Asymptotic Results for a Markov-Modulated Risk Process with Stochastic Investment

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Abstract

In this paper we consider a Markov-modulated risk model, where the premium rates, claim frequency and the distribution of the claim sizes vary depending on the state of an external Markov chain. The free reserves of the insurer are invested in a risky asset whose prices are modelled by a geometric Brownian motion, with parameters that are also influenced according to the external Markov process. A system of integro-differential equations for the ruin probabilities and for the expected discounted penalty function is derived. Using Laplace transforms and regular variation theory, we investigate the asymptotic behaviour of both quantities for the case of light or heavy tailed claim size distributions. Specifically, within this set up (where we lose the strong Markov property of the risk process), we show that the ruin probabilities decrease asymptotically as a power function in the case of the light tailed claims, whilst for the heavy tails we show that the probabilities of ruin decay either like a power function, depending on the parameters of the investment, or behave asymptotically like the tails of the claim size distributions.

Keywords: Markov-modulated risk process, Investment, Integro-differential equation system, Ruin probabilities, Expected discounted penalty function, Regular variation, Frobenius method for systems.

1 Introduction

The investigation of insurance risk models with stochastic return on investments has attracted a lot of attention in recent years. Stimulated by the paper of Paulsen (1993) and

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Paulsen and Gjessing (1997), where continuous time risk processes in a stochastic economic environment are introduced, many researchers have studied Poisson and renewal risk models with risky investments. Lower and upper bounds, numerical solutions, asymptotics and analytic expressions for the probability of ruin (for some individual classes of the aforementioned models), in the case where the wealth process of an insurance portfolio is invested in a stock (whose prices follow a geometric Brownian motion or are Lévy processes), have been derived by several authors. See for example, among others, Cai (2004), Cai and Xu (2006), Paulsen (1998), Paulsen (2008), Tang and Tsitsiashvili (2003), Tang and Tsitsiashvili (2004) and the references therein. More recently, another extension of the aforementioned problem, where a general two sided jump-diffusion risk model that allows correlation between the two Brownian motions driving the insurance risk and investment return, has been investigated by Yin and Wen (2013) in the presence of a constant dividend and a threshold barrier strategy.

With regards to the asymptotic results of risk models with investments, Paulsen (2002) considers a Lévy risk process compounded by another independent Lévy process and shows asymptotically that, as initial capital increases the ruin probability essentially behaves as a power function of the initial capital. Moreover, Gaier and Grandits (2004) showed, within the context of the classical risk model, that when the claim sizes are regularly varying, then the probability of ruin is also regularly varying, whilst Wei (2009) extended these results into the context of the renewal risk model. More recently, Hult and Lindskog (2011) studied the asymptotic decay of finite time ruin probabilities for an insurance portfolio in the presence of heavy-tailed claims when the prices of the risky investments are given by a quite general semimartingale. In this setting, the ruin problem corresponds to determining hitting probabilities for the solution to a randomly perturbed stochastic integral equation. Additionally, Albrecher et al. (2012) considered a general class of renewal risk models (where the inter-arrival claim times satisfy an ordinary differential equation with constant coefficients) with geometrical Brownian motion investments and, using regular variation theory, they derived a unified analytic method for the asymptotic behaviour of the probability of ruin. For this general class of renewal risk models with investment, explicit results for the asymptotic ruin probability are given in the case of both light and heavy tailed claims.

The common idea that investing in an asset with stochastic returns proves too risky for an insurance portfolio in the classical risk model, the renewal and the Lévy risk models, can be justified mathematically by all the above papers. However, once we move to non-renewal models (in the sense that the surplus process does not renew itself at the claim time epochs), the strong Markov property is lost and the problem becomes cumbersome. The Markov-modulated risk model was first introduced by Janssen (1980) and Reinhard (1984) and has since received much attention in the risk theory literature, including applications in queueing theory, see among others Asmussen (1987), Asmussen et al. (1994) and Asmussen and O'Cinneide (2002). The primary motive of these papers is to enhance the flexibility of the models parameter setting. This is achieved by considering an external Markovian environment process which influences both the claim frequencies and the claim severities. The examples usually given are weather conditions, where the sojourns of the external Markov process could be weather types, or in health insurance where the sojourns of the environment process could be certain types of epidemics (see Asmussen (1989)). Surprisingly, only a few authors have studied non-Poissonian risk models in the presence of an investment strategy. Kötter and Bäuerle (2006) were the first to introduce a Markov-modulated risk process where risk reserves, under a special investment strategy, can be invested into a stock index following a geometric Brownian motion. Within this set up, for a special class of investment policies, they derive results for the adjustment coefficient. A second study within the Markov-modulated framework was made by Diko and Usábel (2011), where they considered a risk model perturbed by diffusion in which the reserves are invested into an asset whose return rate and volatility are time-dependent Markov-modulated. For this model they used Chebyshev's polynomial approximation and Laplace-Carson transforms to obtain a numerical solution for the integro-differential equation system for the risk quantity of interest.

In this paper, we consider a Markov-modulated risk model in which the reserves of the insurance portfolio are continuously invested into an asset whose prices follow a geometrical Brownian motion, which is also influenced by the external Markov chain. For the aforementioned model the Markov property no longer holds and thus the ruin probability is given in terms of an integro-differential equation system. Stimulated by Albrecher et al. (2012), we extend their methodology (using Frobenius method for systems - see Barkatou et al. (2010)) to obtain, using regular variation theory, an explicit asymptotic expression of the ultimate ruin probability and the expected discounted penalty function. Within this non-Poissonian model we are able to show that the ruin probability decreases asymptotically as a power function in the case of the light tailed claims, whilst for the heavy tails we show that the probability of ruin decays either like a power function, depending on the parameters of the investment, or behaves asymptotically like the tails of the claim size distributions. The same kind of results hold for the Gerber-Shiu function. Note that the above matrix based analysis holds for more general non-renewal risk models, such as the Markov Arrival Process (MAP) risk models.

In more details the paper is organised as follows; in Section 2 we introduce a Markov-modulated risk model where the reserves of the insurance portfolio are invested in a risky asset whose price follows a geometrical Brownian motion, in which the drift and volatility parameters are also influenced by the external Markov chain. In Section 3, using the infinitesimal generator argument, we derive an integro-differential equation system for the decompositions of the ruin probabilities. In Section 4, we use Laplace transforms to derive an individual form for the system of ruin probabilities, that will allow an asymptotic analysis in the later sections. In Section 5, we give the general solution for the Laplace system and by using the Frobenius method for matrices, Tauberian theorems and Heaviside principle, we derive explicit asymptotic expressions for the probabilities of ruin. Section 6 discusses an extension of the methodology used for the ruin probabilities to more general ruin-related quantities, namely the Gerber-Shiu function.

2 Markov-modulated risk process with stochastic investment

In this section, we introduce the Markov-modulated Poisson risk model in the presence of risky asset investment, where the premium rate, the claim arrival rate, the distribution of the claim sizes and the parameters of the return on the surplus investment are influenced by an external Markov chain (see also Kötter and Bäuerle (2006) and Diko and Usábel (2011)).

Consider the external environment process $\{J(t)\}_{t\geqslant 0}$, which can be interpreted as the general economic conditions that govern the state of the economy. Suppose $\{J(t)\}_{t\geqslant 0}$ is a homogeneous, irreducible and recurrent continuous time Markov process, with finite state space $E=\{1,2,...,m\}$. Let $\mathbf{Q}=(q_{ij})_{i,j=1}^{m}$, with $q_{ii}=-\sum_{j\neq i}^{m} q_{ij}=-q_i$, for $i\in E$, denote the intensity rate matrix of $\{J(t)\}_{t\geqslant 0}$, with a stationary distribution (which exists and is unique since $\{J(t)\}_{t\geqslant 0}$ is irreducible and has finite state space) given by

$$\pi=(\pi_1,\cdots,\pi_m),\quad \pi_i\geqslant 0,\quad i\in E\quad\text{and}\quad \sum_{i\in E}\pi_i=1.$$

Assume that when $J(t)=i\in E$, the number of claims, namely $N(t)$, occur according to a Poisson process with intensity rate $\lambda_i\in\mathbb{R}^+$. Further assume that the corresponding nonnegative claim amounts, $\{X_k\}_{k\geqslant 1}$, have common distribution function $F_i(x)$, with density $f_i(x)$ and finite mean $\mu_i<\infty$. We will also assume that the premiums are received continuously at a rate $c_i>0$ during the time when $\{J(t)\}_{t\geqslant 0}$ remains in the state $i\in E$. Under the above set up, the corresponding risk model is known as a Markov-modulated Poisson process.

Considering the above assumptions, the insurer's surplus process can be given by

$$U(t)=u+\int_0^t c_{J(s)}ds-\sum_{k=1}^{N(t)}X_k,\qquad t\geqslant 0,$$

where $u\geqslant 0$ is the insurer's initial capital. Let us propose that the insurer invests its surplus into a risky asset, with returns process $\{R_i(t)\}_{t\geqslant 0}$, when $J(t)=i\in E$, which is also influenced by the external Markov process, $\{J(t)\}_{t\geqslant 0}$, and satisfies the stochastic differential equation

$$dR_{J(t)}(t)=a_{J(t)}dt+\sigma_{J(t)}dB(t),$$

where $\{a_{J(t)}\}_{t\geqslant 0}$ is the drift and $\{\sigma_{J(t)}\}_{t\geqslant 0}$ the volatility of the randomness produced by the standard Brownian motion $\{B(t)\}_{t\geqslant 0}$.

Within this framework, the surplus process under risky investment, is given by

$$U(t)=u+\int_0^t c_{J(s)}\,ds-\sum_{k=1}^{N(t)}X_k+\int_0^t U(s-)\,dR_{J(s)}(s),\quad t\geqslant 0. \tag{2.1}$$

This model extends the Markov-modulated risk process introduced by Reinhard (1984) and also the classical risk model, with investment, introduced by Paulsen (1993).

The first time the surplus process of the insurance portfolio falls below zero is referred to as the time of ruin and is denoted by

$$T = \inf\{t \geqslant 0 : U(t) < 0 | U(0) = u\}, \quad (\infty, \text{otherwise}).$$

The probability of ruin, given that the initial environment is in state $i \in E$, with initial capital $u \geqslant 0$, is described by

$$\psi_i(u) = \mathbb{P}\{T < \infty | U(0) = u, J(0) = i\}.$$

Then, the ultimate ruin probability, for the stationary case, is given by

$$\psi(u) = \sum_{k=1}^{m} \pi_k \psi_k(u), \qquad u \geqslant 0. \tag{2.2}$$

3 An integro-differential equation system for the ruin probabilities

The main aim of this section is to derive a system of integro-differential equations for the auxiliary function $\psi_i(u)$, $i \in E$. Before we proceed with the derivation, recall that if $\{X(t)\}_{t\geqslant 0}$ is an Itô diffusion, with $X(0) = x$, satisfying a stochastic differential equation of the form

$$dX(t) = \alpha(X(t))\, dt + r(X(t))\, dB(t),$$

then the infinitesimal generator of $X(t)$ is the operator $\mathcal{A}$, acting on suitable functions f, given by

$$\mathcal{A}f(x) = \lim_{h\to 0} \frac{\mathbb{E}\left[f(X(h))|\, X(0) = x\right] - f(x)}{h} = \alpha(x)\frac{\partial}{\partial x} f(x) + \frac{r^2(x)}{2}\,\frac{\partial^2}{\partial x^2} f(x). \tag{3.1}$$

Using an intuitive infinitesimal argument and methods similar to that in Cai and Xu (2006) and Lu and Tsai (2007), we get the following theorem.

Theorem 1. *For $u \geqslant 0$, the ruin probabilities, $\psi_i(u)$, $i \in E$, satisfy the following integro-differential equation system*

$$\begin{aligned} &\frac{1}{2}\sigma_i^2 u^2 \psi_i''(u) + (a_i u + c_i)\psi_i'(u) + \lambda_i \overline{F}_i(u) \\ &\qquad = (\lambda_i + q_i)\psi_i(u) - \lambda_i \int_0^u \psi_i(u-x)\, dF_i(x) - \sum_{j=1, j\neq i}^{m} q_{ij}\psi_j(u), \end{aligned} \tag{3.2}$$

with boundary conditions

$$\lim_{u\to\infty} \psi_i(u) = 0, \tag{3.3}$$

and

$$c_i\psi_i'(0) - (\lambda_i + q_i)\psi_i(0) + \sum_{j=1, j\neq i}^{m} q_{ij}\psi_j(0) + \lambda_i = 0, \tag{3.4}$$

where $\overline{F}_i(u) = 1 - F_i(u)$, $i \in E$.

Proof. Let

$$Y_i(t) = u + c_i t + \int_0^t Y_i(s-)\, dR_i(s), \quad i \in E, \tag{3.5}$$

be the income process under investment, given we start in state $i \in E$ and experience no claims up to time $t \geqslant 0$. In order to derive an integro-differential equation system for the ruin probabilities $\psi_i(u)$, $i \in E$, we consider the risk process $\{U(t)\}_{t\geqslant 0}$, defined by equation (2.1) in an infinitesimal time interval $(0, h]$. Moreover, given that $J(0) = i \in E$ and $\{N(t)\}_{t\geqslant 0}$ is a Poisson process, there are four cases that could appear in $(0, h]$;

1. No claim and no change in state,
2. No change in state but a claim arrival,
3. No claim but a change in state of the external process,
4. Two or more events occur in the interval $(0, h]$.

Considering the possible events above and noticing, for the second case, it holds that $\psi_i(Y_i(h) - x) = 1$, for $x > Y_i(h)$, we have

$$\begin{aligned}\psi_i(u) =& (1 - \lambda_i h - q_i h)\mathbb{E}(\psi_i(Y_i(h))) \\ &+ \lambda_i h \mathbb{E}\left[\int_0^{Y_i(h)} \psi_i(Y_i(h) - x)\, dF_i(x) + \overline{F}_i(Y_i(h))\right] \\ &+ h\mathbb{E}\left[\sum_{j=1, j\neq i}^{m} q_{ij}\psi_j(Y_i(h))\right] + o(h),\end{aligned}$$

where $o(h)$ is such that, $o(h)/h \to 0$ as $h \to 0$.

Re-arranging the above equation, yields

$$\begin{aligned}(\lambda_i + q_i)\mathbb{E}\left[\psi_i(Y_i(h))\right] =& \frac{\mathbb{E}\left[\psi_i(Y_i(h))\right] - \psi_i(u)}{h} + \lambda_i \mathbb{E}\left[\int_0^{Y_i(h)} \psi_i(Y_i(h) - x)\, dF_i(x) + \overline{F}_i(Y_i(h))\right] \\ &+ \mathbb{E}\left[\sum_{j=1, j\neq i}^{m} q_{ij}\psi_j(Y_i(h))\right] + \frac{o(h)}{h}.\end{aligned}$$

Now, letting $h \to 0$ in the equation above yields that

$$(\lambda_i + q_i)\psi_i(u) = \mathcal{A}\psi_i(u) + \lambda_i \left[\int_0^u \psi_i(u-x)\, dF_i(x) + \overline{F}_i(u) \right] + \sum_{j=1, j\neq i}^{m} q_{ij}\psi_j(u), \qquad (3.6)$$

where $\mathcal{A}$ is the infinitesimal generator, defined in equation (3.1), of the process $Y_i(t)$.

Rewriting equation (3.5) in the form of an Itô diffusion process, and using equation (3.1), we get that the generator of $Y_i(t)$ acting on $\psi_i(u)$ is given by

$$\mathcal{A}\psi_i(u) = (c_i + a_i u)\, \psi_i'(u) + \frac{1}{2}\sigma_i^2 u^2 \psi_i''(u).$$

After substituting this form of the generator into equation (3.6), we obtain the integro-differential equation system (3.2). For the boundary condition (3.4), setting $u = 0$ in the integro-differential equation system (3.2), the result follows immediately.

□

Remark 1. *For $m = 1$, we obtain the integro-differential equation for the classical risk model under risky investment*

$$\frac{1}{2}\sigma^2 u^2 \psi''(u) + (au + c)\psi'(u) + \lambda \overline{F}(u) = \lambda\psi(u) - \lambda \int_0^u \psi(u-x)\, dF(x),$$

as it is given in Constantinescu and Thomann (2004).

4 Laplace transforms

The structure of the integro-differential equation system (3.2) suggests the use of Laplace transforms for the asymptotic analysis of the probability of ruin. Thus, in this section, we will derive a matrix closed form expression for the ruin probability, that will be vital for our next section, where Karamata-Tauberian theorems will be applied to derive the asymptotic ruin results.

Let $\widehat{\psi}_i(s)$, $\widehat{\overline{F}}_i(s)$ and $\widehat{f}_i(s)$ be the Laplace transforms of $\psi_i(u)$, $\overline{F}_i(u)$ and $f_i(u)$, respectively. Taking Laplace transforms on both sides of equation system (3.2), one can see that $\widehat{\psi}_i(u)$ satisfies a second order non-homogeneous ordinary differential equation system, for $i \in E$, given by

$$\begin{aligned} \frac{s^2\sigma_i^2}{2}\widehat{\psi}_i''(s) &+ [s(2\sigma_i^2 - a_i)]\widehat{\psi}_i'(s) \\ &+ [\sigma_i^2 + c_i s - (a_i + q_i) - \lambda_i(1 - \widehat{f}_i(s))]\widehat{\psi}_i(s) + \sum_{j=1, j\neq i}^{m} q_{ij}\widehat{\psi}_j(s) \\ &= c_i\psi_i(0) - \lambda_i\widehat{\overline{F}}_i(s), \end{aligned}$$

or in matrix form

$$s^2\frac{d^2\vec{\widehat{\psi}}(s)}{ds^2}+s\mathbf{A}\frac{d\vec{\widehat{\psi}}(s)}{ds}+\mathbf{B}(s)\vec{\widehat{\psi}}(s)=\mathbf{c}\vec{\psi}(0)-\mathbf{\Lambda}\vec{k}(s), \tag{4.1}$$

with

$$\mathbf{A}=\mathrm{diag}\left(4-\frac{2a_1}{\sigma_1^2},\ldots,4-\frac{2a_m}{\sigma_m^2}\right),$$

$$\begin{aligned}
\vec{\widehat{\psi}}(s)&=[\widehat{\psi}_1(s),\ldots,\widehat{\psi}_m(s)]^T,\\
\vec{\psi}(0)&=[\psi_1(0),\ldots,\psi_m(0)]^T,\\
\mathbf{c}&=\mathrm{diag}(\frac{2c_1}{\sigma_1^2},\ldots,\frac{2c_m}{\sigma_m^2}),\\
\mathbf{\Lambda}&=\mathrm{diag}(\frac{2\lambda_1}{\sigma_1^2},\ldots,\frac{2\lambda_m}{\sigma_m^2}),\\
\vec{k}(s)&=[\widehat{\overline{F}}_1(s),\ldots,\widehat{\overline{F}}_m(s)]^T,
\end{aligned}$$

where the superscript, $(\cdot)^T$, denotes the transpose of a vector/matrix, and

$$\mathbf{B}(s)=\begin{pmatrix}
\frac{2}{\sigma_1^2}Z_1(s) & \frac{2}{\sigma_1^2}q_{1,2} & \cdots & \frac{2}{\sigma_1^2}q_{1,m}\\
\frac{2}{\sigma_2^2}q_{2,1} & \ddots & & \vdots\\
\vdots & & \ddots & \frac{2}{\sigma_{m-1}^2}q_{m-1,m}\\
\frac{2}{\sigma_m^2}q_{m,1} & \cdots & \frac{2}{\sigma_m^2}q_{m,m-1} & \frac{2}{\sigma_m^2}Z_m(s)
\end{pmatrix}, \tag{4.2}$$

with $Z_i(s)=\sigma_i^2+c_is-(a_i+q_i)-\lambda_i(1-\widehat{f}_i(s)), i\in E$.

The form of the non-homogeneous matrix equation (4.1) will be used in the sequel so as to derive asymptotic expressions for $\vec{\widehat{\psi}}(s)$ and consequently for the ultimate ruin probability, namely $\psi(u)$.

5 Asymptotic results for arbitrary claim size distributions

In this section we analyse the asymptotic behaviour of the Laplace transform vectors, for the ruin probabilities, and derive asymptotic expressions using Karamata-Tauberian theorems and Heaviside Principle. In order to achieve this, we first need to draw the solution of the Laplace transform vector, satisfying equation (4.1), in the neighbourhood of their singularities.

Let us define the following m-dimensional vector $\vec{\tilde{\psi}}(s) = \vec{y}(s) = (y_1(s), \ldots, y_m(s))^T$, with $\frac{d}{ds}\vec{\tilde{\psi}}(s) = \vec{y}\,'(s)$ and $\frac{d^2}{ds^2}\vec{\tilde{\psi}}(s) = \vec{y}\,''(s)$ denoting the first and second derivatives respectively, of every element of the vector $\vec{y}(s)$. Then, we can rewrite equation (4.1), as follows

$$s^2\vec{y}\,''(s) + s\mathbf{A}\vec{y}\,'(s) + \mathbf{B}(s)\vec{y}(s) = \vec{h}(s), \tag{5.1}$$

where $\vec{h}(s)$ is the m-dimensional vector, given by

$$\vec{h}(s) = \mathbf{c}\vec{\Psi}(0) - \mathbf{\Lambda}\vec{k}(s). \tag{5.2}$$

By the general methodology of differential equations, equation (5.1) has a general solution of the form

$$\vec{y}(s) = \vec{y}_h(s) + \vec{y}_p(s), \tag{5.3}$$

where $\vec{y}_p(s)$ is a particular solution vector and $\vec{y}_h(s)$ is the associated homogeneous solution vector to the corresponding homogeneous matrix equation of (5.1).

Remark 2. *The corresponding homogeneous equation system of* (4.1) *has a regular singular point at zero and will play a vital role in the formulation of its solution, while the extra term in the non-homogeneous system depends on the Laplace transform of the tail of the claim size distribution.*

For the rest of this section let us consider the analysis of the associated homogeneous equation system and the analysis of the particular solution to the matrix equation (5.1) separately. First, let us consider the associated homogeneous equation of (5.1), which has the form

$$s^2\vec{y}\,''(s) + s\mathbf{A}\vec{y}\,'(s) + \mathbf{B}(s)\vec{y}(s) = \vec{0}, \tag{5.4}$$

where $\vec{0}$ is an m-dimensional vector of zero elements. The form of the second order linear homogeneous differential matrix equation (5.4) and the presence of the regular singular point at $s = 0$, requires that the Frobenius method should be employed to determine the solution. Thus, using similar arguments to Barkatou et al. (2010) we consider that the vector solution to (5.4) is in a Frobenius form for systems, i.e.

$$\vec{y}(s, r) = \sum_{k=0}^{\infty} \vec{g}_k(r) s^{r+k},$$

where $\vec{y}(s, r) = (y_1(s, r), \ldots, y_m(s, r))^T$ and $\vec{g}_k(r) = (g_{1,k}(r), \ldots, g_{m,k}(r))^T$, $k \geqslant 0$, are m-dimensional vectors, where $\vec{g}_0$ is non-zero and the exponent r may be real or complex. Differentiating the above form of the solution vector twice, with respect to s, gives

$$\vec{y}\,'(s,r) = \sum_{k=0}^{\infty}(r+k)\vec{g}_k(r)s^{r+k-1},$$
$$\vec{y}\,''(s,r) = \sum_{k=0}^{\infty}(r+k)(r+k-1)\vec{g}_k(r)s^{r+k-2}.$$

Substituting the above forms of the vectors for $\vec{y}(s,r), \vec{y}\,'(s,r)$and $\vec{y}\,''(s,r)$ into the homogeneous second order matrix differential equation (5.4), yields

$$\sum_{k=0}^{\infty}(r+k)(r+k-1)\vec{g}_k(r)s^{r+k} + \mathbf{A}\sum_{k=0}^{\infty}(r+k)\vec{g}_k(r)s^{r+k} + \mathbf{B}(s)\sum_{k=0}^{\infty}\vec{g}_k(r)s^{r+k} = \vec{0}.$$

Analysing the above equation, we can see that by dividing through by the common term s^r and then setting $s = 0$, all terms with $k > 0$ vanish. Thus, we can deduce that r is the solution of the indicial matrix equation

$$r(r-1)\vec{g}_0(r) + \mathbf{A}r\vec{g}_0(r) + \mathbf{B}(0)\vec{g}_0(r) = \vec{0}. \tag{5.5}$$

Recalling the definition of $\mathbf{B}(s)$ and noting that $\widehat{f}_i(0) = 1$ for all $i \in E$, we can easily see that $\mathbf{B}(0)$ is an $m \times m$ matrix with constant elements, given by

$$\mathbf{B}(0) = \begin{pmatrix} \frac{2}{\sigma_1^2}\left(\sigma_1^2 - (a_1 + q_1)\right) & \frac{2}{\sigma_1^2}q_{1,2} & \cdots & \cdots & \frac{2}{\sigma_1^2}q_{1,m} \\ \frac{2}{\sigma_2^2}q_{2,1} & & & & \\ \vdots & & \ddots & & \vdots \\ \vdots & & & & \frac{2}{\sigma_{m-1}^2}q_{m-1,m} \\ \frac{2}{\sigma_m^2}q_{m,1} & \cdots & \cdots & \frac{2}{\sigma_m^2}q_{m,m-1} & \frac{2}{\sigma_m^2}\left(\sigma_m^2 - (a_m + q_m)\right) \end{pmatrix}.$$

Alternatively, the indicial matrix equation (5.5), may be written as

$$\mathbf{L}(r)\vec{g}_0(r) = \vec{0}, \tag{5.6}$$

where

$$\mathbf{L}(r) = r^2\boldsymbol{I} + (\mathbf{A} - \boldsymbol{I})r + \mathbf{B}(0), \tag{5.7}$$

is an $m \times m$ matrix and $\boldsymbol{I}$ is the m-dimensional identity matrix.

An equation of this form has non-trivial solutions, $\vec{g}_0(r)$, only for $\det(\mathbf{L}(r)) = 0$, known as the characteristic equation. Since the determinant of $\mathbf{L}(r)$ gives a polynomial of degree $2m$, with leading coefficient 1, we have the following Lemma.

Lemma 1. *For $r \in \mathbb{C}$, the characteristic equation,* $\det(\mathbf{L}(r)) = 0$*, has exactly* $2m$ *solutions* $r_1, r_2, \ldots, r_{2m}$.

Now, referring back to Frobenius' method, any set of fundamental solutions to the homogeneous matrix equation (5.4), may be written

$$\vec{y}_i(s) = s^{r_i} \sum_{k=0}^{\infty} \vec{g}_k(r_i) s^k = s^{r_i} \vec{\gamma}_i(s), \quad i = 1, \ldots, 2m, \tag{5.8}$$

where $\vec{y}_i(s) = (y_{i,1}(s), \ldots, y_{i,m}(s))^T = \vec{y}(s, r_i)$ and $\vec{\gamma}_i(s) = (\gamma_{i,1}(s), \ldots, \gamma_{i,m}(s))^T$ are vectors of holomorphic functions with $\vec{\gamma}_i(0) = \vec{g}_0(r_i) \neq 0$. Then, as it will be shown later, since the vector solutions $\vec{y}_i(s)$ are linearly independent, the general solution to equation (5.4) is given by

$$\vec{y}_h(s) = \sum_{i=1}^{2m} \eta_i \vec{y}_i(s) = \eta_1 s^{r_1} \vec{\gamma}_1(s) + \ldots + \eta_{2m} s^{r_{2m}} \vec{\gamma}_{2m}(s), \tag{5.9}$$

where η_i are constant coefficients and $r_i,\ i = 1, \ldots, 2m$ are the solutions to the characteristic equation $\det(\mathbf{L}(r)) = 0$. The linear independence of the solution vectors will be made more apparent in a later section.

In particular, the j-th element of the solution vector, $\vec{y}_h(s)$, is given by

$$y_{h,j}(s) = \sum_{i=1}^{2m} \eta_i y_{i,j}(s) = \eta_1 s^{r_1} \gamma_{1,j}(s) + \ldots + \eta_{2m} s^{r_{2m}} \gamma_{2m,j}(s). \tag{5.10}$$

Having obtained a general solution for the homogeneous solution, it remains to determine the contribution of the particular solution $\vec{y}_p(s)$ of equation (5.3).

To find the particular solution of the differential equation system (5.1), we use the method of variation of parameters, similar to Albrecher et al. (2012). Hence, the particular solution has the following form

$$\vec{y}_p(s) = \sum_{i=1}^{2m} v_i(s) \vec{y}_i(s),$$

where $\vec{y}_i(s), i = 1, \ldots, 2m$ are the solution vectors to the homogeneous equation (5.4), given by equation (5.8), and $v_i(s)$ are scalar coefficients that need to be determined.

By the method of variation of parameters and the use of Cramer's rule, the variables $v_i(s), i = 1 \ldots, 2m$, have the following form

$$v_i(s) = \int_{s_0}^{s} \frac{W_i(t)}{t^2 W(t)}\, dt, \tag{5.11}$$

where s_0 is a small positive constant, $W(s)(\neq 0)$ is the Wronskian (block) determinant given by

$$W(s) = \begin{vmatrix} \vec{y}_1(s) & \vec{y}_2(s) & \ldots & \vec{y}_{2m}(s) \\ \vec{y}'_1(s) & \vec{y}'_2(s) & \ldots & \vec{y}'_{2m}(s) \end{vmatrix},$$

and $W_i(s)$ is a consequence of $W(s)$, with the i-th column replaced with $(\vec{0}, \vec{h}(s))^T$. For example, for $i = 1$, $W_1(s)$ is given by

$$W_1(s) = \begin{vmatrix} \vec{0} & \vec{y}_2(s) & \ldots & \vec{y}_{2m}(s) \\ \vec{h}(s) & \vec{y}'_2(s) & \ldots & \vec{y}'_{2m}(s) \end{vmatrix} = \begin{vmatrix} 0 & y_{2,1} & \ldots & y_{2m,1} \\ \vdots & \vdots & \ldots & \vdots \\ 0 & y_{2,m} & \ldots & y_{2m,m} \\ h_1(s) & y'_{2,1} & \ldots & y'_{2m,1} \\ \vdots & \vdots & \ldots & \vdots \\ h_m(s) & y'_{2,m} & \ldots & y'_{2m,m} \end{vmatrix}.$$

Remark 3. *$W(s) \neq 0$ implies the linear independence of the solutions $\vec{y}_i(s)$.*

Equation (5.11) can be re-written as

$$v_i(s) = \int_{s_0}^{s} \frac{\vec{h}(t)^T \overrightarrow{W_i}(t)}{t^2 W(t)}\, dt, \tag{5.12}$$

where $\vec{h}(s)^T = (h_1(s), \ldots, h_m(s))$ is the transpose vector of that given in (5.2) and $\overrightarrow{W}_i(s) = (W_{i,1}(s), \ldots, W_{i,m}(s))^T$ is a vector of corresponding Wronskian determinants, namely $W_{i,j}(s)$, which are a consequence of $W(s)$, with the i-th column replaced by $(0, \ldots, 0, 1, 0, \ldots, 0)^T$, where the unit is in the $(m + j)$-th row.

After algebraic manipulations, the above equation can be written as

$$v_i(s) = \sum_{k=1}^{m} \int_{s_0}^{s} t^{-r_i-1} h_k(t) \frac{\xi_{i,k}(t)}{\xi(t)}\, dt, \tag{5.13}$$

where $\xi(t)$ and $\xi_{i,k}(t)$, $i = 1, \ldots, 2m$, are holomorphic functions, with $\xi(0) \neq 0 \neq \xi_{i,k}(0)$ (as they are linear combinations of $\gamma_{i,j}(s), i = 1, \ldots, 2m, j \in E$ and their derivatives, for which $\vec{\gamma}_i(0) = \vec{g}_0(r_i) \neq 0$ holds).

Recalling the definition of $\vec{h}(s)$ from equation (5.2), we see that $h_k(s)$ has the form

$$h_k(s) = \frac{2}{\sigma_k^2} \left(c_k \psi_k(0) - \lambda_k \widehat{\overline{F}}_k(s) \right), \quad k \in E,$$

and thus we can write the particular solution to the non-homogeneous second order differential equation system (5.1) as

$$
\begin{aligned}
\vec{y}_p(s) &= \sum_{i=1}^{2m} s^{r_i}\vec{\gamma}_i(s)v_i(s) \\
&= \sum_{i=1}^{2m} s^{r_i}\vec{\gamma}_i(s)\sum_{k=1}^{m}\int_{s_0}^{s} t^{-r_i-1}h_k(t)\frac{\xi_{i,k}(t)}{\xi(t)}\,dt \\
&= \sum_{i=1}^{2m} s^{r_i}\vec{\gamma}_i(s)\sum_{k=1}^{m}\frac{2c_k\psi_k(0)}{\sigma_k^2}\int_{s_0}^{s} t^{-r_i-1}\frac{\xi_{i,k}(t)}{\xi(t)}\,dt \\
&\qquad - \sum_{i=1}^{2m} s^{r_i}\vec{\gamma}_i(s)\sum_{k=1}^{m}\frac{2\lambda_k}{\sigma_k^2}\int_{s_0}^{s} t^{-r_i-1}\widehat{\overline{F}}_k(t)\frac{\xi_{i,k}(t)}{\xi(t)}\,dt. \qquad (5.14)
\end{aligned}
$$

From this equation, we can see that the particular solution, for each element $y_{p,j}(s)$, $j \in E$ of $\vec{y}_p(s)$, has the form

$$
\begin{aligned}
y_{p,j}(s) &= \sum_{i=1}^{2m} s^{r_i}\gamma_{i,j}(s)\sum_{k=1}^{m}\frac{2c_k\psi_k(0)}{\sigma_k^2}\int_{s_0}^{s} t^{-r_i-1}\frac{\xi_{i,k}(t)}{\xi(t)}\,dt \\
&\qquad - \sum_{i=1}^{2m} s^{r_i}\gamma_{i,j}(s)\sum_{k=1}^{m}\frac{2\lambda_k}{\sigma_k^2}\int_{s_0}^{s} t^{-r_i-1}\widehat{\overline{F}}_k(t)\frac{\xi_{i,k}(t)}{\xi(t)}\,dt. \qquad (5.15)
\end{aligned}
$$

From the form of the above equations and using equations (5.3) and (5.10), it is clear that the asymptotic behaviour of $\widehat{\psi}_j(s)$, and thus of $\psi_j(s)$, heavily depends on the roots r_i, $i = 1,\ldots,2m$, of the characteristic equation $\det(\mathbf{L}(r)) = 0$, and the behaviour of $\overline{F}_k(s), k \in E$.

Having determined the general solution of the matrix equation (5.1) (given by equations (5.3), (5.9) and (5.14)), in the subsequent work we will perform an asymptotic analysis using Karamata-Tauberian theorems and the Heaviside Operational principle, for the homogeneous solution and particular solutions respectively. We separate the cases for $y_{h,j}(s)$ and $y_{p,j}(s)$ and consequently $\widehat{\psi}_{h,j}(s)$ and $\widehat{\psi}_{p,j}(s)$ respectively, as follows.

Now, since the Karamata-Tauberian theorems correspond to the asymptotic behaviour of the Laplace-Stieltjes transform of a function, then, for the analysis of $y_{h,j}(s)$ similarly to Albrecher et al. (2012), we introduce the auxiliary functions

$$
U_j(u) = \begin{cases} 0 & \text{if} \quad u < 0 \\ \int_0^u \psi_{h,j}(x)\,dx & \text{if} \quad u \geqslant 0. \end{cases}
$$

Let $\tilde{U}_j(s)$ be the Laplace-Stieltjes transform of $U_j(u)$. Note that the Laplace transform of the ruin probabilities $\psi_{h,j}(u)$, defined as $\widehat{\psi}_{h,j}(s)$, is equivalent to the Laplace-Stieltjes transform of the function $U_j(u)$, i.e.

$$\widehat{\psi}_{h,j}(s) = \mathcal{L}(\psi_{h,j}(u))(s) = \int_0^\infty e^{-su}\psi_{h,j}(u)\,du = \int_0^\infty e^{-su}\,dU_j(u) = \tilde{U}_j(s).$$

The asymptotic behaviour at zero of the homogeneous solutions, given by equation (5.10), describes the asymptotic behaviour at zero of $\widehat{\psi}_{h,j}(s)$, consequently of $\tilde{U}_j(s)$. The slowest decaying power of this linear combination dictates the asymptotic behaviour of the solution as $s \to 0$. In general, this power can be found numerically by evaluating all roots r_i, $i \in E$, to the characteristic equation $\det(\mathbf{L}(r)) = 0$, however, in order to explicitly determine the leading power of this equation we must restrict ourselves to the case where the drift and volatility parameters of the investment process are all equal, i.e. $a_i = a, \sigma_i = \sigma$ for all $i \in E$. Note that this restriction does not affect the Markov-modulated environment of the arrival process which is still influenced by the external environment process. Adopting this modification and using the following two Lemmas, we are able to show that the rate of decay, of the homogeneous solution, is driven by the slowest decaying power, corresponding to the leading power of equation (5.10), which will be determined.

Lemma 2. *The transition rate matrix* $\mathbf{Q}$ *has* 0 *as an eigenvalue and the remaining eigenvalues have negative real parts.*

Proof. Let η be a real positive number greater than the absolute value of all entries of $\mathbf{Q}$, i.e. $\eta > |q_{ij}|$, $\forall i, j \in E$. Now, define the matrix

$$\mathbf{P} = \frac{1}{\eta}\mathbf{Q} + \mathbf{I},$$

with elements

$$p_{ij} = \frac{1}{\eta}q_{ij} + \mathbb{I}_{(i=j)},$$

where $\mathbb{I}_{(\cdot)}$ is an indicator function. Now, since

$$\sum_{j\in E} p_{ij} = \sum_{j\in E}(\frac{1}{\eta}q_{ij} + \mathbb{I}_{(i=j)}) = \frac{1}{\eta}\sum_{j\in E} q_{ij} + \sum_{j\in E}\mathbb{I}_{(i=j)} = 1, \quad i \in E,$$

and

$$p_{ij} = \frac{1}{\eta}q_{ij} \geqslant 0, \quad i \neq j \in E,$$

$$p_{ii} = \frac{1}{\eta}q_{ii} + 1 \geqslant 1 - \frac{1}{\eta}|q_{ii}| > 1 - 1 = 0, \quad \text{since } \eta > |q_{ii}|,$$

the matrix $\mathbf{P}$ is a stochastic matrix.

Now, note that the eigenvalues of $\mathbf{P}$ and $\mathbf{Q}$ are related as follows. If λ is an eigenvalue of $\mathbf{P}$, with right eigenvector $\vec{y}$, then

$$\mathbf{P}\vec{y} = \lambda\vec{y},$$

giving that

$$\mathbf{Q}\vec{y} = (\eta\mathbf{P} - \eta\mathbf{I})\,\vec{y} = \eta\mathbf{P}\vec{y} - \eta\vec{y} = (\eta\lambda - \eta)\vec{y} = \eta(\lambda - 1)\vec{y},$$

which implies that $\eta(\lambda - 1)$ is an eigenvalue of $\mathbf{Q}$. Now, by the Perron-Frobenius theorem, we have that $\lambda_{max} = 1$ is the maximum eigenvalue of $\mathbf{P}$ and the remaining eigenvalues λ are such that $|\lambda| < 1$. Based on the connection between the eigenvalues of $\mathbf{P}$ and $\mathbf{Q}$, for the maximum eigenvalue, namely $\lambda_{max} = 1$, the corresponding eigenvalue of $\mathbf{Q}$ is equal to 0. Thus, in order to complete the Lemma, it remains to prove that the remaining eigenvalues of $\mathbf{Q}$ have negative real parts. The remaining eigenvalues of $\mathbf{P}$ are λ such that $|\lambda| < 1$, which for complex λ implies its real part has absolute value less than 1. Thus, since the eigenvalues of $\mathbf{Q}$ are of the form $\eta(\lambda - 1)$, we have

$$\Re\left(\eta(\lambda - 1)\right) = \eta\left(\Re(\lambda) - 1\right) < 0,$$

since η is real and positive. □

Lemma 3. *For $a_i = a$ and $\sigma_i = \sigma$, for all $i \in E$, the characteristic equation* $\det\left(\mathbf{L}(r)\right) = 0$ *has two roots, $r_1 = -1$ and $r_2 = \frac{2a}{\sigma^2} - 2 = \rho - 1$. The remaining roots all have real parts that lie outside the interval determined by r_1 and r_2.*

Proof. In order to find the roots of the characteristic equation, $\det(\mathbf{L}(r)) = 0$, where $\mathbf{L}(r) = r^2\boldsymbol{I} + (\mathbf{A} - \boldsymbol{I})r + \mathbf{B}(0)$, we need to rewrite $\mathbf{L}(r)$ in a slightly different form. Recalling the forms of the matrices $\mathbf{A}$ and $\mathbf{B}(0)$, and after some algebraic manipulations we have that

$$\mathbf{L}(r) = \alpha(r)\mathbf{I} + \frac{2}{\sigma^2}\mathbf{Q},$$

where $\alpha(r) = r^2 + \left(3 - \frac{2a}{\sigma^2}\right)r + 2 - \frac{2a}{\sigma^2} = (r + 1)\left(r + 2 - \frac{2a}{\sigma^2}\right)$.

Recalling the indicial matrix equation (5.6), and using the above expression of $\mathbf{L}(r)$, equation (5.6) may be written

$$\left(\alpha(r)\boldsymbol{I} + \frac{2}{\sigma^2}\mathbf{Q}\right)\vec{g}_0(r) = \vec{0},$$

or equivalently

$$\frac{2}{\sigma^2}\mathbf{Q}\vec{g}_0(r) = -\alpha(r)\vec{g}_0(r). \tag{5.16}$$

From the above equation we see that $-\alpha(r)$ forms an eigenvalue with respect to the matrix $\frac{2}{\sigma^2}\mathbf{Q}$. Thus, solving

$$\det\left(\mathbf{L}(r)\right) = \det\left(\alpha(r)\mathbf{I} + \frac{2}{\sigma^2}\mathbf{Q}\right) = 0,$$

is equivalent to finding the eigenvalues of the matrix $\frac{2}{\sigma^2}\mathbf{Q}$, which are of the form $-\alpha(r)$.

Using Lemma 2, and since $2/\sigma^2$ is real and positive, it follows that the matrix $\frac{2}{\sigma^2}\mathbf{Q}$ also has 0 as an eigenvalue with remaining eigenvalues having negative real parts. Now, given that $-\alpha(r)$ forms an eigenvalue of $\frac{2}{\sigma^2}\mathbf{Q}$, we have for the 0 eigenvalue that

$$-\alpha(r) = 0,$$

which implies

$$\alpha(r) = (r+1)\left(r + 2 - \frac{2a}{\sigma^2}\right) = 0, \tag{5.17}$$

giving the two roots $r_1 = -1$ and $r_2 = 2a/\sigma^2 - 2 = \rho - 1$. Consequently, $r = r_1$ and $r = r_2$ are two roots of the characteristic equation $\det(\mathbf{L}(r)) = 0$.

To complete our Lemma it remains to prove that the real parts of the remaining roots lie outside the interval determined by r_1 and r_2.

Consider that the eigenvalues of the matrix $\frac{2}{\sigma^2}\mathbf{Q}$ have complex form i.e. they are given by

$$-\alpha_k(r) = u_k + iv_k, \quad k = 1, 2, \ldots, m, \quad k \neq j,$$

where u_k and v_k are real numbers and $-\alpha_j(r)$ is an individual eigenvalue corresponding to the 0 eigenvalue (without the loss of generality).

Using the form of $\alpha(r)$ given in equation (5.17), and the fact that $r_1 = -1$ and $r_2 = \rho - 1$ satisfy equation (5.17), then $\alpha_k(r)$ could be written

$$\alpha_k(r) = (r - r_1)(r - r_2) = -\left(u_k + iv_k\right).$$

Since r can also be complex, i.e. $r = x + iy$, the above equation becomes

$$(x - r_1 + iy)(x - r_2 + iy) = -\left(u_k + iv_k\right).$$

Equating the real terms gives

$$(x - r_1)(x - r_2) - y^2 = -u_k,$$

or alternatively

$$(x - r_1)(x - r_2) = y^2 - u_k.$$

Now, from Lemma 2, we have that the non-zero eigenvalues have negative real parts implying that $u_k < 0$, for $k \neq j$. Therefore $(x - r_1)(x - r_2) > 0$, from which it follows that $(x - r_1)$ and $(x - r_2)$ have the same sign. That is, x is either larger or smaller than both r_1 and r_2.

Note that in the case that r has no imaginary part, i.e. $y = 0$, the same argument holds, meaning that the other real solutions also lie outside the interval determined by r_1 and r_2. This completes our proof. $\square$

Remark 4. *Since r_1 and r_2 correspond to the 0 eigenvalue of $\frac{2}{\sigma^2}\mathbf{Q}$ and hence $\mathbf{Q}$, it follows from equation* (5.16) *that*

$$\mathbf{Q}\vec{g}_0(r_k) = \vec{0}, \quad k = 1, 2.$$

Using the fact that the elements in each row of $\mathbf{Q}$ sum to 0, it is not difficult to see that $\vec{g}_0(r_k) = \beta\vec{e}$, $k = 1, 2$, where $\vec{e}$ is an m-dimensional vector of units and β is arbitrary, let's say $\beta = 1$.

Proposition 1. *Consider the model given by* (2.1) *and assume that $\sigma > 0$. Then, if the ruin probability $\psi(u)$ decays at infinity, we have*

$$\rho = \frac{2a}{\sigma^2} - 1 > 0.$$

Proof. The proof of this proposition will become apparent towards the end of this section. □

Using the two Lemmas above, we can determine the slowest decaying power of the homogeneous solution to the vector equation (5.3), given by equation (5.9). Note that, by Proposition 1 we have $r_1 < r_2$. Now, the boundary condition $\lim_{u\to\infty} \psi_i(u) = 0$, and the use of final value theorem, implies that the coefficients of terms with powers that have real part less than r_1 in equation (5.10), must be zero. Consequently this makes r_1 the slowest decaying power.

Next, we will apply Karamata-Tauberian theorems to find the asymptotic behaviour of the homogeneous solution. It is crucial to observe that by applying the Karamata-Tauberian theorem, in the case that the slowest decaying power of equation (5.9) is r_1, results in the fact that the ruin probabilities converge to a constant, which is in contradiction with the boundary condition (3.3). Hence, it should be clear that the coefficient of s^{r_1}, namely η_1, vanishes.

Based on the above observation, we conclude that eventually the slowest decaying power is r_2. Thus, we are ready now to apply Karamata-Tauberian theorem and the Monotone Density theorem to find the asymptotic behaviour of the homogeneous solution, given by equation (5.9). Since we have concluded the root r_2 represents the slowest decaying power, we have that the individual elements of the homogeneous solution vector, $\vec{y}_h(s)$, behave like

$$\widetilde{U}_j(s) \sim \eta_2 s^{\rho-1}\gamma_{2,j}(s), \quad s \to 0,$$

which is equivalent to

$$U_j(u) \sim \frac{\eta_2 u^{1-\rho}\gamma_{2,j}(1/u)}{\Gamma(2-\rho)}, \quad u \to \infty,$$

by the application of Karamata-Tauberian theorem. Finally, applying the Monotone Density theorem gives

$$\psi_{h,j}(u) \sim \frac{\eta_2(1-\rho)u^{-\rho}\gamma_{2,j}(1/u)}{\Gamma(2-\rho)}, \quad u \to \infty.$$

Note that, since $\rho > 0$ by Proposition 1, $\psi_{h,j}(u)$ decays to zero, as required, and the conclusion is that

$$\psi_{h,j}(u) \sim C u^{-\rho} \gamma_{2,j}(1/u), \quad u \to \infty \tag{5.18}$$

where $C = \frac{\eta_2(1-\rho)}{\Gamma(2-\rho)}$. Alternatively, we have

$$\lim_{u \to \infty} \psi_{h,j}(u) u^{\rho} = C$$

since $\gamma_{2,j}(0) = g_{j,0}(r_2) = 1$ (see Remark 4).

Having completed the asymptotic analysis of the homogeneous part of equation (5.3), it remains to analyse the asymptotic behaviour of the particular solution of the aforementioned equation, namely $\vec{y}_p(s)$. Noticing that the elements of the vector $\vec{y}_p(s)$, given by equation (5.15), strongly depend on the tail of the claim size distribution, below we consider two separate cases.

Depending on the distribution of $\widehat{\overline{F}}_k(s)$ we can identify two cases, similarly to Albrecher et al. (2012):

A. Light tailed claims with exponentially bounded tails. Assume $\widehat{\overline{F}}_k(s)$ has a rightmost singularity at $-\mu_k < 0$, $k \in E$, and $\widehat{\overline{F}}_k(-\mu_k) = \infty$ for each $k \in E$.

B. Heavy tailed claims $\widehat{\overline{F}}_k(-\epsilon) = \infty$, for $\epsilon > 0$, $k \in E$.

Light Tailed claims. Let us first note that if $-\mu_k$ is the rightmost singularity of each $\widehat{\overline{F}}_k(s)$, $k \in E$, then $-\delta$, where $\delta = \min_{k \in E}(\mu_k)$, is the rightmost singularity of the summation of $\widehat{\overline{F}}_k(s)$, $k \in E$. Now, using L'Hopital's rule, we have

$$\begin{aligned}
\lim_{s \to -\delta} \frac{\sum_{k=1}^m \lambda_k \int_{s_0}^s t^{-r_i-1} \widehat{\overline{F}}_k(t)\, dt}{s^{-r_i} \sum_{k=1}^m \lambda_k \widehat{\overline{F}}_k(s)} &= \lim_{s \to -\delta} \frac{\sum_{k=1}^m \lambda_k s^{-r_i-1} \widehat{\overline{F}}_k(s)}{-r_i s^{-r_i-1} \sum_{k=1}^m \lambda_k \widehat{\overline{F}}_k(s) + s^{-r_i} \sum_{k=1}^m \lambda_k \frac{d}{ds} \widehat{\overline{F}}_k(s)} \\
&= \lim_{s \to -\delta} \frac{1}{-r_i + s \frac{\sum_{k=1}^m \lambda_k \frac{d}{ds} \widehat{\overline{F}}_k(s)}{\sum_{k=1}^m \lambda_k \widehat{\overline{F}}_k(s)}} = \frac{1}{-r_i}.
\end{aligned}$$

Thus,

$$\sum_{k=1}^m \lambda_k \int_{s_0}^s t^{-r_i-1} \widehat{\overline{F}}_k(t)\, dt \sim \frac{1}{-r_i} s^{-r_i} \sum_{k=1}^m \lambda_k \widehat{\overline{F}}_k(s), \quad \text{as} \quad s \to -\delta.$$

Then, from equation (5.15), we have

$$\widehat{\psi}_{p,j}(s) \sim \sum_{i=1}^{2m} \left(\frac{1}{-r_i}\right) \gamma_{i,j}(-\delta) \sum_{k=1}^{m} \frac{2c_k\psi_k(0)}{\sigma^2} \frac{\xi_{i,k}(-\delta)}{\xi(-\delta)} - \sum_{i=1}^{2m} \left(\frac{1}{-r_i}\right) \gamma_{i,j}(-\delta) \sum_{k=1}^{m} \frac{2\lambda_k}{\sigma^2} \frac{\xi_{i,k}(-\delta)}{\xi(-\delta)} \widehat{\overline{F}}_k(s), \quad s \to -\delta.$$

Normalise $\xi_{i,k}(-\delta)$ such that $\gamma_{i,j}(-\delta)\frac{\xi_{i,k}(-\delta)}{\xi(-\delta)} = 1$, for all $i = 1, \ldots, 2m$, $k = 1, \ldots, m$. Since $-\delta$ is the rightmost singularity of $\widehat{\psi}_{p,j}(s)$ and the first term of the above equation is analytic in $-\delta$, one can apply the Heaviside Operational Principle (see Abate and Whitt (1997)) to deduce

$$\psi_{p,j}(u) \sim \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_i} \sum_{k=1}^{m} \lambda_k \overline{F}_k(u), \quad u \to \infty. \tag{5.19}$$

Heavy Tailed claims. Using L'Hopital's rule and other limit properties, we have, for each $k \in E$

$$\begin{aligned}
\lim_{s\to 0} \frac{\sum_{k=1}^m \lambda_k \int_{s_0}^s t^{-r_i-1}\widehat{\overline{F}}_k(t)\, dt}{s^{-r_i}\sum_{k=1}^m \lambda_k \widehat{\overline{F}}_k(s)} &= \lim_{s\to -0} \frac{\sum_{k=1}^m \lambda_k s^{-r_i-1}\widehat{\overline{F}}_k(s)}{-r_i s^{-r_i-1}\sum_{k=1}^m \lambda_k \widehat{\overline{F}}_k(s) + s^{-r_i}\sum_{k=1}^m \lambda_k \frac{d}{ds}\widehat{\overline{F}}_k(s)} \\
&= \lim_{s\to 0} \frac{1}{-r_i + s\frac{\sum_{k=1}^m \lambda_k \frac{d}{ds}\widehat{\overline{F}}_k(s)}{\sum_{k=1}^m \lambda_k \widehat{\overline{F}}_k(s)}} = \frac{1}{-r_i}.
\end{aligned}$$

Thus,

$$\sum_{k=1}^{m} \lambda_k \int_{s_0}^{s} t^{-r_i-1}\widehat{\overline{F}}_k(t)\, dt \sim \frac{1}{-r_i} s^{-r_i} \sum_{k=1}^{m} \lambda_k \widehat{\overline{F}}_k(s), \quad \text{as} \quad s \to 0.$$

Then,

$$\widehat{\psi}_{p,j}(s) \sim \sum_{i=1}^{2m} \left(\frac{1}{-r_i}\right) \gamma_{i,j}(0) \sum_{k=1}^{m} \frac{2c_k\psi_k(0)}{\sigma^2} \frac{\xi_{i,k}(0)}{\xi(0)} - \sum_{i=1}^{2m} \left(\frac{1}{-r_i}\right) \gamma_{i,j}(0) \sum_{k=1}^{m} \frac{2\lambda_k}{\sigma^2} \frac{\xi_{i,k}(0)}{\xi(0)} \widehat{\overline{F}}_k(t), \quad s \to 0.$$

Normalise $\xi_{i,k}(0)$ such that $\gamma_{i,j}(0)\frac{\xi_{i,k}(0)}{\xi(0)} = 1$, for all $i = 1, \ldots, 2m$, $k = 1, \ldots, m$. Similarly to previous, the first term is analytic in zero, thus one can apply the Heaviside Operational Principle to deduce

$$\psi_{p,j}(u) \sim \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_i} \sum_{k=1}^{m} \lambda_k \overline{F}_k(u), \quad u \to \infty. \tag{5.20}$$

Now that we have completed the analysis of both the homogeneous and non-homogeneous parts of equation (5.3) we can present the asymptotic behaviour of the general solution, for each $j \in E$, namely $\psi_j(u)$. By combining equations (5.18), (5.19) and (5.20), we have

$$\psi_j(u) \sim Cu^{-\rho}\gamma_{2,j}(1/u) + \frac{2}{\sigma^2}\sum_{i=1}^{2m}\frac{1}{r_i}\sum_{k=1}^{m}\lambda_k\overline{F}_k(u), \quad u \to \infty. \tag{5.21}$$

Consequently, by equation (2.2), we can derive the asymptotic behaviour for the ultimate ruin probability, $\psi(u)$, given by

$$\psi(u) \sim Cu^{-\rho}\sum_{j=1}^{m}\pi_j\gamma_{2,j}\left(1/u\right) + \frac{2}{\sigma^2}\sum_{i=1}^{2m}\frac{1}{r_i}\sum_{k=1}^{m}\lambda_k\overline{F}_k(u), \quad u \to \infty. \tag{5.22}$$

Remark 5. *On the right hand side of equation* (5.22) *we have a summation of light and/or heavy tailed distributions. Now, since for some positive constants* r, n, α_k *and* $c_k\, (k = 1, \dots, n)$

$$\lim_{u\to\infty}\frac{\sum_{k=1}^{n}c_k e^{-\alpha_k u}}{u^{-r}} = \sum_{k=1}^{n}\lim_{u\to\infty}\frac{c_k e^{-\alpha_k u}}{u^{-r}} = 0,$$

we have that the particular solution does not represent a significant asymptotic decay in the case of light tails. However, in the case of heavy tails we have to compare the decay of the power function and the tail of the claim size distributions to determine which one is slower.

Considering all of the above, we obtain the following theorem.

Theorem 2. *Let* $a_i = a$ *and* $\sigma_i = \sigma$*, for all* $i \in E$*. Then, if* $\rho = \frac{2a}{\sigma^2} - 1 > 0$*, the ultimate ruin probability behaves asymptotically as*

$$\psi(u) \sim Cu^{-\rho}\sum_{j=1}^{m}\pi_j\gamma_{2,j}\left(1/u\right) + \frac{2}{\sigma^2}\sum_{i=1}^{2m}\frac{1}{r_i}\sum_{k=1}^{m}\lambda_k\overline{F}_k(u), \quad u \to \infty,$$

where $C = \frac{\eta_2(1-\rho)}{\Gamma(2-\rho)}$.

6 Asymptotic results for the Gerber-Shiu function

In this section our aim is to derive asymptotic results with respect to the expected discounted penalty function, introduced first by Gerber and Shiu (1998). The expected discounted penalty function, also called the Gerber-Shiu function, has been extensively studied in ruin theory since it unifies many risk-related quantities into a single function. In more details, quantities such as the time of ruin T, the deficit at ruin $|U(T)|$, the surplus immediately prior to ruin $U(T-)$ and many others can be explicitly derived from the Gerber-Shiu function [see among others Albrecher et al. (2012), Cai (2004) and Lu and Tsai (2007)].

Let $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot|J(0) = i)$ and $\mathbb{E}_i(\cdot)$ be the expectation with respect to $\mathbb{P}_i$, $i \in E$. Also, let $w(x, y)$, for $x, y \geqslant 0$, be an arbitrary non-negative function representing the penalty at ruin. Then, the Gerber-Shiu function, for $\delta, u \geqslant 0$, is given by

$$\phi_i(u) = \mathbb{E}_i\left[e^{-\delta T} w(U(T-), |U(T)|)\, \mathbb{I}_{(T<\infty)}|U(0) = u\right], \quad i \in E, \tag{6.1}$$

where δ can be considered as a constant force of interest. In particular, when $\delta = 0$ and $w(x, y) = 1$, we have

$$\phi_i(u) = \mathbb{E}_i\left[\mathbb{I}_{(T<\infty)}|U(0) = u\right] = \psi_i(u).$$

In a similar way to the ruin probability we can define the ultimate discounted penalty at ruin, in the stationary case, by

$$\phi(u) = \sum_{j=1}^{m} \pi_j \phi_j(u), \quad j \in E. \tag{6.2}$$

Using similar arguments as in Theorem 2, we have the following theorem.

Theorem 3. *The system of Gerber-Shiu functions, $\phi_i(u)$, satisfy the following integro-differential equation system*

$$\begin{aligned}\frac{1}{2}\sigma_i^2 u^2 \phi_i''(u) &+ (a_i u + c_i)\phi_i'(u) \\ &= (\lambda_i + q_i + \delta)\phi_i(u) - \lambda_i\left[\int_0^u \phi_i(u - x)\, dF_i(x) + w_i(u)\right] - \sum_{j=1, j\neq i}^{m} q_{ij}\phi_j(u),\end{aligned} \tag{6.3}$$

where $w_i(u) = \int_u^\infty w(u, x - u)\, dF_i(x)$, with boundary conditions

$$\lim_{u\to\infty} \phi_i(u) = 0, \tag{6.4}$$

and

$$c_i\phi_i'(0) - (\lambda_i + q_i + \delta)\phi_i(0) + \lambda_i \int_0^\infty w(0, x)\, dF_i(x) + \sum_{j=1, j\neq i}^{m} q_{ij}\phi_j(0) = 0. \tag{6.5}$$

Next, we investigate the asymptotic behaviour of the Gerber-Shiu function using a similar methodology as the one used for the analysis of the ruin probabilities. Thus, letting $\widehat{\phi}_i(s)$

and $\widehat{w}_i(s)$ be the Laplace transforms of $\phi_i(u)$ and $w_i(u)$ respectively, taking the Laplace transforms on both sides of equation (6.3), yields

$$\begin{aligned}\frac{s^2\sigma_i^2}{2}\widehat{\phi}_i''(s) &+ [s(2\sigma_i^2 - a_i)]\widehat{\phi}_i'(s) \\ &+ \left[\sigma_i^2 + c_i s - (a_i + q_i + \delta) - \lambda_i(1 - \widehat{f}_i(s))\right]\widehat{\phi}_i(s) + \sum_{j=1, j\neq i}^{m} q_{ij}\widehat{\phi}_j(s) \\ &= c_i\phi_i(0) - \lambda_i\widehat{w}_i(s), \qquad i \in E. \end{aligned} \tag{6.6}$$

In matrix form, the above equation can be written as

$$s^2\frac{d^2\vec{\widehat{\phi}}(s)}{ds^2} + s\mathbf{A}\frac{d\vec{\widehat{\phi}}(s)}{ds} + \mathbf{V}(s)\vec{\widehat{\phi}}(s) = \mathbf{c}\vec{\phi}(0) - \mathbf{\Lambda}\vec{\widehat{w}}(s), \tag{6.7}$$

where

$$\begin{aligned}\vec{\widehat{\phi}}(s) &= [\widehat{\phi}_1(s), \ldots, \widehat{\phi}_m(s)]^T \\ \vec{\phi}(0) &= [\phi_1(0), \ldots, \phi_m(0)]^T \\ \vec{\widehat{w}}(s) &= [\widehat{w}_1(s), \ldots, \widehat{w}_m(s)]^T,\end{aligned}$$

$\mathbf{V}(s) = \mathbf{B}(s) - \mathrm{diag}(\frac{2\delta}{\sigma_1^2}, \ldots, \frac{2\delta}{\sigma_m^2})$, with $\mathbf{B}(s)$, $\mathbf{A}$, $\mathbf{c}$, $\mathbf{\Lambda}$ all defined as in Section 4.

Note that, the matrix equation (6.7) is of a similar form as the matrix equation (4.1). Therefore, this equation can be solved using similar arguments as the ones used for the analysis of the ruin probabilities, i.e. using the Frobenius method for systems [similar to Barkatou et al. (2010)].

Letting $\vec{\widehat{\phi}}(s) = \vec{x}(s) = (x_1(s), \ldots, x_m(s))^T$ (with corresponding first and second derivative as in the previous section), then equation (6.7) has the form

$$s^2\vec{x}\,''(s) + s\mathbf{A}\vec{x}\,'(s) + \mathbf{V}(s)\vec{x}(s) = \vec{g}(s), \tag{6.8}$$

where $\vec{g}(s) = (g_1(s), \ldots, g_m(s))^T$ is the m-dimensional vector, given by

$$\vec{g}(s) = \mathbf{c}\vec{\phi}(0) - \mathbf{\Lambda}\vec{\widehat{w}}(s).$$

By the general theory of ordinary differential equations, equation (6.8) has a general solution of the following form

$$\vec{x}(s) = \vec{x}_h(s) + \vec{x}_p(s),$$

where $\vec{x}_h(s)$ is the solution to the corresponding homogeneous matrix equation and $\vec{x}_p(s)$ is the associated particular solution.

In the following, the particular vector solution $\vec{x}_p(s)$ and the vector solution $\vec{x}_h(s)$ of the corresponding homogeneous matrix equation of (6.8) will be analysed separately. The associated homogeneous equation system is given by

$$s^2\vec{x}''(s) + s\mathbf{A}\vec{x}'(s) + \mathbf{V}(s)\vec{x}(s) = \vec{0}, \tag{6.9}$$

and hence, by the Frobenius method, we adopt a solution of the form

$$\vec{x}(s, r_\delta) = \sum_{k=0}^{\infty} \vec{b}_k(r_\delta) s^{r_\delta + k}, \tag{6.10}$$

where $\vec{b}_k(r_\delta) = (b_{k,1}(r_\delta), \ldots, b_{k,m}(r_\delta))^T$ is an m-dimensional vector of constants with $\vec{b}_0(r_\delta) \neq \vec{0}$, and r_δ is a solution to the characteristic equation

$$\det\left(\mathbf{L}(s) - \operatorname{diag}\left(\frac{2\delta}{\sigma_1^2}, \ldots, \frac{2\delta}{\sigma_m^2}\right)\right) = 0,$$

where $\mathbf{L}(s)$ is defined in equation (5.7).

Following the same arguments as in Lemma 1, one can see that the characteristic equation has $2m$ roots, namely $r_{\delta,1}, \ldots, r_{\delta,2m}$, therefore the solution to the homogeneous equation system (6.9), by the linear independence of solution vectors, is

$$\vec{x}_h(s) = \sum_{i=1}^{2m} p_i s^{r_{\delta,i}} \vec{\beta}_i(s), \tag{6.11}$$

where p_i's are constant coefficients and $\vec{\beta}_i(s)$ are vectors of holomorphic functions with $\vec{\beta}_i(0) = \vec{b}_0(r_{\delta,i}) \neq \vec{0}$.

To complete the solution of the second order differential equation system (6.8), it remains to find the contribution of the particular solution $\vec{x}_p(s)$. For the particular solution, we again use variation of parameters to obtain

$$\begin{aligned}
\vec{x}_p(s) &= \sum_{i=1}^{2m} s^{r_{\delta,i}} \vec{\beta}_i(s) v_i(s) \\
&= \sum_{i=1}^{2m} s^{r_{\delta,i}} \vec{\beta}_i(s) \sum_{k=1}^{m} \int_{s_0}^{s} t^{-r_{\delta,i}-1} g_k(t) \frac{\theta_{i,k}(t)}{\theta(t)}\, dt \\
&= \sum_{i=1}^{2m} s^{r_{\delta,i}} \vec{\beta}_i(s) \sum_{k=1}^{m} \frac{2c_k\phi_k(0)}{\sigma_k^2} \int_{s_0}^{s} t^{-r_{\delta,i}-1} \frac{\theta_{i,k}(t)}{\theta(t)}\, dt \\
&\qquad - \sum_{i=1}^{2m} s^{r_{\delta,i}} \vec{\beta}_i(s) \sum_{k=1}^{m} \frac{2\lambda_k}{\sigma_k^2} \int_{s_0}^{s} t^{-r_{\delta,i}-1} \widehat{w}_k(t) \frac{\theta_{i,k}(t)}{\theta(t)}\, dt,
\end{aligned} \tag{6.12}$$

from which we get the following form for the j-th element of $\vec{x}_p(s)$

$$x_{p,j}(s) = \sum_{i=1}^{2m} s^{r_{\delta,i}} \beta_{i,j}(s) \sum_{k=1}^{m} \frac{2c_k \phi_k(0)}{\sigma_k^2} \int_{s_0}^{s} t^{-r_{\delta,i}-1} \frac{\theta_{i,k}(t)}{\theta(t)}\, dt \\ - \sum_{i=1}^{2m} s^{r_{\delta,i}} \beta_{i,j}(s) \sum_{k=1}^{m} \frac{2\lambda_k}{\sigma_k^2} \int_{s_0}^{s} t^{-r_{\delta,i}-1} \widehat{w}_k(t) \frac{\theta_{i,k}(t)}{\theta(t)}\, dt, \quad (6.13)$$

for each $j \in E$, where $\theta(t)$ and $\theta_{i,k}(t)$, $i = 1, \ldots, 2m$ are holomorphic functions, with $\theta(0) \neq 0 \neq \theta_{i,k}(0)$ (as they are linear combinations of $\beta_{i,j}(s), i = 1, \ldots, 2m, j \in E$ and their derivative, for which $\vec{\beta}_i(0) = \vec{b}_0(r_{\delta,i}) \neq \vec{0}$ holds).

Following a similar line of logic as in Section 5, we will use Karamata-Tauberian theorem to get an asymptotic expression for $\vec{x}_h(s)$ and Heaviside Principle for $\vec{x}_p(s)$, respectively. For the application of the Karamata-Tauberian theorem, we have to identify the slowest decaying power in equation (6.11). To do this explicitly we will have to adopt the same idea as Section 5. Let $a_i = a$ and $\sigma_i = \sigma$, for all $i \in E$, with no change in the Markovian environment of the claim arrival process.

Note that we now have

$$\mathbf{L}(s) - \operatorname{diag}\left(\frac{2\delta}{\sigma^2}, \ldots, \frac{2\delta}{\sigma^2}\right) = \mathbf{L}(s) - \frac{2\delta}{\sigma^2}\boldsymbol{I} = \alpha_\delta(s)\boldsymbol{I} + \frac{2}{\sigma^2}\mathbf{Q},$$

with $\alpha_\delta(s) = \alpha(s) - \frac{2\delta}{\sigma^2} = s^2 + \left(3 - \frac{2a}{\sigma^2}\right) s + 2 - \frac{2(a+\delta)}{\sigma^2}$.

Following the same arguments as in Lemmas 2 and 3 of Section 5 and noticing that $\alpha_\delta(s) = 0$ has two roots, namely $r_{\delta,i}$, given by

$$r_{\delta,i} = -\frac{2-\rho}{2} \pm \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}, \quad i = 1, 2,$$

where $\rho = \frac{2a}{\sigma^2} - 1$, we have the following Lemma.

Lemma 4. *For $a_i = a$, $\sigma_i = \sigma$, for all $i \in E$, the characteristic equation* $\det\left(\mathbf{L}(s) - \frac{2}{\sigma^2}\mathbf{I}\right) = 0$ *has two roots, $r_{\delta,i}$, $i = 1, 2$ and all remaining roots have real parts that lie outside the interval determined by $r_{\delta,1}$ and $r_{\delta,2}$.*

Remark 6. *It should be clear that for $\delta = 0$, $r_{\delta,1}$ and $r_{\delta,2}$ reduce to r_1 and r_2, respectively of Lemma 2.*

The power $r_{\delta,1} = -\frac{2-\rho}{2} + \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}$ would not produce a decay to zero at infinity resulting in the corresponding coefficient in equation (6.11), namely p_1, vanishing. Thus, the slowest decay power is given by

$$r_{\delta,2} = -\frac{2-\rho}{2} - \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}.$$

The slowest asymptotic behaviour of the solutions to the homogeneous part, given in equation (6.11), for some $j \in E$, is then given by

$$\widehat{\phi}_{h,j}(s) \sim p_2 s^{-\frac{2-\rho}{2} - \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}} \beta_{2,j}(s), \quad s \to 0,$$

which, by Karamata-Tauberian theorem is equivalent to

$$\phi_{h,j}(u) \sim K u^{-\frac{\rho}{2} + \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}} \beta_{2,j}(1/u), \quad u \to \infty,$$

where K is a constant.

It remains to analyse the asymptotic behaviour of the particular solution, given by equation (6.13). As before we have to deal with the two cases of light and heavy tailed distributions.

Light tailed claims. We again consider the right most singularity of $\widehat{w}_k(s)$, namely $-\mu_k$, then, in a similar way to the previous section, we can define the rightmost singularity $-\Delta$, where $\Delta = \min_{k \in E}(\mu_k)$, of the summation of $\widehat{w}_k(s), k \in E$. Applying L'Hopital's rule we have the following:

$$\begin{aligned}
\lim_{s \to -\Delta} & \frac{\sum_{k=1}^m \lambda_k \int_{s_0}^s t^{-r_{\delta,i}-1} \widehat{w}_k(t)\, dt}{s^{-r_{\delta,i}} \sum_{k=1}^m \lambda_k \widehat{w}_k(s)} \\
&= \lim_{s \to -\Delta} \frac{s^{-r_{\delta,i}-1} \sum_{k=1}^m \lambda_k \widehat{w}_k(s)}{-r_{\delta,i} s^{-r_{\delta,i}-1} \sum_{k=1}^m \lambda_k \widehat{w}_k(s) + s^{-r_{\delta,i}} \sum_{k=1}^m \lambda_k \frac{d}{ds} \widehat{w}_k(s)} \\
&= \lim_{s \to -\Delta} \frac{1}{-r_{\delta,i} + s \frac{\sum_{k=1}^m \lambda_k \frac{d}{ds} \widehat{w}_k(s)}{\sum_{k=1}^m \lambda_k \widehat{w}_k(s)}} = \frac{1}{-r_{\delta,i}},
\end{aligned}$$

Then, we have

$$\begin{aligned}
\widehat{\phi}_{p,j}(s) \sim \frac{2}{\sigma^2} \sum_{i=1}^{2m} \left(\frac{1}{-r_{\delta,i}}\right) \sum_{k=1}^m c_k \phi_k(0) \\
+ \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_{\delta,i}} \sum_{k=1}^m \lambda_k \widehat{w}_k(s), \quad s \to -\Delta, \qquad (6.14)
\end{aligned}$$

after normalising the value of $\theta_{i,k}$ such that $\beta_{i,j}(-\Delta)\frac{\theta_{i,k}(-\Delta)}{\theta(-\Delta)} = 1, i = 1,\ldots,2m,\ k = 1\ldots,m$. Applying the Heaviside Operational Principle, we have

$$\phi_{p,j}(u) \sim \frac{2}{\sigma^2}\sum_{i=1}^{2m}\frac{1}{r_{\delta,i}}\sum_{k=1}^{m}\lambda_k w_k(u), \quad u \to \infty. \tag{6.15}$$

Heavy Tailed Claims. A similar argument can be given for the case of heavy tailed claim size distributions to obtain

$$\phi_{p,j}(u) \sim \frac{2}{\sigma^2}\sum_{i=1}^{2m}\frac{1}{r_{\delta,i}}\sum_{k=1}^{m}\lambda_k w_k(u), \quad u \to \infty, \tag{6.16}$$

as long as $-\infty < \frac{d}{ds}\ln\left(\sum_{k=1}^{m}\lambda_k\widehat{w}_k(s)\right)|_{s=0} < \infty, k \in E$.

Finally, by the same method as in Section 5, we can combine the homogeneous and corresponding particular solutions of both light and heavy tailed distributions to obtain the asymptotic behaviour of the ultimate Gerber-Shiu function, $\phi(u)$, given in the following theorem.

Theorem 4. *Let $a_i = a$ and $\sigma_i = \sigma$, for all $i \in E$. Consider that $\rho = \frac{2a}{\sigma^2} - 1 > 0$, assume that $\widehat{w}_i(s)$ exists and that $|\frac{d}{ds}\ln\left(\sum_{k=1}^{m}\lambda_k\hat{w}_i(s)\right)|_{s=0} < \infty,\ i \in E$. Then, the ultimate Gerber-Shiu function, $\phi(u)$, behaves asymptotically as*

$$\phi(u) \sim Ku^{-\frac{\rho}{2}+\sqrt{\left(\frac{-\rho}{2}\right)^2+\frac{2\delta}{\sigma^2}}} + \frac{2}{\sigma^2}\sum_{i=1}^{2m}\frac{1}{r_{\delta,i}}\sum_{k=1}^{m}\lambda_k w_k(u), \quad u \to \infty, \tag{6.17}$$

for some strictly positive constant K.

Remark 7. *From the above theorem we deduce that the asymptotic decay will be given by the slower of the power function or the sum of functions $w_i(u)$, $i \in E$. By the definition of the these functions $w_i(u)$, it is clear that the asymptotic behaviour is dependent on the combination of the penalty function and the claim size distributions, which has been discussed in the previous literature.*

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