Spectral Techniques in Argumentation Framework Analysis

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Abstract. Spectral analysis – the study of the properties of the eigenvalues associated with some matrix derived from an underlying graph form – has proven to offer valuable insights in many domains where graph-theoretic models are prevalent. Abstract argumentation frameworks (AFs) are, of course, one such model and have provided a unifying basis for defining semantic properties related to concepts of "argument acceptability". In this paper we consider the possible benefits of adopting spectral methods as a tool for analysing argumentation structures, presenting a preliminary empirical study of semantics in AFs and properties of the associated spectrum.

Keywords. abstract argumentation frameworks; directed graph spectrum; extension-based semantics

Introduction

A notable feature of formal analytic treatments of Dung's seminal model of abstract argumentation from [16] is the focus on *discrete* methodologies. Typical of such directions has been the exploitation of graph-theoretic structures in defining semantics, e.g. Dung [16], Baroni *et al.* [4,2], and Caminada [12]. Developments seeking to alleviate issues with the highly abstracted form of Dung's approach – such as Amgoud and Cayrol [1], Bench-Capon [7], Brewka and Woltran [9] – similarly embrace discrete mechanisms. While there are exceptions in which continuous measures are, in principle, permitted, e.g. within divers forms of so-called "weighted" frameworks, e.g. Dunne *et al.* [20], Barringer *et al.* [5], and, more directly, in models of *probabilistic* frameworks, such as Li *et al.* [29], it could be argued that the presence of continuous numerical quantities in such is more a consequence of the problems addressed than a direct analytic tool.

The aim of this article is to consider what scope for determining argumentation framework properties may be provided by considering the *spectrum* of the $(0,1)$-matrix[1] defined through the directed graph describing the framework. We review the formal definition of "*graph spectrum*" subsequently, but for the purpose of this introduction it suffices to note that the spectrum of an $n \times n$ matrix

[1] Although the $(0,1)$ structure is a natural choice it is often useful – especially within *directed* graph forms to make use of "transformed" $n \times n$ matrix definitions, one of the most widely used of these being the so-called Laplace operator, see e.g. Bauer [6].

is given by an n-tuple $\langle \lambda_1, \lambda_2, \cdots, \lambda_n \rangle$ of (possibly) complex values[2] corresponding to the *eigenvalues* of the matrix. That is to say for $\chi(A, \lambda)$ the polynomial of degree n in λ, (thus $\chi(A, \lambda) = \sum_{i=0}^{n} c_i \lambda^i$) given[3] by $|\lambda I - A|$ the n (not necessarily distinct) solutions of $\chi(A, \lambda) = 0$. If λ_A is an eigenvalue of the matrix A, then one may find n-tuples, $\underline{x}$, with at least one non-zero component, for which $A\underline{x} = \lambda_A \underline{x}$. Such n-tuples being referred to as *eigenvectors.*

At first inspection it may seem that there is little connection between the rather abstruse notion of eigenvalue (especially when these lie in the complex plane) and the, apparently, more practically grounded concepts offered within established ideas of abstract argumentation semantics. In order to motivate our subsequent study, it is worth reviewing contexts both within computational and other domains where their analysis is known to provide important benefits.

Undoubtedly one of the best known such applications is found in Web search-engines and the mechanisms used to rank pages, see Bryan and Leise [11] for further discussions. Other computational applications building on properties of eigenvalues within a supporting graph structure include pattern matching, e.g. Kirby and Shilovich [28], Shi and Malik [32], power control in communication networks, see e.g. Bertoni [8]. Similarly within empirical studies from the physical sciences concepts such as the *Estrada index* – an invariant defined via the eigenvalues arising from a graph introduced in Estrada [22] – have been argued to have important properties with respect to models of molecular structures, see e.g. Gutman and Graovac [26], Ilić and Stevanović [27]. Finally the use of graph spectra to inform reasoning about combinatorial structures within graphs is well established, e.g. Brouwer and Haemers [10].

The exploitation of what are often referred to as "*spectral techniques*", in the light of their use in other domains, may provide some useful insight into aspects of argumentation frameworks. The principal aim of the current paper is to explore this potential. Our approach is empirical rather than analytic in nature. In particular, we consider evidence for links between divers argumentation structures, e.g. acceptability of arguments with respect to given semantics, existence of extensions containing some number of arguments, etc. and various spectral measures defined on the underlying framework, amongst which are invariants such as the Estrada index, the spectral spread – i.e. the difference between largest and smallest eigenvalue, etc.

Before proceeding with the technical presentation we elaborate on what the aims of our empirical investigations are and, of equal importance, what is *not* being asserted.

The central conceit motivating this paper may, informally, be expressed in the following question: do spectral techniques offer a possible basis for studying structural, especially semantic, properties within abstract argumentation frameworks? In support of a positive answer to this question, we have noted the numerous examples in other computational domains, particularly those wherein di-

[2]In special cases, in particular when the underlying matrix $[a_{ij}]$ is symmetric, its spectrum consists of values drawn from $\mathbb{R}$.

[3]For an $n \times n$ real-valued matrix, A we use $|A|$ to denotes its *determinant*, recalling that a matrix B for which $A \times B = B \times A = I$ exists if and only if $|A| \neq 0$.

rected graphs provide a natural modelling formalism, of spectral analysis providing insight.

Of course, the fact that a given formalism has proven helpful in one arena of study does not imply it will also prove useful within different but superficially similar fields. Nevertheless, it would seem reasonable prior to rejecting outright the notion that "spectral methods have a rôle within the analysis of argumentation frameworks", to consider evidence in its support. In addition, we note recent studies of argumentation frameworks have explored operations on the matrix representation as an approach to capturing particular semantics in terms of matrix properties. Notable here is the recent work of Xu and Cayrol [33].

Thus, our principal aim is not to provide a *full* analytic or even empirical study of the relationships between spectra and argumentation but rather to consider connections between one specialized class of AFs and its spectra. For the class of AFs examined, its behaviour with respect to one argumentation semantics is well-characterized this characterization does not, however, assist computationally: that is to say, the canonical decision questions become no more tractable. In principle, however, given what is already known regarding the structural properties of this class, one might reasonably hope that this *could* in turn be tied with spectral properties. We develop this idea in fuller detail within Section 2 below.

We present background to Dung's abstract AF model and review some elements regarding linear algebra and matrices in Section 1. In Section 2 we outline the basis and motivation underlying the structure of the experimental studies, and report on preliminary findings from these. Conclusions are presented in Section 3.

1. Preliminaries

We begin by recalling the concept of abstract argumentation frameworks and terminology from Dung [16]

Definition 1 *We use* $\mathcal{X}$ *to denote a* finite *set of arguments with* $\mathcal{A} \subseteq \mathcal{X} \times \mathcal{X}$ *the so-called* attack *relationship over these. An* argumentation framework *(*AF*) is a pair* $\mathcal{H} = \langle \mathcal{X}, \mathcal{A} \rangle$*. A pair* $\langle x, y \rangle \in \mathcal{A}$ *is referred to as '*y is attacked by x*' or '*x attacks y*'. Using* S *to denote an arbitrary subset of arguments for* $S \subseteq \mathcal{X}$*,*

$$\begin{array}{lll} S^- & =_{\text{def}} & \{ \; p \; : \; \exists \; q \in S \;\; \textit{such that} \; \langle p, q \rangle \in \mathcal{A} \} \\ S^+ & =_{\text{def}} & \{ \; p \; : \; \exists \; q \in S \;\; \textit{such that} \; \langle q, p \rangle \in \mathcal{A} \} \end{array}$$

We say that: $x \in \mathcal{X}$ *is* acceptable with respect to S *if for every* $y \in \mathcal{X}$ *that attacks* x *there is some* $z \in S$ *that attacks* y*. Given* $S \subseteq \mathcal{X}$*,* $\mathcal{F}(S) \subseteq \mathcal{X}$ *is the set of all arguments that are acceptable with respect to* S*, i.e.*

$$\mathcal{F}(S) \;=\; \{x \in \mathcal{X} \;:\; \forall \; y \; \textit{such that} \; \langle y, x \rangle \in \mathcal{A} \;,\; \exists \; z \in S \; \textit{s.t.} \; \langle z, y \rangle \in \mathcal{A}\}$$

A subset, S*, is* conflict-free *if no argument in* S *is attacked by any other argument in* S*. The* $\subseteq$*-maximal conflict-free sets are referred to as* naive *extensions. A conflict-free set* S *is* admissible *if every* $y \in S$ *is acceptable w.r.t* S*.* S *is a* complete *extension if* S *is conflict-free and should* $x \in \mathcal{F}(S)$ *then* $x \in S$*, i.e. every* *argument that is acceptable to S is a member of S, so that $\mathcal{F}(S) = S$. The set of $\subseteq$-maximal complete extensions coincide with the set of $\subseteq$-maximal admissible sets these being termed* preferred extensions*. The set S is a* stable *extension if S is conflict free and $S^+ = \mathcal{X} \setminus S$. It is a* semi-stable *extension (Caminada [12]) if admissible and has $S \cup S^+$ $\subseteq$-maximal among all admissible sets.*

The grounded extension *of $\langle \mathcal{X}, \mathcal{A} \rangle$ is defined as the $\subseteq$-minimal complete extension.*

We use σ to denote an arbitrary semantics and for a given semantics σ and AF, $\mathcal{H}(\mathcal{X}, \mathcal{A})$, $\mathcal{E}_\sigma(\mathcal{H})$ denotes the set of all subsets of $\mathcal{X}$ that satisfy the conditions specified by σ. We say that σ is a *unique status* semantics if $|\mathcal{E}_\sigma(\mathcal{H})| = 1$ for every AF, $\mathcal{H}$, denoting the unique extension by $E_\sigma(\mathcal{H})$.

We complete this, brief, overview by describing the three canonical decision problems that may be instantiated for a given semantics: *Verification* (VER), *Credulous Acceptance* (CA) and *Sceptical Acceptance* (SA). Formal definitions of these problems for AFs are presented in Table 1.

Table 1. Decision Problems in AFs

Problem Name	Instance	Question
Verification (VER_σ)	$\mathcal{H}(\mathcal{X}, \mathcal{A})$; $S \subseteq \mathcal{X}$	Is $S \in \mathcal{E}_\sigma(\mathcal{H})$?
Credulous Acceptance (CA_σ)	$\mathcal{H}(\mathcal{X}, \mathcal{A})$; $x \in \mathcal{X}$	$\exists\ S \in \mathcal{E}_\sigma(\mathcal{H})$ for which $x \in S$?
Sceptical Acceptance (SA_σ)	$\mathcal{H}(\mathcal{X}, \mathcal{A})$; $x \in \mathcal{X}$	$\forall\ T \in \mathcal{E}_\sigma(\mathcal{H})$ is $x \in T$?

Similarly we have two *function* problems – CONSTRUCT and COUNT –

Table 2. Function Problems in AFs

Problem Name	Instance	Computation
Construction (CONSTRUCT_σ)	$\mathcal{H}(\mathcal{X}, \mathcal{A})$	Return some $S \in \mathcal{E}_\sigma(\mathcal{H})$
Count (COUNT_σ)	$\mathcal{H}(\mathcal{X}, \mathcal{A})$	Return $|\mathcal{E}_\sigma(\mathcal{H})|$

Both of the function problems of Table 2 may be qualified so that instances specify a given argument $x \in \mathcal{X}$. In such cases, one is asked to construct a representative (resp. to count the number of subsets) in $\mathcal{E}_\sigma(\mathcal{H})$ containing the given argument x.

1.1. Review of Matrix Algebra

For an AF, $\mathcal{H} = \langle \mathcal{X}, \mathcal{A} \rangle$ with $|\mathcal{X}| = n$ we denote by $\mathbf{M}^\mathcal{H}$ the $n \times n$ $(0,1)$-matrix with entries m_{ij} defined via $m_{ij} = 1$ if and only if $\langle x_i, x_j \rangle \in \mathcal{A}$. With $\mathbb{C}$ denoting the complex plane,[4] $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of $\mathbf{M}^\mathcal{H}$ if there is some $n \times 1$ vector $\underline{v}$ (with $\underline{v}$ having at least one non-zero component) for which $\mathbf{M}^\mathcal{H}\underline{v} = \lambda\underline{v}$. A witnessing vector $\underline{v}$ for λ is referred to as an *eigenvector* with respect to $\langle \mathbf{M}^\mathcal{H}, \lambda \rangle$.[5]

The tuple

[4] That is pairs $(a, b) \in \mathbb{R}$ defining the complex number $z = a + ib$, $i^2 = -1$.

[5] It is, on occasion, useful to distinguish so-called *right* eigenvectors w.r.t. $\langle \mathbf{M}^\mathcal{H}, \lambda \rangle$ from *left* eigenvectors w.r.t. $\langle \mathbf{M}^\mathcal{H}, \lambda \rangle$: the former being $n \times 1$ vectors, $\underline{v}$ with $\mathbf{M}^\mathcal{H}\underline{v} = \lambda\underline{v}$, the latter $1 \times n$ vectors $\underline{w}$ for which $\underline{w}\mathbf{M}^\mathcal{H} = \lambda\underline{w}$.

$$\sigma(\mathcal{H}) \;=\; \langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle$$

formed by the n eigenvalues of $\mathbf{M}^{\mathcal{H}}$ is called the *spectrum* (of $\mathcal{H}$). The *spectral radius* of $\mathbf{M}^{\mathcal{H}}$, denoted $\rho(\mathbf{M}^{\mathcal{H}})$ is

$$\max\ \{\ |\lambda|\ :\ \lambda \text{ is an eigenvalue of } \mathbf{M}^{\mathcal{H}}\ \}$$

where for $\lambda = a + ib \in \mathbb{C}$, $|\lambda| = +\sqrt{(a^2+b^2)}$. We assume an ordering of the spectrum for $\mathcal{H}$ such that whenever $i \le j$ it holds that $|\lambda_i| - |\lambda_j| \ge 0$ so that the eigenvalues are considered in a non-decreasing order and $|\lambda_1| = \rho(\mathbf{M}^{\mathcal{H}})$. The largest eigenvalue (that is to say, λ_1) will be termed the *dominant* eigenvalue. This (and on occasion its successor λ_2) are the typical focus of spectral treatments.

The *Estrada Index* of $\mathcal{H}$ ([22]), $\mathbf{E}(\mathcal{H})$ is given as

$$\mathbf{E}(\mathcal{H}) \;=\; \sum_{\lambda \in \sigma(\mathcal{H})} \mathrm{e}^{\lambda}$$

We briefly recall some well known properties of eigenvalues in,

Fact 1

a. For an $n \times n$-matrix, $\mathbf{A}$, not necessarily $(0,1)$, let $|\mathbf{A}|$ denote its determinant*, and $\chi_{\mathbf{A}}(x)$ the polynomial of degree n in x defined through $|x\mathbf{I} - \mathbf{A}|$ ($\mathbf{I}$ being the $(0,1)$* identity *matrix with (i,j) entries equal to 1 if and only if $i = j$). The quantity $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{A}$ if and only if λ is a root of $\chi_{\mathbf{A}}(x)$, i.e. $\chi_{\mathbf{A}}(\lambda) = 0$.*[6]

b. For $(a,b) \in \mathbb{R}^2$, $\lambda = a + ib$ is an eigenvalue of $\mathbf{A}$ if and only if $\overline{\lambda} = a - ib$ is an eigenvalue of $\mathbf{A}$.

c. If $\mathbf{A}$ is a symmetric *matrix ($a_{ij} = a_{ji}$ for all $1 \le i, j \le n$) then all eigenvalues of $\mathbf{A}$ lie in $\mathbb{R}$.*

The concepts of eigenvalue and eigenvectors arise with respect to $n \times n$ real-valued matrices: of particular interest are the class of *non-negative* matrices and the subset of these defined by *positive* matrices.

Definition 2 *Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ real-valued matrix. We say that $\mathbf{A}$ is* non-negative *if for each i and j ($1 \le i,\ j \le n$) $a_{ij} \ge 0$. It is a* positive *matrix if every a_{ij} satisfies $a_{ij} > 0$.*

It is obvious for the mapping described that $\mathbf{M}^{\mathcal{H}}$ is always a non-negative matrix, however, an apparent difficulty with this representation is that there is *exactly one* AF, $\mathcal{H}$, that gives rise to a positive matrix: namely the AF in which *every* attack between arguments is present (including self-attacks). There are, however, a large class of $\mathcal{H}$ whose structural properties allow $\mathbf{M}^{\mathcal{H}}$ to be related to positive matrices with consequential benefits.

[6] Eigenvalues corresponding to unique roots of $\chi_{\mathbf{A}}(x)$ are referred to as *simple*, e.g. $\lambda = 1$ is a simple eigenvalue (root) of $(x-1)(x+1)$ but not of $(x-1)(x-1)$.

Definition 3 *Let* $\mathbf{A}$ *be a non-negative* $n \times n$ *matrix. If, for some* $k \in \mathbb{N}$, $\mathbf{A}^k$ *is a positive matrix, then* $\mathbf{A}$ *is said to be* primitive.

If for each $i,\ j\ (1 \le i,\ j \le n)$ *there is some* $k_{ij} \in \mathbb{N}$ *for which* $[\mathbf{A}^{k_{ij}}]_{ij} > 0$ *then* A *is said to be* irreducible.

With regards to irreducible matrices we have the following classic theorem, which has been widely applied in many of the applications described in the introduction.

Theorem 1 *(Perron-Frobenius Theorem [30,24])*

If $\mathbf{A}$ *is an irreducible* $n \times n$ *matrix then,*

PF1. There is (at least one) positive real eigenvalue, λ^A, *of* $\mathbf{A}$ *with positive eigenvectors, that is for which there are associated eigenvectors* $\underline{x}$ *all of whose components are strictly greater than* 0.
PF2. There is a unique *positive and dominant eigenvalue* λ^A_{pf}, *i.e.* $\lambda^A_{pf} = \rho(\mathbf{A})$, *and simple.*
PF3. If $\mathbf{A}\underline{x} = \lambda\underline{x}$ *and* $\underline{x}$ *is* positive *then* $\lambda = \lambda^A_{pf}$.
PF4. If $\mathbf{B} \ge \mathbf{A}$ *and* $\mathbf{B} \ne \mathbf{A}$ *then* $\rho(B) > \lambda^A_{pf}$.[7]
PF5. If $\mathbf{B} \le \mathbf{A}$ *and* $\mathbf{B} \ne \mathbf{A}$ *then* $\rho(B) < \lambda^A_{pf}$

The eigenvector associated with λ^{www}_{pf} where (informally) **www** is the matrix corresponding to web-page connectivity, is central to many web search page-ranking algorithms, cf. the discussion in Bryan and Leise [11].

Thm. 1 applies to $\mathbf{M}^{\mathcal{H}}$ for a wide-ranging class of AFs, whose importance has earlier been demonstrated in Baroni *et al.* [4] and in connection with algorithmic study of the semantics considered in [2].

Fact 2 *If* $\mathcal{H} = \langle \mathcal{X}, \mathcal{A} \rangle$ *is* strongly-connected[8] *then* $\mathbf{M}^{\mathcal{H}}$ *is irreducible.*

To conclude this overview we note that the property stated in Fact 1(b), allows us to show,

Fact 3 *For all* $\mathcal{H}$, $\mathbf{E}(\mathcal{H}) \in \mathbb{R}$.

2. Experiment Structure and Motivation

The experimental framework in essence uses *randomly generated* AFs constructed so that the AF's density (that is the ratio $|\mathcal{A}|/|\mathcal{X}|$) varies. For each randomly constructed AF within a given class a specific semantic property is assessed and comparative figures accumulated over all (generated test instances of relevant size) sharing the property and the average of specific spectral parameters.

For the basis of our empirical overview we focus on three measures: the dominant eigenvalue i.e. $\lambda_1 = \rho(\mathbf{M}^{\mathcal{H}})$; the second largest such eigenvalue (λ_2); and,

[7] For $n \times n$ real matrices $\mathbf{A}, \mathbf{B}$ we say $\mathbf{B} \ge \mathbf{A}$ if and only if $b_{ij} \ge a_{ij}$ for $1 \le i, j \le n$, i.e. the comparison is component-wise.

[8] A directed graph, $\langle X, E \rangle$ is said to be strongly-connected if for all $\langle x_i, x_j \rangle \in X \times X$ there is a directed path of links from E starting in x_i and ending in x_j.

in order, to glean some indication of effects arising from the entire range of $\sigma(\mathcal{H})$, its Estrada index $\mathbf{E}(\mathcal{H})$.

The frameworks of interest are characterized by three parameters, $\langle n, m, k\rangle$ ($\mathcal{F}^{(n,m,k)}$ denoting those AFs with the structure referred to and having these parameters set to $\langle n, m, k\rangle$) so that the entire space of interest is

$$\mathbb{S} \quad = \quad \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=0}^{2^k \binom{n}{k}} \mathcal{F}^{(n,m,k)}$$

The class of AFs examined have a number of important properties which we first summarize in terms of their relationship to argumentation semantics.

A1. There is a polynomial-time computable mapping τ that associates an AF, $\tau(\varphi) \in \mathcal{F}^{(n,m,k)}$ for φ an n variable, m clause, k-CNF formula. For such formulae, the framework, $\tau(\varphi)$ has exactly $2n+m+2$ arguments and $4n+(k+1)m+1$ attacks.

A2. The AF $\tau(\varphi)$ has a *non-empty preferred extension* (which is also a *stable* extension) if and only if its source CNF formula is *satisfiable*.

Regarding properties of $\mathbf{M}^{\tau(\varphi)}$ the important one of interest (in the light of Thm 1) is that the non-negative matrix $\mathbf{M}^{\tau(\varphi)}$ is *irreducible*: the AF $\tau(\varphi)$ being strongly-connected.

Before describing the structure of $\tau(\varphi)$ in greater depth, we emphasize that the random variable involved is not drawn from the space of *all* AFs *per se* but rather a *subset* of these, namely for $|\mathcal{X}| = 2n+m+2$, $|\mathcal{A}| = 4n+(k+1)m+1$,

$$\mathrm{P}[\langle \mathcal{X}, \mathcal{A}\rangle \text{ is chosen}] \quad = \quad \begin{cases} 0 & \text{if } \langle \mathcal{X}, \mathcal{A}\rangle \notin \mathcal{F}^{(n,m,k)} \\ > 0 & \text{if } \langle \mathcal{X}, \mathcal{A}\rangle \in \mathcal{F}^{(n,m,k)} \end{cases}$$

with these likelihoods being essentially uniformly distributed over eligible AFs, i.e. those in $\mathcal{F}^{(n,m,k)}$.[9]

Now, although in general one cannot make inferences about the behaviour of one class of random combinatorial structures (for example, directed graphs) *as a whole* via mappings from a different class of random structures (e.g. k-CNF formulae), this, of course, is *not* what we claim to be the focus of our experiments. It is, rather the case that *should* there be any observable link between spectral aspects of the AFs considered and semantic properties then it may well be the case that such behaviour is evident when the source formulae exhibit specific characteristics.

In the case of random k-CNF formulae, such characteristics have been validated (from initial experimental studies) analytically. For further background we refer the reader to, among others, Chao and Franco [13], Freeman [23], Dunne *et al.* [19].

In particular we have,

[9] Describing the distribution as "uniform" is a slight over-simplification, however, the difference between "true" uniform and that pertinent to the experiments themselves is insignificant.

Fact 4 *Let ψ be drawn uniformly at random from the space of n variable, m clause k-CNF formulae where $k \geq 2$. For each k, there are constants $\langle \theta_k^l, \theta_k^u \rangle \in \mathbb{R}^+$ (with $\theta_k^l \leq \theta_k^u$) such that*

Letting $r = m/n$,

$$\begin{array}{lcl} P[\psi \textit{ is satisfiable}] & \rightarrow & 1 \;\; \textit{if } r < \theta_k^l \\ P[\psi \textit{ is satisfiable}] & \rightarrow & 0 \;\; \textit{if } r > \theta_k^u \end{array}$$

The behaviour indicated becoming increasingly pronounced as the sample space induced by n increases in size. When $k = 2$, that $\theta_2^l = \theta_2^u = 1$ has been proven analytically by Goerdt [25].

The "threshold" behaviours observed in random k-CNF formulae together with the properties of the AF constructed by τ as described in (A2), suggest investigating the following as an initial stage regarding putative connections between spectra and semantics:

> "Is the pattern whereby random k-CNF with few clauses (relative to n) are almost certainly satisfiable whilst those with many clauses are not (the transition from"few" to "many" being witnessed by a constant multiple (θ_k) of n), reflected in spectral properties of the AF defined through τ?"

The cases reported below consider a range of randomly generated 3-CNF using clause-to-variable ratios ranging from almost certainly satisfiable ($r \leq 4$) to almost surely unsatisfiable ($r \geq 5$). Before proceeding to describe these in detail, we conclude this overview by recalling the transformation from k-CNF formulae, φ, to AFs $\tau(\varphi)$.

Definition 4 *Given a k-CNF, φ over propositional variables $Z = \{z_1, \ldots, z_n\}$ and clause set $\{C_1, C_2, \ldots, C_m\}$ the* standard translation *of φ is the* AF*, $\mathcal{H}_\varphi = \langle \mathcal{X}_\varphi, \mathcal{A}_\varphi \rangle$*

$$\begin{array}{lcl} \mathcal{X}_\varphi & = & \{\varphi\} \cup \{C_1, \ldots, C_m\} \cup \{z_1, \ldots, z_n\} \cup \{\neg z_1, \ldots, \neg z_n\} \\ \mathcal{A}_\varphi & = & \{\langle C_j, \varphi \rangle \;:\; 1 \leq j \leq m\} \;\cup\; \{\langle z_i, \neg z_i \rangle, \langle \neg z_i, z_i \rangle \;:\; 1 \leq i \leq n\} \\ & & \cup\; \{\langle y_i, C_j \rangle \;:\; y_i \textit{ is a literal (i.e., } z_i \textit{ or } \neg z_i\textit{) of the clause } C_j\} \end{array}$$

The AF*, $\tau(\varphi)$ is formed from $\mathcal{H}_\varphi$ by adding a new argument, ψ to $\mathcal{X}_\varphi$ with $\mathcal{A}_\varphi$ extended with attacks*

$$\{\langle \varphi, \psi \rangle\} \;\cup\; \bigcup_{i=1}^{n} \{\langle \psi, z_i \rangle,\; \langle \psi, \neg z_i \rangle\}$$

The standard translation (and its variants such as τ) has formed an important device in the complexity analysis of decision problems in argumentation semantics since its introduction by Dimopoulos and Torres [15], e.g. Dunne and Bench-Capon [18], Dunne [17], Dvořák and Woltran [21], etc. For our purposes the important property of $\tau(\varphi)$,demonstrated in [15] is,

Fact 5 *Let φ be any* CNF *formula. The following are equivalent properties respecting φ:*

a. The formula φ is satisfiable.
b. The argument φ in both $\mathcal{H}_\varphi$ and $\tau(\varphi)$ is credulously accepted w.r.t. admissible semantics.
c. The AF *$\tau(\varphi)$ has a* non-empty *preferred extension.*
d. The AF *$\tau(\varphi)$ has a stable extension.*

A series of trials involving the following steps were carried out:

S1. Set n the number of propositional variables.
S2. Set m the number of clauses.
S3. Generate a random m-clause, 3-CNF formula, φ.
S4. Form the AF, $\tau(\varphi)$.
S5. Determine, for the (irreducible) matrix $\mathbf{M}^{\tau(\varphi)}$,

L1 The dominant eigenvalue, $\lambda_1 = \rho(\mathbf{M}^{\tau(\varphi)})$.
L2 The second largest eigenvalue, λ_2.
EE The Estrada index, $\mathbf{E}(\tau(\varphi))$

For reasons of space we focus on the experimental outcomes arising from the behaviour of the dominant eigenvalue.

Fig. 1 shows (x-axis) varying clause-to-variable ratio from $r = 3$ (predominantly satisfiable cases) to $r = 8$ (unsatisfiable) and n ranging from 6 to 16. The 24 specific cases result in $\mathbf{M}^{\tau(\varphi)}$ of dimensions 62×62, 72×72 and 82×82 corresponding to the three curves indicated.

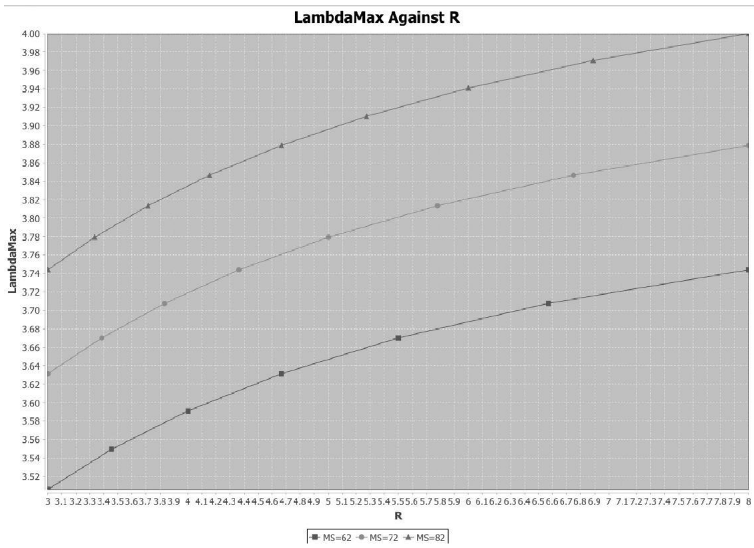

Figure 1. Clause Variable Ratio R vs. Dominant Eigenvalue

In Fig. 2, these ratios are compared against the Estrada Index of the corresponding AF.

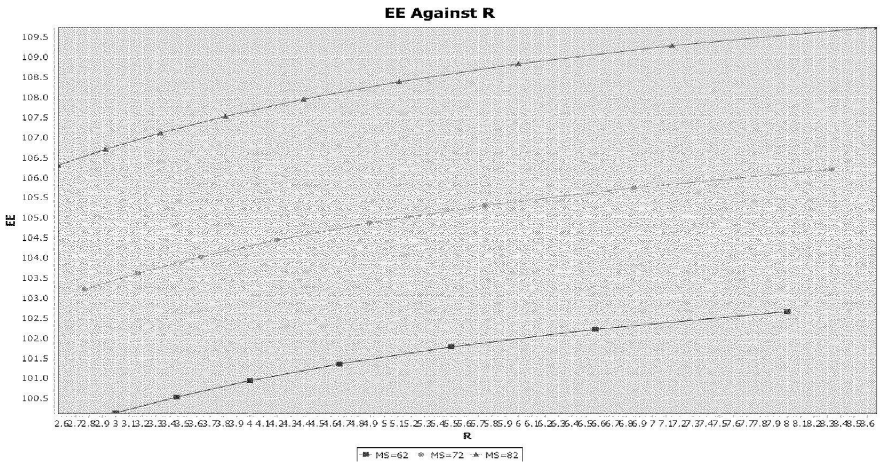

Figure 2. Clause Variable Ratio R vs. Estrada Index

There is some indication that the dominant eigenvalue is dependent on r. The close similarity between the outcomes shown for dominant eigenvalue (Fig. 1) and Estrada index[10] (Fig. 2) could be accounted for by the presence of a large number of very small values in the relevant spectra, so that a significant contribution to $\mathbf{E}(\tau(\varphi))$ is from $(e^{\lambda_1} + e^{\lambda_2})$. We note, however, terms defining $\mathbf{E}(\tau(\varphi))$ that arise from smaller eigenvalues are also important so that estimating $\mathbf{E}(\tau(\varphi))$ as $(e^{\lambda_1} + e^{\lambda_2})$ fails to be accurate.

Of course, these outcomes are very far from being able to argue that $\rho(\mathbf{M}^{\mathcal{H}}) > \alpha$ allows some semantic properties of $\mathcal{H}$ to be deduced. The behaviour, however, does suggest (on the basis of established properties of random CNF formulae) a possible continuation, namely: rather than mapping random 3-CNF to AFs via the standard translation, construct AFs with varying dominant eigenvalues (a non-trivial task) and consider semantic properties of the given AF. This direction is the subject of current work.

3. Conclusions

The use of spectral techniques, while widespread in many fields exploiting graph models, has had comparatively little attention with respect to potential use in studying argumentation frameworks. The primary thesis of this article is that a deeper analysis or the relationship between AF spectra and argumentation properties, such as extension-based semantics, offers possible insights into (among others), algorithm synthesis. In this regard, spectral techniques provide directions well-suited to the consideration of *weighted* frameworks.

We conclude by outlining two (out of many) directions for further research.

[10] Although not shown here, in fact R vs. λ_2 exhibits very similar behaviour.

D1. Cyclic structures in AFs. Several researchers, e.g. Baroni and Giacomin [3], Coste-Marquis *et al.* [14], have observed that directed cycles among arguments (and the parity of such cycles) has a significant influence on argument acceptability and algorithmic behaviour. A well-known relationship between the spectrum of a directed graph, D, and the number of "cyclic paths of length k in D" is that the latter is $\sum_{i=1}^{n} \lambda_i^k$. (Note that this counts *non-simple* cycles). Thus, the spectrum of $\mathcal{H}$ provides information about cycles in $\mathcal{H}$. Notice that, as a consequence, returning to the expression of eigenvalues as roots of a polynomial, it follows that the governing polynomial for acyclic AFs is simply x^n, i.e all eigenvalues are 0.

D2. Argument ranking. A growing area of interest within argumentation has been capturing concepts of argument "strength" and defining "rankings" of arguments, e.g. Pu *et al.* [31], Zhao *et al.* [34]. Many of the problems with "naïve" approaches (e.g. quantifying weakness by the *number* of attackers ignoring the nature of the attack itself) have parallels with naïve approaches to web page-ranking (e.g. using the number of links to a page to determine its importance). Pursuing this analogy suggests that applying consequences of Thm. 1 (the Perron-Frobenius Theorem) – the mechanism underpinning Google's page ranking – offers one technique for exploring argument strength.

In total these and other possibilities suggest that spectral techniques offer, as these have been found to provide in other graph based arenas, a rich potential for effective exploitation applied to abstract argumentation frameworks.

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