The Relationship Between CVaR and Option Prices with a Closed Form Solution

Abstract

Options and CVaR (conditional value at risk) are significant areas of research in their own right, moreover, both are important to risk management and understanding of risk. Despite the importance and the overlap of interests in CVaR and options the literature relating the two is virtually non-existent. In this paper we derive a model-free, simple and closed-form analytic equation that determines the CVaR associated with a put option. This relation is model free and is applicable in complete and incomplete markets. We show that we can account for implied volatility effects using the CVaR risk of options. We show how the relation between options and CVaR has important risk management implications, particularly in terms of integrated risk management and preventing arbitrage opportunities. We conduct numerical experiments to demonstrate obtaining CVaR from empirical options data.

Key words: options, CVaR, VaR, risk management, risk measurement, implied volatility, volatility smile, arbitrage, put-call parity.

1 Introduction and Outline of Paper

Options and risk measures (such as CVaR (conditional value at risk)) have become important areas of research. Options have grown into a multi-trillion dollar industry (Stout, 1999), they are applied in risk management for hedging, they are popular for speculation due to their limited loss and low margin requirements. Risk measures are vital to risk management and the events of the global credit crunch and past financial crises (see for instance (Stonham, 1999)) have highlighted their importance. Poor risk measurement and management can result in bankruptcies and threaten collapses of an entire finance sector (Kabir and Hassan, 2005). CVaR has become a popular risk management tool (Maurer et al., 2009) due to its relation to VaR (value at risk) and it is an informative risk measure.

Despite the importance of options and CVaR in their own right and their applications in risk management, to the best of our knowledge there is no literature relating CVaR and options. A relation between options and CVaR would not only improve our knowledge and practice of risk management but also shed insight on options and CVaR themselves. In this paper we investigate the relation between CVaR and European options by examining the closed form analytic relation that we have derived.

The outline of the paper is as follows: first we derive a closed form analytic relation between CVaR and European options by making use of the relation between the Expected Regret and CVaR risk measures. We also explain some of the significance aspects of the CVaR-option relation. In the next section we investigate the relation between implied volatility and the CVaR of options. In particular, we show how the implied volatility effects, where implied volatility is considered a measure of expensiveness, can be explained when we measure option risk in terms of CVaR.

In the next section we show how theorems and models affecting option pricing (e.g. risk neutral measures) also affect CVaR risk measurement. We show that options and CVaR, contrary to common practice and theory, must be modelled and managed in a consistent way to avoid serious mispricing issues (such as arbitrage opportunities) and enable integrated risk management. This has the significant consequence that, contrary to common practice, we cannot assume CVaR surfaces are more predictable over the short term than compared to the long term. In the next section we conduct numerical experiments on empirical options data, obtaining CVaR over different option strikes. We finally end with a conclusion.

2 The Relation of CVaR and Options

In this section we firstly review CVaR and options, we then derive the closed form analytic relation between CVaR and options, explaining some of the significance aspects of the relation.

2.1 Review of Options and CVaR

As the growth in financial trading of shares and various derivatives has increased over the passed 50 years this has necessitated a need to quantitatively measure and manage an institution's risk (Dowd, 2011). This has been further supported by international regulatory requirements, such as the Basel Accord.

A portfolio's price V(T) at time T consisting of n assets $X_i(T)$ is

$$V(T) \quad = \quad \sum_{i=1}^{n} w_i X_i(T),$$

where i is the asset index where i $\in \{1, ..., n\}$, n is the number of assets in the portfolio, w_i is the number of shares of asset X_i. A portfolio V(T) has its loss Z(T) defined as

$$Z(T) = V(t) - V(T) = \sum_{i=1}^{n} w_i z_i(T),$$

where time t is the time now and the loss of asset i at time T is

$$z_i(T) = X_i(t) - X_i(T).$$

Hence negative losses are gains.

To define CVaR on a portfolio we must define VaR. We define VaR as the loss associated with the quantile β on the loss distribution of Z(T). Mathematically VaR is defined as (Szegö, 2005):

$$F^{\mathbb{P}}(Z(T) \leq VaR) \quad = \quad \beta^{\mathbb{P}},$$

where $F^{\mathbb{P}}(.)$ is the cumulative probability distribution function, under probability measure $\mathbb{P}$. We note that β can refer to the loss distribution quantile not just for a portfolio but also for a single asset (n=1). $CVaR^{\mathbb{P}}$ is defined as (Dowd, 2011):

$$CVaR^{\mathbb{P}} = E^{\mathbb{P}}[Z(T) \mid Z(T) > VaR].$$

In words $CVaR^{\mathbb{P}}$ is the expected loss, given that the loss has exceeded VaR. $CVaR^{\mathbb{P}}$ is bound almost surely for all losses by:

$$CVaR^{\mathbb{P}} \geq VaR. \tag{1}$$

Options provide the ability to hedge out risk: they can insure against losses beyond a chosen threshold. Furthermore, hedging a portfolio's risk with an option is cheaper and more convenient than re-balancing (or liquidating) the portfolio. In addition to hedging applications, options can be more attractive to speculators (rather than purchasing the underlying) due to the limited loss. Also, option trading can normally be executed on a far higher level of leverage compared to trading stocks, therefore offering potentially higher returns for the same initial deposit.

A portfolio option is essentially the same as a single asset option but the single asset is now a portfolio V(t). Following Kwok (Kwok, 1998), under the method of conditional expectations a portfolio call option $L^{\mathbb{P}}(V(t), t, T, K)$ is a call option whose underlying is a portfolio V(t) of assets at time t, evaluated under probability measure $\mathbb{P}$, with expiry T and strike K, where the strike applies to the overall portfolio value V(t) (not to each individual asset). Under risk neutral valuation we can define $L^{\mathbb{Q}}(V(t), t, T, K)$ (Benth, 2004):

$$L^{\mathbb{Q}}(V(t), t, T, K) = e^{-r(T-t)} E^{\mathbb{Q}}[V(T) - K]^+ = e^{-r(T-t)} E^{\mathbb{Q}} \left[\sum_{i=1}^{n} w_i X_i(T) - K \right]^+, \quad (2)$$

where $\mathbb{Q}$ is the risk neutral measure on a portfolio and r is the risk free rate. Similarly a portfolio put option $R^{\mathbb{Q}}(V(t), t, T, K)$ under risk neutral valuation is defined as:

$$R^{\mathbb{Q}}(V(t), t, T, K) \quad = \quad e^{-r(T-t)} E^{\mathbb{Q}}[K - V(T)]^+ = e^{-r(T-t)} E^{\mathbb{Q}} \left[K - \sum_{i=1}^{n} w_i X_i(T) \right]^+ \quad (3)$$

It will also be useful to express a portfolio option without discounting; for a non-discounted option R evaluated under probability measure $\mathbb{P}$, this is denoted by $\widetilde{R}^{\mathbb{P}}$ where

$$\widetilde{R}^{\mathbb{P}}(V(t), t, T, K) \quad = \quad E^{\mathbb{P}}[K - V(T)]^+.$$

Despite the importance and overlap of interests in CVaR and options, the relation between the two has not been investigated but rather each has been investigated separately. The area of options has been significantly developed since the Black-Scholes option pricing model (Black and Scholes, 1973) e.g. to take account of different volatility models (time dependence (Merton, 1976), local volatility (see (Dupire, 1994),(Derman and Kani, 1994),(Cox and Ross, 1976)) and stochastic volatility (see for example (Scott, 1987), (Hull and White, 1987))). Other developments include option pricing under dividends (Musiela and Rutkowski, 2005), transaction costs (Leland, 1985), foreign currencies (Garman and Kohlhagen, 1983), portfolios (Mitra, 2010) and calibration (Mitra and Date, 2010). CVaR research has generally been concerned with modelling various distributions, portfolios, risk factors (e.g. liquidity risk, currency risk) and improving computation. For example, Rockefellar and Uryasev (Rockafellar and Uryasev, 2000) provide a linear program to minimise a portfolio's CVaR by adjusting the asset weights.

In (Boyle et al., 2002), expected shortfall (which is related to CVaR) is used to measure the risk of an unhedged position of derivative instruments, when we use a discrete-time binomial model. In (Pochart and Bouchaud, 2004) the authors develop a Monte Carlo based computational method for pricing and hedging options. Their aim is to minimise hedging errors and this done by minimising expected shortfall.

In (Wylie et al., 2010) expected shortfall and VaR are used as metrics to determine the optimal hedging position in a replicating portfolio (specifically the hedge ratio or Delta). This is applied to minimise hedging risk and they conclude it leads to undesirable behaviour. In (Basak and Shapiro, 2001) VaR and expected shortfall is discussed in terms of optimising a portfolio choice. In (Duffie and Pan, 1997) a good introduction is given to VaR; implied volatility is also discussed but separately from VaR.

The work presented in this paper differs from the previously mentioned papers in two crucial aspects. Firstly, the previously mentioned papers have no discussion on the analytic relation between CVaR and option prices. Secondly, there is no analysis of CVaR and its relation on implied volatility or the implications of CVaR in terms of risk management issues (we note that (Basak and Shapiro, 2001) discusses stock volatility but not implied volatility, which is an entirely different area).

2.2 Derivation of CVaR and Options Relations

To derive the relation between CVaR and options, we must firstly define the Expected Regret risk measure. We then show that CVaR is related to options by relating Expected Regret to put options and then using the relation of CVaR and Expected Regret to relate CVaR to options. Now Expected Regret is the expected loss beyond a threshold loss value $\bar{K}$ with respect to the overall portfolio loss Z(T) (not to the individual assets), at time T, under a probability measure $\mathbb{P}$:

$$ER^{\mathbb{P}}_{\bar{K}}(Z(T)) \quad = \quad E^{\mathbb{P}}[Z(T) - \bar{K}]^{+}.$$

(We note that $\bar{K}$ refers to a value on the loss distribution of Z(T), whereas option strike K refers to a value on an underlying's distribution V(T)). We can choose any value of $\bar{K}$ and chosen correctly enables us to relate Expected Regret to options. If we choose

$$\bar{K} \quad = \quad V(t) - K,$$

then we set the value at which the portfolio loss threshold $\bar{K}$ occurs at the same value as the option strike K, if there were an option on V(t).

We are now ready to state the relation between Expected Regret and an undiscounted portfolio put option.

Proposition 1. *The Expected Regret on a portfolio $ER^{\mathbb{P}}_{\bar{K}}(Z(T))$ is identical to an undiscounted portfolio put option $\widetilde{R}^{\mathbb{P}}(V(t), t, T, K)$ when both are calculated under the same probability measure $\mathbb{P}$, with $K = V(t) - \bar{K}$:*

$$\begin{aligned} ER^{\mathbb{P}}_{\bar{K}}(Z(T)) \quad &= \quad \widetilde{R}^{\mathbb{P}}(V(t), t, T, K), \\ \text{or } ER^{\mathbb{P}}_{\bar{K}}(Z(T)) \quad &= \quad E^{\mathbb{P}}[K - V(T)]^{+}. \end{aligned}$$

Moreover, under the risk neutral measure $\mathbb{Q}$, a portfolio put option $R^{\mathbb{Q}}(V(t), t, T, K)$ is identical to the Expected Regret on a portfolio discounted at the risk free rate r, $e^{-r(T-t)}ER^{\mathbb{Q}}_{\bar{K}}(Z(T))$:

$$e^{-r(T-t)}ER^{\mathbb{Q}}_{\bar{K}}(Z(T)) = R^{\mathbb{Q}}(V(t), t, T, K).$$

Proof: see Appendix, Proof 1.
Hence the Expected Regret risk measure and options are related and so we can calculate either from each other.

Now to relate CVaR to options we make use of the relation of CVaR to Expected Regret. $CVaR^{\mathbb{P}}$ is related to $ER^{\mathbb{P}}_{\bar{K}}(Z(T))$ by (Szegö, 2005):

$$CVaR^{\mathbb{P}} \quad = \quad VaR + \left(\frac{1}{1 - \beta^{\mathbb{P}}}\right) ER^{\mathbb{P}}_{\bar{K}}, \tag{4}$$

where $\bar{K}$= VaR. We note that we have set portfolio loss $\bar{K}$ at VaR (this also means that $K = V(t) - \bar{K}$). Since ER is related to CVaR and we have already established a relation between options and ER, therefore there exists a relation between options and CVaR, which we will now state.

Proposition 2. *Under any probability measure $\mathbb{P}$, the relation between $CVaR^{\mathbb{P}}$ and an undiscounted portfolio put option $\widetilde{R}^{\mathbb{P}}(V(t), t, T, K)$, where $K = V(t) - VaR$, is given by:*

$$\widetilde{R}^{\mathbb{P}}(V(t), t, T, K) \quad = \quad (CVaR^{\mathbb{P}} - VaR).(1 - \beta^{\mathbb{P}}).$$

Moreover, under the risk neutral measure $\mathbb{Q}$, a portfolio put option $R^{\mathbb{Q}}(V(t),t,T,K)$ is related to $CVaR^{\mathbb{Q}}$ by

$$\begin{aligned} R^{\mathbb{Q}}(V(t),t,T,K) &= e^{-r(T-t)}(CVaR^{\mathbb{Q}} - VaR).(1-\beta^{\mathbb{Q}}), \\ &= \frac{\partial R}{\partial K}(CVaR^{\mathbb{Q}} - VaR). \end{aligned}$$

Proof:

First, we know $ER^{\mathbb{P}}_{\bar{K}}$ is an undiscounted put option by Proposition 1

$$ER^{\mathbb{P}}_{\bar{K}}(Z(T)) = E^{\mathbb{P}}[V(T)-K]^{+} = \widetilde{R}^{\mathbb{P}}(V(t),t,T,K), \tag{5}$$

where $\bar{K} = VaR$. If we substitute $\widetilde{R}^{\mathbb{P}}$ for $ER^{\mathbb{P}}_{\bar{K}}(Z(T))$ from equation (5) into equation (4) we have:

$$CVaR^{\mathbb{P}} = VaR + \left(\frac{1}{1-\beta}\right).\widetilde{R}^{\mathbb{P}}(V(t),t,T,K), \tag{6}$$

$$\text{so that } \widetilde{R}^{\mathbb{P}}(V(t),t,T,K) = (CVaR^{\mathbb{P}} - VaR).(1-\beta^{\mathbb{P}}). \tag{7}$$

Additionally, under a risk neutral measure $\mathbb{Q}$ with discounting under the risk free rate r we have:

$$R^{\mathbb{Q}}(V(t),t,T,K) = (e^{-r(T-t)})(CVaR^{\mathbb{Q}} - VaR).(1-\beta^{\mathbb{Q}}). \tag{8}$$

Now by the Breeden-Litzenberger relationship it has been proven (Breeden and Litzenberger, 1978) that

$$\frac{\partial P}{\partial K} = e^{-r(T-t)}F^{\mathbb{Q}}(X(T) \leq K), \tag{9}$$

$$= e^{-r(T-t)}(1-\beta^{\mathbb{Q}}). \tag{10}$$

For a portfolio put R we also have

$$\frac{\partial R}{\partial K} = e^{-r(T-t)}(1-\beta^{\mathbb{Q}})$$

and so we can re-express equation (8) as

$$R^{\mathbb{Q}}(V(t),t,T,K) = \frac{\partial R}{\partial K}(CVaR^{\mathbb{Q}} - VaR).$$

Note that for a single asset put option we have:

$$P^{\mathbb{Q}}(X(t),t,T,K) = (e^{-r(T-t)})(CVaR^{\mathbb{Q}} - VaR).(1-\beta^{\mathbb{Q}}). \tag{11}$$

Alternatively, using the Breeden-Litzenberger relation (Breeden and Litzenberger, 1978) equation (11) becomes

$$P^{\mathbb{Q}}(X(t),t,T,K) = \frac{\partial P}{\partial K}(CVaR^{\mathbb{Q}} - VaR).$$

Remark 1. *As option prices are independent of risk preference we can value them under any probability measure, provided we apply the correct discount rate. Therefore we can also write*

$$R^{\mathbb{P}}(V(t),t,T,K) \;=\; e^{-\tilde{r}(T-t)}(CVaR^{\mathbb{P}} - VaR).(1-\beta^{\mathbb{P}}),$$

for any probability measure $\mathbb{P}$, where $\tilde{r}$ is the correct discount factor for probability measure $\mathbb{P}$.

We now have an equation for relating options and CVaR. One can see that once $CVaR^{\mathbb{P}}, VaR$ and $\beta^{\mathbb{P}}$ are fully specified, we can analytically calculate the put option price required to hedge the CVaR risk. Similarly, in specifying a put option price *implies* a value on the CVaR risk of the underlying at that particular strike (or VaR). This suggests a link between CVaR risk and options, which will be explored later, but also that theories governing options must also affect CVaR, which is important for consistent risk management.

Option prices are sometimes expressed in terms of cumulative probabilities and in such cases it is possible to obtain CVaR more directly. It can be proven that we can re-express $P^{\mathbb{Q}}$ as

$$\begin{aligned} P^{\mathbb{Q}} &= e^{-r(T-t)}E^{\mathbb{Q}}[K - X(T)|K > X(T)], &(12)\\ &= e^{-r(T-t)}E^{\mathbb{Q}}[K|K > X(T)] - e^{-r(T-t)}E^{\mathbb{Q}}[X(T)|K > X(T)], &(13)\\ &= Ke^{-r(T-t)}F^{\mathbb{Q}}(K > X(T)) - X(t)F^{\mathbb{Q}*}(K > X(T)), &(14)\end{aligned}$$

where $\mathbb{Q}*$ is the Martingale probability measure with X(t) as the numeraire asset (that is the Radon-Nikodym derivative is given by $d\mathbb{Q}* = e^{-r(T-t)}(X(T)/X(t))d\mathbb{Q}$). If we rearrange equation (11) we have

$$CVaR^{\mathbb{Q}} = \frac{P^{\mathbb{Q}}e^{r(T-t)}}{1-\beta^{\mathbb{Q}}} + VaR = \frac{P^{\mathbb{Q}}e^{r(T-t)}}{1-\beta^{\mathbb{Q}}} + (X(t) - K). \qquad (15)$$

Therefore substituting equation (14) into equation (15) gives

$$CVaR^{\mathbb{Q}} \;=\; \frac{KF^{\mathbb{Q}}(K > X(T)) - X(t)e^{r(T-t)}F^{\mathbb{Q}*}(K > X(T))}{1-\beta^{\mathbb{Q}}} + (X(t) - K). \qquad (16)$$

Now

$$1 - \beta^{\mathbb{Q}} \;=\; F^{\mathbb{Q}}(K > X(T)), \qquad (17)$$

(see Appendix, Proof 2 for more information). Therefore substituting equation (17) into equation (16) becomes

$$\begin{aligned} CVaR^{\mathbb{Q}} &= \frac{(KF^{\mathbb{Q}}(K > X(T)) - X(t)e^{r(T-t)}F^{\mathbb{Q}*}(K > X(T))}{F^{\mathbb{Q}}(K > X(T))} + (X(t) - K),\\ &= X(t)\left(1 - \frac{e^{r(T-t)}F^{\mathbb{Q}*}(K > X(T))}{F^{\mathbb{Q}}(K > X(T))}\right).\end{aligned}$$

Hence we can obtain closed form expressions for CVaR when options are expressed as in equation (14).

2.3 Significance of the CVaR-Option Relation

The CVaR-option relation (equation (11)) has a number of significant aspects, which we shall now discuss.

Hedging out CVaR Risk

Even though in risk management we can quantify a portfolio's risk by some risk measure (e.g. CVaR, VaR), the choices for reducing risk with respect to some risk measure are limited. One method for reducing risk is to liquidate the portfolio but this can be expensive as we may realise investment losses, incur significant transaction costs, liquidity costs and sacrifice future gains.

We can also reduce risk in a portfolio by purchasing options as this can be a cheaper alternative to liquidation. However, the application of options is currently done without understanding its implication on CVaR; the options are simply bought to provide a 'floor' on losses. This can lead to incorrect risk management of the portfolio to reduce portfolio CVaR. For instance, the options may not be purchased at the correct strike (when measuring risk with respect to CVaR at a given threshold) and we may find that the market option prices are significantly higher than equation (15) implies, hence it may be better not to hedge out risk with options.

If we use equation (15) for hedging out CVaR risk in portfolios we can more effectively manage our portfolio's risk. Firstly, using equation (15) we understand that we can fully remove CVaR risk (at a given threshold) by purchasing put options with strike K by K = V(t)-VaR. This is financially possible because a put option 'insures' losses beyond its strike, hence the CVaR is reduced to 0 because the option will hedge out those losses.

Secondly, equation (15) tells us the cost of hedging out the CVaR risk with put options. As the underlyings and the options are traded in separate markets it is possible for the put options given by equation (15) (for a given CVaR) to differ from traded option values. Consequently, the options market may be charging significantly higher option prices than calculated by equation (15), in which case it may be cheaper to liquidate the portfolio to reduce CVaR. We should note that option prices may significantly differ due to high transaction costs and liquidity effects, which is particularly relevant when markets are in a 'stressed' state.

Put Option Writer's Risk

A put option seller who sells option P is liable for its payoff when it finishes in the money, that is max(K-X(T),0). The seller is therefore interested in the risk he exposes himself to each time he sells an option. In fact selling options is considered significantly riskier than buying options. The equation (15) tells us the CVaR risk a put option seller (also known as the writer) bears, therefore this equation provides a useful risk management tool.

In addition to quantifying the risk a put option writer exposes himself, equation (15) also enables us to understand the variables that affect the writer's risk. For instance, the writer's CVaR risk is not just a function of the option's pricing variables (X(t), K, T, etc.) but also β, hence the loss's distribution also affects the writer's risk. Furthermore, as $0 < \beta < 1$ then the writer's CVaR risk is always a multiple of the option price and the risk increases significantly with small increases in β.

Modelling Applicability, Tractability and Parsimonious Implementation

The CVaR-option relation (equation (11)) is applicable to a wide variety of models and applications. This is because the relation is model independent, hence it is applicable to a wide variety of models and applications. The relation makes no assumptions on distributions, risk preference or market completeness, so it can be applied to a variety of option models and different underlying assets.

The CVaR-option relation (equation (11)) is also an analytically tractable equation, therefore one can analytically derive the CVaR for practically any underlying asset relating to an option model. This is particularly useful as many option pricing models have distribution properties for their underlying assets that make analytical derivation of CVaR practically intractable. To obtain CVaR from option models using equation (11), all we must know are β, VaR(=X(t)-K)) and the risk free rate (X(t),K and r are all observable).

We now give example of obtaining CVaR from the Black-Scholes Model. Under the Black-Scholes option pricing model we price a put option P by (Hull, 2000):

$$P = Ke^{-r(T-t)}\Phi^{\mathbb{P}}(-d_2) - X(t)\Phi^{\mathbb{P}}(-d_1), \tag{18}$$

where d_1, d_2 are given in the Appendix and $\Phi^{\mathbb{P}}(.)$ is the standard normal cumulative distribution function, under the probability measure $\mathbb{P}$. Now under the Black-Scholes model it is known that

$$\begin{aligned} F^{\mathbb{Q}}(K > X(T)) &= \Phi^{\mathbb{P}}(-d_2) \\ \text{and } F^{\mathbb{Q}*}(K > X(T)) &= \Phi^{\mathbb{P}}(-d_1). \end{aligned}$$

This can be verified by comparing terms in equations (18) and (14). Now using equation (18) we have

$$CVaR^{\mathbb{Q}} \;=\; X(t)\left(1 - \frac{e^{r(T-t)}\Phi^{\mathbb{P}}(-d_1)}{\Phi^{\mathbb{P}}(-d_2)}\right). \tag{19}$$

Another advantage of the CVaR-option relation is that we do not require complex parameter estimation techniques to estimate CVaR from options or vice versa. As mentioned previously, to obtain CVaR from options, we must know VaR(=X(t)-K)) and r and all of these are empirically observable so no estimation is required. To determine β there exist many parsimonious techniques for quantile estimation and some option pricing models may analytically specify β (e.g. Heston model (Heston, 1993)). One can also apply the model free Breeden-Litzenberger relationship (equation (9)) to determine the risk neutral $\beta^{\mathbb{Q}}$ directly from options data.

Observability of CVaR Risk

Most risk measures are not observable in the market, consequently they must be calculated from some model. This can cause risk management problems because models generally do not accurately reflect the market's view on risk (regardless of the risk measure). For instance, a risk model may determine risk to be quite low over the next few days, however the market's view may be the complete opposite.

Using equation (11) implies we can observe the market's view of CVaR from traded option data (and estimating β). This is advantageous because we can take into account the market's view on risk, rather than relying purely on an institution's model. Additionally, as the market is generally considered to take into account forward looking information, it can be considered a better predictor of risk compared to institutional models.

3 CVaR and its Relation to Implied Volatility

The implied volatility is defined as the volatility consistent with the quoted option price and parameters using a Black-Scholes option pricing model. Implied volatility is a critical value in option models, in fact in industry options are sometimes priced by implied volatility as the input (Gkamas, 2001) (rather than as the output). The implied volatility has been observed to vary with K and T and its variation with K (for a given T) is known as the smile effect.

The implied volatility is considered a measure of option expensiveness, where expensiveness is defined in terms of price and risk. This has been investigated by numerous researchers due to its importance in option pricing. In this section we show how option expensiveness measured on a CVaR basis accounts for implied volatility effects, since CVaR is a measure of risk. In order to investigate implied volatility in this section we first define a measure of option risk using CVaR. We then show how CVaR is related to implied volatility and accounts for implied volatility as a measure of expensiveness. We also illustrate this using empirical option data.

3.1 CVaR as a Measure of Option Risk

Presently there is little literature on option risk measurement however, we propose measuring an option's risk along the same lines of reasoning we would measure an asset's risk, that is in terms of the quantity and probability of loss. An asset $X(t_1)$ is considered to have a positive loss if $X(t_1) > X(t_2)$ and $t_1 > t_2$. Similarly, we define a put option's positive loss when the option finishes in the money. Hence our measure of put option risk is in terms of the quantity and probability of expiring in the money; losses when the option is out of the money are all 0 and so their probabilities are irrelevant. Thus an option that is more likely to finish in the money is considered riskier.

If we rearrange equation (11) we have

$$\overline{CVaR} = (CVaR^{\mathbb{Q}} - VaR) = \frac{Pe^{r(T-t)}}{(1-\beta^{\mathbb{Q}})},$$

and we can consider $\overline{CVaR}$ as a measure of put option risk. The justification as a risk measure is as follows: firstly, it measures in the money losses and probability. We are not concerned with losses when the condition $X(T) > K$ is true because the put option will not pay out under that condition. $\overline{CVaR}$ is able to measure in the money losses because by definition CVaR measures risk based on the $X(T) < K$ condition. Secondly, the option only pays out K-X(T) (if $X(T) < K$) and not CVaR, therefore we should subtract VaR from CVaR, giving (CVaR-VaR). Therefore (CVaR-VaR) measures the correct loss quantity for risk. The factor $e^{r(T-t)}$ takes into account that longer expiring options have greater (undiscounted) payouts.

The $\overline{CVaR}$ is also a correct option risk measure in that it has the important theoretical property that for *any underlying distribution* and probability measure $\mathbb{P}$ we have

$$(CVaR^{\mathbb{P}}(K_2) - VaR(K_2)) \geq (CVaR^{\mathbb{P}}(K_1) - VaR(K_1)), \tag{20}$$

where $CVaR^{\mathbb{P}}(K)$ is the CVaR associated with a put option with strike K and $K_2 > K_1$. In other words the risk of the option paying out (not the risk of the underlying) increases with K; we should logically expect this because there is more opportunity for the put option to expire in the money with K increasing. Equation (20) is formalised and proved in the following Proposition.

Proposition 3. *For European put options of the same maturity, for a given probability measure $\mathbb{P}$ and any underlying distribution, we have*

$$\begin{aligned} CVaR^{\mathbb{P}}(K_2) &\leq CVaR^{\mathbb{P}}(K_1) \\ \textit{and } (CVaR^{\mathbb{P}}(K_2) - VaR(K_2)) &\geq (CVaR^{\mathbb{P}}(K_1) - VaR(K_1)), \end{aligned}$$

where $K_2 > K_1$, $CVaR^{\mathbb{P}}(K)$ and VaR(K) denote the CVaR and VaR for an option with strike K, respectively.

Proof: see Appendix, Proof 3.

We should recall that CVaR is the expected loss, given the loss *exceeds* VaR. Therefore the $CVaR^{\mathbb{P}}(K_1)$ losses must be higher compared to those from $CVaR^{\mathbb{P}}(K_2)$ because K_1 is a lower threshold and so the losses *must* have exceeded a greater loss value.

3.2 CVaR's Relation to Implied Volatility

Options expensiveness in terms of price and risk can be understood from the definition applied to stocks. A well regarded measure of stock expensiveness in terms of risk is the Sharpe ratio (SR):

$$SR = \frac{r_e}{\varphi} = \frac{\bar{r}}{\varphi} - \frac{r}{\varphi},$$

where $r_e = \bar{r} - r$ is the excess return above the riskless rate, $\bar{r}$ is the stock return, φ is the stock's standard deviation of returns and is considered our measure of risk in SR. Hence the SR measures expensiveness or value for money in terms of a reward (or excess return) to risk ratio. Thus a less "expensive" stock returns a higher amount for a given amount of risk. It is useful to rearrange the SR equation as

$$\frac{\bar{r}}{\varphi} = \frac{r_e}{\varphi} + \frac{r}{\varphi}. \tag{21}$$

Hence the expensiveness or return:risk from the stock ($\bar{r}/\varphi$) is composed of return:risk of the riskless part of the investment (r/φ) and the return:risk of the risky part of the investment (r_e/φ).

Options do not provide a return in the same way as stocks but we can measure an option's expensiveness in terms of its price with respect to the option's risk (note that the price is the discounted expected payoff and so is similar to a "return"). Following the SR we break down the expensiveness into the risky and riskless part as in equation (21) (note that it is common in Finance to break down the risk into a riskless and risky component e.g. CAPM model). We therefore define the price:risk of options λ as:

$$\lambda = \lambda_e + \lambda_r \tag{22}$$

where λ_e is the excess price:risk for taking on additional risk and λ_r is the price:risk for the riskless part of the option.

Now we know from the Breeden-Litzenberger relation (equation (9))

$$\frac{\partial P}{\partial K} = e^{-r(T-t)}(1-\beta^{\mathbb{Q}})$$

and we know by rearranging equation (11)

$$(1-\beta^{\mathbb{Q}}) = \frac{Pe^{r(T-t)}}{(CVaR^{\mathbb{Q}} - VaR)}, \tag{23}$$

therefore

$$\begin{aligned}\frac{\partial P}{\partial K} &= e^{-r(T-t)}\left(\frac{Pe^{r(T-t)}}{(CVaR^{\mathbb{Q}} - VaR)}\right), && (24)\\ &= \frac{P}{(CVaR^{\mathbb{Q}} - VaR)}. && (25)\end{aligned}$$

Therefore $P : (CVaR - VaR)$ accounts for the change in option price with respect to to K. Moreover, the term $P : (CVaR - VaR)$ can be considered as a price:risk ratio for put options: the ratio measures the option price per unit of risk of payout, where option risk is measured in terms of CVaR-VaR.

If we assign $P : (CVaR - VaR)$ as λ then we now have an interpretation of equation (22):

$$\begin{aligned}\lambda &= \frac{\partial P}{\partial K} = \frac{P}{(CVaR - VaR)},\\ \lambda_e &= \nu\frac{\partial \sigma}{\partial K},\\ \lambda_r &= \frac{\partial P}{\partial K}_{\sigma},\end{aligned}$$

where ν is the option vega and $(\cdot)_\sigma$ denotes a partial differential with σ treated as a constant (to be explained later). To understand the assignments of λ_e and λ_r we should understand the partial differential equation $\frac{\partial P}{\partial K}$. Since we are investigating implied volatility effects we must work with the Black-Scholes equation but also take into account the implied volatility's dependence on K: $\sigma = \sigma(K)$. Therefore if we are to partially differentiate P with respect to K we need the product rule of differentiation because $P = P(K, \sigma(K))$ and is not $P = P(K, \sigma)$. For example, Alexander (Alexander, 2001) finds the call option delta hedging parameter by partially differentiating the Black-Scholes model with respect to X but with an X dependent σ thus:

$$\frac{\partial C}{\partial X} = \frac{\partial C}{\partial X}_{\sigma} + \frac{\partial C}{\partial \sigma}\frac{\partial \sigma}{\partial X}.$$

Returning back to the problem, in applying the product rule to $\dfrac{\partial P}{\partial K}$ for $P = P(K, \sigma(K))$ we will obtain two differentials: firstly the differential with respect to K itself:

$$\frac{\partial P(K, \sigma(K))}{\partial K_{\sigma}}. \tag{26}$$

This is the partial differential of P with respect to K and we regard σ as a constant. Secondly, there is a differential with respect to $\sigma(K)$ and this is achieved by regarding K constant but differentiating $\sigma(K)$ with respect to K. In using the chain rule we have:

$$\frac{\partial P}{\partial \sigma}\frac{\partial \sigma}{\partial K}. \tag{27}$$

Therefore the total partial differential equation of P with respect to K is

$$\frac{\partial P}{\partial K} = \frac{\partial P}{\partial K}_{\sigma} + \frac{\partial P}{\partial \sigma}\frac{\partial \sigma}{\partial K}, \tag{28}$$

$$= \frac{\partial P}{\partial K}_{\sigma} + \nu\frac{\partial \sigma}{\partial K}. \tag{29}$$

In terms of calculus we can interpret the total partial differential as follows. Firstly, equation (26) is the partial differential of P with respect to K when $P = P(K, \sigma)$ and not $P = P(K, \sigma(K))$, or alternatively the partial differential with σ as a constant. It is therefore the same differential as in the standard Black-Scholes model under constant volatility, that is

$$\frac{\partial P(K, \sigma(K))}{\partial K_{\sigma}} = \frac{\partial (K e^{-r(T-t)}\Phi(-d_2) - \Phi(-d_1)e^{-r(T-t)})}{\partial K}.$$

Similarly, in terms of calculus we can interpret equation (27) as the partial differential of P with respect to K when $P = P(\sigma(K))$ and not $P = P(K, \sigma(K))$. This is because K is regarded as a constant (except in σ). Therefore this differential gives us the *differential of P with respect to K due to implied volatility changing only* and not the strike, as in equation (26).

Now equation (25) is a model independent equation so we can apply it to any option model. Therefore the differential $\dfrac{\partial P}{\partial K}_{\sigma}$ is also the *price:risk under a constant volatility.*

We will denote the price:risk under constant volatility by λ_r. Similarly equation (27) gives the price:risk for an option model for $P = P(\sigma(K))$. We can also say it gives the change in P with respect to K due to implied volatility effects only, or alternatively the *price:risk due to implied volatility effects only.* We will denote the price:risk due to implied volatility effects only by λ_e.

Therefore the price:risk of options λ, like the SR, consists of a riskless price:risk component and a risky component. The riskless part of the price:risk ratio λ_r is simply the price:risk under a constant volatility model. Note that a constant volatility model corresponds to a flat implied volatility curve and such curves corresponds to no "expensiveness", hence this is consistent. The second term λ_e represents by definition the price:risk obtained for taking on additional risk and it can be seen that it is a function of implied volatility effects. Hence we can say that *implied volatility effects cause changes in the option expensiveness when measured in terms of price:risk, where risk is measured in terms of CVaR.*

To give an example of price:risk calculation let us apply price:risk to the Black-Scholes option pricing model under constant volatility. For Black-Scholes $\dfrac{\partial P}{\partial K}$ is known as the dual delta, the sensitivity of the put price to the strike, which is (Musiela and Rutkowski, 2005)

$$\frac{\partial P}{\partial K} = (\Phi(-d_2))e^{-r(T-t)}.$$

We also know that

$$\begin{aligned} \frac{P}{(CVaR - VaR)} &= (1 - \beta^{\mathbb{Q}})e^{-r(T-t)}, && (30) \\ &= (\Phi(-d_2))e^{-r(T-t)}. && (31) \end{aligned}$$

Hence this implies $\lambda = \lambda_r$ and λ_e=0, that is the price:risk does not differ from the riskless or constant volatility rate. Note that we expect λ_e=0 by calculus because volatility is a constant in the Black-Scholes model.

3.3 Option Expensiveness Comparison Under Implied Volatility

To get an understanding of price:risk under implied volatility effects, we compare the price:risk of a put option under a Black-Scholes model using constant volatility and implied volatility from a typical implied volatility smile curve. Using equation (31) we plotted the price:risk curve over different K (for a given T) on Figure 2, for S & P 500 index options using data from Figlewski's paper (Figlewski, 2009). A plot of the implied volatility smile curve used is given in Figure 1 and the index option's parameters are X(t)=1183, T=72 days, r=2.69% . For the constant volatility price:risk calculations we chose constant volatility to be 15% for all strikes because this is consistent with the average constant volatility on the index.

Figure 1: Implied Volatility Curve for S & P 500 Index Options

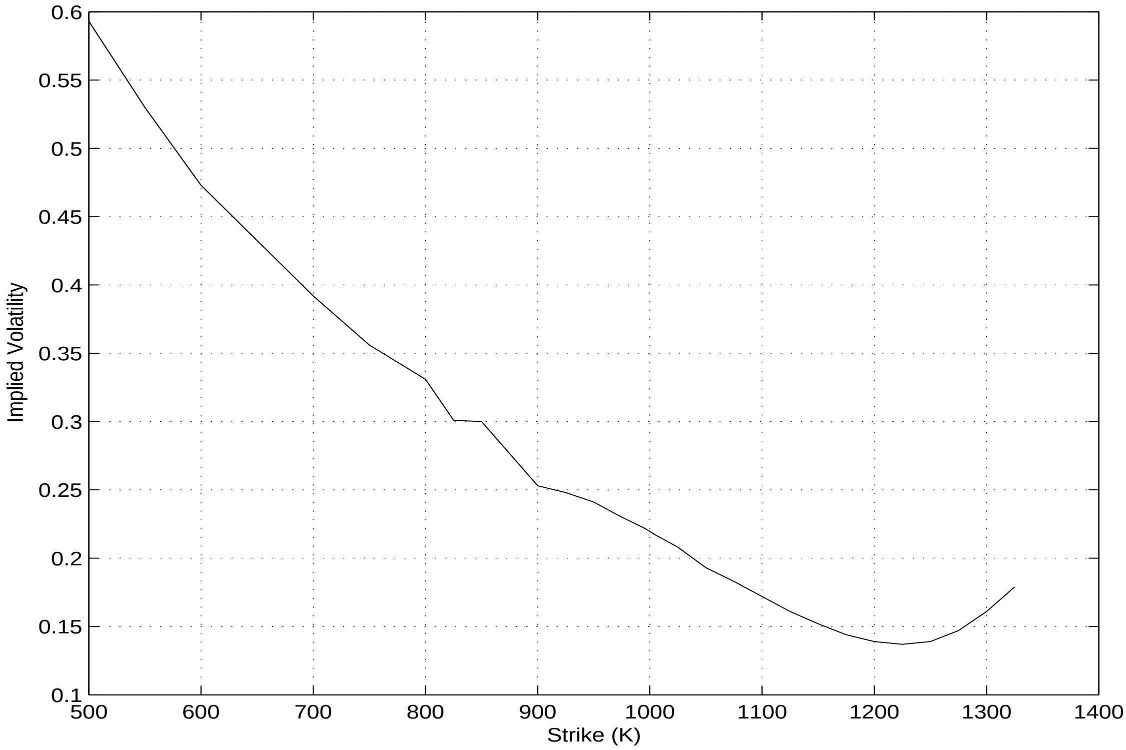
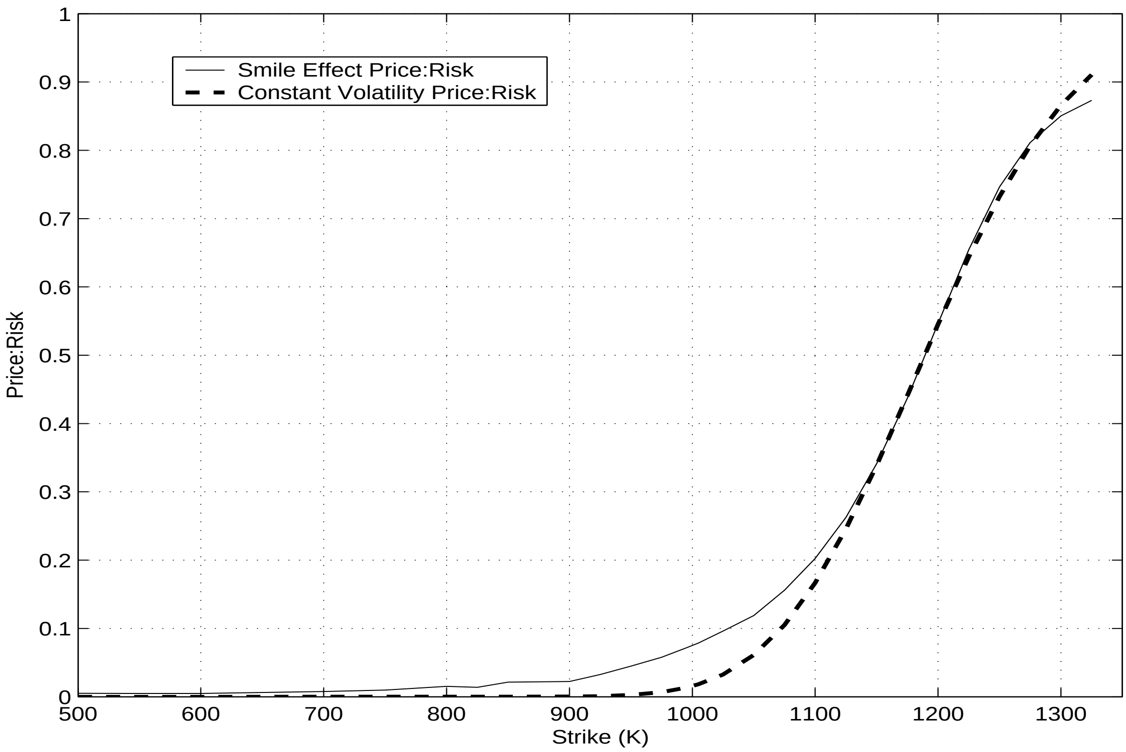


Figure 2: Price:Risk for Options Under Constant and Implied Volatility Smile Effects



The price:risk increases with K despite the riskiness increasing with K (by Proposition 3) because the price increases at a faster rate than the risk. Similarly, the price:risk tends to 0 as K decreases (meaning the option becomes less expensive) because the price decreases at a faster rate than the option risk. It is worth noting that the famous options speculator, Nassim Taleb, is fond of purchasing out of the money puts due to their cheap value.

The inclusion of implied volatility effects increases the price:risk for out of the money options ($K < X(t)$) compared to constant volatility. This is consistent with empirical observations of implied volatility because the implied volatility curve tends to steepen for out of the money options during times of increased perceived risk. In such situations therefore one would expect options to become more expensive to compensate for the additional perceived risk.

From Figure 2 it can be seen that the implied volatility smile effect reduces the price:risk of options compared to the constant volatility price:risk for in the money options ($K > X(t)$). Furthermore, the smile curve has higher (implied) volatility values than the constant volatility for $K > X(t)$, so the option risk (CVaR-VaR) is higher under implied volatility than constant volatility. However, the price charged for the option is higher under constant volatility than compared to options under the smile effect. Hence the price:risk under constant volatility is higher for $K > X(t)$. So, although the implied volatility increases with K for $K > X(t)$, the risk does not increase as quickly as the price, so the price:risk is lower compared to constant volatility. In fact a lower implied volatility for $K > X(t)$ actually increases the price:risk.

4 CVaR, Options and their Implications in Risk Management

Normally risk measurement or management is conducted in isolation from option modelling as there is practically no relation between the two areas. However, we have shown that CVaR and options are analytically related, so it is no longer for the 2 areas to be managed separately. If the 2 areas are managed separately then this can lead to serious mispricing errors e.g. arbitrage. Such possibilities are even more likely considering that both CVaR and options are frequently used in common applications e.g. managing the risk of portfolios.

In this section we discuss how the relation of CVaR and options significantly affect risk management and related issues if they are not consistently modelled with each other. Such a discussion should also be useful for *integrated* risk management; ensuring risk is managed and measured consistently across the whole of a business. We look at arbitrage opportunities that can occur from simple or fundamental modelling inconsistencies (e.g. put-call parity), implications on financial theorems and CVaR surfaces.

4.1 Arbitrage Opportunities Arising from CVaR and Options

In this section we show how the theorems of arbitrage, incompleteness and put-call parity affect CVaR values.

4.1.1 Arbitrage from Distributions and Risk Neutral Measures

Arbitrage is a riskless profit making opportunity and considered a serious case of mispricing; in fact in most theoretical literature we assume arbitrage never occurs. Furthermore, any models that allow arbitrage opportunities are normally not considered usable. For a greater explanation of arbitrage the reader is referred to (Björk, 2004).

Arbitrage in its most basic form occurs whenever we have two prices for the same asset or derivative. Using equation (11) we can obtain option prices directly from some CVaR model and this may differ from an option pricing model (especially as CVaR modelling is done separately to option pricing). Therefore there is a potential for an arbitrage opportunity. More specifically, arbitrage could occur if the underlying's distributions differ under each model or the risk neutral probability measures differ under each model.

Proposition 4. *The market is arbitrage free if the probability measures and the underlying distributions for CVaR and European options are identical.*

Proof:

Case 1: Differing Underlying Distributions

Using a CVaR model we can calculate P from the CVaR-option relation:

$$P^{\mathbb{Q}}(X(t),t,T,K) \;=\; e^{-r(T-t)}(CVaR^{\mathbb{Q}} - VaR)(1-\beta^{\mathbb{Q}}).$$

Let the CVaR model's probability distribution of X(t) be denoted by D_C, the option price using distribution D_C be denoted by $P^{\mathbb{Q}}(D_C)$, the CVaR and VaR obtained from D_C be denoted by respectively $CVaR_C$,VaR_C. Therefore we have

$$P^{\mathbb{Q}}(D_C) \;=\; (e^{-r(T-t)})(CVaR_C^{\mathbb{Q}} - VaR_C).(1-\beta_C).$$

Now let the option model's distribution of X(t) be denoted by D_m and using similar notation we have:

$$P^{\mathbb{Q}}(D_m) \;=\; (e^{-r(T-t)})(CVaR_m^{\mathbb{Q}} - VaR_m).(1-\beta_m^{\mathbb{Q}}).$$

Therefore for option prices to be identical we must have

$$\begin{aligned} P^{\mathbb{Q}}(D_m) &= P^{\mathbb{Q}}(D_C), \\ (e^{-r(T-t)})(CVaR_m^{\mathbb{Q}} - VaR).(1-\beta_m^{\mathbb{Q}}) &= (e^{-r(T-t)})(CVaR_C^{\mathbb{Q}} - VaR).(1-\beta_C^{\mathbb{Q}}). \end{aligned}$$

Therefore for no arbitrage opportunities to exist between $P^{\mathbb{Q}}(D_m)$ and $P^{\mathbb{Q}}(D_C)$, $\forall$ K, T, the distributions D_m and D_C must be identical.

Case 2: Differing Probability Measures

Assume the underlying distributions for both the CVaR and option models are identical. Let $\mathbb{Q}_m$ and $\mathbb{Q}_C$ denote the risk neutral probability measures used with the option pricing model and the CVaR model respectively. Therefore

$$P^{\mathbb{Q}_m}(X(t),t,T,K) \neq P^{\mathbb{Q}_c}(X(t),t,T,K)$$

and so we have two option prices for pricing the same option, which is an arbitrage opportunity. No arbitrage opportunities exist if and only if both probability measures are identical.

The CVaR and option underlying distributions are likely to differ when they modelled separately. For instance if the distribution for CVaR modelling is obtained from stock price data, whereas the underlying distribution for option modelling is obtained from option data (using Breeden and Litzenberger's relation (Breeden and Litzenberger, 1978)), the two distributions may not be identical. This would not only result in different option prices but arbitrage opportunities. An example of case 2 is given at the end of the Appendix for the benefit of the reader.

4.1.2 Put-Call-CVaR Parity

The put-call parity is an important theorem to option pricing for checking the existence of arbitrage opportunities, in fact it is commonly used to check the consistency of new models. According to the put-call parity, no arbitrage opportunities exist if a call C(X(t),t,T,K) and put P(X(t),t,T,K) with the same underlying asset X(t), strike K and expiration T obey the put-call parity relation:

$$P(X(t), t, T, K) \quad = \quad Ke^{-r(T-t)} - X(t) + C(X(t), t, T, K). \tag{32}$$

Using equation (32) we have an equation for arbitrage check between puts and calls, using equation (11) we also have an arbitrage check between CVaR and puts (see previous section), furthermore if we apply (11) to equation (32) we have yet another equation for checking arbitrage between CVaR and call prices:

$$Ke^{-r(T-t)} + C(X(t), t, T, K) - X(t) \quad = \quad (e^{-r(T-t)})(CVaR^{\mathbb{Q}} - VaR).(1-\beta).$$

We therefore now have a "three way" check for arbitrage between puts, calls and CVaR: using equations (11), (32) and the previous call option-CVaR equation. We must check between all 3 relations to prevent arbitrage; we cannot simply check for arbitrage using equation (32) only. Furthermore, as equations (32) and (11) do not require any stringent assumptions, all 3 relationships are applicable to a wide range of option and CVaR models.

The CVaR-call option relation is particularly useful in markets where one would like to obtain the market's view of CVaR on X(t) but there do not exist liquidly traded put options. This is a practical possibility as call options are generally traded more frequently than put options and for a wider range of assets. Additionally, as call options tend to suffer less from liquidity effects compared to put options (due to higher trading volume) we can therefore obtain a more accurate CVaR value without it being distorted by liquidity effects.

4.2 Second Fundamental Theorem of Finance and CVaR

In the previous section we have given conditions under which arbitrage possibilities can occur between CVaR and options, hence such conditions should be taken into account in CVaR and option modelling or risk management. We would also like to know the conditions under which it is also possible for more than 1 CVaR value to exist without allowing arbitrage opportunities to occur. The Second Fundamental Theorem of Finance gives us an insight into this condition and this theorem is considered as important as the principle of no arbitrage.

If we assume there is no arbitrage opportunity between CVaR and options, then the risk neutral measures for CVaR and option prices should be identical (this has been proven earlier). The Second Fundamental Theorem of Finance states that if the market is arbitrage free, then in a complete market we have a unique risk neutral measure and so a unique arbitrage-free option price exists. Therefore it follows from the Second Fundamental Theorem of Finance that CVaR is unique when the market is arbitrage free and complete. This is because there will be a unique risk neutral measure for options and so a unique risk neutral measure for CVaR, therefore a unique value for CVaR.

The Second Fundamental Theorem of Finance also states that in an incomplete and arbitrage free market, we will have a number of arbitrage free option prices (one for each risk neutral measure). Therefore, in an incomplete and arbitrage free market we will have a range of CVaR values because we will have a range of risk neutral measures, without violating arbitrage. The Second Fundamental Theorem of Finance is therefore important to risk measurement and management. We now formalise this as a proposition.

Proposition 5. *(Second Fundamental Theorem of Finance Extension to CVaR) Assume that the option market is arbitrage free. We also assume there exists no arbitrage opportunities between CVaR and options. The CVaR's value is therefore unique if and only if the option market is complete.*

The issue of understanding the conditions under which more than 1 CVaR can exist (without violating arbitrage) is important to risk management, as it allows us to understand when risk management is undertaken correctly. For instance, if we wish to ensure that we have consistent CVaR calculations across an entire bank's operations it would be advisable to use complete (and arbitrage free) models as it would ensure that a unique CVaR exists across an entire bank.

The previous idea is in fact applied to option pricing models in industry (but not CVaR models). In other words, many banks use arbitrage free and complete market option pricing models to prevent the existence of more than one risk neutral measure within a bank e.g. many banks use local volatility rather than stochastic volatility models because local volatility retain completeness (unlike stochastic volatility).

Another important reason that the banks ensure that a unique risk neutral measure exists is that it prevents "internal" arbitrage opportunities. If a company uses an incomplete market model then it is possible for more than one risk neutral measure to exist and two separate sections in a company may use different risk neutral measures. Hence it is potentially possible that the same company may price the same option with two different prices, leading to "internal" arbitrage opportunities. This issue is also important to CVaR risk management because it means that 2 different sections can have 2 entirely different CVaR values for an identical risks, leading to inconsistent risk management.

4.3 CVaR Surface Modelling and the Relation to Option Surfaces

CVaR surface graphs plot CVaR values over different holding periods and thresholds (VaR values). They are important because they enable managers to plan ahead of potential risks they may face over time and at different probabilities. Normally the surface graphs are obtained from some CVaR model, without reference to any option models.

Using equation (11) it is possible to obtain CVaR surfaces from option surfaces, where the holding period equals the option's maturity and the threshold relates to the strike (note that the β can be obtained from the Breeden-Litzenberger equation). If we are to have no arbitrage conditions and integrated risk management then there should be no discrepancies between option and CVaR surface modelling. This has a number of implications to CVaR surface modelling, which we will now discuss.

Firstly, the standard method of modelling CVaR surfaces is to assume CVaR follows some tractable function over all thresholds (or equivalently strike prices) and holding periods (or equivalently T); see (Dowd, 2011) for examples. This generally makes plotting CVaR over all holding periods and thresholds quite straightforward. For example the distribution governing CVaR maybe assumed to be lognormal, therefore the CVaR surface would be a smooth, concave surface which increases with time, analytic equations exist to plot the surfaces.

Although modelling CVaR surfaces using tractable functions is convenient, it would be significantly inconsistent with option surface models or empirically observed option surfaces. Option surface models give the prices for different K and T. Option surface modelling is generally non-trivial as it incorporates the various factors that influenced them. For example, (Schonbucher, 1999) models option surfaces by modelling the implied volatility for every single option using a separate stochastic differential equation for each of them.

Empirically, option surfaces are known to vary with T and K and the relationships are not necessarily stable with time or market conditions. For example, the term structure of implied volatility generally follows a concave relation with T increasing, however it has been empirically observed that this relation breaks down during distressed markets. Therefore, this suggests current CVaR modelling is inconsistent with actual CVaR (or option) dynamics.

Secondly, the CVaR-option relation challenges the view about the short term behaviour of CVaR or risk. Presently in risk management literature it is generally accepted that the portfolio's risk does not change significantly over short holding periods (Dowd, 2011). This assumption is inconsistent with empirical and theoretical literature on option surface modelling. Over short expiries it has been well documented that options are far less predictable than longer maturity options, therefore short holding period CVaR are less predictable than longer term holding period CVaR.

From a fundamental point of view, the factors governing the unpredictable nature of short expiry options would also apply to short holding period CVaR. The short maturity options are affected by a greater number of factors compared to long maturity options. For example, short expiry options are affected more than longer maturity ones by a company announcement. Such a factor would also affect CVaR modelling over short and long term holding periods in a similar way.

It is also well known that short expiry options are known to be influenced by "irrational" and sentiment based factors more than longer expiry options. In fact, short expiry implied volatility values are considered a gauge of market sentiment. We should therefore expect short term CVaR holding periods to be more unpredictable. It would therefore be more beneficial for risk managers to monitor short term holding period figures, rather than longer term holding periods.

5 Numerical Experiments

In this section we calculate the risk neutral CVaR and price:risk ratio from empirical option data using the equations from previous sections. We calculate the CVaR and price:risk for a range of K without making any distribution assumptions and then under the Black-Scholes model assumption. The option data and interpolation results that are required for the results were kindly supplied by Jackwerth and used in the paper (Jackwerth, 2004). We then discuss our results.

5.1 Method

In this section we calculate the risk neutral CVaR under a Black-Scholes model assumption and then without any distribution assumptions. The Black-Scholes implied risk neutral CVaR or the "implied CVaR" is calculated using equation (19), where the volatility applied is the implied volatility (the volatility consistent with quoted option prices). Similarly, the Black-Scholes price:risk is calculated using equation (31) and we use the implied volatility . Assuming we can observe P,r,T,K and X(t) for our option, the implied volatility can be determined using various software.

The distribution independent risk neutral CVaR is obtained using equation (11) and for the price:risk we use equation (23). Since P,r,T,K and X(t) are observable they do not need to be calculated, however to obtain $\beta^{\mathbb{Q}}$ using the Breeden-Litzenberger relation (equation (9)) we require option data and some interpolation method to obtain the partial differentials.

Applying interpolation methods is very common for option data, for example Monteiro et al. (Monteiro et al., 2008) apply a cubic spline method. However, it is well known that the cumulative distribution obtained is highly sensitive to interpolation methods. Also option prices tend to suffer from significant illiquidity effects ((Nordén, 2003), (Pinder, 2003)) and there is evidence that bid-ask spreads are a function of K and T (see for instance Pinder (Pinder, 2003), George and Longstaff (George and Longstaff, 1993)). Therefore obtaining unbias or representative option data can be problematic.

Since the aim of the numerical experiments are to calculate CVaR from option prices, we do not want our results to be subject to bias due to the chosen interpolation method, options data issues and so forth. We therefore chose to use the risk neutral density data obtained from (interpolated) empirical option data that was already published in a paper (Jackwerth, 2004). We performed numerical experiments using option data and the (interpolated) risk neutral density in (Jackwerth, 2004) for

38-day FTSE-100 index options, quoted on the 14th October 2003, with X(t)=4327 and r=3.5%. The reader is referred to (Jackwerth, 2004) for more information on the interpolation method.

We chose (Jackwerth, 2004) for our source of the risk neutral density and option prices because it provides a sufficient number of (interpolated) option prices and risk neutral densities over a range of K. Furthermore, (Jackwerth, 2004) provides option prices and risk neutral densities far from X(t), which many interpolation methods cannot provide due to the difficulty in estimating tail probabilities. Also the risk neutral probabililty distribution obtained from Jackwerth's interpolation method is an acceptable and realistic one, in terms of its shape and does not admit negative probabilities. Other interpolation methods provide unusual or unrealistic risk neutral distributions.

5.2 Results and Analysis

We present the results of our numerical experiments using data and the calculated risk neutral densities from (Jackwerth, 2004) for the FTSE-100 index options. The implied volatility curve used for the numerical experiments is given in Figure 3, which exhibits a smile over K.

One can observe from Figure 4 that the distribution free CVaR decreases as K increases, which is consistent with Proposition 3, and for $K >> X(t)$ then CVaR is virtually constant as K increases. The price:risk ratio graphs behave as theoretically expected, increasing with K. The "implied price to risk" shows that the Black-Scholes model underestimates the true price:risk for $K > X(t)$ but overestimates for $K < X(t)$.

The "implied CVaR" is consistent with Proposition 3, however Proposition 3 does not necessarily apply to "implied CVaR" because Proposition 3 applies to an asset with a single distribution. The "implied CVaR" calculates the CVaR, where at each K we have different lognormal distribution associated with each implied volatility. In fact for "implied CVaR" a decrease in CVaR due to K can be compensated by the increase in CVaR due to implied volatility changing. We notice that the "implied CVaR" curve is far smoother and well behaved than the distribution free CVaR. This is probably due to the interpolation method applied in (Jackwerth, 2004).

In Figure 4 the Black-Scholes CVaR does not significantly differ from the distribution free CVaR for $K > X(t)$, however for $K < X(t)$ the distribution free CVaR increases far more than the Black-Scholes CVaR. This implies that the Black-Scholes model of options significantly underestimates the CVaR risk of the underlying assets for low K, even when implied volatility effects are taken into account. This is con-

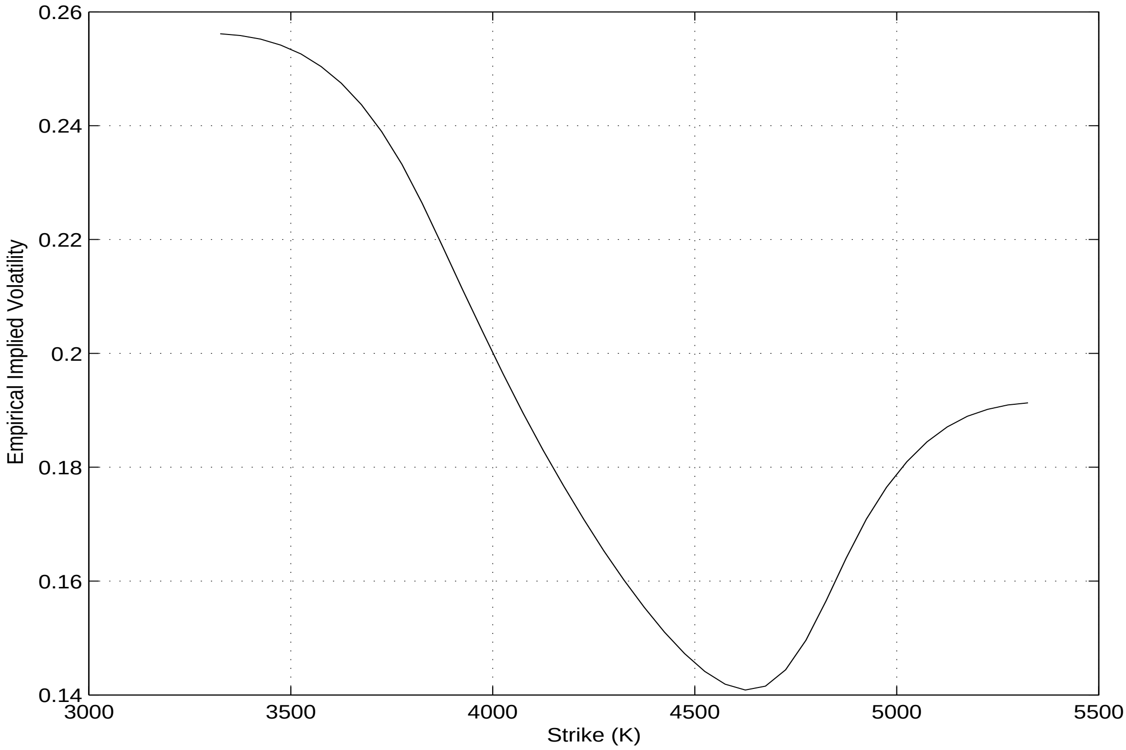


Figure 3: Empirical Implied Volatilities for FTSE-100 Index Options

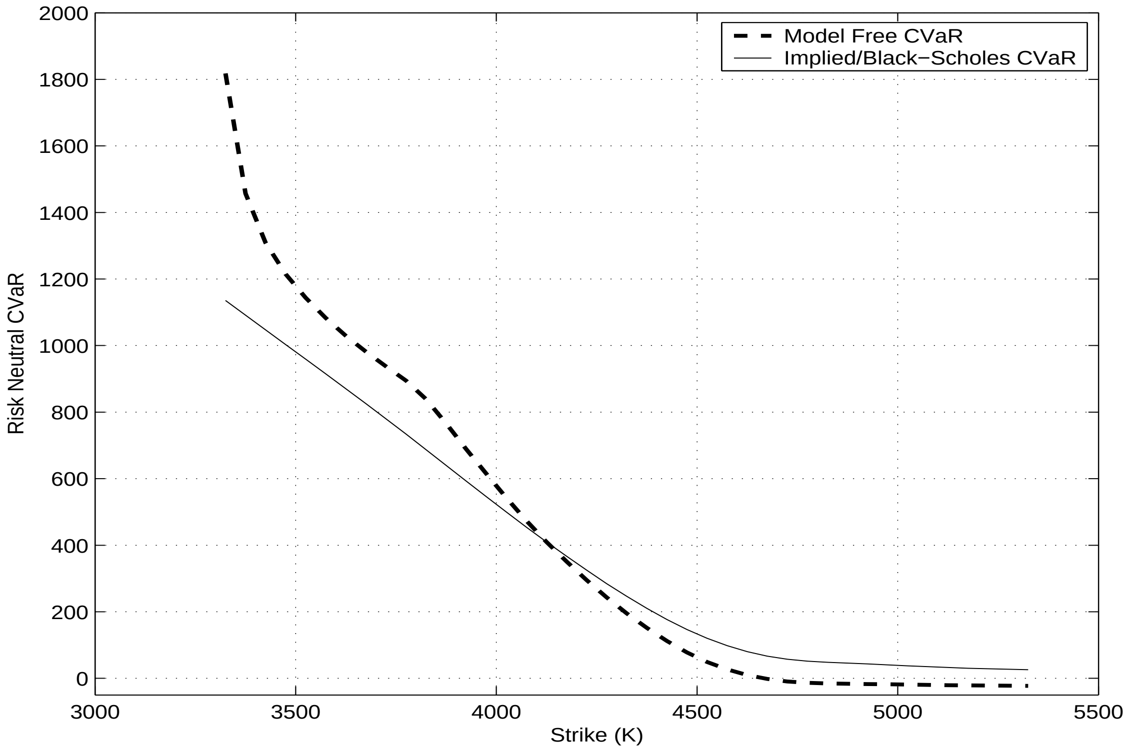


Figure 4: Risk Neutral CVaR Obtained from FTSE-100 Index Options

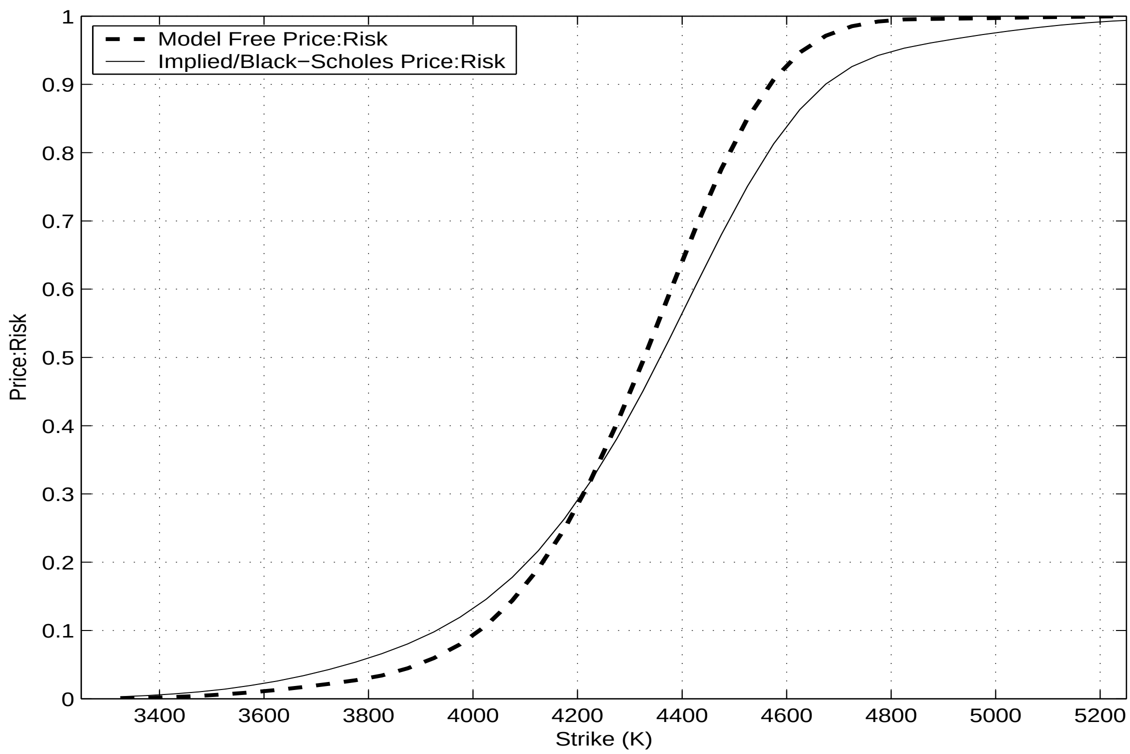


Figure 5: Price:Risk Ratio Obtained from FTSE-100 Index Options

sistent with empirical evidence, which suggests that the Black-Scholes model of asset distributions significantly underestimates fat tails or losses. Hence we can expect the true or distribution free CVaR to be higher than that obtained from "implied CVaR".

6 Conclusion

In this paper we derived a simple, closed form and analytic relationship between CVaR and European options. We discussed the significance of the equation and relationship, in particular with respect to implied volatility and risk management. We showed that we can account for implied volatility smile effects when we measure risk in terms of CVaR. We have demonstrated that CVaR cannot be calculated independently from options, otherwise there may exist arbitrage opportunities. We showed CVaR and VaR surfaces should be modelled like option price surfaces and contrary to common practice, are not predictable over short time periods.

Future areas of work should involve applying CVaR to other areas of option research, such as different option models (options with dividends, options with stochastic interest rates). We should also see how options models and theories can improve upon CVaR modelling, such as option models for implied volatility surfaces, option models under transaction costs, as well as investigating further option risk and CVaR.

7 Appendix

7.1 Table of Symbols

Symbol	Variable
C	European call option
F(.)	Cumulative probability distribution function
K	Option strike
$\bar{K}$	$\bar{K} = V(t) - K$
L	Portfolio call option
n	Number of assets in portfolio
P	European put option
$\mathbb{Q}$	Risk neutral measure
R	Portfolio put option
r	Riskless rate
t	Time
T	Option expiry time
V(t)	Portfolio price at time t
w_i	Number of shares in asset i
X(t)	Stock price at time t
Z(t)	Portfolio loss at time t
$z_i(t)$	Loss of asset i at time t
β	Quantile for VaR such that $F(Z(T) \leq VaR) = \beta$
λ	price:risk ratio of put options
ν	Black-Scholes hedging parameter vega

7.2 Proofs

Proof 1:

$$ER_{\bar{K}}^{\mathbb{P}}(Z(T)) \quad = \quad E^{\mathbb{P}}\left[\sum_{i=1}^{n} w_i z_i(T) - \bar{K}\right]^{+}.$$

We can re-express $\bar{K}$ in terms of the portfolio price:

$$\bar{K} = V(t) - K = \sum_{i=1}^{n} w_i X_i(t) - K.$$

$$
\begin{aligned}
\text{Hence } \left(\sum_{i=1}^{n} w_i z_i(T) - \bar{K}\right) &= \sum_{i=1}^{n} w_i(X_i(t) - X_i(T)) - (w_i X_i(t) - K), \\
&= \sum_{i=1}^{n} -w_i X_i(T) + K, \\
&= K - \sum_{i=1}^{n} w_i X_i(T), \\
\Rightarrow \left[\sum_{i=1}^{n} w_i z_i(T) - \bar{K}\right]^+ &= \left[K - \sum_{i=1}^{n} w_i X_i(T)\right]^+.
\end{aligned}
$$

We recall $\bar{K}, w_i, K, X_i(t)$ and V(t) are all unaffected by measure changes therefore:

$$
\begin{aligned}
E^{\mathbb{P}}\left[\sum_{i=1}^{n} w_i z_i(T) - \bar{K}\right]^+ &= E^{\mathbb{P}}\left[K - \sum_{i=1}^{n} w_i X_i(T)\right]^+, \\
&= E^{\mathbb{P}}[K - V(T)]^+, \\
\text{so that } ER_{\bar{K}}^{\mathbb{P}}(Z(T)) &= \widetilde{R}^{\mathbb{P}}(V(t), t, T, K).
\end{aligned}
$$

Additionally, if we choose risk neutral probability measure $\mathbb{Q}$ we have

$$
ER_{\bar{K}}^{\mathbb{Q}}(Z(T)) = \widetilde{R}^{\mathbb{Q}}(V(t), t, T, K),
$$

therefore discounting the option at the risk free rate $e^{-r(T-t)}$ we have

$$
e^{-r(T-t)} ER_{\bar{K}}^{\mathbb{Q}}(Z(T)) = R^{\mathbb{Q}}(V(t), t, T, K).
$$

Note that for a single asset we have:

$$
(e^{-r(T-t)}) ER_{\bar{K}}^{\mathbb{Q}}(X(T)) = P(X(t), t, T, K).
$$

Remark 2. *As options are independent of risk preference we can value them under any probability measure, provided we apply the correct discount rate. Therefore we can also write*

$$
e^{-\tilde{r}(T-t)} ER_{\bar{K}}^{\mathbb{P}}(Z(T)) = R^{\mathbb{P}}(V(t), t, T, K),
$$

for any probability measure $\mathbb{P}$, where $\tilde{r}$ is the correct discount factor for probability measure $\mathbb{P}$.

Proof 2:

$$
\begin{aligned}
1 - \beta^{\mathbb{Q}} &= F^{\mathbb{Q}}(z(t) > VaR), \\
&= F^{\mathbb{Q}}(X(t) - X(T) > X(t) - K), \\
&= F^{\mathbb{Q}}(K > X(T)).
\end{aligned}
$$

Proof 3:

$$VaR(K_2) = X(t) - K_2 \text{ and } VaR(K_1) = X(t) - K_1,$$

Therefore

$$VaR(K_2) \quad < \quad VaR(K_1). \tag{33}$$

From equation (1) we have

$$\begin{aligned} CVaR^{\mathbb{P}}(K_2) &\leq VaR(K_2) \text{ and} \quad (34)\\ CVaR^{\mathbb{P}}(K_1) &\leq VaR(K_1). \quad (35)\\ \text{so by equation (33) we have } CVaR^{\mathbb{P}}(K_2) &\leq CVaR^{\mathbb{P}}(K_1). \quad (36) \end{aligned}$$

Therefore using equations (33) and (36) we have

$$(CVaR^{\mathbb{P}}(K_2) - VaR(K_2)) \quad \geq \quad (CVaR^{\mathbb{P}}(K_1) - VaR(K_1)).$$

7.3 Black-Scholes Equation Variables

$$\begin{aligned} d_1 &= \frac{ln(X(t)/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}, \\ d_2 &= \frac{ln(X(t)/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}, \\ &= d_1 - \sigma\sqrt{(T-t)}. \end{aligned}$$

7.4 Differing Risk Neutral Probability Measures

To give an example of different risk neutral measures existing for the same empirical or real world stock price process X(t), we assume X(t) follows Heston's model Heston (1993):

$$\begin{aligned} dX/X &= \mu dt + \sigma dW_1^{\mathbb{P}}, \\ d\sigma^2 &= \alpha(m - \sigma^2)dt + \kappa\sigma dW_2^{\mathbb{P}}, \\ corr(dW_1^{\mathbb{P}}, dW_2^{\mathbb{P}}) &= \rho dt, \end{aligned}$$

where α, m, κ are constants, W_1 and W_2 are 2 independent Wiener processes and corr(,) denotes correlation. The equations (under measure $\mathbb{P}$) provide the price process observed under the empirical or the real world measure. To change the probability measure from $\mathbb{P}$ to the risk neutral measure $\mathbb{Q}$ we have:

$$\begin{aligned} dW_1^{\mathbb{P}} &= dW_1^{\mathbb{Q}} + \psi dt, \\ dW_2^{\mathbb{P}} &= dW_2^{\mathbb{Q}} + \upsilon(X, \sigma, t)dt, \\ \psi &= \frac{\mu - r}{\sigma}, \\ corr(dW_1^{\mathbb{Q}}, dW_2^{\mathbb{Q}}) &= \rho dt. \end{aligned}$$

Hence the risk neutral process is given by

$$\begin{aligned} dX/X &= rdt + \sigma dW_1^{\mathbb{Q}}, \\ d\sigma^2 &= \alpha(m - \sigma^2)dt + \kappa\sigma(dW_2^{\mathbb{Q}} + \upsilon(X, \sigma, t)dt). \end{aligned}$$

Under the risk neutral measure $\upsilon(X, \sigma, t)$ can be chosen arbitrarily and is not unique, hence the risk neutral process is not unique. Hence for the option pricing model let us define the risk neutral measure $\mathbb{Q}_m$ with

$$\begin{aligned} \upsilon(X, \sigma, t) &= \eta_1\sigma, \\ \psi &= \frac{\mu - r}{\sigma}, \end{aligned}$$

where η_1 is a constant. For the CVaR model let us define the risk neutral measure $\mathbb{Q}_c$ with

$$\begin{aligned} \upsilon(X, \sigma, t) &= \eta_2\sigma, \\ \psi &= \frac{\mu - r}{\sigma}, \end{aligned}$$

where η_2 is a constant and $\eta_1 \neq \eta_2$. Hence the risk neutral process for the option pricing model is given by

$$\begin{aligned} dX/X &= rdt + \sigma dW_1^{\mathbb{Q}}, \\ d\sigma^2 &= \alpha(m - \sigma^2)dt + \kappa\sigma(dW_2^{\mathbb{Q}} + \eta_1\sigma dt), \end{aligned}$$

and for the CVaR model

$$\begin{aligned} dX/X &= rdt + \sigma dW_1^{\mathbb{Q}}, \\ d\sigma^2 &= \alpha(m - \sigma^2)dt + \kappa\sigma(dW_2^{\mathbb{Q}} + \eta_2\sigma dt). \end{aligned}$$

Hence the 2 models give different risk neutral processes but have the same empirical (or $\mathbb{P}$ measure) stock price process.

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