}} + \left(\mathbf{C} - \mathbf{B}\mathbf{F}^T\right)\dot{\mathbf{x}} + \left(\mathbf{K} - \mathbf{B}\mathbf{G}^T\right)\mathbf{x} = \mathbf{0} \tag{2}$$

where $\mathbf{B} \in \mathbb{R}^{n\times q}$ is the force distribution matrix, $\mathbf{F}$ and $\mathbf{G} \in \mathbb{R}^{n\times q}$ are velocity and displacement feedback control gain matrices respectively.

The dynamic stiffness matrix of the closed-loop system is denoted by,

$$\hat{\mathbf{\Gamma}}(s) = \mathbf{M}s^2 + \left(\mathbf{C} - \mathbf{B}\mathbf{F}^T\right)s + \left(\mathbf{K} - \mathbf{B}\mathbf{G}^T\right) \tag{3}$$

Correspondingly, the closed-loop receptance matrix is the inverse of the dynamic stiffness,

$$\hat{\mathbf{H}}(s) = \hat{\mathbf{\Gamma}}^{-1}(s) \tag{4}$$

where $(\hat{\bullet})$ denotes the closed-loop system.

A list of integers $(n_1, n_2, \cdots, n_v)$ is called a partition of n if $n_i \geq 1, i=1, 2, \cdots, v$ and $\sum_{i=1}^{v} n_i = n$.

The closed-loop dynamic system is said to be block diagonal with respect to the partition $(n_1, n_2, \cdots, n_v)$ if the receptance matrix takes the form,

$$\hat{\mathbf{H}}(s) = \begin{bmatrix} \hat{\mathbf{H}}_{11}(s) & & & \mathbf{0} \\ & \hat{\mathbf{H}}_{22}(s) & & \\ & & \ddots & \\ \mathbf{0} & & & \hat{\mathbf{H}}_{vv}(s) \end{bmatrix} \tag{5}$$

where $\hat{\mathbf{H}}_{ii}(s) \in \mathbb{Q}_p^{n_i \times n_i}(s), i = 1, 2, \cdots, v$, $\mathbb{Q}_p^{n_i \times n_i}(s)$ is the ring of proper rational functions. In the special case when $v = n$ and $n_i = 1$, $i=1, 2, \cdots, v$, the closed-loop system is said to be diagonally decoupled. For a linear system with closed-loop receptance (5), the dynamic behaviour may be expressed as,

$$\hat{\mathbf{H}}(s)\mathbf{f}(s) = \mathbf{x}(s) \tag{6}$$

where $\mathbf{f}(s)$ and $\mathbf{x}(s)$ are the external forces and displacement responses respectively, indicating that a substructure with receptance matrix $\hat{\mathbf{H}}_{ii}(s)$ is independent of other substructures under the external force $\mathbf{f}(s)$.

In the paper, we show how a multi degree of freedom linear structure can be decoupled into two independent substructures ($v = 2$ and $n_1 + n_2 = n$) by a new block decoupling algorithm. The algorithm can be extended straightforwardly to the case of multiple independent substructures and also diagonal decoupling in terms of physical coordinates.

3. Pole placement by the method of receptances

Multi-input active vibration control is proposed in this article by pole placement using the method of receptances proposed by Ram and Mottershead [32]. They showed that when the open-loop system is controllable, there exists a solution to the system of equations,

$$\begin{bmatrix} \mu_1 \mathbf{w}_1^T & \mathbf{w}_1^T \\ \mu_2 \mathbf{w}_2^T & \mathbf{w}_2^T \\ \vdots & \vdots \\ \mu_{2n} \mathbf{w}_{2n}^T & \mathbf{w}_{2n}^T \end{bmatrix} \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \\ \vdots \\ \boldsymbol{\alpha}_{2n}^T \end{pmatrix} \tag{7}$$

thereby assigning closed-loop eigenvalues $\{\mu_k\}_{k=1}^{2n}$, closed under conjugation, by the application of displacement and velocity feedback control gains $\mathbf{F}, \mathbf{G}$.

Terms appearing in (7) are given by Ram and Mottershead [32] as,

$$\mathbf{w}_k = \alpha_{\mu_k,1} \mathbf{r}_{\mu_k,1} + \alpha_{\mu_k,2} \mathbf{r}_{\mu_k,2} + \cdots + \alpha_{\mu_k,q} \mathbf{r}_{\mu_k,q} = \mathbf{R}_{\mu_k} \boldsymbol{\alpha}_k, \ \ k = 1, 2, \cdots, 2n \tag{8}$$

$$\mathbf{R}_{\mu_k} = \begin{bmatrix} \mathbf{r}_{\mu_k,1} & \mathbf{r}_{\mu_k,2} & \cdots & \mathbf{r}_{\mu_k,q} \end{bmatrix} = \mathbf{H}_{\mu_k} \mathbf{B}; \quad \mathbf{H}_{\mu_k} = \begin{bmatrix} \mathbf{M} \mu_k^{\ 2} + \mathbf{C} \mu_k + \mathbf{K} \end{bmatrix}^{-1} \tag{9}$$

$$\boldsymbol{\alpha}_k = \begin{bmatrix} \alpha_{\mu_k,1} & \alpha_{\mu_k,2} & \cdots & \alpha_{\mu_k,q} \end{bmatrix}^T \tag{10}$$

and $\alpha_{\mu_k,j}$ are arbitrary parameters. $\mathbf{H}_{\mu_k}$ are open-loop receptance matrices which may be measured experimentally and $\mathbf{w}_k$ are the closed-loop right eigenvectors. Constraints may be applied at the j^{th} degree of freedom of the k^{th} mode by the choice of $\alpha_{\mu_k,j}$,

$$\mathbf{e}_j^T \mathbf{w}_k = \mathbf{e}_j^T \mathbf{R}_{\mu_k} \boldsymbol{\alpha}_k = 0 \tag{11}$$

where $\mathbf{e}_j$ denotes the j-th unit vector.

It is assumed in the following sections that equation (7) is solvable and the closed-loop eigenvalues are closed under conjugation - to ensure strictly real $\mathbf{F}, \mathbf{G}$.

4. Block-decoupling control for undamped structures

To illustrate the idea of block-decoupling control, we begin with the problem of undamped systems. The equations of motion of the open-loop and closed-loop n degree-of-freedom undamped systems ($\mathbf{C}=\mathbf{0}$, $\mathbf{F}=\mathbf{0}$) may be written as ,

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{K}\mathbf{x}=\mathbf{0} \tag{12}$$

and

$$\mathbf{M}\ddot{\mathbf{x}}+\left(\mathbf{K}-\mathbf{B}\mathbf{G}^{T}\right)\mathbf{x}=\mathbf{0} \tag{13}$$

The closed-loop eigenvalue problem is

$$\left(\left(\mathbf{K}-\mathbf{B}\mathbf{G}^{T}\right)-\eta_k\mathbf{M}\right)\mathbf{w}_k=\mathbf{0}, \qquad k=1,2,\cdots,n \tag{14}$$

or

$$\left(\mathbf{K}-\mathbf{B}\mathbf{G}^{T}\right)\mathbf{W}-\mathbf{M}\mathbf{W}\mathbf{\Lambda}=\mathbf{0} \tag{15}$$

We consider the problem of block decoupling the closed-loop system with respect to the partition $\left(n_1 \quad n_2\right)$. By choice of parameters $\alpha_{\mu_k,j}$ to satisfy equation (11), modal degree of freedom constraints may be imposed on right eigenvector $\mathbf{w}_k$,

$$\begin{aligned} w_{jk}&=0,\ j=n_1+1, n_1+2,\cdots,n, \quad k=1,2,\cdots,n_1 \\ w_{jk}&=0,\ j=1,2,\cdots,n_1, \quad k=n_1+1,n_1+2,\cdots,n \end{aligned} \tag{16}$$

where w_{jk}, is the j^{th} entry of the k^{th} right eigenvector of the closed-loop system. This leads to the block-diagonal matrix of mode shapes,

$$\mathbf{W}=\begin{bmatrix}\mathbf{W}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{W}_{22}\end{bmatrix} \tag{17}$$

and $\mathbf{W}\in\mathbb{C}^{n\times n}$, $\mathbf{W}_{11}\in\mathbb{C}^{n_1\times n_1}$, $\mathbf{W}_{22}\in\mathbb{C}^{n_2\times n_2}$ and zero matrices inside are of proper dimension.

Lemma 1: The closed-loop stiffness and receptance matrices will be block diagonal with partition $\left(n_1 \quad n_2\right)$ with assigned closed-loop system eigenvalues when the closed-loop right eigenvector matrix $\mathbf{W}$ is block diagonal with the same partition and $\mathbf{M}$ is a lumped mass matrix.

Proof: Since the system is controllable, distinct eigenvalues $\{\mu_1, \mu_2, \cdots, \mu_n\}$ may be assigned with block-diagonal constraints on $\mathbf{W}$ by the method of receptances using equation (7) (described in full by Ram and Mottershead [32]). When block-diagonal $\mathbf{W}$ has partition $(n_1 \quad n_2)$, then

$$\mathbf{W}^{-1} = \begin{bmatrix} \mathbf{W}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22}^{-1} \end{bmatrix} \tag{18}$$

Hence, from equation (15), the closed-loop stiffness matrix

$$\left(\mathbf{K} - \mathbf{B}\mathbf{G}^T\right) = \mathbf{M}\mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1} \tag{19}$$

is block diagonal with respect to the partition $(n_1 \quad n_2)$. Consequently, the closed-loop dynamic stiffness and receptance matrices are block diagonal and the system is block decoupled.

□

Remark 1: Equation (19) admits the use of a block diagonal mass matrix $\mathbf{M}$ with partition $(n_1 \quad n_2)$. However, for reasons of physical practicality, we discuss only the case of the diagonal (lumped) mass matrix.

□

Therefore, the block-decoupling vibration control algorithm for undamped systems may be summarised as:

1. Decouple the open-loop undamped system to form two uncoupled substructures. This is achieved by the imposition of modal degree of freedom constraints (16) on the closed-loop right eigenvectors $\mathbf{w}_k$ by the choice of parameters $\alpha_{\mu_k,j}$ to satisfy equation (11).
2. Assign desired eigenvalues $\mathbf{\Lambda}_{11} = diag\left(\mu_k\right)_{k=1}^{n_1}$ and $\mathbf{\Lambda}_{22} = diag\left(\mu_k\right)_{k=n_1+1}^{n}$ to the two substructures by the choice of control gain matrix $\mathbf{G}$ based on the method of receptances using equation (7).

The eigenpairs $\{\mathbf{\Lambda}_{11} \quad \mathbf{W}_{11}\}$ and $\{\mathbf{\Lambda}_{22} \quad \mathbf{W}_{22}\}$ are then assigned to the two independent substructures respectively.

If $\mathbf{W}$ is block diagonal with partition $(n_1, n_2, \cdots, n_v)$, then it is straightforward to show that the closed-loop stiffness matrix is also block diagonal with respect to the partition $(n_1, n_2, \cdots, n_v)$. The system becomes strictly diagonal when $v = n$.

4.1. Example 1

Consider the two degree-of-freedom mass-spring system,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

The open-loop eigenvalues are,

$$\lambda_1 = 0.3820$$
$$\lambda_2 = 2.6180$$

and the eigenvector matrix is,

$$\mathbf{\Phi} = \begin{bmatrix} -0.5257 & -0.8507 \\ -0.8507 & 0.5257 \end{bmatrix}.$$

Now, a two-input proportional feedback controller is used to decouple the system into two independent single-degree-of-freedom systems. The prescribed eigenvalues of the two independent subsystems are

$$\mu_1 = 0.5$$
$$\mu_2 = 3.0.$$

According to the above analysis, modal nodal constraints are imposed to the closed-loop right eigenvectors so that the second entry of the first eigenvector and the first entry of the second eigenvector are zero. The force distribution matrix is chosen as,

$$\mathbf{B} = [b_1 \quad b_2] = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}.$$

To impose the required modal nodal constraints, the parameters $\alpha_{\mu_k, j}$ are chosen as,

$$\alpha_{1,1} = -\frac{\mathbf{e}_2^T \mathbf{r}_{1,2} \alpha_{1,2}}{\mathbf{e}_2^T \mathbf{r}_{1,1}}; \ \alpha_{2,1} = -\frac{\mathbf{e}_1^T \mathbf{r}_{2,2} \alpha_{2,2}}{\mathbf{e}_1^T \mathbf{r}_{2,1}}$$

Where,

$$\alpha_{1,2} = \alpha_{2,2} = 1, \ \mathbf{e}_2^T = [0 \quad 1], \ \mathbf{e}_1^T = [1 \quad 0]$$

This leads to the matrix of control gains,

$$\mathbf{G} = \begin{bmatrix} 3.25 & -2.5 \\ 0.5 & -1.0 \end{bmatrix}.$$

The resulting closed-loop eigenvalues are found to be,

$$\mu_1 = 0.5$$
$$\mu_2 = 3.0$$

and the eigenvectors are

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The closed-loop systems matrices are

$$\mathbf{M}_{CL} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{K}_{CL} = \begin{bmatrix} 0.5 & 0 \\ 0 & 3.0 \end{bmatrix}.$$

Hence, the closed-loop system is found to be decoupled into two independent single-degree-of-freedom systems with desired eigenvalues.

5. Block-decoupling control for damped structures with lumped masses

The closed-loop right- and left-eigenvalue problems may be written as,

$$\left(\mathbf{M}\mu_k^2 + \left(\mathbf{C} - \mathbf{BF}^T\right)\mu_k + \left(\mathbf{K} - \mathbf{BG}^T\right)\right)\mathbf{w}_k = \mathbf{0}. \tag{20}$$

and,

$$\mathbf{z}_k^T\left(\mathbf{M}\mu_k^2 + \left(\mathbf{C} - \mathbf{BF}^T\right)\mu_k + \left(\mathbf{K} - \mathbf{BG}^T\right)\right) = \mathbf{0},\ \ k = 1, 2, \cdots, 2n \tag{21}$$

By combining all the modes into a single expression the right eigenvalue problem (20) becomes,

$$\mathbf{MW\Lambda}^2 + \left(\mathbf{C} - \mathbf{BF}^T\right)\mathbf{W\Lambda} + \left(\mathbf{K} - \mathbf{BG}^T\right)\mathbf{W} = \mathbf{0} \tag{22}$$

and the left eigenvalue problem (21) is,

$$\mathbf{\Lambda}^2\mathbf{Z}^T\mathbf{M} + \mathbf{\Lambda Z}^T\left(\mathbf{C} - \mathbf{BF}^T\right) + \mathbf{Z}^T\left(\mathbf{K} - \mathbf{BG}^T\right) = \mathbf{0} \tag{23}$$

In these expressions $\mathbf{W} \in \mathbb{C}^{n \times 2n}$ is the matrix of right eigenvectors, $\mathbf{Z} \in \mathbb{C}^{n \times 2n}$ is the matrix of left eigenvectors, $\boldsymbol{\Lambda} = \mathrm{diag}\begin{pmatrix}\mu_1 & \cdots & \mu_{2n}\end{pmatrix} \in \mathbb{C}^{2n \times 2n}$ is the spectral matrix.

We partition matrices $\boldsymbol{\Lambda}$, $\mathbf{W}$ and $\mathbf{Z}$ as follows,

$$\begin{gathered}\boldsymbol{\Lambda} = \begin{bmatrix}\boldsymbol{\Lambda}_1 & \\ & \boldsymbol{\Lambda}_2\end{bmatrix} \\ \boldsymbol{\Lambda}_1 = \mathrm{diag}\begin{pmatrix}\mu_1 & \cdots & \mu_n\end{pmatrix} \in \mathbb{C}^{n \times n}; \quad \boldsymbol{\Lambda}_2 = \mathrm{diag}\begin{pmatrix}\mu_{n+1} & \cdots & \mu_{2n}\end{pmatrix} \in \mathbb{C}^{n \times n}\end{gathered} \tag{24}$$

$$\begin{gathered}\mathbf{W} = \begin{bmatrix}\mathbf{W}_{\mathrm{L}} & \mathbf{W}_{\mathrm{R}}\end{bmatrix} \\ \mathbf{W}_{\mathrm{L}} = \begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n\end{bmatrix} \in \mathbb{C}^{n \times n}; \quad \mathbf{W}_{\mathrm{R}} = \begin{bmatrix}\mathbf{w}_{n+1} & \mathbf{w}_{n+2} & \cdots & \mathbf{w}_{2n}\end{bmatrix} \in \mathbb{C}^{n \times n}\end{gathered} \tag{25}$$

and

$$\begin{gathered}\mathbf{Z} = \begin{bmatrix}\mathbf{Z}_{\mathrm{L}} & \mathbf{Z}_{\mathrm{R}}\end{bmatrix} \\ \mathbf{Z}_{\mathrm{L}} = \begin{bmatrix}\mathbf{z}_1 & \mathbf{z}_2 & \cdots & \mathbf{z}_n\end{bmatrix} \in \mathbb{C}^{n \times n}; \quad \mathbf{Z}_{\mathrm{R}} = \begin{bmatrix}\mathbf{z}_{n+1} & \mathbf{z}_{n+2} & \cdots & \mathbf{z}_{2n}\end{bmatrix} \in \mathbb{C}^{n \times n}\end{gathered} \tag{26}$$

In the case of complex eigenvalues, $\mu_{n+i} = \mu_n^*$ where $(\bullet)^*$ denotes complex conjugation. Real eigenvalues are grouped equally in $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ at the same diagonal locations.

Then by choice of $\alpha_{\mu_k, j}$ in (11), modal degree of freedom constraints on the closed-loop right eigenvectors $\mathbf{w}_k$ may be imposed,

$$\begin{aligned} w_{jk} &= w_{j(k+n)} = 0, \qquad && j = n_1+1, n_1+2, \cdots, n, \;\; k = 1, 2, \cdots, n_1 \\ w_{jk} &= w_{j(k+n)} = 0, \qquad && j = 1, 2, \cdots, n_1, \;\; k = n_1+1, n_1+2, \cdots, n \end{aligned} \tag{27}$$

Thus, $\mathbf{W}_{\mathrm{L}}$ and $\mathbf{W}_{\mathrm{R}}$ are block diagonalised with respect to the partition $\begin{pmatrix}n_1 & n_2\end{pmatrix}$ as,

$$\mathbf{W}_{\mathrm{L}} = \begin{bmatrix}\mathbf{W}_{\mathrm{L}11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{\mathrm{L}22}\end{bmatrix}; \quad \mathbf{W}_{\mathrm{R}} = \begin{bmatrix}\mathbf{W}_{\mathrm{R}11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{\mathrm{R}22}\end{bmatrix} \tag{28}$$

where $\mathbf{W}_{\mathrm{L}11}, \mathbf{W}_{\mathrm{R}11} \in \mathbb{C}^{n_1 \times n_1}$ and $\mathbf{W}_{\mathrm{L}22}, \mathbf{W}_{\mathrm{R}22} \in \mathbb{C}^{n_2 \times n_2}$.

We now write equations (22) and (23) in first-order form as,

$$\mathbf{AX} = \mathbf{X}\boldsymbol{\Lambda} \tag{29}$$

$$\mathbf{Y}^T\mathbf{A} = \mathbf{\Lambda}\mathbf{Y}^T \tag{30}$$

where (from Appendix 1),

$$\mathbf{X} = \begin{pmatrix} \mathbf{W} \\ \mathbf{W\Lambda} \end{pmatrix}; \quad \mathbf{Y} = \begin{pmatrix} \mathbf{Y}_{\mathrm{U}} \\ \mathbf{Y}_{\mathrm{L}} \end{pmatrix} = \begin{pmatrix} \left(\mathbf{K} - \mathbf{B}\mathbf{G}^T\right)^T \mathbf{Z}\mathbf{\Lambda}^{-1} \\ \mathbf{MZ} \end{pmatrix}$$
$$\mathbf{Y}_{\mathrm{U}} = \left[\mathbf{y}_{\mathrm{U}1} \quad \mathbf{y}_{\mathrm{U}2} \quad \cdots \quad \mathbf{y}_{\mathrm{U}2n}\right]; \quad \mathbf{Y}_{\mathrm{L}} = \left[\mathbf{y}_{\mathrm{L}1} \quad \mathbf{y}_{\mathrm{L}2} \quad \cdots \quad \mathbf{y}_{\mathrm{L}2n}\right] \tag{31}$$

and,

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\left(\mathbf{K} - \mathbf{B}\mathbf{G}^T\right) & -\mathbf{M}^{-1}\left(\mathbf{C} - \mathbf{B}\mathbf{F}^T\right) \end{bmatrix} \tag{32}$$

Pre-multiplying and post-multiplying equations (29) and (30) by $\mathbf{Y}^T$ and $\mathbf{X}$ respectively -lead to,

$$\mathbf{Y}^T\mathbf{A}\mathbf{X} = \mathbf{\Lambda}\mathbf{Y}^T\mathbf{X} = \mathbf{Y}^T\mathbf{X}\mathbf{\Lambda} \tag{33}$$

It can be seen from (33) that $\mathbf{Y}^T\mathbf{X}$ commutes with $\mathbf{\Lambda}$ so that,

$$\mathbf{Y}^T\mathbf{X} = \mathbf{D} \tag{34}$$

where $\mathbf{D} \in \mathbb{C}^{2n\times 2n}$ is diagonal.

Then by normalising the left and right eigenvectors,

$$\mathbf{Y}^T\mathbf{X} = \mathbf{I} \tag{35}$$

or,

$$\mathbf{Y}^T = \mathbf{X}^{-1} \tag{36}$$

where $\mathbf{I}$ is the identity matrix.

From equations (28) and (31),

$$\mathbf{X} = \begin{bmatrix} \mathbf{W}_{\mathrm{L}11} & \mathbf{0} & \mathbf{W}_{\mathrm{R}11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{\mathrm{L}22} & \mathbf{0} & \mathbf{W}_{\mathrm{R}22} \\ \mathbf{W}_{\mathrm{L}11}\mathbf{\Lambda}_{(1)11} & \mathbf{0} & \mathbf{W}_{\mathrm{R}11}\mathbf{\Lambda}_{(2)11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{\mathrm{L}22}\mathbf{\Lambda}_{(1)22} & \mathbf{0} & \mathbf{W}_{\mathrm{R}22}\mathbf{\Lambda}_{(2)22} \end{bmatrix} \tag{37}$$

where

$$\mathbf{\Lambda}_1 = \begin{bmatrix} \mathbf{\Lambda}_{(1)11} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_{(1)22} \end{bmatrix}; \ \mathbf{\Lambda}_2 = \begin{bmatrix} \mathbf{\Lambda}_{(2)11} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_{(2)22} \end{bmatrix} \tag{38}$$

and $\mathbf{\Lambda}_{(1)11}, \mathbf{\Lambda}_{(2)11} \in \mathbb{C}^{n_1 \times n_1}$; $\mathbf{\Lambda}_{(1)22}, \ \mathbf{\Lambda}_{(2)22} \in \mathbb{C}^{n_2 \times n_2}$.

The matrix $\mathbf{Y}^T$ may be written as

$$\mathbf{Y}^T = \begin{bmatrix} \mathbf{Y}_{\mathrm{U}}^T & \mathbf{Y}_{\mathrm{L}}^T \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{\mathrm{UL}}^T & \mathbf{Y}_{\mathrm{LL}}^T \\ \mathbf{Y}_{\mathrm{UR}}^T & \mathbf{Y}_{\mathrm{LR}}^T \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{\mathrm{UL11}}^T & \mathbf{Y}_{\mathrm{UL21}}^T & \mathbf{Y}_{\mathrm{LL11}}^T & \mathbf{Y}_{\mathrm{LL21}}^T \\ \mathbf{Y}_{\mathrm{UL12}}^T & \mathbf{Y}_{\mathrm{UL22}}^T & \mathbf{Y}_{\mathrm{LL12}}^T & \mathbf{Y}_{\mathrm{LL22}}^T \\ \mathbf{Y}_{\mathrm{UR11}}^T & \mathbf{Y}_{\mathrm{UR21}}^T & \mathbf{Y}_{\mathrm{LR11}}^T & \mathbf{Y}_{\mathrm{LR21}}^T \\ \mathbf{Y}_{\mathrm{UR12}}^T & \mathbf{Y}_{\mathrm{UR22}}^T & \mathbf{Y}_{\mathrm{LR12}}^T & \mathbf{Y}_{\mathrm{LR22}}^T \end{bmatrix} \tag{39}$$

where

$$\begin{gathered} \mathbf{Y}_{\mathrm{U}} = \begin{bmatrix} \mathbf{Y}_{\mathrm{UL}} & \mathbf{Y}_{\mathrm{UR}} \end{bmatrix}, \mathbf{Y}_{\mathrm{L}} = \begin{bmatrix} \mathbf{Y}_{\mathrm{LL}} & \mathbf{Y}_{\mathrm{LR}} \end{bmatrix}, \\ \mathbf{Y}_{\mathrm{UL}} = \begin{bmatrix} \mathbf{Y}_{\mathrm{UL11}} & \mathbf{Y}_{\mathrm{UL12}} \\ \mathbf{Y}_{\mathrm{UL21}} & \mathbf{Y}_{\mathrm{UL22}} \end{bmatrix}, \mathbf{Y}_{\mathrm{LL}} = \begin{bmatrix} \mathbf{Y}_{\mathrm{LL11}} & \mathbf{Y}_{\mathrm{LL12}} \\ \mathbf{Y}_{\mathrm{LL21}} & \mathbf{Y}_{\mathrm{LL22}} \end{bmatrix}, \\ \mathbf{Y}_{\mathrm{UR}} = \begin{bmatrix} \mathbf{Y}_{\mathrm{UR11}} & \mathbf{Y}_{\mathrm{UR12}} \\ \mathbf{Y}_{\mathrm{UR21}} & \mathbf{Y}_{\mathrm{UR22}} \end{bmatrix}, \ \mathbf{Y}_{\mathrm{LR}} = \begin{bmatrix} \mathbf{Y}_{\mathrm{LR11}} & \mathbf{Y}_{\mathrm{LR12}} \\ \mathbf{Y}_{\mathrm{LR21}} & \mathbf{Y}_{\mathrm{LR22}} \end{bmatrix} \end{gathered} \tag{40}$$

with $\mathbf{Y}_{\mathrm{UL}}, \mathbf{Y}_{\mathrm{UR}}, \mathbf{Y}_{\mathrm{LL}}, \mathbf{Y}_{\mathrm{LR}} \in \mathbb{C}^{n \times n}$; $\mathbf{Y}_{\mathrm{UL11}}, \mathbf{Y}_{\mathrm{LL11}}, \mathbf{Y}_{\mathrm{UR11}}, \mathbf{Y}_{\mathrm{LR11}} \in \mathbb{C}^{n_1 \times n_1}$; $\mathbf{Y}_{\mathrm{UL12}}, \mathbf{Y}_{\mathrm{LL12}}, \mathbf{Y}_{\mathrm{UR12}}, \mathbf{Y}_{\mathrm{LR12}} \in \mathbb{C}^{n_1 \times n_2}$; $\mathbf{Y}_{\mathrm{UL21}}, \mathbf{Y}_{\mathrm{LL21}}, \mathbf{Y}_{\mathrm{UR21}}, \mathbf{Y}_{\mathrm{LR21}} \in \mathbb{C}^{n_2 \times n_1}$; $\mathbf{Y}_{\mathrm{UL22}}, \mathbf{Y}_{\mathrm{LL22}}, \mathbf{Y}_{\mathrm{UR22}}, \mathbf{Y}_{\mathrm{LR22}} \in \mathbb{C}^{n_2 \times n_2}$.

Lemma 2: The closed-loop damping and stiffness matrices will be block diagonal with partition $\begin{pmatrix} n_1 & n_2 \end{pmatrix}$ with assigned closed loop eigenvalues when the closed-loop right eigenvector matrices $\mathbf{W}_{\mathrm{R}}$ and $\mathbf{W}_{\mathrm{L}}$ are block diagonal with the same partition and $\mathbf{M}$ is a lumped mass matrix.

Proof: Since the system is controllable, distinct eigenvalues $\{\mu_1, \mu_2, \cdots, \mu_{2n}\}$, may be assigned with block-diagonal constraints on $\mathbf{W}$ by the method of receptances using equation (7), described in full by Ram and Mottershead [32].

By using elementary transformations, the right eigenvector matrix $\mathbf{X}$ may be expressed as,

$$\tilde{\mathbf{X}} = \begin{bmatrix} \mathbf{W}_{\mathrm{L11}} & \mathbf{W}_{\mathrm{R11}} & \mathbf{0} & \mathbf{0} \\ \mathbf{W}_{\mathrm{L11}}\mathbf{\Lambda}_{(1)11} & \mathbf{W}_{\mathrm{R11}}\mathbf{\Lambda}_{(2)11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{W}_{\mathrm{L22}} & \mathbf{W}_{\mathrm{R22}} \\ \mathbf{0} & \mathbf{0} & \mathbf{W}_{\mathrm{L22}}\mathbf{\Lambda}_{(1)22} & \mathbf{W}_{\mathrm{R22}}\mathbf{\Lambda}_{(2)22} \end{bmatrix} \tag{41}$$

The left eigenvector matrix $\mathbf{Y}$ may then be written, using the relationship (36), as

$$\tilde{\mathbf{Y}} = \begin{bmatrix} \mathbf{Y}^T_{\mathrm{UL11}} & \mathbf{Y}^T_{\mathrm{LL11}} & \mathbf{Y}^T_{\mathrm{UL21}} & \mathbf{Y}^T_{\mathrm{LL21}} \\ \mathbf{Y}^T_{\mathrm{UR11}} & \mathbf{Y}^T_{\mathrm{LR11}} & \mathbf{Y}^T_{\mathrm{UR21}} & \mathbf{Y}^T_{\mathrm{LR21}} \\ \mathbf{Y}^T_{\mathrm{UL12}} & \mathbf{Y}^T_{\mathrm{LL12}} & \mathbf{Y}^T_{\mathrm{UL22}} & \mathbf{Y}^T_{\mathrm{LL22}} \\ \mathbf{Y}^T_{\mathrm{UR12}} & \mathbf{Y}^T_{\mathrm{LR12}} & \mathbf{Y}^T_{\mathrm{UR22}} & \mathbf{Y}^T_{\mathrm{LR22}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathbf{W}_{\mathrm{L11}} & \mathbf{W}_{\mathrm{R11}} \\ \mathbf{W}_{\mathrm{L11}}\mathbf{\Lambda}_{(1)11} & \mathbf{W}_{\mathrm{R11}}\mathbf{\Lambda}_{(2)11} \end{bmatrix}^{-1} & \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \begin{bmatrix} \mathbf{W}_{\mathrm{L22}} & \mathbf{W}_{\mathrm{R22}} \\ \mathbf{W}_{\mathrm{L22}}\mathbf{\Lambda}_{(1)22} & \mathbf{W}_{\mathrm{R22}}\mathbf{\Lambda}_{(2)22} \end{bmatrix}^{-1} \end{bmatrix} \tag{42}$$

so that,

$$\begin{gathered} \mathbf{Y}^T_{\mathrm{UL12}} = \mathbf{0},\ \mathbf{Y}^T_{\mathrm{LL12}} = \mathbf{0},\ \mathbf{Y}^T_{\mathrm{UR12}} = \mathbf{0},\ \mathbf{Y}^T_{\mathrm{LR12}} = \mathbf{0} \\ \mathbf{Y}^T_{\mathrm{UL21}} = \mathbf{0},\ \mathbf{Y}^T_{\mathrm{LL21}} = \mathbf{0},\ \mathbf{Y}^T_{\mathrm{UR21}} = \mathbf{0},\ \mathbf{Y}^T_{\mathrm{LR21}} = \mathbf{0} \end{gathered} \tag{43}$$

Therefore

$$\mathbf{Y}^T = \begin{bmatrix} \mathbf{Y}^T_{\mathrm{UL11}} & \mathbf{0} & \mathbf{Y}^T_{\mathrm{LL11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^T_{\mathrm{UL22}} & \mathbf{0} & \mathbf{Y}^T_{\mathrm{LL22}} \\ \mathbf{Y}^T_{\mathrm{UR11}} & \mathbf{0} & \mathbf{Y}^T_{\mathrm{LR11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^T_{\mathrm{UR22}} & \mathbf{0} & \mathbf{Y}^T_{\mathrm{LR22}} \end{bmatrix} \tag{44}$$

with

$$\mathbf{Y}^T_{\mathrm{LL}} = \begin{bmatrix} \mathbf{Y}^T_{\mathrm{LL11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^T_{\mathrm{LL22}} \end{bmatrix} \text{ and } \mathbf{Y}^T_{\mathrm{LR}} = \begin{bmatrix} \mathbf{Y}^T_{\mathrm{LR11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^T_{\mathrm{LR22}} \end{bmatrix} \tag{45}$$

block diagonal with respect to the partition $\left(n_1 \quad n_2\right)$.

From (31),

$$\mathbf{Y}_{\mathrm{LL}} = \mathbf{M}\mathbf{Z}_{\mathrm{L}}; \quad \mathbf{Y}_{\mathrm{LR}} = \mathbf{M}\mathbf{Z}_{\mathrm{R}} \tag{46}$$

Since $\mathbf{M}$ is the lumped mass matrix it follows from equation (46) that $\mathbf{Z}_{\mathrm{L}}$ and $\mathbf{Z}_{\mathrm{R}}$ are block diagonal with respect to the partition $\begin{pmatrix} n_1 & n_2 \end{pmatrix}$.

It is known that the receptance matrix may be expressed as,

$$\begin{aligned}\hat{\mathbf{H}}(s) &= \begin{bmatrix} \mathbf{W}_{\mathrm{L}} & \mathbf{W}_{\mathrm{R}} \end{bmatrix} \begin{bmatrix} (\mathbf{I}_n s - \mathbf{\Lambda}_1)^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I}_n s - \mathbf{\Lambda}_2)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{\mathrm{L}}^T \\ \mathbf{Z}_{\mathrm{R}}^T \end{bmatrix} \\ &= \mathbf{W}_{\mathrm{L}} (\mathbf{I}_n s - \mathbf{\Lambda}_1)^{-1} \mathbf{Z}_{\mathrm{L}}^T + \mathbf{W}_{\mathrm{R}} (\mathbf{I}_n s - \mathbf{\Lambda}_2)^{-1} \mathbf{Z}_{\mathrm{R}}^T \end{aligned} \tag{47}$$

so that $\hat{\mathbf{H}}(s)$ is block diagonal with respect to the partition $\begin{pmatrix} n_1 & n_2 \end{pmatrix}$: so too is the dynamic stiffness matrix, i.e. the inverse of $\hat{\mathbf{H}}(s)$.

When $s = 0$,

$$\hat{\mathbf{\Gamma}}(0) = \left(\mathbf{K} - \mathbf{B}\mathbf{G}^T\right) \tag{48}$$

which shows that the closed-loop stiffness matrix is block diagonal with respect to the partition $\begin{pmatrix} n_1 & n_2 \end{pmatrix}$. The dynamic stiffness may be recast as

$$\hat{\mathbf{\Gamma}}(s) = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} s^2 + \begin{bmatrix} \left(\mathbf{C} - \mathbf{B}\mathbf{F}^T\right)_{11} & \left(\mathbf{C} - \mathbf{B}\mathbf{F}^T\right)_{12} \\ \left(\mathbf{C} - \mathbf{B}\mathbf{F}^T\right)_{21} & \left(\mathbf{C} - \mathbf{B}\mathbf{F}^T\right)_{22} \end{bmatrix} s + \begin{bmatrix} \left(\mathbf{K} - \mathbf{B}\mathbf{G}^T\right)_{11} & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{K} - \mathbf{B}\mathbf{G}^T\right)_{22} \end{bmatrix} \tag{49}$$

so that,

$$\left(\mathbf{C} - \mathbf{B}\mathbf{F}^T\right)_{12} s = 0 \text{ and } \left(\mathbf{C} - \mathbf{B}\mathbf{F}^T\right)_{21} s = 0 \tag{50}$$

for arbitrary s. Hence the closed-loop damping matrix $\left(\mathbf{C} - \mathbf{B}\mathbf{F}^T\right)$ is block diagonal with respect to the partition $\begin{pmatrix} n_1 & n_2 \end{pmatrix}$.

Thus, if the sub-matrices of the right eigenvector, $\mathbf{W}_{\mathrm{L}}$ and $\mathbf{W}_{\mathrm{R}}$, are block diagonal with respect to the partition $\begin{pmatrix} n_1 & n_2 \end{pmatrix}$, then the closed-loop damping and stiffness matrices will also be block decoupled with respect to the partition $\begin{pmatrix} n_1 & n_2 \end{pmatrix}$.

□

Remark 2: Equations (46) admit the use of a block diagonal mass matrix $\mathbf{M}$ with partition $\left(n_1 \quad n_2\right)$. For the same reasons as given before, we only consider the case of the diagonal (lumped mass matrix).
□

Therefore, the block-decoupling vibration control algorithm for damped systems may be summarised as:

1. Decouple the open-loop damped system into two uncoupled substructures. This is achieved by the imposition of modal degree of freedom constraints (27) on the closed-loop right eigenvectors $\mathbf{w}_k$ by choice of parameters $\alpha_{\mu_k, j}$ to satisfy equation (11).
2. Assign desired eigenvalues $\left\{\mathbf{\Lambda}_{(1)11} \quad \mathbf{\Lambda}_{(2)11}\right\} = \left\{diag\left(\mu_k\right)_{k=1}^{n_1} \quad diag\left(\mu_k\right)_{k=n+1}^{n+n_1}\right\}$ and $\left\{\mathbf{\Lambda}_{(1)22} \quad \mathbf{\Lambda}_{(2)22}\right\} = \left\{diag\left(\mu_k\right)_{k=n_1+1}^{n} \quad diag\left(\mu_k\right)_{k=n+n_1+1}^{2n}\right\}$ to the two substructures by the choice of control gain matrices $\mathbf{F}, \mathbf{G}$ based on the method of receptances described by (7).

The eigenpairs $\left\{\left[\mathbf{\Lambda}_{(1)11} \quad \mathbf{\Lambda}_{(2)11}\right] \quad \left[\mathbf{W}_{\mathrm{L}11} \quad \mathbf{W}_{\mathrm{R}11}\right]\right\}$ and $\left\{\left[\mathbf{\Lambda}_{(1)22} \quad \mathbf{\Lambda}_{(2)22}\right] \quad \left[\mathbf{W}_{\mathrm{L}22} \quad \mathbf{W}_{\mathrm{R}22}\right]\right\}$ are then assigned to the two independent substructures respectively.

If $\mathbf{W}_{\mathrm{L}}$ and $\mathbf{W}_{\mathrm{R}}$ are block diagonal with respect to the partition $\left(n_1, n_2, \cdots, n_v\right)$, then the closed-loop stiffness and damping matrices are also block diagonal with respect to the partition $\left(n_1, n_2, \cdots, n_v\right)$. The system becomes strictly diagonal when $v = n$.

5.1. Example 2

Consider the three degree-of-freedom system shown in Fig. 1.

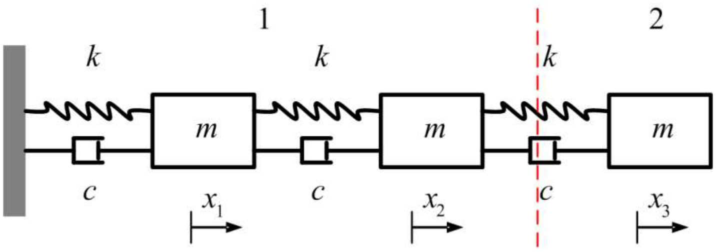

Fig. 1 The three degree-of-freedom system

The system matrices of the open-loop system are ,

$$\mathbf{M} = m\mathbf{I},\ \mathbf{C} = c\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } \mathbf{K} = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

If $m=1$, $c=1$ and $k=5$, the open-loop eigenvalues are,

$$\begin{aligned} \lambda_{1,4} &= -0.0990 \pm 0.9902\mathrm{i} \\ \lambda_{2,5} &= -0.7775 \pm 2.6777\mathrm{i}\,. \\ \lambda_{3,6} &= -1.6235 \pm 3.6877\mathrm{i} \end{aligned}$$

Now, the block decoupling control method is used to decouple the three degree-of-freedom system into two independent substructures as shown in Fig. 1 . The eigenvalues of the first substructure are prescribed as,

$$\begin{aligned} \mu_{1,4} &= -0.1 \pm 1.0\mathrm{i} \\ \mu_{2,5} &= -0.8 \pm 2.8\mathrm{i} \end{aligned}$$

and the second substructure

$$\mu_{3,6} = -1.6 \pm 3.7\mathrm{i}\,.$$

Modal degree of freedom constraints are imposed on the right eigenvectors of the closed-loop system so that the first two entries of the eigenvectors of the last mode and the last entry of the eigenvectors corresponding to the first two modes are zero. The three inputs are used and the force distribution matrix is chosen as

$$\mathbf{B} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 2 & 0 & 0 \end{bmatrix}.$$

The parameters $\alpha_{\mu_k,j}$ are chosen as,

$$\alpha_{\mu_k,2} = 0.5, \qquad \alpha_{\mu_k,3} = 1, \qquad k = 1,2,4,5$$

$$\alpha_{\mu_k,3} = 1, \qquad k = 3,6$$

$$\alpha_{\mu_k,1} = \mathrm{inv}\left(\mathbf{r}_{\mu_k,1(3)}\right) \times \left(-\mathbf{r}_{\mu_k,2(3)}\alpha_{\mu_k,2} - \mathbf{r}_{\mu_k,3(3)}\alpha_{\mu_k,3}\right), \quad k = 1,2,4,5$$

$$\begin{pmatrix} \alpha_{\mu_k,1} \\ \alpha_{\mu_k,2} \end{pmatrix} = \mathrm{inv}\left(\mathbf{R}_{\mu_k(1:2,1:2)}\right) \times \left(-\mathbf{r}_{\mu_k,3(1:2)}\alpha_{\mu_k,3}\right), k = 3,6.$$

and the control gains are found to be,

$$\mathbf{G} = \begin{bmatrix} 0 & 3.0559 & 6.1117 \\ -2.5 & -1.9910 & -3.9820 \\ -5.6250 & 10.6250 & -5.2083 \end{bmatrix} \text{ and } \mathbf{F} = \begin{bmatrix} 0 & 0.6617 & 1.3235 \\ -0.5 & -0.4420 & -0.8840 \\ -1.1000 & 2.1000 & -1.0333 \end{bmatrix}.$$

The closed-loop system is found to have eigenvalues,

$$\begin{aligned} \mu_{1,4} &= -0.1 \pm 1.0\mathrm{i} \\ \mu_{2,5} &= -0.8 \pm 2.8\mathrm{i} \\ \mu_{3,6} &= -1.6 \pm 3.7\mathrm{i} \end{aligned}$$

and eigenvectors

$$\mathbf{w}_1 = \begin{pmatrix} 0.7122+0.0948\mathrm{i} \\ 0.8198+0.1091\mathrm{i} \\ 0 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 0.0683+0.1660\mathrm{i} \\ 0.1245+0.2759\mathrm{i} \\ 0 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ -0.1062 - 0.2082\mathrm{i} \end{pmatrix},$$

$$\mathbf{w}_4 = \mathbf{w}_1^*, \mathbf{w}_5 = \mathbf{w}_2^*, \mathbf{w}_6 = \mathbf{w}_3^*$$

The closed-loop system matrices are

$$\mathbf{M}_{CL} = \mathbf{I},\ \mathbf{C}_{CL} = \begin{bmatrix} -3.2939 & 3.0359 & 0 \\ -5.6322 & 5.0939 & 0 \\ 0 & 0 & 3.2000 \end{bmatrix} \text{ and } \mathbf{K}_{CL} = \begin{bmatrix} -14.4469 & 13.4281 & 0 \\ -26.3910 & 23.9371 & 0 \\ 0 & 0 & 16.2500 \end{bmatrix}$$

which are decoupled to form two independent substructures with desired eigenvalues.

6. The number of actuators and sensors

We have seen that the application of modal degree of freedom constraints to block diagonalise the right eigenvector matrix with respect to the partition $(n_1 \quad n_2)$ will cause the closed-loop stiffness and damping matrices to be block decoupled with the same partition. Ram and Mottershead [32] showed that the number of required control inputs should be no less than $1+\max\{n_1, n_2\}$. In this section, it will be shown that the number of required control inputs may be reduced for structures with banded damping and stiffness matrices with semi-bandwidth t.

For practical engineering structures, the connections between components are in general localised. If discretised by finite element methods, it appears that the damping and stiffness matrices are banded with non-zero entries confined to a diagonal band and the coupling in general exists between adjacent degrees of freedom. Hence, the original structure may be decoupled into two independent substructures if the coupling effect is eliminated in the connection of the two substructures.

Consider a n degree-of-freedom system structure whose dynamic stiffness matrix is banded with equal lower and upper semi-bandwidth t, $1 \le t \le \min\{n_1, n_2\}$, as shown in Fig. 2.

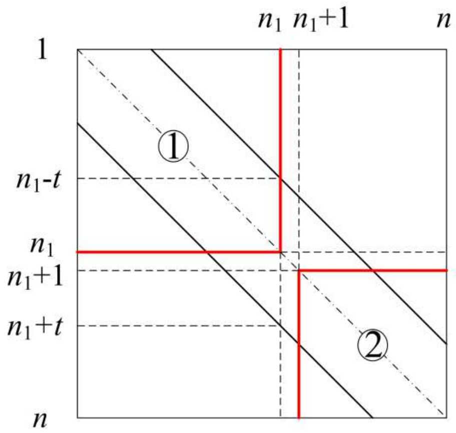

Fig. 2 The banded dynamic stiffness matrix

Now, the control objective is to decouple the structure into independent substructure 1 of dimension $n_1 \times n_1$ and substructure 2 of dimension $n_2 \times n_2$. It can be seen that the two substructures are only locally coupled from the $(n_1 - t + 1)^{\text{th}}$ degree of freedom to the $(n_1 + t)^{\text{th}}$ degree of freedom. Hence, the two substructures can be decoupled if the cross-coupling from the $(n_1 - t + 1)^{\text{th}}$ degree of freedom to the $(n_1 + t)^{\text{th}}$ degree of freedom is removed by using feedback control. This may be achieved by applying neutralising feedback forces from the $(n_1 - t + 1)^{\text{th}}$ degree of freedom to the $(n_1 + t)^{\text{th}}$ degree of freedom.

Lemma 3: The n degree of freedom open-loop dynamic system with lumped mass and banded damping and stiffness matrices having equal lower and upper semi-bandwidth $1 \le t \le \min\{n_1, n_2\}$ may always be decoupled into two independent subsystems when $2t$ actuators are located at the coupled degrees of freedom and the number of inputs $q \ge 2t$.

Proof: Let us begin by assuming there are $2t$ actuators at the coupled degrees of freedom. The force distribution matrix $\mathbf{B}$ may then be written as

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}_{(n_1-t)\times q} \\ \tilde{\mathbf{B}}_{2t\times q} \\ \mathbf{0}_{(n_2-t)\times q} \end{bmatrix} \text{ and } q \ge 2t \tag{51}$$

where $\tilde{\mathbf{B}}_{2t\times q}$ is real parameter matrix chosen so that all open-loop eigenvalues are controllable.

We have seen that the closed-loop damping and stiffness matrices become block diagonal when the right eigenvector matrices $\mathbf{W}_{\mathrm{L}}$ and $\mathbf{W}_{\mathrm{R}}$ are made block diagonal with the same partition $(n_1 \quad n_2)$ by choice of parameters $\boldsymbol{\alpha}_k \in \mathbb{C}^{q\times 1}$, $k = 1,2,\cdots,2n$. From equations (9) and (11), $\boldsymbol{\alpha}_k$ should be chosen such that,

$$\mathbf{w}_{k(n_1+1:n)} = \mathbf{H}_{\mu_k(n_1+1:n,:)}\mathbf{B}\boldsymbol{\alpha}_k = \mathbf{H}_{\mu_k(n_1+1:n,\, n_1-t+1:n_1+t)}\tilde{\mathbf{B}}_{2t\times q}\boldsymbol{\alpha}_k = \mathbf{0}_{n_2\times 1},\;\; k = 1,2,\cdots,n_1,n+1,n+2,\cdots,n+n_1 \tag{52}$$

and

$$\mathbf{w}_{k(1:n_1)} = \mathbf{H}_{\mu_k(1:n_1,:)}\mathbf{B}\boldsymbol{\alpha}_k = \mathbf{H}_{\mu_k(1:n_1,\, n_1-t+1:n_1+t)}\tilde{\mathbf{B}}_{2t\times q}\boldsymbol{\alpha}_k = \mathbf{0}_{n_1\times 1},\;\; k = n_1+1,n_1+2,\cdots,n,n+n_1+1,\, n+n_1+2,\cdots,2n \tag{53}$$

Strang and Nguyen [33] showed if a symmetric matrix $[\bullet]$ is banded with semi-bandwidth t, then above the t-th subdiagonal every submatrix of $[\bullet]^{-1}$ has rank $\le t$, and below the t-th superdiagonal every submatrix of $[\bullet]^{-1}$ has rank $\le t$. Therefore

$$\operatorname{rank}\left(\mathbf{H}_{\mu_k(n_1+1:n,\, n_1-t+1:n_1+t)}\right) \le t \tag{54}$$

and

$$\operatorname{rank}\left(\mathbf{H}_{\mu_k(1:n_1,\, n_1-t+1:n_1+t)}\right) \le t \tag{55}$$

Since

$$\operatorname{rank}\left(\mathbf{H}_{\mu_k(n_1+1:n,\, n_1-t+1:n_1+t)}\tilde{\mathbf{B}}_{2t\times q}\right) \le \operatorname{rank}\left(\mathbf{H}_{\mu_k(n_1+1:n,\, n_1-t+1:n_1+t)}\right) \le t \tag{56}$$

and

$$\text{rank}\left(\mathbf{H}_{\mu_k(1:n_1,\, n_1-t+1:\, n_1+t)}\tilde{\mathbf{B}}_{2t\times q}\right) \le \text{rank}\left(\mathbf{H}_{\mu_k(1:n_1,\, n_1-t+1:n_1+t)}\right) \le t \tag{57}$$

it follows that there always exists nontrivial $\boldsymbol{\alpha}_k$ satisfying equations (52) and (53)

$$\boldsymbol{\alpha}_k = \mathbf{null}\left(\mathbf{H}_{\mu_k(n_1+1:n,\, n_1-t+1:n_1+t)}\tilde{\mathbf{B}}_{2t\times q}\right)\boldsymbol{\gamma},\;\; k = 1, 2, \cdots, n_1, n+1, n+2, \cdots, n+n_1 \tag{58}$$

and

$$\boldsymbol{\alpha}_k = \mathbf{null}\left(\mathbf{H}_{\mu_k(1:n_1,\, n_1-t+1:\, n_1+t)}\tilde{\mathbf{B}}_{2t\times q}\right)\boldsymbol{\gamma},\;\; k = n_1+1, n_1+2, \cdots, n, n+n_1+1,\, n+n_1+2, \cdots, 2n \tag{59}$$

where $\boldsymbol{\gamma}$ is an arbitrary non-zero vector.

□

Remark 3: It may be proved similarly that if $\min\{n_1, n_2\} < t < \max\{n_1, n_2\}$, then $\min\{n_1, n_2\} + t$ actuators are sufficient for decoupling control of the open-loop system.

□

Lemma 4: A necessary condition for block decoupling is that sensors should be placed at the coupled degrees of freedom of the system.

Proof: The force distribution and control gain matrices may be partitioned as

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{bmatrix},\quad \mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \end{bmatrix} \text{ and } \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \\ \mathbf{G}_3 \end{bmatrix} \tag{60}$$

Let the degrees of freedom associated with $\mathbf{B}_2$, $\mathbf{F}_2$ and $\mathbf{G}_2$ be the coupled degrees of freedom. If there are no sensors placed on the coupled degrees of freedom, then $\mathbf{F}_2 = \mathbf{0}$ and $\mathbf{G}_2 = \mathbf{0}$ and

$$\mathbf{B}\mathbf{F}^T = \begin{bmatrix} \mathbf{B}_1\mathbf{F}_1^T & \mathbf{B}_1\mathbf{F}_2^T & \mathbf{B}_1\mathbf{F}_3^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_3\mathbf{F}_1^T & \mathbf{B}_3\mathbf{F}_2^T & \mathbf{B}_3\mathbf{F}_3^T \end{bmatrix} \text{ and } \mathbf{B}\mathbf{G}^T = \begin{bmatrix} \mathbf{B}_1\mathbf{G}_1^T & \mathbf{B}_1\mathbf{G}_2^T & \mathbf{B}_1\mathbf{G}_3^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_3\mathbf{G}_1^T & \mathbf{B}_3\mathbf{G}_2^T & \mathbf{B}_3\mathbf{G}_3^T \end{bmatrix} \tag{61}$$

Consequently, the coupling between the coupled degrees of freedom cannot be eliminated by feedback control.

□

When the mass matrix is diagonal and the damping and stiffness matrices are banded, certain degrees of freedom may be free of actuation and the eigenvalues can be assigned exactly by using full state feedback, which is illustrated in the following example.

6.1. Example 3.

Consider a five-degree-of-freedom system with matrices

$$\mathbf{M}=\mathbf{I},\ \mathbf{C}=\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \text{ and } \mathbf{K}=\begin{bmatrix} 20 & -10 & & & \\ -10 & 15 & -5 & & \\ & -5 & 10 & -5 & \\ & & -5 & 10 & -5 \\ & & & -5 & 5 \end{bmatrix}.$$

The open-loop eigenvalues are

$$\begin{aligned} \lambda_{1,6} &= -0.0462 \pm 0.7706\mathrm{i} \\ \lambda_{2,7} &= -0.3550 \pm 2.0897\mathrm{i} \\ \lambda_{3,8} &= -0.9051 \pm 3.0365\mathrm{i} \\ \lambda_{4,9} &= -1.6249 \pm 3.7653\mathrm{i} \\ \lambda_{5,10} &= -1.5688 \pm 5.0190\mathrm{i} \end{aligned}$$

The open-loop system is to be decoupled into two uncoupled subsystems. The first subsystem consists of the first three degrees of freedom with prescribed eigenvalues,

$$\begin{aligned} \mu_{1,6} &= -0.05 \pm 0.60\mathrm{i} \\ \mu_{2,7} &= -0.35 \pm 1.80\mathrm{i} \\ \mu_{3,8} &= -0.90 \pm 2.80\mathrm{i} \end{aligned}$$

and the second subsystem consists of the last two degrees of freedom with prescribed eigenvalues,

$$\begin{aligned} \mu_{4,9} &= -1.42 \pm 3.50\mathrm{i} \\ \mu_{5,10} &= -1.90 \pm 3.90\mathrm{i}. \end{aligned}$$

Modal degree of freedom constraints are imposed on the right eigenvectors so that the first three entries of the eigenvectors corresponding to the last two modes and the last two entries of the eigenvectors corresponding to the first three modes are zero. The semi-bandwidth of the damping and stiffness matrices is one. Hence, it is possible to have the first two and the last degrees of freedom free actuation. Here, two inputs are used and the force distribution matrix is chosen as,

$$\mathbf{B} = \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 5 & 4 \\ 0 & 0 \end{bmatrix}$$

The parameters $\alpha_{\mu_k,j}$ are chosen as,

$$\boldsymbol{\alpha}_k = \mathbf{null}\left(\mathbf{R}_{\mu_k(4:5,1:2)}\right)\mathbf{e}_1, \quad k = 1,2,3,6,7,8$$

and,

$$\boldsymbol{\alpha}_k = \mathbf{null}\left(\mathbf{R}_{\mu_k(1:3,1:2)}\right)\mathbf{e}_1, \qquad k = 4,5,9,10$$

where $\mathbf{e}_1$ is the 1^{st} unit vector.

The control gains are found to be,

$$\mathbf{F} = \begin{bmatrix} 3.3447 & -4.1809 \\ -0.2537 & 0.3172 \\ -2.6000 & 3.0000 \\ -0.5467 & -0.2267 \\ -2.0508 & 1.0254 \end{bmatrix} \text{ and } \mathbf{G} = \begin{bmatrix} -75.1354 & 93.9193 \\ 63.8172 & -79.7715 \\ -19.7563 & 23.4453 \\ -2.6953 & -1.1523 \\ -10.2043 & 5.1021 \end{bmatrix}$$

and the closed-loop eigenvalues are,

$$\mu_{1,6} = -0.05 \pm 0.60\text{i}$$
$$\mu_{2,7} = -0.35 \pm 1.80\text{i}$$
$$\mu_{3,8} = -0.90 \pm 2.80\text{i}$$
$$\mu_{4,9} = -1.42 \pm 3.50\text{i}$$
$$\mu_{5,10} = -1.90 \pm 3.90\text{i}$$

with eigenvectors,

$$\mathbf{w}_1=\begin{pmatrix}0.1970+0.0071i\\0.3869+0.0131i\\0.7384\\0\\0\end{pmatrix},\mathbf{w}_2=\begin{pmatrix}0.0057-0.1784i\\-0.0026-0.2974i\\-0.0630-0.3240i\\0\\0\end{pmatrix},\mathbf{w}_3=\begin{pmatrix}-0.0113-0.1808i\\-0.0651-0.2025i\\-0.0706+0.1435i\\0\\0\end{pmatrix},$$

$$\mathbf{w}_4=\begin{pmatrix}0\\0\\0\\-0.0846-0.2086i\\0.0452+0.1130i\end{pmatrix},\mathbf{w}_5=\begin{pmatrix}0\\0\\0\\-0.0926+0.1900i\\0.0340-0.0681i\end{pmatrix}$$

$$\mathbf{w}_6=\mathbf{w}_1^*,\mathbf{w}_7=\mathbf{w}_2^*,\mathbf{w}_8=\mathbf{w}_3^*,\mathbf{w}_9=\mathbf{w}_4^*,\mathbf{w}_{10}=\mathbf{w}_5^*$$

The closed-loop system matrices are,

$$\mathbf{M}=\mathbf{I}$$

$$\mathbf{C}_{CL}=\begin{bmatrix}2&-1&0&0&0\\-1&2&-1&0&0\\5.0170&-1.3806&-1.4000&0&0\\0&0&0&5.6400&5.1525\\0&0&0&-1&1\end{bmatrix}$$

$$\mathbf{K}_{CL}=\begin{bmatrix}20&-10&0&0&0\\-10&15&-5&0&0\\-112.7032&90.7257&-17.1343&0&0\\0&0&0&28.09&25.6128\\0&0&0&-5&5\end{bmatrix}$$

Thus, two independent subsystems are achieved as desired with given eigenvalues.

7. Decoupling of linear structures with banded mass matrix

In the preceding analysis the mass matrix was assumed to be diagonal (or lumped). This is an unrealistic assumption and in this section we seek to replace it with the more practical representation of a banded mass matrix. The coupling between system degrees of freedom may reasonably be assumed to be localised, as in the case of the finite-element consistent mass matrix. Here we introduce acceleration feedback (in addition to displacement and velocity feedback) to decouple the linear dynamic system with inertia interaction.

In this case, the equations of motion of the closed-loop system may be expressed as,

$$\left(\mathbf{M}-\mathbf{B}\mathbf{D}^{T}\right)\ddot{\mathbf{x}}+\left(\mathbf{C}-\mathbf{B}\mathbf{F}^{T}\right)\dot{\mathbf{x}}+\left(\mathbf{K}-\mathbf{B}\mathbf{G}^{T}\right)\mathbf{x}=\mathbf{0} \tag{62}$$

where $\mathbf{D}$, $\mathbf{F}$ and $\mathbf{G} \in \mathbb{R}^{n\times q}$ are the acceleration, velocity and displacement feedback gain matrices respectively. $\mathbf{B}\in \mathbb{R}^{n\times q}$ is the force distribution matrix. If the open-loop dynamic stiffness matrix $\mathbf{Z}(s)=\mathbf{M}s^{2}+\mathbf{C}s+\mathbf{K}$ is of semi-bandwidth t , $1\le t\le \min\{n_1, n_2\}$, then the minimum number of inputs is $q=2t$ and the force distribution matrix may be given by (51).

$$\mathbf{B}=\begin{bmatrix}\mathbf{0}_{(n_1-t)\times 2t}\\ \tilde{\mathbf{B}}_{2t\times 2t}\\ \mathbf{0}_{(n_2-t)\times 2t}\end{bmatrix} \tag{63}$$

where $\tilde{\mathbf{B}}_{2t\times 2t}$ is chosen to be invertible.

If the acceleration gain matrix is of the form,

$$\mathbf{D}=\begin{bmatrix}\mathbf{0}_{(n_1-t)\times 2t}\\ \tilde{\mathbf{D}}_{2t\times 2t}\\ \mathbf{0}_{(n_2-t)\times 2t}\end{bmatrix} \tag{64}$$

then,

$$\mathbf{B}\mathbf{D}^{T}=\begin{bmatrix}\mathbf{0}_{(n_1-t)\times(n_1-t)} & \mathbf{0}_{(n_1-t)\times 2t} & \mathbf{0}_{(n_1-t)\times(n_2-t)}\\ \mathbf{0}_{2t\times(n_1-t)} & \tilde{\mathbf{B}}_{2t\times 2t}\tilde{\mathbf{D}}^{T}_{2t\times 2t} & \mathbf{0}_{2t\times(n_2-t)}\\ \mathbf{0}_{(n_2-t)\times(n_1-t)} & \mathbf{0}_{(n_2-t)\times 2t} & \mathbf{0}_{(n_2-t)\times(n_2-t)}\end{bmatrix} \tag{65}$$

The open-loop mass matrix may be written as

$$\mathbf{M}=\begin{bmatrix}\mathbf{M}_{(1:n_1-t,\,1:n_1-t)} & \mathbf{M}_{(1:n_1-t,\,n_1-t+1:n_1)} & \mathbf{0}_{(1:n_1-t,\,n_1+1:n_1+t)} & \mathbf{0}_{(1:n_1-t,\,n_1+t+1:n)}\\ \mathbf{M}_{(n_1-t+1:n_1,\,1:n_1-t)} & \left[\begin{matrix}\mathbf{M}_{(n_1-t+1:n_1,\,n_1-t+1:n_1)}\\ \mathbf{M}_{(n_1+1:n_1+t,\,n_1-t+1:n_1)}\end{matrix}\right. & \left.\begin{matrix}\mathbf{M}_{(n_1-t+1:n_1,\,n_1+1:n_1+t)}\\ \mathbf{M}_{(n_1+1:n_1+t,\,n_1+1:n_1+t)}\end{matrix}\right] & \mathbf{0}_{(n_1-t+1:n_1,\,n_1+t+1:n)}\\ \mathbf{0}_{(n_1+1:n_1+t,\,1:n_1-t)} & & & \mathbf{M}_{(n_1+1:n_1+t,\,n_1+t+1:n)}\\ \mathbf{0}_{(n_1+t+1:n,\,1:n_1-t)} & \mathbf{0}_{(n_1+t+1:n,\,n_1-t+1:n_1)} & \mathbf{M}_{(n_1+t+1:n,\,n_1+1:n_1+t)} & \mathbf{M}_{(n_1+t+1:n,\,n_1+t+1:n)}\end{bmatrix} \tag{66}$$

where we denote the central sub-matrix as,

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{M}_{(n_1-t+1:n_1,\, n_1-t+1:n_1)} & \mathbf{M}_{(n_1-t+1:n_1,\, n_1+1:n_1+t)} \\ \mathbf{M}_{(n_1+1:n_1+t,\, n_1-t+1:n_1)} & \mathbf{M}_{(n_1+1:n_1+t,\, n_1+1:n_1+t)} \end{bmatrix} \tag{67}$$

Acceleration feedback is now applied to modify $\mathbf{M}_1$ such that

$$\mathbf{M}_1 - \tilde{\mathbf{M}}_1 = \tilde{\mathbf{B}}\tilde{\mathbf{D}}^T \tag{68}$$

where

$$\tilde{\mathbf{M}}_1 = \begin{bmatrix} \tilde{\mathbf{M}}_{(n_1-t+1:n_1,\, n_1-t+1:n_1)} & \mathbf{0}_{(n_1-t+1:n_1,\, n_1+1:n_1+t)} \\ \mathbf{0}_{(n_1+1:n_1+t,\, n_1-t+1:n_1)} & \tilde{\mathbf{M}}_{(n_1+1:n_1+t,\, n_1+1:n_1+t)} \end{bmatrix} \tag{69}$$

is prescribed to be symmetric and to make the closed-loop mass matrix $\tilde{\mathbf{M}}$ nonsingular.

$$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{M}_{(1:n_1-t,\, 1:n_1-t)} & \mathbf{M}_{(1:n_1-t,\, n_1-t+1:n_1)} & \mathbf{0}_{(1:n_1-t,\, n_1+1:n_1+t)} & \mathbf{0}_{(1:n_1-t,\, n_1+t+1:n)} \\ \mathbf{M}_{(n_1-t+1:n_1,\, 1:n_1-t)} & \tilde{\mathbf{M}}_{(n_1-t+1:n_1,\, n_1-t+1:n_1)} & \mathbf{0}_{(n_1-t+1:n_1,\, n_1+1:n_1+t)} & \mathbf{0}_{(n_1-t+1:n_1,\, n_1+t+1:n)} \\ \mathbf{0}_{(n_1+1:n_1+t,\, 1:n_1-t)} & \mathbf{0}_{(n_1+1:n_1+t,\, n_1-t+1:n_1)} & \tilde{\mathbf{M}}_{(n_1+1:n_1+t,\, n_1+1:n_1+t)} & \mathbf{M}_{(n_1+1:n_1+t,\, n_1+t+1:n)} \\ \mathbf{0}_{(n_1+t+1:n,\, 1:n_1-t)} & \mathbf{0}_{(n_1+t+1:n,\, n_1-t+1:n_1)} & \mathbf{M}_{(n_1+t+1:n,\, n_1+1:n_1+t)} & \mathbf{M}_{(n_1+t+1:n,\, n_1+t+1:n)} \end{bmatrix} \tag{70}$$

From equation (68), the acceleration feedback gain submatrix $\tilde{\mathbf{D}}$ is seen to be given by,

$$\tilde{\mathbf{D}} = \left(\mathbf{M}_1 - \tilde{\mathbf{M}}_1\right)\left(\tilde{\mathbf{B}}^T\right)^{-1} \tag{71}$$

Now, the eigenvalue problem associated with the closed-loop linear system becomes

$$\left(\mathbf{M}\mu_k^2 + \mathbf{C}\mu_k + \mathbf{K}\right)\mathbf{w}_k = \mathbf{B}\left(\mu_k^2\mathbf{D}^T + \mu_k\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{w}_k, \quad k = 1, 2, \cdots, 2n \tag{72}$$

Then,

$$\mathbf{w}_k = \alpha_{\mu_k,1}\mathbf{r}_{\mu_k,1} + \alpha_{\mu_k,2}\mathbf{r}_{\mu_k,2} + \cdots + \alpha_{\mu_k,q}\mathbf{r}_{\mu_k,q} = \mathbf{R}_{\mu_k}\boldsymbol{\alpha}_k^T \tag{73}$$

where

$$\alpha_{\mu_k,j} = \left(\mu_k^2\mathbf{d}_j^T + \mu_k\mathbf{f}_j^T + \mathbf{g}_j^T\right)\mathbf{w}_k, \ k = 1, 2, \cdots, 2n, \quad j = 1, 2, \cdots, q \tag{74}$$

$$\mathbf{R}_{\mu_k} = \left(\mathbf{M}\mu_k^2 + \mathbf{C}\mu_k + \mathbf{K}\right)^{-1}\mathbf{B}, \quad k = 1, 2, \cdots, 2n \tag{75}$$

and $\alpha_{\mu_k, j}$ are arbitrary variables and $\boldsymbol{\alpha}_k^T$ are nonzero vectors.

Equations (74) may be rewritten as,

$$\beta_{\mu_k, j} = \alpha_{\mu_k, j} - \mu_k^2 \mathbf{d}_j^T \mathbf{w}_k = \left(\mu_k \mathbf{f}_j^T + \mathbf{g}_j^T\right)\mathbf{w}_k, \; k = 1, 2, \cdots, 2n, \quad j = 1, 2, \cdots, q \tag{76}$$

or,

$$\boldsymbol{\beta}_k^T = \boldsymbol{\alpha}_k^T - \mu_k^2 \mathbf{w}_k^T \mathbf{D}, \quad k = 1, 2, \cdots, 2n \tag{77}$$

where,

$$\boldsymbol{\beta}_k^T = \begin{bmatrix} \beta_{\mu_k,1} & \beta_{\mu_k,2} & \cdots & \beta_{\mu_k,q} \end{bmatrix} \tag{78}$$

Hence, the velocity and displacement feedback control gains are obtained by solving,

$$\mathbf{P}\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} = \mathbf{A} \tag{79}$$

where,

$$\mathbf{P} = \begin{bmatrix} \mu_1 \mathbf{w}_1^T & \mathbf{w}_1^T \\ \mu_2 \mathbf{w}_2^T & \mathbf{w}_2^T \\ \vdots & \vdots \\ \mu_{2n} \mathbf{w}_{2n}^T & \mathbf{w}_{2n}^T \end{bmatrix}, \; \mathbf{A} = \begin{bmatrix} \boldsymbol{\beta}_1^T \\ \boldsymbol{\beta}_2^T \\ \vdots \\ \boldsymbol{\beta}_{2n}^T \end{bmatrix}, \tag{80}$$

The closed-loop system will be block decoupled when modal degree of freedom constraints (27) are imposed on $\mathbf{w}_k$ in (80). It is seen that the decoupling algorithm is basically similar to that presented in Section 5 except for the additional of acceleration feedback to generate a block diagonal closed-loop mass matrix.

7.1. Example 4

Consider the structure shown in Fig. 3, which consists of a beam of length $5l = 5\,\text{m}$ fixed at both ends. Assume the cross section of the beam to be rectangular with width $b = 2\text{cm}$ and height $h = 1\text{cm}$

respectively and the material of the beam to be steel with Young's modulus, $E = 2.0\times10^{11}\,\text{Pa}$, and mass density, $\rho = 7{,}800\ \text{kg/m}^3$.

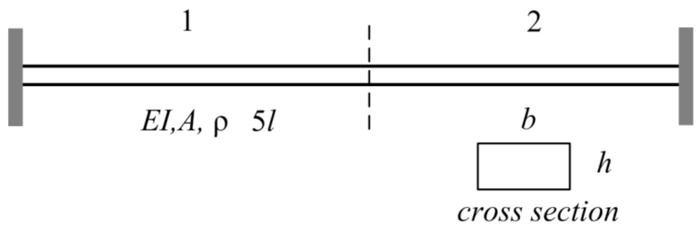

Fig. 3 A beam with both ends fixed

The beam is discretised into five beam elements of equal length, with the joints undergoing lateral and rotational displacements. The consistent-mass matrix is employed to include the inertia coupling effect. That is, the mass matrix of each beam element $\mathbf{M}_e$ is

$$\mathbf{M}_e = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}.$$

For the sake of illustration, proportional damping $\mathbf{C} = \zeta_1\mathbf{M} + \zeta_2\mathbf{K}$, ($\zeta_1 = 0.001$ and $\zeta_2 = 0.0002$) is assumed.

The open-loop eigenvalues are

$$\begin{aligned} &\lambda_{1,9} = 10^2\times(-0.0002\pm0.1309\text{i}); \quad \lambda_{2,10} = 10^2\times(-0.0013\pm0.3620\text{i}); \\ &\lambda_{3,11} = 10^2\times(-0.0051\pm0.7167\text{i}); \quad \lambda_{4,12} = 10^2\times(-0.0143\pm1.1940\text{i}); \\ &\lambda_{5,13} = 10^2\times(-0.0402\pm2.0045\text{i}); \quad \lambda_{6,14} = 10^2\times(-0.0891\pm2.9830\text{i}); \\ &\lambda_{7,15} = 10^2\times(-0.1940\pm4.4003\text{i}); \quad \lambda_{8,16} = 10^2\times(-0.3882\pm6.2186\text{i}). \end{aligned}$$

Now, as shown in Fig. 3, the beam is to be decoupled such that beam 1 of length $2.5l$ with prescribed eigenvalues

$$\begin{aligned} &\mu_{1,9} = 10^2\times(-0.001\pm0.12\text{i}); \quad \mu_{2,10} = 10^2\times(-0.002\pm0.38\text{i}); \\ &\mu_{3,11} = 10^2\times(-0.007\pm0.60\text{i}); \quad \mu_{4,12} = 10^2\times(-0.02\pm1.00\text{i}); \end{aligned}$$

is independent from beam 2 of length $2.5l$ with eigenvalues.

$$\mu_{5,13} = 10^2 \times (-0.04 \pm 2.20\text{i}); \ \mu_{6,14} = 10^2 \times (-0.09 \pm 3.50\text{i});$$
$$\mu_{7,15} = 10^2 \times (-0.22 \pm 4.00\text{i}); \ \mu_{8,16} = 10^2 \times (-0.40 \pm 6.00\text{i}).$$

The closed-loop mass submatrix is given as,

$$\tilde{\mathbf{M}}_1 = \tilde{\mathbf{M}}_{(3:6,3:6)} = \begin{bmatrix} 1.2 & 0.2 & 0 & 0 \\ 0.2 & 0.1 & 0 & 0 \\ 0 & 0 & 1.2 & 0.2 \\ 0 & 0 & 0.2 & 0.1 \end{bmatrix}$$

and the force distribution matrix is chosen to be,

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -18 & -5 & 7 \\ -1 & 8 & 3 & 17 \\ -19 & -9 & -6 & -2 \\ -4 & 1 & 5 & -21 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the acceleration feedback gain matrix is found to be

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0001 & 0.0126 & -0.0496 & -0.0089 \\ 0.0003 & 0.0178 & -0.0334 & -0.0066 \\ 0.0059 & -0.0065 & -0.0044 & 0.0070 \\ 0.0111 & 0.0066 & -0.0112 & -0.0011 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By using the proposed method, the arbitrary parameters are chosen as

$$\boldsymbol{\alpha}_k = \mathbf{null}\left(\mathbf{R}_{\mu_k(5:8,1:4)}\right)\mathbf{e}_1, \quad k = 1, \cdots 4, 9, \cdots, 12$$

and

$$\boldsymbol{\alpha}_k = \mathbf{null}\left(\mathbf{R}_{\mu_k(1:4,1:4)}\right)\mathbf{e}_1, \quad k = 5, \cdots, 8, 13, \cdots, 16$$

where $\mathbf{e}_1$ is the 1^{st} unit vector.

The velocity and displacement feedback matrices are found to be,

$$\mathbf{F} = \begin{bmatrix} -0.0437 & -0.1278 & 0.3051 & 0.0749 \\ -0.2646 & -1.8709 & 3.3882 & 0.7680 \\ 0.3407 & 2.3008 & -4.0814 & -0.9461 \\ -0.0180 & -0.0095 & 0.1289 & 0.0273 \\ -5.1846 & 18.4621 & -64.9532 & 2.4458 \\ -1.5823 & 4.5051 & -16.0528 & 0.6276 \\ -22.9302 & 66.8830 & -237.2810 & 9.0499 \\ 0.1412 & -0.5029 & 1.7722 & -0.0678 \end{bmatrix}$$

$$\mathbf{G} = 10^5 \times \begin{bmatrix} -0.0012 & 0.0026 & -0.0003 & 0.0003 \\ -0.0014 & 0.0123 & -0.0132 & -0.0023 \\ 0.0030 & -0.0111 & 0.0135 & 0.0012 \\ -0.0004 & 0.0043 & -0.0018 & -0.0005 \\ 0.4696 & -0.1181 & 0.5902 & -0.0221 \\ 0.1085 & -0.0226 & 0.1165 & -0.0031 \\ 1.5801 & -0.4320 & 2.0866 & -0.0720 \\ -0.0121 & 0.0035 & -0.0168 & 0.0006 \end{bmatrix}$$

and the closed-loop matrices are,

$$\mathbf{M}_{CL} = \begin{bmatrix} 1.1589 & 0 & 0.2006 & -0.0483 & & & & \\ 0 & 0.0297 & 0.0483 & -0.0111 & & & & \\ 0.2006 & 0.0483 & 1.2000 & 0.2000 & & & & \\ -0.0483 & -0.0111 & 0.2000 & 0.1000 & & & & \\ & & & & 1.2000 & 0.2000 & 0.2006 & -0.0483 \\ & & & & 0.2000 & 0.1000 & 0.0483 & -0.0111 \\ & & & & 0.2006 & 0.0483 & 1.1589 & 0 \\ & & & & -0.0483 & -0.0111 & 0 & 0.0297 \end{bmatrix}$$

$$
\mathbf{K}_{CL} = 10^6 \times \begin{bmatrix} 0.0080 & 0 & -0.0040 & 0.0020 & & & & \\ 0 & 0.0027 & -0.0020 & 0.0007 & & & & \\ 0.0006 & 0.0154 & -0.0066 & 0.0072 & & & & \\ -0.0006 & -0.0015 & 0.0031 & 0.0005 & & & & \\ & & & & 1.1436 & 0.2551 & 3.8470 & -0.0279 \\ & & & & -0.1419 & -0.0165 & -0.5212 & 0.0051 \\ & & & & -0.0040 & -0.0020 & 0.0080 & 0 \\ & & & & 0.0020 & 0.0007 & 0 & 0.0027 \end{bmatrix}
$$

$$
\mathbf{C}_{CL} = 10^3 \times \begin{bmatrix} 0.0016 & 0 & -0.0008 & 0.0004 & & & & \\ 0 & 0.0005 & -0.0004 & 0.0001 & & & & \\ -0.0020 & -0.0220 & 0.0286 & 0.0003 & & & & \\ -0.0008 & -0.0084 & 0.0103 & -0.0003 & & & & \\ & & & & -0.3156 & -0.0846 & -1.2401 & 0.0091 \\ & & & & 0.3369 & 0.0831 & 1.2174 & -0.0091 \\ & & & & -0.0008 & -0.0004 & 0.0016 & 0 \\ & & & & 0.0004 & 0.0001 & 0 & 0.0005 \end{bmatrix}
$$

with the prescribed eigenvalues. The two independent beams are obtained with given eigenvalues.

8. Conclusion

In the theoretical study reported here, it is found that block diagonalisation of the system damping and stiffness matrices is achievable when the open-loop eigenvalues are controllable. In the case of velocity and displacement feedback, the mass matrix is practically restricted to the diagonal (lumped mass) form. This restriction can be lifted to allow for bandedness of the mass matrix when acceleration feedback is included together with velocity and displacement feedback. In both cases the closed-loop system is decoupled to form independent substructures and it is demonstrated that eigenvalues can be assigned to the substructures separately. The procedure is based on eigenstructure assignment using the method of receptances. In the case of banded system matrices, the number of actuators required can be reduced to twice of the semi-bandwidth.

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Appendix One: Left eigenvalue problem

From equation (30)

$$\mathbf{y}_k^T\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\left(\mathbf{K}-\mathbf{B}\mathbf{G}^T\right) & -\mathbf{M}^{-1}\left(\mathbf{C}-\mathbf{B}\mathbf{F}^T\right)\end{bmatrix}=\mu_k\mathbf{y}_k^T \tag{81}$$

where $\mathbf{y}_k^T=\left(\mathbf{y}_{\mathrm{U}k}^T \quad \mathbf{y}_{\mathrm{L}k}^T\right)$. Thus,

$$\left(-\mathbf{y}_{\mathrm{L}k}^T\mathbf{M}^{-1}\left(\mathbf{K}-\mathbf{B}\mathbf{G}^T\right) \quad \mathbf{y}_{\mathrm{U}k}^T-\mathbf{y}_{\mathrm{L}k}^T\mathbf{M}^{-1}\left(\mathbf{C}-\mathbf{B}\mathbf{F}^T\right)\right)=\left(\mu_k\mathbf{y}_{\mathrm{U}k}^T \quad \mu_k\mathbf{y}_{\mathrm{L}k}^T\right) \tag{82}$$

$$\mu_k\mathbf{y}_{\mathrm{U}k}^T=-\mathbf{y}_{\mathrm{L}k}^T\mathbf{M}^{-1}\left(\mathbf{K}-\mathbf{B}\mathbf{G}^T\right) \tag{83}$$

and

$$\mu_k\mathbf{y}_{\mathrm{L}k}^T=\mathbf{y}_{\mathrm{U}k}^T-\mathbf{y}_{\mathrm{L}k}^T\mathbf{M}^{-1}\left(\mathbf{C}-\mathbf{B}\mathbf{F}^T\right) \tag{84}$$

Combining equations (83) and (84) leads to

$$\left(\mathbf{y}_{\mathrm{L}k}^T\mathbf{M}^{-1}\right)\mathbf{M}\mu_k^2+\left(\mathbf{y}_{\mathrm{L}k}^T\mathbf{M}^{-1}\right)\left(\mathbf{C}-\mathbf{B}\mathbf{F}^T\right)\mu_k+\left(\mathbf{y}_{\mathrm{L}k}^T\mathbf{M}^{-1}\right)\left(\mathbf{K}-\mathbf{B}\mathbf{G}^T\right)=\mathbf{0} \tag{85}$$

Hence $\mathbf{y}_{\mathrm{L}k}^T\mathbf{M}^{-1}$ is the left eigenvector associated with μ_k, i.e.

$$\mathbf{y}_{\mathrm{L}k}^T\mathbf{M}^{-1}=\mathbf{z}_k^T \tag{86}$$

References

[1] B. Morgan, Jr., The synthesis of linear multivariable systems by state-variable feedback, Automatic Control, IEEE Transactions on, 9 (1964) 405-411.

[2] P.L. Falb, W. Wolovich, Decoupling in the design and synthesis of multivariable control systems, Automatic Control, IEEE Transactions on, 12 (1967) 651-659.

[3] E. Gilbert, The Decoupling of Multivariable Systems by State Feedback, SIAM Journal on Control, 7 (1969) 50-63.

[4] J. Descusse, J.F. Lafay, M. Malabre, Solution to Morgan's problem, Automatic Control, IEEE Transactions on, 33 (1988) 732-739.

[5] P.N. Paraskevopoulos, F.N. Koumboulis, A new approach to the decoupling problem of linear time-invariant systems, Journal of the Franklin Institute, 329 (1992) 347-369.

[6] J.W. Howze, Necessary and sufficient conditions for decoupling using output feedback, Automatic Control, IEEE Transactions on, 18 (1973) 44-46.

[7] M.J. Denham, A necessary and sufficient condition for decoupling by output feedback, Automatic Control, IEEE Transactions on, 18 (1973) 535-536.
[8] J. Descusse, A necessary and sufficient condition for decoupling using output feedback, International Journal of Control, 31 (1980) 833-840.
[9] A. Morse, W. Wonham, Decoupling and Pole Assignment by Dynamic Compensation, SIAM Journal on Control, 8 (1970) 317-337.
[10] M.M. Bayoumi, T. Duffield, Output feedback decoupling and pole placement in linear time-invariant systems, Automatic Control, IEEE Transactions on, 22 (1977) 142-143.
[11] P.N. Paraskevopoulos, F.N. Koumboulis, Decoupling and pole assignment in generalised state space systems, Control Theory and Applications, IEE Proceedings D, 138 (1991) 547-560.
[12] S. Sato, P.V. Lopresti, On the generalization of state feedback decoupling theory, Automatic Control, IEEE Transactions on, 16 (1971) 133-139.
[13] P.N. Paraskevopoulos, S.G. Tzafestas, Group decoupling theory for a generalized linear multivariable control system, International Journal of Systems Science, 6 (1975) 239-248.
[14] H. Hikita, Block decoupling and arbitrary pole assignment for a linear right-invertible system by dynamic compensation, International Journal of Control, 45 (1987) 1641-1653.
[15] J. Descusse, Block noninteracting control with (non)regular static state feedback: A complete solution, Automatica, 27 (1991) 883-886.
[16] G. Basile, G. Marro, A state space approach to noninteracting controls, Ricerchi di Automatica, 1 (1970) 68-77.
[17] S.M. Sato, P.V. Lopresti, New results in multivariable decoupling theory, Automatica, 7 (1971) 499-508.
[18] M. Malabre, J.A. Torres-Munoz, Block Decoupling by Precompensation Revisited, IEEE Transactions on Automatic Control, 52 (2007) 922-926.
[19] M.J. Hautus, M. Heyman, I.E.E.E. Trans, autom. Control, 28 (1983) 823.
[20] C. Commault, J.M. Dion, J.A. Torres, Minimal structure in the block decoupling problem with stability, Automatica, 27 (1991) 331-338.
[21] Q.-G. Wang, Decoupling with internal stability for unity output feedback systems, Automatica, 28 (1992) 411-415.
[22] Q.-G. Wang, Y. Yang, Transfer function matrix approach to decoupling problem with stability, Systems & Control Letters, 47 (2002) 103-110.
[23] Q.-G. Wang, Decoupling Control, Heidelberg : Springer, 2006.
[24] L. Ching-An, Necessary and sufficient conditions for existence of decoupling controllers, Automatic Control, IEEE Transactions on, 42 (1997) 1157-1161.
[25] Q.-G. Wang, Decoupling Control in: Lecture Notes in Control and Information Sciences: 285, Heidelberg : Springer, 2006., 2006.
[26] E.C. Zacharenakis, Input-output decoupling and disturbance rejection problems in structural analysis, Computers & Structures, 55 (1995) 441-451.
[27] E.C. Zacharenakis, On the input-output decoupling with simultaneous disturbance attenuation and h∞ optimization problem in structural analysis, Computers & Structures, 60 (1996) 627-633.
[28] Q.S. Li, J.Q. Fang, A.P. Jeary, D.K. Liu, Decoupling control law for structural control implementation, International Journal of Solids and Structures, 38 (2001) 6147-6162.
[29] C. He, A.J. Laub, V. Mehrmann, Placing plenty of poles is pretty preposterous, in: DFG-Forschergruppe Scientific Parallel Computing, Preprint 95-17, Fak. f. Mathematik. TU Chemnitz-Zwickau, D-09107, Chemnitz, FRG, 1995.
[30] B.N. Datta, F. Rincón, Feedback stabilization of a second-order system: A nonmodal approach, Linear Algebra and its Applications, 188–189 (1993) 135-161.
[31] D.J. Inman, Active modal control for smart structures, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 359 (2001) 205-219.
[32] Y.M. Ram, J.E. Mottershead, Multiple-input active vibration control by partial pole placement using the method of receptances, Mechanical Systems and Signal Processing, 40 (2013) 727-735.

[33] G. Strang, T. Nguyen, The interplay of ranks of submatrices, SIAM Review, 46 (2004) 637-646.