Efficient Option Risk Measurement With Reduced Model Risk

by

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Abstract

Risk measurement of options that is computationally efficient is important to research and industry. Currently, there exist few methods and they have significant model risk, which negates their risk management purpose. In this paper we propose a new approach to computationally efficient option risk measurement. This is achieved using the idea of a replicating portfolio and coherent risk measurement, rather than relying on mathematical approximations (as is currently done).

We find our replicating portfolio approach to option risk measurement provides fast computation by practically eliminating nonlinear computational operations. We reduce model risk by mostly using inputs that are observable data, we do not admit arbitrage opportunities for complex option portfolios, nor ignore liquidity risk or model misspecification, and enables portfolio optimisation. We also conduct numerical experiments to validate our new approach.

Key words: Options, model risk, contingent claims, risk measurement, delta method, portfolios, liquidity risk, option trading strategies, static replication.

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1 Introduction and Outline of Paper

Computationally efficient risk measures of options are of paramount importance to research and industry, especially with the progressive increase in options trading and hedging. The events of the global credit crisis and past financial crises have demonstrated the necessity for adequate option risk management and measurement; poor risk measurement and management can result in bankruptcies and threaten collapses of an entire finance sector (Kabir and Hassan, 2005). This is further exacerbated by the nonlinear losses associated with options and low margin requirements for options trading which magnify losses. Although there exists a large body of literature on asset risk measures (e.g. stocks and bonds) there is very little literature on option risk measurement.

To measure option risk requires Monte Carlo simulation of its loss distribution as there is typically no analytic solution. However, this can be computationally time consuming even for the simplest option pricing models because it requires computation of nonlinear functions (relating to the option pricing equation). Such long computation times are unsuitable for many financial applications e.g. high frequency trading. Consequently, this has led to the development of more computationally efficient methods of option risk measurement.

To improve the computation speed of option risk, the typical approach has been to apply some mathematical approximation to the option's loss distribution (e.g. Delta method). However, such computational improvements have been generally achieved at the cost of model risk, that is unforeseen losses associated with using a model e.g. calibration errors, implementation errors etc.. Since the purpose of such models are to measure or manage risk, such model risks defeat the purpose of the models and represents a significant flaw.

Model risk is becoming increasingly important in risk management due to the increasing reliance on models in the financial industry and its potential to cause significant losses. For instance, model risk has been cited as a partial cause in past financial crises. Many institutions prefer to use models with lower model risk than models that are theoretically more consistent. For instance, single factor interest rate models are preferred to multi-factor ones despite their theoretical consistency. Multi-factor models may be more realistic at explaining interest rate movements but they can result in higher estimation errors compared to single factor models. Consequently, single factor models are commonly preferred in industry.

In this paper we approach option risk measurement from a new direction. Rather than pursuing approximation methods, we model option risk in terms of its replicating portfolio and measuring the risk of this portfolio. This replicating portfolio method practically eliminates the requirement for calculating nonlinear operations for option risk and so provides faster computation times. Moreover, our replicating portfolio approach has lower model risk compared to other computationally efficient option risk models. The replicating portfolio method does not admit arbitrage opportunities for portfolios containing put and call options (unlike other models), our method also has lower calibration risk, it can take into account liquidity risks and model misspecification, it can model the option risk of option portfolios without losing computational tractability and enables portfolio optimisation.

The outline of the paper is as follows: firstly we introduce option risk measurement and review current computationally efficient methods for measuring option risk. In the next section we then introduce our replicating portfolio approach to risk measurement. We then discuss the advantages of the replicating portfolio approach with respect to computational efficiency and model risk. We then conduct numerical experiments and finally end with a conclusion.

2 Introduction to Option Risk Measurement and Related Literature

In this section we introduce risk measurement, review the current literature on computationally efficient methods for option risk measurement and their relation to model risk. We denote a risk measure by $\rho(.)$ and measuring risk by $\rho(Z)$, where Z(t) for the purpose of this paper is simply the loss distribution associated with some asset or derivative. For example

$$Z(t) = C(0) - C(t),$$

where C(0) and C(t) represent the call option price at time now and time t respectively.

A popular industry risk measure is VaR (Szegö, 2005) (that is $F(Z(t) \leq VaR) = \beta$, where F(.) is the cumulative probability distribution function and β is a cumulative probability associated with threshold value VaR on the loss distribution of Z(t)). A significant milestone in risk measurement was achieved when Artzner et al. (Artzner et al., 1997) proposed the coherency axioms: axioms that risk measures $\rho(.)$ should obey to correctly measure risk. The coherency axioms are included in the Appendix for reference.

To measure option risk we apply some risk measure to the loss distribution governing $C(0) - C(\delta t)$, where $C(\delta t)$ is the option value at some future time step δt. Whereas for stocks it is possible to model the loss distribution in order to apply some risk measure, for options the key difficulty in option risk measurement resides in modelling the loss distribution of $C(0) - C(\delta t)$ and in a computationally efficient method.

The current literature on option risk measurement is limited, particularly for computationally efficient methods. The "brute-force" approach is to use the "full valuation method" (Christoffersen, 2003). This involves Monte Carlo simulation of $S_i(\delta t)$ using some stock price model (e.g. geometric Brownian motion), where i denotes the index of the simulation sample. The option price value associated with $S_i(\delta t)$, that is $C_i(\delta t)$, is then calculated. The algorithm for the full valuation method is given in the Appendix for the Black-Scholes option pricing model C(S(t),t,T,r,σ,K), which is also defined in the Appendix.

The advantage of the full valuation method is that its accuracy can be improved by increasing the number of simulations. The key disadvantage is that it is highly time consuming, due to requiring computational calculations of $C_i(\delta t)$ for each simulated value $S_i(\delta t)$. This is because $C_i(\delta t)$ requires computing nonlinear terms (e.g. the Black-Scholes option pricing model requires calculating $\Psi(d_1)$ and $\Psi(d_2)$ for each $S(\delta t)$) , which is computationally time consuming. The computation time increases further for portfolios of options.

The high computation time incurred by the full valuation method has led to the development of alternative option risk methods with faster computation times. The most popular option risk method is the Delta method (Britten-Jones and Schaefer, 1998). The call option's delta Δ can give the option loss distribution of δC by approximation:

$$\begin{aligned} \Delta &\approx \frac{\delta C}{\delta S}, \\ \delta C &\approx \Delta \delta S. \end{aligned} \tag{1}$$

We obtain δS by simulating $S(\delta t)$ as we would under the full valuation method.

The main advantage of the Delta method is that it has a significantly lower computation time than the full valuation method (Christoffersen, 2003) as it mainly consists of computing linear operations. To improve the accuracy of the Delta method the Delta-Gamma method has been introduced, which takes a Taylor expansion of δC up to squared terms (Christoffersen, 2003):

$$\delta C \approx \Delta \delta S + \frac{\gamma}{2}(\delta S)^2, \text{ where } \gamma = \frac{\partial^2 C}{\partial S^2}.$$

In addition to the Delta and Delta-Gamma method, other less well-known option risk methods exist which apply approximation methods. For example, one method is to apply the Cornish-Fisher approximation (Christoffersen, 2003), where we assume the underlying return distribution is Gaussian with mean 0 and constant variance. Using a quadratic approximation we can obtain the first 3 moments of the distribution of δC, we can then approximately calculate VaR using a Cornish-Fisher approach. Other researchers have also applied moment matching and approximations to measure option risk by VaR e.g. (El-Jahel et al., 1999). The Delta-Gamma method has been developed in terms of a Cornish-Fisher expansion in (Jaschke, 2002); in (Glasserman et al., 2001) Delta-Gamma is used to provide more efficient Monte Carlo simulated estimates of VaR; in (Siven et al., 2009) Delta-Gamma is used along with Fourier inversions to calculate VaR.

In (Sorwar and Dowd, 2010) a simulation-lattice computational method is proposed. This enables one to estimate risk for various option positions, for a range of options (including exotic options and early exercise feature) as well as important underlying distribution features, such as heavy tails. However, such a computational method is computationally intensive and so does not offer fast computation, which is the focus of our paper.

In (Hao and Yang, 2011) option risk is measured but under the assumption of a regime switching stock price process. Also, the risk measurement is restricted to scenario based risk measures, hence its applications (and accuracy) are limited. In (Broda, 2012) computable expressions for risk are given, however this is restricted to the expected shortfall risk measure and that portfolios follow an elliptic multivariate t-distribution.

For all the option risk methods mentioned, the computational speed is improved at the expense of model risk typically increasing. Model risk is defined as the risk of working with a potentially incorrect model, which leads to unexpected losses. The types of model risks incurred (to achieve improved computational speed) can be increased calculation error, increased calibration errors or violation of fundamental principles of Finance e.g. arbitrage (to be addressed in later sections).

Model risk is a key problem in Finance; model errors can result in significant losses (e.g. Long Term Capital Management), they are playing an increasingly important role in industry and institutions are becoming ever more reliant on them for a variety of purposes. In option risk models, model risk is a particularly important issue because such models are used for risk management purposes. Hence it is important that such models have low model risks to prevent the models themselves incorrectly measuring risk or becoming a source of risk themselves.

To give an example of increased model risk, the Delta-Gamma method is theoretically more accurate than the Delta method however the Delta-Gamma method requires calculation of γ. For many option pricing models γ may not be available in analytic form and so can only be calculated by computational methods, which can distort accuracy and increase computation time. In fact it should be noted that computationally evaluating second order partial differential equations (such as γ) in general can be inaccurate. Hence the model risk (and computational efficiency) of the Delta-Gamma method may be worse than the Delta method despite its theoretical advantage. Furthermore, there is no longer a linear relation between δS and δC (unlike in the Delta method), which significantly complicates valuing portfolios with options and portfolio optimisation (see later sections for more details).

The current literature on model risk is limited in finance, although the area is currently growing. Consequently, the literature including model risk and computationally efficient option risk methods is non-existent to the best of our knowledge. In (Kerkhof et al., 2010), model risk is taken into account to determine capital reserves for banks. In particular, estimation risk, identification and misspecification models risks are addressed and combined with standard risk measures such as Value at Risk. In (Kondo and Saito, 2012), a Bayesian method is proposed for measuring model risk for the insurance loss ratio. This method makes specific distribution assumptions and is focussed around Value at Risk calculations, rather than application to any specific risk measure.

In (Alexander and Sarabia, 2012) develop a method for calculating model risk with respect to quantile risk measurement only. This allows institutions to adjust capital reserves to meet potential losses arising from model risk. In Schmeiser et al. (2012) analyses model risk with respect to solvency measures in the insurance sector. In all the aforementioned articles, there is no explicit address of model risk with respect to option risk measurement. In (Guillaume and Schoutens, 2012) model risk is investigated specifically with respect to calibration risk for vanilla and exotic options. However no reference is made with respect to computationally efficient option risk methods.

3 Option Risk Measurement by Replicating Portfolio

As can be seen from the previous section, option risk methods are typically based on some approximation method and can incur significant model risk, which is an important issue as such models are used for risk management purposes. In this section, we show that we can measure option risk with a computationally efficient method by taking a different approach: using its replicating portfolio. This also provides significant model risk advantages.

In this section we first explain how our replicating portfolio method provides computational advantages in measuring option risk; we also show this method has computational advantages for a portfolio of options and portfolio optimisation. We also discuss key model risk advantages of our method, specifically put-call parity consistency, lower implementation risk, calibration risk and can take into account model misspecification and liquidity risk. It should also be noted that the replicating portfolio method can be applied to any contingent claim with a replicating portfolio and not just options.

We note that the usage of a replicating portfolio implies that one has a complete market. Although the Black-Scholes model and the replicating portfolio method do not take into account incompleteness, both models are sufficiently accurate approximations to incomplete market models for the purpose of faster computation. In fact in (Fouque et al., 2000) it is noted that the Black-Scholes equation is an accurate approximation to option prices for at the money options, when incompleteness arises from stochastic volatility. Moreover, the Delta and Delta-Gamma methods both assume a Black-Scholes and complete market model for their approximations. Hence the replicating portfolio method is not any more deficient in complete market assumptions than competing methods.

3.1 Option Risk Calculation Method

The key insight of Black and Scholes (Black and Scholes, 1973) is that we can represent a European option by a replicating portfolio V(t), based on a no arbitrage argument. A replicating portfolio V(t) consists of $\phi_1(t)$ number of shares in the underlying of the option and $\phi_2(t)$ number of units in a riskless bond:

$$V(t) = \phi_1(t)S(t) - \phi_2(t)B(t), \tag{2}$$

where B(t) is the price of a riskless bond at time t (see Appendix for full equation). The negative sign for bonds means we short $\phi_2(t)B(t)$ bonds rather than purchase them. In the case of the Black-Scholes equation we have $\phi_1(t) = \Delta(t)$.

We achieve computationally efficient risk measurement of options by using its replicating portfolio for risk measurement and applying the coherency axioms. This allows the elimination of nonlinear operations in the computational calculation of option risk and so significantly reduces computation time. We now state this in our theorem.

Theorem 1. *For a coherent risk measure $\rho(.)$ the risk of an option or any contingent claim replicated by a replicating portfolio $(\Delta(t)S(t), \phi_2(t)B(t))$ is given by*

$$\rho(dC(t)) = \Delta(t)\rho(dS(t)) + (\Delta(t)S(t) - C(t))rdt. \tag{3}$$

Hence it can be seen from equation (3) *that, excluding* $\rho(.)$*, the number of operations that are a nonlinear function of dS(t) is zero.*

Proof:

$$
\begin{aligned}
dC(t) &= \Delta(t)dS(t) - \phi_2(t)dB(t) \text{ by self-financing property,} && (4)\\
\rho(dC) &= \rho(\Delta(t)dS(t) - \phi_2(t)dB(t)), && (5)\\
&= \rho(\Delta(t)dS(t)) + \phi_2(t)dB(t) \text{ by translation invariance axiom,} && (6)\\
&= \Delta(t)\rho(dS(t)) + \phi_2(t)dB(t) \text{ by homogeneity axiom,} && (7)
\end{aligned}
$$

since dB(t)=rB(t)dt then we have

$$\rho(dC(t)) = \Delta(t)\rho(dS(t)) + \phi_2(t)B(t)rdt,$$

substituting

$$\phi_2(t)B(t) = \Delta(t)S(t) - C(t),$$

we have

$$\rho(dC(t)) = \Delta(t)\rho(dS) + (\Delta S(t) - C(t))rdt.\blacksquare$$

To be able to understand our method it is important to understand the variables that are functions of $S(\delta t)$, since such (non-linear) functions significantly increase computation time as they must re-calculated for every simulated value of $S(\delta t)$. This is achieved by understanding the principles relating to a replicating portfolio, namely no arbitrage and self-financing.

The replicating portfolio V(t) is an adapted process to C(t); it has identical values to C(t) for all t, assuming the market is arbitrage free (see the Appendix for a definition). Therefore

$$C(t) = V(t), \forall t \leq T.$$

This also implies the risk of V(t) and C(t) must be identical because their loss functions must be identical. In other words, we have:

$$\rho(C(t)) = \rho(V(t)), \forall t \leq T.$$

A replicating portfolio also has the important property that it must be self-financing. This is not normally important in option theory, however for the purposes of option risk measurement it is important. By self-financing we have:

$$
\begin{aligned}
dV(t) &= \phi_1(t)dS(t) - \phi_2(t)dB(t), && (8)\\
&= \phi_1(t)dS(t) - \phi_2(t)rB(t)dt. && (9)
\end{aligned}
$$

In terms of option risk, the key part of equation (8) is that neither $\phi_1(t)$ nor $\phi_2(t)$ change when we calculate dV (or dC), for they are constant. Therefore to determine the option loss distribution associated with dC we do not need to calculate $\phi_1(t)$ and $\phi_2(t)$ for each simulated value of $S(\delta t)$. This is because both $\phi_1(t)$ and $\phi_2(t)$ are functions of S(0) but not $S(\delta t)$. If V(t) were not a self-financing portfolio then we would have under standard differentiation (Kwok, 1998)

$$dV(t) \;\; = \;\; \phi_1(t)dS(t) - \phi_2(t)dB(t) + d\phi_1(t)S(t) - d\phi_2(t)B(t).$$

This equation would significantly complicate computational calculation of option price changes because we would need to simulate changes in $d\phi_1$ and $d\phi_2$, in addition to dS. In such a case it may be better to use the full valuation method instead.

The reasons that both $\phi_1(t)$ and $\phi_2(t)$ are functions of S(0) but not $S(\delta t)$ in equation (8) are financial and mathematical. Mathematically the theory is related to forward differences in stochastic differentials (the reader is referred to (Björk, 2004) for a thorough discussion). Essentially, if we were to discretise equation (9) we would have (Jarrow and Turnbull, 1996)

$$\delta V \;\; \approx \;\; \phi_1(t)(S(t+\delta t) - S(t)) - \phi_2(t)rB(t)\delta t.$$

At time t we have only observed S(t) and not $S(t+\delta t)$; $\phi_1(t)$ remains unchanged *during* the time period t to $t+\delta t$. After time $t+\delta t$ (so δt has elapsed), $S(t+\delta t)$ has been observed *and then* we adjust the number of shares and bonds to give the new values $\phi_1(S(t+\delta t), t+\delta t)$ and $\phi_2(S(t+\delta t), t+\delta t)$. From a financial point of view, we cannot have $\phi_1(t)$ (or $\phi_2(t)$) changing until we observe S(t) because the number of stocks and bonds we trade depend on the stock price we actually observe now.

In conclusion we can say that $\Delta(t), C(t)$ and S(t) are not functions of dS (they are constants in dC) due to the self-financing property. Hence in calculating $\rho(dC(t))$ we do not need to re-calculate them and perform nonlinear operations for each simulated value of dS. In fact other than calculating the option Δ there are no nonlinear operations and Δ is only calculated once during the entire simulation, hence does not represent a significant computational operation. The replicating portfolio method is therefore a computational efficient method of calculating option risk.

It is also worth pointing out that Δ is generally calculated for contingent claims even if no risk measurement is conducted, hence it generally imposes no additional computational or analytical 'cost'. We also note that $\Delta(t)$ and $\phi_2(t)$ represent the number of units of stocks and bonds respectively (and are constants) so we can apply the homogeneity axiom and take them outside $\rho(.)$ (see equation (7)). They are also normally given in equation and so do not increase computation (alternative expressions for them are given in the Appendix).

The replicating portfolio method is also able to achieve computational efficiency without sacrificing accuracy. The replicating portfolio option risk measure is based on equation (9); this equation is an identity for dC, therefore it is identical to dC for all states of the world and is not an approximation. We can therefore always increase accuracy by increasing the number of simulations and reducing δt to produce results of δC equivalent to that of the full valuation method. On the other hand, to increase the accuracy of the full valuation method involves increasing the number of simulations and so the number of nonlinear operations, which is computationally expensive.

It is worth noting in passing that in the past 10-20 years trading has become increasingly dominated by automated or computerised trading (rather than fundamental based trading) in many markets. Consequently, many trades are opened and closed on scales of the order of milliseconds (see for example algorithmic trading). Hence even marginal improvements in computing times can make the difference between profit or loss trades.

The Delta method (and other computational methods) are *fundamentally* limited in accuracy because they are approximations. For instance, the Delta and Delta-Gamma methods are taken from an approximation of the Taylor series expansion of δC; in order to achieve full accuracy we require the Taylor series to an infinite series expansion with increasingly more nonlinear terms (which cause increasing computational cost). The Delta and Delta-Gamma methods will therefore never reach a fully accurate calculation of δC as that of the full valuation method, regardless of the number of simulations executed. Such inaccuracies can be particularly important in high volume trading (e.g. high frequency trading), where minor inaccuracies can be magnified and cause unforeseen trading losses.

3.2 Portfolios with Options: Option Risk Calculation and Optimisation

In (Christoffersen, 2003) the Delta and Delta-Gamma methods are discussed in terms of their computational efficiency for calculations involving portfolios contain n units of a stock and options on the same stock (underlying) with the same or different K and T. We now do the same for the replicating portfolio method and show that it retains computational efficiency when applied to such portfolios, furthermore, the replicating portfolio also has computational benefits in portfolio optimisation.

For a portfolio D(t) containing n units of a stock and an option on the same stock (underlying)

$$D(t) = nS(t) + C(S(t)),$$

using the replicating portfolio method to model the change in the portfolio's value $\delta D(t)$ we have

$$\delta D(t) \quad \approx \quad (n + \Delta)\delta S(t) + \phi_2 \delta B.$$

For a full detail of the proof, please see the Appendix. Hence the change in the portfolio's value involves linear operations and so is not computationally expensive. Furthermore, if D(t) is extended to include a set of n options with different K and T (but on the same underlying stock), that is

$$D(t) \quad = \quad \sum_{i=1}^{n} C_i(S(t), K_i, T_i),$$

then by the replicating portfolio method we have

$$\delta D \quad \approx \quad \sum_{i=1}^{n} \phi_{1n} \delta S + \phi_{2n} \delta B.$$

Now if we assume we have a more complex portfolio:

$$D(t) = \sum_{i=1}^{n} v_i S_i + \sum_{j=1}^{m} v_j C_i(S_i(t), K_i, T_i),$$

where v_i and v_j represent the number of units stocks and options respectively and m equals the total number of different stocks. Using the same modelling assumption employed by the Delta method for modelling such portfolios we assume all stocks and options are uncorrelated, therefore using the replicating portfolio approach we have

$$\delta D(t) \approx \sum_{i=1}^{n} v_i \delta S_i + \sum_{j=1}^{m} v_j \Delta_j \delta S_j(t) + \phi_{2j} \delta B.$$

In all cases, the replicating portfolio method still retains computational efficiency as we do not need to re-calculate any non-linear terms with each simulation, hence it is not computationally costly.

In addition to computing portfolio value changes, the replicating portfolio method offers computational advantages in portfolio optimisation. Portfolios are optimised computationally (rather than analytically) by adjusting stock and option portfolio weights. We would therefore like to optimise the problem for a portfolio L(t) containing N stocks and M options

$$\max_{w_i, w_j \forall i,j} f(dL(t)) = \sum_{i=1}^{N} w_i dS_i(t) + \sum_{j=1}^{M} w_j dC_j(t),$$

where w_i, w_j are the stock and option weights respectively. The inclusion of options in L(t) means the optimisation of f(dL(t)) is nonconvex, which is non-trivial. Firstly there exist fewer algorithms for nonconvex optimisation, so there may not exist an optimisation solution. Secondly, an optimal solution that is found may be locally optimal but not necessarily globally optimal.

If one were able to replace options with a linear expression then one would have a linear optimisation, which is highly desirable as they enable powerful and well-developed algorithms to be applied (such as linear programming and stochastic programming) to large portfolios. Linear optimisation of L(t) is possible by using the replicating portfolio approach to options. Therefore we would have:

$$\begin{aligned}\max_{w_i, w_j \forall i,j} f(dL(t)) &= \sum_{i=1}^{N} w_i dS_i(t) + \sum_{j=1}^{M} w_j(\Delta_j(t)(dS_j(t)) + (\Delta_j(t)S_j(t) - C_j(t))rdt), \\ &= \sum_{i=1}^{N} w_i dS_i(t) + \sum_{j=1}^{M} w_j \Delta_j(t) dS_j(t).\end{aligned}$$

The last line is possible because $((\Delta_j(t)S_j(t) - C_j(t))rdt)$ is a constant and so does not affect the optimisation (other than in the possible case there are linear constraints imposed in the optimisation).

3.3 Calibration and Implementation Risk

A key model risk that is frequently incurred is calibration risk, that is unexpected losses arising from incorrect model calibration. In fact in industry, local volatility models are preferred to stochastic volatility models due to the lower calibration risk. Another important area of model risk is implementation risk, that is unexpected losses arising from implementing the model for use e.g. approximation errors, model assumptions etc.. We will now explain how our replicating portfolio method has significant model risk advantages compared to other computationally efficient option risk methods.

Firstly, other than Δ the remaining parameters in equation (3) (that is S(t),C(t) and r) are *observable variables and so do not require calculation or calibration to the market.* Furthermore the calculation of Δ is a function of observable variables (except volatility). Therefore the number of parameters that could cause model risk are significantly limited; our method requires no more observability than the observability required for the Black-Scholes model itself, which is considered a highly observable model. Additionally, the limited calibration required increases the stability of calibration because fewer parameters mean that calibration error is less likely to change with time.

Secondly, the estimation error will be lower for extreme values using the replicating portfolio method compared to other methods. It is important to be able to measure the risk of extreme losses, in fact the most important aspect of risk management is concerned with extreme loss management. However, managing risk under extreme values poses a number of significant problems: firstly, many option risk measures cannot value at extreme values because they are only valid over small changes e.g. the Delta method. Secondly, there may not exist sufficient observations to confidently estimate extreme values and thirdly, fitting the correct distribution to stock data can be non-trivial (Dowd, 2011). Consequently, determining extreme option or underlying values by Monte Carlo simulation may not be feasible. Finally, the inability to accurately fit distributions (or accurately estimate) for extreme values means that we cannot provide an analytical solution to extreme value risks e.g. VaR.

In risk management theory, one applies EVT (extreme value theory) to determine the risk of extreme values on stocks. Our replicating portfolio method can also be applied to EVT to obtain risk measures. From (3) it can be seen that we can obtain extreme risk measurement values of $\rho(dC)$ from extreme risk measurement of $\rho(dS)$ and other observable or tractable parameter estimates. In other models this is not necessarily possible. For instance in the Delta-Gamma model to obtain extreme value measurements in dC we would require simulations of dS and dS^2, hence any estimation errors in simulation would be squared. Such errors would be magnified further when calculating extreme values using Extreme Value Theory. Additionally, we must multiply the dS^2 term by γ, which is a partial derivative and so is difficult to accurately compute or estimate, leading to higher potential extreme value errors.

Thirdly, our replicating portfolio method reduces implementation risk by its tractable computational implementation. Other than the observable variables, all that is required is the Δ calculation, which is computationally tractable. It can be numerically evaluated (e.g. for non-trivial S(t) processes or contingent claims) with sufficient level of accuracy e.g. binomial tree or finite difference methods. For example, for an American option we can easily calculate the Δ using a binomial tree method. The Δ calculation would not form part of the Monte Carlo simulation; the Δ is always a one-off calculation and so the replicating portfolio method still remains computationally efficient. For a basket option the replicating portfolio method only requires calculation of ϕ_1 for each asset in the portfolio (and ϕ_2 can be deduced using equations (26) and (25)).

To implement the replicating portfolio method we require Δ and this is generally analytically possible to determine for a range of option pricing models and contingent claims. Other risk measuring methods are not as easy to implement. For instance, the Delta-Gamma method requires calculating γ and this is a second order partial derivative. Such a derivative may not be easily available and may be analytically intractable to derive, especially for complex contingent claims or non-trivial S(t) processes. Furthermore, it is well known that numerical computation can be intractable for second order partial derivatives, leading to inaccurate calculations. In the case of a basket option the Gamma method requires a second order partial differential equation for each asset in the portfolio (by applying multivariate Taylor's Theorem), which can become intractable for large portfolios.

The implementation risk is reduced further as the replicating portfolio is also analytically more tractable compared to other option risks, in particular we can analytically derive $\rho(dC)$ from $\rho(dS)$ and using equation (3). For instance, we can easily calculate VaR using equation (3):

$$VaR(dC(t)) = \Delta(t)VaR(dS(t)) + (\Delta(t)S(t) - C(t))rdt,$$

(we note that VaR only fails as a coherent risk measure in terms of subadditivity, hence the previous equation is applicable to VaR). Furthermore, if we assume dS follows geometric Brownian motion then VaR will be the VaR for a Gaussian distribution (for which many equations exist), multiplied by $\Delta(t)$, with its centre shifted by the drift term and the expression $(\Delta(t)S(t) - C(t))rdt$.

The ability to analytically calculate VaR(dC(t)) and $\rho(dC)$ for various coherent risk measures reduces model risk because when can analytically verify the model's risk or error, whereas other option risk methods do not enable this. Furthermore, analytical tractability of the replicating portfolio method enables one to analytically determine $\rho(dC)$ if we need to approximate dS. For instance, we may wish to determine VaR(dC(t)) for non-trivial dS processes and so VaR(dS) may be analytically intractable. However, if we apply an approximation of dS we could derive an analytical solution for VaR(dS).

If one were to apply another method we would not necessarily be able to derive analytical solutions for any risk measure. For instance, if we wished to determine VaR using the Delta-Gamma method then

$$VaR(dC) = VaR\left(\Delta\delta S + \frac{\gamma}{2}(\delta S)^2\right).$$

This would not be a tractable method of measuring risk by VaR or any other risk measure. Firstly, the measurement of VaR(dC) is a function of γ, which can difficult to accurately determine for some contingent claims. Secondly, the VaR measurement is now on a non-trivial distribution: the distribution obtained from adding the distributions of δS and $(\delta S)^2$. There may not exist any analytical solution for the overall distribution, let alone the VaR equation (or any risk measure) for such a distribution. Furthermore, computational implementation to obtain VaR or any other risk measure would be computationally expensive.

Fourthly, the replicating portfolio method reduces implementation risk and general model risk by having a parsimonious model of few modelling and unrestrictive assumptions. The replicating portfolio method is based on an identity for δC using the self-financing property and arbitrage free assumption (both of these are not restrictive assumptions). Our method is not restricted to any risk measure, particular assets or distributions. Other risk methods make restrictive assumptions about stock price distributions, numerous variables (e.g. state of the economy) and apply to particular risk measures only (e.g. VaR).

3.4 Arbitrage Free Option Modelling: Put-Call Parity Consistency

The put-call parity is an important theorem between calls C(S(t),t,T,r,K) and puts P(S(t),t,T,r,K); it is model independent, holds under a range of conditions and its disobedience is considered a serious mispricing. An explanation of the put-call parity is given in the Appendix 9.

The replicating portfolio method must obey the put-call parity by construction and so does not admit arbitrage opportunities arising from this (for completeness we give the proof in the Appendix). However, other option risk methods violate the put-call parity, specifically the Delta and the Delta-Gamma methods (two of the most popular option risk models), and so allow arbitrage opportunities. We will now prove this.

Lemma 1. *For any given underlying and any option pricing model, the Delta and Delta-Gamma methods do not obey the put-call parity. Therefore the Delta and Delta-Gamma methods admit arbitrage opportunities in porfolios containing at least any two of the following: put option, call option or shares in the underlying.*

Proof:
By Delta-Gamma method we have

$$\delta P \approx \Delta_p \delta S + \frac{\gamma^2}{2}(\delta S)^2, \tag{10}$$

where Δ_p is the option delta for a put.
By the put-call parity we also have

$$\begin{aligned} P &= Ke^{-r(T-t)} - S(t) + C, && (11) \\ dP &= dC + d(Ke^{-r(T-t)}) - dS, && (12) \\ \delta P &\approx \delta C + \delta(Ke^{-r(T-t)}) - \delta S. && (13) \end{aligned}$$

By the Delta-Gamma method we can express δC as

$$\delta C \approx \Delta \delta S + \frac{\gamma^2}{2}(\delta S)^2. \tag{14}$$

Also for any option pricing model it is known that

$$\Delta_p = \Delta - 1. \tag{15}$$

Now if we substitute δC from equation (14) into equation (13) then we have

$$\begin{aligned} \delta P &\approx (\Delta \delta S + \frac{\gamma^2}{2}(\delta S)^2) + \delta(Ke^{-r(T-t)}) - \delta S, && (16) \\ &\approx \delta S(\Delta - 1) + \frac{\gamma^2}{2}(\delta S)^2 + \delta(Ke^{-r(T-t)}), && (17) \\ &\approx \delta S \Delta_p + \frac{\gamma^2}{2}(\delta S)^2 + \delta(Ke^{-r(T-t)}) \text{ by equation (15).} && (18) \end{aligned}$$

Hence equation (10) and equation (18) are not equal and so does not obey put-call parity and so allows arbitrage. ■

Remark 1. *The Delta method is a special case of the Delta-Gamma method and it can be easily seen by the same proof that the Delta method allows arbitrage opportunities.*

An explanation of the proof is as follows: if the Delta-Gamma method obeyed the put-call parity then substitution of an equation or expression from the put-call parity equation should give the same equation for δP, that is equations (10) and equation (18) should be equal. However these 2 equations are not equal and so this implies the put-call parity is not obeyed. An example of the Delta-Gamma method giving arbitrage opportunities in the put-call parity is given in the Appendix.

The inability for some option risk methods to obey the put-call parity has significant consequences upon the applicability and risk management. Firstly, such methods cannot be used for option trading strategies and static replication of exotic derivatives. Option trading strategies (e.g. a butterfly, a strip and a strangle to name a few) involve purchasing a range of put and call options on the same underlying (Hull, 2000). This portfolio of options is bought in such a way as to construct a net position that will benefit from a particular movement in the underlying. Static replication involves using a portfolio of plain vanilla European puts and calls to hedge an exotic derivative (Derman et al., 1995). Both option trading strategies and exotic derivatives hedging are becoming increasingly popular in industry and so important to risk manage.

The existence of modelling methods allowing arbitrage opportunities to occur also encourages 'internal' arbitrage opportunities. This is when 1 department within an institution takes advantage of the mispricing of derivatives and securities by another department (within the same institution) to enable riskless profit taking. Internal arbitrage is a significant problem and in some institutions they do not use particular models to prevent this occurring (Alexander, 2001). The replicating portfolio method eliminates the possibility of internal arbitrage opportunities as it will always guarantee obeying the put-call parity.

Finally, option risk methods that do not obey the put-call parity can give different risk measurements on the same portfolio. For instance, in equations (29) and (28) we have 2 different values for exactly the same option, which would give 2 different risk measurements for the same option. This can lead to inconsistent risk management of the same portfolio.

3.5 Model Misspecification

Model misspecification is becoming an increasing important factor in model risk, for instance modelling volatility as a constant instead of as a stochastic or time varying variable. A popular method for addressing misspecification is the banded parameter model (Wilmott et al., 1998); in this model we subsume the misspecification into an appropriate variable and allow this variable's value to vary between a maximum and minimum limit. For example, if we choose the variable volatility then its value will be allowed to vary between the limits $\sigma^- < \sigma < \sigma^+$; alternatively we could have chosen r so that r would be bounded by so that $r^- < r < r^+$. Using the banded model we can determine worst and best case scenarios for option risk, which are extremely useful as they are frequently used in risk management.

We would like to be able to use the banded parameter model in option risk modelling to take into account model misspecification risk. However, this may not possible as the banded parameter method is only applicable under the assumption that there are no arbitrage opportunities with the model. Therefore the Delta and Delta-Gamma methods would not be applicable as they allow arbitrage opportunities through the put-call parity.

Using the replicating portfolio method to option risk, it is possible to apply the banded parameter model because the replicating portfolio method is based on assuming no arbitrage conditions. For the purposes of option risk measurement we will restrict our attention to varying volatility in bands (first proposed by (Avellaneda et al., 1995)) because it is a common source of model misspecification. In the banded parameter model, the worst and best case scenarios are not simply obtained by using the lowest and highest volatility values but applying arbitrage principles.

The Black-Scholes equation is derived on the assumption that it constructs a riskless hedge; for a call option we have

$$dC - \Delta dS = -\phi_2 dB.$$

Now to avoid arbitrage opportunities we assume the return on the worst case replicating portfolio earns the riskless rate, that is

$$\begin{aligned} \min_{\sigma^- < \sigma < \sigma^+} (dC - \Delta dS) &= -\phi_2 dB, \\ \min_{\sigma^- < \sigma < \sigma^+} d\Pi &= -\phi_2 dB, \end{aligned}$$

where $d\Pi = dC - \Delta dS$. Our objective is

$$\min_{\sigma^- < \sigma < \sigma^+} \left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2 \gamma}{2} \right).$$

It can be shown that we minimise $d\Pi$ if $\sigma = \sigma^+$ for $\gamma > 0$ and $\sigma = \sigma^-$ for $\gamma < 0$. Therefore to find the best case option risk measurement we use $\sigma = \sigma^+$ if $\gamma < 0$ and $\sigma = \sigma^-$ if $\gamma > 0$; for the worst case option risk measurement we would use $\sigma = \sigma^+$ for $\gamma > 0$ and $\sigma = \sigma^-$ for $\gamma < 0$.

3.6 Liquidity Risk

An increasingly important component of model risk is liquidity risk, which is the potential cost of transactions ($\mathcal{T}$). Transaction costs can form a significant part of risk because they can substantially increase the losses incurred in trading, they are not known with certainty, they can vary with trading volume, the state of the economy and the market size to name a few factors.

Although there exist many liquidity models for stocks, currently there do not exist many liquidity models for options. One model by Krakovsky (Krakovsky, 1999) prices liquidity costs into options by modifying the partial differential equation governing the option pricing equation. However the resulting partial differential equation has no analytic solution, so it must be solved computationally, which is computationally expensive. Krakovsky's model (Krakovsky, 1999) also ignores bid-ask spreads changing with time, which is an important factor in liquidity risk.

There is currently no apparent method of adapting option risk models to take into account liquidity risk. For example, the Delta method does not measure transaction costs, nor provide any method for measuring liquidity risk over short time intervals δt or δC. Using the replicating portfolio method of option risk modelling it is possible to take into account liquidity risk because the replicating portfolio principle is frequently utilised in other financial models (such as liquidity models), hence our option risk method can be adapted.

One popular and well known liquidity model is Leland's transaction cost model (Leland, 1985). This can be applied to our replicating portfolio model of option risk because Leland's model is based on the replicating portfolio principle, unlike other option risk models. In Leland's model the transaction costs $\mathcal{T}$ are proportional to the total value of the underlying transacted:

$$\mathcal{T} = S(t)n(t)k,$$

where n(t) is the number of units (e.g. shares for equities) bought or sold at time t and k/2 is the transaction cost for one share (sold or bought).

To model δC with transaction costs we apply Leland's model:

$$\delta C \approx \Delta \delta S(t) - rB(t)\delta t - \frac{k}{2}|\delta\Delta|S(t),$$

where the last term represents the transaction cost. It has been shown by Leland that

$$\frac{k}{2}|\delta\Delta|S \approx \frac{\sigma^2}{2}\overline{L}S^2\gamma\delta t,$$

where the Leland number $\overline{L}$ is

$$\overline{L} = \sqrt{\left(\frac{2}{\pi}\right)}\left(\frac{k}{\sigma\sqrt{\delta t}}\right).$$

Hence our option risk model with liquidity risk is

$$\rho(\delta C) \approx \Delta\rho(\delta S) + rB(t)\delta t + \frac{\sigma k}{\sqrt{2\pi}}S^2\gamma\sqrt{\delta t}. \tag{19}$$

We note from equation (19) that in order to measure option risk with liquidity risk there is no significant increase in the level of computation. This is because the last term in equation (19) is not a function of $S(t+\delta t)$ but $S(t)$; hence it does not require recalculation for each simulated $S(t+\delta t)$. We also notice from equation (19) that the option risk measurement with liquidity risk does not require significant parameter estimation. In fact, most of the parameters contributing to the transaction costs can be observed or calculated from observable variables. Furthermore, Leland's model (Leland, 1985) and the replicating portfolio method are both derived without admitting arbitrage opportunities, which is important to model risk and preventing internal arbitrage opportunities.

4 Numerical Experiments

In this section we conduct numerical experiments to demonstrate and validate the replicating portfolio method of measuring option risk. To gauge the performance of the replicating portfolio method we also conducted numerical experiments on the Delta method to act as a fair benchmark. In this section first we explain our method, present the results of our experiments and then discuss them.

4.1 Method

In this section we conducted two numerical experiments. Firstly, we conducted a numerical experiment to measure the computation time of the replicating portfolio method against the Delta and full valuation methods. Secondly we evaluated the accuracy of the Delta and replicating portfolio methods in determining changes in option prices. All the numerical experiments were executed on a 1.61 GHz computer, with 992MB RAM, running Matlab version 6.5.

For the computation time experiment we measured the time taken to compute the distribution of the change in call option price δC under a Black-Scholes model. The time measured was for a δC distribution consisting of one million samples. To obtain one million sample points we required one million random samples of $\delta S = S(\delta t) - S(0)$. The δS random samples were obtained by generating the distribution of $S(\delta t)$ under the Black-Scholes model (geometric Brownian motion).

Using the samples of δS we calculated δC: for the full valuation method we applied the method outlined in the Appendix, for the Delta method we used equation (1) and for the replicating portfolio method we used equation (3). We note that the choice of Black-Scholes parameters K,T, etc. do not affect any of the computation times. The Black-Scholes option pricing equation along with other Black-Scholes parameters (e.g. option delta) did not require implementation as they are already available in the

Matlab financial toolbox. The entire experiment was repeated ten times to obtain an average value of computation times. The results are presented in the next section.

In the second experiment we compared the accuracy of the Delta method against the replicating portfolio method. This was done by calculating δC over one day (although any time period could have been chosen) and comparing the methods' accuracies over a range of K and σ. We chose our range of K for $|K/S(0)-1| \leq 10\%$ to test well beyond the range of actively traded options; the range of K for actively traded options tend to be within a range of $|K/S(0) - 1| \leq 3\%$ (Fouque et al., 2000) and beyond this range option prices tend to suffer from significant liquidity effects (Fouque et al., 2000). We also tested a range of volatility values σ from 5% to 20%. The typical volatility for an index is $\sigma = 10\%$, with $\sigma = 20\%$ considered to be high volatility (possibly occurring during a financial crisis).

The range of dS/S was chosen to be $\pm 1\%$, $\pm 2\%$ and $\pm 5\%$ to reflect possible price changes in the underlying under different scenarios. Since a 10% return is the average return over one year for an index (Hull, 2000), a range of dS/S of $\pm 5\%$ in one day reflects a scenario of a large price change. A $\pm 1\%$ price change would be considered a normal price change and so reflects a typical price change scenario. A $\pm 2\%$ change would be considered a significant change, although a possible scenario.

The following option input parameters were chosen to reflect the typical values an option may take (although any values could have been chosen): r=5%; T=100 days; S(0)=1000. We compared the accuracy of each method using the percentage relative error, taking the full valuation method as our correct answer. For example, for the Delta method the percentage relative error was calculated as

$$\frac{|v_{FVM} - v_D|}{|v_{FVM}|} \times 100,$$

where v_{FVM} is the δC calculated by the full valuation method and v_D was δC calculated by the Delta method. A similar equation was applied to the replicating portfolio method. To calculate the average relative percentage error we took the average of these results over 1000 samples for each K and σ.

4.2 Results

Table 1: Computation Time for Calculating δC (Seconds)

Experiment	Delta Method	Replicating Portfolio	Full Valuation Method
1	166.375	166.812	504.312
2	167.609	166.578	508.312
3	165.485	168	510.297
4	167.734	168.078	510.25
5	167.453	167.453	507.469
6	167.282	167.437	533.313
7	167.469	167.906	512.562
8	169.437	168.546	532.89
9	169.656	166.859	510.015
10	167.657	166.703	511.984
Average	167.4157	167.4372	514.1404

Table 2: Average Relative Percentage Error for ± 1% Stock Range

Strike	$\sigma = 5\%$		$\sigma = 10\%$		$\sigma = 15\%$		$\sigma = 20\%$	
K	R	D	R	D	R	D	R	D
900	0	15.6	1.25	15.46	6.2	20.43	16.29	33.18
950	0.48	15.26	5.96	21.61	13.34	27.99	24.01	39.43
1000	15.84	53.68	14.55	29.40	24.21	39.76	38.58	55.57
1050	25.89	40.62	27.2	42.21	38.3	54.78	46.78	62.15
1100	45.79	58.99	47.61	64.87	68.79	90.29	136.94	171.62

Table 3: Average Relative Percentage Error for ± 2% Stock Range

Strike	$\sigma = 5\%$		$\sigma = 10\%$		$\sigma = 15\%$		$\sigma = 20\%$	
K	R	D	R	D	R	D	R	D
900	0.00	7.27	0.91	8.26	3.32	9.97	6.35	12.84
950	0.61	7.84	3.68	9.82	7.82	15.08	10.74	17.38
1000	8.3	13.35	9.47	15.53	14.89	22.94	24.19	34.42
1050	27.57	32.26	17.40	23.37	24.41	33.28	26.64	34.96
1100	52.05	56.61	37.78	47.86	24.98	31.39	111.86	139.58

Table 4: Average Relative Percentage Error for ± 5% Stock Range

Strike	$\sigma = 5\%$		$\sigma = 10\%$		$\sigma = 15\%$		$\sigma = 20\%$	
K	R	D	R	D	R	D	R	D
900	0.02	3.09	1.39	4.86	2.92	4.82	4.21	6.22
950	1.86	5.10	5.26	7.00	6.28	8.14	6.96	8.94
1000	19.74	21.05	13.46	15.32	12.31	14.98	20.56	29.90
1050	63.38	64.71	24.51	26.19	16.58	18.44	14.16	16.17
1100	117.07	118.13	38.89	41.10	24.55	26.95	19.79	22.20

Note: R denotes the replicating portfolio method and D denotes the Delta method.

4.3 Discussion

The numerical experiments in Table 1 demonstrate that full valuation is computationally far more expensive than the Delta method; it takes approximately three times as long. The experiments also confirm that the replicating portfolio method is significantly less time consuming (computationally) than the full valuation method, in fact its computation time is practically identical to the Delta method.

We expect both the Delta and replicating portfolio methods to have far lower computation times than the full valuation method because they require practically no calculation of nonlinear functions (other than for the one-off calculation of the option's Delta). The full valuation on the other hand must calculate the Black-Scholes equation (highly nonlinear function) for each sampled stock price. The replicating portfolio method therefore provides a significant saving in computation time, at a time comparable to the Delta method.

The savings in computation time become particularly important as we increase the number of options in a portfolio and the frequency with which the portfolio is valued during the day. Hence it can be seen that the full valuation method becomes increasingly impractical compared to the Delta and replicating portfolio methods. Additionally, we have used the Black-Scholes model to value the options, for which there exist many optimised computational implementations. For other option pricing models (e.g. with different underlying processes) the full valuation method will increase computation time.

The numerical experiments in Tables 2-4 demonstrate that the replicating portfolio is more accurate than the Delta method and most importantly, this is achieved with little additional computation time. The numerical experiments demonstrate that the replicating portfolio outperforms the Delta method over all K, σ and price ranges. This is because the additional terms obtained from the replicating portfolio argument (and are not present in the Delta method) enable more accurate modelling. To be more specific, the replicating portfolio takes into account the change in option price more accurately due to taking into account the impact of the option variables time and riskless rate affecting option prices. This is not achieved in the Delta method.

We observe that both the Delta and replicating portfolio increase in error as we increase with K. This is because the Black-Scholes call Delta is $\Delta = \Psi(d_1)$, therefore Δ decreases as K increases. As we multiply dS by Δ in both the Delta method and replicating portfolio equation, a reduction in Δ has the effect of changes in stock price being unable to model changes in option price. However, it should be noted in all cases the replicating portfolio method still provides a lower error than the Delta method.

The effect of increasing σ generally has the impact of increasing the error of both the Delta and replicating portfolio method. However the influence of σ is less predictable due to its relation with Δ. Since $\Delta = \Psi(d_1)$ for call options it can be shown that σ has no monotonic increasing or decreasing relation with Δ whilst all other parameters are kept constant. Moreover, whilst we vary K and σ both will affect Δ, which in turn will affect the accuracies of our method.

The numerical experiments show that the replicating portfolio method is more accurate over all K, σ and price ranges than the Delta method. In portfolios containing a range of options at different K (e.g. option trading strategies or static replicating portfolios) therefore the replicating portfolio offers a more accurate modelling method than the Delta method. Additionally, the replicating portfolio method does not admit arbitrage opportunities to occur in the modelling, unlike the Delta method. Most importantly, all these advantages are achieved without increasing computation cost, which is the main purpose behind such methods. Hence our replicating portfolio method is better suited to valuing portfolios of contingent claims than the Delta method.

In conclusion our numerical experiments show that our computational method is significantly faster than the full valuation method and has a computation time comparable to the delta method, underlining the fast computation of our method. Secondly, our computation method is more accurate than the delta method for a range of volatilities, strike and expiries. Hence the negligible increase in computation time using our computation method is worthwhile given the significant gain in accuracy. Additionally, our method does not admit arbitrage opportunities and other model risk errors.

5 Conclusion

In this paper we have proposed a new method of measuring option risk (and other contingent claims) using the replicating portfolio method. We have shown that this method provides a fast computation of options by practically eliminating the requirement for evaluating nonlinear functions. This has resulted in computation times that are practically identical to the Delta method.

We have shown that the replicating portfolio approach provides many significant model risk advantages; one key advantage is that the model does not allow arbitrage opportunities for complex portfolios of options (unlike other methods). Furthermore, our model has lower calibration risk (only requiring observable market data to be implemented (except Δ)), it has parsimonious implementation and fewer model assumptions which reduce implementation risk. Unlike other methods, the replicating portfolio method can be applied to current models to take into account important model risk factors (e.g. liquidity risk and model misspecification). Another key advantage is that our method enables linear optimisation of portfolios containing options.

We conducted numerical experiments on our replicating portfolio method to validate our method. These results have demonstrated that the replicating portfolio method computes changes in option prices in times practically identical to those of the Delta method whilst also giving lower relative error. In conclusion, we believe the replicating portfolio offers significant modelling and computational advantages over alternative modelling methods and this will be of significant interest to Academics and industry professionals.

In terms of future possible areas of research, this includes developing the computational method for exotic options, such as energy options and barrier options; the method could also be extended to other derivatives by application of a replicating portfolio argument. Another area for future research is to develop the method with relaxed Black-Scholes modelling assumptions, such as the explicit inclusion of taxes, dividends and non-constant interest rates. Finally, the risk measurement method could be extended to real options analyses, where a replicating portfolio existed, as risk measurement is important for corporate finance applications.

6 Appendix

6.1 Appendix 1: Black-Scholes Equation

The Black-Scholes option pricing model is given by

$$C(S(t), t, T, r, \sigma, K) = S(t)\Psi(d_1) - Ke^{-r(T-t)}\Psi(d_2), \tag{20}$$

$$\text{where } d_1 = \frac{\ln(S(t)/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}, \tag{21}$$

$$d_2 = \frac{\ln(S(t)/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}},$$

$$= d_1 - \sigma\sqrt{(T-t)}.$$

In C(S(t),t,T,r,σ,K) t is the time at which C is being priced, T is the expiration date, $\Psi(\cdot)$ is the standard normal cumulative distribution function and K is the strike price.

6.2 Appendix 2: Algorithm for Full Valuation Method

Let index $i = \{1, 2, .., n\}$ where n is the number of Monte Carlo simulations.

1. Calculate initial option price C(0): C(S(0),T,r,σ,K).
   Set i=1.

2. Simulate $S(\delta t)$ from S(0).
   Denote simulated value for iteration i as $S_i(\delta t)$.

3. Calculate option price $C_i(\delta t)$ using $S_i(\delta t)$: $C_i(S_i(\delta t), \delta t, T, r, \sigma, K)$.

4. Calculate loss: C(0)-$C_i(\delta t)$.

5. Increment i.
   If i=n+1 then stop, otherwise goto step 2.

6.3 Appendix 3: Bond Price Equation

The price of a bond B(t) is given by (Björk, 2004)

$$B(t) = B(0)exp\left(\int_0^t rdt\right), \tag{22}$$

$$dB = rB(t)dt. \tag{23}$$

6.4 Appendix 4: Arbitrage Definition

An arbitrage possibility in a financial market is a portfolio $V(t)$ such that:

- $V(0) \leq 0$;
- $V(T) \geq 0$ almost surely and
- $E[V(T)] \geq 0$.

In words, arbitrage is an event where it is possible to make a profit without the possibility of incurring a loss. We note that the assumption of no arbitrage in real markets is not a stringent assumption because the existence of arbitrage is typically a symptom of a highly dysfunctional market, which most studied markets do not exhibit.

6.5 Appendix 5: Expressions for Call Option Terms

For a call option it can be shown that (Baxter and Rennie, 1996)

$$\begin{aligned} \phi_1(t) &= \Delta(t), && (24) \\ B(t)\phi_2(t) &= Ke^{-r(T-t)}\Psi(d_2). && (25) \end{aligned}$$

Alternatively we can express $B(t)\phi_2(t)$ as

$$\begin{aligned} B(t)\phi_2(t) &= \Delta S(t) - C(t), && (26) \\ &= \phi_1(t)S(t) - C(t). && (27) \end{aligned}$$

6.6 Appendix 6: Proof for Option Portfolio

$$\delta D(t) \approx (n+\Delta)\delta S(t) + \phi_2 \delta B.$$

We have

$$\begin{aligned} \delta D(t) &\approx n\delta S(t) + \delta C(S(t)), \\ &\approx n\delta S(t) + \phi_1 dS + \phi_2 \delta B, \\ &\approx (n+\phi_1)\delta S(t) + \phi_2 \delta B, \\ &\approx (n+\Delta)\delta S(t) + \phi_2 \delta B. \end{aligned}$$

6.7 Appendix 7: Proof of Put-Call Parity of Replicating Portfolio Method

A put option can be replicated by Δ_p units of shares and a long position in $\phi_{2p}(t)$ units in bonds.

$$\begin{aligned} P &= \Delta_p S(t) + \phi_{2p}(t)B(t), \\ dP &= \Delta_p dS(t) + \phi_{2p}(t)dB(t), \\ \delta P &\approx \Delta_p \delta S(t) + \phi_{2p}(t)rB(t)\delta t(t). \end{aligned} \tag{28}$$

We also have

$$\begin{aligned} \phi_{2p}(t)B(t) &= Ke^{-r(T-t)} - \phi_2(t)B(t), \\ \phi_{2p}(t)dB(t) &= d(Ke^{-r(T-t)}) - \phi_2(t)dB(t). \end{aligned}$$

By put-call parity we have

$$\begin{aligned} \delta P &\approx \delta C + \delta(Ke^{-r(T-t)}) - \delta S, \\ &\approx (\Delta\delta S - \phi_2(t)\delta B(t)) + \delta(Ke^{-r(T-t)}) - \delta S, \\ &\approx \Delta_p\delta S + (\delta(Ke^{-r(T-t)}) - \phi_2(t)\delta B(t)), \\ &\approx \Delta_p\delta S + \phi_{2p}(t)\delta B(t). \end{aligned} \tag{29}$$

Hence equations (29) and (28) are equal, obeying the put-call parity.

6.8 Appendix 8: Coherency Axioms for Risk Measurement

A risk measure $\rho(.)$ is coherent if it is:

1. monotonic: if $Z_1 \leq Z_2$ then $\rho(Z_1) \geq \rho(Z_2)$;
2. homogeneous: $\rho(\kappa Z_1) = \kappa\rho(Z_1)$, where $\kappa \in \mathbb{R}^+$ is a positive constant;
3. translation invariant: $\rho(Z_1 + \nu) = \rho(Z_1) - \nu$, where $\nu \in \mathbb{R}$ is a constant (or a riskless bond portfolio);
4. subadditive: $\rho(Z_1 + Z_2) \leq \rho(Z_1) + \rho(Z_2)$.

6.9 Appendix 9: Put-Call Parity

The put-call parity states that (assuming the market is arbitrage free) a call C(S(t),t,T,r,K) and put P(S(t),t,T,r,K) with the same S(t), K and T obey the relation:

$$P(S(t), t, T, r, K) = Ke^{-r(T-t)} - S(t) + C(S(t), t, T, r, K). \tag{30}$$

6.10 Appendix 10: Example of Internal Arbitrage By Put-Call Parity

Let us assume there are 2 departments, H1 and H2, in 1 company. Department H1 creates a portfolio M consisting of buying put option P, and short selling a call option C. Hence:

$$M = P - C,$$

and

$$\delta M = \delta P - \delta C.$$

Both P and C are identical in terms of option parameters, that is T, r, K, σ, S(t) etc.. By the Delta-Gamma method:

$$\delta P \approx \Delta_p \delta S + \frac{\gamma^2}{2}(\delta S)^2,$$

where Δ_p is the option delta for a put and δC is

$$\begin{aligned} \delta C &\approx \Delta \delta S + \frac{\gamma^2}{2}(\delta S)^2, \\ &\approx (\Delta_p + 1)\delta S + \frac{\gamma^2}{2}(\delta S)^2. \end{aligned}$$

The previous line is possible because for any option pricing model it is known that $\Delta = \Delta_p + 1$. Therefore

$$\begin{aligned} \delta M &= \delta P - \delta C, \\ &= (\Delta_p \delta S + \frac{\gamma^2}{2}(\delta S)^2) - ((\Delta_p + 1)\delta S + \frac{\gamma^2}{2}(\delta S)^2), \\ &= -\delta S. \end{aligned}$$

Hence department H1 expects $-\delta S$ payoff from its portfolio M. Now department H2 can sell H1 the portfolio M. By the put-call parity:

$$\begin{aligned} P &= Ke^{-r(T-t)} - S(t) + C, \\ P - C &= Ke^{-r(T-t)} - S(t). \end{aligned}$$

So H2's payout from the above portfolio will be:

$$\begin{aligned} dP - dC &= d(Ke^{-r(T-t)}) - dS, \\ \delta P - \delta C &\approx Ke^{-r(T-\delta t)} - \delta S. \end{aligned}$$

Now H2 will only need to pay out $-\delta S$ to H1 for the portfolio it sold to H1 because H1 is expecting $-\delta S$ from its model. Hence using the last equation we see that H2 will always be able to make a riskless profit of $Ke^{-r(T-\delta t)}$, regardless of the value of S(t). Hence this represents an arbitrage opportunity.

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