Stability Analysis for Delayed Neural Networks Considering Both Conservativeness and Complexity

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***Abstract*—This paper investigates delay-dependent stability for continuous neural networks with a time-varying delay. The paper aims at deriving a new stability criterion considering tradeoff between conservativeness and calculation complexity. A new Lyapunov-Krasovskii functional (LKF) with simple augmented terms and delay-dependent terms is constructed and its derivative is estimated by several techniques, including free-weighting matrix and inequality estimation methods. Then the influence of the techniques used on the conservativeness and the complexity is analyzed one by one. Moreover, useful guidelines for improving criterion and future work are briefly discussed. Finally, the advantages of the proposed criterion compared with the existing ones are verified based on three numerical examples.**

***Index Terms*—Delayed neural networks, delay-dependent stability, Lyapunov-Krasovskii functional, conservativeness, calculation complexity**

I. Introduction

NEURAL networks have been successfully applied in image processing, pattern recognition, associative memory, optimization problem, *etc.* [1]–[3]. Since most applications of neural networks are closely dependent on their dynamic behaviors, especially on stability, it is an important job to check stability of the concerned system. During the implementation of artificial neural networks, the finite switching speed of amplifiers and the inherent communication time between neurons inevitably introduce time delays, which might cause oscillation, divergence, and even instability. Therefore, the stability of the delayed neural networks (DNNs) is an important problem and has received considerable attention. The delay-dependent stability criteria have obtained more attention since they are less conservative and the delays encountered in neural networks are usually not very big.

A. Brief review of related researches

Delay-dependent criteria are usually derived via the Lyapunov theory and have a certain degree of conservativeness, *how to reduce the conservativeness* is the main research direction in recent years. The conservativeness is usually indexed via the acceptable delay region provided by the corresponding criteria [65]. The conservativeness-reducing has been achieved mainly from two phases: choosing a candidate Lyapunov-Krasovskii functional (LKF) and estimating its derivative [47]. The research of delay-dependent stability of the DNNs is briefly reviewed at first from those two phases.

On one hand, most of the LKFs constructed to discuss the DNNs can be summarized as the following form:

$$V_F(t) = V_{NQ}(t) + V_{IQ}(t) + V_{AF}(t) \tag{1}$$

where $V_{NQ}(t)$, $V_{IQ}(t)$, and $V_{AF}(t)$ are the non-integral quadratic term, the integral quadratic term, and the activation function based term, respectively. Obviously, the more general form of the LKF is, the less conservativeness of the criterion is. Thus, constructing a more general LKF is an effective way to reduce conservativeness. The following summarizes the researches based on the different construction of LKF (1).

(1) The non-integral quadratic term $V_{NQ}(t)$

- simple form only containing current state vector [4]–[10], [13]–[15], [21]–[30], [35]–[39], [47]–[50], [69];
- augmented form including current state vector, delayed state vector, and integral of state vectors, *etc.* [11], [12], [16]–[20], [31]–[34], [40]–[46].

(2) The integral quadratic term $V_{IQ}(t)$

- based on domain of integration (the upper/lower limit of integration): simple form using whole delay interval (e.g. $\int_{t-d(t)}^{t}$, $\int_{t-h}^{t}$ *etc.*) [4]–[8], [11], [17], [24]–[36], [40]–[46], [68]; and delay-partitioning-based form using delay subintervals (e.g. $\int_{t-\frac{i}{m}h}^{t-\frac{i+1}{m}h}$ [9], [15], [19]–[23], [37]–[39], [48]–[50], [73], $\int_{t-\frac{i}{m}h}^{t}$ [12]–[14], and $\int_{t-\frac{i}{m}d(t)}^{t-\frac{i+1}{m}d(t)}$ [10], [16], [18], [47], [72]).
- based on integrand: simple form [4]–[10], [26], [27], [29], [30], [36], [37], [40], [41], and augmented form [11]–[25], [28], [31]–[35], [38], [39], [42]–[50], [73].
- based on multiple integration: simple form containing single and double integral terms [4]–[9], [12]–[19], [21]–[27], [30], [36]–[40], [45], [47]–[50]; and improved form with triple integral [10], [11], [20], [28], [29], [32]–[35], [43] and/or quadruple integral terms [42], [44].

(3) The activation function based term $V_{AF}(t)$

- simple form without the information of the slope of activation function [4]–[9], [11], [14]–[18], [23], [24], [30]–[32], [39];

This work is supported partially by the National Natural Science Foundation of China under Grant Nos. 61125301, 51428702 and 61210011.

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- improved form containing slope information, σ_i^+, σ_i^- [10], [13], [19]–[22], [25]–[29], [33]–[38], [41]–[49].

On the other hand, to describe the final criterion in the form of linear matrix inequalities (LMIs), the estimation operation is necessary during the treating of the LKF's derivative. The original form of the derivative can be summarized as follows:

$$\dot{V}_F(t) = V_{NI}(t) + V_I(t) \tag{2}$$

where $V_{NI}(t)$ and $V_I(t)$ are the sum of the non-integral terms and the integral terms, respectively. The estimation of $\dot{V}_F(t)$ refers to the following three key steps:

(1) Using new terms to estimate the integral terms $V_I(t)$

- recombination method: rewriting the original integral term to obtain related non-integral terms via adding zero-value terms, mainly including He's free-weighting matrix (FWM) technique based on Newton-Leibniz formula [4]–[7], [10], [11], [14], [18], [20], [22]–[24], [30], [32], [47], [48] and Kwon's FWM-based zero-value equality based on Integration-By-Part [33], [34], [42]–[45];
- bounding method: replacing the original integral terms directly by non-integral terms via inequalities, such as, Jensen's inequality [8], [12]–[17], [19], [21], [23], [26]–[29], [33]–[39], [42]–[45], [47]–[50], Wirtinger's inequality [41], [43], [46], [67], [73], and other inequalities [10], [25], [29], [31], [41].

(2) Dealing with $d(t)$ and $h-d(t)$

- enlarging both $d(t)$ and $h-d(t)$ to upper bound, h [4], [5], [8], [12], [14]–[17], [21], [26], [27], [36], [40], [50];
- using the improved FWM approach [5]–[7], [10], [11], [20], [22]–[24], [26], [39], [47], [48];
- applying convex combination technique for $d(t)$ and $h-d(t)$ included in the numerator [9], [19], [25], [26], [28], [30]–[32], [41], [43], [47] and reciprocally convex combination technique for $d(t)$ and $h-d(t)$ included in the denominator [13], [33]–[35], [37], [38], [41]–[49];
- Injecting the $d(t)$ and $h-d(t)$ via defining $v_1 = \int_{t-d(t)}^{t} \frac{x(s)}{d(t)} ds$ and $v_2 = \int_{t-h}^{t-d(t)} \frac{x(s)}{d(t)} ds$ [22], [43].

(3) Adding terms via the information of activation function

- simple form considering the whole interval of the slope $[\sigma_i^-, \sigma_i^+]$ [4]–[32], [34]–[42], [45]–[50]:
- slope-partitioning-based form introducing subintervals of the slope $[\sigma_i^-, \frac{\sigma_i^- + \sigma_i^+}{2}] \cup [\frac{\sigma_i^- + \sigma_i^+}{2}, \sigma_i^+]$ [33], [43], [44].

Unlike most publications, in which the researches are usually surveyed from the method-applied point of view, the above discussion firstly summaries the background based on the key steps of criterion-deriving. The methods are commonly developed only for single step, and some useful treatments without specific name may be ignored, thus the above summarization can review the background more comprehensive and systematic than the method-based review does. More importantly, both the conservativeness and the complexity of the final criterion are dependent on all treatments, the above summarization gives mainly optional treatments for each step, and it helps researchers to choose suitable treatments by simultaneously considering the target (aim at reducing conservativeness only or both conservativeness and complexity) and the characteristic of different treatments.

B. Problems need more investigation

Although various techniques have been developed for discussing the DNNs, there still exists problems that require further investigation, as summarized by the following aspects.

(1) Objective: Most researches focus on reducing conservativeness, while the calculation complexity is only considered in a few literature, in which the ones just discuss how to avoid too many decision variables by using inequality technique, instead of the FWM approach, to estimate derivative of the LKFs and choosing suitable number of delay subintervals for delay-partition method. However, there is no result on the consideration of tradeoff between the conservativeness and the complexity. In fact, during the application of LMI-based criteria to large-scale physical systems, the calculation complexity is also a very important issue [55]–[59]. A criterion with too much complexity will become useful only for small scale numerical examples [65]. The criteria considering both the conservativeness and the complexity need more investigation.

(2) Construction of the LKF: The recent researches tend to use LKFs with more general form to achieve the conservativeness-reducing, while the techniques for constructing such LKFs usually increase both decision variables to be determined and the dimension of the LMI-based conditions, which are two key factors related to calculation complexity [60]. For delay-partitioning-based constructing method, the opposite relationship between the conservativeness and the complexity can be predicted, which depends on the number of subintervals partitioned, and two subintervals are advised and used in recent years [9], [10], [16], [20], [22], [37], [38], [47]–[49]. For augmentation based constructing method, however, the relationship between the conservativeness/complexity and the information of augmentation (which state-based vector and how many vectors are used for augmenting) is not very obvious. Moreover, the work [47] finds that the new LKF, developed in [27] via introducing slope information and used in [37], [38], [48], [49], has no contribution on conservativeness-reducing (see Section III in [47]). Thus, when the complexity becomes an issue in consideration, the analysis of contribution for different treatments should be further investigated to achieve conservativeness-reducing with lower calculation cost.

(3) Estimation of the LKF's derivative: Various techniques have been applied for this problem, and five types commonly used are compared with each other in our previous work [47]. Recently, two new techniques, including Wirtinger's inequality considered as the most effective way to direct estimate single integral terms [51] and the Kwon's FWM-based zero-value equality used widely in recently [33], [34], [42]–[44], have been developed to further improve the results. The previous work usually improves the results by using a better technique, while it may be difficult to judge which is better for some existing techniques [47] and there may exist drawbacks for each technique. Then a problem arises, i.e., whether one can use several techniques together to realize respective advantages and also to overcome their drawbacks, while few related results have been obtained.

(4) Evaluation the contribution of techniques: The criteria are usually obtained by using many techniques, including the

techniques for constructing LKF and the ones for estimating its derivative. Most publications commonly check the advantages of the criteria compared with the existing ones from the combined effect of all techniques. It cannot review different levels of contributions from each technique. However, when the complexity is in consideration, the investigation of the contributions of each technique is necessary in order to find which technique can reduce the conservativeness obviously but not introduce too much complexity.

C. Contributions of the paper

This paper further investigates the stability of continuous DNNs and aims to find possible solutions of the problems mentioned above. The detailed contributions of the paper are summarized as follows:

(1) Objective: In this paper, the main attention is paid to derive a stability criterion considering the tradeoff between the conservativeness and the calculation complexity, since it is an important problem for the LMI-based method [65]. Moreover, for the complexity of criterion, beside the number of decision variables, the dimension of the LMI-based conditions is also taken into account during the construction of LKF, since it is a key factor for calculation complexity and is even more important than the number of the decision variables [60].

(2) Construction of the LKF: A new augmented LKF constructing method is discussed in consideration of the tradeoff between the conservativeness and the calculation complexity. Firstly, several parts of simple form LKF (the non-integral, single integral, and double integral terms) are augmented via introducing additional vectors, and two new delay-dependent non-integral terms are introduced firstly. Secondly, the reasons of those treatments are discussed one by one theoretically and the contributions to reduce conservativeness are verified via numeral examples.

(3) Estimation of the LKF's derivative: How to estimate the LKF's derivative considering both the conservativeness and the complexity via the combination of several effective techniques is investigated. Firstly, for the estimation of single integral term with augmented integrand, Wirtinger's inequality and Kwon's zero-value equality methods are used together to avoid the drawbacks that will be caused if only the one of them is used. Secondly, for the treatments of double integral terms and $d(t)/h_d(t)$ information, the combination of different techniques is used to make the enlargement procedure and the number of decision variables as small as possible.

(4) Evaluation the contribution of techniques: The contributions of the techniques used are analyzed from theory analysis and example verification. On one hand, simple theory analysis is carried out to show how each technique provide contributions. On the other hand, based on three examples, three indexes which indicate the improvements of the conservativeness- and complexity-reducing, are calculated, and it can be found, from the results, which technique has more contribution and is worthy of more deeply study.

The remainder of the paper is organized as follows. Section II gives the problem formulation and preliminary. In section III, a new stability criterion is derived by using several new techniques; the contributions of techniques are analyzed one by one based on theory analysis; and some techniques that can provide further improvements and future work are summarized. In Section IV, three numerical examples are considered to demonstrate the benefits of the proposed criterion. Conclusions are given in Section V.

Notations: Throughout this paper, the superscripts T and -1 mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices; $\|\cdot\|$ refers to the Euclidean vector norm; $P>0$ (≥ 0) means that P is a real symmetric and positive-definite (semi-positive-definite) matrix; diag$\{\cdots\}$ denotes a block-diagonal matrix; symmetric term in a symmetric matrix is denoted by $*$; and $\mathrm{Sym}\{X\}=X+X^T$.

II. Problem formulation and preliminary

This section describes the problem to be investigated and gives related preliminary.

A. Problem formulation

Consider the following generalized DNNs [25], [47]:

$$\dot{y}(t)=-Ay(t)+W_0g(Wy(t))+W_1g(Wy(t-d(t)))+J \quad (3)$$

where $y(t)=[y_1(t)\ y_2(t)\ \cdots\ y_n(t)]^T$ is the state vector associated with the n neutrons; $g(\cdot)=[g_1(\cdot)\ g_2(\cdot)\ \cdots\ g_n(\cdot)]^T$ represents the neutron activation function with $g(0)=0$; $A=\mathrm{diag}\{a_1,a_2,\cdots,a_n\}>0$; W, W_0 and W_1 are the connection weight matrices; $J=[J_1\ J_2\ \cdots\ J_n]^T$ is a vector representing the bias; and $d(t)$ is a time-varying delay satisfying

$$0\leq d(t)\leq h,\quad \dot{d}(t)\leq\mu \quad (4)$$

The neuron activation function is assumed to be bounded and satisfies the following condition:

$$\sigma_i^-\leq\frac{g_i(s_1)-g_i(s_2)}{s_1-s_2}\leq\sigma_i^+,\quad s_1\neq s_2, i=1,2,\cdots,n \quad (5)$$

where σ_i^- and σ_i^+ are known real constants.

Based on the assumption on the activation function, there exists an equilibrium point y^* for the neural network, i.e., $0=-Ay^*+W_0g(Wy^*)+W_1g(Wy^*)+J$. Using transformation $x(t)=y(t)-y^*$ [65], one can shift the equilibrium point y^* of (3) to the origin and rewrite system (3) as [30]:

$$\dot{x}(t)=-Ax(t)+W_0f(Wx(t))+W_1f(Wx(t-d(t))) \quad (6)$$

where $f(.)=[f_1(.)\ f_2(.)\ \cdots\ f_n(.)]^T$ and $f_i(W_{2i}x(t))=g_i(W_{2i}x(t)+W_{2i}y^*)-g_i(W_{2i}y^*)$ with $f_i(0)=0$ and W_{2i} denoting the i-th row vector of the matrix W. Then,

$$\frac{f_i(s_1)-f_i(s_2)}{s_1-s_2}=\frac{g_i(s_1+W_{2i}y^*)-g_i(s_2+W_{2i}y^*)}{s_1+W_{2i}y^*-(s_2+W_{2i}y^*)}$$

Thus, it follows from (5) and $f_i(0)=0$ that [33], [34]

$$\sigma_i^-\leq\frac{f_i(s_1)-f_i(s_2)}{s_1-s_2}\leq\sigma_i^+,\quad s_1\neq s_2 \quad (7)$$

$$\sigma_i^-\leq\frac{f_i(s)}{s}\leq\sigma_i^+,\quad s\neq 0 \quad (8)$$

This paper aims to derive a delay-dependent stability criterion of DNN (3) to determine the acceptable delay region guaranteeing the stability of the DNN. The conservativeness and the calculation complexity will be considered simultaneously during the deriving of the criterion.

B. Preliminary

The following notations and lemmas are introduced at first for simplifying the expression of subsequent parts:

$$
\begin{aligned}
&h_d(t) := h - d(t), && f(t) := f(Wx(t))\\
&x_d(t) := x(t-d(t)), && f_d(t) := f(Wx(t-d(t)))\\
&x_h(t) := x(t-h), && f_h(t) := f(Wx(t-h))\\
&v_1(t) := \int_{t-d(t)}^{t} \tfrac{x(s)}{d(t)} ds, && v_2(t) := \int_{t-h}^{t-d(t)} \tfrac{x(s)}{h_d(t)} ds\\
&v_3(t) := \int_{t-d(t)}^{t} f(s) ds, && v_4(t) := \int_{t-h}^{t-d(t)} f(s) ds\\
&v_5(t) := d(t) v_1(t), && v_6(t) := h_d(t) v_2(t)\\
&\varepsilon_1(t) := [x^T(t)\ f^T(t)]^T, && \varepsilon_2(t) := [x^T(t)\ \dot{x}^T(t)]^T\\
&\varepsilon_3(t) := [x^T(t)\ \textstyle\int_{t-h}^{t} x^T(s) ds]^T, && \varepsilon_4(t) := [x^T(t)\ \ v_2^T(t)]^T\\
&\xi_1(t) := [x^T(t)\ x_d^T(t)\ x_h^T(t)]^T, && \xi_2(t) := [f^T(t)\ f_d^T(t)\ f_h^T(t)]^T\\
&\xi_3(t) := x_d(t) + x(t) - 2v_1(t), && \xi_4(t) := x_h(t) + x_d(t) - 2v_2(t)
\end{aligned}
$$

$$
\begin{aligned}
&\xi(t) := [\xi_1^T(t)\ \ \xi_2^T(t)\ \ v_1^T(t)\ \ v_2^T(t)]^T\\
&e_s := [-A\ \ 0\ \ 0\ \ W_0\ \ W_1\ \ 0\ \ 0\ \ 0]\\
&e_0 := [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]\\
&e_i := [0_{n\times(i-1)n}\ \ I_{n\times n}\ \ 0_{n\times(8-i)n}],\ i = 1, 2, \cdots, 8\\
&\xi_d(t) := [x^T(t)\ \ v_2^T(t)\ \ \varepsilon_1^T(t-d(t))]^T,\ e_{d0} := [0\ \ 0\ \ 0\ \ 0]\\
&e_{di} := [0_{n\times(i-1)n}\ \ I_{n\times n}\ \ 0_{n\times(4-i)n}],\ i = 1, 2, \cdots, 4\\
&\Sigma_1 := \mathrm{diag}\{\sigma_1^+, \cdots, \sigma_n^+\}, \quad \Sigma_2 := \mathrm{diag}\{\sigma_1^-, \cdots, \sigma_n^-\}\\
&\Sigma := \mathrm{diag}\left\{\max\{|\sigma_1^+|, |\sigma_1^-|\}, \cdots, \max\{|\sigma_n^+|, |\sigma_n^-|\}\right\}
\end{aligned}
$$

Lemma 1: (Jensen's inequality [63], [64]) For any matrix $R \in \mathcal{R}^{n\times n}$, $R = R^T > 0$, scalars $a < b$, vector $\omega : [a, b] \mapsto \mathcal{R}^n$ such that the integration concerned are well defined, then

$$(b-a)\int_a^b \omega^T(s) R \omega(s) ds \geq \Big(\int_a^b \omega(s) ds\Big)^T R \Big(\int_a^b \omega(s) ds\Big) \tag{9}$$

$$\frac{(b-a)^2}{2}\int_a^b \int_\theta^b \omega^T(s) R \omega(s) ds d\theta \geq \Big(\int_a^b \int_\theta^b \omega(s) ds d\theta\Big)^T R \Big(\int_a^b \int_\theta^b \omega(s) ds d\theta\Big) \tag{10}$$

Lemma 2: (Extended Wirtinger's inequality [51]) For any matrix $R \in \mathcal{R}^{n\times n}$, $R = R^{\mathrm{T}} > 0$, any differentiable function $\omega : [a, b] \mapsto \mathcal{R}^n$, the following inequality holds

$$\int_a^b \dot{\omega}^{\mathrm{T}}(s) R \dot{\omega}(s) \mathrm{d}s \geq \frac{1}{b-a}\begin{bmatrix}\varsigma_1\\ \varsigma_2\end{bmatrix}^T \begin{bmatrix} R & 0\\ 0 & 3R\end{bmatrix}\begin{bmatrix}\varsigma_1\\ \varsigma_2\end{bmatrix} \tag{11}$$

where $\varsigma_1 = \omega(b) - \omega(a)$ and $\varsigma_2 = \omega(b) + \omega(a) - 2\int_a^b \frac{\omega(s)}{b-a} ds$.

Lemma 3: (Reciprocally convex combination lemma [61]) For any vectors β_1 and β_2, symmetric matrix R, any matrix S, and real scalar $0 \leq \alpha \leq 1$ satisfying $\begin{bmatrix} R & S\\ * & R\end{bmatrix} \geq 0$, then the following inequality holds,

$$\frac{1}{\alpha}\beta_1^T R \beta_1 + \frac{1}{1-\alpha}\beta_2^T R \beta_2 \geq \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}^T \begin{bmatrix} R & S\\ * & R\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \tag{12}$$

III. A NEW STABILITY ANALYSIS METHOD

This section presents a new method for deriving stability criterion from two aspects, including the LKF constructing and its derivative estimating. Then, the influence of each technique used on the conservativeness and the complexity is discussed in detail, guidelines of improving criteria are summarized, and brief discussions of techniques that can further improve results and future work are also given.

A. Derivation of stability criterion

This part will derive a criterion based on Lyapunov theory step by step, that is, after constructing a candidate LKF, the asymptotical stability criterion is component of the conditions guaranteeing the positive and the decreasing of the LKF.

Step 1: *Construct a candidate LKF.* Firstly, a commonly used simple LKF candidate (before augmented) is given:

$$\bar{V}(t) = \bar{V}_1(t) + \bar{V}_2(t) + \bar{V}_3(t) + V_4(t) + V_5(t) \tag{13}$$

where

$$
\begin{aligned}
\bar{V}_1(t) &= x^T(t)\bar{P}x(t)\\
\bar{V}_2(t) &= \int_{t-d(t)}^{t}\big[x^T(s)\bar{Q}_{111}x(s) + f^T(s)\bar{Q}_{122}f(s)\big]ds\\
&\quad + \int_{t-h}^{t-d(t)}\big[x^T(s)\bar{Q}_{211}x(s) + f^T(s)\bar{Q}_{222}f(s)\big]ds\\
\bar{V}_3(t) &= h\int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)\bar{Z}\dot{x}(s)dsd\theta\\
V_4(t) &= \int_{t-h}^{t}\int_{\theta}^{t}\int_{s}^{t}\dot{x}^T(u)Q_3\dot{x}(u)dudsd\theta\\
V_5(t) &= 2\sum_{i=1}^{n}\int_0^{W_i x}\big[\lambda_{1i}(\sigma_i^+ s - f_i(s)) + \lambda_{2i}(f_i(s) - \sigma_i^- s)\big]ds
\end{aligned}
$$

and $\bar{P}$, $\bar{Q}_{i11}$, $\bar{Q}_{i22}, i = 1, 2$, $\bar{Z}$, and Q_3 are the symmetric matrices; and $\Lambda_i = \mathrm{diag}\{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in}\}, i = 1, 2$ are the symmetric diagonal matrices.

Secondly, a new LKF candidate is constructed based on $\bar{V}(t)$. On the one hand, by using the augmented idea proposed in [62], three parts of the above LKF are augmented as:

$$
\begin{aligned}
V_1(t) &= \varepsilon_3^T(t)P\varepsilon_3(t)\\
V_2(t) &= \int_{t-d(t)}^{t}\varepsilon_1^T(s)Q_1\varepsilon_1(s)ds + \int_{t-h}^{t-d(t)}\varepsilon_1^T(s)Q_2\varepsilon_1(s)ds\\
V_3(t) &= h\int_{-h}^{0}\int_{t+\theta}^{t}\varepsilon_2^T(s)Z\varepsilon_2(s)dsd\theta
\end{aligned}
$$

where P, $Q_i, i = 1, 2$, and Z are the symmetric matrices, and

$$Z = Z_a + Z_b, \quad Z_a = \begin{bmatrix} Z_1 & 0\\ 0 & Z_2\end{bmatrix}, \quad Z_b = \begin{bmatrix} Z_3 & Z_{12}\\ * & Z_4\end{bmatrix} \tag{14}$$

On the other hand, based on the idea of [52]–[54], two time-varying delay dependent terms are introduced:

$$V_6(t) = d(t)x^T(t)P_1x(t) + h_d(t)\varepsilon_4^T(t)P_2\varepsilon_4(t) \tag{15}$$

where $P_1 \in \mathcal{R}^{n\times n}$ and $P_2 \in \mathcal{R}^{2n\times 2n}$ are symmetric matrices.

Finally, the following augmented LKF will be applied:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t) + V_6(t) \tag{16}$$

Step 2: *Summarize the conditions to guarantee the positive of the LKF.* The candidate LKF will be positive via letting each term of the LKF be positive, i.e.,

$$P > 0; \quad P_i > 0; \quad \Lambda_i > 0, i = 1, 2 \tag{17}$$

$$Q_j > 0, j = 1, 2, 3; \quad Z = \begin{bmatrix} Z_1 + Z_3 & Z_{12}\\ * & Z_2 + Z_4\end{bmatrix} > 0 \tag{18}$$

That is, for a sufficient scalar $\epsilon_1 > 0$

$$(17),(18) \Rightarrow V(t) \geq \epsilon_1 ||x(t)||^2 \tag{19}$$

Step 3: *Derive the conditions for the negative of the LKF derivative.* Firstly, differentiating $V(t)$ considering (6) yields

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) + \dot{V}_5(t) + \dot{V}_6(t) \tag{20}$$

where

$$\begin{aligned}
\dot{V}_1(t) &= 2\varepsilon_3^T(t)P\dot{\varepsilon}_3(t) \\
&= 2\begin{bmatrix} x(t) \\ d(t)v_1(t) + h_d(t)v_2(t) \end{bmatrix}^T P \begin{bmatrix} \dot{x}(t) \\ x(t) - x_h(t) \end{bmatrix} \\
\dot{V}_2(t) &= \varepsilon_1^T(t)Q_1\varepsilon_1(t) - \varepsilon_1^T(t-h)Q_2\varepsilon_1(t-h) \\
&\quad + [1-\dot{d}(t)]\varepsilon_1^T(t-d(t))[Q_2 - Q_1]\varepsilon_1(t-d(t)) \\
\dot{V}_3(t) &= \varepsilon_2^T(t)[h^2 Z]\varepsilon_2(t) - h\int_{t-h}^{t} \varepsilon_2^T(s)Z\varepsilon_2(s)ds \\
&= \varepsilon_2^T(t)[h^2 Z]\varepsilon_2(t) - h\int_{t-h}^{t} \varepsilon_2^T(s)(Z_a + Z_b)\varepsilon_2(s)ds \\
\dot{V}_4(t) &= \frac{h^2}{2}\dot{x}^T(t)Q_3\dot{x}(t) - \int_{t-h}^{t}\int_{\theta}^{t} \dot{x}^T(s)Q_3\dot{x}(s)dsd\theta \\
&= \frac{h^2}{2}\dot{x}^T(t)Q_3\dot{x}(t) - \int_{t-d(t)}^{t}\int_{\theta}^{t} \dot{x}^T(s)Q_3\dot{x}(s)dsd\theta \\
&\quad - \int_{t-h}^{t-d(t)}\int_{\theta}^{t-d(t)} \dot{x}^T(s)Q_3\dot{x}(s)dsd\theta \\
&\quad - h_d(t)\int_{t-d(t)}^{t} \dot{x}^T(s)Q_3\dot{x}(s)ds \\
\dot{V}_5(t) &= 2\left\{[\Sigma_1 Wx(t) - f(t)]^T\Lambda_1 + [f(t) - \Sigma_2 Wx(t)]^T\Lambda_2\right\}W\dot{x}(t) \\
\dot{V}_6(t) &= \dot{d}(t)x^T(t)P_1x(t) + 2d(t)x^T(t)P_1\dot{x}(t) \\
&\quad - \dot{d}(t)\varepsilon_4^T(t)P_2\varepsilon_4(t) + 2h_d(t)\begin{bmatrix} x(t) \\ v_2(t) \end{bmatrix}^T P_2 \begin{bmatrix} \dot{x}(t) \\ \dot{v}_2(t) \end{bmatrix} \\
&= \dot{d}(t)\left[x^T(t)P_1x(t) - \varepsilon_4^T(t)P_2\varepsilon_4(t)\right] + 2d(t)x^T(t)P_1\dot{x}(t) \\
&\quad + 2h_d(t)\begin{bmatrix} x(t) \\ v_2(t) \end{bmatrix}^T P_2 \begin{bmatrix} \dot{x}(t) \\ \frac{(1-\dot{d}(t))x_d(t) - x_h(t) + \dot{d}(t)v_2(t)}{h_d(t)} \end{bmatrix}
\end{aligned}$$

Secondly, combining similar terms and estimating them. For estimating the $\dot{V}(t)$, the following conditions are assumed:

$$Z_i > 0, i = 1,2,\ Q_3 \geq 0,\ \Phi_1 = \begin{bmatrix} Z_2 & S_1 \\ * & Z_2 \end{bmatrix} \geq 0 \tag{21}$$

$$\Phi_{i+1} = \begin{bmatrix} hZ_3 & hZ_{12} + R_i \\ * & hZ_4 \end{bmatrix} > 0,\ i = 1,2 \tag{22}$$

$$\Phi_4 = \begin{bmatrix} Z_2 + Z_4 & S_2 \\ * & Z_2 + Z_4 \end{bmatrix} > 0 \tag{23}$$

where $S_i, i = 1,2$ are any $n \times n$ matrices. Three methods are applied to estimate three single integral terms in $\dot{V}(t)$. For the Z_a-dependent term, using (21) and Lemmas 1-3 yields

$$\begin{aligned}
&-h\int_{t-h}^{t} \varepsilon_2^T(s)Z_a\varepsilon_2(s)ds \\
&= -h\int_{t-d(t)}^{t} x^T(s)Z_1x(s)ds - h\int_{t-h}^{t-d(t)} x^T(s)Z_1x(s)ds \\
&\quad - h\int_{t-d(t)}^{t} \dot{x}^T(s)Z_2\dot{x}(s)ds - h\int_{t-h}^{t-d(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds \\
&\leq -d(t)v_1^T(t)[hZ_1]v_1(t) - h_d(t)v_2^T(t)[hZ_1]v_2(t) \\
&\quad - \frac{h}{d(t)}\begin{bmatrix} x(t) - x_d(t) \\ \xi_3(t) \end{bmatrix}^T \begin{bmatrix} Z_2 & 0 \\ 0 & 3Z_2 \end{bmatrix} \begin{bmatrix} x(t) - x_d(t) \\ \xi_3(t) \end{bmatrix} \\
&\quad - \frac{h}{h_d(t)}\begin{bmatrix} x_d(t) - x_h(t) \\ \xi_4(t) \end{bmatrix}^T \begin{bmatrix} Z_2 & 0 \\ 0 & 3Z_2 \end{bmatrix} \begin{bmatrix} x_d(t) - x_h(t) \\ \xi_4(t) \end{bmatrix}
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\leq -d(t)v_1^T(t)[hZ_1]v_1(t) - h_d(t)v_2^T(t)[hZ_1]v_2(t) \\
&\quad - \frac{h}{d(t)}(x(t) - x_d(t))^T Z_2(x(t) - x_d(t)) - 3\begin{bmatrix} \xi_3(t) \\ \xi_4(t) \end{bmatrix}^T \Phi_1 \begin{bmatrix} \xi_3(t) \\ \xi_4(t) \end{bmatrix} \\
&\quad - \frac{h}{h_d(t)}(x_d(t) - x_h(t))^T Z_2(x_d(t) - x_h(t))
\end{aligned} \tag{25}$$

For Z_b-dependent term, the following FWM-based zero-value term is defined based on Integration-By-Part [33], [34]:

$$\begin{aligned}
0 &= x^T(t)R_1x(t) - x_d^T(t)R_1x_d(t) - 2\int_{t-d(t)}^{t} x^T(s)R_1\dot{x}(s)ds \\
&\quad + x_d^T(t)R_2x_d(t) - x_h^T(t)R_2x_h(t) - 2\int_{t-h}^{t-d(t)} x^T(s)R_2\dot{x}(s)ds
\end{aligned} \tag{26}$$

where $R_i, i = 1,2$ are the $n \times n$ symmetric matrices. Then, adding it into the Z_b-dependent term in $\dot{V}_3(t)$ yields

$$\begin{aligned}
&-h\int_{t-h}^{t} \varepsilon_2^T(s)Z_b\varepsilon_2(s)ds \\
&= x^T(t)R_1x(t) + x_d^T(t)(R_2 - R_1)x_d(t) - x_h^T(t)R_2x_h(t) \\
&\quad - \int_{t-d(t)}^{t} \varepsilon_2^T(s)\Phi_2\varepsilon_2(s)ds - \int_{t-h}^{t-d(t)} \varepsilon_2^T(s)\Phi_3\varepsilon_2(s)ds
\end{aligned} \tag{27}$$

where $\Phi_i, i = 2,3$ are defined in (22). Then, based on (22), using Lemma 1 to estimate Φ_2- and Φ_3-dependent terms yields

$$\begin{aligned}
&-\int_{t-d(t)}^{t} \varepsilon_2^T(s)\Phi_2\varepsilon_2(s)ds - \int_{t-h}^{t-d(t)} \varepsilon_2^T(s)\Phi_3\varepsilon_2(s)ds \\
&\leq -\frac{1}{d(t)}\begin{bmatrix} d(t)v_1(t) \\ x(t) - x_d(t) \end{bmatrix}^T \Phi_2 \begin{bmatrix} d(t)v_1(t) \\ x(t) - x_d(t) \end{bmatrix} \\
&\quad - \frac{1}{h_d(t)}\begin{bmatrix} h_d(t)v_2(t) \\ x_d(t) - x_h(t) \end{bmatrix}^T \Phi_3 \begin{bmatrix} h_d(t)v_2(t) \\ x_d(t) - x_h(t) \end{bmatrix} \\
&= -d(t)v_1^T(t)(hZ_3)v_1(t) - 2v_1^T(t)[hZ_{12} + R_1](x(t) - x_d(t)) \\
&\quad - h_d(t)v_2^T(t)(hZ_3)v_2(t) - 2v_2^T(t)[hZ_{12} + R_2](x_d(t) - x_h(t)) \\
&\quad - \frac{h}{d(t)}(x(t) - x_d(t))^T Z_4(x(t) - x_d(t)) \\
&\quad - \frac{h}{h_d(t)}(x_d(t) - x_h(t))^T Z_4(x_d(t) - x_h(t))
\end{aligned} \tag{28}$$

Based on the (23) and Lemma 3, the sum of the $\frac{h}{d(t)}$- and $\frac{h}{h_d(t)}$-dependent terms in (25) and (28) can be estimated as:

$$-\begin{bmatrix} x(t) - x_d(t) \\ x_d(t) - x_h(t) \end{bmatrix}^T \Phi_4 \begin{bmatrix} x(t) - x_d(t) \\ x_d(t) - x_h(t) \end{bmatrix} \tag{29}$$

For Q_3-dependent single integral term, using $\frac{-h_d(t)}{d(t)} \leq \frac{-h_d(t)}{h}$ and Lemma 2 yields

$$-h_d(t)\int_{t-d(t)}^{t} \dot{x}^T(s)Q_3\dot{x}(s)ds \tag{30}$$

$$\leq -\frac{h_d(t)}{h}\begin{bmatrix} x(t) - x_d(t) \\ \xi_3(t) \end{bmatrix}^T \begin{bmatrix} Q_3 & 0 \\ 0 & 3Q_3 \end{bmatrix} \begin{bmatrix} x(t) - x_d(t) \\ \xi_3(t) \end{bmatrix} \tag{31}$$

Based on (21), using Lemma 1 to estimate two double integral terms in $\dot{V}_4(t)$ yields

$$\begin{aligned}
&-\int_{t-d(t)}^{t}\int_{\theta}^{t}\dot{x}^T(s)Q_3\dot{x}(s)dsd\theta-\int_{t-h}^{t-d(t)}\int_{\theta}^{t-d(t)}\dot{x}^T(s)Q_3\dot{x}(s)dsd\theta\\
&\leq -\frac{2}{d^2(t)}\int_{t-d(t)}^{t}\int_{\theta}^{t}\dot{x}^T(s)dsd\theta\ Q_3\int_{t-d(t)}^{t}\int_{\theta}^{t}\dot{x}(s)dsd\theta\\
&\quad -\frac{2}{h_d^2(t)}\int_{t-h}^{t-d(t)}\int_{\theta}^{t-d(t)}\dot{x}^T(s)dsd\theta Q_3\int_{t-h}^{t-d(t)}\int_{\theta}^{t-d(t)}\dot{x}(s)dsd\theta\\
&=-\left[x(t)-v_1(t)\right]^T(2Q_3)\left[x(t)-v_1(t)\right]\\
&\quad-\left[x_d(t)-v_2(t)\right]^T(2Q_3)\left[x_d(t)-v_2(t)\right]
\end{aligned}\tag{32}$$

The $\dot{d}(t)$-dependent terms in $\dot{V}(t)$ can be combined and estimated as follows:

$$\dot{d}(t)\xi_d^T(t)\Phi_5\xi_d(t)\leq\mu\xi_d^T(t)\Phi_5\xi_d(t)$$

where

$$\begin{aligned}
\Phi_5=&\begin{bmatrix}e_{d3}\\e_{d4}\end{bmatrix}^T(Q_1-Q_2)\begin{bmatrix}e_{d3}\\e_{d4}\end{bmatrix}+e_{d1}^TP_1e_{d1}-\begin{bmatrix}e_{d1}\\e_{d2}\end{bmatrix}^TP_2\begin{bmatrix}e_{d1}\\e_{d2}\end{bmatrix}\\
&-\mathrm{Sym}\left\{\begin{bmatrix}e_{d1}\\e_{d2}\end{bmatrix}^TP_2\begin{bmatrix}e_{d0}\\e_{d3}-e_{d2}\end{bmatrix}\right\}>0
\end{aligned}\tag{33}$$

Thirdly, by taking into account the assumption of the activation function, (7) and (8), the following inequalities hold:

$$\begin{aligned}
h_i(s)&:=2\left[\Sigma_3Wx(s)-f(s)\right]^TH_i\left[f(s)-\Sigma_2Wx(s)\right]\geq0\\
u_i(s_1,s_2)&:=2\left[\Sigma_3W(x(s_1)-x(s_2))-(f(s_1)-f(s_2))\right]^TU_i\\
&\quad\times\left[(f(s_1)-f(s_2))-\Sigma_2W(x(s_1)-x(s_2))\right]\geq0
\end{aligned}$$

where

$$H_i=\mathrm{diag}\{h_{1i},h_{2i},\cdots,h_{ni}\}\geq0,i=1,2,3\tag{34}$$

$$U_j=\mathrm{diag}\{u_{1j},u_{2j},\cdots,u_{nj}\}\geq0,j=1,2,3\tag{35}$$

Thus, the following inequalities hold:

$$h_1(t)+h_2(t-d(t))+h_3(t-h)\geq0\tag{36}$$

$$u_1(t,t-d(t))+u_2(t-d(t),t-h)+u_3(t,t-h)\geq0\tag{37}$$

Fourthly, using (20), (25)-(33), (36), and (37) and combining the $d(t)$- and $h_d(t)$-dependent terms yield

$$\dot{V}(t)\leq\xi^T(t)\Gamma(d(t))\xi(t)\tag{38}$$

where

$$\begin{aligned}
\Gamma(d(t))&=\Xi_0+\mu\Xi_1+d(t)\Xi_2+h_d(t)\Xi_3\\
\Xi_0&=\mathrm{Sym}\left\{\begin{bmatrix}e_1\\e_0\end{bmatrix}^TP\begin{bmatrix}e_s\\e_1-e_3\end{bmatrix}+\begin{bmatrix}e_1\\e_8\end{bmatrix}^TP_2\begin{bmatrix}e_0\\e_2-e_3\end{bmatrix}\right\}\\
&\quad+\begin{bmatrix}e_1\\e_4\end{bmatrix}^TQ_1\begin{bmatrix}e_1\\e_4\end{bmatrix}-\begin{bmatrix}e_3\\e_6\end{bmatrix}^TQ_2\begin{bmatrix}e_3\\e_6\end{bmatrix}+\begin{bmatrix}e_2\\e_5\end{bmatrix}^T(Q_2-Q_1)\begin{bmatrix}e_2\\e_5\end{bmatrix}\\
&\quad+\begin{bmatrix}e_1\\e_s\end{bmatrix}^T(h^2Z)\begin{bmatrix}e_1\\e_s\end{bmatrix}+e_s^T\Big(\frac{h^2Q_3}{2}\Big)e_s\\
&\quad+\mathrm{Sym}\{[(\Sigma_1We_1-e_4)^T\Lambda_1+(e_4-\Sigma_2We_1)^T\Lambda_2]We_s\}\\
&\quad-3\begin{bmatrix}e_1+e_2-2e_7\\e_2+e_3-2e_8\end{bmatrix}^T\Phi_1\begin{bmatrix}e_1+e_2-2e_7\\e_2+e_3-2e_8\end{bmatrix}\\
&\quad+e_1^TR_1e_1+e_2^T(R_2-R_1)e_2-e_3^TR_2e_3\\
&\quad-\begin{bmatrix}e_1-e_2\\e_2-e_3\end{bmatrix}^T\Phi_4\begin{bmatrix}e_1-e_2\\e_2-e_3\end{bmatrix}\\
&\quad-2(e_1-e_7)^TQ_3(e_1-e_7)-2(e_2-e_8)^TQ_3(e_2-e_8)\\
&\quad-\mathrm{Sym}\left\{\sum_{i=1}^{2}e_{i+6}^T(hZ_{12}+R_i)(e_i-e_{i+1})\right\}\\
&\quad+\sum_{i=1}^{3}\mathrm{Sym}\left\{(\Sigma_1e_i-e_{i+3})^TH_i(e_{i+3}-\Sigma_2e_i)\right\}\\
&\quad+\sum_{i=1}^{2}\mathrm{Sym}\left\{[\Sigma_1(e_i-e_{i+1})-(e_{i+3}-e_{i+4})]^TU_i\right.\\
&\qquad\left.\times[(e_{i+3}-e_{i+4})-\Sigma_2(e_i-e_{i+1})]\right\}\\
&\quad+\mathrm{Sym}\left\{[\Sigma_1(e_1-e_3)-(e_4-e_6)]^TU_3\right.\\
&\qquad\left.\times[(e_4-e_6)-\Sigma_2(e_1-e_3)]\right\}\\
\Xi_1&=\begin{bmatrix}e_2\\e_5\end{bmatrix}^T(Q_1-Q_2)\begin{bmatrix}e_2\\e_5\end{bmatrix}+e_1^TP_1e_1-\begin{bmatrix}e_1\\e_8\end{bmatrix}^TP_2\begin{bmatrix}e_1\\e_8\end{bmatrix}\\
&\quad-\mathrm{Sym}\left\{\begin{bmatrix}e_1\\e_8\end{bmatrix}^TP_2\begin{bmatrix}e_0\\e_2-e_8\end{bmatrix}\right\}\\
\Xi_2&=\mathrm{Sym}\left\{\begin{bmatrix}e_0\\e_7\end{bmatrix}^TP\begin{bmatrix}e_s\\e_1-e_3\end{bmatrix}+e_1^TP_1e_s\right\}-he_7^T(Z_1+Z_3)e_7\\
\Xi_3&=\mathrm{Sym}\left\{\begin{bmatrix}e_0\\e_8\end{bmatrix}^TP\begin{bmatrix}e_s\\e_1-e_3\end{bmatrix}+\begin{bmatrix}e_1\\e_8\end{bmatrix}^TP_2\begin{bmatrix}e_s\\e_0\end{bmatrix}\right\}-he_8^T(Z_1+Z_3)e_8\\
&\quad-\frac{1}{h}\begin{bmatrix}e_2-e_1\\e_1+e_2-2e_7\end{bmatrix}^T\begin{bmatrix}Q_3&0\\0&3Q_3\end{bmatrix}\begin{bmatrix}e_2-e_1\\e_1+e_2-2e_7\end{bmatrix}
\end{aligned}$$

Finally, by using convex combination and following the treatment in [47], the $\dot{V}(t)$ is negative if the follows hold:

$$\Phi_6:=-\Gamma(d(t))|_{d(t)=0}=-\Gamma(0)>0\tag{39}$$

$$\Phi_7:=-\Gamma(d(t))|_{d(t)=h}=-\Gamma(h)>0\tag{40}$$

Therefore, summarize the required conditions of Step 3 yields

$$(21)-(23),(33)-(35),(39),(40)\Rightarrow\dot{V}(t)<-\epsilon_2||x(t)||^2,\epsilon_2>0\tag{41}$$

Step 4: *Summarize the stability criterion.* Based on the above discussions, the following theorem can be summarized by combining conditions (19) and (41):

Theorem 1: For given scalars h and μ, if there exist $n\times n$ symmetric matrices $Z_i,i=1,2,3,4,Q_3,P_1,R_1,R_2$; $2n\times2n$ symmetric matrices P, P_2, Q_1,Q_2; $n\times n$ diagonal matrices Λ_1,Λ_2, U_j, $H_j,j=1,2,3$; and $n\times n$ matrices Z_{12}, S_1,S_2, such that the following LMIs hold

$$P>0;Z=\begin{bmatrix}Z_1+Z_3&Z_{12}\\*&Z_2+Z_4\end{bmatrix}>0\tag{42}$$

$$P_i>0,\ Z_i\geq0,\ \Lambda_i\geq0,i=1,2;\ Q_j\geq0,j=1,2,3\tag{43}$$

$$U_k\geq0;H_k\geq0;k=1,2,3\tag{44}$$

$$\Phi_l>0,l=1,2,\cdots,7$$

where Φ_1 is given in (21), Φ_2 and Φ_3 are given in (22), Φ_4 is given in (23), Φ_5 is given in (33), $\Phi_i,i=6,7$ are given in (39) and (40), respectively, then DNN (3) with time delay satisfying (4) and activation function satisfying (5) is asymptotically stable.

B. Discussions on the techniques used for criterion-deriving

This subsection gives some discussions on the techniques, whose influence on the conservativeness and the complexity is analyzed in detail.

(1) From the LKF-constructing point of view: Unlike the work of [12]–[16], [18]–[23], [37]–[39], [47]–[50], this paper does not follow the delay-partitioning idea to construct LKF, since the relationship between the conservativeness and the complexity can be easily predicted and two subintervals are advised in recent years [9], [10], [16], [20], [22], [37], [38], [47]–[49]. Meanwhile, the recent researches based on augmenting LKFs [16]–[20], [31]–[34], [40]–[45] just simply increase the complexity of the LKFs via lots of vectors, which greatly increases the calculation time due to too many decision matrices introduced. The main techniques of LKF-constructing in this paper, which are concerned with both the conservativeness and the complexity, are listed as follows:

- **Technique 1.1:** augmenting the original non-integral term, $\bar{V}_1(t)$, in (13) by introducing new vector $\int_{t-h}^{t} x(s)ds$;
- **Technique 1.2:** augmenting the original double integral term, $\bar{V}_3(t)$, in (13) by adding vector $x(s)$, especially the introduction of the Z_{12}-dependent cross term;
- **Technique 1.3:** introducing the delay-dependent terms, $V_6(t)$, based on the idea of [52]–[54].

The advantages of techniques related are analyzed as follows:

Firstly, the advantage of Technique 1.1 is analyzed from LMIs (39) and (40). The feasibility of those LMIs is dependent on the cross terms existing in $\xi^T(t)\Gamma(d(t))\xi(t)$, i.e.,

$$\xi^T(t)e_i^T X_{ij}(d(t))e_j\xi(t),\ i,j \in \{1,2,\cdots,8\} \tag{45}$$

where $X_{ij}(d(t))$ is the related matrix for cross terms, which consists of matrices to be determined and the system matrices. It is clear that the feasibility of the LMIs will be increased if all possible cross terms exist, i.e.

$$X_{ij}(d(t)) \neq 0, \quad \forall i,j \in \{1,2,\cdots,8\} \tag{46}$$

When Technique 1.1 is not used (i.e., $P = \mathrm{diag}\{\bar{P}, 0\}$), all existing cross terms are summarized as follows:

$$X_{ij}(d(t)) \neq 0 \ when \ \begin{cases} i=1,2,3 & j=1,2,\cdots,8; \\ i=4,5,6 & j=1,2,\cdots,6; \\ i=7,8 & j=1,2,3,7,8 \end{cases}$$

When Technique 1.1 is used, the connection between $v_{1,2}(t)$ and $\dot{x}(t)$ is constructed, which leads to the cross term $v_i^T(t)P_{22}\dot{x}(t)$, and due to $\dot{x}(t)$ including $f(t)$ and $f_d(t)$, thus the following new cross terms are obtained:

$$\xi^T(t)e_i^T X_{ij}(d(t))e_j\xi(t) \neq 0 \ when \ \begin{cases} i=\mathbf{4},\mathbf{5} & j=7,8; \\ i=7,8 & j=\mathbf{4},\mathbf{5} \end{cases}$$

The new cross terms introduced by the usage of Technique 1.1 provide more freedom and help to reduce the conservativeness. To verify its contribution clearly, the criterion for the case without such technique is given as follows:

Corollary 1: The stability criterion obtained from Theorem 1 by setting $P = \mathrm{diag}\{\bar{P}, 0\}$.

Remark 1: In the existing literature using augmented LKFs [17]–[20], [32]–[34], [40]–[45], the simple non-integral term in $\bar{V}_1(t)$ is usually augmented by simply introducing lots of state-based vectors, which greatly increases the number of decision variables and the dimension of the obtained LMI-based condition (The usage of $x(t–d(t))$ and $x(t–h)$ introduces $\dot{x}(t–d(t))$ and $\dot{x}(t–h)$ into $\xi(t)$ in (38), thus the dimension of $\Gamma(d(t))$ will increase). As a result, the calculation complexity will be greatly increased. Since the complexity should be considered, this paper provides a guideline to augmented this term, i.e., checking the status of cross terms existing in LMI-based conditions and introducing necessary augmented vectors. In fact, this guideline can be used to explain the contribution of the (37), which is firstly developed in [34] and is helpful to reduce conservativeness, which is shown from comparison of Theorems 1 and 2 in [34]. It is easy to find that the (37) introduces the cross terms $\xi^T(t)e_i^T X_{ij}(d(t))e_j\xi(t), i = 1,2,4,5, j = 6; i = 2, j = 4; i = 3, j = 4,5$, which do not appear if (37) is not used.

Secondly, the advantage of Technique 1.2 is analyzed based on LMI (22), i.e., $\Phi_{2,3} > 0$. If Technique 1.2 is not used, then there is no Z_{12}-dependent cross term, i.e., $Z_b = \mathrm{diag}\{Z_3, Z_4\}$. Thus, the conditions $\Phi_i > 0, i = 2,3$ will reduce to

$$\bar{\Phi}_i = \begin{bmatrix} hZ_3 & R_{i-1} \\ * & hZ_4 \end{bmatrix} > 0, i = 2,3 \tag{47}$$

The non-diagonal blocks of $\bar{\Phi}_i$ and Φ_i are respectively

$$\bar{\Phi}_i: \ R_{i-1}, i = 2,3 \tag{48}$$

$$\Phi_i: \ hZ_{12} + R_{i-1}, i = 2,3 \tag{49}$$

Obviously, the former is required to be strict symmetric to guarantee the holding of (26), while the later can be non-symmetric. That is to say, the usage of Technique 1.2 relaxes the constraint condition. To simplify the verification of this technique, the following corollary is given.

Corollary 2: The stability criterion obtained from Theorem 1 by setting $Z_{12} = 0$.

Remark 2: In [33], [34], there are some LMI conditions that require the non-diagonal terms be symmetrical, thus the results are more conservativeness. In fact, the above analysis gives another guideline to construct LKFs, namely, checking the constraint of the each location in LMI-based conditions and augmenting suitable quadratic terms to relax the constraint.

Thirdly, the advantage of Technique 1.3 is analyzed from the point of view of the matrix in the non-integral terms in the LKF. When Technique 1.3 is not applied, all non-integral terms in LKF are given as

$$V_1(t) = \varepsilon_3^T(t)P\varepsilon_3(t) = \begin{bmatrix} x(t) \\ v_1(t) \\ v_2(t) \end{bmatrix}^T \hat{P} \begin{bmatrix} x(t) \\ v_1(t) \\ v_2(t) \end{bmatrix} \tag{50}$$

where

$$\hat{P} = \begin{bmatrix} P_{11} & d(t)P_{12} & h_d(t)P_{12} \\ * & d^2(t)P_{22} & h_d(t)d(t)P_{22} \\ * & * & h_d^2(t)P_{22} \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix}$$

When the delay-dependent terms $V_6(t)$ is introduced via Technique 1.3, all non-integral terms in LKF become as

$$V_1(t) + V_6(t) = \begin{bmatrix} x(t) \\ v_1(t) \\ v_2(t) \end{bmatrix}^T \check{P} \begin{bmatrix} x(t) \\ v_1(t) \\ v_2(t) \end{bmatrix} \tag{51}$$

where

$$\check{P} = \hat{P} + \begin{bmatrix} d(t)P_1 + h_d(t)P_{211} & 0 & h_d(t)P_{212} \\ * & 0 & 0 \\ * & * & h_d(t)P_{222} \end{bmatrix}, P_2 = \begin{bmatrix} P_{211} & P_{212} \\ * & P_{222} \end{bmatrix}$$

It can be found, from the comparison of $\check{P}$ and $\hat{P}$, that $\check{P}$ has more general form than $\hat{P}$ does since the constraint of some locations is relaxed, such as location (1,1), $P_{11} = d(t)\frac{P_{11}}{h} + h_d(t)\frac{P_{11}}{h}$ in $\hat{P}$ is relaxed by $d(t)\frac{P_{11}+hP_1}{h} + h_d(t)\frac{P_{11}+hP_{211}}{h}$ in $\check{P}$, which will lead to the cross terms in obtained criterion with more general form. To simplify the verification of this technique, the following corollary is given.

Corollary 3: The stability criterion obtained from Theorem 1 by setting $P_i = 0, i = 1, 2$.

Remark 3: Compared with the similar quadratic terms used in [66], $V_7(t) = \xi_1^T(t)(d(t)P_1 + h_d(t)P_2)\xi_1(t)$, which is developed mainly for reducing conservativeness, the $V_6(t)$ is improved in consideration of both the complexity and the conservativeness. Firstly, the vectors $x(t-d(t))$ and $x(t-h)$ are excluded to avoid increasing the dimension of the LMI conditions, as mentioned in Remark 1; Secondly, the P_2-dependent term is included the $v_2(t)$, the usage of which does not increase the dimension of the LMI conditions and provides many new across terms, $\xi^T(t)e_i^T X_{ij}(d(t))e_j\xi(t), i = 1, 2, \cdots, 5, j = 8$; In order to avoiding the difficulty in dealing with the $\dot{d}(t)$-dependent terms, the P_1-dependent term of $V_6(t)$ is just chose as single form, instead of an augmented form similar to the P_2-dependent term. In addition, the $V_6(t)$ is more general than the similar one used in [31], which can only lead to the P_1-dependent term.

(2) From the estimation of LKF's derivative point of view: This part analyzes the techniques used for estimating the derivative of the LKF, including two techniques for single integral terms, a new way for triple integral term, and four techniques for treating $d(t)$ and $h-d(t)$.

Firstly, the single integral term (Z-dependent term) is treated by applying two techniques, shown as follows:

- **Technique 2.1:** applying Jensen's and Wirtinger's inequalities to evaluate the Z_a-dependent term;
- **Technique 2.2:** introducing Kwon's FWM-based zero equality to treat the Z_b-dependent term.

As mentioned in Section I, two types of methods have been proposed for treating single integral terms. The one is the inequality-based bounding technique. The other is the recombination technique based on FWM, including He's and Kwon's zero-value equalities, i.e.,

$$\text{He's}: \ 0 = \eta_1^T(t)\Omega(N)\eta_1(t) + 2\int_a^b \eta_2^T(t)N\dot{x}(s)ds \quad (52)$$

$$\text{Kwon's}: \ 0 = \eta_3^T(t)\Omega(R)\eta_3(t) + 2\int_a^b x^T(s)R\dot{x}(s)ds \quad (53)$$

He's zero-value equality is developed based on Newton-Leibniz formula, and the problem is that the optimal dimension of N is difficult to determine [47]; while the Kwon's zero-value equality is presented based on the Integration-By-Parts, and its drawback is that the symmetric requirement of the R is too strict. Theorem 1 uses both Wirtinger's inequality and Kwon's zero-value equality, and such treatment avoids their drawbacks. Specifically, Z-dependent term is divided into two parts, Z_a- and Z_b-dependent terms. On one hand, due to the separating of Z_{12}-dependent cross term from Z_a-dependent term, it can be estimated via both Jensen's and Wirtinger's inequalities. On the other hand, due to the existing of Z_{12}-dependent cross term in Z_b-dependent term, the strict constraint condition in (47) caused by symmetric requirement of the R, is relaxed (see the analysis for Corollary 2). To verify the contribution of those two techniques, the criteria obtained only using the one of them are given. The criterion only using Technique 2.1 can be obtained by letting $Z_b = 0, R_{1,2} = 0$; and the criterion only using Technique 2.2 can be obtained by letting $Z_a = 0, S_1 = 0$.

Corollary 4: The stability criterion obtained from Theorem 1 by setting $Z_{3,4} = 0, Z_{12} = 0, R_{1,2} = 0$.

Corollary 5: The stability criterion obtained from Theorem 1 by setting $Z_i = 0, i = 1, 2, S_1 = 0$.

Remark 4: Recently, the double integral term in the form of $V_3(t) = h\int_{-h}^{0}\int_{t+\theta}^{t}\varepsilon_2^T(s)Z\varepsilon_2(s)dsd\theta$ is usually introduced into an augmented LKF [16], [28], [31], [32], [34], and Wirtinger's inequality is known as the most effective technique for estimating single integral term, $h\int_{t-h}^{t}\dot{x}^T(s)Z_{22}\dot{x}(s)ds$, [51]. However, the integral term, $h\int_{t-h}^{t}\varepsilon_2^T(s)Z\varepsilon_2(s)ds$, in $\dot{V}_3(t)$ is difficultly estimated directly by using Wirtinger's inequality due to the existing of $x^T(s)Z_{12}\dot{x}(s)$. In order to overcome this problem, this paper divides Z into two parts, Z_a and Z_b. Then, non-diagonal matrix, Z_{12}, can be treated using Kwon's FWM-based equality method, and the diagonal matrices, $Z_{11} = Z_1 + Z_3$ and $Z_{22} = Z_2 + Z_4$, are respectively treated by using inequality bounding and Kwon's FWM-based equality methods (The optimal separation of Z_{11}, Z_{22} can be achieved since the optimal values of $Z_i, i = 1, 2, 3, 4$ can be found via LMI).

Secondly, for the double integral term (Q_3-dependent term), two types of methods have been used in the recent researches:

- Non-partition/one-step-enlargement method [10], [20], [29], [32]–[34], [44], non-partition means the domain of integration is $[t-h, t]$, and one-step enlargement means only one time enlargement is carried out, i.e., Q_3-dependent term is treated by

$$\begin{aligned} &-\int_{t-h}^{t}\int_{\theta}^{t}\dot{x}^T(s)Q_3\dot{x}(s)dsd\theta \\ &\leq -\frac{2}{h^2}\int_{t-h}^{t}\int_{\theta}^{t}\dot{x}^T(s)dsd\theta \ Q_3 \int_{t-h}^{t}\int_{\theta}^{t}\dot{x}(s)dsd\theta \end{aligned} \quad (54)$$

- One-partition/two-step-enlargement method [28], [35], one partition means the domain of integration is $[-h, -d(t)] \cup [-d(t), 0]$, and two-step enlargement means just two times enlargement are carried out, i.e., Q_3-dependent term is treated by

$$\begin{aligned} &-\int_{t-h}^{t}\int_{\theta}^{t}\dot{x}^T(s)Q_3\dot{x}(s)dsd\theta \\ =&-\int_{-d(t)}^{0}\int_{t+u}^{t}\dot{x}^T(s)Q_3\dot{x}(s)dsdu - \int_{-h}^{-d(t)}\int_{t+u}^{t}\dot{x}^T(s)Q_3\dot{x}(s)dsdu \\ \leq& -\frac{2}{d^2(t)}\int_{-d(t)}^{0}\int_{t+u}^{t}\dot{x}^T(s)dsdu \ Q_3 \int_{-d(t)}^{0}\int_{t+u}^{t}\dot{x}(s)dsdu \\ &-\frac{2}{h^2-d^2(t)}\int_{-h}^{-d(t)}\int_{t+u}^{t}\dot{x}^T(s)dsdu \ Q_3 \int_{-h}^{-d(t)}\int_{t+u}^{t}\dot{x}(s)dsdu \\ \leq& -\frac{2}{h^2}\begin{bmatrix} \int_{-d(t)}^{0}\int_{t+\theta}^{t}\dot{x}(s)dsd\theta \\ \int_{-h}^{-d(t)}\int_{t+\theta}^{t}\dot{x}(s)dsd\theta \end{bmatrix}^T \begin{bmatrix} Q_3 & S_0 \\ * & Q_3 \end{bmatrix} \begin{bmatrix} \int_{-d(t)}^{0}\int_{t+\theta}^{t}\dot{x}(s)dsd\theta \\ \int_{-h}^{-d(t)}\int_{t+\theta}^{t}\dot{x}(s)dsd\theta \end{bmatrix}. \end{aligned}$$

where S_0 is any $n \times n$ matrix. Obviously, the former does not consider the information of $d(t)$, and the later requires multi-step enlargement. Moreover, both of them lead to $d^2(t)$-dependent terms due to existing of $\int_{-d(t)}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)dsd\theta Q_3 \int_{-d(t)}^{0}\int_{t+\theta}^{t}\dot{x}(s)dsd\theta$. In this paper, Q_3-dependent term is divided into three parts via considering the $d(t)$, each of which needs one step enlargement (see (30) and (32)), i.e., it is a one-partition/one-step-enlargement method.

Thirdly, four techniques are used to treat the $d(t)$ and $h_d(t)$ in the deriving procedure, summarized as follows:

- In (25) and (29): Lemma 3 is used to deal with the $d(t)$- and $h_d(t)$-dependent terms, which only introduces two $n \times n$ matrices S_1 and S_2, less than the FWM approach does [6], [7], [11], [24];
- In (30), $-\frac{h_d(t)}{d(t)}$ is enlarged into $-\frac{h_d(t)}{h}$;
- In (32): The $d(t)$ and $h - d(t)$ are separated from the matrix to be determined (Q_3) and injected into $v_i(t), i = 1, 2$, and this technique is simpler and more effective than methods requiring enlargement;
- In (39) and (40): The related terms are treated by the convex combination technique to avoid the enlargement and the introducing of the extra matrix.

C. Some guidelines for criterion-deriving

This subsection summarizes simple guidelines to derive more effective criterion from different considerations. Assume the derivative of the simple LKF, $\bar{V}(t)$, can be estimated as

$$\dot{\bar{V}}(t) < \vartheta_0^T(t)\Theta_0(d(t))\vartheta_0(t) \tag{55}$$

where $\vartheta_0(t)$ is state-based vector, for DNN (3), vectors $\xi_1(t)$ and $\xi_2(t)$, which must appear in $\dot{\bar{V}}_2(t)$, are basic components of $\vartheta_0(t)$, and the usage of the Wirtinger's inequality, currently the most effective bounding technique, must lead to $v_1(t)$ and $v_2(t)$. Therefore, the $\vartheta_0(t)$ with the lowest dimension should be $\xi(t)$ as used in this paper. Clearly, the final stability criterion requires $\Theta_0(d(t)) < 0$, and the conservativeness-reducing is mainly achieved by making it more relaxable.

1) Only considering the conservativeness: based on the existing literature and the discussion in this paper, the possible ways that can reduce conservativeness are summarized as follows:

- Ways well-studied: Constructing LKFs with more general forms, and estimating the LKF's derivative with less enlargement.
- ways summarized in this paper: Introducing absent cross terms into LMI conditions (Technique 1.1), and weakening the constraint of matrices in LMI conditions (Technique 1.2).

By applying those treatments, the (55) will be replaced by

$$\dot{V}_{d/a}(t) \leq \begin{bmatrix}\vartheta_0(t)\\ \vartheta_1(t)\end{bmatrix}^T \left\{\begin{bmatrix}\Theta_0(d(t)) & 0\\ 0 & 0\end{bmatrix} + \Theta_1(d(t))\right\}\begin{bmatrix}\vartheta_0(t)\\ \vartheta_1(t)\end{bmatrix} \tag{56}$$

where $\vartheta_1(t)$ is caused by the delay-partition terms (for example $x(t-h/2)$ etc.) and the augmented terms (for example, $x(t-d(t))$ and $x(t-h)$ etc.) By adding $\Theta_1(d(t))$, the original condition $\Theta_0(d(t)) < 0$ may be relaxed so as to achieve the conservativeness-reducing.

2) Considering both the conservativeness and the complexity: in this paper, a useful guideline to derive criterion for this case is summarized as follows:

- Avoiding the extending of the $\vartheta_0(t)$ as much as possible during LKF-constructing and its derivative estimating;
- Introducing new cross terms that are less constraint and/or that are absent in original condition, $\Theta_0(d(t)) < 0$.

Then the (55) will be replaced by

$$\dot{V}_p(t) \leq \vartheta_0^T(t)\left[\Theta_0(d(t)) + \Theta_2(d(t))\right]\vartheta_0(t) \tag{57}$$

where $\Theta_2(d(t))$ is added to relax condition $\Theta_0(d(t)) < 0$ and reduce the conservativeness.

D. Techniques for further improvement and future work

This paper has further investigated some problems mentioned in our previous publication [47], including investigation of an effective augmented LKF and discussion the combination effect of different methods during estimating the derivative of LKF, and has given a new stability criterion (Theorem 1). By considering the tradeoff between the conservativeness and the complexity, many techniques, which are reviewed in Section I and benefit to conservativeness-reducing, have not been applied during the deriving of the criterion. When the researchers pay more attention to the conservativeness than to the complexity, many techniques can be applied to achieve this aim and some of them are listed as follows:

1) From the LKF point of view: Constructing more general LKFs, such as augmented LKFs, delay-partition-based LKFs, and LKFs with more multiple integral terms and/or activation function information.

2) From the LKF estimation point of view: Relaxing the positive conditions of the LKF by requiring the sum of all or some terms, instead of each term, together be positive [71].

3) From the LKF's derivative estimation point of view: Dividing the activation function into several subintervals similar to the delay-partition method [33], [34] (Note that the usage of this technique in some existing literature is unsuitable, see Remark 5 for detail). Choosing different β_1 and β_2 during the usage of Lemma 3, for example, the $\frac{h}{d(t)}$- and $\frac{h}{h_d(t)}$-dependent terms in (24) can be estimated directly via setting $\beta_1 = \begin{bmatrix}x(t)-x_d(t)\\ \xi_3(t)\end{bmatrix}$ and $\beta_2 = \begin{bmatrix}x_d(t)-x_h(t)\\ \xi_4(t)\end{bmatrix}$, but not estimated after being divided.

4) From the requirement of LMI-based conditions point of view: Introducing cross terms with less constraint to provide more free connections among different state-based vectors, for example, it can be found, from the discussion of Technique 1.1, that there still does not exist connection between $f_h(t)$ and $v_i(t), i = 1, 2$, and it also can be found, from the deriving of Theorem 1, that the cross terms among $f(t)$, $f_d(t)$, and $f_h(t)$ are only introduced in (37), where the related matrices $U_i, i = 1, 2, 3$ are required to be diagonal and symmetric. Thus, introducing new cross terms with more freedom on those connections may be helpful.

Remark 5: A new treatment that activation function partition is assumed to belong to two parts has been developed in

[33], [34]. Specifically, the activation function is divided into two subintervals, i.e., $\sigma_i^- \leq \frac{f_i(s)}{s} \leq \frac{\sigma_i^+ + \sigma_i^-}{2}$ and $\frac{\sigma_i^+ + \sigma_i^-}{2} \leq \frac{f_i(s)}{s} \leq \sigma_i^+$, then two cases are respectively discussed during the estimation of the derivative of LKF. However, it seems that such treatment is unsuitable. Since different elements of the activation function may belong to different subintervals, namely, when the one of element, $f_i(s)$, belongs to $\left[\sigma_i^-, \frac{\sigma_i^+ + \sigma_i^-}{2}\right]$, the other $n-1$ elements, $f_j(s), j \neq i$, may be bounded by $\left[\sigma_j^-, \frac{\sigma_j^+ + \sigma_j^-}{2}\right]$ or $\left[\frac{\sigma_j^+ + \sigma_j^-}{2}, \sigma_j^+\right]$, thus it is n^2 cases, but not two cases, that should be discussed.

Beside the above technique improvement that can be further investigated, there are other issues related to application extension that can be further studied, for example,

- This paper only considers the cast that the time-varying delay satisfies (4), how to derive the further results for the case that lower and upper bounds of $\dot{d}(t)$ are all available and the case that the lower bound of $d(t)$ is non-zero will be investigated.
- Other problems, such as robust stability, exponential stability, synchronization, etc., can be investigated by using the techniques proposed.

IV. Numerical examples

In this section, the advantages of the proposed criterion are discussed based on three typical numerical examples from the conservativeness and the complexity points of view. The stability criteria published in recent years, especially the ones obtained by augmented LKF, are used for the comparison.

A. Parameters of the DNNs

Consider DNN (3) with three different sets of parameters:

- Example 1:

$$A = \mathrm{diag}\{1.5, 0.7\}, \quad W = \mathrm{diag}\{1, 1\}$$
$$W_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix}$$
$$g(y) = \begin{bmatrix} 0.3\tanh(y_1) \\ 0.8\tanh(y_2) \end{bmatrix}, \quad J = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$$

It can be found that $\sigma_1^+ = 0.3,\ \sigma_2^+ = 0.8;\ \sigma_1^- = \sigma_2^- = 0$.

- Example 2:

$$A = \mathrm{diag}\{7.3458, 6.9987, 5.5949\}$$
$$W = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7290 & -2.6334 & -20.1300 \end{bmatrix}$$
$$W_0 = \mathrm{diag}\{0, 0, 0\}, W_1 = \mathrm{diag}\{1, 1, 1\}$$
$$g(y) = \begin{bmatrix} 0.3680\tanh(y_1) \\ 0.1795\tanh(y_2) \\ 0.2876\tanh(y_3) \end{bmatrix} \quad J = \begin{bmatrix} 0.1 \\ 0.6 \\ 0.3 \end{bmatrix}$$

It can be found that $\sigma_1^+ = 0.3680,\ \sigma_2^+ = 0.1795,\ \sigma_2^+ = 0.2876;\ \sigma_1^- = \sigma_2^- = \sigma_3^- = 0$.

- Example 3:

$$A = \mathrm{diag}\{1.2769, 0.6231, 0.9230, 0.4480\}$$
$$W_0 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix}$$
$$W = \mathrm{diag}\{1, 1, 1, 1\}$$
$$W_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}$$
$$g(y) = [0.1137\tanh(y_1),\ 0.1279\tanh(y_2),\ 0.7994\tanh(y_3),\ 0.2368\tanh(y_4)]^T$$
$$J = [0.4, 0.2, 0.3, 0.1]^T$$

It can be found that $\sigma_1^+ = 0.1137,\ \sigma_2^+ = 0.1279,\ \sigma_3^+ = 0.7994, \sigma_4^+ = 0.2368;\ \sigma_1^- = \sigma_2^- = \sigma_3^- = \sigma_4^- = 0$.

B. Calculation results

The results of calculation complexities and delay upper bounds for different criteria are given in this subsection.

1) The calculation complexity: Since the calculation time of LMIs is dependent on the maximal order of the LMIs (MOL) and the total number of the scalar decision variables (NDV), those factors are considered as the indexes of calculation complexity [60]. Table I lists the calculation complexities of the related criteria for the general DNN and the DNNs with given parameters, respectively, where 'n' is the dimension of the system; and 'Ex.', 'Th.', and 'Co.' indicate Example, Theorem, and Corollary, respectively.

2) The delay upper bound: The acceptable maximal upper bounds (AMUBs) of delay for different μ obtained by different methods are listed in Table II, where '—' indicates that the AMUBs for the corresponding cases are not given in literature.

Note that the calculated results given in [33], [34] have not been given in the tables, since an unsuitable treatment has been applied to derive the criteria in those literature, as mentioned in Remark 5.

C. Discussions

This subsection gives some discussions and summarizations based on the above calculated results.

1) The indexes indicating the improvements: In order to easily estimate the improvements of the proposed method, the follow three indexes are defined at first.

- *The percentage of the conservativeness-reducing (PCR):* the percentage of the increasing of the value calculated by Theorem 1 compared with the ones calculated by other criteria, i.e.,

$$PCR = \frac{AMUB_{Theorem\ 1} - AMUB_{other\ criteria}}{AMUB_{other\ criteria}} \times 100\%$$

- *The percentage of the NDV-decreasing (PND):* the percentage of the decreasing of the NDV of Theorem 1 compared with that of other criteria, i.e.,

$$PND = \frac{NDV_{Theorem\ 1} - NDV_{other\ criteria}}{NDV_{other\ criteria}} \times 100\%$$

- *The percentage of the MOL-decreasing (PMD):* the percentage of the decreasing of the MOL of Theorem 1 compared with that of other criteria, i.e.,

$$PMD = \frac{MOL_{Theorem\ 1} - MOL_{other\ criteria}}{MOL_{other\ criteria}} \times 100\%$$

Based on Tables I and II, the PCRs, PNDs, and PMDs are calculated and parts of results are listed in Table III.

TABLE I
THE CALCULATION COMPLEXITIES OF DIFFERENT CRITERIA

Criteria	DNN (3)		Ex. 1 ($n=2$)		Ex. 2 ($n=3$)		Ex. 3 ($n=4$)	
	NDV	MOL	NDV	MOL	NDV	MOL	NDV	MoL
[47] (Th.4)	$12.5n^2+23.5n$	$8n$	97	16	183	24	294	32
[28] (Th.1)	$63.5n^2+11.5n$	$13n$	277	26	606	39	1062	52
[31] (Th.1)	$30.5n^2+8.5n$	$11n$	139	22	300	33	522	44
[45] (Co.1)	$26n^2+17n$	$12n$	138	24	285	36	484	48
[43] (Co.1)	$38n^2+19n$	$11n$	190	22	399	33	684	44
Corollary 1	$13.5n^2+14.5n$	$8n$	83	16	165	24	274	32
Corollary 2	$14n^2+16n$	$8n$	88	16	174	24	288	32
Corollary 3	$11.5n^2+13.5n$	$8n$	73	16	144	24	238	32
Corollary 4	$12n^2+14n$	$8n$	76	16	150	24	248	32
Corollary 5	$13n^2+15n$	$8n$	82	16	162	24	268	32
Theorem 1	$15n^2+16n$	$8n$	92	16	183	24	304	32

TABLE II
THE AMUBS h FOR VARIOUS μ

Criteria	μ (Ex. 1)				μ (Ex. 2)			μ (Ex. 3)		
	0.40	0.45	0.50	0.55	0.0	0.10	0.50	0.10	0.50	0.90
[5], [11], [20], [22]	<5.3	<4.5	<4.2	<4.0	—	—	—	—	—	—
[8], [24], [32]	—	—	—	—	<1.6	<1.0	<0.43	—	—	—
[5], [6], [16], [17], [26]	—	—	—	—	—	—	—	<3.5	<2.6	<2.2
[15], [23], [25], [38]	—	—	—	—	—	—	—	<4.0	<2.9	<2.5
[11], [34], [48], [49]	—	—	—	—	—	—	—	<4.2	<3.2	<2.7
[47] (Th.4, $\rho=0.5$)	4.5023	3.7588	3.5472	3.4885	1.7683	1.0426	0.4313	3.8739	2.7415	2.3011
[28] (Th.1)	4.5543	3.8364	3.5583	3.4110	—	—	—	3.623	2.965	2.352
[31] (Th.1)	5.1029	4.1100	3.6855	3.4434	—	—	—	3.4984	2.7243	2.2029
[45] (Co.1)	—	—	—	—	1.8764	1.1127	0.4464	—	—	—
[43] (Co.1)	—	—	—	—	1.5575	0.9430	0.4417	—	—	—
Corollary 1	5.6504	4.7596	4.4276	4.2450	1.3327	0.8417	0.4327	3.9055	3.0997	2.6944
Corollary 2	7.5919	6.6339	6.2829	6.0999	1.8874	1.1023	0.4500	4.2729	3.0666	2.7687
Corollary 3	7.4203	6.6190	6.3428	6.2095	1.5857	0.9567	0.4432	4.1838	3.1510	2.8347
Corollary 4	7.5049	6.5563	6.2069	6.0237	1.8835	1.1001	0.4489	4.2732	3.0666	2.7648
Corollary 5	5.4065	4.6401	4.3715	4.2224	1.4877	0.8994	0.4330	3.8554	3.0278	2.6526
Theorem 1	7.6697	6.7287	6.4126	6.2569	1.8886	1.1163	0.4506	4.2993	3.1577	2.8371

TABLE III
THE PCRS, PNDS AND PMDS FOR VARIOUS CASES

Criteria	Ex. 1 (%)					Ex. 2 (%)					Ex. 3 (%)				
	$\text{PCR}_{\mu\in\{0.4,0.45,0.5\}}$			PND	PMD	$\text{PCR}_{\mu\in\{0.0,0.1,0.5\}}$			PND	PMD	$\text{PCR}_{\mu\in\{0.1,0.5,0.9\}}$			PND	PMD
[47] (Th.4)	70.35	79.01	80.77	-5.15	0	6.80	7.06	4.47	0.00	0	10.98	15.18	23.29	3.40	0
[28] (Th.1)	68.40	75.39	80.21	-66.78	-38.46	—	—	—	—	—	18.66	6.49	20.62	-71.37	-38.46
[31] (Th.1)	50.30	63.71	73.99	-33.81	-27.27	—	—	—	—	—	22.89	15.90	28.78	-41.76	-27.27
[45] (Co.1)	—	—	—	—	—	0.65	0.32	0.94	-35.78	-33.33	—	—	—	—	—
[43] (Co.1)	—	—	—	—	—	21.25	18.37	2.01	-54.13	-27.27	—	—	—	—	—
Corollary 1	35.73	41.37	44.83	10.84	0	41.71	32.62	4.13	10.90	0	10.08	1.87	5.29	10.94	0
Corollary 2	1.02	1.42	2.06	4.54	0	0.06	1.27	0.13	5.17	0	0.61	2.97	2.47	5.55	0
Corollary 3	3.36	1.65	1.10	26.02	0	19.10	16.68	1.66	27.08	0	2.76	0.21	0.08	27.73	0
Corollary 4	2.19	2.62	3.31	21.05	0	0.27	1.47	0.37	22.00	0	0.61	2.97	2.61	22.58	0
Corollary 5	41.86	45.01	46.69	12.19	0	26.94	24.11	4.06	12.96	0	11.51	4.29	6.95	13.43	0

2) Comparison and analysis: This part observes the comparisons from the above three tables and gives some analysis.

On one hand, the comparisons of the results obtained by Theorem 1 and the existing criteria show the improvements of the proposed method.

- The AMUBs calculated by Theorem 1 are bigger than the ones provided by other criteria for all cases, and the PCRs are over 5% for most cases, which show that Theorem 1 has less conservativeness.
- The NDVs and MOLs of Theorem 1 are smaller than the ones of other criteria for most cases (except for two cases of Th.4 in [47]), and PNDs and PMDs are all bigger than 25% for those cases, which shows that Theorem 1 will spend less time on the calculation and is more suitable for high-dimension systems.
- Although the NDV of Th.4 in [47] for EX. 3 is smaller than that of Theorem 1 ($294 < 304$), the usage of Th.4 in [47] requires to tune a parameter, ρ, which is a time-consuming procedure.

On the other hand, the comparisons of the results obtained by theorem and corollaries verify the benefits of the techniques discussed in Section III.B.

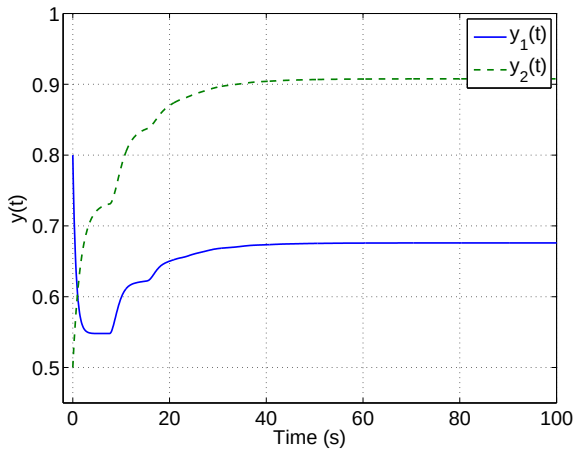

Fig. 1. State trajectories of the DNN of Example 1.

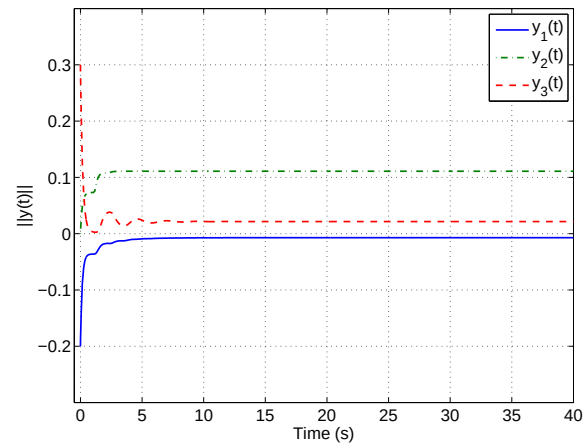

Fig. 2. State trajectories of the DNN of Example 2.

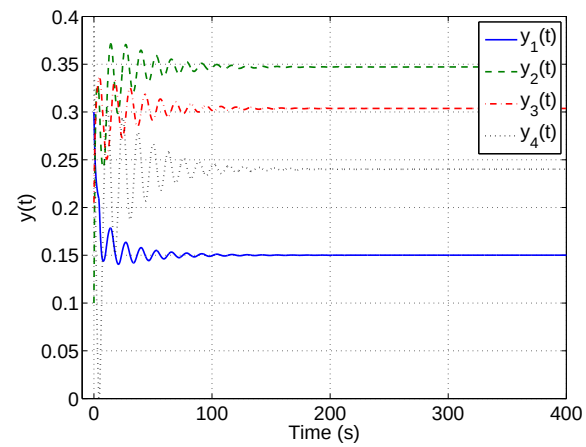

Fig. 3. State trajectories of the DNN of Example 3.

- The AMUBs by Theorem 1 are bigger than the ones by corollaries, which show that techniques mentioned in Section III.B indeed reduces the conservativeness. The results obtained by Theorem 1 are obviously bigger than the ones of Corollaries 1, 3, and 5, that is to say, the contribution of Techniques 1.1, 1.3, and 2.1 are more obvious.
- The MOLs for corollaries and theorem are same. From five corollaries, it is found that Technique 1.3 introduces the most additional decisions, the related PND for any dimension, n, must be small than 30.5% due to $\frac{15n^2+16n-(11.5n^2+13.5n)}{11.5n^2+19.5n} < \frac{3.5n^2}{11.5n^2} < 30.5\%$.
- By considering the PCRs and the PNDs, it can be found that Techniques 1.1 and 2.1 are more effective and worthy of further investigation.

D. Simulation verification

From the parameters of the DNNs, the equilibrium points of them can be obtained as $y^* = [0.6760, 0.9077]^T$, $y^* = [0.0071, 0.1110, 0.0216]^T$, and $y^* = [0.1501, 0.3471, 0.3037, 0.2401]^T$, respectively. From Table II, the DNNs are stable for the cases: Example 1, $\mu = 0.4$, and $h = 7.6697$; Example 2, $\mu = 0.1$, and $h = 1.1163$; and Example 3, $\mu = 0.1$, and $h = 4.2993$. Thus, simulation studies for the following three cases are given:

- Example 1: $y(t) = [0.8, 0.5]^T$, $t \in [-7.6697, 0]$; $d(t) = 7.2697 + 0.4\sin(t)$;
- Example 2: $y(t) = [0.2, 0.5, 0.1]^T$, $t \in [-1.1163, 0]$; $d(t) = 1.0163 + 0.8\sin(t)$;
- Example 3: $y(t) = [0.3, 0.1, 0.2, 0.4]^T$, $t \in [-4.2993, 0]$; $d(t) = 4.1993 + 0.1\sin(t)$;

The responses of the DNNs are shown in Figs. 1-3, and the results show that the DNNs are stable at their equilibrium points, which verifies the effectiveness of the proposed methods.

V. Conclusions

This paper has further investigated the delay-dependent stability of continuous DNNs, and a new criterion has been obtained via considering the tradeoff between the conservativeness and the calculation complexity. A new augmented LKF with delay-dependent terms has been developed from the point of view of the constraint of LMI-based conditions. An effective method to estimate the LKF's derivative has been investigated based on the combination of several techniques. The influence of techniques used has been analyzed in theory, and some useful guidelines for criterion-deriving has been summarized. The advantages of the proposed criterion have been verified via three numerical examples.

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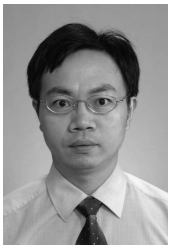

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