

EXCHANGEABLE BRAIDS

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§1. Introduction

Representations of the unknot as a closed braid have been studied variously by Stallings, Goldsmith and others, and can be used in describing a number of constructions for fibred knots, [G], [M].

To represent a knot $K \subset S^3$ as a closed braid a disjoint unknotted curve L must be chosen as axis. The complement $S^3 - L$ is then a product, $D^2 \times S^1$. If we can choose a projection $p_L : S^3 - L \rightarrow S^1$ which gives a regular n -fold covering of S^1 when restricted to K then K is called a closed braid relative to axis L . It is common in this case to view K as the 'closure', $\hat{\beta}$, of an n -string braid β , an element of the braid group B_n , see Birman [B]. The essential geometric information about the braid is contained in the link $K \cup L$, from which the element $\beta \in B_n$ can be recovered up to conjugacy in B_n , [M].

In this paper I shall study links $K \cup L$ for which K is a closed braid relative to L , and in addition L is a closed braid relative to K . I propose to call such a link exchangeably braided. A braid $\beta \in B_n$ for which $\hat{\beta} \cup \text{axis} (= K \cup L)$ is exchangeably braided, and hence isotopic to $\text{axis} \cup \hat{\gamma}$ for some other braid $\gamma \in B_n$, will be termed exchangeable.

Two features of an exchangeable braid $\beta \in B_n$ are then apparent.

1. The closure $\hat{\beta}$ is unknotted, for it is required to form the axis for the closed braid $\hat{\gamma}$.
2. There is a disc D spanning $\hat{\beta}$ which meets the axis, L , in exactly n points.

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This follows by choosing D as any one of the level discs for a projection $S^3 - \hat{\beta} \rightarrow S^1$ under which L is seen as a closed braid relative to $\hat{\beta}$. (The existence of such a projection can be interpreted as straightening out $\hat{\beta}$ so that L appears to wind 'monotonically' about it.)

Here $n = lk(\hat{\beta}, L)$ is the minimum possible number of transverse intersections of L with a disc spanning $\hat{\beta}$, since the linking number is equal to the algebraic number of these intersections, taken with their sign. Property 2 can then be thought of as requiring the 'geometric' linking number of $\hat{\beta}$ and L to equal their algebraic linking number.

An exchangeable braid will satisfy rather more than Property 2, for the level discs of the projection above provide a whole 1-parameter family of spanning discs, each meeting L in n points.

Stallings, [S], describes for each n a finite set of braids on n strings with unknotted closure, which I shall call Stallings braids. These and their conjugates form a proper subset of the braids with unknotted closure [K]. In this paper I shall show that exchangeable braids, up to conjugacy, form a proper subset of the set of Stallings braids.

This follows from two main results.

Theorem 1 Let $\beta \in B_n$. Then β is conjugate to a Stallings braid if and only if it satisfies properties 1 and 2 above; i.e. $\hat{\beta}$ is unknotted, and is spanned by a disc meeting the axis in exactly n points.

Corollary 1.1 Every exchangeable braid is conjugate to a Stallings braid.

Corollary 1.2 There are only a finite number of exchangeably braided links with linking number n , since each link can then be realised as the closure of a Stallings n -braid, with its axis, and there are only finitely many such braids.

Theorem 2

There exist Stallings braids which are not exchangeable.

An exchangeable braid β determines a link $\beta \cup L = K \cup \gamma$, and, up to conjugacy, another braid γ whose closure is L . It is not always true that β and γ are conjugate, so the link $K \cup L$ is not necessarily interchangeable in the sense there is an isotopy interchanging the components.

Goldsmith's construction of fibred knots starts with an unknotted closed braid β with axis L , and observes that in the k -fold cyclic cover of S^3 branched over β , (which is again S^3), the preimage of L will be a fibred knot or link, the fibres being simply the preimages of the family of discs which span L and meet β in n points.

If β is exchangeable this preimage is simply the closure of the k^{th} power of the corresponding braid γ . Now the braid γ is itself exchangeable, for the roles of β and γ can be exchanged.

Remark If β is exchangeable then β^k is fibred for each k .

Goldsmith extends her construction to the case where L is a 'generalised axis' for an unknotted curve K . The curve L is called a 'generalised axis' for K if (a) the complement of L is fibred and (b) there is a fibre projection $p_L : S^3 - L \rightarrow S^1$ under which K covers S^1 .

It would be interesting to study exchangeability in the wider context of links $K \cup L$ where each component forms a generalised axis for the other. The closure of a homogeneous braid, with its axis, gives an example of such a link.

§2. Characterisation of Stallings braids

Stallings defines the elementary braid $\sigma_{i,j} \in B_n$, for $1 \leq i < j \leq n$ by interchanging the i^{th} and j^{th} strings with a single positive crossing, in front of any intermediate strings, and leaving all other strings alone, see fig. 1.

A Stallings braid is defined to be a product of $n - 1$ elementary braids, or their inverses, whose closure has only one component.

Lemma 1 Every Stallings braid has unknotted closure, with a spanning disc meeting the axis in n points.

Proof Construct a spanning surface for the closure of a product of elementary braids as prescribed by Stallings. To do this, start with the closure of the trivial braid spanned by n disjoint discs labelled $1, \dots, n$ corresponding to the string labelling, and each meeting the axis in one point. The complement of an open tubular neighbourhood of the axis is a closed solid torus $D^2 \times S^1$ which can be arranged to meet these n discs in n annuli, each of the form $\text{arc} \times S^1$. Imagine the discs stacked vertically, and refer to the D^2 factor in $D^2 \times S^1$ as horizontal. Where the elementary braid $\sigma_{i,j}$ or its inverse occurs in the given product connect the i^{th} and j^{th} discs by a ribbon in $D^2 \times S^1$, avoiding the other $n - 2$ discs and any previous ribbons. Choose the core of the ribbon to lie at one horizontal level, and arrange the cores in order along the S^1 factor according to the order of the elementary braids in the product. The ribbon itself is given a left or right-handed half-twist, so that no edge meets the same horizontal level more than once, see fig. 2. Up to isotopy the ribbons can be made to lie arbitrarily close to their core levels, so that different ribbons lie in completely different levels. Each ribbon is determined up to isotopy by i, j and the sign of the twist.

Together with the n discs they form an oriented surface spanning the closure of the given braid and meeting the axis in n points.

For a Stallings braid this surface will be connected and from its Euler characteristic it must be a disc, meeting the axis as required. I shall refer to such a spanning surface as a Stallings disc.

Lemma 2 Any braid $\beta \in B_n$ whose closure is spanned by a disc meeting the axis in n points is conjugate to a Stallings braid.

Proof It will be enough to exhibit a Stallings braid whose closure is isotopic to $\hat{\beta}$ in the complement of the axis L . Such a braid is available if we can produce a disc spanning $\hat{\beta}$ made up from n discs each meeting L in one point, joined by $n - 1$ ribbons each with a horizontal core and a single half-twist. The n discs taken in order around L then prescribe the labelling of strings in the braid, and the ribbons determine the elementary braids in the product, with their order determined by the core levels.

Take a disc D spanning $\hat{\beta}$ which meets L in c_1, \dots, c_n . Under some projection $p_L : S^3 - L \rightarrow S^1$ the curve $\hat{\beta}$ is a regular n -fold cover, while small circles around each c_i can be chosen which project homeomorphically. We can assume that curves parallel to $\hat{\beta}$ in some collar also cover S^1 regularly, so that two pieces of a level curve for p_L never meet at a point of $\hat{\beta}$, or of $L \cap D$.

Isotop D , leaving a neighbourhood of $D \cap L$ and $\hat{\beta}$ unchanged, so that the number of non-degenerate critical points for p_L on $D - L$ is as small as possible, and study the behaviour of p_L on $D - L$.

The level curve through any saddle point locally forms a cross. All four arms of this cross may continue to meet $\hat{\beta}$ or $D \cap L$, forming an essential saddle, or two of these arms may meet, forming an inessential saddle.

Where the number of critical points is minimal there can be no inessential saddles. For the two arms which meet provide a closed curve C in D . Now the intersections of L with D all have the same sense, so we can calculate the degree of p_L when restricted to any closed curve in $D - L$ by counting the number of points of $D \cap L$ enclosed by the curve. The map p_L is constant, and hence of degree 0, on the curve C . The subdisc of D bounded by C then contains no points of L . D can now be isotoped in the complement of L across a ball bounded by this subdisc and the disc bounded by C in the critical level so as to cancel the saddle with a local extremum, possibly after an innermost disc argument, as in [M2] or [Bo].

It follows that there will be no local extrema either among the minimal number of critical points, otherwise as the level of p_L is changed the expanding system of level circles surrounding the critical point will eventually encounter an inessential saddle - they cannot reach either $\hat{\beta}$ or $D \cap L$ intact, since two pieces of a level curve never meet at such points.

Every non-critical component of a level curve must then be an arc. Furthermore, these arcs must each join a point of $D \cap L$ to a point of $\hat{\beta}$. For if such an arc were to join two points of $\hat{\beta}$ then the direction around $\hat{\beta}$ of increase of level at the two points would be in opposite senses, while level increases monotonically on travelling in one sense around $\hat{\beta}$. Similarly two points p, q of $D \cap L$ cannot be joined, for we can choose small circles around p and q on which p_L changes monotonically. Since L crosses D in the same sense at p as at q , the direction of increase of p_L around each of these circles will be in the same sense relative to their orientation in D . The two circles cannot then be joined by a non-critical level curve of p_L .

The surface $D - L$ has Euler characteristic $1 - n$. Since the function p_L is suitably behaved on the boundary and round the missing points it follows that p_L has exactly $n - 1$ saddle points. These

saddles are all essential. One pair of opposite arms must join points of $\hat{\beta}$ while the other pair joins two (distinct) points of $D \cap L$, otherwise some neighbouring level curve would join points of the wrong kind.

Look now at the graph Γ embedded in D whose vertices are the points of $D \cap L$ and whose edges are the $n - 1$ pairs of opposite arms of the saddles joining such points. The graph Γ cannot separate D , for there will be points in every component of $D - \Gamma$ which are joined to $\hat{\beta}$ by a non-critical arc, necessarily lying itself in $D - \Gamma$. Hence by virtue of its Euler characteristic Γ must be a tree.

Construct a neighbourhood of Γ in D consisting of a disc round each vertex joined by narrow ribbons around each edge. These ribbons can be chosen so that p_L is monotone on their edges.

The boundary of this neighbourhood is then exhibited as the closure of a Stallings braid with axis L , where the ribbon cores arise from the critical curves joining points of $D \cap L$, and the sign of the half-twist depends on the direction of the positive normal to D at the critical point. Since $\hat{\beta}$ is isotopic through $D - L$ to this curve Lemma 2 is established.

The characteristic of Stallings braids given in Theorem 1 is then complete.

§3. Non-exchangeable braids

The simplest example of a non-exchangeable Stallings braid is the braid $\omega = \sigma_{2,4} \sigma_{2,3} \sigma_{1,3} = \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1 \in B_4$, whose closure is shown in figure 2. Here σ_i , $i = 1, \dots, n-1$ denote the standard generators of the braid group B_n , where σ_i is the elementary braid $\sigma_{i,i+1}$ which interchanges the strings i and $i+1$. The elementary braid $\sigma_{i,j}$ can be written in terms of the standard generators as $\sigma_i^{-1} \sigma_{i+1}^{-1} \dots \sigma_{j-2}^{-1} \sigma_{j-1} \sigma_{j-2} \dots \sigma_i$, or equally $\sigma_{j-1} \sigma_{j-2} \dots \sigma_i \sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1}$ as illustrated in figure 1. The non-exchangeability of ω follows from a calculation of the two variable Alexander polynomial $\Delta(x, t)$ for the link $\hat{\omega} \cup \text{axis } L$, where t, x denote generators of $H_1(S^3 - (\hat{\omega} \cup L))$ represented by meridians of $\hat{\omega}, L$ respectively.

This calculation can be made most readily by a slight extension of Birman's observations on the Alexander polynomial of a closed braid, [B], which is proved in the final section of this paper.

Theorem 3 Suppose that $\beta \in B_n$, with reduced Burau matrix $B(t) \in GL(n-1, \mathbb{Z}\mathbb{Z}[t, t^{-1}])$. Then the Alexander polynomial, $\Delta(x, t)$, of the braided link $\hat{\beta} \cup \text{axis}$ is given by $\Delta(x, t) = \det(B(t) - xI)$, the characteristic polynomial of $B(t)$, where x, t are represented by meridians of the axis and $\hat{\beta}$ respectively.

Remark The reduced Burau matrix $B(t)$ is the image of β under the reduced Burau representation $\rho : B_n \rightarrow GL(n-1, \mathbb{Z}\mathbb{Z}[t, t^{-1}])$, see [B]. In this representation

$$\rho(\sigma_i) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 0 & 0 \\ & & t & -t & 1 \\ & & 0 & 0 & 1 \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \leftarrow i^{\text{th}} \text{ row}$$

truncated appropriately where $i = 1$ or $i = n-1$.

The polynomial $\Delta(x, t)$ is defined up to multiplication by $\pm x^r t^s$. Its extreme powers of x for the braided link in Theorem 3 are x^{n-1} with coefficient 1 and x^0 with coefficient $(-1)^{n-1} \det B(t)$. Now $\det \rho(\sigma_i) = -t$, so $\det B(t) = (-t)^{|\beta|}$, where $|\beta|$ = algebraic number of crossings in β , counting +1 for each σ_j and -1 for each σ_j^{-1} . Even after multiplication by a unit $\pm x^r t^s$, the coefficients of the extreme powers of x will each remain a power of t ,

If a braid β is to be exchangeable then $\hat{\beta} \cup \text{axis} = K \cup L$ say, must also be braided relative to K , and so its Alexander polynomial will have a similar form with the roles of x and t reversed. Thus the coefficients of the extreme powers of t must be powers of x , and further, the 'degree' in t , that is the difference in degree of the extreme powers of t , must also be $n - 1 = \text{lk}(K, L) - 1$.

Cororollary 3.1 The Stallings braid ω above is not exchangeable.

Proof Its reduced Burau matrix $\omega(t)$ is

$$\omega(t) = \begin{pmatrix} 1 - t & -t^{-1} & t^{-1} \\ 2t - 3t^2 + 2t^3 - t^4 & -2 + t & -t^{-1} + 3 - 3t + t^2 \\ t^2 - 2t^3 + t^4 - t^5 & -t + t^2 & t - 2t^2 + t^3 \end{pmatrix},$$

having characteristic polynomial $\det (xI - \omega(t)) = x^3 + p_1(t)x^2 + p_2(t)x + t^3$, where $p_1(t) = -\text{tr } \omega(t) = 1 - t + 2t^2 - t^3$. This polynomial has degree 3 in t , but the coefficients of the extreme powers of t are not powers of x . For example, the coefficient of t^3 is $1 + ax - x^2$, where a is its coefficient in $p_2(t)$. Consequently the axis of $\hat{\omega}$ is not braided relative to $\hat{\omega}$.

In trying to decide whether a Stallings braid β is exchangeable it can be helpful to look at a schematic diagram, in which the n strings are drawn vertically, and $\sigma_{i,j}^{\pm 1}$ is represented by a horizontal line with a sign attached, joining string i to string j and lying in front of the intervening strings. The braid ω is represented in this way in figure 3.

This representation corresponds closely to the graph Γ constructed at the end of §2 from a disc D meeting an unknotted curve L in n points, when $K = \partial D$ is braided relative to L , and $n = \text{lk}(L, K)$. In such a representation the vertical strings correspond to small circles, one around each vertex of Γ in D , i.e. each intersection of L with D , and the horizontals correspond to the edges of Γ outside these circles.

A Stallings disc D for β can be constructed by clothing the horizontal lines with half-twisted ribbons to join discs spanning the vertical strings. The axis L will meet D once in each vertical disc, and is then separated into n arcs, L_1, \dots, L_n . From the embedding of $D \cup L$ in S^3 we can decide whether L is braided relative to $\hat{\beta}$, and so whether β is exchangeable, as follows.

Construction Split S^3 open along D to give $D \times I$, i.e. choose an explicit map $q : D \times I \rightarrow S^3$ which identifies only $D \times \{0\}$ and $D \times \{1\}$ with D , and $\partial D \times I$ with ∂D . Then $D \times I$ contains n arcs, $A_i = q^{-1}(L_i)$, formed from the pieces of the axis L .

Theorem 4 L is braided relative to ∂D if and only if there is an isotopy of these arcs in $D \times I$, rel boundary, to n arcs running monotonically from $D \times \{0\}$ to $D \times \{1\}$.

Proof Such an isotopy, followed by projection to I , will determine via q a projection from $S^3 - \partial D$ to S^1 , whose restriction to L is a regular n -fold covering map.

Conversely, if L is braided relative to ∂D , and $p : S^3 - \partial D \rightarrow S^1$ is a suitable projection then there is an isotopy of S^3 carrying $D_0 \cup L$ to $D \cup L$, where $D_0 = p^{-1}(1)$. This isotopy is analogous to one between two minimal genus spanning surfaces for a fibred knot, and can be found by examining the inverse images of D_0 , D and L in the universal cover of $S^3 - \partial D$.

Remark

(Th. 10.2) We can deduce an algebraic criterion for exchangeability from the fact, [H], that the inclusion $(D \times \{0\}, n \text{ points}) \subset (D \times I, n \text{ arcs})$ induces an isomorphism of fundamental groups if and only if the arcs lie monotonically up to isotopy rel boundary. On returning to D in S^3 , this result combined with Theorem 4 gives the following test:

Corollary 4.1

Let L meet a disc D in $n = \text{lk}(L, \partial D)$ points. Then L is braided relative to ∂D if and only if the inclusion $D^+ - L \subset S^3 - (D \cup L)$ induces a fundamental group isomorphism, where D^+ denotes a translate of D through a small distance in the direction of the positive normal to D .

Corollary 4.2

With L, D as in 4.1, a necessary, but not sufficient, condition for L to be braided relative to ∂D is that any k of the arcs A_i can be isotoped to lie monotonically in $D \times I$. Consequently $\pi_1(S^3 - (D \cup k \text{ arcs of } L))$ must be free on k generators if L is braided.

We can use 4.2 in the simplest case $k = 1$ to construct Stallings braids, based on a diagram such as figure 4, which are not exchangeable for any choice of signs in the ribbons. This is in contrast to the braid ω

in figure 2, where a change of signs on the ribbons to give $v = \sigma_{2,4}^{-1} \sigma_{2,4}^{-1} \sigma_{1,3}^{-1}$ yields an exchangeable braid. (See figure 8 and the end of §4.)

Theorem 5 None of the braids $\sigma_{2,4}^{\pm 1} \sigma_{3,5}^{\pm 1} \sigma_{4,6}^{\pm 1} \sigma_{1,5}^{\pm 1} \sigma_{3,6}^{\pm 1} \in B_6$ are exchangeable.

Proof The axis is broken into six arcs by a Stallings disc D , and for one of these arcs L_1 we have $\pi_1(S^3 - (D \cup L_1)) \not\cong \mathbb{Z}$. The arcs L_i are shown in figure 5. In the calculation of π_1 we can replace $D \cup L_1$ by the curve in figure 6, which forms a non-trivial knot.

§4. Compound braids

By way of giving some sufficient geometric conditions for exchangeability, I shall describe two ways in which a Stallings braid may decompose into simpler Stallings braids, and prove that the original braid is exchangeable if and only if its constituents are.

(a) Murasugi Sums

The first decomposition is a counterpart of the generalised plumbing, or 'Murasugi sum' of two surfaces, originally described in [Mu]. Here the construction is extended to apply to pairs, (Stallings disc, axis), in a similar vein to the description given in [Ga] or [M3].

Given a Stallings braid $\beta \in B_n$, with Stallings disc D and axis L , I shall say that (D, L) is a Murasugi sum of (D_1, L_1) and (D_2, L_2) if (i) D_1 and D_2 are subdiscs of D each lying in one half-space of \mathbb{R}^3 , and meeting only in a disc $D_0 = D_1 \cap D_2$ in the common plane \mathbb{R}^2 , (ii) the axis L meets the separating plane \mathbb{R}^2 in just two points, $d \in D_0$ and $c \notin D_0$, (iii) the disc D_1 forms a Stallings disc with axis L_1 , where L_1 consists of the part of L in one half-space, completed by an unknotted arc cd in the other half-space, and similarly for (D_2, L_2) .

This sort of decomposition can be seen when none of the horizontal bands in the disc for β pass over the k^{th} string, say, i.e. no generators $\sigma_{i,j}$, with $i < k < j$ occur in β . Then the vertical plane \mathbb{R}^2 containing the k^{th} vertical disc will separate the Stallings disc D spanning $\hat{\beta}$ into two discs, D_1 consisting of the first k vertical discs with the ribbons joining them, and D_2 consisting of the last $n - k + 1$ vertical discs and joining ribbons. These form Stallings discs, relative to the curve L as axis, for braids β_1 on the first k strings, and β_2 on the last $n - k + 1$ strings, see for example figure 7.

Theorem 6 The Stallings braid β is exchangeable if and only if β_1 and β_2 are exchangeable.

Corollary 6.1 Every Stallings braid $\beta \in B_n$ in which one string, the k^{th} say, is joined to each of the others by a ribbon is exchangeable.

Proof If $k \neq 1, n$ then β is a Murasugi sum of two braids with the same property, on fewer strings. The result then follows by induction on n , using Theorem 6 and the fact that the two Stallings braids on two strings are exchangeable.

Otherwise let $k = 1$, ($k = n$ is similar). The conjugate braid $\beta^1 = \gamma^{-1}\beta\gamma$, where $\gamma = \sigma_{n-1} \dots \sigma_2 \sigma_1$, given by moving the last string of β over the others to become the first, is again a Stallings braid of the same form as β , with $k = 2$, and has thus already been shown to be exchangeable.

Proof of Theorem 6 Let D be the Stallings disc for $\hat{\beta}$. Write c_1, \dots, c_n for the points of $D \cap L$ in order along L , so that the separating plane \mathbb{R}^2 meets L in $c_k = d$, and in a point c between c_n and c_1 . Suppose first that β_1 and β_2 are exchangeable. We must show that there is a fibration for $S^3 - \partial D$ in which L lies monotonically. Now the construction in [M3] of a fibration for $S^3 - \partial F$ where F is a Murasugi sum of two fibre surfaces F_1 and F_2 will apply here to give a fibration of $S^3 - \partial D$ arising explicitly from the given fibrations of $S^3 - \partial D_1$ and $S^3 - \partial D_2$. It follows from this construction that each arc $c_i c_{i+1}$, for $1 \leq i < n$, and also the arcs cc_1 and $c_n c$ lie

monotonically in this fibration, since their counterparts did in the constituent fibrations. Hence L lies monotonically in the fibration, and so β is exchangeable.

Conversely, suppose that β is exchangeable. Choose an arc α in D_2 joining c_k to c_n . This arc α followed by the arc on L from c_n to c_1 gives an arc from c_k to c_1 which can be altered slightly by pushing gradually off D_2 so as to give an arc from c_k to c_1 which lies monotonically in the fibration of $S^3 - \partial D$. The closed curve C made up of this arc, with the arcs $c_1c_2, \dots, c_{k-1}c_k$ of L is then braided relative to ∂D .

I claim that $\partial D \cup C$ is isotopic to $\hat{\beta}_1 \cup$ axis, showing that β_1 is exchangeable.

It is clear that $D \cup C$ is isotopic to $D_1 \cup C$, since C does not meet $D - D_1$. It remains to show that C is isotopic, rel D_1 , to the axis for $\hat{\beta}_1 = \partial D_1$. Up to isotopy rel D_1 , the curve C can be taken as α together with the part of L from c_n through c to c_k . It will then be enough to show that the arc $\alpha \cup c_n c$ is unknotted in the half-space which contained D_2 , for this piece can then be moved to form the remaining part of the axis. Complete this arc to a curve C' by adjoining the arc cc_1 from L and an arc γ in D_1 joining c_1 to c_k . This curve C' then consists of one arc $c_n c_1$ on the axis of D with ends joined by an arc lying in D , and hence unknotted, by the exchangeability of β , since its complement will have free fundamental group (Corollary 4.2). Hence $\alpha \cup c_n c$ is unknotted in its half-space, since it forms a connected summand of the unknotted curve C' . This completes the proof of Theorem 6.

(b) Satellites

The other possible decomposition of a Stallings braid which I would like to describe is related to the construction of satellite knots and links. Generally, to construct a satellite of a link $L = L_1 \cup \dots \cup L_r$ we start with another link, $C = C_1 \cup \dots \cup C_k$, in which one unknotted component, C_k say, is selected. Choose one component, L_1 say, of the original link, and replace a solid torus neighbourhood V of L_1 by the complementary solid torus W to a neighbourhood of C_k . This replacement is by means of a faithful homeomorphism $h : W \rightarrow V$, i.e. one which carries a longitude of W to a longitude of V . The satellite link then consists of $h(C_1) \cup \dots \cup h(C_{k-1}) \cup L_2 \cup \dots \cup L_r$, and contains a 'splitting torus', $T = \partial V = h(\partial W)$.

I shall only be concerned here with the case $r = k = 2$ in which $C_1 \cup C_2$ and $L_1 \cup L_2$ each consist of a closed braid, together with its axis. It is not difficult to see that the resulting satellite $h(C_1) \cup L_2$ is a closed braid with axis L_2 on mn strings, when the constituent braids have m and n strings respectively. In fact a converse of this can be proved, either as a special case of results about fixed satellite links, or by bare-hands isotopy of a splitting torus for the satellite. The result can be stated as follows.

Theorem 7 A satellite link $K_1 \cup K_2 = h(C_1) \cup L_2$ arising from links $C_1 \cup C_2$ and $L_1 \cup L_2$ consists of a closed braid K_1 with axis K_2 if and only if C_1 is a closed braid with axis C_2 and L_1 is a closed braid with axis L_2 .

As a consequence we have the following result about exchangeable braids.

Theorem 8 Let the satellite link $K_1 \cup K_2$ consist of a closed braid β with its axis. Then β is exchangeable if and only if β_1 and β_2 are exchangeable, where $C_1 \cup C_2 = \beta_1 \cup \text{axis}$, $L_1 \cup L_2 = \beta_2 \cup \text{axis}$, as in Theorem 7.

Proof If β is exchangeable then $K_1 \cup K_2$ forms a closed braid K_2 with axis K_1 . The splitting torus must be unknotted, so the link K can also be viewed as a satellite constructed from the links C and L with the roles of C_2 and L_1 interchanged and also the roles of C_1 and L_2 . By Theorem 7 it follows that C and L also consist of closed braids C_2 , L_2 with axes C_1 and L_1 respectively; thus β_1 and β_2 are exchangeable. Conversely, if β_1 and β_2 are exchangeable the homeomorphism $h : W \rightarrow V$ used in constructing the satellite, which carries the exterior of C_2 to the neighbourhood of L_1 can be extended to a homeomorphism $h : S^3 \rightarrow S^3$, with h^{-1} carrying the exterior of L_1 to a neighbourhood of C_2 . The satellite of C constructed using h^{-1} , which consists of $h^{-1}(L_2) \cup C_1$, also forms a closed braid with axis C_1 , since C_2 and L_2 are braided relative to C_1 , L_1 respectively. This satellite, however, is equivalent under h to the link $L_2 \cup h(C_1)$, which forms the original link $K_2 \cup K_1$. Thus K_1 is braided relative to K_2 , and β is exchangeable.

One of the simplest non-trivial examples of this construction is illustrated by the braid $v \in B_4$ shown in figure 8, where the link $\hat{v} \cup \text{axis}$ results from the satellite construction with $\beta_1^{-1} = \beta_2 = \sigma_1 \in \beta_2$. Examples of this sort with $\beta_1 \neq \beta_2^{\pm 1}$ can be used to show that exchangeable

braids do not always yield symmetric links, i.e. there need not be a isotopy interchanging the two components of the link $\hat{\beta} \cup \text{axis}$ for an exchangeable braid β .

Conclusion It would be nice to have an effective geometric test to decide which Stallings braids are exchangeable. Although I have developed some necessary and some sufficient conditions for exchangeability there is still a considerable gap between the two.

As a final section I include a brief review of the properties of the Alexander polynomial for a link, and its calculation, leading to the relation with the Burau matrices as described in Theorem 3.

§5 Alexander polynomials of closed braids

In Fox's theory of Alexander polynomials, $[F1, F2]$, a presentation of a group G , with n generators and r relations, yields an $r \times n$ matrix A with entries in the group ring $\mathbb{Z}[H]$, where H is the abelian group $G/[G, G]$, written multiplicatively. The ideals $E_k(A)$ in $\mathbb{Z}[H]$ generated by the $(n - k) \times (n - k)$ minors of A are invariants of the group G . In the case when G is the group of a link with μ components the group H is free abelian of rank μ .

Fox shows that the ideal

$$E_1(G) = D \quad \mu = 1$$

$$= D.I \quad \mu > 1,$$

where I is the augmentation ideal, and D is a principal ideal, with generator Δ . The element $\Delta \in \mathbb{Z}[H]$, determined up to unit multiple, can be viewed as an integer polynomial in μ variables, and is defined to be the Alexander polynomial of the link. Fox shows, in $[F2, (6.4)]$, that if the column of A corresponding to a generator $g \in G$ is deleted to give a matrix B , then $E_0(B)$ is a principal ideal, with generator

$$\Delta.(\langle g \rangle - 1), \quad \text{for } \mu > 1,$$

$$\text{or} \quad \Delta. \frac{\langle g \rangle - 1}{t - 1}, \quad \text{for } \mu = 1, \quad \text{where}$$

$\langle g \rangle \in H$ is the image of g , and t is a generator of H in the case $\mu = 1$.

One immediate consequence is that if $r = n - 1$, we can find Δ readily as $\det B \cdot \frac{1}{\langle g \rangle - 1}$, when $\mu > 1$ and B is given as above by deleting a column of A .

Proof of Theorem 3

A closed braid $\hat{\beta}$ on n strings with axis L forms a link with group G generated by t_1, \dots, t_n and x and relations $\beta(t_i) = x^{-1}t_ix$ for each i , where β is an automorphism of the free group F_n (see [B]).

The group F_n appears here as $\pi_1(D^2 - n \text{ pts.})$, where D^2 is a spanning disc for L , meeting $\hat{\beta}$ in n points.

For a knot $\hat{\beta}$ the abelianisation $H = G/[G, G]$ is free abelian on two generators $t = \langle t_i \rangle$, $i = 1, \dots, n$, corresponding to meridians of $\hat{\beta}$, and $x = \langle x \rangle$, corresponding to meridians of L .

Birman shows that the matrix $\left(\frac{\partial \beta(t_i)}{\partial t_j} \right)_H = \tilde{B}(t)$ of free derivatives evaluated in $\mathbb{Z}[H]$ is just the full $(n \times n)$ Burau matrix of the braid $\beta \in B_n$.

Applying Fox's free calculus to the presentation of G given above yields the $n \times (n + 1)$ matrix $A = \left(\tilde{B}(t) - xI_n \mid \begin{smallmatrix} t - 1 \\ \vdots \\ t - 1 \end{smallmatrix} \right)$, where the last column corresponds to the generator $x \in G$. The Alexander polynomial

$\Delta(x, t)$ of $\hat{\beta} \cup L$ is then given by deleting this last column, so that

$\Delta(x, t) = \det(\tilde{B}(t) - xI_n)/x - 1$. Now $\tilde{B}(t)$ is conjugate in $GL(n, \mathbb{Z}[t, t^{-1}])$ to $\left(\begin{smallmatrix} B(t) & \vdots \\ 0 & 1 \end{smallmatrix} \right)$ where $B(t) \in GL(n - 1, \mathbb{Z}[t, t^{-1}])$ is the reduced Burau matrix of β , [B]. Hence $\det(B(t) - xI_{n-1}) = \Delta(x, t)$ as claimed in Theorem 3.

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Figure 1

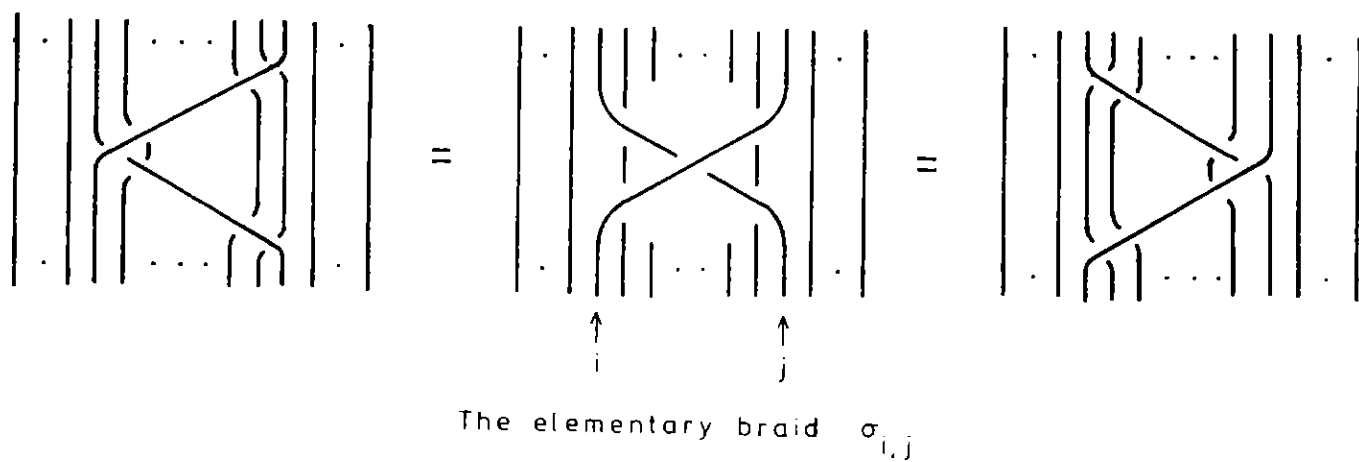
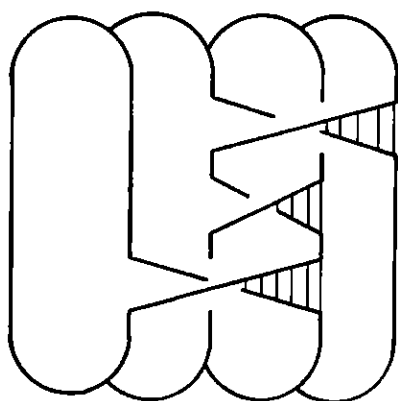


Figure 2



A Stallings disc for $w = \sigma_{2,4} \sigma_{2,3} \sigma_{1,3}$

Figure 3

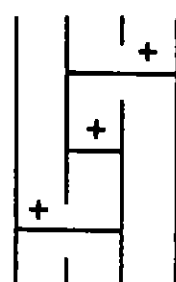


Figure 4

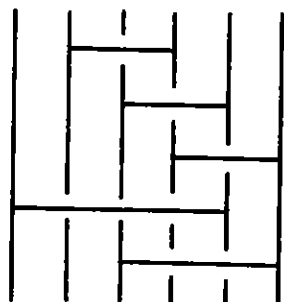


Figure 6

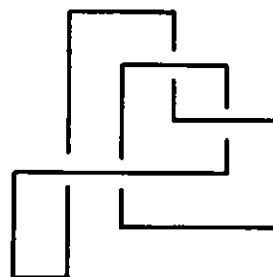


Figure 5

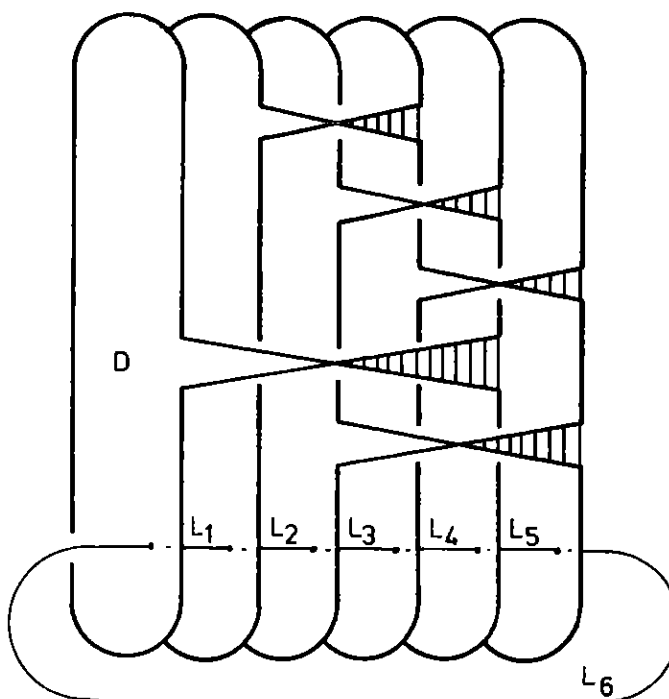
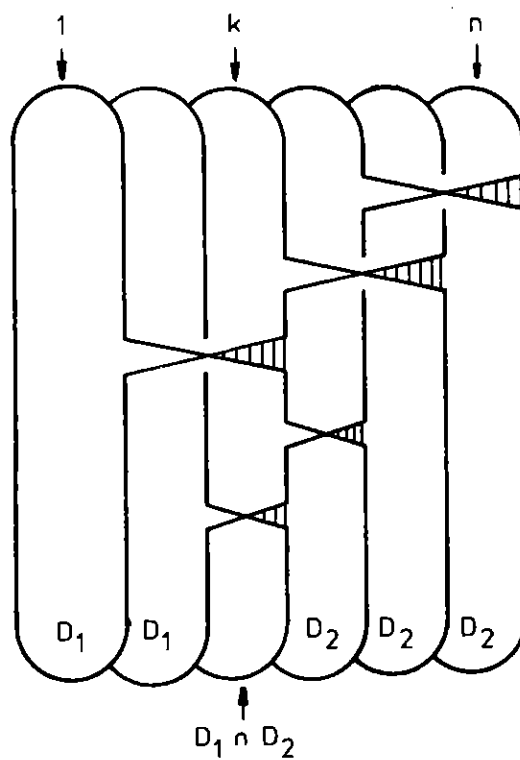
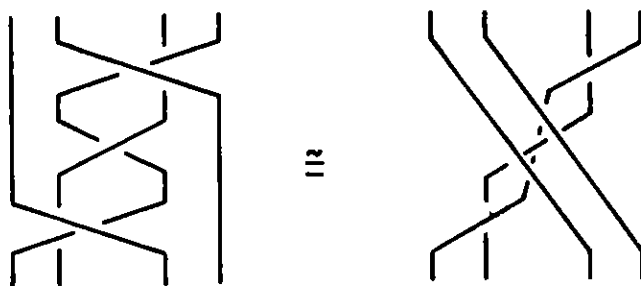


Figure 7



A Murasugi sum of two Stallings braids

Figure 8



The Stallings braid $v = \sigma_{2,4}^{-1} \sigma_{2,3} \sigma_{1,3}^{-1}$