Optimal vibration control of beams subjected to a mass moving at constant speed

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Abstract

This paper studies optimal control of vibration of a beam excited by a moving mass. One important background of this work is vehicle-bridge interaction. As this is a time-varying system, some methods suitable for time-invariant systems are not always effective and will lead to suboptimal solutions when applied.

In this particular vibration problem, the terminal time instant when the moving mass leaves the beam and the moving-mass as the source of excitation are known. This particularity allows to express this problem in a very simple way as a fixed terminal-time optimal control problem. In this paper the limitations of the practical implementation of the control solution are discussed in relation to different performance indices and actuation strategies. Numerical results obtained by using several control methods (time-invariant, time-variant with or without bounds on the control force) are analysed and compared. It is shown that for particular actuator locations the use of time-varying control strategy instead of a time invariant strategy is necessary. The approach of formulating the system equation in an augmented form put forward in this paper is shown to yield accurate results at lower cost than the conventional time-dependent Riccati equation method. This approach is expected to be applicable to optimal control of vibration of other more complicated time-variant systems.

Keywords

Optimal control, vibration, beam, moving mass, time-variant system, Riccati equation

1. Introduction

Vibration control of flexible structures has gained much attention as the requirement of safety and comfort are more difficult to achieve at the same time when dimensional and weight restrictions are imposed. With respect to these the control design method needs to take into account all the possible information that characterizes the structural dynamic behaviour under particular loading conditions. The last decades have seen important achievements in both theoretical and experimental structural control. There is a large body of works where different control methods have been investigated and their feasibility in control of flexible structures assessed (Soong, 1990; Preumont, 2002; Alkhatib and Golnaraghi, 2003; Tatavolu and Panchumarthy, 2012). Among these the most common are the modal control method (Balas, 1978; Inman, 2001), methods based on the linear quadratic formulation (Abdel-Rohman et al. 1980), different algorithms for pole relocation or cancellation for symmetric (Ram and Mottershead, 2007) and asymmetric systems (Ouyang, 2010) or methods of robust control for the case where the uncertainties in the system are taken into account (Liu et al., 2010). A modal control method applicable to time-varying systems was presented in (Deng et al., 2010). This method was based on real-time identification. The identified model was subsequently used to design the controller.

Although the linear quadratic control-based design methods are very popular it has been observed that their effectiveness in minimization of the peak-response amplitude is somehow limited (Wu and Soong, 1996). Alternative formulations can provide a better response reduction and in this respect Wu and Soong (1996) studied the applicability of a bang-bang controller for vibration control. A bang-bang controller only provides two extreme values of the control force; the maximum or the minimum with an abrupt switch action between these two states. For this reason it provides a more rapid action than the linear quadratic controller which has to follow the state variation. In fact it has long been reported that a modification of the linear quadratic control laws can be beneficial for many control objectives. In Kim et al. (1997) a quadratic index designed to minimize the effects of the reaction forces on a beam was presented and compared with the effects of a controller designed by using a quadratic index in state variables. A non-quadratic performance index for an optimal control problem was also studied in Shamma and Xiong (1997).

The vibration control of beams under a moving mass/load is a particular example of control of flexible structures. This problem is generally studied in the context of bridge-vehicle/pedestrian interaction and due to particular aspects of the supporting structure dynamics many solutions proposed for vibration reduction are passive (Raftoyiannis and Michaltsos, 2012; Pierson et al., 2013; Muscolino and Palmeri, 2007; Casado et al., 2011; Younesian et al., 2006) but the background of the theoretical model allows wider practical applications, for instance, the stability conditions for catenary-pantograph systems (Lee et al., 2012) where the loss of contact creates problems of electric energy collection, or simplified models for overhead cranes dynamics. For general moving-load problems, please refer to, for example, Ouyang (2011).

Sung (2002) studied the vibration of a beam traversed by a moving mass with two piezoelectric actuators at different locations determined by the minimisation of an optimal quadratic cost functional. The controller was seen to perform well under unknown disturbances. The control of the dynamic interaction of a beam-moving mass system based on a quadratic performance index was presented in (Nikkhoo et al., 2007) where a displacement-velocity feedback controller assuming the full knowledge of the state variables was synthesised via an algebraic Riccati equation that took into account the time variation of the system matrices. A similar problem was treated in (Zarfam et al., 2012) where based on the same performance index a set of constant gain state feedback controllers were designed for minimization of the lateral deflection of a beam resting on an elastic foundation. Xiong et al. (1999) studied active control of power flow during vibration of a beam under moving load. Yau (2007) investigated the dynamic response of a maglev vehicle travelling over a series of guideway girders experiencing ground support settlement. Qian and Tang (2008) used a delayed controller to address the problem of instability induced by a moving force. A proportional-integral controller tuned with Ziegler–Nichols (Z–N) method was used to regulate the electromagnetic forces between the magnetic-wheels and guide-rail. Mofid et al. (2012) also simulated optimal control of a beam subjected to multiple masses with piezo-ceramic actuators using time-invariant Riccati matrix.

The control problem characteristic of this type of dynamic systems can be split into two subproblems (Tsao et al., 2001; Stancioiu et al., 2012): one of these concerns a time-varying system and corresponds to the time duration when the mass moves on the beam and the other one results in a time-invariant system and models the system after the mass moves off the beam. The first of these two subproblems concerning the time-varying vibration control of structures subjected to moving loads has some particular characteristics that make the optimal control theory easier to apply as compared to a general structural control problem. The equations governing the motion although with time-varying coefficients have the advantage of being linear. One other advantage, essential to this approach, is the existence of a given terminal time which is the time when the mass leaves the beam. The optimal problem can be formulated as a fixed terminal time problem (Naidu, 2003). It is also possible to take into account the constraints on the control force magnitude (Kirk, 1970, Kamien et al., 1991) due to limitations of the existing actuation devices.

The main idea of this paper is that once a problem of optimal control has been defined, the optimal solution associated with it is only optimal with respect to the performance index corresponding to the particular problem but it may not be the “optimal solution” a structural engineer looks for. For instance a quadratic performance index in output theoretically emphasises the importance of the control when the magnitude of the output increases, but for a linear quadratic problem the solution is given by a constant gain matrix which is determined beforehand and may not take into account all of the specific aspects of the dynamics of the solution. If for a beam-moving mass interaction problem the beam’s mid-span displacements are of interest the control engineer may think of a time-varying performance index that boosts the effect of the control when the moving mass passes near the middle of the span or alternatively may think of a performance index that reduces or boosts the control action when the moving mass passes above the actuation point (Stancioiu et al., 2013; Pisarski et al., 2010).

In this respect a series of optimal control methods that can be applied to active vibration control of time-varying systems are analysed with emphasis on the possible performance indexes that define in an appropriate manner the optimality conditions as well as the practical implementation of the methods. The focus is not only in making the best possible use of the mathematical apparatus that underlays the theory in order to define an appropriate performance index which best describes the physical problem but also to find a practical mean of implementation of a control solution that is able to approach as close as possible the optimal theoretical solution. An optimal solution in this study is in the form of a control-output pair and a comparison criterion is established based on the output response improvement in relation to the control effort.

2. Theoretical Aspects

One of the aims of this study is to briefly review some concepts and objectives of structural control from an optimal control point of view and to set them within a proper mathematical formulation specific of optimal control system theory. In this respect the concepts like “bang-bang” control are derived from the appropriate objective performance index established based on the control requirements. Another reason why this approach is preferred is that most control synthesis methods used in structural dynamics are particularly designed or adapted for time-invariant systems but the problem of minimization of the vibration response of a beam subjected to a moving mass is a time varying control problem.

2.1 Optimal control of differential systems

The problem of optimal control of a continuous system in a very general formulation consists of finding a control $\mathbf{u}(t)$ within a specified function space and the associated state variable $\mathbf{x}(t)$ as continuous time functions such that the performance functional:

$$J(u) = f_0\big(\mathbf{x}(t_\mathrm{f})\big) + \int_{t_0}^{t_\mathrm{f}} f\big(t, \mathbf{x}(t), \mathbf{u}(t)\big)\mathrm{d}t \tag{1}$$

subjected to:

$$\mathbf{x}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)) \tag{2}$$
$$\mathbf{x}(t_0) = \mathbf{x_0}$$

with f and $\mathbf{g}$ continuously differentiable in all the arguments, reaches its minimum. An optimal solution of system equations (2) is given by the pair $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ which realizes the minimum of the performance index.

The necessary conditions for the existence of solution can be determined using the minimum principle (Naidu, 2003; Kirk, 1970). With respect to this the optimization conditions are expressed in terms of the Hamiltonian function:

$$H\big(t,\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\big) = f\big(t,\mathbf{x}(t),\mathbf{u}(t)\big) + \boldsymbol{\lambda}^{\mathrm{T}}(t)\mathbf{g}(t,\mathbf{x}(t),\mathbf{u}(t)) \tag{3}$$

where $\boldsymbol{\lambda}(t)$ is the vector of piecewise continuous functions (costate variables).

The main advantage of introducing the Hamiltonian is that it changes the problem from finding the minimum of a functional $J(u)$ to finding the minimum of a function $H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda})$ and allows expressing the necessary conditions for extremum in a simpler manner as:

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}} = -\frac{\partial f}{\partial \mathbf{x}} - \boldsymbol{\lambda}^{\mathrm{T}}\frac{\partial \mathbf{g}}{\partial \mathbf{x}} \tag{4}$$

with terminal costates such that $\boldsymbol{\lambda}(t_f) = \partial f_0\big(\mathbf{x}(t_f)\big)/\partial \mathbf{x}$.

The optimality condition can be expressed as

$$H\big(t,\mathbf{x}^*(t),\mathbf{u}(t),\boldsymbol{\lambda}^*(t)\big) \geq H\big(t,\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t)\big) \tag{5}$$

where the optimal values for states $\mathbf{x}^*(t)$ and control $\mathbf{u}^*(t)$ are assumed.

When the differentiability of the control is altered, for instance if the control function is constrained within a specified rectangle $\mathbf{a} \leq \mathbf{u} \leq \mathbf{b}$ the optimal control of the minimization problem can be written as a piecewise continuous function (Kamien, 1991) :

$$\mathbf{u}^* = \begin{cases} \mathbf{a} & \dfrac{\partial H}{\partial \mathbf{u}} > 0 \\ \underset{\mathbf{a}\leq\mathbf{u}\leq\mathbf{b}}{\arg} \dfrac{\partial H}{\partial \mathbf{u}} = 0 & \dfrac{\partial H}{\partial \mathbf{u}} = 0 \\ \mathbf{b} & \dfrac{\partial H}{\partial \mathbf{u}} < 0 \end{cases} \tag{6}$$

2.2 Optimal control of linear systems

For a linear structural system the plant equations (2) can be expressed in state-space form as:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{f}(t) \tag{7}$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

where the input $\mathbf{f}(t)$ represents a force vector that acts as an input disturbance to the system.

When dealing with flexible structures, the control requirements could be stated as the minimization of the dynamic system's response. In this respect one natural control objective would be the output vibration amplitude minimization which will lead to a performance index like in (8).

$$J(\mathbf{x},\mathbf{u}) = \mathbf{x}^{\mathrm{T}}(t_{\mathrm{f}})\mathbf{S}\mathbf{x}(t_{\mathrm{f}}) + \frac{1}{2}\int_0^{t_{\mathrm{f}}} \mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t)\,\mathrm{d}t \tag{8}$$

When referring to the structure subjected to the action of a moving mass a control objective formulated this way could represent the minimization of the deflection/acceleration response of the supporting structure (beam) at certain positions.
On the other hand the limitations of the practical actuation solutions will constrain the control force $\mathbf{u}(t)$ within some specified bounds. Therefore the control function that solves problem (7) and (8) needs a constraint on the realizable control input action $|\mathbf{u}(t)| < \mathbf{U}$.

The required output may be a linear combination of some or all of the state variables $\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)$ and thus the weighting performance matrix will become $\mathbf{Q} = \mathbf{C}^{\mathrm{T}}\mathbf{C}$. The trade-off between the control performance and the control effort is determined by the bounds on the control $\mathbf{U}$. When the interest is directly in the state variables the $\mathbf{C}$ matrix is the identity matrix. It should be pointed out that for a moving load control problem matrix $\mathbf{C}$ could be a time dependent matrix $\mathbf{C}(vt)$ corresponding to a moving coordinate output reponse $\mathbf{y}(t) = \mathbf{C}(vt)\,\mathbf{x}(t)$.

Based on the minimum principle given in (5) a solution for this optimal control problem (7-8) can be determined by using the Hamiltonian function (3). After simplifications the optimality condition becomes:

$$\mathbf{u}(t)\,\mathbf{B}^{\mathrm{T}}(t)\boldsymbol{\lambda}^*(t) \geq \mathbf{u}^*(t)\,\mathbf{B}^{\mathrm{T}}(t)\boldsymbol{\lambda}^*(t) \tag{9}$$

In the case that on the interval $[0, t_f]$ there exists no subinterval on which the function $\boldsymbol{\lambda}^*(t)\,\mathbf{B}(t)$ is zero, the optimal control is given by a piecewise constant function of time (bang-bang control)

$$\mathbf{u}^*(t) = -U\,\mathrm{sgn}(\mathbf{B}^{\mathrm{T}}(t)\cdot\boldsymbol{\lambda}^*(t)) \tag{10}$$

and takes only two values $\mp U$. This control function has the advantage of making direct use of the maximum control force required and this way it is possible to produce a significant peak-response reduction. The control function (10) is discontinuous and many numerical methods have difficulties dealing with this problem.

Although the performance index (8) and the control constraint given above express naturally occurring requirements for a quite important number of dynamical systems including some structural systems, the possibility of practically implementing a piecewise control solution like the one given in (10) depends heavily on the actuation system and even finding a solution presents mathematical difficulties. Wu and Soong (1996) presented a suboptimal solution for this problem whereas the control is determined via a Lyapunov equation. The solution proposed has the advantage of avoiding an off-line solution of the costate system.
A natural extension of this optimal problem is obtained by using a quadratic performance index both in control and state variables in (11).

$$J(\mathbf{x},\mathbf{u}) = \mathbf{x}^{\mathrm{T}}(t_{\mathrm{f}})\mathbf{S}\mathbf{x}(t_{\mathrm{f}}) + \frac{1}{2}\int_{0}^{t_{\mathrm{f}}} \left(\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)\right) \mathrm{d}t \tag{11}$$

This performance index (11) represents a trade-off between the requirement of a small state variation and a small control effort and it is capable of achieving either a limited control effort or a tight control by balancing the values of the elements of the states-weighted matrix $\mathbf{Q}$ and the control-weighted matrix $\mathbf{R}$. In a similar manner to the previously presented case by changing the values of $\mathbf{R}$ the control amplitude levels can be also limited to fit within prescribed bounds $|\mathbf{u}(t)| < \mathbf{U}$. The major difference in using the control-weighted matrix $\mathbf{R}$ to limit the control effort is that the control function is now continuous; on the other hand the quick access to high values of the control forces characteristic of the bang-bang control is slowed down by the requirement that the control follows first the state variables instead of jumping directly to the maximum value possible.

One important advantage when using a performance index with a quadratic term in control is that when the control function is unconstrained, for a linear system like (7) the associated Hamiltonian is pointwise differentiable with respect to control variable and the optimality condition takes a simpler form

$$\frac{\partial H}{\partial u} = \mathbf{R}\mathbf{u} + \mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda} = 0 \tag{12}$$

When the control function values are limited to a finite interval, $\mathbf{a} \leq \mathbf{u} \leq \mathbf{b}$ the control force that will minimize the index in (11) is:

$$\mathbf{u}^{*}(t) = \begin{cases} \mathbf{a} & \mathbf{R}\mathbf{u}(t) + \mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) > 0 \\ -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) & \mathbf{R}\mathbf{u}(t) + \mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) = 0 \\ \mathbf{b} & \mathbf{R}\mathbf{u}(t) + \mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) < 0 \end{cases} \tag{13}$$

On the other hand the Hamiltonian associated with this performance index and plant equations (7) is pointwise differentiable with respect to costate variables and the necessary conditions for extremum (4) can be expressed as

$$\dot{\boldsymbol{\lambda}}(t) = -\mathbf{Q}\mathbf{x}(t) - \mathbf{A}^{\mathrm{T}}(t)\boldsymbol{\lambda}(t) \tag{14}$$
$$\boldsymbol{\lambda}(t_{\mathrm{f}}) = \mathbf{S}\mathbf{x}(t_{\mathrm{f}})$$

Equation (14) along with the state equations (7) and the optimality condition constitute a two-point boundary value problem whose solution gives the optimal state variables $\mathbf{x}^{*}(t)$.

The main disadvantage of both the above control methods is that they rely on an off-line time variable solution of the costate equation (14) which should be fed into the controller using a look-up table. From a mathematical point of view the state and costate variables are coupled due to the quadratic nature of the state term $\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t)$ in the performance index and this leads to a two-point boundary value problem.

2.3. Linear quadratic regulation

A particular case arises when the control function is formulated with a quadratic index as (11) and the control is not restricted to a specified rectangle. The control can be expressed as a linear function of the states

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}(t)\,\mathbf{x}(t) \tag{15}$$

where matrix $\mathbf{P}(t)$ is obtained by solving backward the following differential Riccati equation:

$$\dot{\mathbf{P}}(t) = -\mathbf{A}\mathbf{P}(t) - \mathbf{A}^{\mathrm{T}}\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}(t) - \mathbf{Q} \tag{16}$$

with final condition: $\mathbf{P}(t_{\mathrm{f}}) = \mathbf{S}$. Equation (16) is completely independent of the states of the system and can be solved separately.

The most important advantage of this method is that it allows for the control to be expressed as a state-feedback function (15) and transforms the system from an open-loop control system into a closed-loop control system

$$\begin{gathered}\dot{\mathbf{x}}(t) = \left(\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}(t)\right)\mathbf{x}(t)\\ \mathbf{x}(0) = \mathbf{x}_0\end{gathered} \tag{17}$$

but still the Riccati matrix should be calculated off-line.

By using this approach the control function is synthesised based on the system equations regardless of the effect of any force acting at the system input. In section 3.2 a simple change of variable will be introduced which will allow for the forcing component $\mathbf{f}(t)$, in this case the time-varying modal force to be taken into account during the design process.

2.4. Control method based on step by step calculation of the Riccati equation

All the control methods considered above require at least an off-line calculation of the Riccati matrix equation or of the costate equations. A way around this was suggested by the state-dependent Riccati equation technique (Cloutier, 1997; Souza and Gonzales, 2012). This is an on-line method that can be applied to particular nonlinear systems and consists of solving an algebraic Riccati equation at discrete time steps along the state trajectory. As it cannot take into account the transversality conditions it cannot provide a globally optimal solution.

Instead of solving backward off-line the time-varying Riccati equation (16), the algorithm proposed here in a manner similar to the state-dependent Riccati equation technique mimics the linear quadratic problem solution for time-invariant systems and consists of solving at every time step t_i the algebraic Riccati equation

$$\mathbf{P}\,\mathbf{A}(t_i) + \mathbf{A}^{\mathrm{T}}(t_i)\mathbf{P} - \mathbf{P}\mathbf{B}(t_i)\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}(t_i)\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{18}$$

which subsequently gives the control force $\mathbf{u}(t_i) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}(t_i)\mathbf{P}\mathbf{x}(t_i)$ at time instant t_i.

3. Control function synthesis for the moving mass problem

For a general structural dynamic problem where a system given in state-space form (7) is subjected to an uncertain external load acting over an indefinite period of time the applicability of the synthesis of an optimal control as presented in section 2 is somehow restricted.

When the supporting structure is modelled as a beam under the action of a moving mass the most relevant aspect of dynamic interaction is that in modal coordinates, the dynamic system is a time-varying system and the simple application of a method specific to a time-invariant system will end-up with a suboptimal control system whose response although improved may still be far away from the optimal one. In this respect any knowledge of a time-varying optimal solution synthesised using an appropriate performance index will be of advantage as it lets the control engineer know how close the proposed solution is to the optimal one.

From a theoretical point of view the problem of vibration control particularized to a moving load problem introduces a fixed terminal time $t_{\mathrm{f}} = L/v$ which enables a formulation of the optimal control problem in the way it was presented in Section 2. Moreover, the excitation function which depends on the moving mass m and moving speed v is known. This allows the synthesis of the control action taking the excitation force into account by means of a simple change of variables (26).

As a disadvantage even though an optimal solution $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ exists and it is determined as a continuous or piecewise continuous function, depending on the actuation system, it may not be practically possible to implement it. In this case the optimal pair $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ provides a theoretical solution that solves the problem within a given setup and this optimal pair can be subsequently set as a target for a practical realization.

Another aspect concerns the practical difficulty of dealing with a solution based on state variables. The state variables are not always completely determined and usually require a state observer included within the controller. Some of the control design solutions presented in this study can easily take this into account, but this aspect will not be considered herein.

3.1 Dynamic equations of a beam-moving mass system

In modal coordinates, the general system of equations governing the dynamics of a beam of length L traversed by a mass m traveling at constant speed v at any time t within the interval $[0, t_{\mathrm{f}}]$ with $t_{\mathrm{f}} = L/v$, modelled using Euler-Bernoulli theory and under the assumption there is no contact loss is (Ouyang, 2011; Stancioiu et al., 2011):

$$(\mathbf{M} + \mathbf{\Delta M}(t))\ddot{\mathbf{q}} + (\mathbf{D} + \mathbf{\Delta D}(t))\dot{\mathbf{q}} + (\mathbf{K} + \mathbf{\Delta K}(t))\mathbf{q} = -m\mathrm{g}\boldsymbol{\psi}(vt) \tag{19}$$

The system is assumed at rest for $t \leq 0$.

Eq. (19) represents a system with time-varying matrix coefficients. The time-invariant matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ can be expressed as functions of the modal shape vectors $\boldsymbol{\psi}(x)$, mass per unit length ρA , damping ρAc and stiffness EI (Stancioiu et al., 2011):

$$\mathbf{M} = \rho A \int_0^L \boldsymbol{\psi}(x) \cdot \boldsymbol{\psi}(x)\mathrm{d}x, \quad \mathbf{D} = \rho Ac \int_0^L \boldsymbol{\psi}(x) \cdot \boldsymbol{\psi}(x)\mathrm{d}x, \ \mathbf{K} = EI \int_0^L \boldsymbol{\psi}(x) \cdot \boldsymbol{\psi}''''(x)\mathrm{d}x \tag{20}$$

The time dependent matrix coefficients that appear in (19) are explicitly defined by (Stancioiu et al., 2011):

$$\mathbf{\Delta M}(t) = m\boldsymbol{\psi}(vt)\boldsymbol{\psi}^{\mathrm{T}}(vt), \quad \mathbf{\Delta D}(t) = 2mv\boldsymbol{\psi}(vt)\boldsymbol{\psi}'^{\mathrm{T}}(vt), \quad \mathbf{\Delta K}(t) = mv^2\boldsymbol{\psi}(vt)\boldsymbol{\psi}''^{\mathrm{T}}(vt) \tag{21}$$

and depend directly on the moving mass and its position vt.

The prime symbol defines the derivative of the modal vector with respect to the moving coordinate $s(t) = vt$:

$$\psi'(t) = \frac{\partial \psi}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}t} \tag{22}$$

After time instant t_{f} the beam vibrates freely and the system of equations governing the dynamics of the free vibrations $t > t_{\mathrm{f}}$ is:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0 \tag{23}$$

with $(\mathbf{q}^{\mathrm{T}}(t_{\mathrm{f}}) \quad \dot{\mathbf{q}}^{\mathrm{T}}(t_{\mathrm{f}}))^{\mathrm{T}}$ the terminal states of system (1) as initial conditions.

3.2. Control problem for a beam-moving mass system

As already mentioned in section 3.1 the dynamics of the beam subjected to a moving mass and accordingly the control problem can be mathematically separated in two subproblems. When the mass moves on the beam, the control problem is a time-varying optimal control problem with fixed terminal time where methods of optimal control as presented in section 2 can be used. When the mass leaves the beam a linear time-invariant control problem for the free beam vibration could be formulated and methods like LQR or pole placement can be used. In this study the control objective is to minimize the beam's deflection response during the moving load action.

The dynamic equations associated with the moving mass problem (19) for $t \leq t_{\mathrm{f}}$ when the beam is controlled by a set of k controllers positioned at coordinates: $x_{\mathrm{a}1}$, $x_{\mathrm{a}2}$,… $x_{\mathrm{a}k}$ should be modified to

$$(\mathbf{M} + \boldsymbol{\Delta}\mathbf{M}(t))\ddot{\mathbf{q}} + (\mathbf{D} + \boldsymbol{\Delta}\mathbf{D}(t))\dot{\mathbf{q}} + (\mathbf{K} + \boldsymbol{\Delta}\mathbf{K}(t))\mathbf{q} = -mg\boldsymbol{\psi}(vt) - \sum_{i=1}^{k} \boldsymbol{\psi}(x_{\mathrm{a}i})\mathbf{u}_i \tag{24}$$

by adding on the right hand side of equation (19) the term describing the control action.

When these equations are written in state-space form (7), the time variable coefficient matrices are:

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -(\mathbf{M} + \boldsymbol{\Delta}\mathbf{M}(t))^{-1}(\mathbf{K} + \boldsymbol{\Delta}\mathbf{K}(t)) & -(\mathbf{M} + \boldsymbol{\Delta}\mathbf{M}(t))^{-1}(\mathbf{D} + \boldsymbol{\Delta}\mathbf{D}(t)) \end{bmatrix} \tag{25}$$

$$\mathbf{B}(t) = \begin{bmatrix} \mathbf{0}_{n\times k} \\ -(\mathbf{M} + \boldsymbol{\Delta}\mathbf{M}(t))^{-1}[\boldsymbol{\psi}(x_{\mathrm{a}1}) \quad \dots \quad \boldsymbol{\psi}(x_{\mathrm{a}k})] \end{bmatrix}$$

with $\mathbf{x}^{\mathrm{T}}(t) = [\mathbf{q}(t) \quad \dot{\mathbf{q}}(t)]$ as state variable.

Another particularity of the moving mass problem is that the excitation term which is given by the modal force $mg\boldsymbol{\psi}(vt)$ is known and can be considered when the control law is determined. Working with the state-space system determined by the matrices described in (25) will not allow taking this advantage into account but using a change of variables the system matrices can be augmented to

$$\mathbf{A}(t) = \tag{26}$$

$$\begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} \\ -(\mathbf{M} + \boldsymbol{\Delta}\mathbf{M}(t))^{-1}(\mathbf{K} + \boldsymbol{\Delta}\mathbf{K}(t)) & -(\mathbf{M} + \boldsymbol{\Delta}\mathbf{M}(t))^{-1}(\mathbf{D} + \boldsymbol{\Delta}\mathbf{D}(t)) & -(\mathbf{M} + \boldsymbol{\Delta}\mathbf{M}(t))^{-1}\boldsymbol{\psi}(vt) \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & 0 \end{bmatrix}$$

$$\mathbf{B}(t) = \begin{bmatrix} \mathbf{0}_{n\times k} \\ -(\mathbf{M} + \boldsymbol{\Delta}\mathbf{M}(t))^{-1}[\boldsymbol{\psi}(x_{\mathrm{a}1}) \quad \dots \quad \boldsymbol{\psi}(x_{\mathrm{a}k})] \\ 0 \end{bmatrix}$$

This formulation includes the right hand side of the equation (19) as a new state variable q_1. The state variable vector becomes $\mathbf{x}^{\mathrm{T}}(t) = [\mathbf{q}(t) \quad \dot{\mathbf{q}}(t) \quad q_1]$ and in S.I. has $[\mathbf{q}(0) \quad \dot{\mathbf{q}}(0) \quad 9.81 \times m]$ as initial conditions. It can be seen that the newly introduced state-variable q_1 is a constant and equals $9.81 \times m$ which represents exactly the weight of the moving mass *m*. The advantage of working with the augmented system (26) is that the required control action will now take into account directly the excitation force rather than considering it as a disturbance. At this point it should be made clear that matrix **B** in both representations depends on the location of the actuators and this not only affects the design process but also can affect to a large extent the controllability of the system.

One other aspect relating to deflection control of a beam subjected to a moving load is that not only it is a time-varying problem but also the control requirements may ask for time dependent weighting matrices. For instance, the objective may be formulated in terms of beam's deflection at the moving coordinate. In this case the output is $\mathbf{y}(t) = \mathbf{C}(vt)\mathbf{q}(t)$ with $\mathbf{C}(vt) = \boldsymbol{\psi}(vt)$ and $\mathbf{Q} = \mathbf{C}^{\mathrm{T}}(vt)\,\mathbf{C}(vt)$. A similar case was studied in (Stancioiu and Ouyang, 2013) where the weighting matrices were allowed to vary following the position of the mass on the beam in order to make a better use of the control effort.

4. Numerical Example

In order to assess the performance indexes presented in section 2 and their suitability to address a structural control problem of the type presented in section 3, the methods of optimal control are theoretically tested for the minimization of deflection response of a single span simply-supported beam subjected to the action of a point mass m moving at a constant speed v along the span length. The geometrical and dynamical characteristics of the beam structure are: span length L=1m, mass per length unit ρA=0.22 kg×m^{-1} and flexural rigidity EI=3.66 Nm^{-2}. A constant modal damping coefficient ζ=0.01 is assumed throughout. For a simply-supported beam, the i^{th} mode $\psi_i(x) = \sin(i\,\pi\,x/L)$.

In the following analysis only the first three modes are used as increasing the number of modes considered for analysis over two do not improve the accuracy of the beam deflection results. The addition of the third mode was necessary for control function $\mathbf{u}(t)$ convergence but even so this has no real implications on the accuracy of the deflection response.

The assessment of the control action improvement on the response will be based on the level of reduction of the mid-span deflection amplitude of the supporting structure. Unfortunately a comparison criterion is not easy to define even though the control methods are based on similar performance indexes. In what follows different design methods will be compared by inspection of both time histories of the response and required control force action to achieve this response. For instance, if the same order of the amplitude response reduction will be obtained with different levels of control effort the better method will be considered the one with the lowest level of control effort.

4.1. Unbounded control action

The methods presented in section 2 for time-varying systems are used for the first part of the motion $t \leq t_{\mathrm{f}}$ to obtain the deflection response of the controlled system (24). The solution is completed for the free vibration part of the motion with the deflection response obtained for the time-invariant part of the beam's dynamics with a control function synthesised from the linear quadratic control problem as $\mathbf{u}(t) = \mathbf{K}_{\mathrm{g}}\,\mathbf{x}(t)$. The constant gain control matrix $\mathbf{K}_{\mathrm{g}}$ is determined for the time-invariant system with matrices:

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}; \qquad \mathbf{B}(t) = \begin{bmatrix} \mathbf{0}_{n\times k} \\ -\mathbf{M}^{-1}[\boldsymbol{\psi}(x_{\mathrm{a}1}) \quad \dots \quad \boldsymbol{\psi}(x_{\mathrm{a}k})] \end{bmatrix} \tag{27}$$

and then used to control the beam's dynamic response from time instant t_{f} when the mass steps off the beam until t=2.5L/v.

In most of the cases considered the actuation points are located at two fixed positions on the beam $x_{\mathrm{a}1}$= 0.157 m and $x_{\mathrm{a}2}$=0.811 m (Fig 1A). The effects of the controller action on the deflection response are shown at the mid-span position $x = L/2$. Another case when the actuators are positioned due to constructive restrictions on one side of the beam at $x_{\mathrm{a}1}$= 0.11 m and $x_{\mathrm{a}2}$=0.15 m is studied for some of the control methods, as well as a case when only one actuator is positioned at $x_{\mathrm{a}1}$= 0.13 (Fig. 1B).

Figure 1. Actuators positions for a single span beam example

A. two actuators positioned on both sides at locations $x_{\mathrm{a}1}$ and $x_{\mathrm{a}2}$

B. one actuator positioned at x_{a}

By defining the error and control weighting matrices given in the quadratic performance index as $\mathbf{Q} = \mathrm{diag}(1000,100,10,10)$ and $\mathbf{R} = \mathrm{diag}(0.1, 0.1)$ for the time interval when the mass is on the beam the constant gain matrix obtained for a time-invariant linear quadratic formulation is given by

$$\mathbf{K}_{\mathrm{g}}(x_{\mathrm{a}1}, x_{\mathrm{a}2}) = \begin{bmatrix} 8.6 & 2.6 & 7.1 & 6.64 \\ 9.1 & 2.4 & 7.04 & -6.59 \end{bmatrix} \tag{28}$$

Based on the knowledge of the particular dynamic behaviour exhibited by a single-span supporting structure whereas the response in displacements is mainly influenced by the first mode, in defining the weighting matrix **Q** the aim is to emphasise the importance of the first mode. The control weighting matrix **R** is made small enough to allow for the required control action.

Formula (28) shows that although the gain matrix is constant, it is determined based on given actuation positions and any change of the actuation set of points $\{x_{\mathrm{a}1}, x_{\mathrm{a}2}\}$ will alter the elements of matrix **B** and implicitly the constant gain values. Using this matrix a constant gain feedback control for system (27) can be determined as $\mathbf{u}(t, x_{\mathrm{a}1}, x_{\mathrm{a}2}) = \mathbf{K}_g(x_{\mathrm{a}1}, x_{\mathrm{a}2})\, \mathbf{x}(t)$.

For the time-varying system, gain matrix $\mathbf{K}_{\mathrm{g}}(x_{\mathrm{a}1}, x_{\mathrm{a}2}, t) = \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}(x_{\mathrm{a}1}, x_{\mathrm{a}2})\mathbf{P}(t)$ has a time dependency and is determined after solving backward the matrix Riccati equation (16) for $\mathbf{P}(t)$ with terminal conditions given by the terminal cost matrix $\mathbf{S} = \mathrm{diag}(280,2900,0.25,0.1)$. Considering the augmented system (26) matrices **Q** and **S** are also augmented with a new line and a new column $\mathbf{Q} = \mathrm{diag}(1000,100,10,10,0)$ and $\mathbf{S} = \mathrm{diag}(280, 2900, 0.25, 0.1, 0.12)$. In what follows the dependency of the gain matrix on the positions will not be explicitly shown.

Figure 2 compares the variable time control gain matrix $\mathbf{K}_{\mathrm{g}}(t)$ corresponding to the two actuators**,** obtained for the augmented system (26) with weighting matrices **Q** and **R**, and terminal cost matrix **S** as given above with constant gain matrix $\mathbf{K}_{\mathrm{g}}$ obtained for the time-invariant system (27). The last column takes into account the effect of the moving constant force $-mg\boldsymbol{\psi}(vt)$ from (19) which for the new state-space representation becomes a state-variable.

Figure 2. Time varying $\mathbf{K}_{\mathrm{g}}(t)$ (blue) vs. time-invariant (green) control gain matrix $\mathbf{K}_{\mathrm{g}}$**;** and mean values of the time varying elements of $\mathbf{K}_{\mathrm{g}}(t)$ **(**red dashed line).

The mean values for the time-varying gain matrix are plotted along with the constant gain matrix obtained for the time-invariant system. It is seen that there is a small difference which may lead to the conclusion that the time-varying system needs a slightly different control action from the one provided by the time-invariant system.

The terminal cost matrix **S** is selected in this case to give a small settling time even though an increased terminal cost would have brought the terminal states very close to zero thus resulting in small free vibration amplitude. After the time instant $t_{\mathrm{f}} = L/v$ when the mass leaves the beam, the beam vibrates freely as a time-invariant system (27). At this time instant the controller is switched to a new control action synthesised by adopting a linear quadratic index with a control weighting matrix changed to $\mathbf{R} = \mathrm{diag}(5, 5)$ which results in a reduced control effort as the beam is no longer under direct action of the moving load. In this case a high terminal cost $\mathbf{x}^{\mathrm{T}}(t_{\mathrm{f}})\mathbf{S}\mathbf{x}(t_{\mathrm{f}})$ would have resulted in very small values for the initial conditions of the free vibration equation at t_{f}.

Figure 3. Mid-point deflection of the beam
A. Time-invariant quadratic control (red-continuous) vs. uncontrolled system (blue-continuous)
B. Time-invariant quadratic control (red-continuous) time-varying system control without taking into account the moving mass (green-dashed) and time-varying system control taking into account the moving mass (black-dotted)

The performance improvement obtained using a controller designed either by a time-invariant or a time-varying linear quadratic performance index appear similar (Fig. 3). In this case the time-varying controller is synthesised with a smaller terminal cost and a final deflection amplitude $w(L, t_{\mathrm{f}})$ of about 1 cm results, which subsequently leads to a slightly higher amplitude of the free vibration deflection response.

The difference between the linear time-invariant control method and the time-varying method without using the contact force during the design process is seen by analysing Figure 4 where the control action is plotted for the time history of the moving load action. It is seen that the control action variation obtained for the time-invariant system require a high magnitude at the end of the time interval when the mass is on the beam. In comparison the first actuator needs a smaller control force amplitude but most importantly in the case in which the contact force is taken into account the control and the augmented state-space representation (26) is used, force variation is approximately within ±2N.

Figure. 4 Control force time variation, linear time-invariant system (red-continuous), time-varying system (green- dashed), time-varying augmented system (black-dotted)

The most important observation is that by taking into account the contribution of the moving mass in the design process (26) the effect of the controller is significantly improved and a drop of about 30% of amplitude is achieved (Fig. 6 A).

If as a result of constructive restrictions the actuators need to be only on one side of the beam's span, the difference between the constant gain controllers' and the time-varying gain controllers' contributions become more significant during the moving mass action (Figure 5) and the time dependence of the system can no longer be neglected.

Figure 5. Mid-point deflection of the beam, actuators on one side of the beam's span
A. Time-invariant quadratic control (red) vs. uncontrolled system (blue)
B. Time-invariant quadratic control (red-continuous) time-varying system control without taking into account the moving mass (green-dashed) and time-varying system control taking into account the moving mass (black-dotted)

The linear time-invariant control method (Fig.5 A, Fig. 6 B) brings no significant improvement to the deflection amplitude (97.9%) compared with the uncontrolled case whereas the time-varying technique undergoes a drop in the amplitude level of about 20% (Fig.5 B, Fig. 6 B).

The use of only one actuator at position x=0.113 m which would be a more realistic setup instead of two actuators on one side, would modify the bar chart in Figure 6. B to the levels: 96.1% for time-invariant system, 86% for time-varying system and 72.8% for time-varying augmented system (26).

Figure 6. Comparison of the control action's effect on the maximum mid-span deflection amplitude for two cases considered
A. actuators positioned on both sides of the beam
B. actuators positioned on one side of the beam
Un – uncontrolled system, TI – time-invariant LQR based method, TV – time-varying LQR based method, Aug – time-varying method with contact force included in controller design phase

4.2. Bounded control action

All the numerical results presented in section 4.1 are based on the assumption that any value of the control force can be achieved. From a mathematical point of view this makes the Hamiltonian function point-wise differentiable with respect to the control variable and allows the use of the Riccati equation (16) which in the end leads to a feedback representation of the control system.

In reality the physical system may require a bounded control action due to inherent limitations of the existing actuation solutions. These limitations are taken into account in the design phase by limiting the control action to realizable values of the actuation force. Looking at Figure 5 it can be seen that both the time-invariant controller and the time-varying controller require values of the control force amplitude above about -4 N and therefore one could ask if it is possible to achieve the same effect with a control force amplitude within a prescribed interval centred to zero.

In section 2 the possibility of a constraint in control is presented for two idealizations, the first of them for a quadratic index in state-variables (8) and bounds in control $|\mathbf{u}(t)| < \mathbf{U}$ which results in a "bang-bang" control action (10) and the second of them for a quadratic performance index both in control and state-variables (11).

The problems involving bounds on control are mathematically more demanding as it requires solving a two-point boundary value problem. One particular difficulty is also brought about by the discontinuity of the "bang-bang" control (10) which makes the Jacobian of the system (7) and (14) singular. One way around this is to approximate the control given by (10) with a continuous function, for instance

$$\mathbf{u}^*(t) = -U_0 \operatorname{sgn}\left(\mathbf{B}^{\mathrm{T}}(t)\boldsymbol{\lambda}^*(t)\right) \cong -U_0 \operatorname{atan}\left(\alpha\, \mathbf{B}^{\mathrm{T}}(t)\, \boldsymbol{\lambda}^*(t)\right)/\beta \qquad (29)$$

where constant α is chosen to adjust the approximation (a greater α leads to a sharper rise) and β is a normalization constant. The disadvantage of this approximation is that for very small values of the elements of $\mathbf{B}^{\mathrm{T}}(t)\, \boldsymbol{\lambda}^*(t)$ the values obtained for the control force action are below the expected values $\pm U_0$.

Figure 7. Time history of the mid-span deflection obtained with bounded control

A. uncontrolled system (blue-continuous); quadratic index with bounded control (green-dashed)

B. quadratic index, bang-bang control (black-continuous); quadratic index with bounded control (green-dashed); quadratic index, time-invariant (red-dotted)

The output responses obtained for the optimal performance indices (8) and (11) when the control action is bounded $|u_i(t)| \leq 1.2$ are plotted in figure 7 for the time-invariant system (27) with quadratic index (11) and the uncontrolled case. It has to be mentioned that in the case that the problem is solved by using a two-point boundary value method the moving contact force is taken into account. Both bounded control methods give similar results. By analysing figure 8 it is possible to see that for most of the time the control is "on" for both cases. The time intervals where a difference between the two control actions occurs are very small and can be neglected. This is the reason why there is no obvious advantage or disadvantage in using any of the two bounded control methods for this example.

Figure 8. Control force action for bounded control methods, quadratic index, bang-bang control (black-continuous); quadratic index with bounded control (green-dashed); quadratic index, time-invariant (red-dotted)

A. first actuator position; B. second actuator position

One of the main disadvantages of the control design methods presented in this subsection is that they result in an open-loop control system whereas the control depends on the solution of the costate variables which should be solved off-line and then fed into the system.

4.3 Suboptimal technique based on algebraic Riccati equation

In section 2.4 a solution method based on the step by step calculation of the algebraic Riccati equation (18) has been advanced. The method is similar to the state-dependent Riccati equation technique which was studied in connection with a particularly shaped nonlinear system that can be transformed into a state-dependent linear form (Cloutier, 1997):

$$\dot{\mathbf{x}}(t) = f(t, \mathbf{x}, \mathbf{u}) = \mathbf{A}(\mathbf{x})\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t) \tag{30}$$

The system that describes the moving mass problem is linear but the time variation of the matrix coefficients makes an optimal control solution difficult to obtain. The method proposed consists of using at every time step an optimal control determined as $\mathbf{u}(t_i) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}(t_i)\mathbf{P}\mathbf{x}(t_i)$ with matrix $\mathbf{P}$ determined from the algebraic Riccati equation (18). The final solution is not optimal as this division in time segments does not satisfy the optimality principle but this constitutes an on-line method where the feedback control gain is adjusted at every time step according to time evolution of the system. In equation (18) from which $\mathbf{P}(t_i)$ is determined matrices $\mathbf{A}(t_i)$ and $\mathbf{B}(t_i)$ have elements changing at every time step. The method also allows for the control to be bounded as the control action $\mathbf{u}(t_i)$ is determined at every time step and can be tested against the bounds and adjusted.

Figure 9A shows the deflection responses obtained using this method compared with the time-invariant method and the time-varying method for the augmented system. It can be seen that the controlled response at mid-span is comparable to the one obtained with the time-invariant method, but this technique allows for the weighting matrices to be variable as well which will give a better response to the control requirements (Stancioiu and Ouyang, 2013). For instance the shape of the mid-span deflection response of a single span beam shows that large deflection occurs at time instant of 0.2 sec, when the mass moves near to the mid-span. Therefore it seems natural to increase the value of **Q** matrix elements around that time instant.

Figure 9B shows that the effects of this alteration results in a better response reduction around 0.2 sec and leaves the response outside this time interval virtually unchanged. In fact the response can be made to approach the same level of reduction obtained using the optimal solution for the augmented system (26).

Figure 9. Time history of the mid-span deflection obtained with the step by step method compared to the time-invariant and time-varying control design methods, moving speed v = 2.5 m/s
red-continuous(Time Invariant) – time invariant, black-dotted(Time Varying Aug)-time-varying with contact force action, dark grey-dashed (Time Varying RE) – step by step Riccati equation method constant performance index, light grey-continuous (Time Varying RE var) – step by step Riccati equation method variable performance index

The improvement achieved with the step by step Riccati equation technique when it is compared with other methods becomes more visible as the speed of the moving load increases. At a moving speed of 5m/s instead of 2.5 m/s the maximum mid-span deflection of the system with a controller designed using the step by step method has more than 10% reduction (Figure 10).

Figure 10. Time history of the mid-span deflection obtained with the step by step method compared to the time-invariant and time-varying control design methods, moving speed v = 5m/s
Red-continuous (Time Invariant) – time invariant, black-dotted(Time Varying Aug)-time-varying with contact force action, dark grey-dashed (Time Varying RE) – step by step Riccati equation method constant performance index

Similarly to the cases studied before when only one actuator is fitted under the beam at position x_{a}= 0.11m the effect of the controller designed using this method becomes more significant (Figure 11).

Figure 11. Time history of the mid-span deflection obtained with the step by step method compared to the time-invariant and time-varying control design methods, moving speed v = 2.5m/s, one actuator at position x_a = 0.11 m
red-continuous (Time Invariant) – time invariant, black-dotted (Time Varying Aug)-time-varying with contact force action, dark grey-dashed (Time Varying RE) – step by step Riccati equation method constant performance index

5. Conclusions

This paper studies optimal control of vibration of a beam traversed by a mass moving at a constant speed. As a time-varying system, its vibration is nonstationary and control synthesis requires suitable methods. In order to take advantage of the specifics of the problem, the system equation is cast in an augmented form which considers the moving load effect and is amenable to theoretical and numerical treatment. Several control methods based on different practical requirements are given a mathematical meaning and are compared through numerical simulation. One particular aspect studied concerns the influence of the number and location of the actuators. It is shown that the actuator position influences to a great extent the control design method. In this respect, when the actuators are placed only at one end of the supporting structure the control method needs to take into account the time-varying character of the problem.

It is found through numerical simulation that time-varying control always leads to greater reduction of vibration than time-invariant control and different performance indices affect the effectiveness of the optimal control solution. Another finding of this study is that including the moving mass contact force as a state variable into a time-varying system can significantly improve the system response under control and hence proves that the approach based on the augmented system is useful.

One of the difficulties involved in using optimal control methods lies in their implementation. Generally a more complex problem such as bounded control requires an open-loop control strategy. A good compromise to this problem can be made by using a step by step Riccati equation method, whereas the control is determined at any time step from the time-varying system matrices by solving a Riccati equation. This method mimics the solution of a linear quadratic problem and is inspired by the state-dependent Riccati equation technique. It is shown to give a viable alternative solution to the problem of bounded control as well as to the problem with time-varying performance weighting matrices.

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B. quadratic index, bang-bang control (black-continuous); quadratic index with bounded control (green-dashed); quadratic index, time-invariant (red-dotted)

Figure 8. Control force action for bounded control methods, quadratic index, bang-bang control (black-continuous); quadratic index with bounded control (green-dashed); quadratic index, time-invariant (red-dotted)

A. first actuator position; B. second actuator position

Figure 9. Time history of the mid-span deflection obtained with the step by step method compared to the time-invariant and time-varying control design methods, moving speed v = 2.5 m/s

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Figure 10. Time history of the mid-span deflection obtained with the step by step method compared to the time-invariant and time-varying control design methods, moving speed v = 5m/s

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Figure 11. Time history of the mid-span deflection obtained with the step by step method compared to the time-invariant and time-varying control design methods, moving speed v = 2.5m/s, one actuator at position x_{a} = 0.11 m

red-continuous (Time Invariant) – time invariant, black-dotted (Time Varying Aug)-time-varying with contact force action, dark grey-dashed (Time Varying RE) – step by step Riccati equation method constant performance index