NNNLO and All-Order Corrections to Splitting and Coefficient Functions in Deep-Inelastic Scattering

Thesis submitted in accordance with the requirements of
the University of Liverpool for the degree of Doctor in Philosophy

by

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Declaration

I hereby declare that the work described in this thesis is the result of my own research activities unless reference is given to others. None of this material has been previously submitted to this or any other university. All work was carried out in the Department of Mathematical Sciences during the period October 2012 to August 2016.

Contributions to this work have previously been published or are awaiting publication elsewhere in

- [1] B. Ruijl, T. Ueda, J. A. M. Vermaseren, J. Davies and A. Vogt, *First Forcer results on deep-inelastic scattering and related quantities* in *13th DESY Workshop on Elementary Particle Physics: Loops and Legs in Quantum Field Theory (LL2016) Leipzig*, 2016. `arXiv:1605.08408`. Accepted: PoS(LL2016)071.

- [2] J. Davies, A. Vogt, S. Moch and J. A. M. Vermaseren, *Non-singlet coefficient functions for charged-current deep-inelastic scattering to the third order in QCD* in *Proceedings, 24th International Workshop on Deep-Inelastic Scattering and Related Subjects (DIS 2016)*, 2016. `arXiv:1606.08907`. Accepted: PoS(DIS2016)059.

- [3] J. Davies, S. Moch, J. A. M. Vermaseren and A. Vogt, *Perturbative QCD corrections to charged-current and polarized DIS structure functions at higher orders*, *In preparation*, 2016. `arXiv:16MM.XXXXX`.

- [4] J. Davies, C. H. Kom and A. Vogt, *Generalized double-logarithmic small-x resummation in inclusive deep-inelastic scattering*, *In preparation*, 2016. `arXiv:16MM.XXXXX`.

- [5] J. Davies, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt *Large-n_f contributions to the four-loop splitting functions in QCD*, *In preparation*, 2016. `arXiv:16MM.XXXXX`.

To my parents,

who encouraged my love of physics from an early age.

Abstract

This thesis describes several calculations of quantities describing the deep-inelastic scattering (DIS) of leptons and hadrons, within the framework of massless perturbative quantum chromodynamics. The third order (NNNLO) contributions to the coefficient functions $C^-_{2,ns}$, $C^-_{L,ns}$ and $C^-_{3,ns}$, which describe charged-current ($W^{\pm}$-exchange) DIS in the linear combination $W^+ - W^-$ are presented. Complementing existing results for the $W^+ + W^-$ combination, these new results complete the third-order description of charged-current DIS. The results are presented both as compact parametrizations and exact expressions. The corrections are found to be small for experimentally relevant values of the Bjorken-x variable.

The behaviour of the DIS structure functions in the small-x limit is considered. By finding a suitable functional form with which to describe them, it is possible to use the results of existing fixed-order perturbative calculations to resum the leading small-x double logarithms of the coefficient functions and splitting functions to all orders in the strong coupling constant α_s. All-order descriptions of the leading three double logarithms are discussed and presented for both coefficient functions and splitting functions.

Finally, the results of recent advances in the fourth-order computation of the Mellin moments of structure functions are used to reconstruct expressions for the general Mellin-N dependence of the large-n_f parts of the fourth-order contributions to the splitting functions. The software package `FORCER` is able to compute a sufficient number of Mellin moments to determine the N dependence of the n_f^2 terms of the non-singlet splitting functions, and the n_f^3 terms of the singlet splitting functions. The resulting expressions are in agreement with, and extend, various existing computations found in the literature.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

The Large Hadron Collider (LHC) at CERN is the highest energy particle collider ever constructed. Like the high-energy colliders of the past it collides hadrons, which are bound states of quarks, anti-quarks, and gluons. Being composite particles, the interactions of hadrons are very complex. In order to accurately interpret data collected by such experiments, we must have a good theoretical description of their interactions with other particles. The framework of Quantum Chromodynamics (QCD) forms this description and is used for the computations of this thesis. In particular we focus on the high energy regime of QCD, which is perturbative. That is, quantities of interest can be expanded as a series in some small parameter, the strong coupling constant α_s.

The ever-increasing precision of experimental data demands that we compute more and more perturbative corrections in order to provide sufficiently precise predictions of measured quantities. A crucial theoretical input for hadron colliders is the Parton Distribution Functions (PDFs). These functions describe the particle content of the colliding hadrons and must be determined from experimental data. To perform an accurate determination of the PDFs it is necessary to compute, as precisely as possible, how external particles interact with the individual partons which form the hadron.

For this we rely on Deep-Inelastic Scattering (DIS), the high-energy interaction of leptons and hadrons. Involving just a single hadron, this provides a "clean" (both experimentally and theoretically) environment in which to study the effects of QCD. Quantities that we are able to compute within the framework of DIS are universal to all hadron reactions and are thus useful, in addition to their obvious application to lepton-hadron colliders such as HERA, to proton-proton and proton-anti-proton machines such as the LHC and Tevatron.

The structure of this thesis is as follows. In Chapter 2 we review the formalism of QCD in the context of DIS, outlining the theoretical description of leptons scattering from partons within a hadron by means of the exchange of a gauge boson. We describe a prescription for the separation of the high-energy regime from the low-energy physics of the hadronic bound state, which cannot be described by perturbation theory.

Chapter 3 concerns a high-order calculation describing a particular type of DIS, lepton-hadron scattering by the exchange of a charged electro-weak boson; a W^+ or a W^-. Such an exchange allows one to consider the scattering of neutrinos from hadrons. For these calculations one must consider the linear combinations of $W^+ + W^-$ and $W^+ - W^-$ scattering; only the $W^+ + W^-$ combination is currently known at third order in QCD. We complete the description of these interactions to the third order by computing the $W^+ - W^-$ combination.

In Chapter 4 we turn to a different style of calculation; a *resummation* of quantities describing DIS in certain kinematic limits. Despite being perturbative, there are regions in which the convergence of QCD quantities can be spoiled by large logarithms of kinematic parameters. The determination of these logarithms to all orders in the strong coupling parameter α_s aims to develop a better understanding of the behaviour of the quantities in these regions. It also provides predictions which cross-check higher fixed-order calculations.

Finally in Chapter 5, we begin a project to determine the so-called *splitting functions* of perturbative QCD at the fourth order in α_s. At the time of writing, very few calculations have been performed to this order in QCD. So far we only have analytic expressions for certain, structurally more simple, terms and aim to produce numerical approximations for the rest in the near future. However, the eventual complete calculation of the splitting functions at fourth order will allow for a reduction of the theoretical uncertainties of PDFs determined from experimental data.

Chapter 2

Formalism

Deep-Inelastic Scattering (DIS) is the process in which a lepton scatters from a hadron,

$$l(k) + h(P) \to l'(k') + X. \tag{2.1}$$

This reaction is depicted in Fig. 2.1, to leading order in Quantum Electrodynamics (QED). An incoming lepton of (four-)momentum k exchanges a boson of momentum $q = k - k'$ with a hadronic state carrying momentum P. The hadronic state breaks apart during the interaction, yielding an unspecified hadronic final state X; we consider only *inclusive* DIS processes in this thesis, in which we sum over all possible states X.

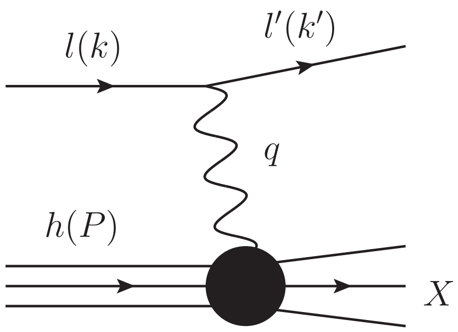

Figure 2.1: *Deep-Inelastic Scattering. A lepton l scatters from a hadron h, via the exchange of a virtual boson carrying momentum q. The hadron breaks apart into some hadronic system X.*

The exchanged boson may be a photon (γ), a Z-boson or a Higgs-boson (so called Neutral Current (NC) reactions) or a $W^{\pm}$-boson (Charged Current (CC) reactions). Since the exchanged boson is space-like, q^2 is negative. It is useful to define a positive quantity $Q^2 = -q^2$. We also define the Bjorken-x parameter, which takes values between 0 and 1 and is given by

$$x = \frac{Q^2}{2P \cdot q}. \tag{2.2}$$

For $x = 1$, the invariant mass of the hadronic final state X is equal to that of the incoming hadron; this is *elastic* scattering. $x \to 1$ is called the large-x or *threshold* limit. Small values of x correspond to large momentum transfer between the incoming lepton and the hadronic final state. $x \to 0$ is the *high-energy* limit.

To leading order in the electromagnetic coupling ($\alpha_{\rm em}$) the cross section for the process can be written as the product of two tensors; one describing the lepton side of the interaction ($L_{\mu\nu}$), and one describing the hadron side ($W^{\mu\nu}$),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}y} = \frac{2\pi y \alpha_{\rm em}^2}{Q^4} L^{\mu\nu} W_{\mu\nu}. \tag{2.3}$$

where $y = P \cdot q / P \cdot k$ and also takes values between 0 and 1. The lepton tensor is given by

$$L^{\mu\nu} = 2\left(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu} k \cdot k'\right). \tag{2.4}$$

We do not consider corrections that are higher order in $\alpha_{\rm em}$ here, on the grounds that it is very small compared to the strong coupling, $\alpha_{\rm s}$. We decompose the hadron tensor in terms of scalar *hadron structure functions*, which are the coefficients of the Lorentz-invariant structures built out of the available vectors P and q such that the electromagnetic current is conserved (which requires that $q^\mu W_{\mu\nu} = 0$). The standard definition is (see e.g. the PDG [6])

$$\begin{aligned} W_{\mu\nu} = &\left(P_\mu - \frac{(P\cdot q) q_\mu}{q^2}\right)\left(P_\nu - \frac{(P\cdot q) q_\nu}{q^2}\right)\frac{1}{P\cdot q} F_2(x,Q^2) \\ &+ \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x,Q^2) + i\epsilon_{\mu\nu\rho\sigma}\frac{P^\rho q^\sigma}{2P\cdot q} F_3(x,Q^2), \end{aligned} \tag{2.5}$$

or alternatively in terms of the longitudinal structure function defined as $F_L = F_2 - 2xF_1$,

$$\begin{aligned} W_{\mu\nu} = &\left(-g_{\mu\nu} - P_\mu P_\nu \frac{4x^2}{q^2} - (P_\mu q_\nu + P_\nu q_\mu)\frac{2x}{q^2}\right)\frac{1}{2x} F_2(x,Q^2) \\ &+ \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{1}{2x} F_L(x,Q^2) + i\epsilon_{\mu\nu\rho\sigma}\frac{P^\rho q^\sigma}{2P\cdot q} F_3(x,Q^2), \end{aligned} \tag{2.6}$$

This is the combination used throughout this thesis. The structure function $F_3(x, Q^2)$ is only present in the case of CC DIS ($W^\pm$ exchange) or NC DIS (for Z^0 exchange only), where we have axial terms in the vertex factors. It vanishes in the case of electromagnetic interactions since the ϵ-tensor is contracted with the (symmetric) lepton tensor ($L^{\mu\nu}$) of Eq. (2.4).

2.1 The Parton Model

We now assume some further structure for the process. Let the hadron consist of non-interacting *partons*. The probing boson scatters from one of these partons. The cross section *at the parton level* is thus given by

$$\frac{\mathrm{d}\hat{\sigma}_i}{\mathrm{d}x\mathrm{d}y} \sim L^{\mu\nu} \hat{W}_{i\,\mu\nu} \tag{2.7}$$

where the lepton tensor is as defined in Eq. (2.4) and the "hatted" symbols refer to parton-level quantities, which carry a label i that specifies the parton species under consideration. We assume the struck parton to carry a fraction $\xi \in (0,1)$ of the hadron's momentum P; thus it carries no transverse component, it is collinear with the hadron. The parton-level tensor decomposition for $\hat{W}_{i\,\mu\nu}$ is the same as Eq. (2.6) but is written in terms of hatted *parton-level structure functions* $\hat{F}_i$.

The hadron structure functions are related to their parton-level equivalents by integrating over all possible values of the momentum fraction ξ and summing over all parton species i. In the electromagnetic case,

$$F_a(x,Q^2) = \sum_i e_i^2 \int_x^1 \frac{\mathrm{d}\xi}{\xi} \tilde{f}_i(\xi) \hat{F}_{a,i}\left(\frac{x}{\xi}, Q^2\right) = \sum_i e_i^2 \left[\tilde{f}(\xi) \otimes \hat{F}_{a,i}(\xi, Q^2)\right](x). \quad (2.8)$$

The parton-level structure functions have been weighted by PDFs $\tilde{f}_i(\xi)$ which describe the momentum distribution of the parton species i within the hadron, as a function of the momentum fraction ξ. This integral is the *Mellin convolution* of $\tilde{f}_i$ and $\hat{F}_{a,i}$, as defined in Eq. (A.9), which we will denote by the symbol $\otimes$. The result of the convolution is a function of x but we will suppress this in the following discussion, as well as the dependence of each of the convoluted functions on the convolution variable ξ.

The parton-level cross section can now be computed using the standard tools of perturbation theory, since we have separated the long-distance behaviour of the hadron from the hard interaction,

$$l(k) + p(\xi P) \to l'(k') + p'(\xi P + q), \quad (2.9)$$

as depicted in Fig. 2.2.

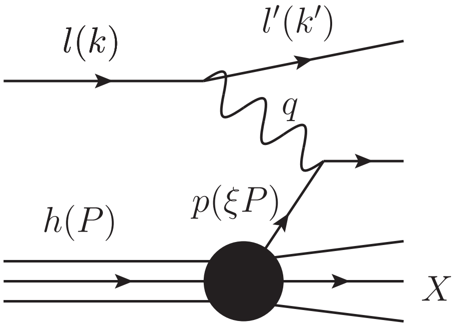

Figure 2.2: *Deep-Inelastic Scattering in the parton model. We assume that the lepton scatters from some parton within the hadron, which carries a fraction ξ of the hadron's momentum P.*

2.2 QCD Corrections to the Parton Model

We now identify the partons of the previous discussion with the quarks and gluons of QCD. Within the framework of perturbative QCD, we can compute corrections to the parton-level cross section as a series in the strong coupling constant $a_{\rm s} = \alpha_{\rm s}/4\pi$.

By making such corrections, we introduce divergences in the structure functions. These may originate in loop integrals or from so-called *mass singularities* (or *collinear singularities*), which occur when two particles become collinear. Consider Fig. 2.3, for which the quark propagator will have $(p-r)^2$ in its denominator (denoting the quark momentum by $p = \xi P$). Since we assume that the quarks are massless ($p^2 = 0$), this is equal to $-2|\vec{p}||\vec{r}|(1-\cos\vartheta)$. As $\vartheta \to 0$ this denominator $\to 0$, producing a singularity in the amplitude. Such singularities involving *final-state* particles and infra-red singularities due to loop integrals cancel since we consider only *inclusive* DIS and thus sum over all possible final states. That these singularities cancel is guaranteed by the Kinoshita-Lee-Nauenberg (KLN) theorem [7,8].

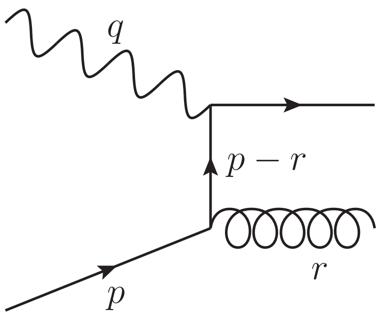

Figure 2.3: *QCD corrections to the boson-parton interaction produce singularities in the amplitude when particles become collinear with initial- or final-state particles.*

The structure functions appear directly in the expression for the cross section; they must therefore be finite, since the cross section is an experimentally measurable quantity. The parton-level cross section must be renormalized in order to obtain physically meaningful predictions. We use the framework of *dimensional regularization* [9], in which we work in $D = 4 - 2\varepsilon$ dimensions. The divergences described above (from *initial-state* mass singularities and ultraviolet singularities of loop integrals) manifest as poles in ε in the limit $\varepsilon \to 0$ (so $D \to 4$). An arbitrary scale μ^2 is introduced to keep the strong coupling constant, $\alpha_{\rm s,bare}$, dimensionless. From Eq. (2.8) we have that (omitting the sum over j and the factors of e_j^2)

$$F_a(x, Q^2) = \hat{F}_{a,j}\left(a_{\rm s,bare}, \frac{Q^2}{\mu^2}, \varepsilon\right) \otimes \tilde{f}_j, \tag{2.10}$$

where $\hat{F}_{a,i}$ has picked up dependence on μ^2 and ε, and $a_{\rm s,bare}$ denotes the un-renormalized strong coupling.

The first step in our renormalization procedure is to renormalize the coupling constant $a_{\rm s,bare}$. This removes the ultraviolet divergences due to the loop integrals, introducing a renormalization scale $\mu_{\rm r}^2$. The relation between the bare and renormalized coupling is given by

$$a_{\rm s,bare} = a_{\rm s} - a_{\rm s}^2\frac{\beta_0}{\varepsilon} + a_{\rm s}^3\left(\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{\varepsilon}\right) - a_{\rm s}^4\left(\frac{\beta_0^3}{\varepsilon^3} - \frac{7\beta_0\beta_1}{6\varepsilon^2} + \frac{\beta_2}{3\varepsilon}\right) + \mathcal{O}(a_{\rm s}^5), \qquad (2.11)$$

where the coefficients of the QCD beta-function are given to fourth order in $a_{\rm s}$ by [10, 11, 12, 13]

$$\begin{aligned}\beta_{QCD} &= -a_{\rm s}\varepsilon - a_{\rm s}^2\beta_0 - a_{\rm s}^3\beta_1 - a_{\rm s}^4\beta_2 + \mathcal{O}(a_{\rm s}^5),\\ \beta_0 &= \frac{11}{3}C_A - \frac{2}{3}n_f\\ \beta_1 &= \frac{34}{3}C_A^2 - 2C_F n_f - \frac{10}{3}C_A n_f\\ \beta_2 &= \frac{2857}{54}C_A^3 + C_F^2 n_f - \frac{205}{18}C_F C_A n_f - \frac{1415}{54}C_A^2 n_f + \frac{11}{9}C_F n_f^2 + \frac{79}{54}C_A n_f^2. \qquad (2.12)\end{aligned}$$

The coefficients of the next two terms of this expansion, β_3 and β_4, have been computed in [14, 15, 16] but they are not required by the calculations of this thesis. The $SU(N)$ fundamental and adjoint Casimirs C_F and C_A have the values 4/3 and 3 in QCD. n_f is the number of participating massless quark flavours. Setting the arbitrary scale μ^2 of dimensional regularization to $\mu_{\rm r}^2$, Eq. (2.10) becomes

$$F_a(x, Q^2) = \hat{F}_{a,j}\left(a_{\rm s}(\mu_{\rm r}^2), \frac{Q^2}{\mu_{\rm r}^2}, \varepsilon\right) \otimes \tilde{f}_j. \qquad (2.13)$$

The only remaining divergences are due to the collinearity of initial state particles. We deal with these using *mass factorization.* We assume that one can factorize $\hat{F}_{a,i}$ into two functions, one which is finite in the $\varepsilon \to 0$ limit and one which contains the poles. This is not a unique procedure; rather it depends on a *factorization scheme* which specifies exactly what is to be included in each function. We have that

$$F_a(x, Q^2) = C_{a,i}^{\rm scheme}\left(a_{\rm s}(\mu_{\rm r}^2), \frac{Q^2}{\mu_{\rm r}^2}, \frac{\mu_{\rm f}^2}{\mu_{\rm r}^2}, \varepsilon\right) \otimes Z_{ij}^{\rm scheme}\left(a_{\rm s}(\mu_{\rm r}^2), \frac{\mu_{\rm f}^2}{\mu_{\rm r}^2}, \frac{1}{\varepsilon}\right) \otimes \tilde{f}_j, \qquad (2.14)$$

where $C_{a,i}^{\rm scheme}$ is called a *coefficient function* (sometimes also a *Wilson coefficient*) and $Z_{ij}^{\rm scheme}$ a *renormalization matrix* (sometimes also a *transition function*). The separation occurs at a scale $\mu_{\rm f}^2$. The dependence of Z_{ij} on $1/\varepsilon$ is to denote that Z_{ij} contains only pole terms in ε.

The simplest choice of factorization scheme is called *Minimal Subtraction* (MS) [17], in which we absorb *only* the ε-pole terms of $\hat{F}_a$ into the renormalization matrix. Throughout this thesis, we use the *Modified* Minimal Subtraction scheme ($\overline{\rm MS}$) in which we also absorb ubiquitous factors of $\ln 4\pi$ and γ_E (the Euler-Mascheroni constant) into $Z_{ij}^{\overline{\rm MS}}$. From here on we will not typeset the scheme tags on the coefficient functions and renormalization matrices, but one should bear in mind that these functions always depend on this choice. One should also bear in mind, then, that throughout this thesis where we use the symbol ε we in fact mean some $\varepsilon'(\varepsilon, \ln 4\pi, \gamma_E)$ which $\to 0$ as $\varepsilon \to 0$.

We are free to set (without loss of generality, in the sense that the scale dependence can be restored in the results, see for e.g. [18]) the arbitrary renormalization and factorization scales μ_{r}^2 and μ_{f}^2 to the energy scale Q^2, yielding

$$F_a(x,Q^2) = C_{a,i}\left(a_{\mathrm{s}}(Q^2),\varepsilon\right) \otimes Z_{ij}\left(a_{\mathrm{s}}(Q^2),\frac{1}{\varepsilon}\right) \otimes \tilde{f}_j. \tag{2.15}$$

We can now renormalize the "bare" PDFs $\tilde{f}_j$ in such a way that the renormalization matrix is absorbed into their definition, leaving us with a finite expression for the hadron structure functions as $\varepsilon \to 0$. That is, we define the renormalized (finite, but scheme-dependent) PDF

$$f_i\left(a_{\mathrm{s}}(Q^2)\right) = Z_{ij}\left(a_{\mathrm{s}}(Q^2),\frac{1}{\varepsilon}\right) \otimes \tilde{f}_j \tag{2.16}$$

and so

$$F_a(x,Q^2) = C_{a,i}\left(a_{\mathrm{s}}(Q^2),\varepsilon\right) \otimes f_i\left(a_{\mathrm{s}}(Q^2)\right). \tag{2.17}$$

In the equations above Z_{ij} and by extension the renormalized PDF f_i do not carry the label "a" of the structure functions. This is an important point; although we are describing DIS here, we claim that *all* interactions with hadrons should depend on these *universal* PDFs. When we determine QCD corrections to Z_{ij} we are computing quantities that are useful not just in DIS, but in *all* hadron interactions.

Comparing Eq. (2.10) and Eq. (2.17), we can see that this procedure has introduced a dependence on a_{s} of the PDF f_i, which the bare PDF $\tilde{f}_j$ did not have. We have said that the PDF is non-perturbative and so cannot be computed, but we can describe the dependence of f_i on the energy scale Q^2. Suppressing all function arguments, we have that

$$\frac{\mathrm{d}f_i}{\mathrm{d}\ln Q^2} = \frac{\mathrm{d}Z_{ij}}{\mathrm{d}\ln Q^2} \otimes \tilde{f}_j = \frac{\mathrm{d}Z_{ik}}{\mathrm{d}\ln Q^2} \otimes \left({Z^{-1}}_{kj} \otimes f_j\right) = \underbrace{\left[\frac{\mathrm{d}Z_{ik}}{\mathrm{d}\ln Q^2} \otimes {Z^{-1}}_{kj}\right]}_{P_{ij}} \otimes f_j, \tag{2.18}$$

where the evolution kernels $P_{ij}(a_{\mathrm{s}}(Q^2))$ are the *Splitting Functions* of QCD. Equation (2.18) is called the *DGLAP evolution equation* in the literature [19, 20, 21]. We mentioned above that the PDFs are universal to all hadron interactions, so the splitting functions must be also. Using Eq. (2.18) one can take a PDF determined from the experimental measurement of the structure functions at a particular energy scale, and evolve it to a different energy scale for use with, say, a different experiment.

If we perform a Mellin transform of (any of) the above equations the convolutions reduce to simple products, somewhat simplifying the notation. See Appendix A.3 for a definition and discussion of the Mellin transform. In Mellin space, we define the *anomalous dimensions* γ_{ij} of the PDFs as (in line with the historic convention)

$$\frac{\mathrm{d}f_i(N)}{\mathrm{d}\ln Q^2} = P_{ij}(N) f_j(N) = -\gamma_{ij}(N) f_j(N), \tag{2.19}$$

and we will use the terms "splitting function" and "anomalous dimension" interchangeably throughout this thesis. The splitting functions/anomalous dimensions can be expanded in the QCD coupling, with coefficients defined by

$$P_{ij}(x,a_\mathrm{s}) = \sum_{n=1}^{\infty} a_\mathrm{s}^n\, P_{ij}^{(n-1)}(x) \qquad \text{and} \qquad \gamma_{ij}(N,a_\mathrm{s}) = \sum_{n=1}^{\infty} a_\mathrm{s}^n\, \gamma_{ij}^{(n-1)}(N). \tag{2.20}$$

The coefficient functions of Eq. (2.17) can also be expanded in a_s, and additionally in positive powers of ε. We define the expansions of the coefficient functions as

$$C_a\left(x, a_\mathrm{s}(Q^2)\right) = \sum_{i=0}^{\infty} a_\mathrm{s}^i\, c_a^{(i)}(x) \tag{2.21}$$

and

$$C_a\left(x, a_\mathrm{s}(Q^2), \varepsilon\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_\mathrm{s}^i\, \varepsilon^j\, c_a^{(i,j)}(x), \tag{2.22}$$

where in Eq. (2.21) the dimensional regularization parameter ε has been set to 0. Note that the arguments x in the expansions above are convoluted over, as in Eq. (2.8). They are not the Bjorken-x variable, but we nonetheless call them x in line with the literature.

From Eq. (2.18) and Eq. (2.19) it follows that

$$-\gamma_{ij} = \frac{\mathrm{d}Z_{ik}(N)}{\mathrm{d}\ln Q^2} Z^{-1}(N)_{kj} = \beta(a_\mathrm{s})\frac{\mathrm{d}Z_{ik}(N)}{\mathrm{d}a_\mathrm{s}} Z^{-1}(N)_{kj}, \tag{2.23}$$

where we have used that $\mathrm{d}a_\mathrm{s}/\mathrm{d}\ln Q^2 = \beta(a_\mathrm{s})$. Equation (2.23) can be solved order-by-order in a_s to determine Z_{ij} in terms of the expansion coefficients of γ_{ij}. The result to a_s^4 is as follows,

$$\begin{aligned}
Z = {} & 1 + a_\mathrm{s}\frac{1}{\varepsilon}\gamma^{(0)} \\
& + a_\mathrm{s}^2\Big\{\frac{1}{2\varepsilon^2}(\gamma^{(0)}-\beta_0)\gamma^{(0)} + \frac{1}{2\varepsilon}\gamma^{(1)}\Big\} \\
& + a_\mathrm{s}^3\Big\{\frac{1}{6\varepsilon^3}(\gamma^{(0)}-\beta_0)(\gamma^{(0)}-2\beta_0)\gamma^{(0)} \\
& \qquad + \frac{1}{6\varepsilon^2}\Big[(\gamma^{(0)}-2\beta_0)\gamma^{(1)} + (\gamma^{(1)}-\beta_1)2\gamma^{(0)}\Big] + \frac{1}{3\varepsilon}\gamma^{(2)}\Big\} \\
& + a_\mathrm{s}^4\Big\{\frac{1}{24\varepsilon^4}(\gamma^{(0)}-\beta_0)(\gamma^{(0)}-2\beta_0)(\gamma^{(0)}-3\beta_0)\gamma^{(0)} \\
& \qquad + \frac{1}{24\varepsilon^3}\Big[(\gamma^{(0)}-2\beta_0)(\gamma^{(0)}-3\beta_0)\gamma^{(1)} + (\gamma^{(0)}-3\beta_0)(\gamma^{(1)}-\beta_1)2\gamma^{(0)} \\
& \qquad + (\gamma^{(0)}-\beta_0)(\gamma^{(1)}-2\beta_1)3\gamma^{(0)}\Big] + \frac{1}{24\varepsilon^2}\Big[(\gamma^{(0)}-3\beta_0)2\gamma^{(2)} \\
& \qquad + (\gamma^{(1)}-2\beta_1)3\gamma^{(1)} + (\gamma^{(2)}-\beta_2)6\gamma^{(0)}\Big] + \frac{1}{4\varepsilon}\gamma^{(3)}\Big\} + \mathcal{O}(a_\mathrm{s}^5),
\end{aligned} \tag{2.24}$$

where the symbols are to be interpreted as matrices and the arguments (N) have been suppressed. To perform the mass factorization, one equates an unfactorized parton-level structure function $\hat{F}_a(N, a_\mathrm{s}, \varepsilon)$ (which contains poles in ε) with the product $C_{a,i}(N, a_\mathrm{s}, \varepsilon)\, Z_{ij}(N, a_\mathrm{s}, \frac{1}{\varepsilon})$. Order-by-order in a_s, the anomalous dimension expansion coefficients are determined from the coefficients of the ε poles of $\hat{F}_a(N, a_s, \varepsilon)$ and the coefficient function expansion coefficients from the remaining finite terms. This matrix equation forms a system of equations which must be mass factorized together.

In this way, high order corrections to the DIS-specific coefficient functions and the universal anomalous dimensions/splitting functions are determined from the perturbative computation of the parton-level structure functions of DIS.

2.3 Parton Distribution Functions

We now discuss what "parton species" are present in the hadron, i.e. what values the sum over i in Eq. (2.17) should run over and what the PDFs f_i are. In principle, the hadron has PDFs associated with all quarks f_i, anti-quarks $\bar{f}_i$ and with the gluon g. This makes the matrix equation of DGLAP evolution (Eq. (2.19)) a system of $(2n_f + 1)$ coupled equations, where n_f denotes the number of (massless) quarks considered. In the CC case of Chapter 3 we take n_f to be even due to the considerations of Section 2.4. We can simplify the description somewhat by noting a few symmetries.

Quark-gluon and gluon-quark splittings are independent of the quark flavour. We must have then, that $P_{q_i g} = P_{q_j g} = P_{\bar{q}_i g} = P_{\bar{q}_j g}$ and so define

$$P_{qg} = n_f P_{q_i g} = n_f P_{\bar{q}_i g} \tag{2.25}$$

(i.e. a gluon splitting to one of n_f quark-anti-quark pairs) and that

$$P_{gq} = P_{gq_i} = P_{g\bar{q}_i} \tag{2.26}$$

(any quark flavour radiates a gluon in the same way). By defining the *singlet distribution*

$$q_s = \sum_{i=1}^{n_f} f_i + \bar{f}_i, \tag{2.27}$$

the DGLAP evolution equation can be reduced to a system of just two coupled equations,

$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2} \begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix}. \tag{2.28}$$

Differences in quark and anti-quark PDFs additionally must decouple from the gluon PDF during evolution. We form three different so-called *non-singlet* combinations which evolve independently,

$$q_{ns}^{\mathrm{V}} = \sum_{i=1}^{n_f} f_i - \bar{f}_i \qquad \text{and} \qquad q_{ns,ij}^{\pm} = f_i \pm \bar{f}_i - (f_j \pm \bar{f}_j), \tag{2.29}$$

the *valence* and *flavour-asymmetric* distributions. Their evolution is governed by the non-singlet splitting functions P_{ns}^{V} and $P_{ns}^{\pm}$.

With the exception of q_{ns}^{V} (called non-singlet to align with most of the literature), the labels "singlet" and "non-singlet" refer to the transformation properties of the PDFs under the SU(n_f) flavour symmetry of massless QCD. The PDFs q_s and q_{ns}^{V} are invariant under the switching of up- and down-type quarks and anti-quarks. The PDFs $q_{ns,ij}^{\pm}$ rather pick up a $(-)$ sign under such a switch and are thus called non-singlet.

The mass-factorized structure functions defined by Eq. (2.17) can be written in terms of these PDFs. For example, considering just the u and d quarks in the electromagnetic case,

$$F_a = C_{a,q}\left(\frac{4}{9}(u+\bar{u}) + \frac{1}{9}(d+\bar{d})\right) + \langle e^2\rangle C_{a,g}\, g \tag{2.30}$$

which can be rearranged to give

$$\begin{aligned} F_a &= \frac{5}{18}C_{a,q}(u+\bar{u}+d+\bar{d}) + \frac{1}{6}C_{a,ns}\left(u+\bar{u}-(d+\bar{d})\right) + \frac{5}{18}C_{a,g}\, g \\ &= \langle e^2\rangle C_{a,q}\, q_s + \frac{1}{6}C_{a,ns}\, q_{ns,ud}^{+} + \langle e^2\rangle C_{a,g}\, g, \end{aligned} \tag{2.31}$$

where $\langle e^2\rangle$ denotes the average squared charge of the participating quarks. The coefficient function associated with $q_{ns,ud}^{+}$ has inherited the "ns" label, and is not equal to $C_{2,q}$ at higher orders. The various structure functions that we consider later can be written in terms of the four PDF combinations defined in Eq. (2.27) and Eq. (2.29).

2.4 The Optical Theorem and Forward Compton Amplitudes

In the preceding sections, we have defined a framework which describes DIS processes. It separates the non-perturbative physics of the hadronic bound state from the perturbative hard scattering of the lepton and a constituent parton within the hadron. This allows us to consider QCD corrections to the hard interaction, and we have discussed how one can renormalize these parton-level hard scattering cross sections. We now discuss how we will compute them in the framework of massless perturbative QCD.

We proceed, not by squaring amplitudes and computing phase-space integrals, but via the *optical theorem*. This relates the squared amplitude to a *forward Compton amplitude*, shown in Fig. 2.4. We define this forward Compton amplitude $\hat{T}_{\mu\nu}$ (hatted quantities still denote parton-level objects) such that

$$\hat{W}_{\mu\nu} = \frac{1}{\pi}\mathrm{Im}\hat{T}_{\mu\nu}. \tag{2.32}$$

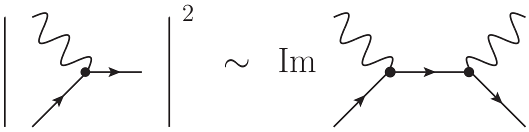

Figure 2.4: *The optical theorem relates a squared matrix element to the imaginary part of a forward Compton amplitude.*

Instead of the phase-space integrals of the usual description, we now must compute loop integrals. $\hat{T}_{\mu\nu}$ has the same tensor decomposition as $W_{\mu\nu}$ (Eq. (2.6)),

$$\hat{T}_{\mu\nu}(z,Q^2) = \left(-g_{\mu\nu} - p_\mu p_\nu \frac{4z^2}{q^2} - (p_\mu q_\nu + p_\nu q_\mu)\frac{2z}{q^2}\right)\frac{1}{2z}\hat{T}_2(z,Q^2) + \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{1}{2z}\hat{T}_L(x,Q^2) + i\epsilon_{\mu\nu\rho\sigma}\frac{p^\rho q^\sigma}{2p\cdot q}\hat{T}_3(z,Q^2), \tag{2.33}$$

written in terms of the parton momentum $p = \xi P$ and the parton-level Bjorken variable $z = x/\xi$. The (forward) structure functions can be projected out of this tensor using the following projectors (in $D = 4 - 2\varepsilon$ dimensions),

$$\begin{aligned} \frac{1}{2z}\hat{T}_2 &= -\left(\frac{1}{(2-2\varepsilon)}g^{\mu\nu} + \frac{q^2}{(p\cdot q)^2}\frac{(3-2\varepsilon)}{(2-2\varepsilon)}p^\mu p^\nu\right)\hat{T}_{\mu\nu}, \\ \frac{1}{2z}\hat{T}_L &= -\frac{q^2}{(p\cdot q)^2}p^\mu p^\nu \hat{T}_{\mu\nu}, \\ \hat{T}_3 &= \frac{i}{(1-2\varepsilon)(1-\varepsilon)}\frac{\epsilon^{\mu\nu\rho'\sigma'}p_{\rho'}q_{\sigma'}}{p\cdot q}\hat{T}_{\mu\nu}. \end{aligned} \tag{2.34}$$

It is instructive to consider the tree-level Compton amplitudes. Suppose we aim to compute the parton-level quark structure function $\frac{1}{2z}\hat{F}_{2,q}$. There are two contributing forward diagrams, shown in Fig. 2.5. Denoting the quark spinor as $u(p)$ (which can be

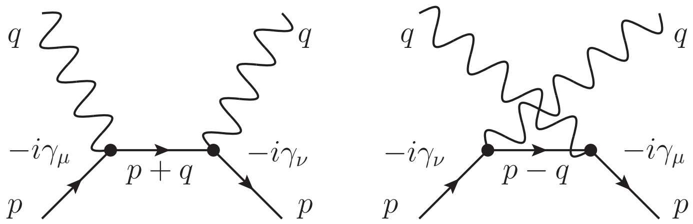

Figure 2.5: *The leading-order foward diagrams contributing to photon-quark scattering.*

any quark or anti-quark here), the contributions to the forward amplitude are

$$\bar{u}(p)(-i\gamma_\nu)\frac{\not{p}+\not{q}}{(p+q)^2}(-i\gamma_\mu)u(p) \quad \text{and} \quad \bar{u}(p)(-i\gamma_\mu)\frac{\not{p}-\not{q}}{(p-q)^2}(-i\gamma_\nu)u(p) \tag{2.35}$$

for the left and right (crossed) diagrams. The projector for $\frac{1}{2z}\hat{T}_{2,q}$ in Eq. (2.34) has two Lorentz structures. Contracting the second ($p^\mu p^\nu$) with Eq. (2.35) yields 0 since we take the quark to be massless ($p^2 = 0$) (this implies that $\frac{1}{2z}\hat{T}_{L,q} = 0$ at tree level, see below). Contracting (in 4 dimensions) with $-\frac{1}{2}g^{\mu\nu}$, averaging over the quark spin and tracing over the gamma matrices yields

$$\frac{1}{2z}\hat{T}_{2,q} = \frac{1}{2}\left(-\frac{4p\cdot q}{(p+q)^2} \quad + \quad \frac{4p\cdot q}{(p-q)^2}\right). \tag{2.36}$$

Assuming q^2 is large, we may expand the propagators to get

$$\begin{aligned}\frac{1}{2z}\hat{T}_{2,q} &= \frac{2p\cdot q}{q^2}\left(-\frac{1}{1+\frac{2p\cdot q}{q^2}} \quad + \quad \frac{1}{1-\frac{2p\cdot q}{q^2}}\right) \\ &= \sum_{N=0}^{\infty}\left(\frac{2p\cdot q}{q^2}\right)^{N+1}\left[-(-1)^N \quad + \quad 1\right],\end{aligned} \tag{2.37}$$

and using that the square-bracketed combination vanishes for even N,

$$\frac{1}{2z}\hat{T}_{2,q} = 2\sum_{\text{odd N}}\left(\frac{2p\cdot q}{q^2}\right)^{N+1} = 2\sum_{\text{even N}}\left(\frac{2p\cdot q}{q^2}\right)^{N} = 2\sum_{\text{even N}}\left(\frac{1}{z}\right)^{N}. \tag{2.38}$$

It now only remains to connect this expression with the parton-level structure function $\frac{1}{2z}\hat{F}_{2,q}$. In the kinematic region of DIS $0 < z < 1$ but the sum in Eq. (2.38) does not converge here. A dispersion relation in the complex z-plane allows us to analytically continue this result to the physical region of DIS, and determine the *even-N Mellin moments* of the structure function,

$$\hat{F}_{2,q}(N,Q^2) = \int_0^1 \mathrm{d}z z^{N-1}\frac{1}{2z}\hat{F}_{2,q}(z,Q^2), \tag{2.39}$$

as the coefficients of $2(1/z)^N$ in the sum of Eq. (2.38). See Appendix A.5 for a brief explanation. We have, then, that $\hat{F}_{2,q}(N,Q^2) = 1$ at tree level, or $\delta(1-z)$ in z-space.

We find that $\frac{1}{2z}\hat{T}_{L,q} = 0$ at tree level, since the projector produces only terms proportional to $p^2 = 0$ (this is the Callan-Gross relation). $\frac{1}{2z}\hat{T}_{3,q} = 0$ at all orders in a_{s}, since the antisymmetric ϵ-tensor of the projector (Eq. (2.34)) is contracted with a Lorentz structure that is symmetric in its indices.

In the CC case, the situation is a little different. For any given incoming quark or anti-quark, a crossed diagram (corresponding to the right-hand side of Fig. 2.5) must have the *oppositely-charged W boson* due to charge conservation at the vertices. The diagrams for an incoming d quark, for example, are shown in Fig. 2.6. The vertex factors are (proportional to) $-i\gamma_\rho P_L = -i\gamma_\rho(1-\gamma_5)/2$ for initial state quarks or $-i\gamma_\rho P_R = -i\gamma_\rho(1+\gamma_5)/2$ for initial state anti-quarks.

The contributions to the amplitudes of these two diagrams are proportional to

$$\bar{u}(p)(-i\gamma_\nu P_L)\frac{\not{p}+\not{q}}{(p+q)^2}(-i\gamma_\mu P_L)u(p) \quad \text{and} \quad \bar{u}(p)(-i\gamma_\mu P_L)\frac{\not{p}-\not{q}}{(p-q)^2}(-i\gamma_\nu P_L)u(p) \tag{2.40}$$

for initial state quarks and, for initial state anti-quarks,

$$v(p)(-i\gamma_\nu P_R)\frac{\not{p}+\not{q}}{(p+q)^2}(-i\gamma_\mu P_R)\bar{v}(p) \quad \text{and} \quad v(p)(-i\gamma_\mu P_R)\frac{\not{p}-\not{q}}{(p-q)^2}(-i\gamma_\nu P_R)\bar{v}(p). \tag{2.41}$$

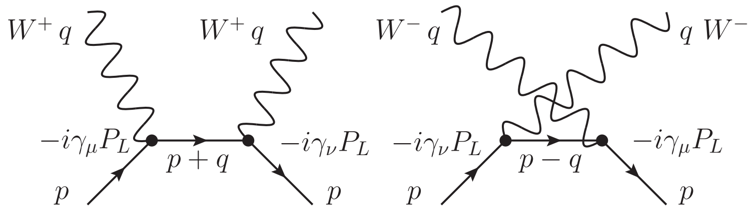

Figure 2.6: *The leading-order foward diagrams contributing to $W^\pm$-down-quark scattering.*

If we sum the two diagrams (so we compute the linear combination $W^+ + W^-$ of $W^\pm$ scattering), contract with $-\frac{1}{2}g^{\mu\nu}$, average over quark spin and trace over the gamma matrices we find (expanding the propagators as in Eq. (2.37))

$$\frac{1}{2z}\hat{T}^{W^+ + W^-}_{2,q} = \frac{1}{2}\left(-\frac{2p\cdot q}{(p+q)^2} + \frac{2p\cdot q}{(p-q)^2}\right) = \sum_{\text{even N}}\left(\frac{1}{z}\right)^N . \tag{2.42}$$

Again we find that $\frac{1}{2z}\hat{T}^{W^+ + W^-}_{L,q} = 0$ but now, due to the presence of the γ_5 matrix in the vertex factors, there is a non vanishing contribution to $\hat{T}^{W^+ + W^-}_{3,q}$ given by

$$\hat{T}^{W^+ + W^-}_{3,q} = \frac{1}{2}\left(-\frac{4p\cdot q}{(p+q)^2} - \frac{4p\cdot q}{(p-q)^2}\right) = 2\sum_{\text{odd N}}\left(\frac{1}{z}\right)^N , \tag{2.43}$$

and the same expression with an overall $(-)$ sign if we are considering an initial state anti-quark (due to the P_R in place of P_L in the vertex factors). Unlike the expressions for $\frac{1}{2z}\hat{T}_{2,q}$ and $\frac{1}{2z}\hat{T}^{W^+ + W^-}_{2,q}$ above, in the expression for $\hat{T}^{W^+ + W^-}_{3,q}$ the sum runs over *odd* values of N.

We can also form a linearly independent combination in which we subtract, rather than add, diagrams involving a W^- boson. Following the same steps as above, we have

$$\begin{aligned}
\frac{1}{2z}\hat{T}^{W^+ - W^-}_{2,q} &= \frac{1}{2}\left(-\frac{2p\cdot q}{(p+q)^2} - \frac{2p\cdot q}{(p-q)^2}\right) = \sum_{\text{odd N}}\left(\frac{1}{z}\right)^N , \\
\frac{1}{2z}\hat{T}^{W^+ - W^-}_{L,q} &= 0, \\
\hat{T}^{W^+ - W^-}_{3,q} &= \frac{1}{2}\left(-\frac{4p\cdot q}{(p+q)^2} + \frac{4p\cdot q}{(p-q)^2}\right) = 2\sum_{\text{even N}}\left(\frac{1}{z}\right)^N ,
\end{aligned} \tag{2.44}$$

where again, the expression for $\hat{T}^{W^+ - W^-}_{3,q}$ picks up an overall $(-)$ sign for an initial state anti-quark. The sums run over different N values for this $W^+ - W^-$ combination. One must consider these linear combinations $W^+ \pm W^-$ in order to map onto either even-N or odd-N Mellin moments. We are interested in both combinations, since in principle an experiment can determine which W boson was exchanged in an interaction, or be set up to only exchange one of the bosons.

The above considerations apply also at higher orders, and to the structure functions for interactions with a gluon inside the hadron. Although we do not discuss it here, one may also refer to the Operator Product Expansion (OPE) of the currents interacting with the parton. This procedure yields the same conclusions regarding the computation of even-N and odd-N Mellin moments for various structure functions. For a discussion see, for example, [22, 23, 24].

For the quark parton-level structure functions we define also the "non-singlet" (ns) coefficient functions, as briefly discussed below Eq. (2.31). In the $W^+ - W^-$ case there are *only* the ns parton-level structure functions, which are convoluted with the flavour-asymmetric PDFs $q^{\pm}_{ns,ij}$ of Eq. (2.29). These structure functions will carry an ns label throughout Chapter 3.

In summary then,

- For *Electromagnetic DIS*, one computes even-N Mellin moments for the parton-level structure functions $\hat{F}_{2,q}$, $\hat{F}_{2,g}$, $\hat{F}_{L,q}$, $\hat{F}_{L,g}$.
- For *CC DIS in the $W^+ + W^-$ combination*, one computes even-N Mellin moments for the parton-level structure functions $\hat{F}^{W^+ + W^-}_{2,q}$, $\hat{F}^{W^+ + W^-}_{2,g}$, $\hat{F}^{W^+ + W^-}_{L,q}$, $\hat{F}^{W^+ + W^-}_{2,g}$ and odd-N Mellin moments for $\hat{F}^{W^+ + W^-}_{3,q}$.
- For *CC DIS in the $W^+ - W^-$ combination*, one computes odd-N Mellin moments for the parton-level structure functions $\hat{F}^{W^+ - W^-}_{2,q}$, $\hat{F}^{W^+ - W^-}_{L,q}$ and even-N Mellin moments for $\hat{F}^{W^+ - W^-}_{3,q}$.
- For *Higgs-exchange DIS*, one computes even-N Mellin moments for the parton-level structure functions $\hat{F}_{\phi,q}$, $\hat{F}_{\phi,g}$, which are a useful theoretical probe; one considers the direct coupling of a scalar boson to the gluon. This allows the determination of the "lower row" of the splitting function matrix, P_{gq} and P_{gg} of Eq. (2.28).

The 3rd bullet point is the topic of Chapter 3, where we compute the third-order corrections to these $W^+ - W^-$ CC parton-level structure functions. The 1st, 2nd and 4th points are the topic of Chapter 5, where we compute Mellin moments of the fourth-order corrections to (parts of) these structure functions. Chapter Chapter 4 concerns coefficient functions and splitting functions related to the structure functions of the 1st, 3rd and 4th bullet points.

Chapter 3

Third-Order QCD Corrections to Charged-Current Deep-Inelastic Scattering

Full, analytic expressions for third-order QCD corrections to most quantities describing massless DIS have already been computed. Here we provide a set of references for the convenience of the reader. The non-singlet anomalous dimensions $\gamma_{ns}^{(2),\pm}$ and $\gamma_{ns}^{(2),\mathrm{v}}$ were computed in [25] and the whole singlet system $\gamma_{ij}^{(2)}$ in [26]. The corrections to the non-singlet and singlet coefficient functions $c_2^{(3)}$ and $c_L^{(3)}$ were presented in [27] and to the CC coefficient function $c_{3,ns}^{(3),W^+ + W^-}$ in [28]. The third-order corrections to the Higgs-exchange coefficient functions $c_\phi^{(3)}$ (which are useful only as a theoretical tool, as explained in Section 2.4) were presented in [29].

In this chapter we consider third-order corrections to the other CC combination, $W^+ - W^-$, as discussed in Section 2.4. Some calculations of these functions exist in the literature, in the form of numerical approximations. These are discussed in more detail in Section 3.1.1. The analytic computation of these coefficient functions presented in this chapter completes the third-order description of CC DIS in massless QCD. Some results of this chapter have been published in [2] and will be published in [3].

3.1 Introduction

Here we repeat some of the formalism outlined in Chapter 2. We define the structure functions for this "$W^+ - W^-$" case as follows,

$$\begin{aligned} F_{i,ns}^{W^+-W^-} &= C_{i,ns}^{W^+-W^-} \otimes Z_{ns}^- \otimes \tilde{q}_{ns}^- = C_{i,ns}^{W^+-W^-} \otimes q_{ns}^-, \qquad (i = 2, L) \\ F_{3,ns}^{W^+-W^-} &= C_{3,ns}^{W^+-W^-} \otimes Z_{ns}^+ \otimes \tilde{q}_{ns}^+ = C_{3,ns}^{W^+-W^-} \otimes q_{ns}^+. \end{aligned} \tag{3.1}$$

The anomalous dimensions $\gamma_{ns}^{\pm}$ are defined in terms of the $Z_{ns}^{\pm}$ matrices,

$$-\gamma_{ns}^{\pm} = \frac{\mathrm{d}Z_{ns}^{\pm}}{\mathrm{d}\ln Q^2} \left(Z_{ns}^{\pm}\right)^{-1}. \tag{3.2}$$

We define the short-hand $C_{i,ns}^{\pm} = C_{i,ns}^{W^+ \pm W^-}$ for use throughout this chapter. It should be mentioned that this notation is different to some of the literature. For example, [28] defines $C_{3,ns}^{W^+ + W^-} = C_{3,-}$, where the $(-)$ label is referring to the fact that the quantity is based on odd-N Mellin moments. Throughout this thesis, the label rather refers to whether the coefficient function is for the $W^+ + W^-$ or $W^+ - W^-$ combination. The anomalous dimensions $\gamma_{ns}^{\pm}$ are both already known to third order in $a_{\rm s}$ [25] (the label of $\gamma_{ns}^{\pm}$ does *not* refer to the combination $W^+ \pm W^-$ but to their evolution of the PDFs $q_{ns}^{\pm}$, see Eq. (3.1)). The calculation outlined here will reproduce them, providing a strong check of the consistency of the results for the coefficient functions.

There are a few aspects of renormalization relevant here which were not discussed in Chapter 2. The general procedure is much the same; we renormalize the strong coupling $a_{\rm s,bare}$ and mass factorize the remaining (collinear) poles in ε into the bare PDF, producing a physical result. The structure function $F_{3,ns}^{W^+ - W^-}$, however, is slightly more complicated (as would be $F_{3,ns}^{W^+ + W^-}$). As discussed in Section 2.4, the vertices in the diagrams for this structure function contain the projectors $\frac{1}{2}(1-\gamma_5)$ or $\frac{1}{2}(1+\gamma_5)$ where the $W^{\pm}$ bosons couple to quarks or anti-quarks. One must consider carefully how to treat the intrinsically 4-dimensional γ_5 in $D = 4 - 2\varepsilon$ dimensions. Here, as in for e.g. [28,30,31,32], we use the "Larin scheme" [33] in which one makes the replacement

$$\gamma_\mu \gamma_5 \to \frac{i}{6} \epsilon_{\mu\nu\rho\sigma} \gamma^\nu \gamma^\rho \gamma^\sigma. \tag{3.3}$$

This can be contracted in the usual way with the ϵ-tensor of the projector of Eq. (2.34), outside of the D-dimensional renormalization operation, yielding contractions of the metric tensor which can be defined in D dimensions. The use of this scheme violates the axial Ward identity, incurring the additional ($\overline{\rm MS}$) renormalization factors Z_A and Z_5. These factors are computed to $a_{\rm s}^3$ in [33,34] and are given by

$$\begin{aligned} Z_A = 1 &+ \frac{a_{\rm s}^2}{\varepsilon}\left(\frac{22}{3}C_A C_F - \frac{4}{3}n_f C_F\right) + 64 a_{\rm s}^3 \left(\frac{C_F}{432\varepsilon^2}\left[44 C_A n_f - 121 C_A^2 - 4n_f^2\right]\right. \\ &\left. + \frac{C_F}{2592\varepsilon}\left[1789 C_A^2 - 1386 C_F C_A + 144 C_F n_f\right] - 416 C_A n_f + 4 n_f^2\right) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} Z_5 = 1 &- a_{\rm s} C_F (4 + 10\varepsilon + [22 - 2\zeta_2]\varepsilon^2) + a_{\rm s}^2 C_F \Big(22 C_F - \frac{107}{9} C_A + \frac{2}{9} n_f \\ &+ \varepsilon\Big[C_F(132 - 48\zeta_3) + C_A(-\frac{7229}{54} + 48\zeta_3) + n_f \frac{331}{27}\Big]\Big) \\ &+ 64 a_{\rm s}^3 \Big(C_F^3(-\frac{185}{96} + \frac{3}{2}\zeta_3) + C_F^2 C_A(\frac{2917}{864} - \frac{5}{2}\zeta_3) + C_F C_A^2(-\frac{2147}{1728} + \frac{7}{8}\zeta_3) \\ &+ C_F^2 n_f(-\frac{31}{864} - \frac{1}{6}\zeta_3) + C_F C_A n_f(\frac{89}{1296} + \frac{1}{6}\zeta_3) + \frac{13}{1296} C_F n_f^2\Big). \end{aligned} \tag{3.5}$$

See [35] for discussion on the implementation of this scheme in a computationally efficient manner. After multiplication by these factors, one may proceed with the mass factorization of the parton-level structure functions.

3.1.1 Existing Results

CC DIS in the $W^+ - W^-$ combination has been fully computed only to the second order in a_s [23, 30]. First results on the third-order corrections were obtained in [36], in the form the first five odd-N (even-N) Mellin moments of the third-order coefficient function contributions $c_{2,ns}^{(3),W^+-W^-}$ and $c_{L,ns}^{(3),W^+-W^-}$ ($c_{3,ns}^{(3),W^+-W^-}$). Of course, an inverse Mellin transform to produce an x-space expression requires not just a few moments but the analytic all-N dependence of the function.

However, given a few moments one can produce an approximation of the exact x-space result by choosing a suitable functional ansatz and fitting coefficients to reproduce the known moments. This procedure is performed in [37]. The sixth moments of the $W^+ - W^-$ coefficient functions were presented in [38] and used as a verification of the approximations. At large values of x, such approximations prove to be reasonably accurate.

To second order, these coefficient functions and their opposite-sign ($W^+ + W^-$) counterparts have the same large-x behaviour. It is helpful to define and consider the *differences* between the $W^+ + W^-$ and $W^+ - W^-$ coefficient functions, which are therefore suppressed at large x. The approximations of [37] are made, not directly to the $W^+ - W^-$ coefficient functions, but to these differences. We define

$$\begin{aligned} \delta C_i &= C_i^{W^+ + W^-} - C_i^{W^+ - W^-} \qquad (i = 2, L), \\ \delta C_3 &= C_3^{W^+ - W^-} - C_3^{W^+ + W^-}, \end{aligned} \tag{3.6}$$

where we always form the difference as the even-N minus the odd-N quantity. This difference must be formed in x space after the appropriate inverse Mellin transform of the even-N and odd-N parts. Additionally, these differences are formed with the caveat that the so-called "fl_{02}"-flavour-class diagrams (in which both bosons couple to a closed, internal quark loop, see Fig. 3.1) of the $C_{3,ns}^{W^+ + W^-}$ coefficient function are removed. This flavour class does not contribute in the $W^+ - W^-$ case, which is proportional only to the flavour asymmetric PDFs.

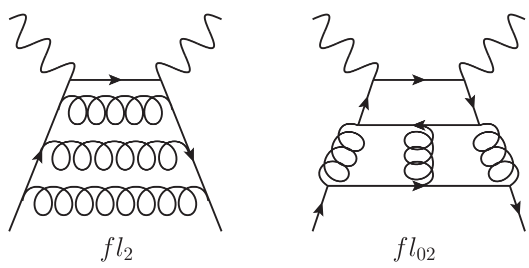

Figure 3.1: *Representative three-loop diagrams for the diagram classes fl_2 and fl_{02} of CC DIS. In fl_{02} diagrams neither boson couples to the external quark line; they both couple to the same internal quark loop.*

These approximations are plotted in Figs. 3.2 and 3.3. They have been used in the analysis of [39] in which the $\mathrm{N^3LO}$ corrections to the cross-section for Higgs production via vector-boson fusion have been estimated. Although authors state that the additional uncertainty in the cross-section incurred due to the use of these approximations is very small, an exact result is always preferable if possible. There has also been a lot of recent progress on the computation of massive quark corrections to DIS, see [40] for an overview. These computations require knowledge of the massless coefficient functions, motivating their complete calculation here.

3.2 Software and Calculation

We take a moment here to outline the software used in the computation of these third-order corrections. Due to the large number of Feynman diagrams contributing at this order, much automation is required. First the diagrams are generated by `QGRAF` [41], which produces 3633 diagrams. These are further processed by a `FORM` [42] script known as `convdia`, whose role is to simplify the `QGRAF` output and bring it into a form suitable for further processing. (Throughout this thesis, references to `FORM` really mean scripts run with the parallel implementation, `TFORM` [43], which provides large reductions in wall-time when running on multi-core computers.) Where possible, diagrams are combined into "meta-diagrams"; collections of diagrams of the same topology, colour factors and flavour class, differing only in the particle type of various lines. This procedure produces just 233 meta-diagrams for each of $F_{2,ns}^{W^+-W^-}$ and $F_{L,ns}^{W^+-W^-}$ and 198 meta-diagrams for $F_{3,ns}^{W^+-W^-}$, greatly reducing the time required to complete computations.

These meta-diagrams are the input for further tools. `MINCER` [44, 45] is a package which computes Mellin moments of the parton-level structure functions for fixed values of N. The diagram database used here is much smaller than that of [36] due to a greatly improved version of `convdia`. Between this and access to more significant computational resources, we have extended the fixed-moment `MINCER` calculation from the first 6 to the first 15 moments of each of the $W^+ - W^-$ structure functions. As well as to verify this new, smaller, diagram database against the previous calculations, these moments were used to attempt a reconstruction of the all-N expressions in the style of Chapter 5. This approach was unsuccessful for the most difficult terms (those proportional to n_f^0) and all discussion of this method is deferred until Chapter 5 where it is used to reconstruct other quantities.

The diagram database was also used with an in-house "all-N" code, which can compute an analytic result directly from the diagrams. This is the code which was used in other third-order computations of DIS structure functions and is described briefly in [27]. This code is what ultimately completed the calculations here, although the colours factors that were successfully reconstructed from Mellin moments of course agree with the full results. The `MINOS` database facility [46] handles the automation of both these and the `MINCER` calculations of the diagram sets.

3.3 Results

Here we present the results of the calculations outlined in the previous sections. To reduce the length of the typeset expressions, we present only the even-N−odd-N differences as defined in Eq. (3.6). The exact expressions are nonetheless rather lengthy and are deferred until Appendix A.7, in which they are presented in x space in terms of the harmonic polylogarithms defined in Appendix A.2.

The parametrizations presented here are accurate to within 0.1% of the exact expressions for $x \in (10^{-6}, 1)$, to within 1% for $x \in (10^{-8}, 10^{-6})$ and to within 3% for $x \in (10^{-10}, 10^{-8})$. They are not intended for use outside of this range. The n_f dependence is retained as a symbol and the colour factors C_A and C_F are set to their SU(3) values of 3 and 4/3 respectively. We also define the following abbreviations,

$$\begin{aligned} X_1 &= (1-x), \\ L_1 &= \log X_1 = \log(1-x), \\ L_0 &= \log x, \end{aligned} \tag{3.7}$$

to make the typesetting a little more compact. These parametrizations are obtained by choosing a suitable x-space functional form (small-x and large-x logarithms and interpolating polynomial terms) and fitting the coefficients to the exact result using `MINUIT` [47].

$$\begin{aligned} \delta c^{(3)}_{2,ns} = &+\big(+ 273.59 - 44.95x - 73.56x^2 + 40.68x^3 + 0.1356L_0^5 + 8.483L_0^4 \\ &\quad + 55.90L_0^3 + 120.67L_0^2 + 388.0L_0 - 329.8L_0L_1 - xL_0(316.2 + 71.63L_0) \\ &\quad + 46.30L_1 + 5.447L_1^2\big)X_1 \\ &+\big(- 19.093 + 12.97x + 36.44x^2 - 29.256x^3 - 0.76L_0^4 - 5.317L_0^3 - 19.82L_0^2 \\ &\quad - 38.958L_0 - 13.395L_0L_1 + xL_0(14.44 + 17.74L_0) + 1.395L_1\big)X_1 n_f \\ &+\big(- 0.0008 + 0.0001n_f\big)\delta(1-x), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \delta c^{(3)}_{L,ns} = &+\big(- 620.53 - 394.5x + 1609x^2 - 596.2x^3 + 0.217L_0^3 + 62.18L_0^2 + 208.47L_0 \\ &\quad - 482.5L_0L_1 - xL_0(1751 - 197.5L_0) + 105.5L_1 + 0.442L_1^2\big)X_1^2 \\ &+\big(- 6.500 - 12.435x + 23.66x^2 + 0.914x^3 + 0.015L_0^3 - 6.627L_0^2 - 31.91L_0 \\ &\quad - xL_0(5.711 + 28.635L_0)\big)X_1^2 n_f, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \delta c^{(3)}_{3,ns} = &+\big(- 553.5 + 1412.5x - 990.3x^2 + 361.1x^3 + 0.1458L_0^5 + 9.688L_0^4 \\ &\quad + 90.62L_0^3 + 83.684L_0^2 - 602.32L_0 - 382.5L_0L_1 - xL_0(2.805 + 325.92L_0) \\ &\quad + 133.5L_1 + 10.135L_1^2\big)X_1 \\ &+\big(- 16.777 + 77.78x - 24.81x^2 - 28.89x^3 - 0.7714L_0^4 - 7.701L_0^3 \\ &\quad - 21.522L_0^2 - 7.897L_0 - 16.17L_0L_1 + xL_0(43.21 + 67.04L_0) \\ &\quad + 1.519L_1\big)X_1 n_f \\ &+\big(- 0.0029 + 0.00006n_f\big)\delta(1-x), \end{aligned} \tag{3.10}$$

The delta-functions enhance the accuracy if these expressions are used to compute approximate Mellin moments or convolutions, which require numerical integrations up to the $x = 1$ endpoint.

The QCD corrections to the Paschos-Wolfenstein relation, which we will discuss briefly in Section 3.4, require the second ($N = 2$) Mellin moment of these coefficient function differences. As discussed in Section 2.4, the even-N moments of $C_{2,ns}^{W^+-W^-}$ and $C_{L,ns}^{W^+-W^-}$ are not directly accessible to these N-space computations. Since we now have the *exact* x-space expressions for these coefficient functions we are able to perform an even-N Mellin transform and thus obtain the "unnatural" even-N moments. The exact expressions at $N = 2$ are given by

$$\begin{aligned}
\delta c^{(3)}_{2,ns}(N=2) = &+ C_F\bigg(- \frac{1496939}{43740} - \frac{4958}{243}\zeta_2 + \frac{160852}{405}\zeta_3 + \frac{4520}{9}\zeta_3\zeta_2 - \frac{8}{3}\zeta_3^2 \\
&\qquad - \frac{309253}{135}\zeta_4 + 48\zeta_4\zeta_2 + \frac{15616}{9}\zeta_5 - \frac{7093}{9}\zeta_6\bigg) \\
&+ C_A\bigg(- \frac{1482179}{1944} + \frac{358747}{486}\zeta_2 - \frac{910861}{405}\zeta_3 - \frac{11764}{9}\zeta_3\zeta_2 + \frac{368}{3}\zeta_3^2 \\
&\qquad + \frac{181501}{45}\zeta_4 - 48\zeta_4\zeta_2 + \frac{1028}{9}\zeta_5 + \frac{2161}{9}\zeta_6\bigg) \\
&+ n_f\bigg(\frac{552223}{7290} - \frac{23362}{243}\zeta_2 + \frac{155744}{405}\zeta_3 + \frac{704}{9}\zeta_3\zeta_2 - \frac{53594}{135}\zeta_4 - \frac{896}{9}\zeta_5\bigg)
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
\delta c^{(3)}_{L,ns}(N=2) = &+ C_F\bigg(\frac{45284}{1215} - \frac{1316}{27}\zeta_2 + \frac{12536}{135}\zeta_3 + 32\zeta_3\zeta_2 - \frac{1664}{45}\zeta_4 - \frac{224}{3}\zeta_5\bigg) \\
&+ C_A\bigg(\frac{8119}{162} - \frac{3046}{27}\zeta_2 - \frac{22028}{135}\zeta_3 + 32\zeta_3\zeta_2 + \frac{12644}{45}\zeta_4 - \frac{176}{3}\zeta_5\bigg) \\
&+ n_f\bigg(\frac{3374}{405} + \frac{136}{27}\zeta_2 + \frac{1232}{135}\zeta_3 - \frac{1072}{45}\zeta_4\bigg) .
\end{aligned} \tag{3.12}$$

In these expressions we retain the full dependence on the colour factors C_A, C_F and n_f, and suppress an overall "non-planar" colour factor of $C_F\,(C_A - 2C_F)$ in both expressions. This overall factor is a prediction of [48, 49] and implies the vanishing of these expressions in the large-N_c limit (N_c being the number of colours). Also of note is the appearance of the irrational constants ζ_2 and ζ_6. These do not appear in the "natural" moments of either $c^{(3),+}_{i,ns}$ or $c^{(3),-}_{i,ns}$, for $i = 2, L, 3$.

The numerical values of these moments (which include the overall colour factor combination of $C_F\,(C_A - 2C_F)$) with C_F and C_A set to their QCD values of 4/3 and 3 respectively are

$$\delta c^{(3)}_{2,ns}(N=2) = -20.40014403 + 0.7220159109\, n_f \tag{3.13}$$

and

$$\delta c^{(3)}_{L,ns}(N=2) = -24.77551732 + 0.8013314149\, n_f . \tag{3.14}$$

The approximate values of $\delta c^{(3)}_{2,ns}(N=2)$ and $\delta c^{(3)}_{L,ns}(N=2)$ computed in [37] prove to be very accurate. They have errors of just 0.5% and 0.07% respectively, and the exact values are within the quoted uncertainty. We thus expect that the conclusions regarding the Paschos-Wolfenstein relation, i.e. that the third-order QCD corrections are very small, will remain unchanged by the inclusion of the exact values of the second Mellin moments.

The new exact results are plotted alongside the approximations of [37] in Figs. 3.2 and 3.3. While the new lines (labelled: $s = -\,(ex.)$) fall within the band formed by the approximations ($s = -\,(A, B)$), we see that for small values of x the approximations are rather unreliable. Indeed, they describe the exact results to within a 5% error only for x values above 0.12, 0.14 and 0.16 for $c^{(3),-}_{i,ns}$, $i = 2, L, 3$ respectively. The ($s = +\,(ex.)$) lines are $c^{(3),+}_{i,ns}$ and are plotted for comparison with the new results. The plots show the common large-x behaviour of $c^{(3),+}_{i,ns}$ and $c^{(3),-}_{i,ns}$, which will be discussed in more detail below.

It is worth pointing out that the even-N function $c^{(3),-}_{3,ns}$ is not approximated as well as the odd-N $c^{(3),-}_{2,ns}$ and $c^{(3),-}_{L,ns}$. This is because the small-x behaviour is governed by the small-N behaviour, particularly for N values close to the pole at $N = 0$. Since the odd-N moments $N = 1, 3, \ldots$ are closer to this pole than the even-N moments $N = 2, 4, \ldots$ they are better able to constrain the small-x behaviour.

Despite these small-x inaccuracies, the approximations are more useful than they first appear. It is not the coefficient functions themselves that are of experimental relevance but the their convolution with a PDF. As can be seen from its definition (Eq. (A.9)), in the Mellin convolution integrand when one function is evaluated at small values the other is evaluated at large values. Thus, the inaccurate small-x region of the coefficient function approximations are multiplied by the (small) large-x part of the PDF. [37] deems the convolution of the approximations to be reliable for x values as low as 10^{-3}.

Note that in Fig. 3.3 as well as the plots of Sections 3.3.1 to 3.3.4, the curves for $C^{+}_{3,ns}$ do not have their fl_{02} contributions. This allows a “like-for-like” comparison with the $C^{-}_{3,ns}$ curves. Fig. 3 of [28] shows $C^{+}_{3,ns}$ with and without the fl_{02} contribution and the paper contains a discussion of its effects.

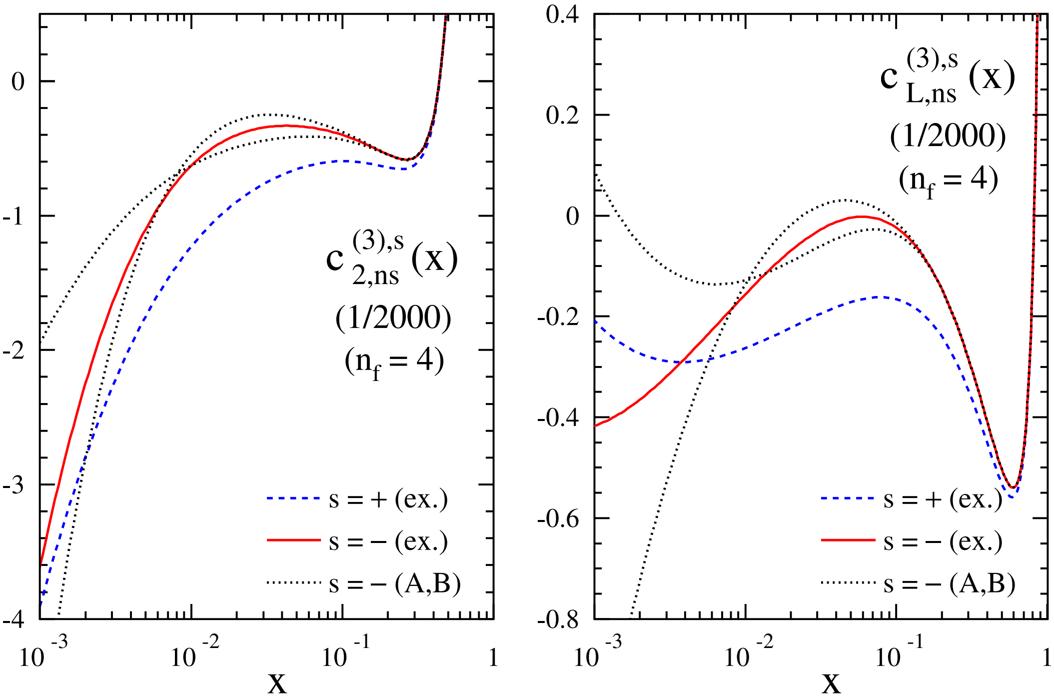

Figure 3.2: *The exact (labelled: (ex.)) third-order coefficient function contributions* $c_{2,ns}^{(3),+}$, $c_{2,ns}^{(3),-}$ *and* $c_{L,ns}^{(3),+}$, $c_{L,ns}^{(3),-}$, *plotted with four massless flavours. The colour factors* C_A *and* C_F *take their QCD values of* 3 *and* 4/3*. The curves labelled* (A, B) *are the previous approximations. An overall factor of* $(1/2000) \approx 1/(4\pi)^3$ *is included to approximately convert the result to a series in* α_s.

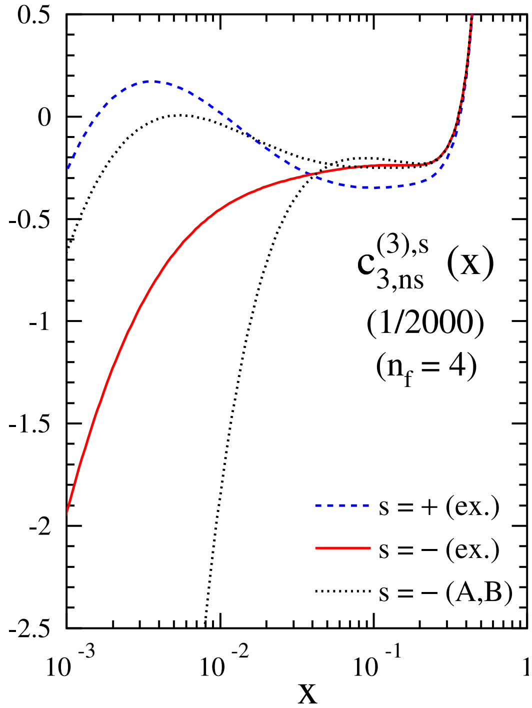

Figure 3.3: *As Fig. 3.2, for* $c_{3,ns}^{(3),+}$, $c_{3,ns}^{(3),-}$.

3.3.1 Perturbative Stability of the Coefficient Functions

In Fig. 3.4 ($i = 2$), Fig. 3.5 ($i = L$) and Fig. 3.6 ($i = 3$) we show the perturbative expansions of both $C^+_{i,ns}(x)$ and $C^-_{i,ns}(x)$. They allow us to assess the stability of the perturbative expansion in the strong coupling constant a_s and investigate how this stability depends on x.

As can been seen in all three pairs of plots, the second-order corrections (labelled as "NNLO" in Figs. 3.4 and 3.6 and as "NLO" in Fig. 3.5) are rather large for small values of x; the lines diverge significantly from the first-order expressions at x values as large as 10^{-2}. One cannot claim to have a good understanding of the coefficient functions with these contributions alone. The third-order corrections (labelled as "N^{3}LO" in Figs. 3.4 and 3.6 and as "NNLO" in Fig. 3.5) do much to improve the situation. We observe rather small corrections to the second-order lines over much of the x range plotted.

More quantitatively, for $C^+_{2,ns}$ and $C^-_{2,ns}$ the N^{3}LO curves correct the NNLO curves by less than 3% in the regions $\big(2.1 \times 10^{-7} < x < 0.74\big)$ and $\big(5.8 \times 10^{-8} < x < 0.75\big)$ respectively. $C^+_{3,ns}$ and $C^-_{3,ns}$ display rather similar behaviour. The N^{3}LO curves correct the NNLO curves by less than 3% in the regions $\big(3.5 \times 10^{-8} < x < 0.74\big)$ and $\big(9.6 \times 10^{-8} < x < 0.74\big)$ respectively. $C^+_{L,ns}$ and $C^-_{L,ns}$ converge less well by comparison. The NNLO curves correct the NLO curves by less than 5% only in the regions $(0.13 < x < 0.92)$ and $(0.0072 < x < 0.92)$ respectively, and by less than 11% in the regions $\big(8.7 \times 10^{-5} < x < 0.97\big)$ and $\big(1.0 \times 10^{-5} < x < 0.97\big)$ respectively. A 100% correction is reached at x values as "large" as 2.4×10^{-7} and 8.3×10^{-8}. The reason for this reduced convergence is the lack of a tree-level a_s^0 contribution, which adds 1 to the value of $C^\pm_{2,ns}$ and $C^\pm_{3,ns}$.

It should be stated that these plots demonstrate a rather ideal scenario, with a very low α_s value of 0.12. This is the value of α_s around the scale of the $W^\pm$ mass, relevant to high-energy neutrino scattering. A more typical α_s value of, say, 0.2 would yield somewhat less well-converging curves. Nonetheless, the new third-order corrections provide the first opportunity to assess the convergence of these coefficient functions and to assess the vales of x for which they can be considered reliable.

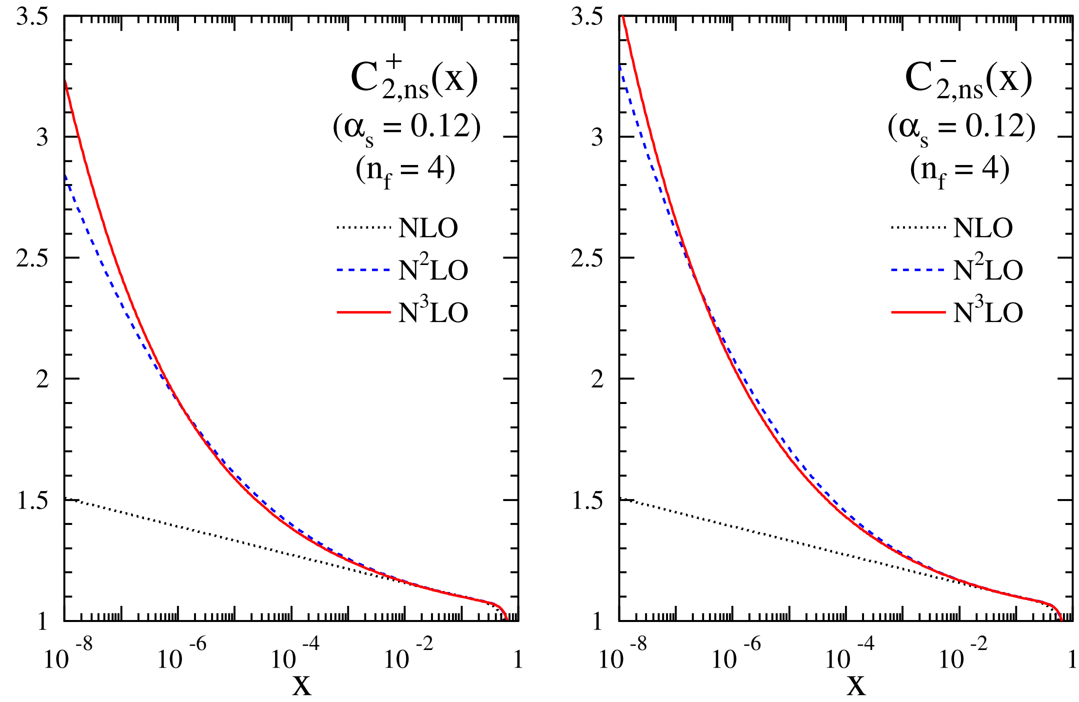

Figure 3.4: *The perturbative expansion of the coefficient functions $C^{+}_{2,ns}$ and $C^{-}_{2,ns}$ to third order, plotted with four massless flavours and an α_s value of* 0.12*. The colour factors C_A and C_F take their QCD values of* 3 *and* 4/3*.*

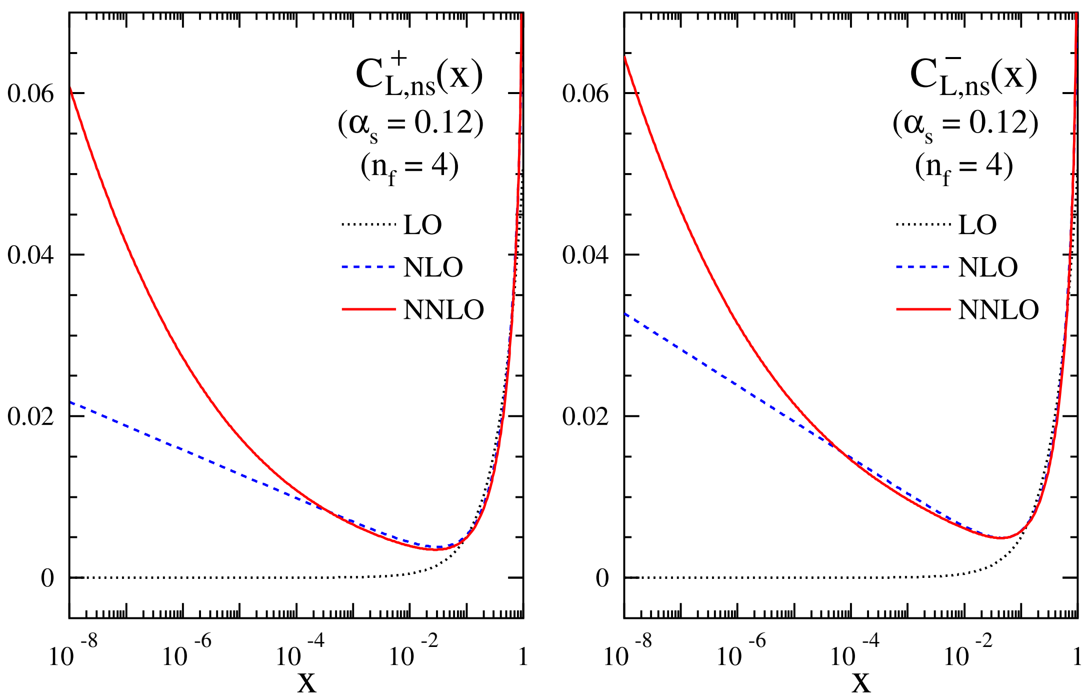

Figure 3.5: *As Fig. 3.4, for the coefficient functions $C^{+}_{L,ns}$ and $C^{-}_{L,ns}$.*

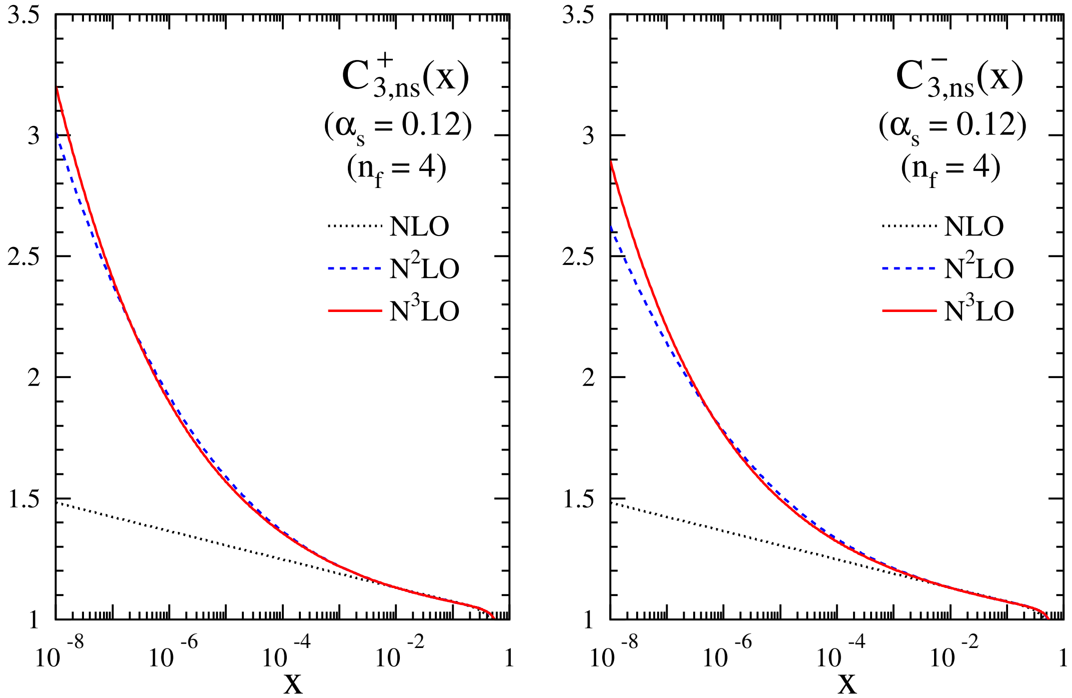

Figure 3.6: *As Fig. 3.4, for the coefficient functions* $C^+_{3,ns}$ *and* $C^-_{3,ns}$.

3.3.2 Small-x Behaviour of the Coefficient Functions

We now consider the behaviour of the coefficient functions in the small-x limit. First, we present expressions for the functions in this limit, which give the dominant behaviour in terms of powers of small-x logarithms $L_0 = \ln(x)$.

We find for $c^{(3),-}_{2,ns}$,

$$c^{(3),-}_{2,ns}\Big|_{L_0^5} = +\frac{2}{5}C_A^2C_F + \frac{53}{30}C_F^3 - \frac{29}{15}C_AC_F^2\,,$$

$$c^{(3),-}_{2,ns}\Big|_{L_0^4} = +\frac{46}{27}C_AC_Fn_f - \frac{31}{18}C_F^2n_f + \frac{23}{12}C_F^3 + \frac{247}{36}C_AC_F^2 - \frac{193}{27}C_A^2C_F\,,$$

$$c^{(3),-}_{2,ns}\Big|_{L_0^3} = -\frac{92}{81}C_Fn_f^2 + \left(13 - \frac{710}{9}\zeta_2\right)C_F^3 - \frac{1873}{81}C_F^2n_f + \frac{652}{27}C_AC_Fn_f + \left(\frac{14183}{162} + \frac{220}{3}\zeta_2\right)C_AC_F^2 - \left(\frac{7117}{81} + \frac{64}{9}\zeta_2\right)C_A^2C_F\,,$$

$$c^{(3),-}_{2,ns}\Big|_{L_0^2} = -\frac{496}{81}C_Fn_f^2 - \left(\frac{2945}{27} + \frac{62}{3}\zeta_2\right)C_F^2n_f + \left(\frac{3652}{27} - \frac{112}{9}\zeta_2\right)C_AC_Fn_f + \left(\frac{1307}{6} - \frac{1025}{3}\zeta_2 - 318\zeta_3\right)C_F^3 + \left(\frac{14119}{54} + 281\zeta_2 + 344\zeta_3\right)C_AC_F^2 - \left(\frac{34115}{81} - \frac{352}{9}\zeta_2 + \frac{212}{3}\zeta_3\right)C_A^2C_F\,,$$

$$c^{(3),-}_{2,ns}\Big|_{L_0^1} = -\left(\frac{1204}{81} - \frac{16}{3}\zeta_2\right)C_Fn_f^2 - \left(\frac{43207}{162} + \frac{70}{27}\zeta_2 + \frac{644}{9}\zeta_3\right)C_F^2n_f + \left(\frac{31280}{81} - \frac{2752}{27}\zeta_2 + \frac{112}{9}\zeta_3\right)C_AC_Fn_f + \left(\frac{182801}{324} + \frac{21349}{27}\zeta_2 + \frac{4862}{9}\zeta_3 + \frac{1636}{3}\zeta_4\right)C_AC_F^2$$

$$
\begin{aligned}
&+\left(\frac{3265}{4}-\frac{3028}{3}\zeta_2-\frac{1666}{3}\zeta_3-\frac{2230}{3}\zeta_4\right)C_F^3\\
&-\left(\frac{101635}{81}-\frac{7930}{27}\zeta_2-\frac{80}{9}\zeta_3+208\zeta_4\right)C_A^2C_F
\end{aligned}
$$

$$
\begin{aligned}
c_{2,ns}^{(3),-}\Big|_{L_0^0} =& -\left(\frac{11170}{729}-\frac{232}{27}\zeta_2-\frac{32}{27}\zeta_3\right)C_Fn_f^2\\
&-\left(\frac{45371}{108}-\frac{7016}{81}\zeta_2-\frac{434}{27}\zeta_3-\frac{988}{9}\zeta_4\right)C_F^2n_f\\
&+\left(\frac{374105}{729}-\frac{16832}{81}\zeta_2-\frac{1756}{27}\zeta_3-8\zeta_4\right)C_AC_Fn_f\\
&+\left(\frac{20147}{24}-\frac{4355}{3}\zeta_2-\frac{1090}{3}\zeta_3+\frac{2824}{3}\zeta_3\zeta_2-\frac{4016}{3}\zeta_4-1120\zeta_5\right)C_F^3\\
&+\left(\frac{358787}{216}+\frac{76246}{81}\zeta_2+\frac{391}{27}\zeta_3-1176\zeta_3\zeta_2+\frac{5240}{9}\zeta_4+\frac{3158}{3}\zeta_5\right)C_AC_F^2\\
&-\left(\frac{1496305}{729}-\frac{42629}{81}\zeta_2-\frac{10784}{27}\zeta_3-\frac{724}{3}\zeta_3\zeta_2+178\zeta_4+\frac{560}{3}\zeta_5\right)C_A^2C_F\,.
\end{aligned}
\tag{3.15}
$$

For $c_{3,ns}^{(3),-}$ we find

$$
c_{3,ns}^{(3),-}\Big|_{L_0^5} = -\frac{1}{2}C_F^3\,,
$$

$$
c_{3,ns}^{(3),-}\Big|_{L_0^4} = +\frac{91}{54}C_F^2n_f+\frac{15}{4}C_F^3-\frac{1001}{108}C_AC_F^2\,,
$$

$$
\begin{aligned}
c_{3,ns}^{(3),-}\Big|_{L_0^3} =& -\frac{92}{81}C_Fn_f^2+\frac{143}{27}C_F^2n_f+\frac{1012}{81}C_AC_Fn_f+\left(\frac{71}{3}+\frac{262}{3}\zeta_2\right)C_F^3\\
&-\left(\frac{2783}{81}-20\zeta_2\right)C_A^2C_F-\left(\frac{2353}{54}+64\zeta_2\right)C_AC_F^2\,,
\end{aligned}
$$

$$
\begin{aligned}
c_{3,ns}^{(3),-}\Big|_{L_0^2} =& -\left(\frac{47}{6}-\frac{719}{3}\zeta_2-\frac{646}{3}\zeta_3\right)C_F^3-\frac{712}{81}C_Fn_f^2-\left(\frac{1909}{81}+\frac{266}{9}\zeta_2\right)C_F^2n_f\\
&+\left(\frac{18989}{162}-\frac{1129}{9}\zeta_2-64\zeta_3\right)C_AC_F^2+\left(\frac{9572}{81}-8\zeta_2\right)C_AC_Fn_f\\
&-\left(\frac{29596}{81}-114\zeta_2\right)C_A^2C_F\,,
\end{aligned}
$$

$$
\begin{aligned}
c_{3,ns}^{(3),-}\Big|_{L_0^1} =& -\left(\frac{109}{12}+\frac{340}{3}\zeta_2-418\zeta_3+\frac{1654}{3}\zeta_4\right)C_F^3-\left(\frac{1684}{81}-\frac{16}{3}\zeta_2\right)C_Fn_f^2\\
&-\left(\frac{12047}{162}+\frac{794}{9}\zeta_2+132\zeta_3\right)C_F^2n_f+\left(\frac{9676}{27}-\frac{760}{9}\zeta_2+\frac{128}{3}\zeta_3\right)C_AC_Fn_f\\
&+\left(\frac{228649}{324}+\frac{5987}{9}\zeta_2+\frac{130}{3}\zeta_3+\frac{460}{3}\zeta_4\right)C_AC_F^2\\
&-\left(\frac{104450}{81}-\frac{1996}{9}\zeta_2-32\zeta_3+60\zeta_4\right)C_A^2C_F\,,
\end{aligned}
$$

$$
\begin{aligned}
c_{3,ns}^{(3),-}\Big|_{L_0^0} =& -\left(\frac{8974}{729}-\frac{328}{27}\zeta_2-\frac{32}{27}\zeta_3\right)C_Fn_f^2\\
&+\left(\frac{5153}{24}-\frac{817}{3}\zeta_2+\frac{1706}{3}\zeta_3-904\zeta_3\zeta_2-\frac{1246}{3}\zeta_4-832\zeta_5\right)C_F^3\\
&+\left(\frac{226739}{729}-\frac{4172}{27}\zeta_2+\frac{2276}{27}\zeta_3+\frac{88}{3}\zeta_4\right)C_AC_Fn_f\\
&+\left(\frac{234227}{648}+\frac{83612}{81}\zeta_2+\frac{18301}{27}\zeta_3+520\zeta_3\zeta_2-\frac{2191}{9}\zeta_4+\frac{1910}{3}\zeta_5\right)C_AC_F^2\\
&-\left(\frac{1938467}{1458}-\frac{8614}{27}\zeta_2+\frac{2392}{27}\zeta_3+\frac{436}{3}\zeta_3\zeta_2+43\zeta_4+\frac{152}{3}\zeta_5\right)C_A^2C_F\,.
\end{aligned}
\tag{3.16}
$$

Finally, for $c^{(3),-}_{L,ns}$ we find

$$c^{(3),-}_{L,ns}\Big|_{L_0^3} = +\frac{92}{3}C_F^3 - \frac{104}{3}C_AC_F^2 + 8C_A^2C_F\,,$$

$$c^{(3),-}_{L,ns}\Big|_{L_0^2} = +6C_AC_F^2 + 16C_AC_Fn_f - 20C_F^2n_f - 44C_A^2C_F + 64C_F^3\,,$$

$$\begin{aligned}c^{(3),-}_{L,ns}\Big|_{L_0^1} &= -\frac{32}{9}C_Fn_f^2 + \frac{368}{3}C_AC_Fn_f - \frac{1280}{9}C_F^2n_f - \Big(248 + 352\zeta_2\Big)C_F^3\\ &\quad - \left(\frac{3580}{9} + 56\zeta_2\right)C_A^2C_F + \left(\frac{5264}{9} + 352\zeta_2\right)C_AC_F^2\,,\end{aligned}$$

$$\begin{aligned}c^{(3),-}_{L,ns}\Big|_{L_0^0} &= -\left(\frac{3304}{27} + 32\zeta_2\right)C_F^2n_f + \left(\frac{5264}{27} - \frac{16}{3}\zeta_2\right)C_AC_Fn_f\\ &\quad + \left(\frac{6016}{27} + 688\zeta_2 + 464\zeta_3\right)C_AC_F^2 - \left(\frac{6310}{27} + \frac{284}{3}\zeta_2 + 160\zeta_3\right)C_A^2C_F\\ &\quad - \Big(1016 + 544\zeta_2 + 192\zeta_3\Big)C_F^3\,. \qquad (3.17)\end{aligned}$$

As is the case with the $c^{(3),+}_{i,ns}$ coefficient functions, $c^{(3),-}_{L,ns}$ has a maximum power of L_0 that is two below that of $c^{(3),-}_{2,ns}$ and $c^{(3),-}_{3,ns}$.

The convergence of these leading logarithms on the exact expressions is best demonstrated with the plots Figs. 3.7 to 3.9. As above, we plot both the even-N and odd-N coefficient functions. We show lines for the *Leading Logarithmic (LL)* approximation (labelled L_0^5 for $c^{(3),\pm}_{2,ns}$ and $c^{(3),\pm}_{3,ns}$, and L_0^3 for $c^{(3),\pm}_{L,ns}$), the *Next-to-Leading Logarithmic (NLL)* approximation which is the sum of the two highest power logarithms (labelled $+L_0^4$ for $c^{(3),\pm}_{2,ns}$ and $c^{(3),\pm}_{3,ns}$, and $+L_0^2$ for $c^{(3),\pm}_{L,ns}$) and so on.

It is clear that the first few logarithmic approximations do not provide a good description of the exact expressions over the plotted range. For $c^{(3),\pm}_{2,ns}$ and $c^{(3),\pm}_{3,ns}$ we appear to need a N^{3}LL approximation, and for $c^{(3),\pm}_{L,ns}$ a NNLL approximation, to achieve reasonable accuracy.

In Chapter 4, we will discuss the all-order resummation of small-x leading logarithms for various DIS quantities, including $C^-_{3,ns}$. Looking at the results here, we cannot hope that these resummations can have any *direct* phenomenological applications, since knowledge of just the highest few logarithmic contributions appears to be insufficient to approximate the exact function, even for small values of x. Indeed, the problem is worse at higher orders; the tower of logarithms grows ever higher with the power of a_s. A fixed number of logarithms captures less and less of the behaviour. In addition new flavour structures can appear, the behaviour of which cannot possibly be predicted from lower-order information. For example, the fl_{02} diagrams discussed below Eq. (3.6) have a large effect on $c^{(3),+}_{3,ns}$ at small-x. Nonetheless, the resummations of Chapter 4 will be useful for more theoretical reasons and these will be discussed in detail later.

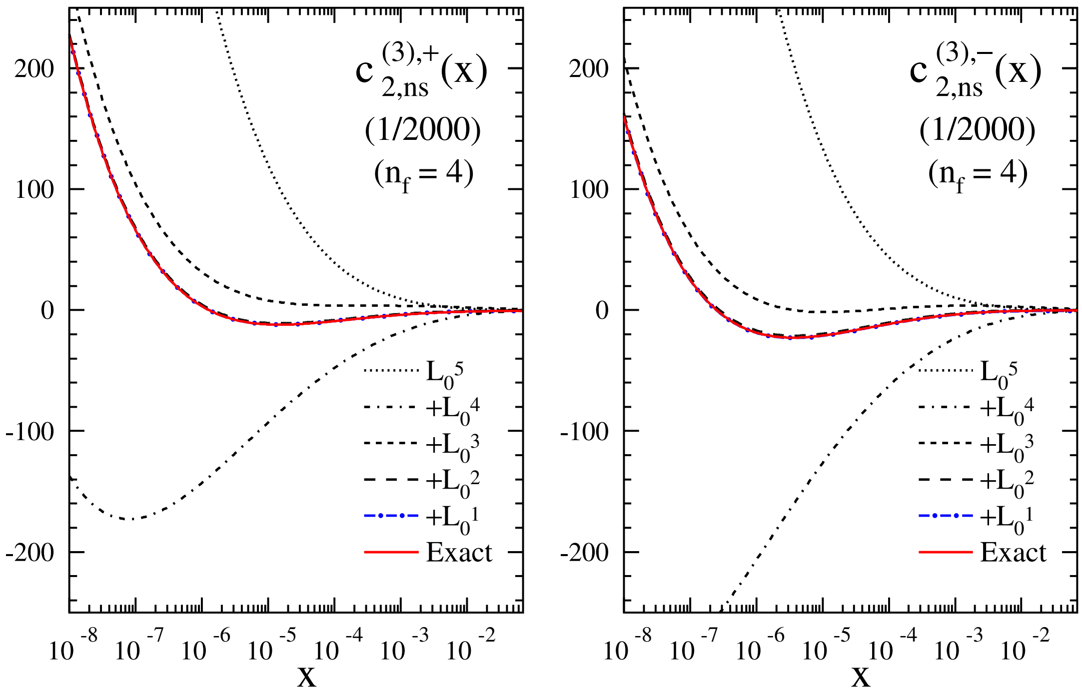

Figure 3.7: *The small-x behaviour of the third-order coefficient function contributions* $c^{(3),\pm}_{2,ns}$*, plotted alongside their logarithmic approximations. The curves are plotted for four massless flavours, and the colour factors* C_A *and* C_F *taking their QCD values of* 3 *and* 4/3*. An overall factor of* $(1/2000) \approx (1/(4\pi)^3)$ *is included to approximately convert the result to a series in* α_s*.*

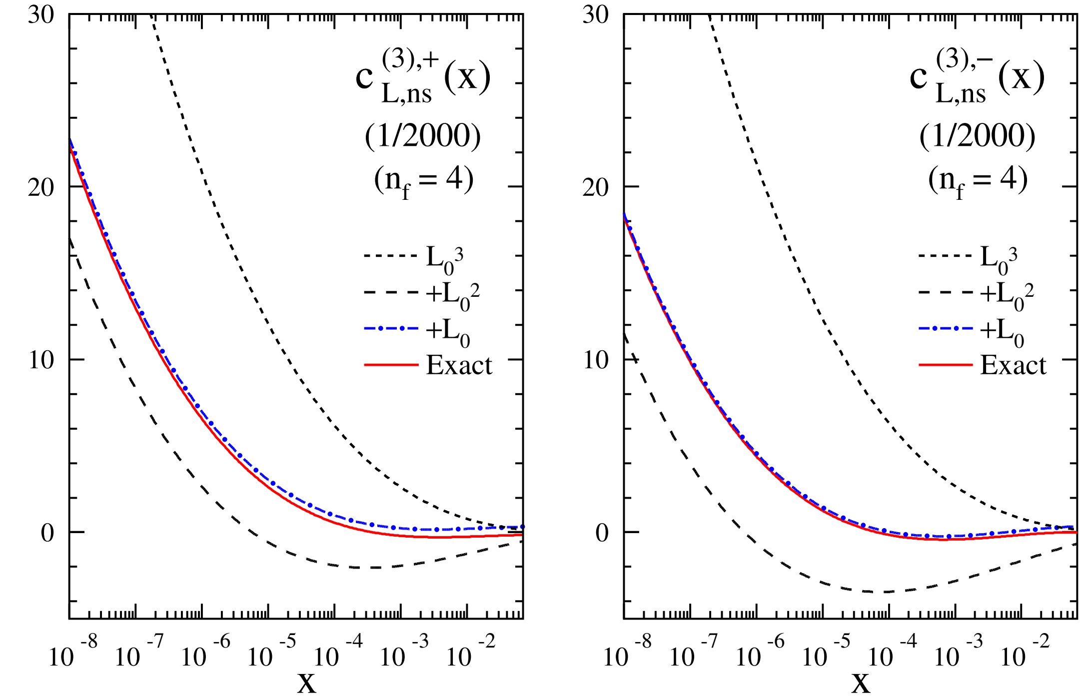

Figure 3.8: *As Fig. 3.7, for* $c^{(3),\pm}_{L,ns}$*.*

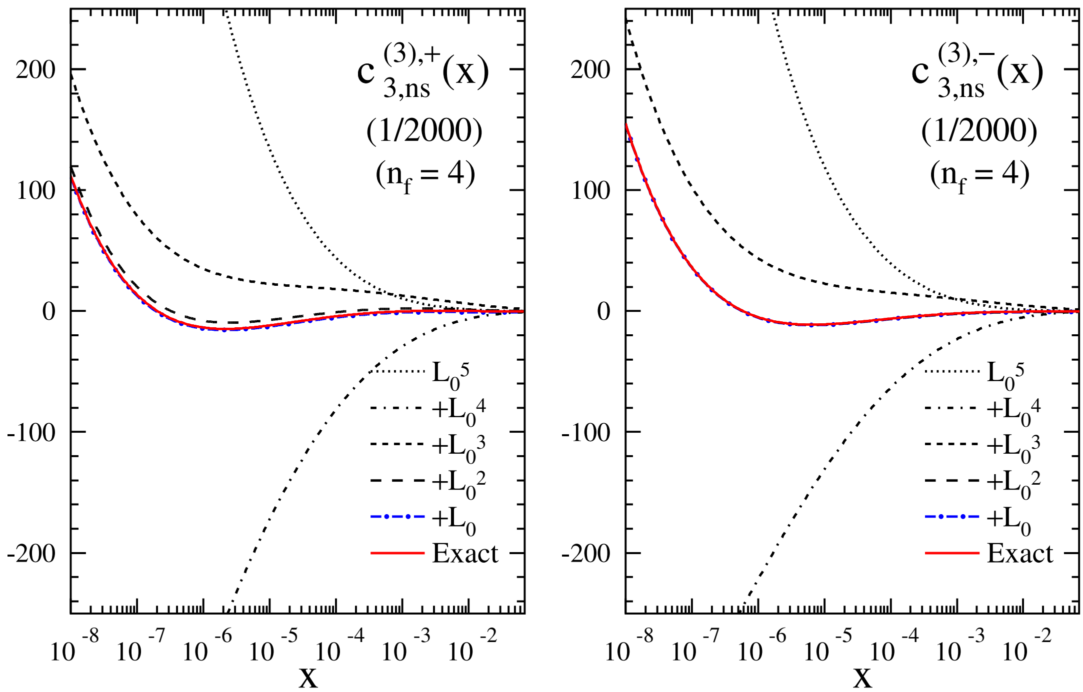

Figure 3.9: *As Fig. 3.7, for* $c^{(3),\pm}_{3,ns}$.

3.3.3 Large-x Behaviour of the Coefficient Functions

Here we describe the large-x behaviour of the new results. This is done most compactly by giving the large-x expressions for the even-N−odd-N differences as defined in Eq. (3.6). As in Eqs. (3.11) and (3.12) we suppress the overall colour factor combination $C_F\,(C_A - 2C_F)$ in the typesetting. We have for $\delta c^{(3)}_{2,ns}$,

$$\begin{aligned}
\delta c^{(3)}_{2,ns}\Big|_{L_1^2} &= \Big[\Big(12 - 8\zeta_2\Big)\,C_F\Big]\,(1-x) + \mathcal{O}\left((1-x)^2\right)\,,\\
\delta c^{(3)}_{2,ns}\Big|_{L_1^1} &= \Big[\Big(-50 - 48\zeta_3 + 68\zeta_2\Big)\,C_F + \Big(-\frac{712}{9} + 64\zeta_3 - \frac{16}{3}\zeta_2\Big)\,C_A\\
&\quad + \Big(\frac{16}{9}\Big)\,n_f\Big]\,(1-x) + \mathcal{O}\left((1-x)^2\right)\,,\\
\delta c^{(3)}_{2,ns}\Big|_{L_1^0} &= \Big[\Big(-\frac{212}{3} + \frac{256}{5}\zeta_2^2 - 12\zeta_3 - \frac{104}{3}\zeta_2\Big)\,C_F + \Big(-\frac{856}{27} + \frac{32}{3}\zeta_3 + \frac{80}{9}\zeta_2\Big)\,n_f\\
&\quad + \Big(\frac{7780}{27} - \frac{112}{5}\zeta_2^2 - \frac{272}{3}\zeta_3 - \frac{464}{9}\zeta_2\Big)\,C_A\Big]\,(1-x) + \mathcal{O}\left((1-x)^2\right)\,.
\end{aligned} \tag{3.18}$$

Similarly, for $\delta c^{(3)}_{3,ns}$,

$$\begin{aligned}
\delta c^{(3)}_{3,ns}\Big|_{L_1^2} &= \Big[\Big(-20 + 8\zeta_2\Big)\,C_F\Big]\,(1-x) + \mathcal{O}\left((1-x)^2\right)\,,\\
\delta c^{(3)}_{3,ns}\Big|_{L_1^1} &= \Big[\Big(158 + 48\zeta_3 - 100\zeta_2\Big)\,C_F + \Big(\frac{152}{9} - 64\zeta_3 + \frac{80}{3}\zeta_2\Big)\,C_A\\
&\quad + \Big(\frac{16}{9}\Big)\,n_f\Big]\,(1-x) + \mathcal{O}\left((1-x)^2\right)\,,
\end{aligned}$$

$$\delta c^{(3)}_{3,ns}\Big|_{L_1^0} = \Bigg[\Big(- 28 - 96\zeta_2^2 + 88\zeta_3 + 172\zeta_2\Big)\, C_F + \left(\frac{608}{9} - \frac{64}{3}\zeta_3 - \frac{160}{9}\zeta_2\right) n_f$$
$$+ \left(-\frac{5600}{9} + \frac{192}{5}\zeta_2^2 + \frac{496}{3}\zeta_3 + \frac{784}{9}\zeta_2\right) C_A\Bigg] (1-x)^2 + \mathcal{O}\left((1-x)^3\right) . \tag{3.19}$$

Due to the longitudinal projection another factor $(1 - x)$ relative to Eq. (3.18) and Eq. (3.19) appears for $\delta c^{(3)}_{L,ns}$,

$$\delta c^{(3)}_{L,ns}\Big|_{L_1^2} = \Bigg[\Big(- 32 + 16\zeta_2\Big)\, C_F\Bigg] (1-x)^2 + \mathcal{O}\left((1-x)^3\right) ,$$
$$\delta c^{(3)}_{L,ns}\Big|_{L_1^1} = \Bigg[\Big(240 + 64\zeta_3 - 184\zeta_2\Big)\, C_F + \Big(96 - 96\zeta_3 + 32\zeta_2\Big)\, C_A\Bigg] (1-x)^2$$
$$+ \mathcal{O}\left((1-x)^3\right) ,$$
$$\delta c^{(3)}_{L,ns}\Big|_{L_1^0} = \Bigg[\Big(- 28 - 96\zeta_2^2 + 88\zeta_3 + 172\zeta_2\Big)\, C_F + \left(\frac{608}{9} - \frac{64}{3}\zeta_3 - \frac{160}{9}\zeta_2\right) n_f$$
$$+ \left(-\frac{5600}{9} + \frac{192}{5}\zeta_2^2 + \frac{496}{3}\zeta_3 + \frac{784}{9}\zeta_2\right) C_A\Bigg] (1-x)^2 + \mathcal{O}\left((1-x)^3\right) . \tag{3.20}$$

The coefficient functions $C^+_{i,ns}$ and $C^-_{i,ns}$ display the usual large-x double-logarithmic enhancement in their third order contributions. The differences $\delta c^{(3)}_{i,ns}$ show much cancellation, however. They are suppressed by two powers of $(1 - x)$ compared to the functions that form them, and their maximum power of L_1 is lower by 3. The leading large-x behaviour of $c^{(3),-}_{i,ns}$ is thus the same as that of $c^{(3),+}_{i,ns}$ for $i = 2, 3, L$.

3.3.4 Perturbative Stability of the Structure Functions

We now investigate to what extent these new third-order corrections to the coefficient functions affect *structure functions*. As explained in Section 2.1, the structure functions are a convolution of the coefficient functions and non-perturbative PDFs. The PDFs are determined by fitting (rather complicated) functions to experimental data. This is a highly non-trivial procedure with many research groups adopting different approaches and assumptions. Here we do not choose any particular PDF with which to convolute our coefficient functions but rather use a simple but sufficiently realistic function, intended to suitably represent the general shape of real PDFs. We use, as in [25, 27, 37],

$$xf(x) = \sqrt{x}(1-x)^3. \tag{3.21}$$

This form is inspired at small-x and large-x by Regge theory and counting rules for quark distributions. See for e.g. [50, 51] for a discussion.

We plot in Figs. 3.10 to 3.12 the following six structure functions,

$$F^{W^+\pm W^-}_{i,ns} = C^{W^+\pm W^-}_{i,ns} \otimes f, \qquad (i = 2, L, 3), \tag{3.22}$$

normalized to the value of the PDF, i.e. we plot $F_{i,ns}^{W^+ \pm W^-}/f$. Of course, each of the PDFs in these equations should really be different, as discussed at the end of Section 2.4. We use the same PDF everywhere here to facilitate an easy comparison of the convolution of the different coefficient functions. There is a small technicality in computing these convolutions. The coefficient functions contain plus distributions and delta functions which must be handled carefully. The procedure is described in Appendix A.4.

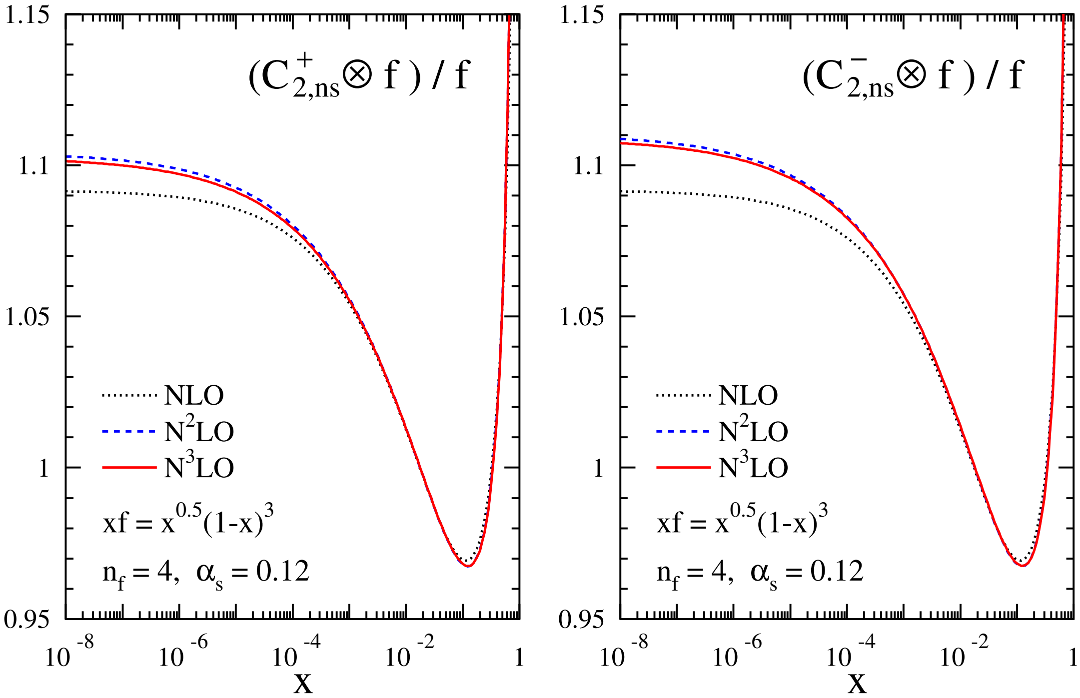

Figure 3.10: *The perturbative expansion of the structure functions* $F_{2,ns}^{W^+ \pm W^-}$ *to third order, using a reference distribution* $xf = \sqrt{x}(1-x)^3$. *The curves are plotted with four massless flavours,* C_A *and* C_F *taking their QCD values of* 3 *and* 4/3, *and an* α_s *value of* 0.12. *The lines are normalized to* $f(x)$ *for plotting purposes.*

Successive contributions to the a_s expansion for these convolutions converge better than the coefficient functions themselves. As pointed out near the end of Section 3.3, the convolution with a PDF suppresses the effect of the small-x region of the coefficient functions. This is exactly the region in which the third-order corrections to the coefficient functions diverge significantly from the second-order corrections. The N^3LO contributions to $F_{2,ns}^{W^+ \pm W^-}$ and $F_{3,ns}^{W^+ \pm W^-}$ thus correct the NNLO contributions by less than 1% in the range $(10^{-8} < x < 0.82)$. The NNLO contributions to $F_{L,ns}^{W^+ \pm W^-}$, as with the associated coefficient functions, converge less well. Even so they correct the NLO contributions by less than 3% in the range $(10^{-8} < x < 0.12)$. All six structure functions of CC DIS therefore appear to be stable for x values relevant to current collider experiments [6], when a_s^3 corrections are included.

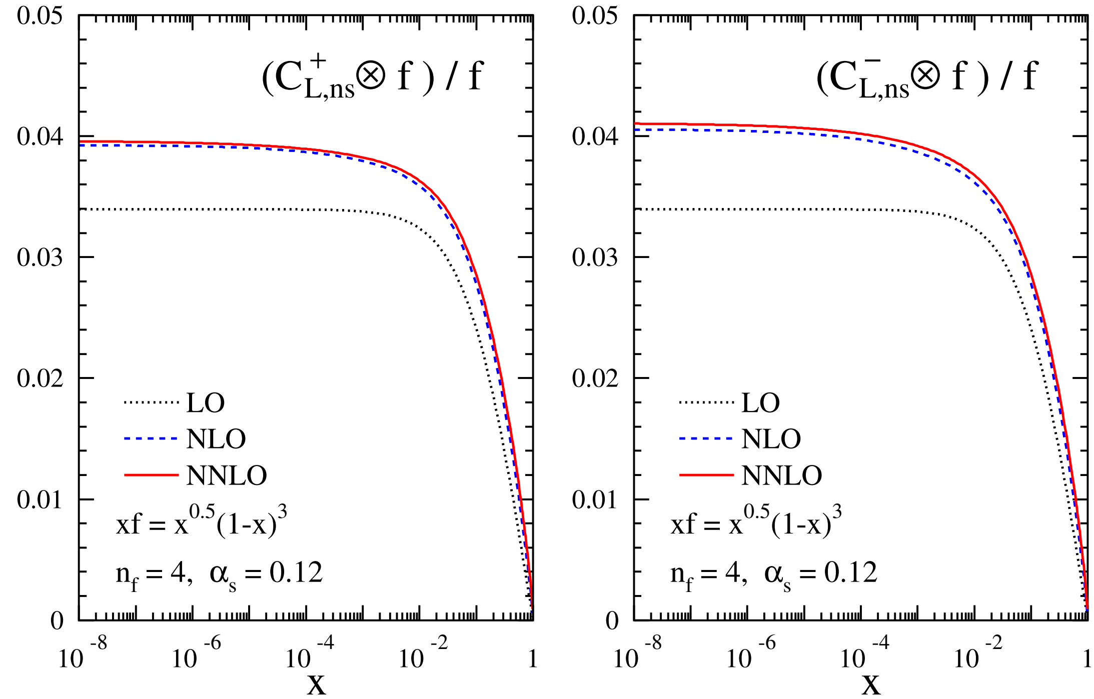

Figure 3.11: *As Fig. 3.10, for the structure functions* $F_{L,ns}^{W^+ \pm W^-}$.

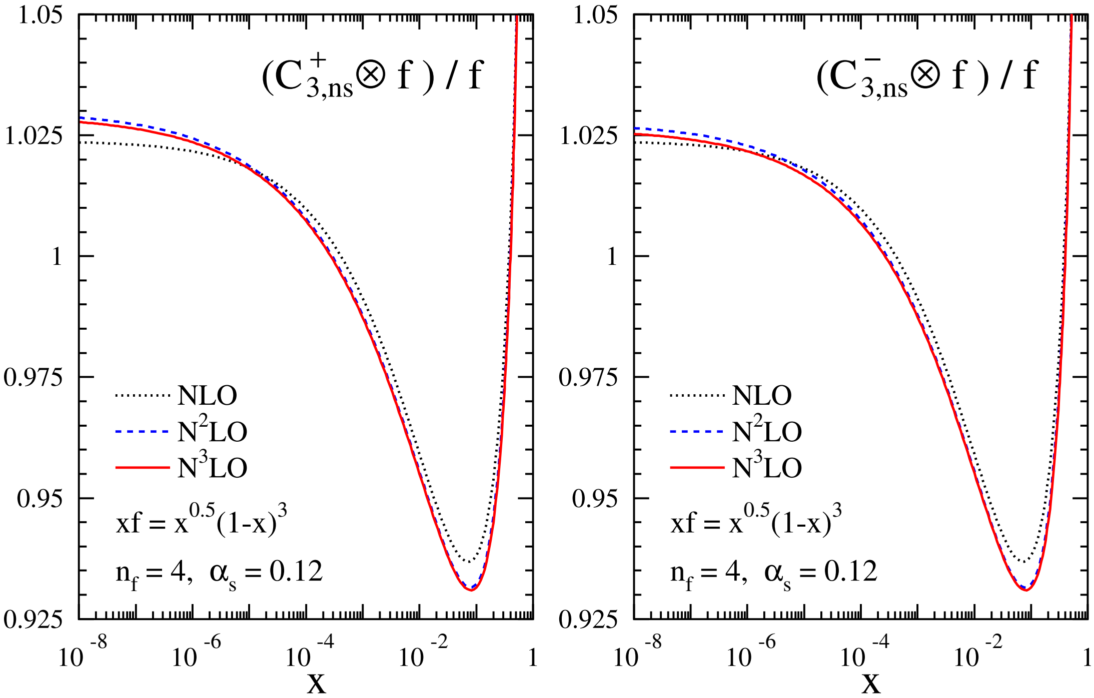

Figure 3.12: *As Fig. 3.10, for the structure functions* $F_{3,ns}^{W^+ \pm W^-}$.

3.4 Phenomenological Application: The Paschos-Wolfenstein Relation

The NuTeV experiment caused excitement some years ago due to a measurement of the weak mixing angle, $\sin^2\vartheta_W$, which was 3σ above standard model predictions [52]. Dubbed the "NuTeV anomaly", this measurement of $\sin^2\vartheta_W$ was determined via the *Paschos-Wolfenstein relation* [53], the ratio of neutral-current to charged-current neutrino-nucleon scattering:

$$R_{PW}^{-} = \frac{\sigma(\nu N \to \nu X) - \sigma(\bar{\nu} N \to \bar{\nu} X)}{\sigma(\nu N \to l^- X) - \sigma(\bar{\nu} N \to l^+ X)}. \tag{3.23}$$

This discrepancy motivates, in [37], the consideration of the QCD corrections to this ratio. Beyond leading order in QCD, Eq. (3.23) may be written as [37, 54]

$$\begin{aligned} R_{PW}^{-} = \frac{1}{2} - \sin^2\vartheta_W + \frac{u^- - d^- + c^- - s^-}{u^- + d^-} & \left\{ 1 - \frac{7}{3}\sin^2\vartheta_W + \left(\frac{1}{2} - \sin^2\vartheta_W\right) \times \right. \\ & \left. \frac{8}{9}\frac{\alpha_s}{\pi}\left[1 + 1.689\alpha_s + \left(3.61792 - \frac{9}{256\pi^2}\delta c_{2,ns}^{(3)}(2) + \frac{9}{1024\pi^2}\delta c_{L,ns}^{(3)}(2)\right)\alpha_s^2\right]\right\} \\ & + \mathcal{O}\left(\frac{1}{(u^- + d^-)^2}\right) + \mathcal{O}\left(\alpha_s^4\right). \end{aligned} \tag{3.24}$$

The symbols $q^- = \int_0^1 \mathrm{d}x x(q - \bar{q})$ are the second moments of the valence distributions of the quark flavours, and we have expanded in inverse powers of the dominant combination $(u^- + d^-)$. The quantities $\delta c_{2,ns}^{(3)}(2)$ and $\delta c_{L,ns}^{(3)}(2)$ are known exactly from the new results of this chapter, and were given in Eqs. (3.11) and (3.12). We mentioned that the numerical values of these moments (given in Eqs. (3.13) and (3.14)) are very close to the approximations of [37] and as such, our conclusions about R_{PW}^{-} do not change. The coefficient of α_s^3 in Eq. (3.24) is given by 3.66109, compared to the previous approximation of 3.661 ± 0.002 (an error of just 0.009%). R_{PW}^{-} is thus stable under QCD corrections. The third-order contribution increases the square-bracketed combination by 16% and the curly-bracketed combination by just 1%.

3.5 Conclusions

In this chapter, we have computed the coefficient functions of CC DIS for the linear combination $W^+ - W^-$. Along with the existing results for the $W^+ + W^-$ combination (see [27,28]) we have completed the description of CC DIS at the third order in massless QCD. The main results of this chapter have been provided in terms of the difference between the $W^+ + W^-$ and $W^+ - W^-$ coefficient functions. Compact, yet accurate, parametrizations were given in Eqs. (3.8) to (3.10) and the exact results are given in Appendix A.7. `FORTRAN` and `FORM` files for these parametrizations and exact results will be included with the `arXiv` source of the article [3].

We have found that by including these third-order corrections, the perturbative expansion of these coefficient functions appears to be stable for the experimentally relevant range of x. This was found to be especially true of the CC structure functions $F^{\pm}_{i,ns}$, $i = 2, L, 3$, determined by convolution the new results of this chapter with a reference distribution.

We also investigated the behaviour of the coefficient functions in the small-x and large-x limits, paying particular attention to the extent to which the small-x double-logarithmic approximations converge on the exact curves. The leading logarithms of $c^{(3),-}_{3,ns}$ given by Eq. (3.16) form the input for some of the computations of Chapter 4, in which we consider the resummation of these double logarithms to all orders of perturbation theory.

Knowledge of the exact x-space expression allowed us to evaluate the second Mellin moment of the differences between the $W^+ + W^-$ and $W^+ - W^-$ coefficient functions, $\delta c^{(3)}_{2,ns}$ and $\delta c^{(3)}_{L,ns}$, which contribute to the third-order QCD corrections to the Paschos-Wolfenstein relation. We concluded that these corrections are of negligible effect.

Chapter 4

Resummation of Small-x Double Logarithms in Deep-Inelastic Scattering

4.1 Introduction

While high-order corrections to the anomalous dimensions and coefficient functions, such as those of Chapter 3 and Chapter 5, allow us to describe DIS with great precision, they do not do so for the entire kinematic range. For large and small values of the parameter x, (that is, in the limits $x \to 1$ and $x \to 0$) we find powers of logarithms of $(1-x)$ and x which can spoil the convergence of the series. For any fixed value of $a_{\rm s}$, one can of course find a value of x for which $\ln(1-x)$ or $\ln x$ dominates $a_{\rm s}$ raised to any power, as demonstrated by the plots and discussion of Section 3.3.2 above. We also showed that knowledge of just a few of the leading logarithmic contributions to the coefficient functions does not give a good approximation of their true value.

However, a systematic study and all-$a_{\rm s}$-order determination of the leading logarithms is mathematically interesting and provides predictions for the limiting behaviour of higher fixed-order corrections, allowing us to check future calculations. Indeed, quantities computed here provide checks of the fourth-order contributions to the anomalous dimensions computed in Chapter 5. Knowledge of the endpoint behaviour also provides additional constraints when one attempts to approximate a function based on a small number of Mellin moments.

The method of this chapter is related to that of [55], in which the leading three *large-x* double logarithms were determined to all orders in $a_{\rm s}$ via the assumption of an all-order form for un-mass-factorized DIS structure functions. Similar resummations of both large-x [56] and small-x [57] double logarithms have also been performed in the context of semi-inclusive annihilation. The procedure here is similar but applies to the small-x limit of the DIS structure functions.

4.1.1 Small-x Expansion

In this chapter we will be referring to the small-x limit of various quantities. Based on the known fixed-order expressions, we summarize here the leading behaviour of the anomalous dimensions and the DIS coefficient functions in x space. For the non-singlet (even-N) anomalous dimension

$$\begin{aligned}\gamma_{ns}^{(n),+} &= + x^0\left(\ln^{2n} x + \ln^{2n-1} x + \cdots + \text{const}\right)\\ &\quad + x^1\left(\ln^{2n} x + \ln^{2n-1} x + \cdots + \text{const}\right) + \mathcal{O}\left(x^2\right), \end{aligned} \tag{4.1}$$

and for the singlet system,

$$\begin{aligned}\gamma_{ij}^{(n)} &= + \frac{1}{x}\left(\ln^{n-1} x + \ln^{n-2} x + \cdots + \text{const}\right)\\ &\quad + x^0\left(\ln^{2n} x + \ln^{2n-1} x + \cdots + \text{const}\right)\\ &\quad + x^1\left(\ln^{2n} x + \ln^{2n-1} x + \cdots + \text{const}\right) + \mathcal{O}\left(x^2\right), \qquad (i,j = q,g). \end{aligned} \tag{4.2}$$

For the (even-N) coefficient functions,

$$\begin{aligned}C_{a,ns}^{(n\geq 1)} &= + x^0\left(\ln^{2n-1-\delta_{aL}} x + \ln^{2n-2-\delta_{aL}} x + \cdots + \text{const}\right)\\ &\quad + x^1\left(\ln^{2n-1-\delta_{aL}} x + \ln^{2n-2-\delta_{aL}} x + \cdots + \text{const}\right) + \mathcal{O}\left(x^2\right), \qquad (a = 2,3,L), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}C_{a,i}^{(n\geq 1)} &= + \frac{1}{x}\left(\ln^{n-2} x + \ln^{n-3} x + \cdots + \text{const}\right)\\ &\quad + x^0\left(\ln^{2n-1-\delta_{aL}} + \ln^{2n-2-\delta_{aL}} + \cdots + \text{const}\right) + \mathcal{O}\left(x^1\right), \;\; (a = 2,3,L,\; i = q,g) \end{aligned} \tag{4.4}$$

where $\delta_{aL} = 1$ if $a = L$ and $\delta_{aL} = 0$ otherwise. Finally for the scalar-exchange coefficient functions,

$$\begin{aligned}C_{\phi,i}^{(n\geq 1)} &= + \frac{1}{x}\left(\ln^{2n-1} x + \ln^{2n-2} x + \cdots + \text{const}\right)\\ &\quad + x^0\left(\ln^{2n-3} + \ln^{2n-4} + \cdots + \text{const}\right) + \mathcal{O}\left(x^1\right), \qquad (i = q,g). \end{aligned} \tag{4.5}$$

For the singlet functions the leading terms are single-logarithmically enhanced $\frac{1}{x}$ (with the exception of $C_{\phi,i}$ which is double-logarithmically enhanced). These terms are *not* considered here. They are resummed by the the BFKL formalism, see for e.g. [58, 59, 60, 61]. Rather, we consider here the *sub*-leading x^0 double-logarithmic terms. It is not inconceivable that for some intermediate values of x they in fact dominate the formally leading $\frac{1}{x}$ terms due to their double, rather than single, logarithms.

All calculations here are performed in Mellin-N space, so we note that the Mellin transform of the leading small-x terms has the form

$$\int_0^1 \mathrm{d}x\, x^{N-1} x^m \ln^k x = \frac{(-1)^k k!}{(N+m)^{k+1}}. \tag{4.6}$$

The $x^0 \ln^k x$ terms of interest here are thus expressed as poles in N as $N \to 0$. The highest powers of $1/N$ are the leading terms – the "leading logarithms" in x-space – which we label LL. The second-highest powers of $1/N$ are labelled NLL, and so on.

We find that for even-N based quantities, this method applies to the $x^0, x^2, \ldots$ double logarithms only, and for odd-N based quantities it applies just to the $x^1, x^3, \ldots$ double logarithms. We do not consider the odd-N based quantities in this chapter, so the resummations here focus on the non-singlet parton-level structure functions $\hat{F}_{a,ns}$ ($a = 2, L$) and $\hat{F}^{-}_{3,ns}$ and the singlet parton-level structure functions $\hat{F}_{a,i}$ where $a = 2, L, \phi$ and $i = q, g$.

4.2 Method

We begin with the expression for an un-mass-factorized structure function, as discussed above in Section 2.2. In particular, we deal with the quantity $\hat{F}_a$ of Eq. (2.13) and Eq. (2.14) (the "parton-level" structure function) given by

$$\hat{F}_a(N, a_{\rm s}, \varepsilon) = C_a(N, a_{\rm s}, \varepsilon)\, Z(N, a_{\rm s}, \tfrac{1}{\varepsilon}), \tag{4.7}$$

and recall that the coefficient function C_a contains only terms which are finite in the limit $\varepsilon \to 0$ and the renormalization matrix Z contains only poles in ε. In a typical fixed-order calculation the next step is to absorb the renormalization matrix Z into the bare PDF yielding a finite result for the structure function,

$$F_a(N, a_{\rm s}, \varepsilon) = C_a(N, a_{\rm s}, \varepsilon)\, Z(N, a_{\rm s}, \varepsilon)\, \tilde{f} = C_a(N, a_{\rm s}, \varepsilon)\, f. \tag{4.8}$$

Since the renormalization matrix Z is related to the anomalous dimension of the PDF by

$$-\gamma = \frac{{\rm d}Z}{{\rm d}\ln Q^2} Z^{-1} = \beta(a_{\rm s}) \frac{{\rm d}Z}{{\rm d}a_{\rm s}} Z^{-1} \tag{4.9}$$

we can compute, order by order, a perturbative expansion of Z in terms of the expansion coefficients of γ. Such an expansion is given to $a_{\rm s}^4$ by Eq. (4.10). High-order corrections to this matrix have been computed (to NNLL accuracy only, many non-contributing terms are discarded during computation) using `FORM`, to $a_{\rm s}^{30}$ for the 2×2 matrix case and to $a_{\rm s}^{60}$ for the scalar case. These calculations become very computationally demanding, although they are not the bottleneck of the calculations of this chapter. The mass factorization of the all-order expressions that we obtain for the parton-level structure functions is more difficult and limits how deeply we can push the expansions here.

For the convenience of the reader we repeat Eq. (2.24) here,

$$
\begin{aligned}
Z = 1 &+ a_\mathrm{s}\frac{1}{\varepsilon}\gamma^{(0)} \\
&+ a_\mathrm{s}^2\Big\{\frac{1}{2\varepsilon^2}(\gamma^{(0)}-\beta_0)\gamma^{(0)} + \frac{1}{2\varepsilon}\gamma^{(1)}\Big\} \\
&+ a_\mathrm{s}^3\Big\{\frac{1}{6\varepsilon^3}(\gamma^{(0)}-\beta_0)(\gamma^{(0)}-2\beta_0)\gamma^{(0)} \\
&\qquad + \frac{1}{6\varepsilon^2}\Big[(\gamma^{(0)}-2\beta_0)\gamma^{(1)} + (\gamma^{(1)}-\beta_1)2\gamma^{(0)}\Big] + \frac{1}{3\varepsilon}\gamma^{(2)}\Big\} \\
&+ a_\mathrm{s}^4\Big\{\frac{1}{24\varepsilon^4}(\gamma^{(0)}-\beta_0)(\gamma^{(0)}-2\beta_0)(\gamma^{(0)}-3\beta_0)\gamma^{(0)} \\
&\qquad + \frac{1}{24\varepsilon^3}\Big[(\gamma^{(0)}-2\beta_0)(\gamma^{(0)}-3\beta_0)\gamma^{(1)} + (\gamma^{(0)}-3\beta_0)(\gamma^{(1)}-\beta_1)2\gamma^{(0)} \\
&\qquad + (\gamma^{(0)}-\beta_0)(\gamma^{(1)}-2\beta_1)3\gamma^{(0)}\Big] + \frac{1}{24\varepsilon^2}\Big[(\gamma^{(0)}-3\beta_0)2\gamma^{(2)} \\
&\qquad + (\gamma^{(1)}-2\beta_1)3\gamma^{(1)} + (\gamma^{(2)}-\beta_2)6\gamma^{(0)}\Big] + \frac{1}{4\varepsilon}\gamma^{(3)}\Big\} + \mathcal{O}(a_\mathrm{s}^5). \qquad (4.10)
\end{aligned}
$$

Note that the *highest order* ε-poles at each power of a_s always have coefficients which depend on the *lowest order* contributions to the anomalous dimension and beta function. This is a very important point; it means that given the nth-a_s-order contributions to γ and β we can determine the highest n poles of Z to *all* orders in a_s. The multiplication of Z by C_a (as in Eq. (4.7)) of course introduces expansion coefficients of C_a into the coefficients of the ε poles, but we can make the same observation; N^{n-1}LO knowledge of the coefficient functions and anomalous dimensions determines the highest n poles of $\hat{F}_a$ to all orders in a_s.

In the phase-space integrals of the second-order calculations of [30, 62], one can see that the 2- and 3-particle phase spaces behave as x^ε and $x^{2\varepsilon}$ in the small-x limit. A second order calculation would have, in addition to diagrams with 3-particle final states, diagrams with a 2-particle final state and a virtual correction. At small-x, we thus have behaviour of the form $x^\varepsilon + x^{2\varepsilon}$. We take inspiration from this, and also from the large-x resummations of DIS quantities [63] as well as large-x [56] and small-x [57] resummations in the context of semi-inclusive e^+e^- annihilation. We assume the un-mass-factorized structure functions to have a small-x structure of the form

$$
\hat{F}_a(x)\Big|_{a_\mathrm{s}^n} = \frac{1}{\varepsilon^{2\tilde{n}-1}}\sum_{l=0}^{\tilde{n}-1} x^{(\tilde{n}-l)\varepsilon}\left(A_a^{(n,l)} + \varepsilon B_a^{(n,l)} + \varepsilon^2 C_a^{(n,l)} + \cdots\right), \qquad (4.11)
$$

where $\tilde{n} = (n-1)$ when considering $\hat{F}_{L,ns}$ and $\tilde{n} = n$ when considering $\hat{F}_{2,ns}$ and $\hat{F}^-_{3,ns}$. The sum over l provides terms proportional to $x^{n\varepsilon}, \ldots, x^\varepsilon$. The coefficients $A_a^{(n,l)}$, $B_a^{(n,l)}$ and $C_a^{(n,l)}$ correspond to the LL, NLL and NNLL small-x contributions to $\hat{F}_a$. Taking the Mellin transform of Eq. (4.11) we find

$$
\hat{F}_a(N)\Big|_{a_\mathrm{s}^n} = \frac{1}{\varepsilon^{2\tilde{n}-1}}\sum_{l=0}^{\tilde{n}-1} \frac{1}{N+(\tilde{n}-l)\varepsilon}\left(A_a^{(n,l)} + \varepsilon B_a^{(n,l)} + \varepsilon^2 C_a^{(n,l)} + \cdots\right), \qquad (4.12)
$$

which is the form used throughout the computations of this chapter; the small-x limit becomes the small-N limit in Mellin space.

This structure contains *double poles* in ε, of the form $a_s^n\varepsilon^{-2\tilde{n}+1}$. In Section 2.2 we briefly discussed how the KLN theorem guarantees the cancellation of infra-red and final-state collinear poles in the structure function, leaving just the initial state collinear poles (these are the poles we remove using the mass factorization procedure). This implies that the double-pole terms of Eq. (4.11) and Eq. (4.12) *must have coefficients of zero.* This requirement constrains the possible values of the unknown coefficients $A_a^{(n,l)}$, $B_a^{(n,l)}$ and $C_a^{(n,l)}$. Along with our all-a_s-order knowledge of the highest n single poles of $\hat{F}_a$, we have enough relations to determine the unknown coefficients at arbitrarily high values of n. Since Eq. (4.12) describes $\hat{F}_a$ to all orders in ε for any particular n, we can claim to know the LL, NLL and NNLL contributions to $\hat{F}_a$ to *all orders in both* a_s *and* ε.

4.2.1 An Example: The LL Resummation of $\hat{F}_{2,ns}$

In this section, we discuss the resummation of the leading (LL) behaviour of $\hat{F}_{2,ns}$ in detail. We then describe how one can deduce all-a_s-order expressions for the anomalous dimension γ_{ns}^{+} and coefficient function $C_{2,ns}$ by mass factorizing $\hat{F}_{2,ns}$ at very high a_s orders. In section Section 4.3, all-a_s-order expressions for the LL, NLL and NNLL contributions to the anomalous dimension γ_{ns}^{+} and coefficient functions $C_{2,ns}$, $C_{L,ns}$ and $C_{3,ns}^{-}$ will be given.

We begin by considering the product of the expansions of the coefficient function and renormalization matrix. By inserting the expansions Eqs. (2.22) and (4.10) into Eq. (4.7) we have (noting that some terms which do not contribute at the NNLL level have already been discarded in this expression; specifically those proportional to β_0^3 or $\beta_1, \beta_2, \ldots$)

$$
\begin{aligned}
\hat{F}_{2,ns} = &+1 \\
&+ a_s\Big\{ +\varepsilon^{-1}\gamma_{ns}^{(0)} + \varepsilon^{0}c_{2,ns}^{(1,0)} + \varepsilon^{1}c_{2,ns}^{(1,1)} + \varepsilon^{2}c_{2,ns}^{(1,2)} + \cdots \Big\} \\
&+ a_s^2\Big\{ -\frac{1}{2}\varepsilon^{-2}\Big(\gamma_{ns}^{(0)}\beta_0 - {\gamma_{ns}^{(0)}}^2\Big) + \frac{1}{2}\varepsilon^{-1}\Big(2c_{2,ns}^{(1,0)}\gamma_{ns}^{(0)} + \gamma_{ns}^{(1)}\Big) \\
&\qquad + \varepsilon^{0}\Big(c_{2,ns}^{(1,1)}\gamma_{ns}^{(0)} + c_{2,ns}^{(2,0)}\Big) + \varepsilon^{1}\Big(c_{2,ns}^{(1,2)}\gamma_{ns}^{(0)} + c_{2,ns}^{(2,1)}\Big) + \cdots \Big\} \\
&+ a_s^3\Big\{ +\frac{1}{6}\varepsilon^{-3}\Big(2\gamma_{ns}^{(0)}\beta_0^2 - 3{\gamma_{ns}^{(0)}}^2\beta_0 + {\gamma_{ns}^{(0)}}^3\Big) \\
&\qquad - \frac{1}{6}\varepsilon^{-2}\Big(3c_{2,ns}^{(1,0)}\gamma_{ns}^{(0)}\beta_0 - 3c_{2,ns}^{(1,0)}{\gamma_{ns}^{(0)}}^2 - 3\gamma_{ns}^{(0)}\gamma_{ns}^{(1)} + 2\gamma_{ns}^{(1)}\beta_0\Big) \\
&\qquad + \frac{1}{6}\varepsilon^{-1}\Big(3c_{2,ns}^{(1,0)}\gamma_{ns}^{(1)} - 3c_{2,ns}^{(1,1)}\gamma_{ns}^{(0)}\beta_0 + 3c_{2,ns}^{(1,1)}{\gamma_{ns}^{(0)}}^2 + 6c_{2,ns}^{(2,0)}\gamma_{ns}^{(0)} + 2\gamma_{ns}^{(2)}\Big) \\
&\qquad + \frac{1}{2}\varepsilon^{0}\Big(c_{2,ns}^{(1,1)}\gamma_{ns}^{(1)} - c_{2,ns}^{(1,2)}\gamma_{ns}^{(0)}\beta_0 + c_{2,ns}^{(1,2)}{\gamma_{ns}^{(0)}}^2 + 2c_{2,ns}^{(2,1)}\gamma_{ns}^{(0)} + 2c_{2,ns}^{(3,0)}\Big) + \cdots \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+ a_{\mathrm{s}}^4\Big\{ + \frac{1}{24}\varepsilon^{-4}\Big(11\gamma_{ns}^{(0)^2}\beta_0^2 - 6\gamma_{ns}^{(0)^3}\beta_0 + \gamma_{ns}^{(0)^4}\Big) \\
&\quad + \frac{1}{12}\varepsilon^{-3}\Big(4c_{2,ns}^{(1,0)}\gamma_{ns}^{(0)}\beta_0^2 - 6c_{2,ns}^{(1,0)}\gamma_{ns}^{(0)^2}\beta_0 + 2c_{2,ns}^{(1,0)}\gamma_{ns}^{(0)^3} + 3\gamma_{ns}^{(0)^2}\gamma_{ns}^{(1)} \\
&\qquad\qquad - 7\gamma_{ns}^{(0)}\gamma_{ns}^{(1)}\beta_0 + 3\gamma_{ns}^{(1)}\beta_0^2\Big) \\
&\quad + \frac{1}{24}\varepsilon^{-2}\Big(12c_{2,ns}^{(1,0)}\gamma_{ns}^{(0)}\gamma_{ns}^{(1)} - 8c_{2,ns}^{(1,0)}\gamma_{ns}^{(1)}\beta_0 + 8c_{2,ns}^{(1,1)}\gamma_{ns}^{(0)}\beta_0^2 - 12c_{2,ns}^{(1,1)}\gamma_{ns}^{(0)^2}\beta_0 \\
&\qquad\qquad + 4c_{2,ns}^{(1,1)}\gamma_{ns}^{(0)^3} - 12c_{2,ns}^{(2,0)}\gamma_{ns}^{(0)}\beta_0 + 12c_{2,ns}^{(2,0)}\gamma_{ns}^{(0)^2} + 8\gamma_{ns}^{(0)}\gamma_{ns}^{(2)} \\
&\qquad\qquad + 3\gamma_{ns}^{(1)^2} - 6\gamma_{ns}^{(2)}\beta_0\Big) \\
&\quad + \frac{1}{12}\varepsilon^{-1}\Big(4c_{2,ns}^{(1,0)}\gamma_{ns}^{(2)} + 6c_{2,ns}^{(1,1)}\gamma_{ns}^{(0)}\gamma_{ns}^{(1)} - 4c_{2,ns}^{(1,1)}\gamma_{ns}^{(1)}\beta_0 + 4c_{2,ns}^{(1,2)}\gamma_{ns}^{(0)}\beta_0^2 \\
&\qquad\qquad - 6c_{2,ns}^{(1,2)}\gamma_{ns}^{(0)^2}\beta_0 + 2c_{2,ns}^{(1,2)}\gamma_{ns}^{(0)^3} + 6c_{2,ns}^{(2,0)}\gamma_{ns}^{(1)} - 6c_{2,ns}^{(2,1)}\gamma_{ns}^{(0)}\beta_0 \\
&\qquad\qquad + 6c_{2,ns}^{(2,1)}\gamma_{ns}^{(0)^2} + 12c_{2,ns}^{(3,0)}\gamma_{ns}^{(0)} + 3\gamma_{ns}^{(3)}\Big) + \cdots\Big\} \\
&+ \mathcal{O}(a_{\mathrm{s}}^5). \qquad (4.13)
\end{aligned}
$$

Only the first four ε terms at each order have been typeset here. With the exception of $\gamma_{ns}^{(3)}$ (appearing on the last line), the quantities appearing in this expansion are all known from existing fixed-order calculations to third order in a_{s}. Their three leading small-N terms are as follows,

$$
\begin{aligned}
c_{2,ns}^{(1,0)} &= + 2N^{-2} + 3N^{-1} - [5 + 2\zeta_2] \\
c_{2,ns}^{(1,1)} &= - 2N^{-3} - 3N^{-2} + [5 + 3\zeta_2]N^{-1} \\
c_{2,ns}^{(1,2)} &= + 2N^{-4} + 3N^{-3} - [5 + 3\zeta_2]N^{-2} \\
c_{2,ns}^{(2,0)} &= + 10C_FN^{-4} + (18C_F - 5\beta_0)N^{-3} + (10C_A + 6\beta_0 - [17 + 24\zeta_2]C_F)N^{-2} \\
c_{2,ns}^{(2,1)} &= - 26C_FN^{-5} - (50C_F - 13\beta_0)N^{-4} - \Big(\frac{70}{3}C_A + \frac{32}{3}\beta_0 - [47 + 68\zeta_2]C_F\Big)N^{-3} \\
c_{2,ns}^{(3,0)} &= + 60C_F^2N^{-6} + \Big(134C_F^2 - \frac{182}{3}\beta_0C_F\Big)N^{-5} - \Big(120C_A^2\zeta_2 - \frac{5}{3}\beta_0C_F - \frac{46}{3}\beta_0^2 \\
&\quad + [30 + 524\zeta_2]C_F^2 - \Big[\frac{260}{3} + 384\zeta_2\Big]C_AC_F\Big)N^{-4}, \qquad (4.14)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{ns}^{(0)} &= - 2N^{-1} - 1 - [2 - 4\zeta_2]N \\
\gamma_{ns}^{(1)} &= - 4C_FN^{-3} - (4C_F - 2\beta_0)N^{-2} + \Big(\frac{20}{3}C_A + \frac{22}{3}\beta_0 - (4 + 8\zeta_2)C_F\Big)N^{-1} \\
\gamma_{ns}^{(2)} &= - 16C_F^2N^{-5} - (24C_F^2 - 12\beta_0C_F)N^{-4} + \Big(60C_A^2\zeta_2 - \frac{64}{3}\beta_0C_F - 2\beta_0^2 \\
&\quad - \Big[\frac{80}{3} + 192\zeta_2\Big]C_AC_F - (8 - 208\zeta_2)C_F^2\Big)N^{-3}, \qquad (4.15)
\end{aligned}
$$

where an overall factor of C_F has been omitted in both Eq. (4.14) and Eq. (4.15). Inserting these into Eq. (4.13) we find the leading three ε-terms of $\hat{F}_{2,ns}$ at each order, to LL accuracy, to be

$$\begin{aligned}\hat{F}_{2,ns} =& +1\\ &+ a_{\mathrm{s}} C_F \left(-2\frac{N^{-1}}{\varepsilon} + 2N^{-2} - 2\varepsilon N^{-3} \right)\\ &+ a_{\mathrm{s}}^2 C_F^2 \left(2\frac{N^{-2}}{\varepsilon^2} - 6\frac{N^{-3}}{\varepsilon} + 14N^{-4} \right)\\ &+ a_{\mathrm{s}}^3 C_F^3 \left(-\frac{4}{3}\frac{N^{-3}}{\varepsilon^3} + 8\frac{N^{-4}}{\varepsilon^2} - \frac{100}{3}\frac{N^{-5}}{\varepsilon} \right)\\ &+ a_{\mathrm{s}}^4 C_F^4 \left(\frac{2}{3}\frac{N^{-4}}{\varepsilon^4} - \frac{20}{3}\frac{N^{-5}}{\varepsilon^3} + \frac{130}{3}\frac{N^{-6}}{\varepsilon^2} \right) + \mathcal{O}(a_{\mathrm{s}}^5). \end{aligned} \tag{4.16}$$

We of course know these three highest poles to "all" orders in a_{s}, but we do not display beyond a_{s}^4 here. Assuming the LL N-space structure to be (based on Eq. (4.12))

$$\hat{F}_{2,ns}(N)\Big|_{a_{\mathrm{s}}^n} = \frac{1}{\varepsilon^{2n-1}} \sum_{l=0}^{n-1} \frac{1}{N + (n-l)\varepsilon} A_{2,ns}^{(n,l)}, \tag{4.17}$$

one can expand the fraction as

$$\frac{1}{N + (n-l)\varepsilon} = \frac{1}{N}\left(\frac{1}{1 + (n-l)\varepsilon/N} \right) = \frac{1}{N} \sum_{i=0}^{\infty} \left(\frac{-(n-l)\varepsilon}{N} \right)^i \tag{4.18}$$

to obtain

$$\begin{aligned}\hat{F}_{2,ns} =& +1\\ &+ a_{\mathrm{s}} \Bigg(A_{2,ns}^{(1,0)} \frac{N^{-1}}{\varepsilon} - A_{2,ns}^{(1,0)} N^{-2} + A_{2,ns}^{(1,0)} \varepsilon N^{-3} - A_{2,ns}^{(1,0)} \varepsilon^2 N^{-4} + \cdots \Bigg)\\ &+ a_{\mathrm{s}}^2 \Bigg([A_{2,ns}^{(2,0)} + A_{2,ns}^{(2,1)}] \frac{N^{-1}}{\varepsilon^3} + [-2A_{2,ns}^{(2,0)} - A_{2,ns}^{(2,1)}] \frac{N^{-2}}{\varepsilon^2} + [4A_{2,ns}^{(2,0)} + A_{2,ns}^{(2,1)}] \frac{N^{-3}}{\varepsilon}\\ &\qquad + [-8A_{2,ns}^{(2,0)} - A_{2,ns}^{(2,1)}] N^{-4} + \cdots \Bigg)\\ &+ a_{\mathrm{s}}^3 \Bigg([A_{2,ns}^{(3,0)} + A_{2,ns}^{(3,1)} + A_{2,ns}^{(3,2)}] \frac{N^{-1}}{\varepsilon^5} + [-3A_{2,ns}^{(3,0)} - 2A_{2,ns}^{(3,1)} - A_{2,ns}^{(3,2)}] \frac{N^{-2}}{\varepsilon^4}\\ &\qquad + [9A_{2,ns}^{(3,0)} + 4A_{2,ns}^{(3,1)} + A_{2,ns}^{(3,2)}] \frac{N^{-3}}{\varepsilon^3}\\ &\qquad + [-27A_{2,ns}^{(3,0)} - 8A_{2,ns}^{(3,1)} - A_{2,ns}^{(3,2)}] \frac{N^{-4}}{\varepsilon^2} + \cdots \Bigg) + \mathcal{O}(a_{\mathrm{s}}^4). \end{aligned} \tag{4.19}$$

Now we can determine the coefficients $A_{2,ns}^{(i,j)}$. By comparing our two expressions for $\hat{F}_{2,ns}$ (Eqs. (4.16) and (4.19)), we can form systems of equations for the coefficients $A_{2,ns}^{(i,j)}$. These can easily be solved to yield

$$\begin{gathered} A_{2,ns}^{(1,0)} = -2C_F, \\ A_{2,ns}^{(2,0)} = -2C_F^2 \qquad A_{2,ns}^{(2,1)} = 2C_F^2, \\ A_{2,ns}^{(3,0)} = -\frac{2}{3}C_F^3 \qquad A_{2,ns}^{(3,1)} = \frac{4}{3}C_F^3 \qquad A_{2,ns}^{(3,2)} = -\frac{2}{3}C_F^3, \\ \vdots \end{gathered} \tag{4.20}$$

where we only show the coefficients to third order in a_s.

The remaining terms of Eq. (4.16) (two terms per a_s power were *not* used to determine the coefficients) provide a non-trivial verification of the solutions. For each extra power of a_s we have *one* additional coefficient to determine, but there are *two* additional double poles in the expansion of Eq. (4.17) which must vanish. The LL coefficients $A^{(i,j)}_{2,ns}$ are *over-constrained* to all orders in a_s.

We can now claim to know the leading small-x behaviour of $\hat{F}_{2,ns}$ to *all orders in* ε (we can expand the fraction in Eq. (4.17) as deeply as we please) and also to *all orders in* a_s (we can carry out this procedure as far as we please in the a_s expansion). We are limited only by how deeply in the a_s expansion we know Z (Eq. (4.10)). Using this all-order (effectively, "very high order") knowledge of $\hat{F}_{2,ns}$, we can now mass factorize to determine high-a_s-order corrections to the coefficient functions and anomalous dimensions.

4.2.2 All-Order LL Results for $C_{2,ns}$ and γ^+_{ns} in the Small-x Limit

The leading logarithmic "all-a_s" contributions to γ^+_{ns} and $C_{2,ns}$ are found to be

$$\begin{aligned}\gamma^+_{ns} = &-2C_F a_s \frac{1}{N} - 4C_F^2 a_s^2 \frac{1}{N^3} - 16C_F^3 a_s^3 \frac{1}{N^5} - 80C_F^4 a_s^4 \frac{1}{N^7} - 448C_F^5 a_s^5 \frac{1}{N^9} \\ &- 2688C_F^6 a_s^6 \frac{1}{N^{11}} - 16896C_F^7 a_s^7 \frac{1}{N^{13}} - 109824C_F^8 a_s^8 \frac{1}{N^{15}} - 732160C_F^9 a_s^9 \frac{1}{N^{17}} \\ &- 4978688C_F^{10} a_s^{10} \frac{1}{N^{19}} + \mathcal{O}(a_s^{11}) \end{aligned} \tag{4.21}$$

and

$$\begin{aligned}C_{2,ns} = 1 &+ 2C_F a_s \frac{1}{N^2} + 10C_F^2 a_s^2 \frac{1}{N^4} + 60C_F^3 a_s^3 \frac{1}{N^6} + 390C_F^4 a_s^4 \frac{1}{N^8} + 2652C_F^5 a_s^5 \frac{1}{N^{10}} \\ &+ 18564C_F^6 a_s^6 \frac{1}{N^{12}} + 132600C_F^7 a_s^7 \frac{1}{N^{14}} + 961350C_F^8 a_s^8 \frac{1}{N^{16}} \\ &+ 7049900C_F^9 a_s^9 \frac{1}{N^{18}} + 52169260C_F^{10} a_s^{10} \frac{1}{N^{20}} + \mathcal{O}(a_s^{11}). \end{aligned} \tag{4.22}$$

The contributions from a_s^{11} to a_s^{40} have been computed but are not printed here. It was not possible to perform the mass factorization of $\hat{F}_{2,ns}$ to higher order than this with the available computational resources.

The integer coefficients of Eq. (4.21) are given by sequence `A025225` of the Online Encyclopedia of Integer Sequences (OEIS) [64]; $2^n C(n-1)$, where $C(n)$ are the *Catalan numbers* defined by

$$C(n) = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0. \tag{4.23}$$

Thus one can write an all-a_s-order expression for γ^+_{ns} in terms of these coefficients,

$$\gamma^+_{ns} = -N \sum_{i=1}^{\infty} 2^i C(i-1) \left(\frac{C_F a_s}{N^2}\right)^i, \tag{4.24}$$

or noting that the generating function of the coefficients is $c(x) = (1 - \sqrt{1-8x})/2$ we can write a closed-form expression

$$\gamma_{ns}^{+} = -N\ c(C_F a_{\rm s}/N^2) = -N \left(\frac{1 - \sqrt{1 - 8\frac{C_F a_{\rm s}}{N^2}}}{2} \right). \tag{4.25}$$

Alternatively, defining the function (which proves to be slightly more convenient for the NLL and NNLL contributions later)

$$S(\xi) = \sqrt{1 - 4\xi}, \tag{4.26}$$

we can write γ_{ns}^{+} in the form

$$\gamma_{ns}^{+} = \frac{N}{2} \left(S(\xi) - 1 \right), \quad \xi = \frac{2C_F a_{\rm s}}{N^2}. \tag{4.27}$$

This is in agreement with [65]. The integer coefficients of Eq. (4.22) are given by sequence `A004981` of the OEIS,

$$\frac{2^n}{n!} \prod_{k=0}^{n-1} (4k+1), \tag{4.28}$$

which have the generating function $f(x) = (1-8x)^{-1/4}$ or alternatively

$$F(\xi) = (1 - 4\xi)^{-1/4} = S(\xi)^{-1/2}, \tag{4.29}$$

with which we can write that

$$C_{2,ns} = F(\xi), \quad \xi = \frac{2C_F a_{\rm s}}{N^2}. \tag{4.30}$$

Expanding Eqs. (4.27) and (4.30) about $\xi = 0$ recovers the explicit series of Eqs. (4.21) and (4.22).

4.3 NNLL All-Order Results for $C_{2,ns}$, $C_{L,ns}$, $C_{3,ns}^{-}$ and γ_{ns}^{+} in the Small-x Limit

With the fixed-order knowledge available (coefficient functions and anomalous dimensions to $a_{\rm s}^3$), the above procedure is readily extended to the leading *three* logarithmic contributions to the coefficient functions and anomalous dimension for the parton-level structure functions $\hat{F}_{2,ns}$, $\hat{F}_{L,ns}$ and $\hat{F}_{3,ns}^{-}$. We include the coefficients for the next-to-leading ($B_a^{(n,l)}$) and next-to-next-to-leading ($C_a^{(n,l)}$) terms in Eq. (4.12) and assume the following all-order forms,

$$\hat{F}_a(N)\bigg|_{a_{\rm s}^n} = \frac{1}{\varepsilon^{2n-1}} \sum_{l=0}^{n-1} \frac{1}{N + (n-l)\varepsilon} \left(A_a^{(n,l)} + \varepsilon B_a^{(n,l)} + \varepsilon^2 C_a^{(n,l)} \right), \quad (a = 2, 3), \tag{4.31}$$

$$\hat{F}_L(N)\bigg|_{a_{\rm s}^n} = \frac{1}{\varepsilon^{2n-3}} \sum_{l=0}^{n-2} \frac{1}{N + (n-1-l)\varepsilon} \left(A_L^{(n,l)} + \varepsilon B_L^{(n,l)} + \varepsilon^2 C_L^{(n,l)} \right). \tag{4.32}$$

The small-x limits of the coefficient functions used to determine $\hat{F}_{L,ns}$ and $\hat{F}^{-}_{3,ns}$ are given, in the style of Eq. (4.14) and Eq. (4.15), in Appendix A.8.1.

Table 4.1 shows, at each $a_{\rm s}$ order, the requirements to determine the all-order coefficients $A_a^{(n,l)}$, $B_a^{(n,l)}$ and $C_a^{(n,l)}$ $(a = 2, 3)$. The cells are filled as follows:

- 0: a double pole produced by the expansion of Eq. (4.31) which must vanish. The coefficients must combine to give zero.
- R: a single pole whose coefficient is known from the results of fixed-order perturbative calculations. It is required to determine the all-order coefficients.
- V: a single pole whose coefficient is known from the results of fixed-order perturbative calculations. It is *not* required to determine the all-order coefficients, and thus verifies that the all-order coefficients produce the correct numbers.
- P: a previously unknown coefficient, predicted by the all-order coefficients. These predictions extend to *all powers* of ε.

We see that for the LL coefficients, the double-pole zeroes and one single-pole term are sufficient to determine the all-order coefficients. We thus have *two* further terms as verification. At NLL, everything is shifted upwards by one power of ε (c.f. Eq. (4.31)). We thus only have a single term which verifies the all-order coefficients. At the NNLL level, everything is shifted by two powers of ε with respect to the LL and so we have no verification that the all-order coefficients are correct, based on knowledge from fixed-order calculations.

This may seem a little unsatisfactory, but in fact the constraints on these coefficients are a lot stronger than they first appear. Consider the $a_{\rm s}^4\varepsilon^{-1}$ term at the NNLL level which, according to Table 4.1, is an *unverified prediction* of our all-order structure. Looking at Eq. (4.13) we can see that its prediction determines the NNLL contribution of $\gamma_{ns}^{(3),+}$. $\gamma_{ns}^{(3),+}$ appears again in the $a_{\rm s}^5\varepsilon^{-2}, a_{\rm s}^5\varepsilon^{-1}, \ldots$ terms, the $a_{\rm s}^6\varepsilon^{-3}, a_{\rm s}^6\varepsilon^{-2}, \ldots$ terms, the $a_{\rm s}^7\varepsilon^{-4}, a_{\rm s}^7\varepsilon^{-3}, \ldots$ terms and so on. The coefficients of each of these terms has been *independently* predicted by the all-ε-order expressions at each of $a_{\rm s}^5$, $a_{\rm s}^6$, $a_{\rm s}^7$, ... This "unverified" coefficient in fact satisfies an *infinite* number of additional equations (of course, in practice we can only demonstrate this for some finite, computer-limited, value of n).

The crucial point is this: the "clean" mass factorization of a structure function to order $a_{\rm s}^n$ requires the mutual consistency of the first n coefficients of the ε expansion of *every* power of $a_{\rm s}$ up to $a_{\rm s}^n$. Any errors in the determination of lower-$a_{\rm s}$-power higher-ε-power coefficients will break the mass factorization of the poles at higher powers of $a_{\rm s}$.

LL	ε^{-9}	ε^{-8}	ε^{-7}	ε^{-6}	ε^{-5}	ε^{-4}	ε^{-3}	ε^{-2}	ε^{-1}	ε^{0}	ε^{1}	ε^{2}
a_s^1									R	V	V	P
a_s^2							0	R	V	V	P	P
a_s^3					0	0	R	V	V	P	P	P
a_s^4			0	0	0	R	V	V	P	P	P	P
a_s^5	0	0	0	0	R	V	V	P	P	P	P	P

NLL	ε^{-9}	ε^{-8}	ε^{-7}	ε^{-6}	ε^{-5}	ε^{-4}	ε^{-3}	ε^{-2}	ε^{-1}	ε^{0}	ε^{1}	ε^{2}
a_s^1										R	V	P
a_s^2								R	R	V	P	P
a_s^3						0	R	R	V	P	P	P
a_s^4				0	0	R	R	V	P	P	P	P
a_s^5		0	0	0	R	R	V	P	P	P	P	P

NNLL	ε^{-9}	ε^{-8}	ε^{-7}	ε^{-6}	ε^{-5}	ε^{-4}	ε^{-3}	ε^{-2}	ε^{-1}	ε^{0}	ε^{1}	ε^{2}
a_s^1											R	P
a_s^2									R	R	P	P
a_s^3							R	R	R	P	P	P
a_s^4					0	R	R	R	P	P	P	P
a_s^5			0	0	R	R	R	P	P	P	P	P

Table 4.1: *A graphical representation of the expansion of Eq.* (4.31). *The cells marked "0" and "R" are required to determine the all-order coefficients. Cells marked "V" are known from fixed-order perturbative calculations and verify the all-order coefficients. Cells marked "P" are previously unknown coefficients which are predicted by this resummation procedure, and extend to all powers of* ε.

After mass factorization we find that we can write the anomalous dimensions and coefficient functions to all orders in $a_{\rm s}$ in terms of powers of the functions $S(\xi)$ and $F(\xi)$ defined above in Eq. (4.26) and Eq. (4.29). The method here is to choose a basis of powers of these functions, with arbitrary coefficients, and solve for them by Gaussian elimination. We also determine, but do not present here, all-$a_{\rm s}$ forms for the first five ε-power contributions to the coefficient functions.

Omitting the argument of S and F, $\xi = \frac{2C_F a_{\rm s}}{N^2}$ (as above), we have

$$
\begin{aligned}
\gamma^{+}_{ns} = &+ \left[\frac{N}{2}(S-1)\right]_{LL} \\
&+ \left[\frac{a_{\rm s}}{2}\Big((S^{-1}-1)\beta_0 - 2C_F S^{-1}\Big)\right]_{NLL} \\
&- \left[\frac{a_{\rm s}N}{96C_F}\Big(12(S^{-3}+2S^{-1}+13S-96\zeta_2 S^{-1}+144\zeta_2-80\zeta_2 S)C_F^2\right. \\
&\qquad + 16(5S^{-1}-5S+72\zeta_2 S^{-1}-144\zeta_2+72\zeta_2 S)C_A C_F \\
&\qquad - 360(\zeta_2 S^{-1}-2\zeta_2+\zeta_2 S)C_A^2 - 4(3S^{-3}-28S^{-1}+25S)\beta_0 C_F \\
&\qquad \left. + 3(S^{-3}-2S^{-1}+S)\beta_0^2\Big)\right]_{NNLL}
\end{aligned}
\tag{4.33}
$$

and for the coefficient functions,

$$
\begin{aligned}
C_{2,ns} = &+ \Big[F\Big]_{LL} \\
&- \left[\frac{N}{8}\Big(4F^{-1}-3F-F^5\Big)\right. \\
&\quad \left. + \frac{N}{192C_F}\Big(-44F-6F^3+12F^5+5F^7+33F^{-1}\Big)\beta_0\right]_{NLL} \\
&+ \left[\frac{a_{\rm s}}{16}\Big(8F^3+3[3-64\zeta_2]F^5+5F^9-2[37-152\zeta_2](F^{-3}-F^{-1})\xi^{-1}\right. \\
&\quad - 2[125-384\zeta_2]F\Big)C_F + \frac{a_{\rm s}}{192}\Big(340F-15F^3+216F^5+18F^7-60F^9 \\
&\quad - 35F^{11}+232(F^{-3}-F^{-1})\xi^{-1}\Big)\beta_0 + \frac{a_{\rm s}}{6}\Big(5[1-72\zeta_2]F+[5+72\zeta_2]F^5 \\
&\quad + [5-144\zeta_2](F^{-3}-F^{-1})\xi^{-1}\Big)C_A + \frac{a_{\rm s}}{9216C_F}\Big(5111F-632F^3-2093F^5 \\
&\quad - 1232F^7+181F^9+840F^{11}+385F^{13}+1280(F^{-3}-F^{-1})\xi^{-1}\Big)\beta_0^2 \\
&\quad \left. + \frac{15a_{\rm s}}{4C_F}\Big(5F-F^5+2(F^{-3}-F^{-1})\xi^{-1}\Big)\zeta_2 C_A^2\right]_{NNLL},
\end{aligned}
\tag{4.34}
$$

$$
\begin{aligned}
C_{L,ns} = &+ \Big[4a_{\rm s}C_F F\Big]_{LL} \\
&- \left[\frac{a_{\rm s}N}{2}\Big(8F^{-1}+F-F^5\Big)C_F\right. \\
&\quad \left. - \frac{a_{\rm s}N}{48}\Big(15F^{-1}-4F+6F^3-12F^5-5F^7\Big)\beta_0\right]_{NLL} \\
&+ \left[\frac{a_{\rm s}^2}{4}\Big(-2[193-64\zeta_2]F+16F^3+[1-192\zeta_2]F^5+5F^9-2[25+8\zeta_2]F^{-3}\xi^{-1}\right. \\
&\quad + 2[41+8\zeta_2]F^{-1}\xi^{-1}\Big)C_F^2 + \frac{2a_{\rm s}^2}{3}\Big([25-72\zeta_2]F+[5+72\zeta_2]F^5
\end{aligned}
$$

$$+5(F^{-3}-F^{-1})\xi^{-1}\Big)C_AC_F+\frac{a_s^2}{48}\Big(+960F-115F^3+188F^5+38F^7$$
$$-60F^9-35F^{11}+184(F^{-3}-F^{-1})\xi^{-1}\Big)\beta_0C_F-\frac{a_s^2}{2304}\Big(1321F-424F^3$$
$$+269F^5+752F^7-181F^9-840F^{11}-385F^{13}+256(F^{-3}-F^{-1})\xi^{-1}\Big)\beta_0^2$$
$$+15a_s^2(F-F^5)\zeta_2C_A^2\Big]_{NNLL}, \tag{4.35}$$

$$C^-_{3,ns}=+\Big[F\Big]_{LL}$$
$$-\Big[\frac{N}{8}\Big(F-F^5\Big)+\frac{N}{192C_F}\Big(-44F-6F^3+12F^5+5F^7+33F^{-1}\Big)\beta_0\Big]_{NLL}$$
$$+\Big[\frac{a_s}{16}\Big(-2[157-384\zeta_2]F+[1-192\zeta_2]F^5+5F^9$$
$$-2[49-152\zeta_2](F^{-3}-F^{-1})\xi^{-1}\Big)C_F+\frac{a_s}{6}\Big(5[1-72\zeta_2]F+[5+72\zeta_2]F^5$$
$$+[5-144\zeta_2](F^{-3}-F^{-1})\xi^{-1}\Big)C_A+\frac{a_s}{192}\Big(72F+29F^3+292F^5+38F^7$$
$$-60F^9-35F^11+168(F^{-3}-F^{-1})\xi^{-1}\Big)\beta_0+\frac{a_s}{9216C_F}\Big(5111F-632F^3$$
$$-2093F^5-1232F^7+181F^9+840F^{11}+385F^{13}+\frac{1280}{\xi}(F^{-3}-F^{-1})\Big)\beta_0^2$$
$$+\frac{15a_s}{4C_F}\Big(5F-F^5+2(F^{-3}-F^{-1})\xi^{-1}\Big)\zeta_2C_A^2\Big]_{NNLL}. \tag{4.36}$$

As in the previous section, expanding the S and F functions about $\xi=0$ recovers the expansion coefficients for these expressions to any order in a_s. We now show the a_s^4 contribution to γ^+_{ns} explicitly, since it contains a term which features in the calculations of Chapter 5. We have that

$$\gamma^{(3),+}_{ns}(N)=-80\,C_F^4\,N^{-7}-C_F^3\,(160C_F-80\beta_0)\,N^{-6}-C_F^2\Big([128-1600\,\zeta_2]C_F^2$$
$$+80C_F\beta_0+[160+1536\,\zeta_2]C_FC_A\;+24\beta_0^2\;-480\,\zeta_2C_A^2\Big)N^{-5}+\mathcal{O}(N^{-4}), \tag{4.37}$$

where the β_0^2 term is of interest (and has been highlighted) as it contains the n_f^2 dependent term of $\gamma^{(3),+}_{ns}$,

$$-\frac{32}{3}C_F^2n_f^2. \tag{4.38}$$

The a_s^4 predictions for the coefficient functions are

$$c^{(4)}_{2,ns}(N)=+390C_F^4N^{-8}+\Big(1052C_F^4-1822/3\beta_0C_F^3\Big)N^{-7}+\Big(-1560C_A^2C_F^2\zeta_2$$
$$-448\beta_0C_F^3+1951/6\beta_0^2C_F^2+[336-5872\zeta_2]C_F^4$$
$$+[2180/3+4992\zeta_2]C_AC_F^3\Big)N^{-6}+\mathcal{O}(N^{-5}), \tag{4.39}$$

$$c^{(4)}_{L,ns}(N)=+240C_F^4N^{-6}+\Big(472C_F^4-992/3\beta_0C_F^3\Big)N^{-5}+\Big(-1200C_A^2C_F^2\zeta_2$$
$$+56\beta_0C_F^3+460/3\beta_0^2C_F^2-[644+4016\zeta_2]C_F^4$$
$$+[480+3840\zeta_2]C_AC_F^3\Big)N^{-4}+\mathcal{O}(N^{-3}), \tag{4.40}$$

$$c^{(4),-}_{3,ns}(N) = +\,390C_F^4N^{-8} + \Big(780C_F^4 - 1822/3\beta_0C_F^3\Big)N^{-7} + \Big(-1560C_A^2C_F^2\zeta_2 \\ - 8/3\beta_0C_F^3 + 1951/6\beta_0^2C_F^2 - [496 + 5872\zeta_2]C_F^4 \\ + [2180/3 + 4992\zeta_2]C_AC_F^3\Big)N^{-6} + \mathcal{O}(N^{-5}). \tag{4.41}$$

The a_s^5 predictions for both the anomalous dimension and coefficient functions are presented explicitly in Appendix A.9 for future reference.

To compare the numerical size of these logarithmic corrections with the fixed order results, we plot the functions in Figs. 4.1 to 4.4. In the left panel of the figures we show the fixed order corrections, to a_s^3, to the splitting function $P^+_{ns} = -\gamma^+_{ns}$ and the coefficient functions $C_{2,ns}$, $C_{L,ns}$ and $C^-_{3,ns}$. In the right panel, we show the sum of the fixed order corrections and the all-a_s resummation of the leading logarithms. We show the logarithmic approximation achievable with each fixed order, for e.g. LO knowledge allows for a LL resummation, NLO knowledge allows for a NLL resummation, etc.

We see, for all functions plotted, that the logarithmic corrections are large and do not converge. Based on these results, one cannot claim to know any form of "all-order endpoint behaviour", since the leading three logarithms alone are not indicative of any particular behaviour. Despite being of no direct phenomenological use, the corrections are mathematically interesting. The highlighted term of Eq. (4.37) provides a cross-check of the results of Chapter 5, and the other terms will provide cross-checks of future fixed-order calculations.

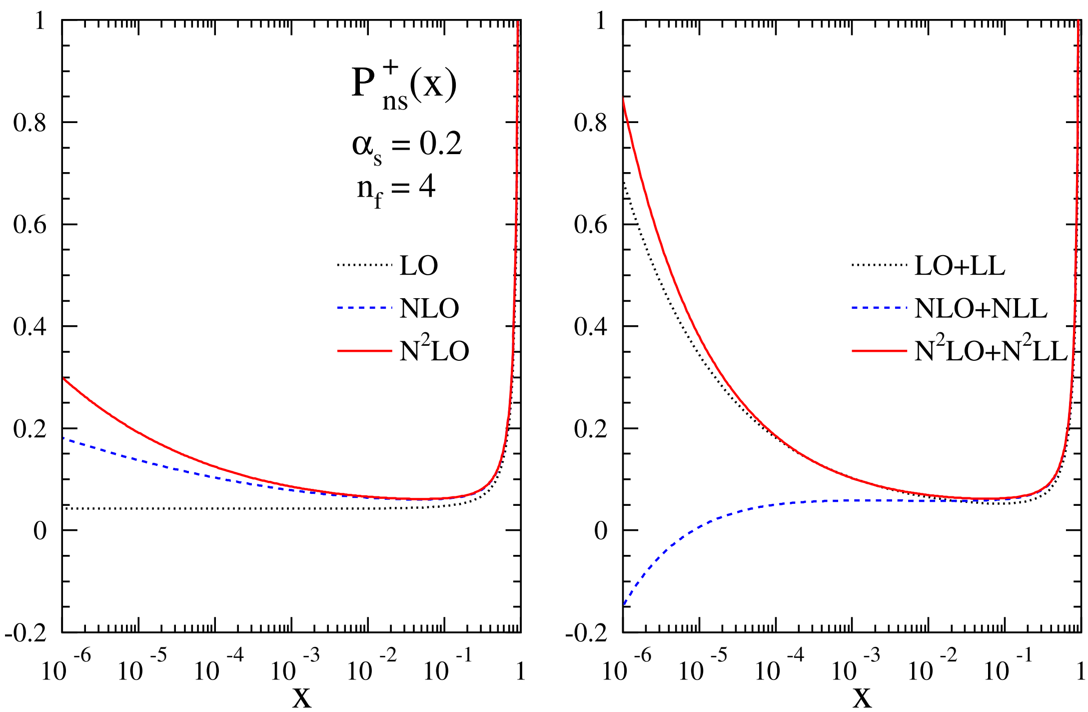

Figure 4.1: *The left panel shows the known fixed-order perturbative corrections to the splitting function P^+_{ns}. The right panel shows the three leading logarithmic corrections to all orders in a_s. The curves are plotted with the colour factors C_A and C_F taking their QCD values of* 3 *and* 4/3*, and with 4 massless flavours.*

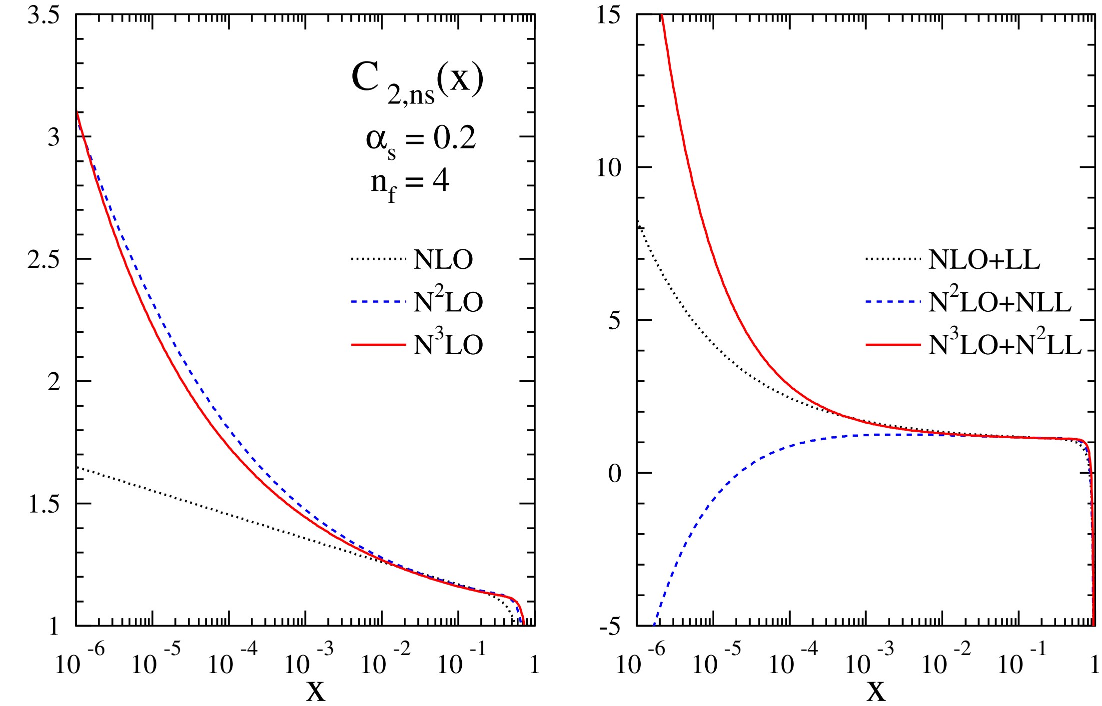

Figure 4.2: *As Fig. 4.1, for the coefficient function* $C_{2,ns}$.

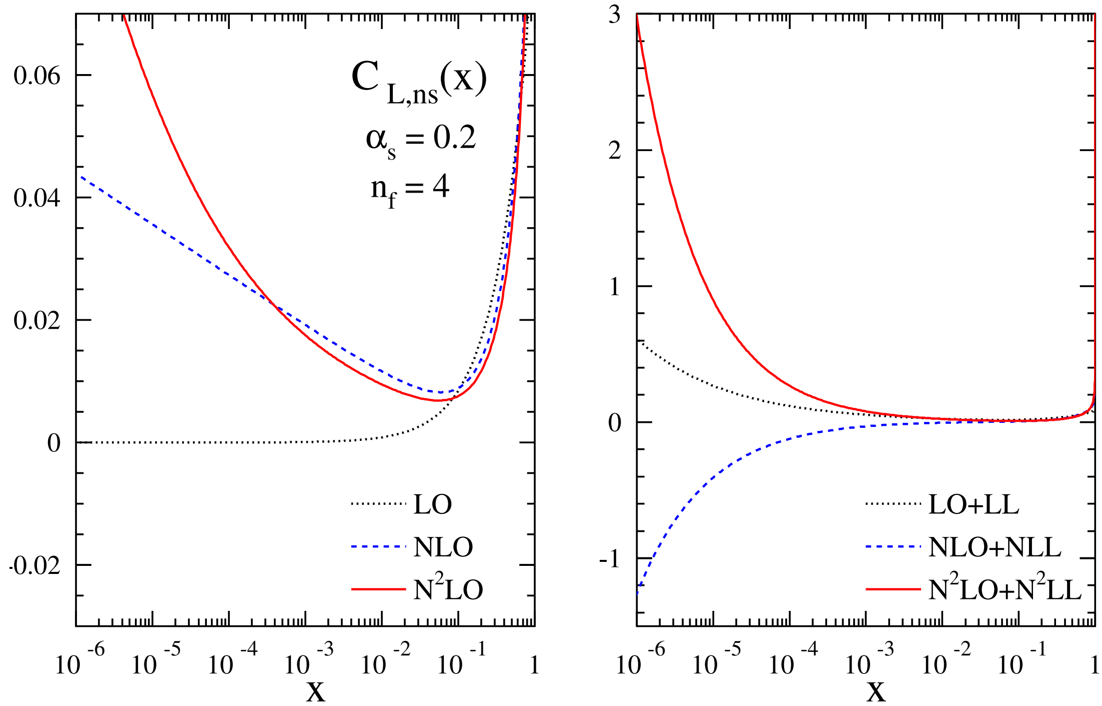

Figure 4.3: *As Fig. 4.1, for the coefficient function* $C_{L,ns}$.

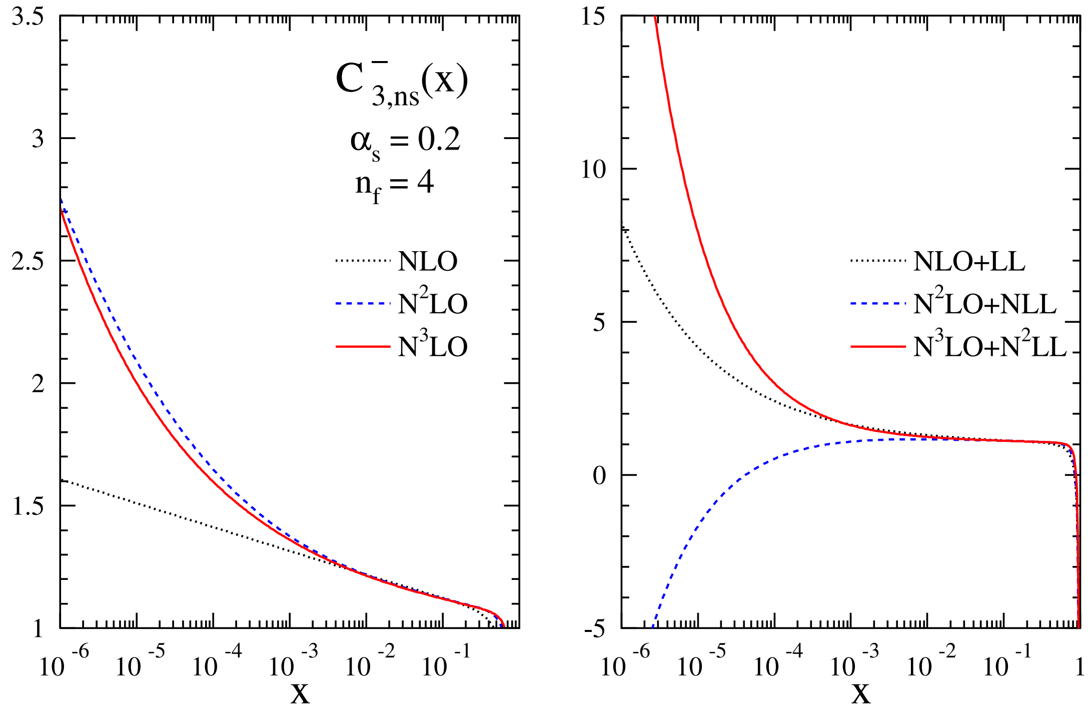

Figure 4.4: *As Fig. 4.1, for the coefficient function* $C^-_{3,ns}$.

4.4 Resummation of the Singlet Structure Functions

We now turn to the discussion of the singlet structure functions $F_{2,q}$, $F_{2,g}$, $F_{L,q}$, $F_{L,g}$, $F_{\phi,q}$ and $F_{\phi,g}$. Here we have at the parton level

$$\begin{aligned}\hat{F}_{2,q} &= C_{2,q}Z_{qq} + C_{2,g}Z_{gq}\\ \hat{F}_{2,g} &= C_{2,q}Z_{qg} + C_{2,g}Z_{gg}\\ \hat{F}_{L,q} &= C_{L,q}Z_{qq} + C_{L,g}Z_{gq}\\ \hat{F}_{L,g} &= C_{L,q}Z_{qg} + C_{L,g}Z_{gg}\\ \hat{F}_{\phi,q} &= C_{\phi,q}Z_{qq} + C_{\phi,g}Z_{gq}\\ \hat{F}_{\phi,g} &= C_{\phi,q}Z_{qg} + C_{\phi,g}Z_{gg}\end{aligned} \tag{4.42}$$

where Z_{ij} satisfies the matrix equation

$$-\begin{pmatrix}\gamma_{qq} & \gamma_{qg}\\ \gamma_{gq} & \gamma_{gg}\end{pmatrix} = \beta(a_s)\frac{\mathrm{d}}{\mathrm{d}a_s}\left[\begin{pmatrix}Z_{qq} & Z_{qg}\\ Z_{gq} & Z_{gg}\end{pmatrix}\right]\begin{pmatrix}Z_{qq} & Z_{qg}\\ Z_{gq} & Z_{gg}\end{pmatrix}^{-1}. \tag{4.43}$$

As in the non-singlet case, the entries of Z can be determined order-by-order in their a_s expansion in terms of the expansion coefficients of the anomalous dimensions. In this case, each entry of Z will depend on the expansion coefficients of all of the entries of the anomalous dimension matrix. For this reason, computing the expansion of Z for the singlet system is significantly more difficult and it is only known (at the NNLL level) to a_s^{30}.

We assume the same small-x structure for these singlet structure functions as in the non-singlet case, given by Eqs. (4.31) and (4.32). In particular, $\hat{F}_{2,i}$ and $\hat{F}_{\phi,i}$ have the same form as $\hat{F}_{2,ns}$ and $\hat{F}_{L,i}$ has the same form as $\hat{F}_{L,ns}$, (where $i = q, g$).

In exactly the same way as in Section 4.2.1, we determine the structure functions to all orders in $a_{\rm s}$ and all orders in ε. Mass factorizing the result gives us all-$a_{\rm s}$-order contributions to the singlet anomalous dimensions and all-$a_{\rm s}$- all-ε-order contributions to the corresponding coefficient functions.

4.4.1 Results

Defining

$$\begin{aligned} \gamma_{qq} &= \gamma_{ns}^{+} + \gamma_{qq,ps} \\ \gamma_{gg} &= \gamma_{ns,gg}^{+} + \gamma_{gg,ps}, \end{aligned} \tag{4.44}$$

where $\gamma_{gg,ns}^{+}$ is a "non-singlet like" quantity describing the pure-C_A terms of "Quantum Gluo-dynamics", from diagrams with an unbroken external gluon line reaching the (scalar) boson. γ_{ns}^{+} was given in Eq. (4.33), and we find that

$$\gamma_{ns,gg}^{+} = \frac{N}{2}(S(\xi') - 1), \quad \xi' = -\frac{4C_A a_{\rm s}}{N^2}. \tag{4.45}$$

For the remaining singlet contributions, a closed-form expression has not been found at the time of writing. The LL terms can be reproduced with the series of Eq. (4.46), but such series have not been found beyond the LL contributions. The NLL and NNLL contributions to high powers of $a_{\rm s}$ will be tabulated in [4].

$$\begin{aligned} \gamma_{qq,ps}^{(n)}(N) &= -C_n \frac{2^{n+1}}{N^{2n+1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-1-2i} (-2)^{i+1+k} (n_f C_F)^{i+1} C_A^k C_F^\rho \binom{k+i}{k} \binom{\rho+i+1}{\rho}, \\ \gamma_{gg,ps}^{(n)}(N) &= -C_n \frac{2^{n+1}}{N^{2n+1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-1-2i} (-2)^{i+1+k} (n_f C_F)^{i+1} C_A^k C_F^\rho \binom{k+i+1}{k} \binom{\rho+i}{\rho}, \\ \gamma_{qg}^{(n)}(N) &= -n_f C_n \frac{2^{n+1}}{N^{2n+1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2i} (-2)^{i+k} (n_f C_F)^{i} C_A^k C_F^\delta \binom{k+i}{k} \binom{\delta+i}{\delta}, \\ \gamma_{gq}^{(n)}(N) &= -\frac{2C_F}{n_f} \gamma_{qg}^{(n)}(N). \end{aligned} \tag{4.46}$$

The presence of the Catalan numbers and factors of 2^{n+1} suggests that these could be written by some generalization of the S function of the non-singlet results, but we have not been able to find a closed form. For the next-to- and next-to-next-to-leading contributions we cannot even find a series representation in the form of Eq. (4.46), although of course we can produce the expansion coefficients to "arbitrarily many" orders in $a_{\rm s}$. It should be noted here that the relation between $\gamma_{qg}^{(n)}$ and $\gamma_{gq}^{(n)}$ holds only at the leading-logarithmic level.

We present explicit predictions for the NNLL behaviour of the singlet anomalous dimensions at fourth order, since some terms feature in the calculations of Chapter 5. These terms of interest are highlighted. It should be noted here that linear combinations of these terms were computed in [66, 67], see Section 5.6.1 for more details. We have that

$$\begin{aligned}
\gamma^{(3)}_{qq}(N) \;=\; & \gamma^{(3),+}_{ns}(N) + n_f\, C_F \Big\{ N^{-7} \Big(640\, C_A^2 - 640\, C_F\, C_A + 480\, C_F^2 - 320\, n_f\, C_F \Big) \\
& + N^{-6} \Big(\frac{2176}{3}\, C_A^2 - \frac{3424}{3}\, C_F\, C_A + \frac{1024}{3}\, C_F^2 + 256\, n_f\, C_F \Big) \\
& + N^{-5} \Big(288\, n_f\, C_A - \frac{15232}{9}\, n_f\, C_F + \frac{32}{9}\, n_f^2 + \frac{8}{3}\, [519 - 524\, \zeta_2]\, C_F^2 \\
& - \frac{8}{9}\, [541 - 1332\, \zeta_2]\, C_F\, C_A + \frac{8}{9}\, [1709 - 192\, \zeta_2]\, C_A^2 \Big) \Big\} + \mathcal{O}(N^{-4}), \qquad (4.47)
\end{aligned}$$

$$\begin{aligned}
\gamma^{(3)}_{qg}(N) \;=\; & n_f \Big\{ N^{-7} \Big(640\, C_A^3 - 320\, C_F\, C_A^2 + 160\, C_F^2\, C_A - 80\, C_F^3 - 640\, n_f\, C_F\, C_A \\
& + 320\, n_f\, C_F^2 \Big) + N^{-6} \Big(\frac{416}{3}\, C_A^3 - 192\, C_F\, C_A^2 + \frac{632}{3}\, C_F^2\, C_A + \frac{32}{3}\, C_F^3 \\
& + \frac{320}{3}\, n_f\, C_A^2 + \frac{1408}{3}\, n_f\, C_F\, C_A - 432\, n_f\, C_F^2 \Big) + N^{-5} \Big(\frac{32}{9}\, n_f^2\, C_A \\
& - \frac{2224}{27}\, n_f^2\, C_F + \frac{32}{27}\, [148 + 81\, \zeta_2]\, n_f\, C_A^2 - \frac{2}{3}\, [557 - 1448\, \zeta_2]\, C_F^3 \\
& + \frac{40}{27}\, [1711 + 108\, \zeta_2]\, C_A^3 - \frac{8}{9}\, [2951 + 300\, \zeta_2]\, n_f\, C_F\, C_A \\
& - \frac{8}{27}\, [6427 - 3960\, \zeta_2]\, C_F\, C_A^2 + \frac{2}{27}\, [6707 - 19368\, \zeta_2]\, C_F^2\, C_A \\
& + \frac{4}{27}\, [13583 - 3600\, \zeta_2]\, n_f\, C_F^2 \Big) \Big\} + \mathcal{O}(N^{-4}), \qquad (4.48)
\end{aligned}$$

$$\begin{aligned}
\gamma^{(3)}_{gq}(N) \;=\; & C_F \Big\{ N^{-7} \Big(-1280\, C_A^3 + 640\, C_F\, C_A^2 - 320\, C_F^2\, C_A + 160\, C_F^3 \\
& + 1280\, n_f\, C_F\, C_A - 640\, n_f\, C_F^2 \Big) + N^{-6} \Big(-\frac{4160}{3}\, C_A^3 + 1280\, C_F\, C_A^2 \\
& - \frac{2800}{3}\, C_F^2\, C_A + 320\, C_F^3 - \frac{640}{3}\, n_f\, C_A^2 + 640\, n_f\, C_F\, C_A - \frac{800}{3}\, n_f\, C_F^2 \Big) \\
& + N^{-5} \Big(-\frac{64}{9}\, n_f^2\, C_A + \frac{12256}{27}\, n_f^2\, C_F + \frac{4}{3}\, [25 - 1248\, \zeta_2]\, C_F^3 \\
& - \frac{64}{27}\, [542 + 81\, \zeta_2]\, n_f\, C_A^2 + \frac{16}{3}\, [817 + 164\, \zeta_2]\, n_f\, C_F\, C_A \\
& - \frac{16}{27}\, [1969 + 936\, \zeta_2]\, C_F\, C_A^2 - \frac{16}{27}\, [3871 + 2340\, \zeta_2]\, C_A^3 \\
& - \frac{8}{27}\, [7747 - 2448\, \zeta_2]\, n_f\, C_F^2 + \frac{4}{27}\, [8633 + 12672\, \zeta_2]\, C_F^2\, C_A \Big) \Big\} \\
& + \mathcal{O}(N^{-4}), \qquad (4.49)
\end{aligned}$$

$$\begin{aligned}
\gamma^{(3)}_{gg}(N) \;=\; & N^{-7} \Big(-1280\, C_A^4 + 1920\, n_f\, C_F\, C_A^2 - 640\, n_f\, C_F^2\, C_A + 160\, n_f\, C_F^3 \\
& - 320\, n_f^2\, C_F^2 \Big) + N^{-6} \Big(-\frac{640}{3}\, C_A^4 - \frac{1280}{3}\, n_f\, C_A^3 - \frac{1856}{3}\, n_f\, C_F\, C_A^2 \\
& - \frac{256}{3}\, n_f\, C_F^2\, C_A - \frac{64}{3}\, n_f\, C_F^3 + \frac{640}{3}\, n_f^2\, C_F\, C_A + \frac{1472}{3}\, n_f^2\, C_F^2 \Big) \\
& + N^{-5} \Big(-\frac{128}{3}\, n_f^2\, C_A^2 + \frac{4768}{9}\, n_f^2\, C_F\, C_A - \frac{19904}{9}\, n_f^2\, C_F^2 + \frac{32}{9}\, n_f^3\, C_F \\
& - \frac{128}{3}\, [20 + 9\, \zeta_2]\, n_f\, C_A^3 - 32\, [137 + 64\, \zeta_2]\, C_A^4 + \frac{8}{3}\, [195 - 148\, \zeta_2]\, n_f\, C_F^3 \\
& - \frac{8}{9}\, [1997 - 756\, \zeta_2]\, n_f\, C_F^2\, C_A + \frac{8}{3}\, [2751 + 688\, \zeta_2]\, n_f\, C_F\, C_A^2 \Big) + \mathcal{O}(N^{-4}). \qquad (4.50)
\end{aligned}$$

Similarly, we can describe the leading-logarithmic contributions to the coefficient functions with a series but not with a closed-form expression. Defining

$$
\begin{aligned}
C_{2,q} &= C_{2,ns} + C_{2,ps}, \\
C_{L,q} &= C_{L,ns} + C_{L,ps}, \\
C_{\phi,g} &= C^{+}_{\phi,g,ns} + C_{\phi,g,ps},
\end{aligned} \tag{4.51}
$$

we have for the "non-singlet-like" part of $C_{\phi,g}$

$$
C_{\phi,g,ns}(N) = F(\xi'), \tag{4.52}
$$

and for the remaining singlet contributions

$$
\begin{aligned}
c^{(n)}_{2,ps}(N) &= \mathcal{D}_n \frac{2^n}{N^{2n}} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \sum_{k=0}^{n-2-2i} (-2)^{i+1+k} (n_f C_F)^{i+1} C_A^k C_F^{\rho'} \binom{k+i}{k} \binom{\rho'+i+1}{\rho'}, \\
c^{(n)}_{2,g}(N) &= n_f \mathcal{D}_n \frac{2^n}{N^{2n}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-1-2i} (-2)^{i+k} (n_f C_F)^{i} C_A^k C_F^{\delta'} \binom{k+i}{k} \binom{\delta'+i}{\delta'}, \\
c^{(n)}_{L,ps}(N) &= \mathcal{D}_{n-1} \frac{2^{n+1}}{N^{2n-2}} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \sum_{k=0}^{n-2-2i} (-2)^{i+1+k} (n_f C_F)^{i+1} C_A^k C_F^{\rho'} \binom{k+i}{k} \binom{\rho'+i+1}{\rho'}, \\
c^{(n)}_{L,g}(N) &= n_f \mathcal{D}_{n-1} \frac{2^{n+1}}{N^{2n-2}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-1-2i} (-2)^{i+k} (n_f C_F)^{i} C_A^k C_F^{\delta'} \binom{k+i}{k} \binom{\delta'+i}{\delta'}, \\
c^{(n)}_{\phi,q}(N) &= -C_F \mathcal{D}_n \frac{2^{n+1}}{N^{2n}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-1-2i} (-2)^{i+k} (n_f C_F)^{i} C_A^k C_F^{\delta'} \binom{k+i}{k} \binom{\delta'+i}{\delta'}, \\
c^{(n)}_{\phi,g,ps}(N) &= \mathcal{D}_n \frac{2^n}{N^{2n}} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \sum_{k=0}^{n-2-2i} (-2)^{i+1+k} (n_f C_F)^{i+1} C_A^k C_F^{\rho'} \binom{k+i+1}{k} \binom{\rho'+i}{\rho'},
\end{aligned} \tag{4.53}
$$

where $\rho' = n-k-2i-2$, $\delta' = n-k-2i-1$ and the symbol $\mathcal{D}_n$ is defined as

$$
\mathcal{D}_n = \frac{1}{n!} \prod_{k=0}^{n-1} (1+4k) \, . \tag{4.54}
$$

$2^n \mathcal{D}_n$ are the expansion coefficients of the function $F(\xi)$ defined in Eq. (4.29), again hinting at some deeper structure which is worth further investigation in the future.

The explicit predictions for the NNLL behaviour of the a_s^4 contributions to the coefficient functions $C_{2,q}$, $C_{2,g}$, $C_{L,q}$ and $C_{L,g}$ are as follows,

$$
\begin{aligned}
c^{(4)}_{2,q}(N) \;=\; & c^{(4)}_{2,ns}(N) + n_f\, C_F \Big\{ N^{-8} \Big(-3120\, C_A^2 + 3120\, C_F\, C_A - 2340\, C_F^2 \\
& + 1560\, n_f\, C_F \Big) + N^{-7} \Big(-\frac{60872}{9}\, C_A^2 + \frac{86228}{9}\, C_F\, C_A - \frac{7798}{3}\, C_F^2 \\
& + \frac{5216}{9}\, n_f\, C_A - \frac{16688}{9}\, n_f\, C_F \Big) + N^{-6} \Big(+ \frac{9848}{27}\, n_f\, C_A - \frac{952}{9}\, n_f^2 \\
& - \frac{1}{3}\, [16611 - 21752\, \zeta_2]\, C_F^2 + \frac{8}{27}\, [24251 - 20439\, \zeta_2]\, C_F\, C_A \\
& + \frac{2}{27}\, [124393 - 14688\, \zeta_2]\, n_f\, C_F - \frac{2}{27}\, [242611 - 22752\, \zeta_2]\, C_A^2 \Big) \Big\} \\
& + \mathcal{O}(N^{-5})\, ,
\end{aligned} \tag{4.55}
$$

$$
\begin{aligned}
c_{2,g}^{(4)}(N) \;=\; & n_f \Big\{ N^{-8} \Big(-3120\, C_A^3 + 1560\, C_F\, C_A^2 - 780\, C_F^2\, C_A + 390\, C_F^3 \\
& + 3120\, n_f\, C_F\, C_A - 1560\, n_f\, C_F^2 \Big) + N^{-7} \Big(-\frac{35132}{9}\, C_A^3 + \frac{30052}{9}\, C_F\, C_A^2 \\
& - \frac{21101}{9}\, C_F^2\, C_A + \frac{889}{3}\, C_F^3 + \frac{536}{9}\, n_f\, C_A^2 - \frac{2056}{3}\, n_f\, C_F\, C_A + \frac{13778}{9}\, n_f\, C_F^2 \\
& - \frac{2608}{9}\, n_f^2\, C_F \Big) + N^{-6} \Big(-\frac{248}{27}\, n_f^2\, C_A + \frac{20300}{27}\, n_f^2\, C_F \\
& + \frac{52}{3}\, [771 - 41\, \zeta_2]\, n_f\, C_F\, C_A + \frac{1}{6}\, [2453 - 23816\, \zeta_2]\, C_F^3 \\
& - \frac{4}{27}\, [2882 + 1647\, \zeta_2]\, n_f\, C_A^2 + \frac{67}{27}\, [3265 - 1512\, \zeta_2]\, C_F\, C_A^2 \\
& - \frac{1}{27}\, [19957 - 145440\, \zeta_2]\, C_F^2\, C_A - \frac{8}{27}\, [25579 - 9972\, \zeta_2]\, n_f\, C_F^2 \\
& - \frac{10}{27}\, [48911 + 846\, \zeta_2]\, C_A^3 \Big) \Big\} + \mathcal{O}(N^{-5}), \qquad (4.56)
\end{aligned}
$$

$$
\begin{aligned}
c_{L,q}^{(4)}(N) \;=\; & c_{L,ns}^{(4)}(N) + n_f\, C_F \Big\{ N^{-6} \Big(-1920\, C_A^2 + 1920\, C_F\, C_A - 1440\, C_F^2 \\
& + 960\, n_f\, C_F \Big) + N^{-5} \Big(-\frac{24640}{9}\, C_A^2 + \frac{37408}{9}\, C_F\, C_A - \frac{2048}{3}\, C_F^2 \\
& + \frac{2176}{9}\, n_f\, C_A - \frac{13024}{9}\, n_f\, C_F \Big) + N^{-4} \Big(-\frac{5696}{27}\, n_f\, C_A - \frac{128}{3}\, n_f^2 \\
& - \frac{32}{3}\, [49 - 361\, \zeta_2]\, C_F^2 - \frac{224}{27}\, [698 - 207\, \zeta_2]\, C_A^2 \\
& - \frac{8}{27}\, [4913 + 11988\, \zeta_2]\, C_F\, C_A + \frac{16}{27}\, [8461 - 1188\, \zeta_2]\, n_f\, C_F \Big) \Big\} \\
& + \mathcal{O}(N^{-3}), \qquad (4.57)
\end{aligned}
$$

$$
\begin{aligned}
c_{L,g}^{(4)}(N) \;=\; & n_f \Big\{ N^{-6} \Big(-1920\, C_A^3 + 960\, C_F\, C_A^2 - 480\, C_F^2\, C_A + 240\, C_F^3 \\
& + 1920\, n_f\, C_F\, C_A - 960\, n_f\, C_F^2 \Big) + N^{-5} \Big(-\frac{8800}{9}\, C_A^3 + \frac{9248}{9}\, C_F\, C_A^2 \\
& - \frac{8296}{9}\, C_F^2\, C_A - \frac{16}{3}\, C_F^3 - \frac{704}{9}\, n_f\, C_A^2 - \frac{4640}{3}\, n_f\, C_F\, C_A + \frac{13648}{9}\, n_f\, C_F^2 \\
& - \frac{1088}{9}\, n_f^2\, C_F \Big) + N^{-4} \Big(-\frac{64}{27}\, n_f^2\, C_A + \frac{11776}{27}\, n_f^2\, C_F \\
& - \frac{4}{3}\, [115 + 1964\, \zeta_2]\, C_F^3 - \frac{32}{27}\, [263 + 162\, \zeta_2]\, n_f\, C_A^2 \\
& + \frac{16}{3}\, [1231 - 118\, \zeta_2]\, n_f\, C_F\, C_A - \frac{32}{27}\, [6314 - 459\, \zeta_2]\, C_A^3 \\
& + \frac{4}{27}\, [6487 + 24048\, \zeta_2]\, C_F^2\, C_A + \frac{8}{27}\, [8785 - 7722\, \zeta_2]\, C_F\, C_A^2 \\
& - \frac{8}{27}\, [14249 - 5256\, \zeta_2]\, n_f\, C_F^2 \Big) \Big\} + \mathcal{O}(N^{-3}). \qquad (4.58)
\end{aligned}
$$

Explicit expressions for the a_{s}^5 small-x contributions are given in Appendix A.9 for future reference.

4.5 Conclusions

In this chapter, we have computed x^0 double logarithmic small-x contributions to coefficient functions and anomalous dimensions to *all orders* in the strong coupling constant a_{s}. By inspecting the D-dimensional structure of the phase space of existing fixed-order perturbative calculations, we were able to make an assumption for the all-order structure of un-mass-factorized parton-level structure functions which allowed their computation, in the small-x limit, not just to all orders in a_{s} but also to all orders in the dimensional regularization parameter ε. Such knowledge allows for the mass factorization of the structure function to arbitrary order in a_{s}, yielding the all-order expressions for the coefficient functions and anomalous dimensions.

In the non-singlet sector we were able to compute the leading three logarithmic contributions to the coefficient functions $C_{2,ns}$, $C_{L,ns}$ and $C^{-}_{3,ns}$ and to the anomalous dimension γ^{+}_{ns}. We constructed closed all-a_{s}-order expressions for these functions, first by inspecting the coefficients with the help of online resources and then by making suitable guesses of the functional bases required to describe the coefficients.

In the singlet sector we were not able to determine closed-form expressions for the logarithmic corrections to either the coefficient functions or the anomalous dimensions. The LL terms are described by means of series, the overall coefficients of which are related to the expansion coefficients of the functions used for the non-singlet expressions. There are tantalizing hints that a "nice" closed-form expression should be achievable but at the time of writing it has not been found. This will be the topic of future research.

We showed by plotting the non-singlet results that knowledge of just the three leading contributions is insufficient to describe the functions at any reasonable values of x; the leading three all-order corrections do not converge. However this knowledge is nonetheless useful in a more mathematical context. The a_{s}^4 terms of the expressions for the anomalous dimensions computed here provide a cross-check of the results of Chapter 5, increasing our confidence that they are correct.

Chapter 5

Large-n_f Contributions to the Four-Loop QCD Splitting Functions

The first approximations to the third-order contributions to the splitting functions and DIS coefficient functions were determined from a small number of Mellin moments computed [24,68,69,70] with the `MINCER` package [44,45]. These approximations became available some 5 years before a full analytic result was computed, see for e.g. [18, 37, 71, 72].

With the recent development of the `FORCER` package [73, 74] for `FORM`, we are now in a similar position at the four-loop level. `FORCER` is able to compute Mellin moments of the DIS parton-level structure functions to *fourth* order in a_s. Like `MINCER` it implements a parametric reduction of the integrals, yielding results in terms of known master integrals. The usual mass factorization procedure, as described in Chapter 2, yields Mellin moments of the splitting functions (or, anomalous dimensions) and coefficient functions to this order. As might be expected, the calculations of these moments is much more computationally demanding than their third-order counterparts.

At the time of writing, the Mellin moments ($N = 1, 2, \ldots, 6$) have been computed in full [1] for the non-singlet structure functions and moments ($N = 2, 4$) for the singlet structure functions. These moments alone are not sufficient to produce x-space approximations, but more will be available in the near future. Some Mellin moments ($N = 2, 3, 4$) of the non-singlet anomalous dimension have also been computed by other methods (see [75,76,77]) as well as the first moment of $c^{(4),+}_{3,ns}$ [78]. The results of `FORCER` are in agreement.

The topic of this chapter is not the x-space splitting function approximations or even the computation of the Mellin moments of the structure functions (i.e. the internal workings of `FORCER`), but rather the reconstruction of analytic all-N formulae for particular parts of the fourth-order contributions. Further discussions of the results of this chapter will be published in [5].

To third order, the anomalous dimensions can be written in terms of *harmonic sums* (defined in Appendix A.1) and powers of simple "denominator functions" in N, for which we define the notation

$$D_i = \frac{1}{N+i}. \tag{5.1}$$

The *harmonic weight* of a harmonic sum is defined to be the sum of the absolute values of its indices. We define the *overall weight* of a term to be the sum of its harmonic weight and the power of its denominator function, if present.

To third order, the anomalous dimensions $\gamma_{ij}^{(n)}$ contain terms of maximum overall weight $2n+1$. One would expect, then, that the fourth-order contributions $\gamma_{ij}^{(3)}$ can be written in terms of overall weight 7 combinations. The subsets of diagrams with colour factors proportional to powers of n_f contain only terms with harmonic sums of reduced harmonic weight and so have a much smaller potential functional basis than their n_f^0 counterparts. These "large-n_f" diagram subsets are also (by far) the easiest for `FORCER` to compute; they consist of simpler topologies and many 2 and 3 loop diagrams with gluon propagator loop insertions.

Equipped with some number of Mellin moments for the large-n_f terms of the fourth-order anomalous dimensions and "educated guesses" of their functional bases, we aim to compute the analytic all-N expressions for these Mellin moments. This technique was used in the evaluation of the third-order corrections to the polarized (helicity dependent) splitting functions [79]. We will see that some of the expressions below are rather more difficult to solve, but the method is very similar. Another work which has used related techniques to reconstruct analytic formulae from Mellin moments is [80].

For the non-singlet anomalous dimensions $\gamma_{ns}^{(3),\pm}$, the n_f^3 contribution is already known [81]. Here we aim to compute the n_f^2 contributions, for which we have 57 contributing (meta-)diagrams. In the singlet sector we aim to compute the n_f^3 contributions, which are currently unknown except for the linear combinations of [66, 67]. For the singlet structure functions $F_{2,q}$, $F_{2,g}$, $F_{\phi,q}$ and $F_{\phi,g}$ we have just 6, 36, 6 and 70 contributing (meta-)diagrams to fourth-order in a_s. Additionally for $F_{2,g}$ and $F_{\phi,g}$ we must compute 8 and 6 (meta-)diagrams with external ghosts, due to the un-physical gluon helicity projection used by `FORCER`. One may use a physical projection, removing the need for these external ghosts, but this is much more demanding to compute (as demonstrated at three loops in [24]).

These large-n_f diagram sets being small and "easy" to compute makes such analytic reconstructions viable. We will see that nonetheless, these "easy" diagrams become *very* computationally demanding for high values of N. The remaining diagrams (with fewer n_f powers than what we consider here) are sufficiently difficult to compute that finding analytic expressions with the methods of this chapter is impossible, even with a large supercomputer.

The reconstruction procedure, then, is as follows,

- Compute Mellin moments of the large-n_f contributions to the structure functions using `FORCER`. Renormalize and mass factorize the resulting expressions, yielding Mellin moments of the large-n_f terms of the anomalous dimensions.
- Determine bases of functions that should describe them in Mellin space, taking inspiration from the known lower order quantities.
- The moments and these bases form systems of equations, with an unknown coefficient for each basis function. Solve this system (using that they are Diophantine systems, see the following discussion), yielding analytical Mellin space expressions for the anomalous dimensions.

5.1 Defining a Basis

Here we introduce some notation to facilitate the description of the functional structure of lower-order anomalous dimensions and the bases used to determine analytic expressions for the moments of fourth-order anomalous dimensions. We define the following sets of harmonic sums,

$$\begin{aligned}
SW0 &= \{1\},\\
SW1 &= \{\mathrm{S}_1\},\\
SW2 &= \{\mathrm{S}_2, \mathrm{S}_{-2}, \mathrm{S}_{1,1}\},\\
SW3 &= \{\mathrm{S}_3, \mathrm{S}_{-3}, \mathrm{S}_{2,1}, \mathrm{S}_{1,2}, \mathrm{S}_{-2,1}, \mathrm{S}_{1,-2}, \mathrm{S}_{1,1,1}\},
\end{aligned} \tag{5.2}$$

where we *skip* harmonic sums containing indices -1; these are not present in any coefficient function or anomalous dimension to third order. The generalization to a set SWN, i.e. "harmonic sums of harmonic weight N", should be clear. In addition we define sets which skip not just sums containing indices -1, but sums containing *any* negative index. We denote these

$$\begin{aligned}
SW2{+} &= \{\mathrm{S}_2, \mathrm{S}_{1,1}\},\\
SW3{+} &= \{\mathrm{S}_3, \mathrm{S}_{2,1}, \mathrm{S}_{1,2}, \mathrm{S}_{1,1,1}\}.
\end{aligned} \tag{5.3}$$

Again the generalization to $SWN{+}$, "all-positive index harmonic sums of harmonic weight N", should be clear.

We will describe the functional structure of the third-order anomalous dimensions, as well as define bases for the reconstruction of new fourth-order quantities, with tables in the format of Table 5.1.

Harmonic Sums	Denominators
$SW2$	1, $D_i^{1,\ldots,a}$
$SW1$	1, $D_i^{1,\ldots,b}$
$SW0$	1, $D_i^{1,\ldots,c}$

Table 5.1: *The format in which we will define bases of functions for the reconstruction of analytic expressions for the Mellin moments of anomalous dimensions.*

For each entry of the specified harmonic sum set, we include products with the objects in the Denominators column. An entry of 1 is to be interpreted as one might expect – we include the bare sums. A $D_i^{1,\ldots,a}$ is to mean that we include products of the sums with *each* of $D_i^1, D_i^2, \ldots, D_i^{a-1}, D_i^a$. Each element of a basis has its own coefficient, to be determined by the reconstruction procedure.

We must pull some factors out of these coefficients since the algorithm used to fix them requires them to be *integers* (see the discussion in Section 5.2). We will assign these factors based on the overall weight of the term, and refer to them as *coefficient factors*. They will be specified in a second table, in the format of Table 5.2. The required values of these factors will be discussed in Section 5.3.1.

Overall Weight	3	2	1
Coeff. Factors	d	e	f

Table 5.2: *The format in which we will define coefficient factors for bases.*

5.2 Solving Diophantine Equation Systems

In Section 5.1 we briefly alluded to the requirement that the unknown coefficients of our basis should be *integer* coefficients. This is an important point; in general we will not be able to compute a sufficient number of Mellin moments to determine the coefficients in full generality (solution by, say, Gaussian elimination which would allow the coefficients to take *rational* values). As we will see in Section 5.3.1, the denominators of the coefficients of the third-order anomalous dimensions appear in a structured and predictable way. Arranging our basis to make the unknown coefficients integers proves to be quite powerful.

Rather than a general system of linear equations for the coefficients (one equation per computed Mellin moment) we thus have a *Diophantine* system of linear equations; a system of equations with *integer* solutions. One can find solutions to such a system using fewer equations than the number of unknown coefficients to be determined. Of course, these solutions will not necessarily be be unique. We discuss later how we can convince ourselves that a particular solution of a system is the "correct" solution.

A method based on the *Lenstra-Lenstra-Lovász (LLL) lattice reduction* algorithm [82] is used here. Given a basis describing some lattice, the algorithm finds a *short* (in the sense that the vectors have a small norm), *nearly orthogonal*, basis for the same lattice in polynomial time.

The number-theory calculator program `CALC` [83] includes a routine called `AXB` [84] (summarized in [85]), intended to provide short integer solutions to matrix equations $\mathbf{A}\vec{X} = \vec{B}$ using LLL lattice reduction. We provide it with a matrix of our basis elements evaluated at the appropriate N values and a vector of the Mellin moments we wish to reproduce with that basis. Each row is suitably normalized such that the entries are integers. This is the solver used throughout this chapter. If there are few enough coefficients to determine, it perform a Gaussian elimination. We now show an explicit example of the reconstruction of a low-order quantity using this method.

5.2.1 An Example Reconstruction

As a simple, yet demonstrative, example of the method outlined in Section 5.2, consider the determination of the analytic form of the $C_A n_f$ part of $\gamma_{qg}^{(1)}$ from its Mellin moments. It is given by

$$\begin{aligned}\gamma_{qg}^{(1)}\Big|_{C_A n_f} = +\Big[&8(2D_2 - 2D_1 + D_0)\mathrm{S}_{-2} + 8(2D_2 - 2D_1 + D_0)\mathrm{S}_{1,1} + 16(D_2^2 - D_1^2)\mathrm{S}_1 \\ &+ 8(4D_2^3 + 2D_1^3 + D_0^3)\Big]_{OW3} + \Big[\frac{4}{3}(44D_2^2 + 12D_1^2 + 3D_0^2)\Big]_{OW2} \\ &+ \Big[-\frac{4}{9}(20D_{-1} - 146D_2 + 153D_1 - 18D_0)\Big]_{OW1}\end{aligned} \tag{5.4}$$

where the square brackets collect together terms of the same overall weight. Note that the harmonic weight 2 sums come with the same combination of denominator functions, $D_0 - 2D_1 + 2D_2$. This is proportional to the the leading order contribution $\gamma_{qg}^{(0)}$. That this combination appears with the highest weight harmonic sums will be used later to assist in the reconstructions.

Suppose we choose the basis (in the notation of Section 5.1) given in Table 5.3. With the coefficient factors given, the coefficients that we must determine are all integers and we can use `AXB` to attempt a solution.

Harmonic Sums	Denominators
$SW2$	D_0 , D_1 , D_2
$SW1$	$D_0^{1,2}$, $D_1^{1,2}$, $D_2^{1,2}$
$SW0$	$D_0^{1,2,3}$, $D_1^{1,2,3}$, $D_2^{1,2,3}$, D_{-1}

Overall Weight	3	2	1
Coeff. Factors	4	$\frac{2}{3}$	$\frac{1}{9}$

Table 5.3: *A basis for the reconstruction of the $C_A n_f$ terms of $\gamma_{qg}^{(1)}$.*

This basis has 25 unknown integer coefficients. We attempt to determine them from some number of Mellin moments of the function. The first 11 are given below,

$$
\begin{aligned}
\gamma_{qg}^{(1)}\Big|_{C_A n_f}(N=2) &= -\frac{35}{3^3} \\
\gamma_{qg}^{(1)}\Big|_{C_A n_f}(N=4) &= \frac{16387}{2^3\,3^2\,5^3} \\
\gamma_{qg}^{(1)}\Big|_{C_A n_f}(N=6) &= \frac{867311}{2^3\,3^3\,5^1\,7^3} \\
\gamma_{qg}^{(1)}\Big|_{C_A n_f}(N=8) &= \frac{100911011}{2^6\,3^6\,5^3\,7^1} \\
\gamma_{qg}^{(1)}\Big|_{C_A n_f}(N=10) &= \frac{373810079}{2^3\,3^4\,5^2\,7^1\,11^3} \\
\gamma_{qg}^{(1)}\Big|_{C_A n_f}(N=12) &= \frac{653436358741}{2^4\,3^4\,5^2\,7^3\,11^1\,13^3} \\
\gamma_{qg}^{(1)}\Big|_{C_A n_f}(N=14) &= \frac{386324173}{2^6\,3^3\,5^2\,7^3\,11^1} \\
\gamma_{qg}^{(1)}\Big|_{C_A n_f}(N=16) &= \frac{56849473253143}{2^9\,3^6\,5^2\,7^2\,11^1\,17^3} \\
\gamma_{qg}^{(1)}\Big|_{C_A n_f}(N=18) &= \frac{106266207488029}{2^4\,3^6\,5^1\,7^2\,11^1\,13^1\,17^1\,19^3} \\
\gamma_{qg}^{(1)}\Big|_{C_A n_f}(N=20) &= \frac{1006804883130941}{2^3\,3^5\,5^3\,7^3\,11^3\,13^1\,17^1\,19^1} \\
\gamma_{qg}^{(1)}\Big|_{C_A n_f}(N=22) &= \frac{108581251285561567}{2^6\,3^5\,7^2\,11^3\,13^1\,17^1\,19^1\,23^3}.
\end{aligned}
\tag{5.5}
$$

The denominators have been prime factorized, since we will make some observations and arguments based on the prime structure of the denominators in later sections.

We thus have a system of equations like Eq. (5.6) (for $(N=2)$), where C_i denotes the coefficient of basis element i. We have multiplied by appropriate factors to remove all denominators from the equation,

$$\begin{aligned}-560 = {} & 720\, C_{\mathrm{S}_2 D_1} + 540\, C_{\mathrm{S}_2 D_2} + 1080\, C_{\mathrm{S}_2 D_0} - 432\, C_{\mathrm{S}_{-2} D_1} - 324\, C_{\mathrm{S}_{-2} D_2} - 648\, C_{\mathrm{S}_{-2} D_0} \\ & + 1008\, C_{\mathrm{S}_{1,1} D_1} + 756\, C_{\mathrm{S}_{1,1} D_2} + 1512\, C_{\mathrm{S}_{1,1} D_0} + 144\, C_{\mathrm{S}_1 D_1} + 288\, C_{\mathrm{S}_1 D_1^2} \\ & + 108\, C_{\mathrm{S}_1 D_2} + 162\, C_{\mathrm{S}_1 D_2^2} + 216\, C_{\mathrm{S}_1 D_0} + 648\, C_{\mathrm{S}_1 D_0^2} + 48\, C_{D_{-1}} + 16\, C_{D_1} \\ & + 32\, C_{D_1^2} + 64\, C_{D_1^3} + 12\, C_{D_2} + 18\, C_{D_2^2} + 27\, C_{D_2^3} + 24\, C_{D_0} + 72\, C_{D_0^2} + 216\, C_{D_0^3}. \end{aligned} \tag{5.6}$$

AXB correctly determines the 25 basis coefficients here using Mellin moments $N = 2$ to $N = 18$, i.e. by solving just 9 equations. This shows the power of the method; a solution by Gaussian elimination would require Mellin moments to $N = 50$. While it is possible to compute moments this high for anomalous dimensions at second order, it will not be possible to compute enough moments at fourth order for a solution by Gaussian elimination. The vector of coefficients returned by AXB is

$$(\underbrace{2, 6, 72, 8, 88, 584, 4, 24, -612, -80}_{SW0}, \underbrace{0, 0, 4, 0, -4, 0}_{SW1}, \underbrace{2, 4, -4, 2, 4, -4, 0, 0, 0}_{SW2}). \tag{5.7}$$

Suppose we make an incorrect choice of basis to determine this function, for example, we neglect to include the D_{-1} with $SW0$. Again with Mellin moments $N = 2$ to $N = 18$, AXB returns the coefficients

$$\begin{aligned}(& -43, 423, 123, 1492, -102, 1332, 4, 24, -612, -15, 437, 102, -2399, 80, 1700, \\ & -146, 180, -26, -1065, 670, 579, -919, 490, 605).\end{aligned} \tag{5.8}$$

Using more Mellin moments the coefficients start to look even worse, as the solver forces a solution using the inadequate basis. With $N = 2$ to $N = 20$ we find

$$\begin{aligned}(& -178, 4391, -25712, 412, -10348, -6476, 4, 24, -612, -572, 25401, -2178, -5642, \\ & -3526, -20152, -3302, -3161, 6474, -4011, 5092, 3775, -3283, -4617, 11029).\end{aligned} \tag{5.9}$$

We claim that it should be "obvious" that such a solution is incorrect. The correct coefficients should be small (especially since we pull some factors of 2 into the coefficient factors). This should be particularly be the case for the higher weight harmonic sum sets (the right-hand end of the vector in Eq. (5.7)) where also many coefficients should be zero; we typically do not need the full set of higher weight sums.

With the larger systems that we will consider later, bad solutions might be less clear. In particular, solutions where the basis is correct but the number of Mellin moments used is insufficient to determine the correct solution can be harder to distinguish. For this reason we must have a way to satisfactorily verify a potential solution. We will always require a potential solution to correctly reproduce one (or ideally, more than one) Mellin moment beyond those used for its determination.

5.3 Bases for Large-n_f Singlet Anomalous Dimensions

In this section, we choose bases with which we can determine the analytic all-N forms of the large-n_f (n_f^3) contributions to the fourth-order singlet anomalous dimensions. We begin by making a careful investigation of the structure of the large-n_f (n_f^2) contributions to the third-order anomalous dimensions, which will motivate our choices of elements for the fourth-order bases for each entry of the anomalous dimension matrix.

We will then discuss the bases used in detail, as well as any additional assumptions made, for each reconstruction. We give in each case the number of Mellin moments required for the successful solution of the Diophantine equation system as well as how many moments were used as verification of the result. In Section 5.6 we will discuss where the results determined here overlap with other calculations in the literature and show that they agree.

5.3.1 Third Order Structures

We now turn to our investigation of the structures of the large-n_f contributions to the third-order singlet anomalous dimensions, $\gamma_{qq,ps}^{(2)}$, $\gamma_{qg}^{(2)}$, $\gamma_{gq}^{(2)}$ and $\gamma_{gg}^{(2)}$. These are terms with the colour factors $C_F n_f^2$ and $C_A n_f^2$. We introduce the following symbols,

$$\eta = D_0 - D_1, \tag{5.10}$$

$$\eta' = -D_2 + D_{-1}, \tag{5.11}$$

$$\rho = D_0 - 2D_1 + 2D_2, \tag{5.12}$$

which are combinations of denominator function which commonly appear with the highest weight harmonic sums in some of the anomalous dimensions. They are related to the leading order anomalous dimensions $\gamma_{qq}^{(0)}$, $\gamma_{gg}^{(0)}$ and $\gamma_{qg}^{(0)}$. The third-order functional structures are presented below, in Tables 5.4 to 5.9, and some discussion follows the table for each quantity.

Harmonic Sums	Denominators
$SW2+$	$D_0^{1,2}$, $D_1^{1,2}$, D_2 , D_{-1}
$SW1$	$D_0^{1,2,3}$, $D_1^{1,2,3}$, $D_2^{1,2}$, D_{-1}
$SW0$	$D_0^{1,2,3,4}$, $D_1^{1,2,3,4}$, $D_2^{1,2,3}$, D_{-1}

Overall Weight	4	3	2	1
Coeff. Factors	$\frac{16}{3}$	$\frac{8}{9}$	$\frac{8}{27}$	$\frac{8}{27}$

Table 5.4: *The structure of the $C_F n_f^2$ terms of $\gamma_{qq,ps}^{(2)}$.*

$\gamma_{qq,ps}^{(2)}$ contains overall weight 4 objects, but with positive-index harmonic sums of no more than harmonic weight 2. D_{-1} never appears to more than the first power. The maximum power of D_2 is reduced by 1, compared to that of D_0 and D_1.

Harmonic Sums	Denominators
$SW3+$	ρ
$SW2+$	$D_0^{1,2}$, D_1 , D_2
$SW1$	$D_0^{1,2}$, $D_1^{1,2}$, D_2
$SW0$	$D_0^{1,2,3,4,5}$, $D_1^{1,2,3,4,5}$, $D_2^{1,2,3,4}$, D_{-1}

Overall Weight	5	4	3	2	1
Coeff. Factors	32	$\frac{8}{3}$	$\frac{2}{9}$	$\frac{2}{27}$	$\frac{1}{2 \cdot 81}$

Table 5.5: *The structure of the $C_F n_f^2$ terms of $\gamma_{qg}^{(2)}$.*

Harmonic Sums	Denominators
$SW3$	ρ
$SW2$	D_0 , $D_1^{1,2}$, $D_2^{1,2}$
$SW1$	D_0 , $D_1^{1,2,3}$, $D_2^{1,2,3}$
$SW0$	$D_0^{1,2,3,4}$, $D_1^{1,2,3,4}$, $D_2^{1,2,3,4}$, D_{-1}

Overall Weight	4	3	2	1
Coeff. Factors	$\frac{8}{3}$	$\frac{8}{9}$	$\frac{8}{27}$	$\frac{2}{81}$

Table 5.6: *The structure of the $C_A n_f^2$ terms of $\gamma_{qg}^{(2)}$.*

$\gamma_{qg}^{(2)}$ has overall weight 4 elements, except for some pure-denominator-function elements for the $C_F n_f^2$ terms which have an overall weight of 5 (just D_0^5 and D_1^5). D_{-1} appears only without harmonic sums and only to the first power. The highest weight harmonic sums appear only with the denominator function combination ρ, defined in Eq. (5.12). Unlike the $C_A n_f^2$ terms, the $C_F n_f^2$ terms appear only with positive-index harmonic sums. These structures have the largest number of elements of all of the third-order non-singlet anomalous dimensions, so we anticipate that the $C_F n_f^3$ and $C_A n_f^3$ terms of $\gamma_{qg}^{(3)}$ will be the most difficult to reconstruct at fourth order.

Harmonic Sums	Denominators
$SW2+$	D_0, D_1 , D_{-1}
$SW1$	D_0, $D_1^{1,2}$, D_{-1}
$SW0$	D_0, $D_1^{1,2,3}$, D_{-1}

Overall Weight	3	2	1
Coeff. Factors	$\frac{8}{3}$	$\frac{64}{9}$	$\frac{64}{9}$

Table 5.7: *The structure of the $C_F n_f^2$ terms of $\gamma_{gq}^{(2)}$.*

$\gamma_{gq}^{(2)}$ has elements of overall weight 3, with positive-index harmonic sums to harmonic weight 2. D_0 and D_{-1} appear to first power only, with all harmonic sum weights.

Harmonic Sums	Denominators
$SW2+$	$\eta^{1,2}$, η'
$SW1$	$D_0^{1,2,3}$, $D_1^{1,2,3}$, D_2, D_{-1}
$SW0$	1, $D_0^{1,2,3,4}$, $D_1^{1,2,3,4}$, D_2, D_{-1}

Overall Weight	4	3	2	1	0
Coeff. Factors	$\frac{16}{3}$	$\frac{8}{9}$	$\frac{8}{27}$	$\frac{8}{81}$	$\frac{1}{9}$

Table 5.8: *The structure of the* $C_F n_f^2$ *terms of* $\gamma_{gg}^{(2)}$.

Harmonic Sums	Denominators
$SW1$	1, $D_0^{1,2}$, $D_1^{1,2}$, D_2, D_{-1}
$SW0$	1, $D_0^{1,2,3}$, $D_1^{1,2,3}$, D_2, D_{-1}

Overall Weight	3	2	1	0
Coeff. Factors	$\frac{16}{9}$	$\frac{2}{27}$	$\frac{2}{81}$	$\frac{1}{2\cdot 9}$

Table 5.9: *The structure of the* $C_A n_f^2$ *terms of* $\gamma_{gg}^{(2)}$.

Unlike the above, $\gamma_{gg}^{(2)}$ contains harmonic sums which are not multiplied by denominator functions. The $C_F n_f^2$ and $C_A n_f^2$ contributions have terms of overall weight 4 and 3, respectively. D_2 and D_{-1} appear to no more than the first power.

We now make some general observations about the structure of these third-order singlet anomalous dimensions. We will assume that these observations will apply also at fourth order.

- *Coefficient Factors:* We can take factors of two out of most of the coefficients, particularly at high overall weight. It depends which function we are considering, but in general it seems safe to take out an additional factor of two for each increase in overall weight, starting from some minimal factor (which is 2^{-1}, in some cases). Taking these factors out of the coefficients makes them smaller which should help `AXB`, but if too many powers of two are taken out of the coefficients they will no longer be integers.

- *Coefficient Factors:* We *must* take factors of a third out of almost all of the coefficients. Again, it depends which function we are considering, but we must take an additional factor of a third per reduction in overall weight, starting from some minimal factor at maximal overall weight. Occasionally moving from overall weight 2 to 1, or 1 to 0, does not incur and extra factor of a third. If *too few* factors of a third are taken out of the coefficients, they will not be integers.

- *Denominator Functions:* For some anomalous dimensions, the highest weight sums appear only with particular combinations of denominator functions (this is also true below third order). These are the η, η' and ρ defined in Eqs. (5.10) to (5.12). Including just these combinations with the high-weight sums greatly reduces the size of a basis. The denominator function D_{-1} never appears to more than the first power. No denominator functions other than D_0, D_1, D_2, D_{-1} ever appear (this is the case also for the other colour factors).

- *Harmonic Sums:* For some functions, no negative-index harmonic sums appear. Sums with an index of -1 *never* appear (not even with the other colour factors, or at lower orders, or in any of the coefficient functions). Hence the definitions of the sets SWN in Eq. (5.2) do not contain sums with an index of -1; we will assume these sums do not appear at fourth order either.

We now discuss the bases for reconstruction of fourth-order singlet anomalous dimensions. We begin with the lower row of the anomalous dimension matrix, $\gamma_{gq}^{(3)}$ and $\gamma_{gg}^{(3)}$, as it appears that these will require a lower weight basis and thus be easier to solve. We must increase the maximum allowed overall weight by 1 for the fourth-order anomalous dimensions; an extra 2 for the increase in order, but a reduction of 1 for the increase in power of n_f.

For all of the singlet anomalous dimensions considered above there are terms proportional to ζ_3. The overall weight of these terms is reduced by 3 or equivalently, ζ symbols contribute to the harmonic weight of the term (after all, the zeta numbers are just single-index harmonic sums at infinity, $\zeta_i = S_i(\infty)$ for $i > 1$). We can use the same bases for the reconstruction of these ζ_3 terms, but with the highest three weights of basis elements discarded.

5.3.2 A Basis for $\gamma_{gq}^{(3)}$

We assume a basis with a similar structure to Table 5.7, with (positive index) harmonic sums of weight 3 and a maximum overall weight of 4. We allow denominator functions D_0 and D_1 up to to the maximum overall weight, and D_{-1} to a single power only. As in $\gamma_{gq}^{(2)}$, we assume D_2 does not appear. We make a rather relaxed choice of coefficient factors; a generous factor of $(1/3)^6$ is taken from the overall weight 1 coefficients.

Harmonic Sums	Denominators
$SW3+$	D_0 , D_1 , D_{-1}
$SW2+$	$D_0^{1,2}$, $D_1^{1,2}$, D_{-1}
$SW1$	$D_0^{1,2,3}$, $D_1^{1,2,3}$, D_{-1}
$SW0$	$D_0^{1,2,3,4}$, $D_1^{1,2,3,4}$, D_{-1}

Overall Weight	4	3	2	1
Coeff. Factors	$\frac{8}{27}$	$\frac{4}{81}$	$\frac{2}{243}$	$\frac{1}{729}$

Table 5.10: *The basis for the reconstruction of the $C_F n_f^3$ terms of $\gamma_{gq}^{(3)}$.*

This is a basis with 38 unknown coefficients. Mellin moments $N = 2$ to $N = 18$ reconstruct the all-N result, with moments $N = 20$ to $N = 28$ serving as verification of the result. For the ζ_3 terms we reduce the basis overall weight by 3, leaving just D_0, D_1 and D_{-1}. The coefficients can be determined by Gaussian elimination using moments $N = 2$ to $N = 6$, leaving $N = 8$ to $N = 28$ as verification of the solution.

The solution is Eq. (5.31). It proves not to require powers of D_0 above the first in combination with harmonic sums, as observed in $\gamma_{gq}^{(2)}$.

5.3.3 A Basis for $\gamma_{gg}^{(3)}$

$\gamma_{gg}^{(3)}$ has contributions from both $C_F n_f^3$ and $C_A n_f^3$ terms. The third-order structures (Tables 5.8 and 5.9) are rather similar in their lower overall weight contributions. We choose a basis suitable for both, but remove the overall weight 5 terms when solving for the $C_A n_f^3$ moments. For $C_F n_f^3$ we assume the same denominator function structure at harmonic sum weight 3 as the $\gamma_{gg}^{(2)}$ had at sum weight 2; just the combinations η, η^2 and η'. We assume that D_2 and D_{-1} appear only to the first power, and that sub-maximal weight harmonic sums may appear alone.

Further evidence for the reduced overall weight of the $C_A n_f^3$ basis compared to that of $C_F n_f^3$ can be seen by analysing the prime structure of the denominators of the Mellin moments. Consider the $(N = 18)$ Mellin moment of both functions,

$$\gamma_{gg}^{(3)}\Big|_{C_F n_f^3}(N=18) = -\frac{120434323080094241480978616 8123}{2^5\, 3^{12}\, 5^4\, 7^3\, 11^3\, 13^3\, 17^4\, 19^5}, \tag{5.13}$$

$$\gamma_{gg}^{(3)}\Big|_{C_A n_f^3}(N=18) = -\frac{25223004081586999165793 71}{2^7\, 3^{11}\, 5^3\, 7^2\, 11^2\, 13^2\, 17^3\, 19^4}. \tag{5.14}$$

The reduced power of $1/19$ in Eq. (5.14) contribution suggests that $D_1 = 1/(18+1)$ does *not* appear to the fifth power, unlike in Eq. (5.13). This could of course be an “accidental” cancellation with the numerator, but we observe the same pattern in many other Mellin moments (any for which $(N+1)$ is prime). This is highly suggestive that this is a structural feature and not an “accident”.

Harmonic Sums	Denominators
$SW3+$	$\eta^{1,2}$, η'
$SW2+$	1, $D_0^{1,2,3}$, $D_1^{1,2,3}$, D_2, D_{-1}
$SW1$	1, $D_0^{1,2,3,4}$, $D_1^{1,2,3,4}$, D_2, D_{-1}
$SW0$	1, $D_0^{1,2,3,4,5}$, $D_1^{1,2,3,4,5}$, D_2, D_{-1}

Overall Weight	5	4	3	2	1	0
Coeff. Factors	$\frac{16}{9}$	$\frac{8}{27}$	$\frac{4}{81}$	$\frac{2}{243}$	$\frac{1}{729}$	$\frac{1}{2\cdot 729}$

Table 5.11: *The basis for the reconstruction of the $C_F n_f^3$ terms of $\gamma_{gg}^{(3)}$. For the $C_A n_f^3$ terms we use the same basis, but remove elements of overall weight 5.*

This is a basis with 54 unknown coefficients. The $C_F n_f^3$ solution is found using moments $N = 2$ to $N = 26$, with $N = 28$ to $N = 32$ verifying the solution. The $C_F n_f^3 \zeta_3$ terms are determined with moments $N = 2$ to $N = 16$, or by Gaussian elimination using moments $N = 2$ to $N = 28$, after reducing the maximal overall weight of the basis by 3.

Removing the overall weight 5 terms leaves a basis of 34 unknown coefficients. The $C_A n_f^3$ solution is found using moments $N = 2$ to $N = 20$, with $N = 22$ to $N = 28$ verifying the solution, and the $C_A n_f^3 \zeta_3$ terms are determined with moments $N = 2$ to $N = 14$, or by Gaussian elimination with moments $N = 2$ to $N = 28$.

The result is given in Eq. (5.32).

5.3.4 A Basis for $\gamma_{qq,ps}^{(3)}$

For $\gamma_{qq,ps}^{(3)}$ we extend the structure of $\gamma_{qq,ps}^{(2)}$ by one in overall weight; we allow positive-index harmonic sums to harmonic weight 3, in combination with denominator functions to overall weight 5. We maintain the assumption that D_{-1} appears only to the first power, and that D_2 appears with its maximum power reduced by 1 compared to that of D_0 or D_1.

Harmonic Sums	Denominators
$SW3+$	$D_0^{1,2}$, $D_1^{1,2}$, D_2 , D_{-1}
$SW2+$	$D_0^{1,2,3}$, $D_1^{1,2,3}$, $D_2^{1,2}$, D_{-1}
$SW1$	$D_0^{1,2,3,4}$, $D_1^{1,2,3,4}$, $D_2^{1,2,3}$, D_{-1}
$SW0$	$D_0^{1,2,3,4,5}$, $D_1^{1,2,3,4,5}$, $D_2^{1,2,3,4}$, D_{-1}

Overall Weight	5	4	3	2	1
Coeff. Factors	$\frac{8}{9}$	$\frac{4}{27}$	$\frac{2}{81}$	$\frac{1}{243}$	$\frac{1}{2\cdot 243}$

Table 5.12: *The basis for the reconstruction of the $C_F n_f^3$ terms of $\gamma_{qq,ps}^{(3)}$.*

This basis of 69 unknown coefficients can be determined using moments $N = 2$ to $N = 30$, with $N = 32$ to $N = 44$ verifying the solution. The $C_F n_f^3 \zeta_3$ terms are solved by moments $N = 2$ to $N = 14$, or by Gaussian elimination using moments $N = 2$ to $N = 22$. The result is Eq. (5.29).

5.3.5 A Basis for $\gamma_{qg}^{(3)}$

The leading-n_f terms of $\gamma_{qg}^{(2)}$ have higher weight harmonic sums than those of the other singlet anomalous dimensions, so we anticipate the same for $\gamma_{qg}^{(3)}$. This will mean it has by far the largest basis and thus require many more Mellin moments to solve.

Based on the structure of the $C_F n_f^2$ terms of $\gamma_{qg}^{(2)}$ (Table 5.5) we might anticipate positive index harmonic sum and denominator function combinations to overall weight 5 (with harmonic sums to harmonic weight 4), and denominator functions without sums to overall weight 6. We make some observations based on the prime structure of a few Mellin moments of the $C_F n_f^3$ terms of $\gamma_{qg}^{(3)}$ which force us to extend our assumptions a little further.

$$\gamma_{qg}^{(3)}\Big|_{C_F n_f^3}(N=12) = -\frac{1672242508473024481360 3}{2^8\, 3^{12}\, 5\, 7^4\, 11^4\, 13^6}, \tag{5.15}$$

$$\gamma_{qg}^{(3)}\Big|_{C_F n_f^3}(N=26) = +\frac{113200266100470508445879415952337515752014200 01}{2^9\, 3^{20}\, 5^8\, 7^5\, 11^3\, 13^6\, 17^4\, 19^4\, 23^4}, \tag{5.16}$$

$$\gamma_{qg}^{(3)}\Big|_{C_F n_f^3}(N=54) = +\frac{1399917280922149939086993045998420420188570675563 2\cdots}{2^{16}\, 3^{20}\, 5^9\, 7^9\, 11^6\, 13^4\, 17^4\, 19^4\, 23^4\, 29^4\, 31^4\, 37^4\, 41^4\, 43^4\, 47^4\, 53^4}. \tag{5.17}$$

The $\cdots$ signifies that some numerator digits have been truncated. They are unimportant for the present discussion. We observe that:

- The 13^6 of $(N = 12)$ (and also any other N value for which $(N + 1)$ is prime) requires that we include D_1^6 in the basis.
- Assuming the above, the 3^{20} of $(N = 26)$ suggests that we require a coefficient factor of $1/9$ for basis elements of overall weight 6. It can be formed by $D_1^6/9 = 1/(27^6)/9 = 1/(3^{18})/9$.
- If one pushes the moment calculation to a high enough N value, one finds a $1/3^{20}$ at $(N = 54)$. This requires overall weight 6 basis elements which contain powers of D_0 and a coefficient factor of $1/9$. It can be formed by $D_0^6/9 = 1/(2 \cdot 27)^6/9 = 1/(2^6 \cdot 3^{18})/9$, but we also include weight 6 elements with powers of D_0 with all sub-maximal weight harmonic sum sets.

Although the $C_F n_f^2$ terms of $\gamma_{qg}^{(2)}$ contain D_{-1} only *without* harmonic sums, all of the other singlet anomalous dimensions at third order include it in combination with them. We include it with the sub-maximal weight harmonic sums here. We choose for a basis, then,

Harmonic Sums	Denominators
$SW4+$	ρ
$SW3+$	$D_0^{1,2,3}$, $D_1^{1,2,3}$, $D_2^{1,2}$, D_{-1}
$SW2+$	$D_0^{1,2,3,4}$, $D_1^{1,2,3,4}$, $D_2^{1,2,3}$, D_{-1}
$SW1$	$D_0^{1,2,3,4,5}$, $D_1^{1,2,3,4,5}$, $D_2^{1,2,3,4}$, D_{-1}
$SW0$	$D_0^{1,2,3,4,5,6}$, $D_1^{1,2,3,4,5,6}$, $D_2^{1,2,3,4,5}$, D_{-1}

Overall Weight	6	5	4	3	2	1
Coeff. Factors	$\frac{16}{9}$	$\frac{8}{27}$	$\frac{4}{81}$	$\frac{2}{243}$	$\frac{1}{729}$	$\frac{1}{2\cdot 2187}$

Table 5.13: *The basis for the reconstruction of the $C_F n_f^3$ terms of $\gamma_{qg}^{(3)}$.*

This basis has 101 unknown coefficients. The Mellin moments $N = 2$ to $N = 40$ yield a solution, with $N = 42$ to $N = 54$ providing verification. The $C_F n_f^3 \zeta_3$ terms are solved by moments $N = 2$ to $N = 22$, or by $N = 2$ to $N = 50$ using Gaussian elimination, after reducing the maximal overall weight by 3.

The $C_A n_f^2$ terms in $\gamma_{qg}^{(2)}$ include harmonic sums with negative indices, but have a lower maximum overall weight than the $C_F n_f^3$ terms. Assuming the same here increases the size of the $C_A n_f^3$ basis relative to that of $C_F n_f^3$. As above, we begin by analysing the denominator prime structure of the moments to confirm our suspicions.

$$\gamma_{qg}^{(3)}\Big|_{C_A n_f^3}(N=8) = \frac{886247558029}{\mathbf{3^{13}}\ 5^5\ 7^3}, \tag{5.18}$$

$$\gamma_{qg}^{(3)}\Big|_{C_A n_f^3}(N=12) = \frac{894866035734231246739}{2^3\ 3^{10}\ 5^4\ 7^5\ 11^3\ \mathbf{13^5}}, \tag{5.19}$$

$$\gamma_{qg}^{(3)}\Big|_{C_A n_f^3}(N=26) = \frac{409941447682009724129686958033477 93}{2^7\ \mathbf{3^{18}}\ 5^6\ 7^5\ 11^3\ 13^5\ 17^2\ 19^2\ 23^2}, \tag{5.20}$$

$$\gamma_{qg}^{(3)}\Big|_{C_A n_f^3}(N=36) = \frac{312338610317762672764170663884151814931126699 2097833}{2^{13}\ 3^{15}\ 5^5\ 7^4\ 11^4\ 13^4\ 17^4\ 19^5\ \mathbf{23^2\ 29^2\ 31}\ 37^5}, \tag{5.21}$$

$$\gamma_{qg}^{(3)}\Big|_{C_A n_f^3}(N=40) = \frac{179775513227136505984381879121121165459788447773677 3\cdots}{2^{13}\ 3^{13}\ 5^7\ 7^4\ 11^4\ 13^4\ 17^4\ 19^4\ \mathbf{23^2\ 29^2\ 31^2\ 37^2}\ 41^5}, \tag{5.22}$$

$$\gamma_{qg}^{(3)}\Big|_{C_A n_f^3}(N=42) = \frac{215620301970251491890675470554571111691184266240801 2\cdots}{2^{11}\ 3^{13}\ 5^5\ 7^5\ 11^5\ 13^4\ 17^4\ 19^4\ \mathbf{23^2\ 29^2\ 31^2\ 37}\ 41^3\ 43^5}. \tag{5.23}$$

- The 13^5 of $(N = 12)$, along with other N values for which $(N + 1)$ is prime, require D_1^5.
- The 3^{13} of $(N = 8)$ and 3^{18} of $(N = 26)$ suggest that, since we assume no more than D_1^5, we must have a coefficient factor of $1/27$ on the overall weight 5 basis elements. This is also what the $C_F n_f^3$ basis required at overall weight 5.

The prime structures at $N = 36, 40, 42$ serve to demonstrate an (unexplained) curiosity, observed also at lower orders. Primes P in the range $N/2 < P < N - 1$, appear with lower (here, at least 2 lower) powers in the C_A terms compared to the C_F terms. Compare these with Eq. (5.17) above, for which the high primes all appear to the 4th power (since we have harmonic weight 4 sums). It is not that the individual C_A terms lack the ability to produce these primes, but rather that they all cancel among each other when evaluated at a particular N and summed. A systematic way to explain this behaviour would presumably yield some powerful constraints on the basis coefficients we are trying to determine here.

For the $C_A n_f^3$ terms we choose a basis of the form

Harmonic Sums	Denominators
$SW4$	ρ
$SW3$	$D_0^{1,2}$, $D_1^{1,2}$, $D_2^{1,2}$, D_{-1}
$SW2$	$D_0^{1,2,3}$, $D_1^{1,2,3}$, $D_2^{1,2,3}$, D_{-1}
$SW1$	$D_0^{1,2,3,4}$, $D_1^{1,2,3,4}$, $D_2^{1,2,3,4}$, D_{-1}
$SW0$	$D_0^{1,2,3,4,5}$, $D_1^{1,2,3,4,5}$, $D_2^{1,2,3,4,5}$, D_{-1}

Overall Weight	5	4	3	2	1
Coeff. Factors	$\frac{8}{27}$	$\frac{4}{81}$	$\frac{2}{243}$	$\frac{1}{729}$	$\frac{1}{2\cdot 2187}$

Table 5.14: *The basis for the reconstruction of the $C_A n_f^3$ terms of $\gamma_{qg}^{(3)}$.*

It has 125 unknowns. This is too large to yield a solution with the Mellin moments we have been able to compute. We must therefore try some additional assumptions:

- Upon making a large-x expansion of the basis (after inverse Mellin transformation to x space) we note the appearance of terms proportional to the irrational numbers $\ln 2$ and $\mathrm{Li}_4(1/2)$[1]. These do not appear in the large-x expansion of any anomalous dimension computed to date, and we assume the same here. We can therefore form some relations between the coefficients of some of the basis elements such that these irrational terms cancel. We require that

$$2C_{\mathrm{S}_{-3,1}} - 2C_{\mathrm{S}_{1,-3}} + 4C_{\mathrm{S}_{2,-2}} - 4C_{\mathrm{S}_{-2,2}} + C_{\mathrm{S}_{1,1,-2}} - C_{\mathrm{S}_{-2,1,1}} = 0, \tag{5.24}$$

  removing *one* coefficient from the basis.

- We often observe a relationship between the coefficients of $\mathrm{S}_{1,2}$ and $\mathrm{S}_{2,1}$. In the $C_A n_f^2$ terms of $\gamma_{qg}^{(2)}$ we have that $C_{\mathrm{S}_{1,2}} = -C_{\mathrm{S}_{2,1}}$ (in combination with any denominator function), so we assume the same for the fourth-order basis. This removes 7 coefficients. (Such a relationship, where $C_{\mathrm{S}_{1,2}} = \pm C_{\mathrm{S}_{2,1}}$, is also visible in other third- and fourth-order expressions).

[1]The constants $\ln 2 = S_{-1}(\infty)$ and $\mathrm{Li}_4(1/2) = S_{-1,1,1,1}(\infty)$.

These assumptions fix 8 coefficients in total, leaving a basis with 117 coefficients to determine. The Mellin moments $N = 2$ to $N = 44$ yield a solution, with $N = 46$ providing verification. The $C_A n_f^3 \zeta_3$ terms are solved by moments $N = 2$ to $N = 24$ after reducing the maximal overall weight by 3.

The full result for the n_f^3 terms of $\gamma_{qg}^{(3)}$ is given by Eq. (5.30).

5.4 A Basis for the Large-n_f Non-Singlet Anomalous Dimensions

In the non-singlet sector, the leading-n_f contribution to $\gamma_{ns}^{(3),\pm}$ is already known [81], it is given by Eq. (5.28). By computing Mellin moments with `FORCER` we are able to verify this result and also to extend this result to the *next-to-leading-n_f* terms, i.e. terms proportional to the colour factors $C_F^2 n_f^2$ and $C_F C_A n_f^2$. For $\gamma_{ns}^{(3),\pm}$ the computation with `FORCER` is sufficiently easy that such a next-to-leading-n_f reconstruction is possible; this was not the case for the singlet anomalous dimensions discussed above. Even so, the reconstruction is only possible here if one considers some very particular combinations of the colour factors. Rather than writing

$$\gamma_{ns}^{(3),\pm}\Big|_{n_f^2} = C_F^2 \gamma_{ns}^{(3),\pm}\Big|_{C_F^2 n_f^2} + C_F C_A \gamma_{ns}^{(3),\pm}\Big|_{C_F C_A n_f^2}, \tag{5.25}$$

we can form alternative linear combinations of the colour factors,

$$\gamma_{ns}^{(3),\pm}\Big|_{n_f^2} = 2C_F^2 A + C_F \left(C_A - 2C_F\right) B^\pm, \tag{5.26}$$

$$= 2C_F^2 \left(A - B^\pm\right) + C_F C_A B^\pm. \tag{5.27}$$

In the large-N_c limit, the combination $(C_A - 2C_F)$ vanishes. The remaining terms, given by $2C_F^2 A$, should be common to both the even-N $\gamma_{ns}^{(3),+}$ and the odd-N $\gamma_{ns}^{(3),-}$, which we observe at lower orders. By computing even-N moments of $\gamma_{ns}^{(3),+}$ and odd-N moments of $\gamma_{ns}^{(3),-}$ for each of the colour factors $C_F^2 n_f^2$ and $C_F C_A n_f^2$ we can form the combination of Eq. (5.26) and discard terms proportional to $(C_A - 2C_F)$ to obtain both even-N and odd-N moments for $2C_F^2 A$. This provides a sufficient number of moments to reconstruct A *without the value of N becoming too high to compute.*

To reconstruct B^+ (B^-) we can only use even-N (odd-N) moments. However, the a_s^4 diagrams proportional to $C_F^2 n_f^2$ are 2-loop diagrams with 2-loop gluon propagator insertions. These are comparatively easy for `FORCER` to compute. By computing the moments for just the $C_F^2 n_f^2$ diagrams, we can compute even-N moments for $(A - B^+)$ and odd-N moments for $(A - B^-)$ to sufficiently high N values to reconstruct these linear combinations of A and $B^\pm$. Knowing both linear combinations, along with A alone, we can determine both $B^\pm$.

5.4.1 Third-Order Structures

We now consider the functional structure of the same colour factor combinations at third order, the only difference being that we have an overall factor of n_f rather than n_f^2. For the third-order A part and $C_F^2 n_f$ terms of $\gamma_{ns}^{(2),\pm}$ we observe the following structures:

Harmonic Sums	Denominators
$SW4+$	1
$SW3+$	1, η
$SW2+$	1, $\eta^{1,2}$, D_1^2
$SW1$	1, $\eta^{1,2,3}$, D_1^2
$SW0$	1, $\eta^{1,2,3,4}$, $D_1^{2,3}$

Overall Weight	4	3	2	1	0
Coeff. Factors	$\frac{4}{3}$	$\frac{8}{9}$	$\frac{2}{27}$	$\frac{1}{2\cdot 27}$	$\frac{1}{2}$

Table 5.15: *The structure of the A part of $\gamma_{ns}^{(2),\pm}$.*

Harmonic Sums	Denominators
$SW4$	1
$SW3$	1, η
$SW2$	1, $\eta^{1,2}$, D_1^2
$SW1$	1, $\eta^{1,2,3}$
$SW0$	1, $\eta^{1,2,3}$, D_1^2

Overall Weight	4	3	2	1	0
Coeff. Factors	$\frac{8}{3}$	$\frac{8}{9}$	$\frac{4}{27}$	$\frac{2}{27}$	1

Table 5.16: *The structure of the $C_F^2 n_f$ terms of $\gamma_{ns}^{(2),+}$.*

Harmonic Sums	Denominators
$SW4$	1
$SW3$	1, η
$SW2$	1, $\eta^{1,2}$, D_1^2
$SW1$	1, $\eta^{1,2,3}$, D_1^3
$SW0$	1, $\eta^{1,2,3,4}$, $D_1^{2,3,4}$

Overall Weight	4	3	2	1	0
Coeff. Factors	$\frac{8}{3}$	$\frac{8}{9}$	$\frac{4}{27}$	$\frac{2}{27}$	1

Table 5.17: *The structure of the* $C_F^2 n_f$ *terms of* $\gamma_{ns}^{(2),-}$.

We observe that

- The A part has positive-index harmonic sums only.
- Using the combination η, rather than D_0, we never see D_1^1. This reduces the size of the basis.
- We may have to relax the coefficient factors to reconstruct the A piece, compared to those suitable for the $C_F^2 n_f^2$ pieces.
- There are no terms with an overall weight greater than 4. Harmonic sums of weight 4 appear without denominator functions.

5.4.2 A Basis for $\gamma_{ns}^{(3),\pm}$

It is not possible to make any conclusive statements based on the prime structure of the denominators of the moments, other than that we should have weight 5 objects present and that they should have a coefficient factor of at least 1/3. We assume slightly more generous coefficient factors than the primes suggest, along the lines of the reconstruction of the singlet anomalous dimensions, i.e. allowing 1/9 at overall weight 5. Based on the third order structure, it seems we may get away with not adding an extra factor of 1/3 between the overall weight 2 and 1 basis elements, and perhaps even between the overall weight 3 and 2 basis elements. The constant term ($SW0 \cdot 1$) also seems not to require such a generous coefficient factor.

We try the following basis, then, to first reconstruct the $C_F^2 n_f^2$ terms of $\gamma_{ns}^{(3),+}$:

Harmonic Sums	Denominators
$SW5$	1
$SW4$	1, η
$SW3$	1, $\eta^{1,2}$, D_1^2
$SW2$	1, $\eta^{1,2,3}$, $D_1^{2,3}$
$SW1$	1, $\eta^{1,2,3,4}$, $D_1^{2,3,4}$
$SW0$	1, $\eta^{1,2,3,4,5}$, $D_1^{2,3,4,5}$

Overall Weight	5	4	3	2	1	0
Coeff. Factors	$\frac{16}{9}$	$\frac{8}{27}$	$\frac{4}{81}$	$\frac{2}{243}$	$\frac{1}{243}$	$\frac{1}{2\cdot 81}$

Table 5.18: *The basis for the reconstruction of the $C_F^2 n_f^2$ terms of $\gamma_{ns}^{(3),+}$.*

Without further assumptions, this basis contains 139 unknowns. At this point this is the largest basis of any reconstruction described here, largely due to the inclusion of the harmonic sums of weight 5 (of which there are 41). The Mellin moments computed are insufficient to solve the system. As in the reconstruction of the leading-n_f terms of $\gamma_{qg}^{(3)}$ (Section 5.3.5), some additional constraints are required. We assume:

- In the large-N limit, the non-singlet anomalous dimensions should behave as $\ln N$ [86] in the $\overline{\text{MS}}$ scheme. This can be enforced by killing off combinations of basis elements which contribute higher powers of $\ln N$ in the large-N expansion of the basis. Additionally, $\gamma_{ns}^{(2),+}$ and $\gamma_{ns}^{(2),-}$ have the sub-leading behaviour of $\frac{\ln N^2}{N^2}$ (and only with the colour factor C_F^3). We assume that we can allow such behaviour in the $C_F^2 n_f^2$ terms of $\gamma_{ns}^{(3),\pm}$, but kill off combinations of basis elements which go as $\ln N^3$, to all powers in $1/N$. These assumptions reduce the number of unknown coefficients to 123.

- In the large-N limit, there should be no terms proportional to the irrational numbers $\ln 2$, $\text{Li}_4(1/2)$, and $\text{Li}_5(1/2)$[2]. Enforcing the that their coefficients are zero leaves 119 coefficients to determine.

- As with $\gamma_{qg}^{(2)}$, we use the relationship between the coefficients of $\text{S}_{1,2}$ and $\text{S}_{2,1}$ in $\gamma_{ns}^{(2),\pm}$. We set $C_{\text{S}_{1,2}} = C_{\text{S}_{2,1}}$ in combination with any denominator function. This leaves 115 coefficients to determine.

With these additional assumptions, the equation system can be solved with Mellin moments $N = 2$ to $N = 40$, with $N = 42$ serving as a check. After reducing the maximal overall weight of the basis by 3, the $C_F^2 n_f^2 \zeta_3$ terms can be solved using moments $N = 2$ to $N = 10$ or by Gaussian elimination using moments $N = 2$ to $N = 18$. After reducing the maximal overall weight by a further 1, the $C_F^2 n_f^2 \zeta_4$ terms (which do not exist in the n_f^3 reconstructions) can be solved by Gaussian elimination with moments $N = 2$ to $N = 6$.

[2]The constant $\text{Li}_5(1/2) = S_{-1,1,1,1,1}(\infty)$.

Inspired by this result, we adjust the coefficient factors of the basis; it appears we can tighten them such that the coefficients to be determined are significantly smaller. Choosing

Overall Weight	5	4	3	2	1	0
Coeff. Factors	$\frac{32}{9}$	$\frac{16}{27}$	$\frac{8}{81}$	$\frac{4}{81}$	$\frac{2}{81}$	$\frac{1}{27}$

Table 5.19: *Coefficient factors for the reconstruction of the $C_F^2 n_f^2$ terms of $\gamma_{ns}^{(3),-}$.*

with the same basis and assumptions as above, we are able to solve the system for the $C_F^2 n_f^2$ terms of $\gamma_{ns}^{(3),-}$ using Mellin moments $N = 3$ to $N = 37$, with $N = 39$ serving as a check of the result. Similarly, the $C_F^2 n_f^2 \zeta_3$ terms can be solved with moments $N = 3$ to $N = 11$ or by Gaussian elimination with $N = 2$ to $N = 19$ and the $C_F^2 n_f^2 \zeta_4$ terms by Gaussian elimination with $N = 3$ to $N = 7$. (Just for information, these tighter coefficient factors allow for a re-solution of the $C_F^2 n_f^2$ terms of $\gamma_{ns}^{(3),+}$ with two Mellin moments fewer: with $N = 2$ to $N = 36$.)

For the A piece, we keep the same assumptions made above but also remove all harmonic sums which contain negative indices. This vastly reduces the number of unknowns, to just 65. To find a solution, however, we must assume some further structure still. Based on the third-order counterpart to this function, we assume that particular high-weight harmonic sums should not appear in the result:

- At harmonic weight 5, we assume that the sums $S_{1,1,1,2}$, $S_{1,1,2,1}$, $S_{1,2,1,1}$ and $S_{2,1,1,1}$ do not appear. Also we assume that the sums $S_{1,2,2}$, $S_{2,1,2}$ and $S_{2,2,1}$ do not appear.

- At harmonic weight 4, we assume that the sums $S_{1,1,2}$, $S_{1,2,1}$ and $S_{2,1,1}$ do not appear.

These assumptions reduce the basis to just 54 unknowns. Relaxing the coefficient factors to

Overall Weight	5	4	3	2	1	0
Coeff. Factors	$\frac{16}{9}$	$\frac{8}{27}$	$\frac{4}{81}$	$\frac{2}{81}$	$\frac{1}{81}$	$\frac{1}{27}$

Table 5.20: *Coefficient factors for the reconstruction of the A piece of $\gamma_{ns}^{(3),\pm}$.*

allows for a solution using Mellin moments (both even-N and odd-N) $N = 2$ to $N = 17$, with $N = 18, 19, 20$ and $N = 22$ serving as a checks of the result. The ζ_3 terms are then solved with moments $N = 2$ to $N = 7$ or by Gaussian elimination with $N = 2$ to $N = 10$. The ζ_4 terms are solved by Gaussian elimination with moments $N = 2$ to $N = 4$.

5.5 Results

Having discussed the bases and assumptions used for the reconstruction of the analytic N dependence of various quantities in the previous section, we now present the results. They are not especially lengthy so are reproduced in full here. Further verification that they are correct (beyond their reproduction of higher Mellin moments) is discussed in Section 5.6.

It should be noted that these Mellin moment calculations with `FORCER` really push the limits of what is possible, computationally. The hardest diagrams at the highest moment computed of the $C_A n_f^3$ terms of $\gamma_{qg}^{(3)}$ each took around 2 weeks to complete on rather fast machines, and produce some 10TB of intermediate expressions (around 130 billion terms) during the calculation. This also demonstrates the power of `FORM`; no other Computer Algebra System can perform manipulations at this scale.

The wall-time and disk space required increase approximately exponentially with N so to reconstruct, say, the $\mathcal{O}(n_f^2)$ colour factors of the singlet anomalous dimensions is out of the question with current resources. The combined effects of (very much) more computationally demanding moment calculations and larger reconstruction bases requiring yet more moments for solution increase the resource requirements far beyond what could be provided by even a large supercomputer.

There is one remaining viable target for reconstruction; the n_f^2 terms of fl_{02} diagrams which contribute to the evolution of the valence PDF q_{ns}^{V}, defined in Eq. (2.29). The computations would be approximately of the difficulty of the A part of $\gamma_{ns}^{(3),\pm}$ but without the benefit of being able to use both even-N and odd-N moments. A solution is thus estimated to require odd moments of these diagrams to some N value in the 40s. Such a computation would be significantly harder than anything required by the results of this chapter and would certainly require improvements of the efficiency of `FORCER` or some very tight constraints on the basis.

5.5.1 Results for the Singlet Anomalous Dimensions

Here we present the leading contributions to the singlet anomalous dimensions in the large-n_f limit. That is, terms proportional to $C_A n_f^3$ and $C_F n_f^3$. These are the results of the discussions of Sections 5.3.2 to 5.3.5. We display the results in both Mellin-N space and Bjorken-x space and plot the functions in both spaces. Eq. (5.28), necessary to define Eq. (5.29), has been taken from [81].

$$\gamma_{ns}^{(3),\pm}\Big|_{n_f^3} =$$

$$+\, C_F \Big[+ 32/27\,\mathrm{S}_4 \;-\; 160/81\,\mathrm{S}_3 \;-\; 32/81\,\mathrm{S}_2 \;-\; 32/81\,\mathrm{S}_1\,[1 - 6\,\zeta_3] \;-\; 1/81\,(192\,D_0^2$$

$$- 176\, D_0^3 + 48\, D_0^4 - 192\, D_1^2 + 176\, D_1^3 - 48\, D_1^4 - 32\, [2 - 3\, \zeta_3]\, D_0 + 32\, [2 - 3\, \zeta_3]\, D_1 - [131 - 144\, \zeta_3])\Bigg] \quad (5.28)$$

$$\gamma_{qq}^{(3)}\Big|_{n_f^3} = \gamma_{ns}^{(3),\pm}\Big|_{n_f^3} + \gamma_{qq,ps}^{(3)}\Big|_{n_f^3} = \gamma_{ns}^{(3),\pm}\Big|_{n_f^3} + \\ + C_F \Bigg[- 64/27\, \mathrm{S}_{1,1,1}\, (3\, D_0 - 6\, D_0^2 - 3\, D_1 - 6\, D_1^2 - 4\, D_2 + 4\, D_{-1}) \\ + 64/27\, \mathrm{S}_{1,1}\, (11\, D_0 - 13\, D_0^2 + 6\, D_0^3 - 17\, D_1 - 4\, D_1^2 + 12\, D_1^3 + 2\, D_2 + 8\, D_2^2 \\ + 4\, D_{-1}) - 32/81\, \mathrm{S}_1\, (94\, D_0 - 98\, D_0^2 + 87\, D_0^3 - 18\, D_0^4 - 226\, D_1 + 100\, D_1^2 \\ + 111\, D_1^3 - 90\, D_1^4 + 128\, D_2 + 88\, D_2^2 - 48\, D_2^3 + 4\, D_{-1}) + 16/81\, (146\, D_0^3 - 87\, D_0^4 \\ + 18\, D_0^5 - 54\, D_1^3 - 309\, D_1^4 + 198\, D_1^5 + 72\, D_2^2 - 176\, D_2^3 + 96\, D_2^4 \\ - 4\, [1 - 18\, \zeta_3]\, D_{-1} + 2\, [26 + 27\, \zeta_3]\, D_0 - 2\, [59 + 54\, \zeta_3]\, D_0^2 + 4\, [91 - 18\, \zeta_3]\, D_2 \\ - 2\, [206 + 27\, \zeta_3]\, D_1 + 2\, [215 - 54\, \zeta_3]\, D_1^2)\Bigg] \quad (5.29)$$

$$\gamma_{qg}^{(3)}\Big|_{n_f^3} = \\ + C_F \Bigg[- 32/27\, \mathrm{S}_{1,1,1,1}\, \rho + 32/9\, \mathrm{S}_4\, \rho - 32/81\, \mathrm{S}_{1,1,1}\, (71\, D_0 - 30\, D_0^2 \\ + 18\, D_0^3 - 115\, D_1 - 36\, D_1^3 + 42\, D_2 + 24\, D_2^2 - 8\, D_{-1}) + 32/81\, \mathrm{S}_3\, (71\, D_0 - 27\, D_0^2 \\ + 18\, D_0^3 - 109\, D_1 - 36\, D_1^3 + 36\, D_2 + 24\, D_2^2 - 8\, D_{-1}) + 32/81\, [\mathrm{S}_{1,2} + \mathrm{S}_{2,1}]\, (81\, D_0 \\ - 27\, D_0^2 + 18\, D_0^3 - 135\, D_1 - 36\, D_1^3 + 62\, D_2 + 24\, D_2^2 - 8\, D_{-1}) \\ - 16/243\, \mathrm{S}_{1,1}\, (416\, D_0 - 102\, D_0^2 - 72\, D_0^3 - 1633\, D_1 + 90\, D_1^2 - 288\, D_1^3 - 216\, D_1^4 \\ + 1174\, D_2 + 648\, D_2^2 + 288\, D_2^3 + 72\, D_{-1}) - 32/243\, \mathrm{S}_2\, (976\, D_0 - 891\, D_0^2 \\ + 360\, D_0^3 - 216\, D_0^4 + 88\, D_1 - 459\, D_1^2 - 72\, D_1^3 + 540\, D_1^4 - 1101\, D_2 - 852\, D_2^2 \\ - 432\, D_2^3 + 68\, D_{-1}) - 16/729\, \mathrm{S}_1\, (8634\, D_0^2 - 6822\, D_0^3 + 2430\, D_0^4 - 1620\, D_0^5 \\ + 1125\, D_1^2 - 2070\, D_1^3 - 3456\, D_1^4 + 3240\, D_1^5 - 1812\, D_2^2 - 2448\, D_2^3 - 1728\, D_2^4 \\ + 352\, D_{-1} + 24\, [427 + 27\, \zeta_3]\, D_1 - [763 + 648\, \zeta_3]\, D_2 - 12\, [802 + 27\, \zeta_3]\, D_0) \\ + 4/729\, (17370\, D_0^4 - 15012\, D_0^5 - 25992\, D_1^4 + 49464\, D_1^5 - 28512\, D_1^6 - 5280\, D_2^3 \\ - 3456\, D_2^4 + 13824\, D_2^5 + 128\, [31 + 27\, \zeta_3]\, D_{-1} - 6\, [281 - 9936\, \zeta_3]\, D_1 \\ + 72\, [635 - 18\, \zeta_3]\, D_1^2 - 54\, [835 + 144\, \zeta_3]\, D_0^3 + 24\, [959 - 432\, \zeta_3]\, D_2^2 \\ - 6\, [1621 - 2592\, \zeta_3]\, D_1^3 + 24\, [1988 + 459\, \zeta_3]\, D_0^2 - 9\, [7037 + 3852\, \zeta_3]\, D_0 \\ + 2\, [31649 - 14688\, \zeta_3]\, D_2)\Bigg] \\ + C_A \Bigg[32/27\, \rho\, \Big(\mathrm{S}_{1,1,1,1} - \mathrm{S}_{1,1,2} + \mathrm{S}_{1,2,1} + \mathrm{S}_{2,1,1} - \mathrm{S}_{1,3} - \mathrm{S}_{2,2} + \mathrm{S}_{3,1} + 4\, \mathrm{S}_{-4} \\ + 3\, \mathrm{S}_4\Big) - 128/81\, \mathrm{S}_{-3}\, (5\, D_0 - 7\, D_1 + 7\, D_2) - 64/81\, \mathrm{S}_3\, (5\, D_0 - 4\, D_1 - 3\, D_1^2 \\ + 4\, D_2 + 3\, D_2^2) + 64/81\, [\mathrm{S}_{1,2} - \mathrm{S}_{2,1} - \mathrm{S}_{1,1,1}]\, (5\, D_0 - 10\, D_1 + 3\, D_1^2 + 10\, D_2$$

$$
\begin{aligned}
&- 3\, D_2^2) \; - 4/243\, \mathrm{S}_{1,1}\, (316\, D_0 \; - 45\, D_0^2 \; + 144\, D_0^3 \; - 641\, D_1 \; - 354\, D_1^2 \; + 349\, D_2 \\
&+ 792\, D_2^2 \; - 288\, D_2^3 \; - 104\, D_{-1}) \; + 16/243\, \mathrm{S}_{-2}\, (38\, D_0 \; - 10\, D_1 \; + 9\, D_1^2 \; + 28\, D_2) \\
&- 4/243\, \mathrm{S}_2\, (468\, D_0 \; - 45\, D_0^2 \; + 144\, D_0^3 \; - 1659\, D_1 \; + 912\, D_1^2 \; - 576\, D_1^3 \; + 1277\, D_2 \\
&- 168\, D_2^2 \; + 288\, D_2^3 \; - 104\, D_{-1}) \; - 2/729\, \mathrm{S}_1\, (6354\, D_0^2 \; - 3258\, D_0^3 \; + 3456\, D_0^4 \\
&+ 5298\, D_1^2 \; + 648\, D_1^3 \; - 5184\, D_1^4 \; + 15408\, D_2^2 \; + 16992\, D_2^3 \; - 3456\, D_2^4 \; - 128\, D_{-1} \\
&- 6\, [1895 + 864\, \zeta_3]\, D_1 \; - 3\, [2863 - 864\, \zeta_3]\, D_0 \; + [17447 + 5184\, \zeta_3]\, D_2) \\
&+ 2/243\, (554\, D_0^3 \; + 696\, D_0^4 \; + 432\, D_0^5 \; + 8508\, D_1^3 \; - 6816\, D_1^4 \; + 3168\, D_1^5 \; + 2720\, D_2^3 \\
&- 4608\, D_2^4 \; + 2304\, D_2^5 \; - 192\, [2 - 3\, \zeta_3]\, D_{-1} \; + 6\, [125 + 288\, \zeta_3]\, D_1 \\
&- 3\, [269 + 912\, \zeta_3]\, D_2 \; + 2\, [643 - 432\, \zeta_3]\, D_0^2 \; + 8\, [653 - 216\, \zeta_3]\, D_2^2 \\
&- [655 - 432\, \zeta_3]\, D_0 \; - 2\, [2399 + 864\, \zeta_3]\, D_1^2) \Big]
\end{aligned} \tag{5.30}
$$

$$
\begin{aligned}
\gamma_{gq}^{(3)}\Big|_{n_f^3} &= \\
&+ {\color{blue}C_F} \Big[64/27\, \mathrm{S}_{1,1,1}\, (2\, D_0 \; - D_1 \; - 2\, D_{-1}) \; - 64/81\, \mathrm{S}_{1,1}\, (16\, D_0 \; - 8\, D_1 \; + 3\, D_1^2 \\
&\quad - 16\, D_{-1}) \; + 64/81\, \mathrm{S}_1\, (8\, D_0 \; - 4\, D_1 \; + 8\, D_1^2 \; - 3\, D_1^3 \; - 8\, D_{-1}) \; - 64/81\, (4\, D_1^2 \\
&\quad - 8\, D_1^3 \; + 3\, D_1^4 \; - 12\, \zeta_3\, D_0 \; + 6\, \zeta_3\, D_1 \; + 12\, \zeta_3\, D_{-1}) \Big]
\end{aligned} \tag{5.31}
$$

$$
\begin{aligned}
\gamma_{gg}^{(3)}\Big|_{n_f^3} &= \\
&+ {\color{blue}C_F} \Big[64/27\, (3\, D_0 \; - 6\, D_0^2 \; - 3\, D_1 \; - 6\, D_1^2 \; - 4\, D_2 \; + 4\, D_{-1})\, [\mathrm{S}_{1,1,1} \; - \mathrm{S}_{1,2} \; - \mathrm{S}_{2,1} \\
&\quad + \mathrm{S}_3/2] \; + 64/81\, \mathrm{S}_{1,1}\, (57\, D_0 \; + 21\, D_0^2 \; + 18\, D_0^3 \; - 39\, D_1 \; + 12\, D_1^2 \; + 20\, D_2 \\
&\quad - 38\, D_{-1}) \; - 32/81\, \mathrm{S}_2\, (42\, D_0 \; + 69\, D_0^2 \; + 18\, D_0^3 \; - 42\, D_1 \; + 69\, D_1^2 \; - 18\, D_1^3 \; + 70\, D_2 \\
&\quad - 70\, D_{-1}) \; - 32/243\, \mathrm{S}_1\, (429\, D_0 \; + 276\, D_0^2 \; + 207\, D_0^3 \; + 54\, D_0^4 \; - 33\, D_1 \; - 30\, D_1^2 \\
&\quad + 135\, D_1^3 \; - 54\, D_1^4 \; - 26\, D_2 \; - 370\, D_{-1}) \; - 2/243\, (77 \; - 3360\, D_0^3 \; - 1656\, D_0^4 \\
&\quad - 432\, D_0^5 \; - 3840\, D_1^3 \; + 3816\, D_1^4 \; - 1296\, D_1^5 \; - 1296\, [3 + \zeta_3]\, D_1 \\
&\quad - 432\, [11 - 3\, \zeta_3]\, D_0 \; + 96\, [43 - 18\, \zeta_3]\, D_2 \; + 96\, [47 + 18\, \zeta_3]\, D_{-1} \\
&\quad - 24\, [179 + 108\, \zeta_3]\, D_0^2 \; + 24\, [193 - 108\, \zeta_3]\, D_1^2) \Big] \\
&+ {\color{blue}C_A} \Big[4/81\, [\mathrm{S}_2 - 2\, \mathrm{S}_{1,1}]\, (33\, D_0 \; + 48\, D_0^2 \; - 33\, D_1 \; + 48\, D_1^2 \; + 52\, D_2 \; - 52\, D_{-1}) \\
&\quad + 4/243\, \mathrm{S}_1\, (480\, D_0 \; + 456\, D_0^2 \; + 144\, D_0^3 \; - 480\, D_1 \; + 456\, D_1^2 \; - 144\, D_1^3 \; + 527\, D_2 \\
&\quad - 527\, D_{-1} \; - 24\, [1 - 6\, \zeta_3]) \; - 1/243\, (5 \; + 1380\, D_0^2 \; + 912\, D_0^3 \; + 288\, D_0^4 \; + 1380\, D_1^2 \\
&\quad - 912\, D_1^3 \; + 288\, D_1^4 \; + 6\, [229 - 96\, \zeta_3]\, D_0 \; - 6\, [229 - 96\, \zeta_3]\, D_1 \\
&\quad + 4\, [331 - 144\, \zeta_3]\, D_2 \; - 4\, [331 - 144\, \zeta_3]\, D_{-1}) \Big]
\end{aligned} \tag{5.32}
$$

Figures 5.1 and 5.2 show plots of these N-space expressions for the coefficients of n_f^3.

The solid points show the function values for integer Mellin moments. The smooth curves through them are the result of an inverse Mellin transform of the results to x space followed by a numerical evaluation of the Mellin transform integral for non-integer N.

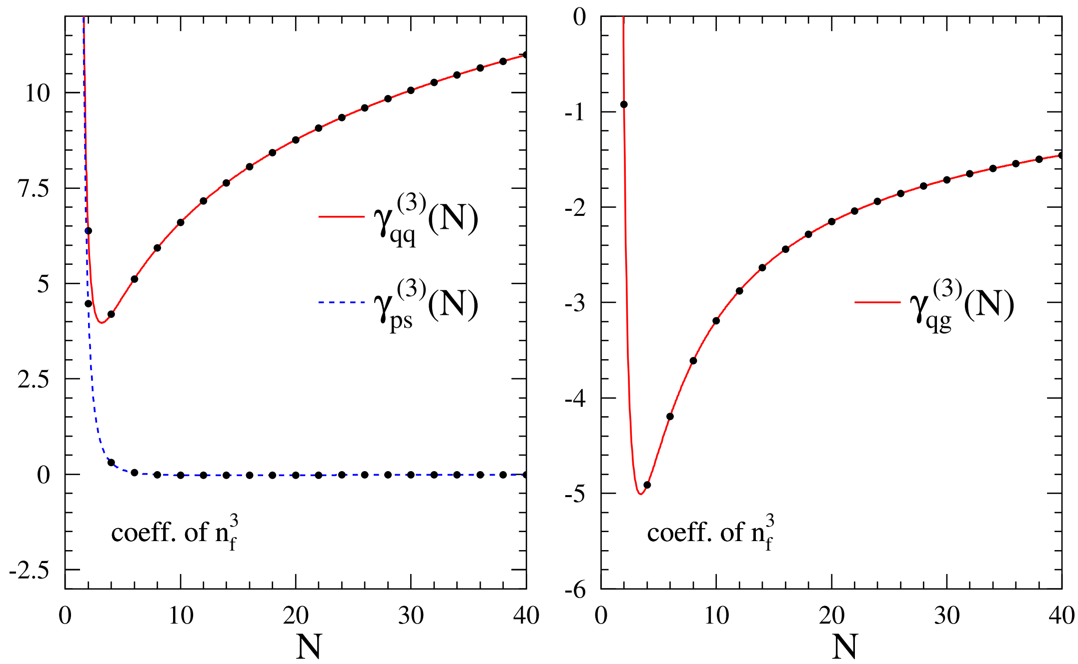

Figure 5.1: *The coefficients of the n_f^3 terms of $\gamma_{qq}^{(3)}$ and $\gamma_{qg}^{(3)}$, plotted in Mellin-N space. The colour factors C_A and C_F have been set to their QCD values of 3 and 4/3 respectively. The solid points shows the values of the (integer) Mellin moments computed by* **FORCER**.

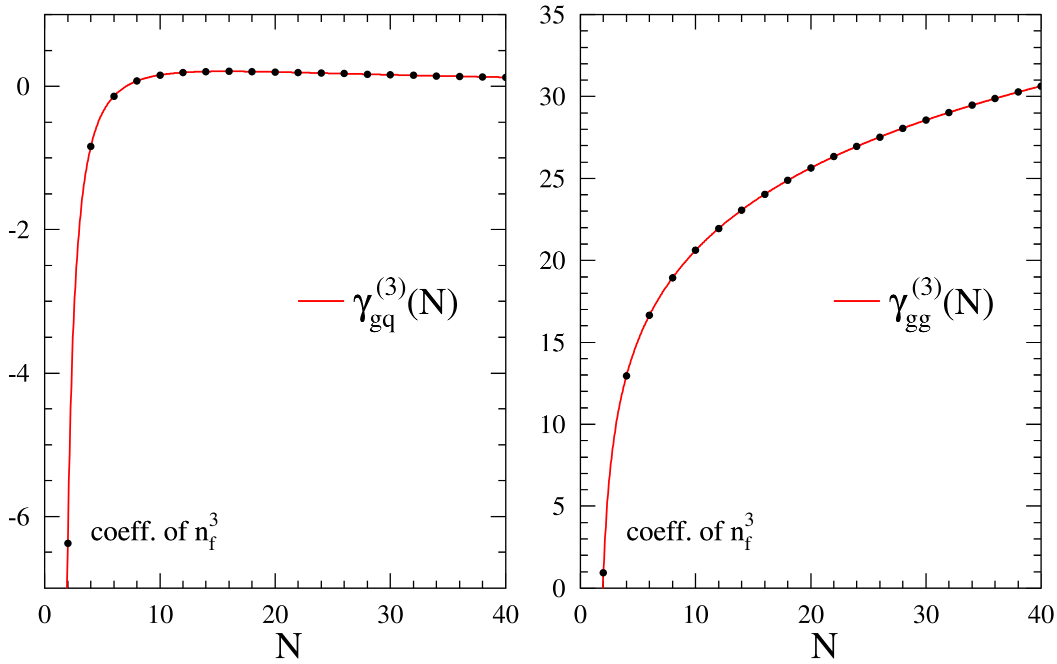

Figure 5.2: *As Fig. 5.1, for the coefficients of the* n_f^3 *terms of* $\gamma_{gq}^{(3)}$ *and* $\gamma_{gg}^{(3)}$*.*

Defining the functions

$$\begin{aligned}
p_{qq}(x) &= 2(1-x)^{-1} - 1 - x \\
p_{qg}(x) &= 1 - 2x + 2x^2 \\
p_{gq}(x) &= 2x^{-1} - 2 + x \\
p_{gg}(x) &= (1-x)^{-1} + x^{-1} - 2 + x - x^2,
\end{aligned} \tag{5.33}$$

we have the same quantities in x-space, presented as splitting functions (i.e. with a relative $(-)$ compared to Eq. (5.28) to (5.32), as defined in Eq. (2.19)).

$$\begin{aligned}
P_{ns}^{(3),\pm}\Big|_{n_f^3} &= \\
&+ C_F \Big[p_{qq}(x) \Big(-\frac{16}{81} + \frac{32}{27}\zeta_3 - \frac{16}{27}\,\mathrm{H}_{0,0,0} - \frac{80}{81}\,\mathrm{H}_{0,0} + \frac{16}{81}\,\mathrm{H}_0 \Big) + x \Big(\frac{16}{27} + \frac{32}{27}\,\mathrm{H}_{0,0} \\
&+ \frac{208}{81}\,\mathrm{H}_0 \Big) + \Big(-\frac{16}{27} - \frac{32}{27}\,\mathrm{H}_{0,0} - \frac{208}{81}\,\mathrm{H}_0 \Big) + \delta(1-x) \Big(-\frac{131}{81} + \frac{32}{81}\zeta_2 + \frac{304}{81}\zeta_3 \\
&- \frac{32}{27}\zeta_4 \Big) \Big]
\end{aligned} \tag{5.34}$$

$$\begin{aligned}
P_{qq}^{(3)}\Big|_{n_f^3} &= P_{ns}^{(3),\pm}\Big|_{n_f^3} + P_{qq,ps}^{(3)}\Big|_{n_f^3} = P_{ns}^{(3),\pm}\Big|_{n_f^3} + \\
&+ C_F \Big[\frac{1}{x} \Big(\frac{64}{27} - \frac{128}{9}\zeta_3 + \frac{256}{27}\,\mathrm{H}_{1,1,1} + \frac{128}{81}\,\mathrm{H}_1 \Big) + (1+x) \Big(-\frac{160}{9}\zeta_4 + \frac{64}{9}\,\mathrm{H}_4 \\
&- \frac{128}{9}\,\mathrm{H}_{3,1} + \frac{128}{9}\,\mathrm{H}_{2,1,1} - \frac{32}{9}\,\mathrm{H}_{0,0,0,0} + \frac{928}{27}\,\mathrm{H}_3 - \frac{464}{27}\,\mathrm{H}_{0,0,0} - \frac{2336}{81}\,\mathrm{H}_{0,0} \\
&- \frac{64}{9}\,\mathrm{H}_{0,0}\zeta_2 - \frac{928}{27}\,\mathrm{H}_0\zeta_2 - \frac{128}{9}\,\mathrm{H}_0\zeta_3 \Big) + x^2 \Big(-\frac{448}{81} + \frac{2432}{81}\zeta_2 + \frac{128}{27}\zeta_3
\end{aligned}$$

$$
\begin{aligned}
&+\frac{256}{27}\,\mathrm{H}_{2,1}-\frac{256}{27}\,\mathrm{H}_{1,1,1}-\frac{2432}{81}\,\mathrm{H}_{2}-\frac{128}{9}\,\mathrm{H}_{1,1}+\frac{2048}{81}\,\mathrm{H}_{0}+\frac{2176}{81}\,\mathrm{H}_{1}\Big)\\
&+\Big(-\frac{320}{27}-\frac{3136}{81}\,\zeta_2-\frac{128}{9}\,\zeta_3-\frac{832}{27}\,\mathrm{H}_{2,1}+\frac{64}{9}\,\mathrm{H}_{1,1,1}+\frac{3136}{81}\,\mathrm{H}_{2}-\frac{320}{9}\,\mathrm{H}_{1,1}\\
&-\frac{1888}{81}\,\mathrm{H}_{0}+\frac{3008}{81}\,\mathrm{H}_{1}\Big)+x\Big(\frac{1216}{81}-\frac{64}{81}\,\zeta_2-\frac{64}{9}\,\zeta_3-\frac{448}{27}\,\mathrm{H}_{2,1}-\frac{64}{9}\,\mathrm{H}_{1,1,1}\\
&+\frac{64}{81}\,\mathrm{H}_{2}+\frac{448}{9}\,\mathrm{H}_{1,1}-\frac{352}{81}\,\mathrm{H}_{0}-\frac{5312}{81}\,\mathrm{H}_{1}\Big)\Big]
\end{aligned}
\tag{5.35}
$$

$$
\begin{aligned}
&P_{qg}^{(3)}\Big|_{n_f^3}=\\
&+{\color{blue}C_F}\Big[\frac{1}{x}\Big(\frac{4352}{729}-\frac{512}{27}\,\zeta_3-\frac{256}{81}\,\mathrm{H}_{1,2}+\frac{256}{81}\,\mathrm{H}_{1,0,0}-\frac{256}{81}\,\mathrm{H}_{1,1,0}-\frac{256}{81}\,\mathrm{H}_{1,1,1}\\
&-\frac{2944}{243}\,\mathrm{H}_{1,0}+\frac{128}{81}\,\mathrm{H}_{1,1}+\frac{9088}{729}\,\mathrm{H}_{1}+\frac{256}{81}\,\mathrm{H}_{1}\,\zeta_2\Big)+(1-2\,x)\Big(-\frac{64}{9}\,\zeta_3\,\zeta_2\\
&-\frac{64}{9}\,\zeta_5-\frac{320}{9}\,\mathrm{H}_{5}+\frac{64}{9}\,\mathrm{H}_{3,2}-\frac{256}{9}\,\mathrm{H}_{4,0}-\frac{64}{9}\,\mathrm{H}_{3,0,0}+\frac{64}{9}\,\mathrm{H}_{3,1,0}+\frac{64}{9}\,\mathrm{H}_{3,1,1}\\
&+\frac{32}{9}\,\mathrm{H}_{1,0,0,0}+\frac{32}{27}\,\mathrm{H}_{1,1,1,1}-\frac{64}{9}\,\mathrm{H}_{3}\,\zeta_2+\frac{320}{9}\,\mathrm{H}_{0,0,0}\,\zeta_2+\frac{64}{3}\,\mathrm{H}_{0,0}\,\zeta_3-\frac{128}{9}\,\mathrm{H}_{0}\,\zeta_4\\
&-\frac{64}{9}\,\mathrm{H}_{1}\,\zeta_3\Big)+x\Big(-\frac{253016}{729}+\frac{40384}{243}\,\zeta_2-\frac{31136}{81}\,\zeta_3-\frac{64}{3}\,\zeta_4+\frac{512}{9}\,\mathrm{H}_{4}\\
&+\frac{256}{27}\,\mathrm{H}_{3,0}-\frac{512}{27}\,\mathrm{H}_{3,1}-\frac{64}{27}\,\mathrm{H}_{2,1,1}-\frac{5216}{27}\,\mathrm{H}_{0,0,0,0}-\frac{8480}{81}\,\mathrm{H}_{3}-\frac{160}{3}\,\mathrm{H}_{1,2}\\
&-\frac{1024}{9}\,\mathrm{H}_{2,0}-\frac{4160}{81}\,\mathrm{H}_{2,1}-\frac{17216}{81}\,\mathrm{H}_{0,0,0}+\frac{3488}{81}\,\mathrm{H}_{1,0,0}-\frac{160}{3}\,\mathrm{H}_{1,1,0}\\
&-\frac{3872}{81}\,\mathrm{H}_{1,1,1}-\frac{40384}{243}\,\mathrm{H}_{2}+\frac{2728}{81}\,\mathrm{H}_{0,0}-\frac{512}{9}\,\mathrm{H}_{0,0}\,\zeta_2-\frac{11072}{243}\,\mathrm{H}_{1,0}-\frac{32464}{243}\,\mathrm{H}_{1,1}\\
&+\frac{45632}{243}\,\mathrm{H}_{0}+\frac{8480}{81}\,\mathrm{H}_{0}\,\zeta_2-\frac{320}{27}\,\mathrm{H}_{0}\,\zeta_3+\frac{10336}{243}\,\mathrm{H}_{1}+\frac{160}{3}\,\mathrm{H}_{1}\,\zeta_2\Big)\\
&+x^2\Big(\frac{6568}{729}-\frac{17504}{243}\,\zeta_2+\frac{3904}{81}\,\zeta_3+\frac{128}{3}\,\zeta_4+\frac{1280}{27}\,\mathrm{H}_{4}-\frac{256}{27}\,\mathrm{H}_{2,2}+\frac{1280}{27}\,\mathrm{H}_{3,0}\\
&+\frac{256}{27}\,\mathrm{H}_{3,1}+\frac{256}{27}\,\mathrm{H}_{2,0,0}-\frac{256}{27}\,\mathrm{H}_{2,1,0}-\frac{64}{9}\,\mathrm{H}_{2,1,1}+\frac{320}{9}\,\mathrm{H}_{0,0,0,0}+\frac{64}{9}\,\mathrm{H}_{1,0,0,0}\\
&+\frac{64}{27}\,\mathrm{H}_{1,1,1,1}-\frac{5056}{81}\,\mathrm{H}_{3}+\frac{1984}{81}\,\mathrm{H}_{1,2}-\frac{2624}{27}\,\mathrm{H}_{2,0}-\frac{896}{27}\,\mathrm{H}_{2,1}-\frac{4096}{27}\,\mathrm{H}_{0,0,0}\\
&-\frac{128}{9}\,\mathrm{H}_{1,0,0}+\frac{1984}{81}\,\mathrm{H}_{1,1,0}+\frac{512}{27}\,\mathrm{H}_{1,1,1}+\frac{17504}{243}\,\mathrm{H}_{2}+\frac{256}{27}\,\mathrm{H}_{2}\,\zeta_2+\frac{11488}{81}\,\mathrm{H}_{0,0}\\
&-\frac{1280}{27}\,\mathrm{H}_{0,0}\,\zeta_2+\frac{4832}{27}\,\mathrm{H}_{1,0}+\frac{25120}{243}\,\mathrm{H}_{1,1}+\frac{150800}{729}\,\mathrm{H}_{0}+\frac{5056}{81}\,\mathrm{H}_{0}\,\zeta_2\\
&-\frac{896}{27}\,\mathrm{H}_{0}\,\zeta_3+\frac{120752}{729}\,\mathrm{H}_{1}-\frac{1984}{81}\,\mathrm{H}_{1}\,\zeta_2-\frac{128}{9}\,\mathrm{H}_{1}\,\zeta_3\Big)+\Big(\frac{233108}{729}+\frac{45280}{243}\,\zeta_2\\
&+\frac{7984}{81}\,\zeta_3-\frac{320}{9}\,\zeta_4-\frac{160}{3}\,\mathrm{H}_{4}+\frac{32}{3}\,\mathrm{H}_{2,2}-\frac{1280}{27}\,\mathrm{H}_{3,0}-\frac{128}{27}\,\mathrm{H}_{3,1}-\frac{32}{3}\,\mathrm{H}_{2,0,0}\\
&+\frac{32}{3}\,\mathrm{H}_{2,1,0}+\frac{320}{27}\,\mathrm{H}_{2,1,1}+\frac{2224}{27}\,\mathrm{H}_{0,0,0,0}-\frac{12128}{81}\,\mathrm{H}_{3}+32\,\mathrm{H}_{1,2}-\frac{352}{3}\,\mathrm{H}_{2,0}\\
&+\frac{544}{81}\,\mathrm{H}_{2,1}+\frac{7720}{81}\,\mathrm{H}_{0,0,0}-\frac{2272}{81}\,\mathrm{H}_{1,0,0}+32\,\mathrm{H}_{1,1,0}+\frac{2272}{81}\,\mathrm{H}_{1,1,1}-\frac{45280}{243}\,\mathrm{H}_{2}\\
&-\frac{32}{3}\,\mathrm{H}_{2}\,\zeta_2+\frac{19784}{81}\,\mathrm{H}_{0,0}+\frac{160}{3}\,\mathrm{H}_{0,0}\,\zeta_2-\frac{30464}{243}\,\mathrm{H}_{1,0}+\frac{7424}{243}\,\mathrm{H}_{1,1}+\frac{67328}{243}\,\mathrm{H}_{0}\\
&+\frac{12128}{81}\,\mathrm{H}_{0}\,\zeta_2+\frac{640}{27}\,\mathrm{H}_{0}\,\zeta_3-\frac{52480}{243}\,\mathrm{H}_{1}-32\,\mathrm{H}_{1}\,\zeta_2\Big)\Big]\\
&+{\color{blue}C_A}\Big[\frac{1}{x}\Big(-\frac{448}{729}-\frac{128}{27}\,\zeta_3+\frac{416}{243}\,\mathrm{H}_{1,0}-\frac{416}{243}\,\mathrm{H}_{1,1}-\frac{1504}{729}\,\mathrm{H}_{1}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+(1-2\,x)\Big(-\frac{256}{27}\,\mathrm{H}_{4}-\frac{32}{27}\,\mathrm{H}_{1,3}-\frac{64}{27}\,\mathrm{H}_{3,0}+\frac{64}{27}\,\mathrm{H}_{3,1}+\frac{32}{27}\,\mathrm{H}_{1,1,2}+\frac{32}{27}\,\mathrm{H}_{1,2,0}\\
&+\frac{32}{27}\,\mathrm{H}_{1,2,1}+\frac{32}{9}\,\mathrm{H}_{1,0,0,0}+\frac{32}{27}\,\mathrm{H}_{1,1,0,0}-\frac{32}{27}\,\mathrm{H}_{1,1,1,0}-\frac{32}{27}\,\mathrm{H}_{1,1,1,1}-\frac{724}{81}\,\mathrm{H}_{3}\\
&+\frac{256}{27}\,\mathrm{H}_{0,0}\,\zeta_2+\frac{32}{27}\,\mathrm{H}_{1,0}\,\zeta_2-\frac{32}{27}\,\mathrm{H}_{1,1}\,\zeta_2+\frac{256}{27}\,\mathrm{H}_{1}\,\zeta_3\Big)+x\Big(-\frac{71974}{729}+\frac{4016}{243}\,\zeta_2\\
&-\frac{1352}{81}\,\zeta_3-\frac{544}{27}\,\zeta_4-\frac{256}{27}\,\mathrm{H}_{-1,0,0,0}+\frac{64}{27}\,\mathrm{H}_{0,0,0,0}-\frac{16}{27}\,\mathrm{H}_{-2,0}+\frac{448}{81}\,\mathrm{H}_{1,2}\\
&+\frac{128}{27}\,\mathrm{H}_{2,0}-\frac{40}{9}\,\mathrm{H}_{2,1}-\frac{896}{81}\,\mathrm{H}_{-1,0,0}-\frac{256}{9}\,\mathrm{H}_{0,0,0}-\frac{448}{81}\,\mathrm{H}_{1,0,0}-\frac{448}{81}\,\mathrm{H}_{1,1,0}\\
&-\frac{448}{81}\,\mathrm{H}_{1,1,1}-\frac{464}{27}\,\mathrm{H}_{2}-\frac{160}{243}\,\mathrm{H}_{-1,0}-\frac{3944}{81}\,\mathrm{H}_{0,0}+\frac{524}{27}\,\mathrm{H}_{1,0}-\frac{4484}{243}\,\mathrm{H}_{1,1}\\
&-\frac{9796}{243}\,\mathrm{H}_{0}-\frac{1496}{81}\,\mathrm{H}_{0}\,\zeta_2-\frac{896}{27}\,\mathrm{H}_{0}\,\zeta_3-\frac{1072}{27}\,\mathrm{H}_{1}-\frac{448}{81}\,\mathrm{H}_{1}\,\zeta_2\Big)\\
&+x^2\Big(\frac{72316}{729}+\frac{9548}{243}\,\zeta_2+\frac{2144}{81}\,\zeta_3-\frac{64}{27}\,\mathrm{H}_{1,3}+\frac{64}{27}\,\mathrm{H}_{1,1,2}+\frac{64}{27}\,\mathrm{H}_{1,2,0}+\frac{64}{27}\,\mathrm{H}_{1,2,1}\\
&-\frac{256}{27}\,\mathrm{H}_{-1,0,0,0}+\frac{64}{9}\,\mathrm{H}_{1,0,0,0}+\frac{64}{27}\,\mathrm{H}_{1,1,0,0}-\frac{64}{27}\,\mathrm{H}_{1,1,1,0}-\frac{64}{27}\,\mathrm{H}_{1,1,1,1}+\frac{2080}{81}\,\mathrm{H}_{3}\\
&-\frac{448}{81}\,\mathrm{H}_{1,2}+\frac{416}{81}\,\mathrm{H}_{2,0}-\frac{416}{81}\,\mathrm{H}_{2,1}-\frac{896}{81}\,\mathrm{H}_{-1,0,0}+\frac{800}{27}\,\mathrm{H}_{0,0,0}+\frac{448}{81}\,\mathrm{H}_{1,0,0}\\
&+\frac{448}{81}\,\mathrm{H}_{1,1,0}+\frac{448}{81}\,\mathrm{H}_{1,1,1}-\frac{9548}{243}\,\mathrm{H}_{2}-\frac{448}{243}\,\mathrm{H}_{-1,0}-\frac{13172}{243}\,\mathrm{H}_{0,0}-\frac{3188}{243}\,\mathrm{H}_{1,0}\\
&+\frac{64}{27}\,\mathrm{H}_{1,0}\,\zeta_2+\frac{3316}{243}\,\mathrm{H}_{1,1}-\frac{64}{27}\,\mathrm{H}_{1,1}\,\zeta_2+\frac{17722}{729}\,\mathrm{H}_{0}-\frac{2080}{81}\,\mathrm{H}_{0}\,\zeta_2+\frac{41098}{729}\,\mathrm{H}_{1}\\
&+\frac{448}{81}\,\mathrm{H}_{1}\,\zeta_2+\frac{512}{27}\,\mathrm{H}_{1}\,\zeta_3\Big)+\Big(\frac{6682}{729}+\frac{1412}{81}\,\zeta_2+\frac{256}{81}\,\zeta_3+\frac{16}{9}\,\zeta_4-\frac{128}{27}\,\mathrm{H}_{-1,0,0,0}\\
&-\frac{32}{9}\,\mathrm{H}_{0,0,0,0}-\frac{320}{81}\,\mathrm{H}_{1,2}-\frac{20}{27}\,\mathrm{H}_{2,0}+\frac{20}{27}\,\mathrm{H}_{2,1}-\frac{640}{81}\,\mathrm{H}_{-1,0,0}+\frac{464}{81}\,\mathrm{H}_{0,0,0}\\
&+\frac{320}{81}\,\mathrm{H}_{1,0,0}+\frac{320}{81}\,\mathrm{H}_{1,1,0}+\frac{320}{81}\,\mathrm{H}_{1,1,1}-\frac{1412}{81}\,\mathrm{H}_{2}-\frac{608}{243}\,\mathrm{H}_{-1,0}-\frac{1108}{243}\,\mathrm{H}_{0,0}\\
&-\frac{208}{27}\,\mathrm{H}_{1,0}+\frac{1264}{243}\,\mathrm{H}_{1,1}+\frac{2156}{243}\,\mathrm{H}_{0}+\frac{724}{81}\,\mathrm{H}_{0}\,\zeta_2-\frac{128}{27}\,\mathrm{H}_{0}\,\zeta_3-\frac{590}{27}\,\mathrm{H}_{1}\\
&+\frac{320}{81}\,\mathrm{H}_{1}\,\zeta_2\Big)\Big] \qquad (5.36)
\end{aligned}
$$

$$
\begin{aligned}
&P_{gq}^{(3)}\Big|_{n_f^3}=\\
&+C_F\Big[\frac{1}{x}\Big(-\frac{128}{81}+\frac{256}{27}\,\zeta_3+\frac{128}{27}\,\mathrm{H}_{1,1,1}-\frac{640}{81}\,\mathrm{H}_{1,1}-\frac{128}{81}\,\mathrm{H}_{1}\Big)\\
&+x\Big(\frac{128}{27}\,\zeta_3+\frac{64}{27}\,\mathrm{H}_{1,1,1}-\frac{512}{81}\,\mathrm{H}_{1,1}+\frac{256}{81}\,\mathrm{H}_{1}\Big)+\Big(\frac{128}{81}-\frac{256}{27}\,\zeta_3-\frac{128}{27}\,\mathrm{H}_{1,1,1}\\
&+\frac{640}{81}\,\mathrm{H}_{1,1}+\frac{128}{81}\,\mathrm{H}_{1}\Big)\Big] \qquad (5.37)
\end{aligned}
$$

$$
\begin{aligned}
&P_{gg}^{(3)}\Big|_{n_f^3}=\\
&+C_F\Big[(\frac{1}{x}-x^2)\Big(\frac{128}{9}\,\zeta_3-\frac{256}{27}\,\mathrm{H}_{1,2}-\frac{128}{27}\,\mathrm{H}_{1,0,0}-\frac{256}{27}\,\mathrm{H}_{1,1,0}-\frac{256}{27}\,\mathrm{H}_{1,1,1}\\
&+\frac{1472}{81}\,\mathrm{H}_{1,0}+\frac{256}{27}\,\mathrm{H}_{1}\,\zeta_2\Big)+\frac{1}{x}\Big(-\frac{1088}{243}+\frac{1664}{81}\,\mathrm{H}_{1,1}-\frac{4544}{243}\,\mathrm{H}_{1}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ (1-x)\Big(-\frac{64}{9}\,\mathrm{H}_{1,2} - \frac{32}{9}\,\mathrm{H}_{1,0,0} - \frac{64}{9}\,\mathrm{H}_{1,1,0} - \frac{64}{9}\,\mathrm{H}_{1,1,1} - \frac{64}{9}\,\mathrm{H}_{1,0} + \frac{64}{9}\,\mathrm{H}_{1}\,\zeta_2\Big) \\
&+ (1+x)\Big(-\frac{160}{9}\,\zeta_4 - \frac{64}{9}\,\mathrm{H}_{4} - \frac{128}{9}\,\mathrm{H}_{2,2} - \frac{64}{9}\,\mathrm{H}_{3,0} - \frac{128}{9}\,\mathrm{H}_{3,1} - \frac{64}{9}\,\mathrm{H}_{2,0,0} \\
&- \frac{128}{9}\,\mathrm{H}_{2,1,0} - \frac{128}{9}\,\mathrm{H}_{2,1,1} - \frac{32}{9}\,\mathrm{H}_{0,0,0,0} + \frac{448}{27}\,\mathrm{H}_{2,1} + \frac{128}{9}\,\mathrm{H}_{2}\,\zeta_2 + \frac{64}{9}\,\mathrm{H}_{0,0}\,\zeta_2 \\
&+ \frac{256}{9}\,\mathrm{H}_{0}\,\zeta_3\Big) + x\Big(-\frac{1376}{243} - \frac{2048}{81}\,\zeta_2 + \frac{64}{9}\,\zeta_3 + \frac{928}{27}\,\mathrm{H}_{3} + \frac{928}{27}\,\mathrm{H}_{2,0} + \frac{464}{27}\,\mathrm{H}_{0,0,0} \\
&+ \frac{2048}{81}\,\mathrm{H}_{2} - \frac{1280}{81}\,\mathrm{H}_{0,0} + \frac{64}{3}\,\mathrm{H}_{1,1} - \frac{176}{9}\,\mathrm{H}_{0} - \frac{928}{27}\,\mathrm{H}_{0}\,\zeta_2 + \frac{160}{81}\,\mathrm{H}_{1}\Big) \\
&+ x^2\Big(\frac{1856}{243} + \frac{512}{81}\,\zeta_2 + \frac{256}{27}\,\mathrm{H}_{3} + \frac{256}{27}\,\mathrm{H}_{2,0} + \frac{256}{27}\,\mathrm{H}_{2,1} + \frac{128}{27}\,\mathrm{H}_{0,0,0} - \frac{512}{81}\,\mathrm{H}_{2} \\
&- \frac{1472}{81}\,\mathrm{H}_{0,0} - \frac{512}{81}\,\mathrm{H}_{1,1} - \frac{2368}{243}\,\mathrm{H}_{0} - \frac{256}{27}\,\mathrm{H}_{0}\,\zeta_2 - \frac{2368}{243}\,\mathrm{H}_{1}\Big) + \Big(\frac{608}{243} + \frac{2176}{81}\,\zeta_2 \\
&+ \frac{64}{3}\,\zeta_3 + \frac{736}{27}\,\mathrm{H}_{3} + \frac{736}{27}\,\mathrm{H}_{2,0} + \frac{368}{27}\,\mathrm{H}_{0,0,0} - \frac{2176}{81}\,\mathrm{H}_{2} - \frac{1856}{81}\,\mathrm{H}_{0,0} - \frac{320}{9}\,\mathrm{H}_{1,1} \\
&+ \frac{112}{9}\,\mathrm{H}_{0} - \frac{736}{27}\,\mathrm{H}_{0}\,\zeta_2 + \frac{2144}{81}\,\mathrm{H}_{1}\Big) + \delta(1-x)\Big(\frac{154}{243}\Big)\Big] \\
+ C_A\Big[&p_{gg}(x)\Big(-\frac{32}{81} + \frac{64}{27}\,\zeta_3\Big) + (\tfrac{1}{x} - x^2)\Big(-\frac{208}{81}\,\mathrm{H}_{1,0} - \frac{416}{81}\,\mathrm{H}_{1,1} + \frac{860}{243}\,\mathrm{H}_{1}\Big) \\
&+ \frac{1}{x}\Big(\frac{256}{243}\Big) + (1-x)\Big(\frac{44}{27}\,\mathrm{H}_{1,0} + \frac{88}{27}\,\mathrm{H}_{1,1} - \frac{224}{81}\,\mathrm{H}_{1}\Big) + (1+x)\Big(-\frac{64}{27}\,\mathrm{H}_{3} \\
&- \frac{64}{27}\,\mathrm{H}_{2,0} - \frac{128}{27}\,\mathrm{H}_{2,1} - \frac{32}{27}\,\mathrm{H}_{0,0,0} + \frac{64}{27}\,\mathrm{H}_{0}\,\zeta_2\Big) + x\Big(\frac{206}{243} - \frac{344}{81}\,\zeta_2 + \frac{64}{27}\,\zeta_3 \\
&+ \frac{344}{81}\,\mathrm{H}_{2} + \frac{172}{81}\,\mathrm{H}_{0,0} - \frac{28}{81}\,\mathrm{H}_{0}\Big) + x^2\Big(-\frac{256}{243} - \frac{416}{81}\,\zeta_2 + \frac{416}{81}\,\mathrm{H}_{2} + \frac{208}{81}\,\mathrm{H}_{0,0} \\
&- \frac{860}{243}\,\mathrm{H}_{0}\Big) + \Big(-\frac{206}{243} - \frac{608}{81}\,\zeta_2 + \frac{64}{27}\,\zeta_3 + \frac{608}{81}\,\mathrm{H}_{2} + \frac{304}{81}\,\mathrm{H}_{0,0} - \frac{28}{9}\,\mathrm{H}_{0}\Big) \\
&+ \delta(1-x)\Big(\frac{5}{243}\Big)\Big] \qquad (5.38)
\end{aligned}
$$

Figures 5.3 to 5.6 show these expressions plotted x-space. In each figure, the right-hand panel shows the small-x behaviour of the same curves, including their leading small-x term ($1/x$). In all cases the curves have been multiplied by x for plotting purposes, to suppress the large divergence in the small-x limit. The diagonal splitting functions have additionally been multiplied by $(1-x)$ to suppress a divergence in the large-x limit. In each case, the $1/x$ term becomes a reasonable approximation at the lower end of the plotted x range. The small-x expressions for these functions are presented in full in Appendix A.10. Figure 5.3 clearly shows the end-point dominance of either the pure-singlet (small-x) or non-singlet (large-x) parts of $P_{qq}^{(3)}$.

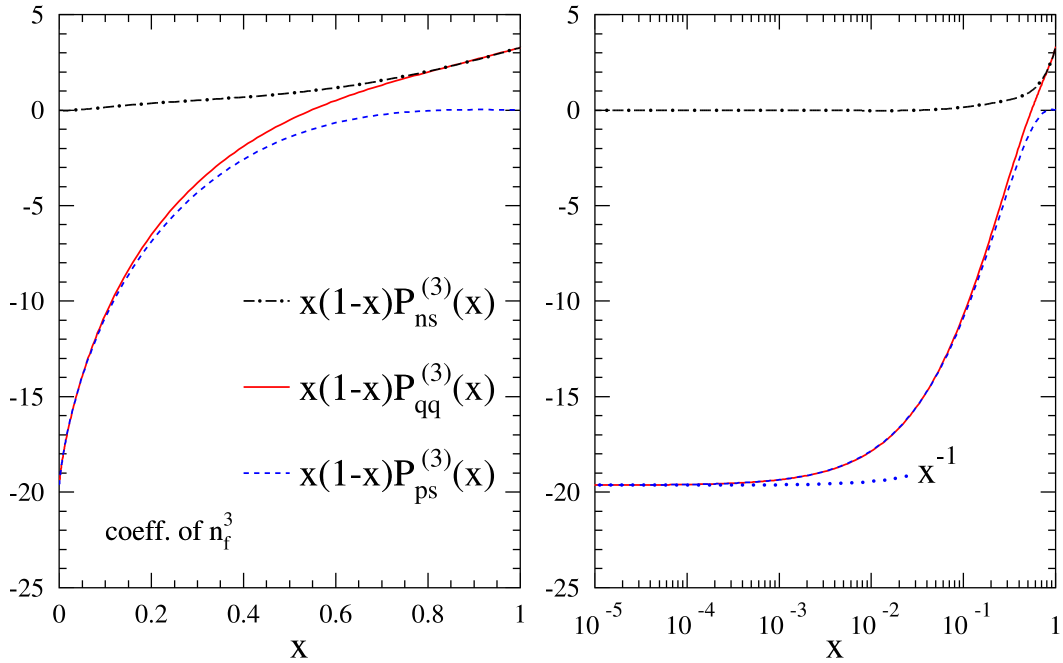

Figure 5.3: *The coefficients of the n_f^3 terms of $P_{qq}^{(3)}$, plotted in x-space. The colour factors C_A and C_F have been set to their QCD values of* 3 *and* 4/3 *respectively. The right-hand panel shows the small-x behaviour of the same curves, including the leading small-x term of $P_{qq,ps}^{(3)}$. The multiplication by $x(1-x)$ is for display purposes, and suppresses the diverging behaviour of the splitting function at each endpoint.*

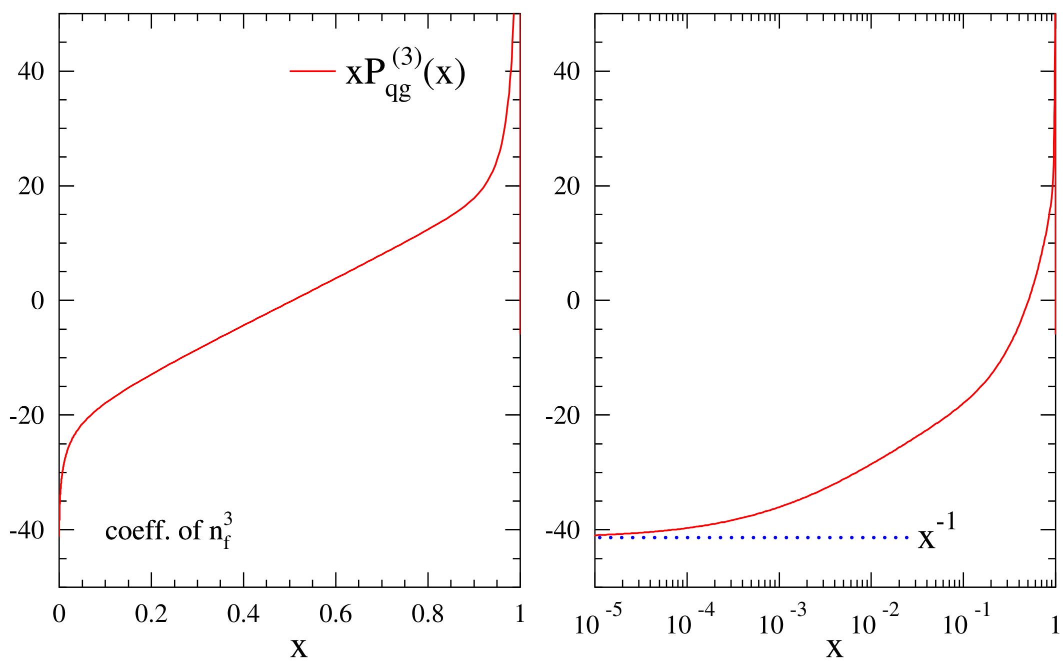

Figure 5.4: *The coefficients of the n_f^3 terms of $P_{qg}^{(3)}$, plotted in x-space. The colour factors C_A and C_F have been set to their QCD values of* 3 *and* 4/3 *respectively. The right-hand panel shows the small-x behaviour of the same curve, including the leading small-x term. The multiplication by x is for display purposes, and suppresses the diverging behaviour of the splitting function near $x = 0$.*

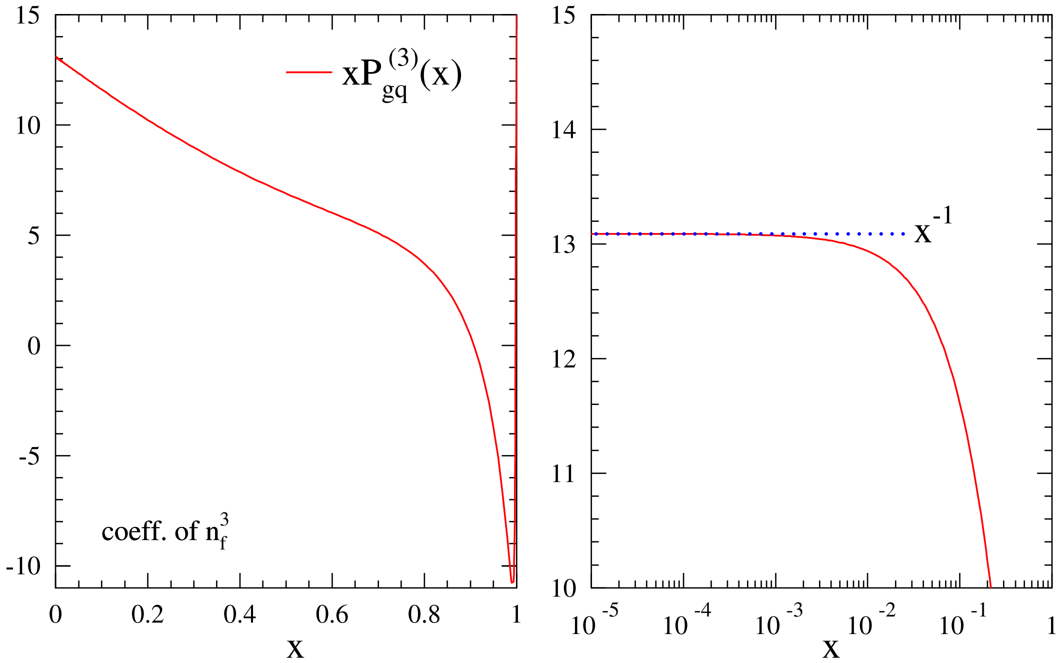

Figure 5.5: *As Fig. 5.4, for the n_f^3 terms of $P^{(3)}_{gq}$.*

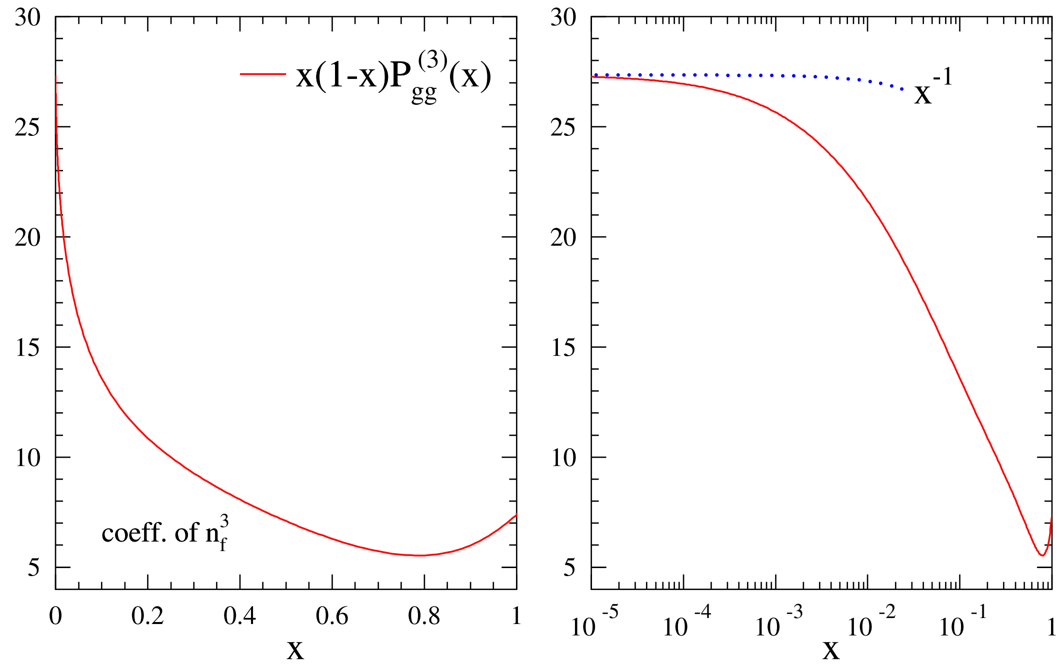

Figure 5.6: *As Fig. 5.3, for the n_f^3 terms of $P^{(3)}_{gg}$.*

5.5.2 Results for the Non-Singlet Anomalous Dimension

Here we present the next-to-leading contributions to the non-singlet anomalous dimensions $\gamma_{ns}^{(3),\pm}$ in the large-n_f limit. The terms proportional to $C_F n_f^3$ are given in N-space by Eq. (5.28) and in x-space by Eq. (5.34). The new results, proportional to $C_F C_A n_f^2$ and $C_F^2 n_f^2$, are given below. As defined in Eq. (5.27) we present the A and $B^\pm$ parts of $\gamma_{ns}^{(3)}$. We also show the (rather compact) difference $\delta B = B^+ - B^-$.

$$\begin{aligned}
2A =\ & \\
& 32/27\Big[- 38/3\,\mathrm{S}_{1,2} + 20\,\mathrm{S}_{1,3} + 6\,\mathrm{S}_{1,4} - 38/3\,\mathrm{S}_{2,1} + 40\,\mathrm{S}_{2,2} - 12\,\mathrm{S}_{2,3} + 6\,\mathrm{S}_{3,1}\,(10 \\
& + \eta) - 24\,\mathrm{S}_{3,2} - 30\,\mathrm{S}_{4,1} + 1/48\,\mathrm{S}_1\,(3392\,\eta - 3656\,\eta^2 + 432\,\eta^3 + 720\,\eta^4 \\
& - 3392\,D_1^2 - 576\,D_1^3 - 1728\,D_1^4 + [2119 + 2880\,\zeta_3 - 1296\,\zeta_4]) - 1/12\,\mathrm{S}_2\,(416\,\eta \\
& - 12\,\eta^2 - 144\,\eta^3 - 768\,D_1^2 + [1259 + 216\,\zeta_3]) + 1/3\,\mathrm{S}_3\,(287 - 12\,\eta + 18\,\eta^2 \\
& - 36\,D_1^2) - 3/2\,\mathrm{S}_4\,(53 + 2\,\eta) + 36\,\mathrm{S}_5 + 1/96\,(944\,\eta^3 - 864\,\eta^5 - 7088\,D_1^3 \\
& - 2736\,D_1^4 - 1728\,D_1^5 + 9\,[127 - 264\,\zeta_3 + 216\,\zeta_4] - 24\,[1705 + 72\,\zeta_3]\,D_1^2 \\
& - 2\,[2275 - 432\,\zeta_3]\,\eta^2 + [20681 - 2880\,\zeta_3 + 1296\,\zeta_4]\,\eta) - 12\,\mathrm{S}_{1,3,1}\Big]
\end{aligned} \tag{5.39}$$

$$\begin{aligned}
B^+ =\ & \\
& 32/27\Big[- 12\,\mathrm{S}_{-4,1} - 6\,\mathrm{S}_{-3,-2} + 2\,\mathrm{S}_{-3,1}\,(10 - 3\,\eta) - 6\,\mathrm{S}_{-2,-2}\,\eta + 2\,\mathrm{S}_{-2,1}\,(10\,\eta \\
& - 3\,\eta^2 + 6\,D_1^2) + 6\,\mathrm{S}_{1,-4} - 20\,\mathrm{S}_{1,-3} + 38/3\,\mathrm{S}_{1,-2} + 6\,\mathrm{S}_{1,1}\,(2\,\eta^2 + \eta^3) - 30\,\mathrm{S}_{1,3} \\
& + 24\,\mathrm{S}_{1,4} + 6\,\mathrm{S}_{2,-3} - 20\,\mathrm{S}_{2,-2} + 9\,\mathrm{S}_{2,3} + 6\,\mathrm{S}_{3,-2} + \mathrm{S}_{3,1}\,(10 + 3\,\eta) - 3\,\mathrm{S}_{3,2} \\
& - 6\,\mathrm{S}_{4,1} - 9\,\mathrm{S}_{-5} + \mathrm{S}_{-4}\,(20 - 3\,\eta) - 1/3\,\mathrm{S}_{-3}\,(19 - 30\,\eta + 9\,\eta^2 - 18\,D_1^2) \\
& + 1/3\,\mathrm{S}_{-2}\,(8\,\eta + 39\,\eta^2 - 96\,D_1^2) + 1/96\,\mathrm{S}_1\,(1584\,\eta - 3672\,\eta^2 + 720\,\eta^3 + 864\,\eta^4 \\
& - 1728\,D_1^2 - 1728\,D_1^3 - 2592\,D_1^4 + [923 + 5760\,\zeta_3 - 2592\,\zeta_4]) + 1/48\,\mathrm{S}_2\,(144\,\eta^2 \\
& + 72\,\eta^3 - [1585 + 864\,\zeta_3]) + 1/12\,\mathrm{S}_3\,(619 + 180\,\eta - 54\,\eta^2 + 108\,D_1^2) \\
& - 1/2\,\mathrm{S}_4\,(73 + 24\,\eta) + 9\,\mathrm{S}_5 - 1/192\,(1392\,\eta^3 - 1584\,\eta^4 + 3168\,D_1^4 \\
& - 3\,[193 - 1584\,\zeta_3 + 1296\,\zeta_4] + 2\,[2447 - 864\,\zeta_3]\,\eta^2 + 4\,[7561 + 864\,\zeta_3]\,D_1^2 \\
& - [15077 - 5760\,\zeta_3 + 2592\,\zeta_4]\,\eta) - 12\,\mathrm{S}_{-3,1,1} - 12\,\mathrm{S}_{-2,1,1}\,\eta + 12\,\mathrm{S}_{1,-3,1} \\
& + 12\,\mathrm{S}_{1,-2,-2} - 40\,\mathrm{S}_{1,-2,1} - 6\,\mathrm{S}_{1,3,1} + 12\,\mathrm{S}_{2,-2,1} + 24\,\mathrm{S}_{1,-2,1,1}\Big]
\end{aligned} \tag{5.40}$$

$$\begin{aligned}
B^- =\ & \\
& 32/27\Big[- 12\,\mathrm{S}_{-4,1} - 6\,\mathrm{S}_{-3,-2} + 2\,\mathrm{S}_{-3,1}\,(10 - 3\,\eta) - 6\,\mathrm{S}_{-2,-2}\,\eta + 2\,\mathrm{S}_{-2,1}\,(10\,\eta \\
& - 3\,\eta^2 + 6\,D_1^2) + 6\,\mathrm{S}_{1,-4} - 20\,\mathrm{S}_{1,-3} + 38/3\,\mathrm{S}_{1,-2} - 6\,\mathrm{S}_{1,1}\,(2\,\eta^2 + \eta^3) - 30\,\mathrm{S}_{1,3} \\
& + 24\,\mathrm{S}_{1,4} + 6\,\mathrm{S}_{2,-3} - 20\,\mathrm{S}_{2,-2} + 9\,\mathrm{S}_{2,3} + 6\,\mathrm{S}_{3,-2} + \mathrm{S}_{3,1}\,(10 + 3\,\eta) - 3\,\mathrm{S}_{3,2} \\
& - 6\,\mathrm{S}_{4,1} - 9\,\mathrm{S}_{-5} + \mathrm{S}_{-4}\,(20 - 3\,\eta) - 1/3\,\mathrm{S}_{-3}\,(19 - 30\,\eta + 9\,\eta^2 - 18\,D_1^2) \\
& + 1/3\,\mathrm{S}_{-2}\,(8\,\eta + 3\,\eta^2 - 18\,\eta^3 - 96\,D_1^2) - 1/96\,\mathrm{S}_1\,(432\,\eta - 1032\,\eta^2 + 240\,\eta^3
\end{aligned}$$

$$+ 288\,\eta^4 - 576\,D_1^2 - 576\,D_1^3 - 864\,D_1^4 - [923 + 5760\,\zeta_3 - 2592\,\zeta_4])$$
$$+ 1/48\,\mathrm{S}_2\,(144\,\eta^2 + 72\,\eta^3 - [1585 + 864\,\zeta_3]) + 1/12\,\mathrm{S}_3\,(619 + 180\,\eta - 54\,\eta^2$$
$$+ 108\,D_1^2) - 1/2\,\mathrm{S}_4\,(73 + 24\,\eta) + 9\,\mathrm{S}_5 + 1/192\,(7280\,\eta^3 - 336\,\eta^4 - 1728\,\eta^5$$
$$- 11136\,D_1^3 - 18144\,D_1^4 + 4608\,D_1^5 + 3\,[193 - 1584\,\zeta_3 + 1296\,\zeta_4]$$
$$- 18\,[583 - 96\,\zeta_3]\,\eta^2 - 4\,[10489 + 864\,\zeta_3]\,D_1^2 + [25541 - 5760\,\zeta_3 + 2592\,\zeta_4]\,\eta)$$
$$- 12\,\mathrm{S}_{-3,1,1} - 12\,\mathrm{S}_{-2,1,1}\,\eta + 12\,\mathrm{S}_{1,-3,1} + 12\,\mathrm{S}_{1,-2,-2} - 40\,\mathrm{S}_{1,-2,1} - 6\,\mathrm{S}_{1,3,1}$$
$$+ 12\,\mathrm{S}_{2,-2,1} + 24\,\mathrm{S}_{1,-2,1,1}\Big] \qquad (5.41)$$

$$\delta B =$$
$$32/27\Big[- 12\,\mathrm{S}_{1,1}\,(2\,\eta^2 + \eta^3) - 6\,\mathrm{S}_{-2}\,(2\,\eta^2 + \eta^3) - \mathrm{S}_1\,(21\,\eta - 49\,\eta^2 + 10\,\eta^3$$
$$+ 12\,\eta^4 - 24\,D_1^2 - 24\,D_1^3 - 36\,D_1^4) + 1/6\,(327\,\eta - 175\,\eta^2 + 271\,\eta^3 - 60\,\eta^4$$
$$- 54\,\eta^5 - 366\,D_1^2 - 348\,D_1^3 - 468\,D_1^4 + 144\,D_1^5)\Big] \qquad (5.42)$$

These N-space expressions are plotted in Fig. 5.7. Note that the curves for the n_f^2 terms of $\gamma_{ns}^{(3),+}$ and $\gamma_{ns}^{(3),-}$ lie almost exactly on top of each other in the right-hand panel. The closed points show the function values of the n_f^2 coefficients of $\gamma_{ns}^{(3),+}$ for even integer N, the open points the function values of the n_f^2 coefficients of $\gamma_{ns}^{(3),-}$ for odd integer N.

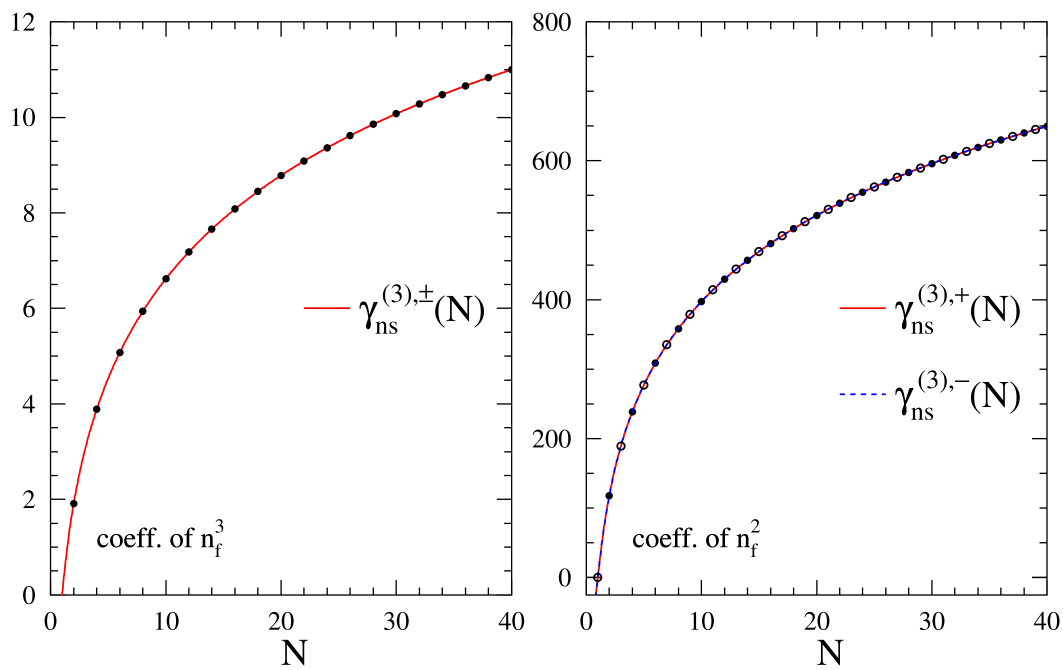

Figure 5.7: *The coefficients of the* n_f^3 *(left-hand panel) and* n_f^2 *(right-hand panel) terms of* $\gamma_{ns}^{(3),+}$ *and* $\gamma_{ns}^{(3),-}$*. The colour factors* C_A *and* C_F *have been set to their QCD values of* 3 *and* 4/3 *respectively. The solid points show the values of the even-integer Mellin moments of* $\gamma_{ns}^{(3),+}$*. The open points show the values of the odd-integer Mellin moments of* $\gamma_{ns}^{(3),-}$*.*

In x-space, these expressions are given by

$$
\begin{aligned}
2A =&\\
&p_{qq}(x)\Bigg(\frac{2119}{81} - \frac{608}{81}\zeta_2 + \frac{1280}{27}\zeta_3 - \frac{112}{9}\zeta_4 + \frac{160}{9}\mathrm{H}_4 - \frac{64}{9}\mathrm{H}_{1,3} + \frac{128}{9}\mathrm{H}_{3,0}\\
&+ \frac{64}{9}\mathrm{H}_{2,0,0} + \frac{64}{3}\mathrm{H}_{0,0,0,0} - \frac{32}{9}\mathrm{H}_{1,0,0,0} + \frac{320}{9}\mathrm{H}_3 + \frac{640}{27}\mathrm{H}_{2,0} + \frac{424}{9}\mathrm{H}_{0,0,0}\\
&+ \frac{320}{27}\mathrm{H}_{1,0,0} + \frac{608}{81}\mathrm{H}_2 + \frac{4592}{81}\mathrm{H}_{0,0} - \frac{160}{9}\mathrm{H}_{0,0}\zeta_2 + \frac{608}{81}\mathrm{H}_{1,0} + \frac{64}{9}\mathrm{H}_{1,0}\zeta_2\\
&+ \frac{5036}{81}\mathrm{H}_0 - \frac{320}{9}\mathrm{H}_0\zeta_2 + \frac{64}{3}\mathrm{H}_0\zeta_3 + \frac{64}{9}\mathrm{H}_1\zeta_3\Bigg) + (1-x)\Bigg(\frac{22916}{81} + 32\zeta_3\\
&+ \frac{64}{9}\mathrm{H}_{1,0,0} + \frac{928}{27}\mathrm{H}_{1,0} + \frac{736}{27}\mathrm{H}_1\Bigg) + x\Bigg(\frac{560}{27}\zeta_2 - \frac{32}{3}\mathrm{H}_{0,0,0,0} - \frac{32}{3}\mathrm{H}_3 - \frac{32}{3}\mathrm{H}_{2,0}\\
&- \frac{296}{9}\mathrm{H}_{0,0,0} - \frac{560}{27}\mathrm{H}_2 - \frac{6016}{81}\mathrm{H}_{0,0} - \frac{5078}{27}\mathrm{H}_0 + \frac{32}{3}\mathrm{H}_0\zeta_2\Bigg) - 48\zeta_2 - \frac{32}{3}\mathrm{H}_{0,0,0,0}\\
&+ \frac{224}{9}\mathrm{H}_3 + \frac{160}{9}\mathrm{H}_{2,0} + \frac{56}{9}\mathrm{H}_{0,0,0} + 48\mathrm{H}_2 + \frac{7424}{81}\mathrm{H}_{0,0} + \frac{17822}{81}\mathrm{H}_0 - \frac{224}{9}\mathrm{H}_0\zeta_2\\
&+ \delta(1-x)\Bigg(-\frac{127}{9} + \frac{10072}{81}\zeta_2 - \frac{1864}{27}\zeta_3 + \frac{320}{9}\zeta_3\zeta_2 - \frac{2584}{27}\zeta_4 + \frac{64}{3}\zeta_5\Bigg) \qquad (5.43)
\end{aligned}
$$

$$
\begin{aligned}
B^+ =&\\
&p_{qq}(x)\Bigg(\frac{923}{162} - \frac{304}{81}\zeta_2 + \frac{160}{3}\zeta_3 - \frac{64}{3}\zeta_4 + \frac{32}{9}\mathrm{H}_4 - \frac{32}{9}\mathrm{H}_{-3,0} - \frac{32}{9}\mathrm{H}_{1,3} + \frac{16}{9}\mathrm{H}_{3,0}\\
&- \frac{64}{9}\mathrm{H}_{1,-2,0} - \frac{16}{3}\mathrm{H}_{2,0,0} + \frac{16}{3}\mathrm{H}_{0,0,0,0} - \frac{128}{9}\mathrm{H}_{1,0,0,0} + \frac{160}{27}\mathrm{H}_3 + \frac{584}{27}\mathrm{H}_{0,0,0}\\
&- \frac{160}{9}\mathrm{H}_{1,0,0} + \frac{2476}{81}\mathrm{H}_{0,0} - \frac{64}{9}\mathrm{H}_{0,0}\zeta_2 + \frac{1585}{81}\mathrm{H}_0 - \frac{320}{27}\mathrm{H}_0\zeta_2 + \frac{112}{9}\mathrm{H}_0\zeta_3\\
&- \frac{64}{9}\mathrm{H}_1\zeta_3\Bigg) + \Bigg(\frac{1}{(1+x)} - \frac{1}{2}(1-x)\Bigg)\Bigg(\frac{608}{81}\zeta_2 - \frac{320}{9}\zeta_3 + \frac{16}{9}\zeta_4 + \frac{128}{9}\mathrm{H}_4\\
&+ \frac{64}{9}\mathrm{H}_{-3,0} - \frac{128}{9}\mathrm{H}_{-2,2} - \frac{128}{9}\mathrm{H}_{-1,3} - \frac{128}{9}\mathrm{H}_{3,1} + \frac{64}{9}\mathrm{H}_{-2,0,0} + \frac{256}{9}\mathrm{H}_{-1,2,1}\\
&+ \frac{64}{9}\mathrm{H}_{-1,0,0,0} - \frac{32}{3}\mathrm{H}_{0,0,0,0} + \frac{640}{27}\mathrm{H}_3 + \frac{640}{27}\mathrm{H}_{-2,0} - \frac{1280}{27}\mathrm{H}_{-1,2} + \frac{640}{27}\mathrm{H}_{-1,0,0}\\
&- \frac{640}{27}\mathrm{H}_{0,0,0} + \frac{128}{9}\mathrm{H}_{-2}\zeta_2 + \frac{1216}{81}\mathrm{H}_{-1,0} + \frac{64}{9}\mathrm{H}_{-1,0}\zeta_2 - \frac{608}{81}\mathrm{H}_{0,0} - \frac{64}{9}\mathrm{H}_{0,0}\zeta_2\\
&+ \frac{1280}{27}\mathrm{H}_{-1}\zeta_2 - \frac{128}{9}\mathrm{H}_{-1}\zeta_3 - \frac{320}{27}\mathrm{H}_0\zeta_2 - \frac{32}{9}\mathrm{H}_0\zeta_3\Bigg) + (1-x)\Bigg(\frac{2374}{27}\\
&- \frac{32}{3}\mathrm{H}_{1,0,0} + \frac{32}{9}\mathrm{H}_{1,0} - \frac{128}{9}\mathrm{H}_{1,1} + \frac{448}{9}\mathrm{H}_1\Bigg) + (1+x)\Bigg(-\frac{128}{9}\mathrm{H}_{-1,2} + \frac{16}{9}\mathrm{H}_{2,0}\\
&- \frac{64}{9}\mathrm{H}_{2,1} + \frac{64}{9}\mathrm{H}_{-1,0,0} + \frac{544}{27}\mathrm{H}_{-1,0} + \frac{128}{9}\mathrm{H}_{-1}\zeta_2\Bigg) + x\Bigg(-\frac{200}{27}\zeta_2 - \frac{320}{9}\zeta_3\\
&+ \frac{112}{9}\mathrm{H}_3 + \frac{32}{3}\mathrm{H}_{-2,0} + \frac{248}{9}\mathrm{H}_2 - \frac{844}{27}\mathrm{H}_{0,0} - \frac{1112}{9}\mathrm{H}_0 - \frac{16}{9}\mathrm{H}_0\zeta_2\Bigg) - \frac{104}{3}\zeta_2\\
&+ \frac{64}{3}\zeta_3 + 16\mathrm{H}_3 + \frac{32}{9}\mathrm{H}_{-2,0} + \frac{104}{3}\mathrm{H}_2 + \frac{188}{9}\mathrm{H}_{0,0} + \frac{664}{9}\mathrm{H}_0 - 16\mathrm{H}_0\zeta_2\\
&+ \delta(1-x)\Bigg(-\frac{193}{54} + \frac{3170}{81}\zeta_2 - \frac{320}{9}\zeta_3 + \frac{80}{3}\zeta_3\zeta_2 - \frac{80}{9}\zeta_4 - \frac{88}{9}\zeta_5\Bigg) \qquad (5.44)
\end{aligned}
$$

$$
\begin{aligned}
B^- =&\\
&p_{qq}(x)\Bigg(\frac{923}{162} - \frac{304}{81}\zeta_2 + \frac{160}{3}\zeta_3 - \frac{64}{3}\zeta_4 + \frac{32}{9}\mathrm{H}_4 - \frac{32}{9}\mathrm{H}_{-3,0} - \frac{32}{9}\mathrm{H}_{1,3} + \frac{16}{9}\mathrm{H}_{3,0}
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{64}{9}\,\mathrm{H}_{1,-2,0} - \frac{16}{3}\,\mathrm{H}_{2,0,0} + \frac{16}{3}\,\mathrm{H}_{0,0,0,0} - \frac{128}{9}\,\mathrm{H}_{1,0,0,0} + \frac{160}{27}\,\mathrm{H}_{3} + \frac{584}{27}\,\mathrm{H}_{0,0,0} \\
& - \frac{160}{9}\,\mathrm{H}_{1,0,0} + \frac{2476}{81}\,\mathrm{H}_{0,0} - \frac{64}{9}\,\mathrm{H}_{0,0}\,\zeta_2 + \frac{1585}{81}\,\mathrm{H}_{0} - \frac{320}{27}\,\mathrm{H}_{0}\,\zeta_2 + \frac{112}{9}\,\mathrm{H}_{0}\,\zeta_3 \\
& - \frac{64}{9}\,\mathrm{H}_{1}\,\zeta_3\Bigg) + \left(\frac{1}{(1+x)} - \frac{1}{2}(1-x)\right)\Bigg(\frac{608}{81}\,\zeta_2 - \frac{320}{9}\,\zeta_3 + \frac{16}{9}\,\zeta_4 + \frac{128}{9}\,\mathrm{H}_{4} \\
& + \frac{64}{9}\,\mathrm{H}_{-3,0} - \frac{128}{9}\,\mathrm{H}_{-2,2} - \frac{128}{9}\,\mathrm{H}_{-1,3} - \frac{128}{9}\,\mathrm{H}_{3,1} + \frac{64}{9}\,\mathrm{H}_{-2,0,0} + \frac{256}{9}\,\mathrm{H}_{-1,2,1} \\
& + \frac{64}{9}\,\mathrm{H}_{-1,0,0,0} - \frac{32}{3}\,\mathrm{H}_{0,0,0,0} + \frac{640}{27}\,\mathrm{H}_{3} + \frac{640}{27}\,\mathrm{H}_{-2,0} - \frac{1280}{27}\,\mathrm{H}_{-1,2} + \frac{640}{27}\,\mathrm{H}_{-1,0,0} \\
& - \frac{640}{27}\,\mathrm{H}_{0,0,0} + \frac{128}{9}\,\mathrm{H}_{-2}\,\zeta_2 + \frac{1216}{81}\,\mathrm{H}_{-1,0} + \frac{64}{9}\,\mathrm{H}_{-1,0}\,\zeta_2 - \frac{608}{81}\,\mathrm{H}_{0,0} - \frac{64}{9}\,\mathrm{H}_{0,0}\,\zeta_2 \\
& + \frac{1280}{27}\,\mathrm{H}_{-1}\,\zeta_2 - \frac{128}{9}\,\mathrm{H}_{-1}\,\zeta_3 - \frac{320}{27}\,\mathrm{H}_{0}\,\zeta_2 - \frac{32}{9}\,\mathrm{H}_{0}\,\zeta_3\Bigg) + (1-x)\Bigg(\frac{11554}{81} \\
& - \frac{128}{9}\,\mathrm{H}_{4} + \frac{128}{9}\,\mathrm{H}_{3,1} - \frac{32}{3}\,\mathrm{H}_{1,0,0} + \frac{32}{9}\,\mathrm{H}_{1,0} + \frac{128}{9}\,\mathrm{H}_{1,1} - \frac{608}{27}\,\mathrm{H}_{1}\Bigg) \\
& + (1+x)\Bigg(\frac{64}{9}\,\mathrm{H}_{-3,0} - \frac{128}{9}\,\mathrm{H}_{-1,2} + \frac{16}{9}\,\mathrm{H}_{2,0} + \frac{64}{9}\,\mathrm{H}_{2,1} + \frac{64}{9}\,\mathrm{H}_{-1,0,0} + \frac{928}{27}\,\mathrm{H}_{-1,0} \\
& + \frac{128}{9}\,\mathrm{H}_{-1}\,\zeta_2 + \frac{32}{3}\,\mathrm{H}_{0}\,\zeta_3\Bigg) + x\Bigg(\frac{1496}{27}\,\zeta_2 - \frac{2464}{27}\,\zeta_3 + 8\,\zeta_4 - \frac{160}{9}\,\mathrm{H}_{0,0,0,0} \\
& + \frac{1168}{27}\,\mathrm{H}_{3} + \frac{32}{9}\,\mathrm{H}_{-2,0} - \frac{736}{27}\,\mathrm{H}_{0,0,0} - \frac{568}{27}\,\mathrm{H}_{2} - \frac{4532}{81}\,\mathrm{H}_{0,0} - \frac{64}{9}\,\mathrm{H}_{0,0}\,\zeta_2 \\
& - \frac{5176}{81}\,\mathrm{H}_{0} - \frac{1072}{27}\,\mathrm{H}_{0}\,\zeta_2\Bigg) + \frac{376}{27}\,\zeta_2 + \frac{1696}{27}\,\zeta_3 + \frac{88}{3}\,\zeta_4 + \frac{32}{3}\,\mathrm{H}_{0,0,0,0} - \frac{784}{27}\,\mathrm{H}_{3} \\
& + \frac{32}{3}\,\mathrm{H}_{-2,0} + \frac{1120}{27}\,\mathrm{H}_{0,0,0} - \frac{376}{27}\,\mathrm{H}_{2} + \frac{6476}{81}\,\mathrm{H}_{0,0} + \frac{128}{9}\,\mathrm{H}_{0,0}\,\zeta_2 + \frac{10808}{81}\,\mathrm{H}_{0} \\
& + \frac{784}{27}\,\mathrm{H}_{0}\,\zeta_2 + \delta(1-x)\Bigg(- \frac{193}{54} + \frac{3170}{81}\,\zeta_2 - \frac{320}{9}\,\zeta_3 + \frac{80}{3}\,\zeta_3\,\zeta_2 - \frac{80}{9}\,\zeta_4 - \frac{88}{9}\,\zeta_5\Bigg)
\end{aligned}
\tag{5.45}
$$

$$
\begin{aligned}
\delta B = {} & \\
& (1-x)\Bigg(\frac{4432}{81} - \frac{128}{9}\,\mathrm{H}_{4} + \frac{128}{9}\,\mathrm{H}_{3,1} + \frac{64}{9}\,\mathrm{H}_{-2,0} + \frac{256}{9}\,\mathrm{H}_{1,1} - \frac{1952}{27}\,\mathrm{H}_{1}\Bigg) \\
& + (1+x)\Bigg(\frac{64}{9}\,\mathrm{H}_{-3,0} + \frac{128}{9}\,\mathrm{H}_{2,1} - \frac{1312}{27}\,\mathrm{H}_{2} + \frac{128}{9}\,\mathrm{H}_{-1,0} + \frac{4832}{81}\,\mathrm{H}_{0} + \frac{32}{3}\,\mathrm{H}_{0}\,\zeta_3\Bigg) \\
& + x\Bigg(\frac{1696}{27}\,\zeta_2 - \frac{1504}{27}\,\zeta_3 + 8\,\zeta_4 - \frac{160}{9}\,\mathrm{H}_{0,0,0,0} + \frac{832}{27}\,\mathrm{H}_{3} - \frac{736}{27}\,\mathrm{H}_{0,0,0} - \frac{2000}{81}\,\mathrm{H}_{0,0} \\
& - \frac{64}{9}\,\mathrm{H}_{0,0}\,\zeta_2 - \frac{1024}{27}\,\mathrm{H}_{0}\,\zeta_2\Bigg) + \frac{1312}{27}\,\zeta_2 + \frac{1120}{27}\,\zeta_3 + \frac{88}{3}\,\zeta_4 + \frac{32}{3}\,\mathrm{H}_{0,0,0,0} \\
& - \frac{1216}{27}\,\mathrm{H}_{3} + \frac{1120}{27}\,\mathrm{H}_{0,0,0} + \frac{4784}{81}\,\mathrm{H}_{0,0} + \frac{128}{9}\,\mathrm{H}_{0,0}\,\zeta_2 + \frac{1216}{27}\,\mathrm{H}_{0}\,\zeta_2
\end{aligned}
\tag{5.46}
$$

The x-space curves for the coefficients of the n_f^2 terms of $P_{ns}^{(3),+}$ and $P_{ns}^{(3),-}$ are plotted in Fig. 5.8. The n_f^3 coefficients have already been plotted in Fig. 5.3. In x-space one starts to see the difference between $P_{ns}^{(3),+}$ and $P_{ns}^{(3),-}$ and this is made clear in the right-hand panel, which shows the small-x behaviour. The two best logarithmic approximations to the curves are also plotted. Here, unlike the singlet functions, there is no $1/x$ term. These leading logarithmic approximations are the terms $L_0+L_0^2+L_0^3+L_0^4$ ($\mathrm{N^3LL}$) and $L_0^2+L_0^3+L_0^4$ (NNLL). The small-x expressions for these functions are presented in full in Appendix A.10.

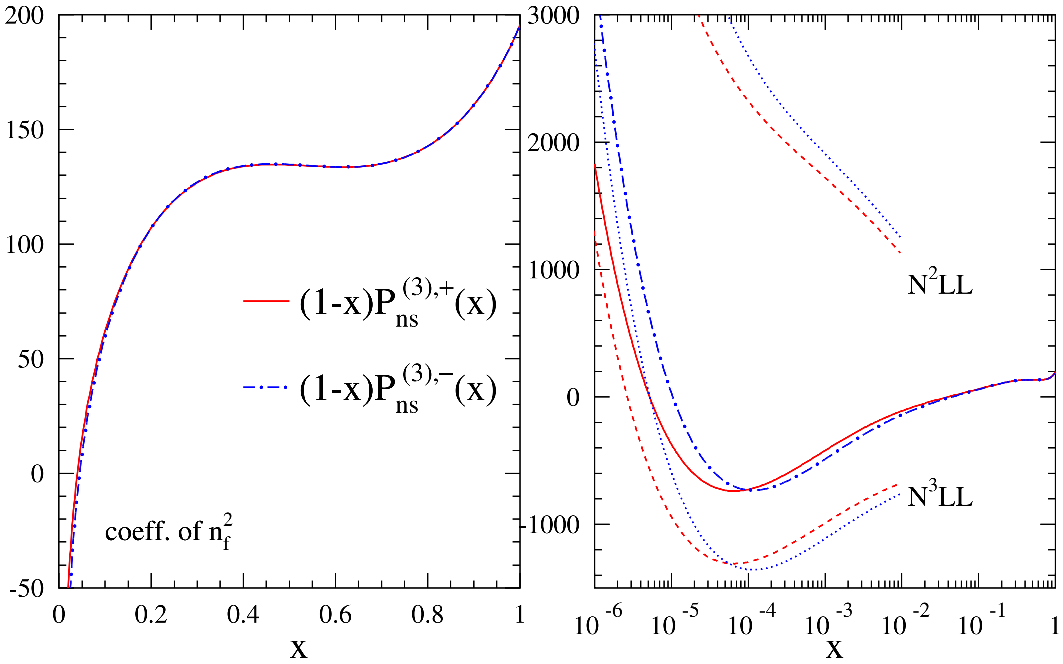

Figure 5.8: *The coefficients of the n_f^2 terms of $P_{ns}^{(3),+}$ and $P_{ns}^{(3),-}$. The colour factors C_A and C_F have been set to their QCD values of 3 and 4/3 respectively. The right-hand panel shows the small-x behaviour of the same curves, alongside their two best logarithmic approximations.*

5.6 Verification

There are some existing results in the literature which overlap with the results presented above. We briefly review it here, to further convince ourselves of the validity of the reconstructed expressions.

5.6.1 Linear Combinations of Large-n_f Singlet Anomalous Dimensions

The papers [66, 67] present large-n_f contributions to linear combinations of the singlet anomalous dimensions computed above. Introducing the notation

$$\begin{aligned}
\gamma_{qq} &= a_1 a_s + (a_{21} n_f + a_{22})\, a_s^2 + \left(a_{31} n_f^2 + a_{32} n_f + a_{33}\right) a_s^3 + \mathcal{O}(a_s^4) \\
\gamma_{qg} &= c_1 n_f a_s + c_2 n_f a_s^2 + \left(c_{31} n_f^2 + c_{32} n_f + c_{33}\right) a_s^3 + \mathcal{O}(a_s^4) \\
\gamma_{gq} &= b_1 a_s + (b_{21} n_f + b_{22})\, a_s^2 + \left(b_{31} n_f^2 + b_{32} n_f + b_{33}\right) a_s^3 + \mathcal{O}(a_s^4) \\
\gamma_{gg} &= (d_{11} n_f + d_{12})\, a_s + (d_{21} n_f + d_{22})\, a_s^2 + \left(d_{31} n_f^2 + d_{32} n_f + d_{33}\right) a_s^3 + \mathcal{O}(a_s^4) \qquad (5.47)
\end{aligned}$$

the following diagonalized anomalous dimensions are computed at leading n_f to all orders in a_s,

$$\lambda_\pm = \frac{1}{2}\left(\gamma_{qq} + \gamma_{gg}\right) \pm \frac{1}{2}\sqrt{\left(\gamma_{qq} - \gamma_{gg}\right)^2 + 4\gamma_{qg}\gamma_{gq}}. \qquad (5.48)$$

In terms of the coefficients of Eq. (5.47) (and their obvious fourth-order extension),

$$\lambda_- = \left(a_1 - \frac{b_1 c_1}{d_{11}}\right) a_s + \left(a_{21} - \frac{b_{21} c_1}{d_{11}}\right) n_f a_s^2 + \left(a_{31} - \frac{b_{31} c_1}{d_{11}}\right) n_f^2 a_s^3$$

$$+\left(a_{41}-\frac{b_{41}c_1}{d_{11}}\right)n_f^3a_s^4+\mathcal{O}\left(n_f^4a_s^5\right),$$

$$\lambda_+=\left(d_{11}n_fd_{12}+\frac{b_1c_1}{d_{11}}\right)a_s+\left(d_{21}+\frac{b_{21}c_1}{d_{11}}\right)n_fa_s^2+\left(d_{31}+\frac{b_{31}c_1}{d_{11}}\right)n_f^2a_s^3$$
$$+\left(d_{41}+\frac{b_{41}c_1}{d_{11}}\right)n_f^3a_s^4+\mathcal{O}\left(n_f^4a_s^5\right). \tag{5.49}$$

By computing the same combinations of coefficients, $\left(a_{41}-\frac{b_{41}c_1}{d_{11}}\right)$ and $\left(d_{41}+\frac{b_{41}c_1}{d_{11}}\right)$, we find that the results of this chapter agree. Unfortunately these combinations do not include the fourth-order corrections to $\gamma_{qg}^{(3)}$ (Eq. (5.30)), which were the hardest to determine and least verified by further Mellin moments.

5.6.2 Fourth-Order Cusp Anomalous Dimension

There have been recent computations of the so-called *cusp anomalous dimension*. The large-N limit of our results for the non-singlet anomalous dimension (with the n_f^3 terms coming from [81]) yield

$$\gamma_{cusp}^{(3)}=C_Fn_f^3\left(-\frac{32}{81}+\frac{64}{27}\zeta_3\right)+C_F^2n_f^2\left(\frac{2392}{81}-\frac{640}{9}\zeta_3+32\zeta_4\right)$$
$$+C_AC_Fn_f^2\left(\frac{923}{81}-\frac{608}{81}\zeta_2+\frac{2240}{27}\zeta_3-\frac{112}{3}\zeta_4\right)+\mathcal{O}\left(n_f\right). \tag{5.50}$$

After taking the large-N_c limit and some conversion of notation, this expression agrees with the n_f^3 and n_f^2 contributions to the results of both [87] and [88].

5.6.3 Large-N Behaviour of Diagonal Anomalous Dimensions

In [89], the large-N structure of the diagonal anomalous dimensions γ_{qq} and γ_{gg} is studied and some predictions of higher order contributions are made, based on lower order coefficients. In the notation (where $a=q,g$)

$$\gamma_{aa}^{(i)}=-A_a^i\ln N+B_a^i-C_a^i\left(\frac{\ln N}{N}\right)+\mathcal{O}\left(\frac{1}{N}\right), \tag{5.51}$$

it is determined that

$$\begin{aligned}C_a^1&=0,\\ C_a^2&=\left(A_a^1\right)^2,\\ C_a^3&=2A_a^1A_a^2,\\ C_a^4&=\left(A_a^2\right)^2+2A_a^1A_a^3\end{aligned} \tag{5.52}$$

in $\overline{\text{MS}}$. That the higher-order C coefficients can be written in terms of the lower-order A coefficients had been previously observed at three loops in [25], i.e. up to C_a^3. The relation for C_a^4 is thus a prediction that we are now able to (partially) verify. We should have that

$$C_q^4=\frac{1216}{81}C_F^2n_f^2+\mathcal{O}(n_f) \tag{5.53}$$

and

$$C_g^4 = 0 + \mathcal{O}(n_f^2). \tag{5.54}$$

The previously known n_f^3 terms of $\gamma_{qq}^{(3)}$ already satisfy Eq. (5.53), in that they give a contribution of zero. The new n_f^3 terms of $\gamma_{gg}^{(3)}$ satisfy Eq. (5.54); they also contribute zero. The new n_f^2 terms of $\gamma_{qq}^{(3)}$ provide the first "non-trivial" verification of this conjecture.

5.6.4 Small-x Double Logarithms of Anomalous Dimensions

In Chapter 4, we computed the leading three small-x logarithms of the anomalous dimensions and coefficient functions of DIS to all orders in their a_s expansions. We thus have existing calculations of the small-x logarithms which serve as a verification of the new fourth-order results. The large-n_f terms of the results in Section 4.3 and Section 4.4.1 are as follows (in N space),

$$\begin{aligned}
\gamma_{qq}^{(3)} &= \left(\frac{32}{9}C_F n_f^3 - \frac{32}{3}C_F^2 n_f^2\right) N^{-5} + \mathcal{O}(N^{-4}),\\
\gamma_{qg}^{(3)} &= \left(\frac{32}{9}C_A n_f^3 - \frac{2224}{27}C_F n_f^3\right) N^{-5} + \mathcal{O}(N^{-4}),\\
\gamma_{gq}^{(3)} &= 0 + \mathcal{O}(N^{-4}),\\
\gamma_{gg}^{(3)} &= \left(\frac{32}{9}C_F n_f^3\right) N^{-5} + \mathcal{O}(N^{-4}),
\end{aligned} \tag{5.55}$$

in agreement with the new fixed-order results of this chapter. For future reference, we provide the complete small-x behaviour of the large-n_f terms of the splitting functions in Appendix A.10.

5.6.5 Large-x Double Logarithms of Anomalous Dimensions

The leading logarithms of the fourth-order anomalous dimensions in the $x \to 1$ limit are also the result of various resummation efforts. In [29], the large-x structure of *physical kernels* is used to predict the leading logarithms of the anomalous dimensions to all orders of the expansion in powers of $(1-x)$. Just as the splitting functions/anomalous dimensions determine the energy-scale evolution of the PDFs, the physical kernels determine the energy scale evolution of the structure functions themselves. They are defined as K where

$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2}F = \frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\Big(Cq\Big) = \left(\beta\frac{\mathrm{d}C}{\mathrm{d}a_s} - C\gamma\right)q = \underbrace{\left[\left(\beta\frac{\mathrm{d}C}{\mathrm{d}a_s} - C\gamma\right)C^{-1}\right]}_{K}F. \tag{5.56}$$

To third order the physical kernels are observed to have *single logarithmic enhancement*, that is, in the large-x limit they have logarithms which go as $a_s^n \ln(1-x)^n$. This is a non-trivial property since the quantities that form them, the anomalous dimensions and coefficient functions of DIS, largely display *double* logarithmic enhancement; they go as $a_s^n \ln(1-x)^{2n}$. These double logarithms cancel when combined to form a physical kernel.

The conjecture that this property should hold to all orders in a_s allows one to determine the double-logarithmic contributions to fourth-order anomalous dimensions and coefficient functions. It is predicted that

$$\begin{aligned}
\gamma^{(3)}_{qq,ps} &= C_F n_f^3 L_1^3 \left(\frac{128}{81x} + \frac{32}{27}(1-x) - \frac{128}{81}x^2 + \frac{64}{27}(1+x)\mathrm{H}_0\right) + \mathcal{O}\left(L_1^2\right) \\
\gamma^{(3)}_{qg} &= (C_A - C_F) n_f^3 L_1^4 \left(-\frac{4}{81}p_{qg}(x)\right) + \mathcal{O}\left(L_1^3\right) \\
\gamma^{(3)}_{gq} &= 0 + \mathcal{O}\left(L_1^3\right) \\
\gamma^{(3)}_{gg} &= -\gamma^{(3)}_{qq,ps},
\end{aligned} \tag{5.57}$$

in agreement with the new results of this chapter.

5.7 Conclusions

The recently developed `FORCER` package has been able to compute low-N Mellin moments of the structure functions of DIS at fourth-order in massless QCD. It has verified and extended previous calculations of some moments of the non-singlet QCD splitting functions, and computed moments of the singlet splitting functions for the first time. Once some additional moments are available it will be possible to produce the first numerical approximations to the fourth-order QCD splitting functions.

In this chapter, we have used `FORCER` to compute a sufficient number of Mellin moments of very particular sets of diagrams (those leading in the colour factor n_f) to perform reconstructions of the analytic N-dependent expressions for the large-n_f a_s^4 contributions to the splitting functions. These results are the first analytic calculations of the n_f^2 terms of the non-singlet splitting functions and the n_f^3 terms of the singlet splitting functions. Where they coincide, we have shown the expressions to be in agreement with existing results in the literature. These expressions can be combined with low-N moments of the remaining colour factors to produce numerical approximations to the fourth-order splitting functions. Such approximations will be the topic of a future publication.

The computations of this chapter exhaust the opportunities to reconstruct analytic expressions for anomalous dimensions from a fixed number of Mellin moments, with the possible exception of the n_f^2 terms of the fl_{02} diagrams contributing to the evolution of the valence PDF q_{ns}^{V}, defined in Eq. (2.29). The computations involved would be more computationally demanding than anything computed for the reconstructions of this chapter, but *may* be possible with further optimization of the `FORCER` package.

Chapter 6

Summary and Outlook

The research presented in this thesis concerns QCD corrections to the deep-inelastic scattering of leptons and hadrons. The framework in which we performed our calculations allows the extraction of the coefficient functions, which are specific to DIS processes, as well as the splitting functions of QCD which are universal to all interactions with hadrons. As such, they are crucial theoretical input for data analysis at current and future collider experiments, such as the LHC and its potential upgrades. This motivates the computation of high-order QCD corrections to these quantities.

In Chapter 3, we considered the scattering of leptons and hadrons via the exchange of a charged boson. As determined in Section 2.4, we must compute parton-level structure functions for the linear combinations of $W^+ + W^-$ exchange and $W^+ - W^-$ exchange. The third-order QCD corrections to the structure functions $F_i^{W^+ + W^-}$ were computed and presented in [27, 28]. For the $W^+ - W^-$ combination, only a numerical approximation based on the first five Mellin moments was available [36, 37]. The main result of Chapter 3 was the computation of the exact expression for these third-order coefficient function contributions, $c^{(3),-}_{2,ns}$, $c^{(3),-}_{L,ns}$ and $c^{(3),-}_{3,ns}$. We investigated how these exact corrections compare to the existing approximations and how they affect the convergence of the perturbative expansions of the coefficient functions themselves and also of the structure functions after convolution with a PDF. We found both the coefficient functions and structure functions to be reasonably well-converging for x values as small as around 10^{-7} for $c^{(3),-}_{2,ns}$ and $c^{(3),-}_{3,ns}$, and around 10^{-4} for $c^{(3),-}_{L,ns}$. We also provided an exact version of the discussion of [37] regarding QCD corrections to the Paschos-Wolfenstein relation. The approximations of the required second Mellin moment proved to be very accurate and the conclusions here were unchanged.

In Chapter 4 we studied the small-x behaviour of both non-singlet and singlet parton-level structure functions. In this limit, the coefficient functions and splitting functions exhibit diverging logarithms which spoil the convergence of the perturbative series. We saw the effects of such logarithms in Chapter 3. Our focus was on the x^0 double logarithmic contributions which give the leading behaviour in the non-singlet cases. In the singlet cases these terms are sub-leading to the $\frac{1}{x}$ single logarithms. We chose a functional form for the parton-level structure functions in this small-x limit, inspired by the terms that appear in the 2 and 3 particle phase-space integrals which were computed for calculation of the second-order coefficient functions [30, 62]. Using the results of fixed-order N^{2}LO calculations, we were able to determine the highest 3 ε-poles of the parton-level structure functions to all orders in a_s. These poles completely determined our functional form and thus allowed us to express the leading small-x double logarithms (to the NNLL level) at *all* orders in both a_s and ε.

The subsequent mass factorization of the parton-level structure functions allowed us to determine the small-x expansion coefficients of the DIS coefficient functions and splitting functions to all orders in a_s. In the non-singlet cases we were able to provide closed-form expressions which give the double logarithms to all orders in a_s. In the singlet case this was not achieved but we noted some features that suggest that this should be possible with some further investigation. We also noted that the procedure should apply to the sub-leading $x^2, x^4, \ldots$ double logarithms of even-N quantities, and to $x^1, x^3, \ldots$ double logarithms of odd-N quantities. While not directly phenomenologically relevant, the resummation of such contributions would provide additional checks of reconstructions, such as those of Chapter 5, by predicting the coefficients of further sub-leading double logarithms. Such resummations will also be the topic of future research.

Finally in Chapter 5 we used a recently developed software package, `FORCER`, to compute a large number of Mellin moments of diagrams contributing to both the non-singlet and singlet structure functions in the large-n_f limit. By investigating the functional structure of the QCD splitting functions at lower orders, we were able to form bases of functions that we assumed to be sufficient to describe the fourth-order contributions. By equating these bases with the computed Mellin moments, we were able to form systems of Diophantine equations for the unknown coefficients of these bases.

By making use of a specialized software package, we were able to solve these Diophantine systems and thus reconstruct analytic expressions for the N dependence of the large-n_f terms of the fourth-order splitting functions. The results given in Section 5.5 are the first analytic expressions for the n_f^2 terms of the non-singlet splitting functions and the n_f^3 terms of the singlet splitting functions at fourth order. In the near future, these reconstructions will be combined with numerical approximations of the remaining colour factors to produce the first numerical approximations of the fourth-order splitting functions.

Appendix

A.1 Harmonic Sums

The *harmonic sums* [90] are used extensively in this thesis when discussing results and computations in Mellin space (see the below discussion of the Mellin transform, Appendix A.3). A harmonic sum is defined by a vector of integers $\vec{m}$. For negative integers we have an alternating sign in the numerator of the sum. For a vector of length one, m, we define

$$S_m(n) = \sum_{i=1}^{n} \frac{1}{i^m} \tag{A.1}$$

and

$$S_{-m}(n) = \sum_{i=1}^{n} \frac{(-1)^i}{i^m}. \tag{A.2}$$

The single-positive-index harmonic sums correspond, if we let $n = \infty$, to positive integer values of the Riemann zeta function ζ_s. Indeed, we find that fixed values of the Riemann zeta function appear in our results.

For a vector of length l the harmonic sums are defined recursively;

$$S_{m_1,m_2,\ldots,m_l}(n) = \sum_{i=1}^{n} \frac{1}{i^{m_1}} S_{m_2,\ldots,m_l}(n) \tag{A.3}$$

and as above

$$S_{-m_1,m_2,\ldots,m_l}(n) = \sum_{i=1}^{n} \frac{(-1)^i}{i^{m_1}} S_{m_2,\ldots,m_l}(n). \tag{A.4}$$

The *harmonic weight* of such a sum is defined as $\sum_{i=1}^{l} |m_i|$.

We will often suppress the argument N in typesetting, to reduce the length of expressions.

A.2 Harmonic Polylogarithms

The *harmonic polylogarithms* [91] are another useful set of functions with which we can describe the results of calculations in perturbation theory. They are related to the harmonic sums (Appendix A.1) via the Mellin transform (Appendix A.3).

As with the harmonic sums, a harmonic polylogarithm is defined by a vector, here with entries $\in \{-1, 0, 1\}$. Defining the three rational functions

$$\begin{aligned} f_{-1}(x) &= \frac{1}{1+x}, \\ f_0(x) &= \frac{1}{x}, \\ f_1(x) &= \frac{1}{1-x}, \end{aligned} \tag{A.5}$$

we have for a vector of length l the recursive definition

$$H_{m_1,m_2,\ldots,m_l}(x) = \int_0^x \mathrm{d}y f_{m_1}(y) H_{m_2,\ldots,m_l}(y). \tag{A.6}$$

There is a caveat to this definition; for an *all-zero* vector of length l, we define

$$H_{0_1,\ldots,0_l}(x) = \frac{1}{l!} \ln^l x. \tag{A.7}$$

We introduce the "shorthand" notation that zero entries in the vector (with the exception of in the last position) are removed and the absolute value of the following entry is increased by one. That is,

$$H_{\underbrace{0,\ldots,0}_{m},\pm 1,\underbrace{0,\ldots,0}_{n},\pm 1,\ldots}(x) = H_{\pm(m+1),\pm(n+1),\ldots}(x). \tag{A.8}$$

We will often suppress the argument x in typesetting, to reduce the length of expressions.

With the above "shorthand" definition, we can define the *weight* of a harmonic sum as either the number of indices in the full vector, or the sum of the absolute values of "shorthand" indices.

A.3 The Mellin Transform and its Inverse

When performing calculations of various quantities of DIS, we often encounter Mellin convolutions of the form

$$(f \otimes g)(x) = \int_x^1 \frac{\mathrm{d}y}{y} f\left(\frac{x}{y}\right) g(y). \tag{A.9}$$

As with other types of convolution, the appropriate integral transform of the functions reduces the convolution to a simple product. In this case, the *Mellin transform* has this property and is defined by

$$M[f(x)](N) = \int_0^1 \mathrm{d}x x^{N-1} f(x). \tag{A.10}$$

The x-space harmonic polylogarithms described above in Appendix A.2 can be written in terms of harmonic sums (Appendix A.1) in Mellin N-space; this is why these classes of functions are particularly useful to us.

The inverse transform is in general rather complicated. It is defined by an integral over N in the complex plane,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{d}N \, x^{-N} f(N). \tag{A.11}$$

Here, since we only deal with fairly restricted classes of functions (the harmonic sums and harmonic polylogarithms) it is possible to construct a database of the inverse transforms of the harmonic sums by forming suitable linear combinations of harmonic polylogarithms such that the *forward* Mellin transform produces the harmonic sum desired.

The routines to perform these transforms are all included within the `FORM` packages `summer` [90] and `harmpol`.

A.4 The Mellin Convolution of Plus-Distributions

In Section 3.3.4 we discussed convolutions between coefficient functions and a PDF. We noted that one must take care to properly convolute terms of the coefficient functions involving plus-distributions, defined by $a_+(x)$ such that

$$\int_0^1 \mathrm{d}x \, a_+(x) f(x) = \int_0^1 \mathrm{d}x \, a(x) \left[f(x) - f(1)\right] \tag{A.12}$$

where f is a regular (analytic) function of x.

To convolute such a plus-distribution with a PDF $xf(x)$ we must compute (see Eq. (A.9))

$$x \left[a_+ \otimes f\right](x) = \int_x^1 \frac{\mathrm{d}y}{y} \, a_+(y) \, x f\left(\frac{x}{y}\right). \tag{A.13}$$

Extending the range and subtracting the "extra" part,

$$= \int_0^1 \mathrm{d}y \, a_+(y) \, \frac{x}{y} f\left(\frac{x}{y}\right) - \int_0^x \mathrm{d}y \, a(y) \frac{x}{y} f\left(\frac{x}{y}\right), \tag{A.14}$$

we can insert the definition of the integral of a plus-distribution into the first term, yielding

$$= \int_0^1 \mathrm{d}y \, a(y) \left[\frac{x}{y} f\left(\frac{x}{y}\right) - x f(x)\right] - \int_0^x \mathrm{d}y \, a(y) \frac{x}{y} f\left(\frac{x}{y}\right). \tag{A.15}$$

Splitting the range of the first integral, the second integral cancels a term in the lower part of the range,

$$= \int_x^1 \mathrm{d}y \, a(y) \left[\frac{x}{y} f\left(\frac{x}{y}\right) - x f(x)\right] + \int_0^x \mathrm{d}y \, a(y) \left[\cancel{\frac{x}{y} f\left(\frac{x}{y}\right)} - x f(x)\right]$$
$$- \int_0^x \mathrm{d}y \, a(y) \cancel{\frac{x}{y} f\left(\frac{x}{y}\right)}, \tag{A.16}$$

$$\boxed{= \int_x^1 \mathrm{d}y \, a(y) \left[\frac{x}{y} f\left(\frac{x}{y}\right) - x f(x)\right] - \int_0^x \mathrm{d}y \, a(y) x f(x)}. \tag{A.17}$$

In Section 3.3.4 we have that

$$a_+(x) = \left[\frac{\ln(1-x)^k}{1-x}\right]_+, \tag{A.18}$$

so the second integral can easily be evaluated by substitution as

$$xf(x)\int_0^x \mathrm{d}y\,\frac{\ln(1-y)^k}{1-y} = -xf(x)\frac{\ln(1-x)^{k+1}}{k+1}. \tag{A.19}$$

In the coefficient functions, we also have terms containing $\delta(1-x)$. For these the convolution integral is trivial,

$$\int_x^1 \mathrm{d}y\,\delta(1-y)\frac{x}{y}f\left(\frac{x}{y}\right) = xf(x). \tag{A.20}$$

Similar to the notation of [92], we decompose the coefficient functions into regular and singular pieces. Let $C_{i,A}$ be the regular piece and $C_{i,B}$ be the plus-distribution piece. Let $C_{i,C}$ be the *integrated* plus-distribution and delta-function piece, that is, the sum of the results of integrals Eq. (A.19) and Eq. (A.20). We have then, that

$$\begin{aligned} x\left[C_i \otimes f\right](x) = &+ \int_x^1 \frac{\mathrm{d}y}{y}\,C_{i,A}(y)\,xf\left(\frac{x}{y}\right) \\ &+ \int_x^1 \mathrm{d}y\,C_{i,B}(y)\left[\frac{x}{y}f\left(\frac{x}{y}\right) - xf(x)\right] \\ &+ C_{i,C}(x)\,xf(x). \end{aligned} \tag{A.21}$$

In the third line of Eq. (A.21) we must be careful to consistently handle the $(-)$ signs between the second integral in Eq. (A.17) and the results of Eqs. (A.19) and (A.20).

A.5 Dispersion Relations

Here we discuss the dispersion relation required to connect Eq. (2.38), an expression for the forward amplitude $\frac{1}{2z}\hat{T}_{2,q}$, to the parton-level structure function $\frac{1}{2z}\hat{F}_{2,q}$. We found that

$$\frac{1}{2z}\hat{T}_{2,q} = 2\sum_{\text{even N}}\left(\frac{1}{z}\right)^N. \tag{A.22}$$

The problem is that this sum converges for $z > 1$, but the physical kinematic region for DIS is the range $0 < z < 1$. Here we follow the reasoning of [93, 94] and Appendix B of [36]. Writing z as $\frac{\nu}{Q^2}$, we consider $\frac{1}{2z}\hat{T}_{2,q}$ as a complex function of ν. It has a branch cut for $\nu > Q^2$ and, since it is an even function of ν, another for $\nu < -Q^2$. Consider the integral

$$I_n = \int \frac{\mathrm{d}\nu}{2\pi i}\frac{1}{\nu^{n+1}}\frac{1}{2z}\hat{T}_{2,q} = 2\int \frac{\mathrm{d}\nu}{2\pi i}\frac{1}{\nu^{n+1}}\sum_{\text{even N}}\left(\frac{\nu}{Q^2}\right)^N \tag{A.23}$$

around a closed contour around the origin (avoiding the branch cuts starting at $\pm Q^2$). By Cauchy's residue theorem, it is given by the residue of the pole at $\nu = 0$; the coefficient of the ν^n term of the sum. Thus,

$$I_n = 2\left(\frac{1}{Q^2}\right)^n . \tag{A.24}$$

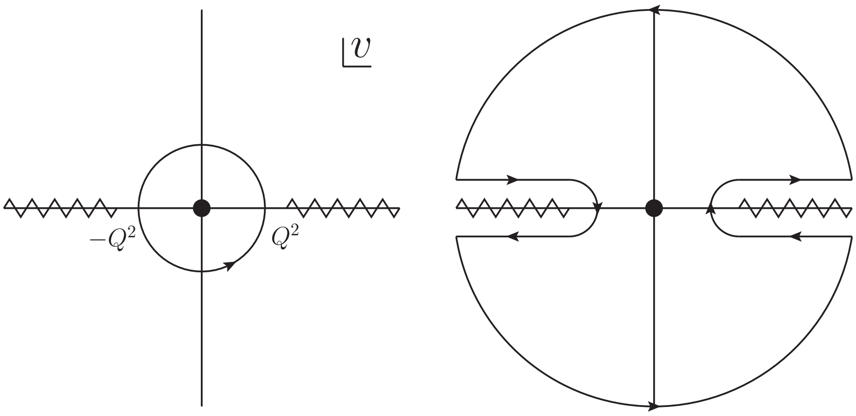

Figure A.1: *The two integration contours of the dispersion integral.*

Alternatively we may consider a deformation of the contour, pushing it out to infinity (but avoiding the branch cuts) as depicted in Fig. A.1. The integral around the curves at infinity and the curves around the poles vanish, leaving us with just the integrals along the straight lines above and below the branch cuts. Each cut gives an equal contribution, given by the discontinuity of $\frac{1}{2z}\hat{T}_{2,q}$ across the cut,

$$\begin{aligned} I_n &= 2\int_{Q^2}^{\infty} \frac{\mathrm{d}\nu}{2\pi i}\frac{1}{\nu^{n+1}}\mathrm{Disc}\left[\frac{1}{2z}\hat{T}_{2,q}\right] \\ &= 2\int_{Q^2}^{\infty} \frac{\mathrm{d}\nu}{2\pi i}\frac{1}{\nu^{n+1}}\, 2i\,\mathrm{Im}\left[\frac{1}{2z}\hat{T}_{2,q}\right] . \end{aligned} \tag{A.25}$$

Making a change of integration variable $\nu \to z = Q^2/\nu$, we find that

$$I_n = 2\left(\frac{1}{Q^2}\right)^n \int_0^1 \mathrm{d}z z^{n-1}\frac{1}{\pi}\mathrm{Im}\left[\frac{1}{2z}\hat{T}_{2,q}\right] , \tag{A.26}$$

and thus by equating Eq. (A.24) and Eq. (A.26) that

$$1 = \int_0^1 \mathrm{d}z z^{n-1}\frac{1}{\pi}\mathrm{Im}\left[\frac{1}{2z}\hat{T}_{2,q}\right] . \tag{A.27}$$

Comparing Eq. (A.27) with our statement of the optical theorem (Eq. (2.32)) we see that this is nothing but the Mellin transform of the parton-level structure function $\frac{1}{2z}\hat{F}_{2,q}$. The Nth Mellin moment of $\frac{1}{2z}\hat{F}_{2,q}$ is thus simply given by the coefficient of $2(1/z)^N$ in the forward Compton amplitude.

A.6 The g-Functions

Splitting functions and coefficient functions in DIS can generally be written using (in Mellin-space) the harmonic sums, possibly multiplied by simple denominators in the

Mellin variable N, such as $\frac{1}{N}$, $\frac{1}{N+1}$ etc.

This is not always the case, however. Starting at 3 loops (a_s^3) we find terms which must be written with *numerator* N dependence. This numerator N dependence appears with very particular combinations of harmonic sums and Riemann zeta values which we call the g-functions. These were first documented in [27]. We extend the definition a little here, to make clear whether we are discussing even- or odd-N functions.

With

$$f(N) = 5\zeta_5 - 2S_{-5} + 4S_{-2}\zeta_3 - 4S_{-2,-3} + 8S_{-2,-2,1} + 4S_{3,-2} - 4S_{4,1} + 2S_5 \tag{A.28}$$

and

$$\begin{aligned} h^E(N) &= \zeta_3 - S_{-3} - S_{-2} + 2S_{-2,1}\,, \\ h^O(N) &= \zeta_3 - S_{-3} + S_{-2} + 2S_{-2,1}\,, \end{aligned} \tag{A.29}$$

we define

$$\begin{aligned} g_1(N) &= Nf(N), \\ g_2(N) &= N^2 f(N), \\ g_3^E(N) &= N^3 f(N) - 2Nh^E(N), \\ g_3^O(N) &= N^3 f(N) + 2Nh^O(N). \end{aligned} \tag{A.30}$$

The forms of $g_1(N)$ and $g_2(N)$ differ in x-space after inverse Mellin transformation, so in the x-space expressions below we define $g_i^E(x)$ and $g_i^O(x)$ (for $i = 1, 2, 3$) as the inverse transforms for even- and odd-N. The x-space expressions are as follows, where the odd-N functions are typeset as the even-N functions plus the odd-N−even-N difference for compactness.

$$\begin{aligned} g_1^E(x) \;=\;& 2\,(1-x)^{-2}\Big[-\zeta_4 + 4\,\mathrm{H}_{-2}\,\zeta_2 - 3\,\mathrm{H}_0\,\zeta_3 + 2\,\mathrm{H}_4 + \mathrm{H}_{0,0,0,0} - 2\,\mathrm{H}_{-2,0,0} \\ & - 4\,\mathrm{H}_{-2,2} - 3\,\mathrm{H}_{0,0}\,\zeta_2 \Big] + 2\,(1-x)^{-1}\Big[-4\,\mathrm{H}_{-2}\,\zeta_2 + 4\,\mathrm{H}_{-1}\,\zeta_2 + 2\,\mathrm{H}_3 \\ & - 2\,\mathrm{H}_4 - \mathrm{H}_{0,0,0,0} + 2\,\mathrm{H}_{-2,0,0} - 2\,\mathrm{H}_{-1,0,0} + \mathrm{H}_{0,0,0} + 4\,\mathrm{H}_{-2,2} - 4\,\mathrm{H}_{-1,2} \\ & + 3\,\mathrm{H}_{0,0}\,\zeta_2 - 3\,(\zeta_2 - \zeta_3)\,\mathrm{H}_0 - (3\,\zeta_3 - \zeta_4)\Big] + 2\,(1+x)^{-2}\Big[\frac{21}{4}\,\zeta_4 + 2\,\mathrm{H}_0\,\zeta_3 \\ & - \mathrm{H}_{0,0,0,0} + 2\,\mathrm{H}_{-3,0} + \mathrm{H}_{0,0}\,\zeta_2\Big] + 2\,(1+x)^{-1}\Big[\mathrm{H}_{0,0,0,0} - \mathrm{H}_{0,0,0} - 2\,\mathrm{H}_{-3,0} \\ & + 2\,\mathrm{H}_{-2,0} - \mathrm{H}_{0,0}\,\zeta_2 + (\zeta_2 - 2\,\zeta_3)\,\mathrm{H}_0 + \frac{1}{4}\,(8\,\zeta_3 - 21\,\zeta_4)\Big] + 2\,\zeta_3 - 8\,\mathrm{H}_{-1}\,\zeta_2 \\ & + 4\,\mathrm{H}_0\,\zeta_2 - 4\,\mathrm{H}_3 + 4\,\mathrm{H}_{-1,0,0} - 4\,\mathrm{H}_{-2,0} + 8\,\mathrm{H}_{-1,2} \end{aligned} \tag{A.31}$$

$$\begin{aligned} g_2^E(x) \;=\;& 4\,(1-x)^{-3}\Big[\zeta_4 - 4\,\mathrm{H}_{-2}\,\zeta_2 + 3\,\mathrm{H}_0\,\zeta_3 - 2\,\mathrm{H}_4 - \mathrm{H}_{0,0,0,0} + 2\,\mathrm{H}_{-2,0,0} \\ & + 4\,\mathrm{H}_{-2,2} + 3\,\mathrm{H}_{0,0}\,\zeta_2\Big] + 4\,(1-x)^{-2}\Big[6\,\mathrm{H}_{-2}\,\zeta_2 - 4\,\mathrm{H}_{-1}\,\zeta_2 - 2\,\mathrm{H}_3 + 3\,\mathrm{H}_4 \\ & + \frac{3}{2}\,\mathrm{H}_{0,0,0,0} - 3\,\mathrm{H}_{-2,0,0} + 2\,\mathrm{H}_{-1,0,0} - \mathrm{H}_{0,0,0} - 6\,\mathrm{H}_{-2,2} + 4\,\mathrm{H}_{-1,2} \end{aligned}$$

$$
\begin{aligned}
& - \frac{9}{2}\,\mathrm{H}_{0,0}\,\zeta_2 \,+ \frac{3}{2}\,(2\,\zeta_2 - 3\,\zeta_3)\,\mathrm{H}_0 \,+ \frac{3}{2}\,(2\,\zeta_3 - \zeta_4)\Big] \\
& + 4\,(1-x)^{-1}\Big[- 2\,\mathrm{H}_{-2}\,\zeta_2 \,+ 4\,\mathrm{H}_{-1}\,\zeta_2 \,+ 2\,\mathrm{H}_3 \,- \mathrm{H}_4 \,- \frac{1}{2}\,\mathrm{H}_{0,0,0,0} \,+ \mathrm{H}_{-2,0,0} \\
& - 2\,\mathrm{H}_{-1,0,0} \,+ \mathrm{H}_{0,0,0} \,+ 2\,\mathrm{H}_{-2,2} \,- 4\,\mathrm{H}_{-1,2} \,+ \frac{3}{2}\,\mathrm{H}_{0,0}\,\zeta_2 \,+ \frac{1}{2}\,(\zeta_2 - 6\,\zeta_3 + \zeta_4) \\
& - \frac{3}{2}\,(2\,\zeta_2 - \zeta_3)\,\mathrm{H}_0\Big] \,+ 4\,(1+x)^{-3}\Big[- \frac{21}{4}\,\zeta_4 \,- 2\,\mathrm{H}_0\,\zeta_3 \,+ \mathrm{H}_{0,0,0,0} \,- 2\,\mathrm{H}_{-3,0} \\
& - \mathrm{H}_{0,0}\,\zeta_2\Big] \,+ 4\,(1+x)^{-2}\Big[- \frac{3}{2}\,\mathrm{H}_{0,0,0,0} \,+ \mathrm{H}_{0,0,0} \,+ 3\,\mathrm{H}_{-3,0} \,- 2\,\mathrm{H}_{-2,0} \\
& + \frac{3}{2}\,\mathrm{H}_{0,0}\,\zeta_2 \,- (\zeta_2 - 3\,\zeta_3)\,\mathrm{H}_0 \,- \frac{1}{8}\,(16\,\zeta_3 - 63\,\zeta_4)\Big] \,+ 4\,(1+x)^{-1}\Big[\mathrm{H}_2 \\
& + \frac{1}{2}\,\mathrm{H}_{0,0,0,0} \,- \mathrm{H}_{0,0,0} \,- \mathrm{H}_{-3,0} \,+ 2\,\mathrm{H}_{-2,0} \,- \mathrm{H}_{-1,0} \,+ \frac{1}{2}\,(2 - \zeta_2)\,\mathrm{H}_{0,0} \\
& + (\zeta_2 - \zeta_3)\,\mathrm{H}_0 \,- \frac{1}{8}\,(12\,\zeta_2 - 16\,\zeta_3 + 21\,\zeta_4)\Big] \,+ \delta(1-x)\Big[(\zeta_2 + \zeta_3)\Big] \,+ 4\,\zeta_2 \\
& - 4\,\mathrm{H}_2 \,+ 4\,\mathrm{H}_{-1,0} \,- 4\,\mathrm{H}_{0,0}
\end{aligned} \qquad \text{(A.32)}
$$

$$
\begin{aligned}
g_3^E(x) \;=\;& 12\,(1-x)^{-4}\Big[- \zeta_4 \,+ 4\,\mathrm{H}_{-2}\,\zeta_2 \,- 3\,\mathrm{H}_0\,\zeta_3 \,+ 2\,\mathrm{H}_4 \,+ \mathrm{H}_{0,0,0,0} \,- 2\,\mathrm{H}_{-2,0,0} \\
& - 4\,\mathrm{H}_{-2,2} \,- 3\,\mathrm{H}_{0,0}\,\zeta_2\Big] \,+ 12\,(1-x)^{-3}\Big[- 8\,\mathrm{H}_{-2}\,\zeta_2 \,+ 4\,\mathrm{H}_{-1}\,\zeta_2 \,+ 2\,\mathrm{H}_3 \\
& - 4\,\mathrm{H}_4 \,- 2\,\mathrm{H}_{0,0,0,0} \,+ 4\,\mathrm{H}_{-2,0,0} \,- 2\,\mathrm{H}_{-1,0,0} \,+ \mathrm{H}_{0,0,0} \,+ 8\,\mathrm{H}_{-2,2} \,- 4\,\mathrm{H}_{-1,2} \\
& + 6\,\mathrm{H}_{0,0}\,\zeta_2 \,- 3\,(\zeta_2 - 2\,\zeta_3)\,\mathrm{H}_0 \,- (3\,\zeta_3 - 2\,\zeta_4)\Big] \,+ 2\,(1-x)^{-2}\Big[28\,\mathrm{H}_{-2}\,\zeta_2 \\
& - 36\,\mathrm{H}_{-1}\,\zeta_2 \,- 18\,\mathrm{H}_3 \,+ 14\,\mathrm{H}_4 \,+ 7\,\mathrm{H}_{0,0,0,0} \,- 14\,\mathrm{H}_{-2,0,0} \,+ 18\,\mathrm{H}_{-1,0,0} \\
& - 9\,\mathrm{H}_{0,0,0} \,- 28\,\mathrm{H}_{-2,2} \,+ 36\,\mathrm{H}_{-1,2} \,- 21\,\mathrm{H}_{0,0}\,\zeta_2 \,- (3\,\zeta_2 - 27\,\zeta_3 + 7\,\zeta_4) \\
& + 3\,(9\,\zeta_2 - 7\,\zeta_3)\,\mathrm{H}_0\Big] \,+ 2\,(1-x)^{-1}\Big[- 4\,\mathrm{H}_{-2}\,\zeta_2 \,+ 12\,\mathrm{H}_{-1}\,\zeta_2 \,+ 2\,\mathrm{H}_2 \,+ 6\,\mathrm{H}_3 \\
& - 2\,\mathrm{H}_4 \,- \mathrm{H}_{0,0,0,0} \,+ 2\,\mathrm{H}_{-2,0,0} \,- 6\,\mathrm{H}_{-1,0,0} \,+ 3\,\mathrm{H}_{0,0,0} \,+ 4\,\mathrm{H}_{-2,2} \,- 12\,\mathrm{H}_{-1,2} \\
& + (1 + 3\,\zeta_2)\,\mathrm{H}_{0,0} \,+ (\zeta_2 - 9\,\zeta_3 + \zeta_4) \,- 3\,(3\,\zeta_2 - \zeta_3)\,\mathrm{H}_0\Big] \\
& + 12\,(1+x)^{-4}\Big[\frac{21}{4}\,\zeta_4 \,+ 2\,\mathrm{H}_0\,\zeta_3 \,- \mathrm{H}_{0,0,0,0} \,+ 2\,\mathrm{H}_{-3,0} \,+ \mathrm{H}_{0,0}\,\zeta_2\Big] \\
& + 12\,(1+x)^{-3}\Big[2\,\mathrm{H}_{0,0,0,0} \,- \mathrm{H}_{0,0,0} \,- 4\,\mathrm{H}_{-3,0} \,+ 2\,\mathrm{H}_{-2,0} \,- 2\,\mathrm{H}_{0,0}\,\zeta_2 \\
& + (\zeta_2 - 4\,\zeta_3)\,\mathrm{H}_0 \,+ \frac{1}{2}\,(4\,\zeta_3 - 21\,\zeta_4)\Big] \,+ 2\,(1+x)^{-2}\Big[- 7\,\mathrm{H}_{0,0,0,0} \,+ 9\,\mathrm{H}_{0,0,0} \\
& + 14\,\mathrm{H}_{-3,0} \,- 18\,\mathrm{H}_{-2,0} \,+ 6\,\mathrm{H}_{-1,0} \,- (3 - 7\,\zeta_2)\,\mathrm{H}_{0,0} \,- (3 + 9\,\zeta_2 - 14\,\zeta_3)\,\mathrm{H}_0 \\
& + \frac{3}{4}\,(4\,\zeta_2 - 24\,\zeta_3 + 49\,\zeta_4)\Big] \,+ 2\,(1+x)^{-1}\Big[- 2\,\mathrm{H}_2 \,+ \mathrm{H}_{0,0,0,0} \,- 3\,\mathrm{H}_{0,0,0} \\
& - 2\,\mathrm{H}_{-3,0} \,+ 6\,\mathrm{H}_{-2,0} \,- 6\,\mathrm{H}_{-1,0} \,+ (2 - \zeta_2)\,\mathrm{H}_{0,0} \,+ (4 + 3\,\zeta_2 - 2\,\zeta_3)\,\mathrm{H}_0 \\
& - \frac{1}{4}\,(4 + 4\,\zeta_2 - 24\,\zeta_3 + 21\,\zeta_4)\Big] \,+ \delta(1-x)\Big[- (\zeta_2 + \zeta_3)\Big] \,+ 2 \,- 2\,\mathrm{H}_0
\end{aligned} \qquad \text{(A.33)}
$$

$$
\begin{aligned}
g_1^O(x) \;=\;& g_1^E(x) + 2\,(1+x)^{-2}\Big[- \frac{21}{2}\,\zeta_4 \,- 4\,\mathrm{H}_0\,\zeta_3 \,+ 2\,\mathrm{H}_{0,0,0,0} \,- 4\,\mathrm{H}_{-3,0} \,- 2\,\mathrm{H}_{0,0}\,\zeta_2\Big] \\
& + 2\,(1+x)^{-1}\Big[- 2\,\mathrm{H}_{0,0,0,0} \,+ 2\,\mathrm{H}_{0,0,0} \,+ 4\,\mathrm{H}_{-3,0} \,- 4\,\mathrm{H}_{-2,0} \,+ 2\,\mathrm{H}_{0,0}\,\zeta_2 \\
& - 2\,(\zeta_2 - 2\,\zeta_3)\,\mathrm{H}_0 \,- \frac{1}{2}\,(8\,\zeta_3 - 21\,\zeta_4)\Big] \,+ 8\,\zeta_3 \,+ 4\,\mathrm{H}_0\,\zeta_2 \,- 4\,\mathrm{H}_{0,0,0} \\
& + 8\,\mathrm{H}_{-2,0}
\end{aligned} \qquad \text{(A.34)}
$$

$$
\begin{aligned}
g_2^O(x) \;=\; & g_2^E(x) + 4\,(1+x)^{-3}\Big[\,\frac{21}{2}\,\zeta_4 \;+4\,\mathrm{H}_0\,\zeta_3 \;-2\,\mathrm{H}_{0,0,0,0} \;+4\,\mathrm{H}_{-3,0} \;+2\,\mathrm{H}_{0,0}\,\zeta_2\Big] \\
& +4\,(1+x)^{-2}\Big[3\,\mathrm{H}_{0,0,0,0} \;-2\,\mathrm{H}_{0,0,0} \;-6\,\mathrm{H}_{-3,0} \;+4\,\mathrm{H}_{-2,0} \;-3\,\mathrm{H}_{0,0}\,\zeta_2 \\
& +2\,(\zeta_2-3\,\zeta_3)\,\mathrm{H}_0 \;+\frac{1}{4}\,(16\,\zeta_3-63\,\zeta_4)\Big] \;+4\,(1+x)^{-1}\Big[-\mathrm{H}_{0,0,0,0} \\
& +2\,\mathrm{H}_{0,0,0} \;+2\,\mathrm{H}_{-3,0} \;-4\,\mathrm{H}_{-2,0} \;+2\,\mathrm{H}_{-1,0} \;-(1-\zeta_2)\,\mathrm{H}_{0,0} \;-2\,(\zeta_2-\zeta_3)\,\mathrm{H}_0 \\
& +\frac{1}{4}\,(4\,\zeta_2-16\,\zeta_3+21\,\zeta_4)\Big] \;+\delta(1-x)\Big[-2\,\zeta_3\Big] \;-4\,\zeta_2 \;-8\,\mathrm{H}_{-1,0} \;+4\,\mathrm{H}_{0,0}
\end{aligned}
\tag{A.35}
$$

$$
\begin{aligned}
g_3^O(x) \;=\; & g_3^E(x) + 12\,(1+x)^{-4}\Big[-\frac{21}{2}\,\zeta_4 \;-4\,\mathrm{H}_0\,\zeta_3 \;+2\,\mathrm{H}_{0,0,0,0} \;-4\,\mathrm{H}_{-3,0} \;-2\,\mathrm{H}_{0,0}\,\zeta_2\Big] \\
& +12\,(1+x)^{-3}\Big[-4\,\mathrm{H}_{0,0,0,0} \;+2\,\mathrm{H}_{0,0,0} \;+8\,\mathrm{H}_{-3,0} \;-4\,\mathrm{H}_{-2,0} \;+4\,\mathrm{H}_{0,0}\,\zeta_2 \\
& -2\,(\zeta_2-4\,\zeta_3)\,\mathrm{H}_0 \;-(4\,\zeta_3-21\,\zeta_4)\Big] \;+2\,(1+x)^{-2}\Big[14\,\mathrm{H}_{0,0,0,0} \;-18\,\mathrm{H}_{0,0,0} \\
& -28\,\mathrm{H}_{-3,0} \;+36\,\mathrm{H}_{-2,0} \;-12\,\mathrm{H}_{-1,0} \;+2\,(3-7\,\zeta_2)\,\mathrm{H}_{0,0} \\
& +2\,(3+9\,\zeta_2-14\,\zeta_3)\,\mathrm{H}_0 \;-\frac{3}{2}\,(4\,\zeta_2-24\,\zeta_3+49\,\zeta_4)\Big] \\
& +2\,(1+x)^{-1}\Big[-2\,\mathrm{H}_{0,0,0,0} \;+6\,\mathrm{H}_{0,0,0} \;+4\,\mathrm{H}_{-3,0} \;-12\,\mathrm{H}_{-2,0} \;+12\,\mathrm{H}_{-1,0} \\
& -2\,(3-\zeta_2)\,\mathrm{H}_{0,0} \;-2\,(4+3\,\zeta_2-2\,\zeta_3)\,\mathrm{H}_0 \;+\frac{1}{2}\,(4+12\,\zeta_2-24\,\zeta_3+21\,\zeta_4)\Big] \\
& +\delta(1-x)\Big[2\,\zeta_3\Big] \;-4 \;+4\,\mathrm{H}_0
\end{aligned}
\tag{A.36}
$$

Despite containing these positive powers of N the expressions are nonetheless finite as $N \to \infty$ or equivalently, $x \to 1$. In the large-x limit they go as powers of $\ln(1-x)$ suppressed by powers of $(1-x)$,

$$
\begin{aligned}
g_1^E(x) \to & [\zeta_2+\zeta_3] - (1-x)[\zeta_2+\zeta_3] + (1-x)^2\left[\frac{5}{8} - \frac{1}{4}\zeta_2 - \frac{1}{2}\zeta_3 - \frac{1}{2}\ln(1-x)\right] \\
& + \mathcal{O}\left((1-x)^3\right)\,, \\
g_2^E(x) \to & \,\delta(1-x)[\zeta_2+\zeta_3] - [\zeta_2+\zeta_3] + (1-x)\left[\frac{3}{4} + \frac{1}{2}\zeta_2 - \ln(1-x)\right] \\
& + (1-x)^2\left[-\frac{9}{8} + \frac{1}{2}\zeta_3 + \frac{1}{4}\zeta_2 + \frac{1}{2}\ln(1-x)\right] + \mathcal{O}\left((1-x)^3\right)\,, \\
g_3^E(x) \to & -\delta(1-x)[\zeta_2+\zeta_3] + \left[\frac{3}{4} + \frac{1}{2}\zeta_2 + \ln(1-x)\right] + (1-x)\left[-\frac{1}{2} + \zeta_3\right] \\
& + (1-x)^2\left[-\frac{7}{24} - \frac{1}{12}\zeta_2 + \frac{1}{2}\zeta_3 - \frac{1}{2}\ln(1-x)\right] + \mathcal{O}\left((1-x)^3\right)\,,
\end{aligned}
\tag{A.37}
$$

$$
\begin{aligned}
g_1^O(x) \to & [\zeta_2-\zeta_3] - (1-x)[\zeta_2-\zeta_3] + (1-x)^2\left[\frac{5}{8} - \frac{1}{4}\zeta_2 + \frac{1}{2}\zeta_3 - \frac{1}{2}\ln(1-x)\right] \\
& + \mathcal{O}\left((1-x)^3\right)\,, \\
g_2^O(x) \to & \,\delta(1-x)(\zeta_2-\zeta_3) - [\zeta_2-\zeta_3] + (1-x)\left[\frac{3}{4} + \frac{1}{2}\zeta_2 - \ln(1-x)\right] \\
& + (1-x)^2\left[-\frac{9}{8} + \frac{1}{4}\zeta_2 - \frac{1}{2}\zeta_3 + \frac{1}{2}\ln(1-x)\right] + \mathcal{O}\left((1-x)^3\right)\,, \\
g_3^O(x) \to & -\delta(1-x)(\zeta_2-\zeta_3) - \left[\frac{5}{4} - \frac{1}{2}\zeta_2 - \ln(1-x)\right] + (1-x)\left[\frac{3}{2} - \zeta_3\right] \\
& + (1-x)^2\left[\frac{41}{24} - \frac{1}{12}\zeta_2 - \frac{1}{2}\zeta_3 - \frac{1}{2}\ln(1-x)\right] + \mathcal{O}\left((1-x)^3\right)\,.
\end{aligned}
\tag{A.38}
$$

It is worth taking the time to describe how an inverse Mellin transform of these quantities can be performed. The numerator N adds an additional complication and the transform cannot be automatically performed by the procedures we usually use to produce x-space expressions, which use routines from the `FORM` package `summer` [90].

We seek a function $g_1(x)$ (this argument applies to both even- and odd-N functions) such that

$$g_1(N) = \int_0^1 \mathrm{d}x x^{N-1}[g_1(x)] = N \int_0^1 \mathrm{d}x x^{N-1}[f(x)]. \tag{A.39}$$

Proceeding by parts

$$g_1(N) = \cancel{N} \left[\frac{x^N}{\cancel{N}} f(x)\right]_0^1 - \cancel{N} \int_0^1 \mathrm{d}x x^{N-1} \left[\frac{x}{\cancel{N}} f'(x)\right], \tag{A.40}$$

and bringing the boundary term inside the integral

$$g_1(N) = \int_0^1 \mathrm{d}x x^{N-1} \left[\delta(1-x) f(1) - x f'(x)\right], \tag{A.41}$$

we have in the square brackets an x-space expression for $g_1(N)$.

We can use our x-space expression for $g_1(N)$ to compute the inverse Mellin transform of $g_2(N)$ in the same way,

$$g_2(N) = \int_0^1 \mathrm{d}x x^{N-1} \left[\delta(1-x) g_1(1) - x g_1'(x)\right]. \tag{A.42}$$

Since it turns out that $f(1) = 0$, we sidestep the issues of evaluating the delta function at 0 and of taking its derivative.

Similarly for $g_3^{E,O}(N)$ we have

$$g_3^{E,O}(N) = \int_0^1 \mathrm{d}x x^{N-1} \left[\delta(1-x)\left(g_2(1) \mp h^{E,O}(1)\right) - x\left(g_2'(x) \mp h^{E,O}(x)\right)\right]. \tag{A.43}$$

Here, delta functions at 0 and delta function derivatives from $g_2(x)$ are exactly cancelled by contributions from $h^{E,O}(x)$ and again we do not have to consider how to treat these.

A.7 Third-Order Coefficient Functions in Charged-Current Deep-Inelastic Scattering

We show here the full x-space results for the coefficient functions discussed in Section 3.3. We typeset only the differences between the even-N and odd-N coefficient functions to save space; the full expressions can be reconstructed by combining these expressions and those presented in [27] and [28]. We repeat the definition of the even-$N-$odd-N differences here for convenience. See Eq. (3.6) for a more detailed discussion. Let

$$\begin{aligned} \delta C_a &= C_{a,ns}^{W^+ + W^-} - C_{a,ns}^{W^+ - W^-}, \qquad a = 2, L, \\ \delta C_3 &= C_{3,ns}^{W^+ - W^-} - C_{3,ns}^{W^+ + W^-}. \end{aligned} \tag{A.44}$$

In terms of the harmonic polylogarithms defined in Appendix A.2 and the g-functions defined in Appendix A.6, the third-order contributions to these coefficient function differences are given by

$$
\begin{aligned}
&\delta c^{(3)}_{2,ns}(x) \; = \\
&C_F\,(C_A - 2\,C_F)\,C_F\biggl(x^2\biggl\{2464\,\mathrm{H}_{-2}\,\zeta_2 \,+832\,\mathrm{H}_3 \,+1856\,\mathrm{H}_4 \,-448\,\mathrm{H}_{-3,0} \\
&\quad -1184/5\,\mathrm{H}_{-2,0} -2368\,\mathrm{H}_{-2,2} -2464\,\mathrm{H}_{-1,-1}\,\zeta_2 -160\,\mathrm{H}_{-1,2} -1664\,\mathrm{H}_{-1,3} \\
&\quad -416\,\mathrm{H}_{1,0}\,\zeta_2 -96\,\mathrm{H}_{1,1}\,\zeta_2 +192\,\mathrm{H}_{1,3} -576/5\,\mathrm{H}_{2,0} -576/5\,\mathrm{H}_{2,1} +192\,\mathrm{H}_{3,0} \\
&\quad +192\,\mathrm{H}_{3,1} +192\,\mathrm{H}_{-2,-1,0} -1664\,\mathrm{H}_{-2,0,0} +192\,\mathrm{H}_{-1,-2,0} -256/5\,\mathrm{H}_{-1,-1,0} \\
&\quad +2368\,\mathrm{H}_{-1,-1,2} +176\,\mathrm{H}_{-1,0,0} -192\,\mathrm{H}_{-1,2,0} -192\,\mathrm{H}_{-1,2,1} +688\,\mathrm{H}_{0,0,0} \\
&\quad -256\,\mathrm{H}_{1,-2,0} -192\,\mathrm{H}_{2,0,0} -192\,\mathrm{H}_{-1,-1,-1,0} +1664\,\mathrm{H}_{-1,-1,0,0} -704\,\mathrm{H}_{-1,0,0,0} \\
&\quad +832\,\mathrm{H}_{0,0,0,0} +128\,\mathrm{H}_{1,0,0,0} -192\,\mathrm{H}_{1,1,0,0} -16/5\,(12 \,-20\,\zeta_3 \,+325\,\zeta_2)\,\mathrm{H}_1 \\
&\quad -96/25\,(52 \,+25\,\zeta_2)\,\mathrm{H}_2 +8/25\,(185 \,-5900\,\zeta_3 +1876\,\zeta_2 \,+1610\,\zeta_2^2) \\
&\quad +16/25\,(293 \,-3050\,\zeta_3 -2825\,\zeta_2)\,\mathrm{H}_0 \,+32/25\,(586 +1475\,\zeta_2)\,\mathrm{H}_{-1,0} \\
&\quad -16/25\,(827 \,+3600\,\zeta_2)\,\mathrm{H}_{0,0} +672/5\,(15\,\zeta_3 +\zeta_2)\,\mathrm{H}_{-1}\biggr\} + x^3\biggl\{ \\
&\quad -4368/5\,\mathrm{H}_{-2}\,\zeta_2 +96/5\,\mathrm{H}_1\,\zeta_2 -144/5\,\mathrm{H}_2\,\zeta_2 -10416/25\,\mathrm{H}_3 -2496/5\,\mathrm{H}_4 \\
&\quad +96\,\mathrm{H}_{-3,0} +10416/25\,\mathrm{H}_{-2,0} +3936/5\,\mathrm{H}_{-2,2} +7008/5\,\mathrm{H}_{-1,-1}\,\zeta_2 \\
&\quad +19872/25\,\mathrm{H}_{-1,2} +960\,\mathrm{H}_{-1,3} -288/5\,\mathrm{H}_{3,0} -288/5\,\mathrm{H}_{3,1} -864/5\,\mathrm{H}_{-2,-1,0} \\
&\quad +2976/5\,\mathrm{H}_{-2,0,0} -576/5\,\mathrm{H}_{-1,-2,0} -16032/25\,\mathrm{H}_{-1,-1,0} -1344\,\mathrm{H}_{-1,-1,2} \\
&\quad +21792/25\,\mathrm{H}_{-1,0,0} +576/5\,\mathrm{H}_{-1,2,0} +576/5\,\mathrm{H}_{-1,2,1} -11376/25\,\mathrm{H}_{0,0,0} \\
&\quad +96/5\,\mathrm{H}_{2,0,0} +576/5\,\mathrm{H}_{-1,-1,-1,0} -960\,\mathrm{H}_{-1,-1,0,0} +384\,\mathrm{H}_{-1,0,0,0} \\
&\quad -1056/5\,\mathrm{H}_{0,0,0,0} +32/25\,(314 \,-825\,\zeta_2)\,\mathrm{H}_{-1,0} -12/25\,(337 \,-1240\,\zeta_2)\,\mathrm{H}_{0,0} \\
&\quad +144/25\,(95\,\zeta_3 +138\,\zeta_2)\,\mathrm{H}_0 -48/25\,(600\,\zeta_3 +581\,\zeta_2)\,\mathrm{H}_{-1} +12/25\,(2010\,\zeta_3 \\
&\quad +337\,\zeta_2 -322\,\zeta_2^2)\biggr\} + \bigl(1/x^2 + 9\,x^3\bigr)\,\biggl\{96/5\,\mathrm{H}_{-2}\,\zeta_2 \,+32/5\,\mathrm{H}_2\,\zeta_2 \\
&\quad -128/15\,\mathrm{H}_{-2,0} -64/5\,\mathrm{H}_{-2,2} -1168/15\,\mathrm{H}_{-1,-1}\,\zeta_2 -3152/75\,\mathrm{H}_{-1,2} \\
&\quad -160/3\,\mathrm{H}_{-1,3} +112/15\,\mathrm{H}_{1,0}\,\zeta_2 +16/5\,\mathrm{H}_{1,1}\,\zeta_2 -32/15\,\mathrm{H}_{1,3} +64/5\,\mathrm{H}_{-2,-1,0} \\
&\quad -64/5\,\mathrm{H}_{-2,0,0} +32/5\,\mathrm{H}_{-1,-2,0} +2512/75\,\mathrm{H}_{-1,-1,0} +224/3\,\mathrm{H}_{-1,-1,2} \\
&\quad -3472/75\,\mathrm{H}_{-1,0,0} -32/5\,\mathrm{H}_{-1,2,0} -32/5\,\mathrm{H}_{-1,2,1} +64/15\,\mathrm{H}_{1,-2,0} \\
&\quad -32/5\,\mathrm{H}_{-1,-1,-1,0} +160/3\,\mathrm{H}_{-1,-1,0,0} -64/3\,\mathrm{H}_{-1,0,0,0} -32/15\,\mathrm{H}_{1,0,0,0} \\
&\quad +32/15\,\mathrm{H}_{1,1,0,0} -4/225\,(1501 \,-3300\,\zeta_2)\,\mathrm{H}_{-1,0} -8/75\,(30\,\zeta_3 -197\,\zeta_2)\,\mathrm{H}_1 \\
&\quad +8/75\,(600\,\zeta_3 +551\,\zeta_2)\,\mathrm{H}_{-1}\biggr\} + \bigl(1/x + 9\,x^2\bigr)\,\biggl\{1888/15\,\mathrm{H}_{-1}\,\zeta_2 \,-16/25\,\mathrm{H}_2 \\
&\quad +832/15\,\mathrm{H}_3 \,-32/3\,\mathrm{H}_{-2,0} -1024/25\,\mathrm{H}_{-1,0} -320/3\,\mathrm{H}_{-1,2} +2672/75\,\mathrm{H}_{0,0} \\
&\quad -32/5\,\mathrm{H}_{1,0} \,-32/5\,\mathrm{H}_{1,1} \,+32/5\,\mathrm{H}_{2,0} \,+32/5\,\mathrm{H}_{2,1} \,+192/5\,\mathrm{H}_{-1,-1,0}
\end{aligned}
$$

$$
\begin{aligned}
& - 256/3\,\mathrm{H}_{-1,0,0} + 352/15\,\mathrm{H}_{0,0,0} - 32/15\,\mathrm{H}_{1,0,0} - 16/25\,(19 - 20\,\zeta_2)\,\mathrm{H}_1 \\
& - 4/225\,(743 + 3720\,\zeta_2)\,\mathrm{H}_0 + 4/225\,(983 - 3420\,\zeta_3 - 2208\,\zeta_2) \Big\} + (1-x) \Big\{ \\
& - 3728/3\,\mathrm{H}_{-3}\,\zeta_2 - 1912\,\mathrm{H}_4 - 248/3\,\mathrm{H}_5 + 1504/3\,\mathrm{H}_{-4,0} + 3728/3\,\mathrm{H}_{-3,0} \\
& + 2080/3\,\mathrm{H}_{-3,2} + 9808/3\,\mathrm{H}_{-2,-1}\,\zeta_2 + 10544/3\,\mathrm{H}_{-2,2} + 5584/3\,\mathrm{H}_{-2,3} \\
& + 3040\,\mathrm{H}_{-1,-2}\,\zeta_2 + 13136/3\,\mathrm{H}_{-1,3} + 1760\,\mathrm{H}_{-1,4} + 856/3\,\mathrm{H}_{1,2} - 448/3\,\mathrm{H}_{1,3} \\
& - 448/3\,\mathrm{H}_{2,2} + 320\,\mathrm{H}_{2,3} - 1948/3\,\mathrm{H}_{3,0} - 2528/3\,\mathrm{H}_{3,1} + 32/3\,\mathrm{H}_{3,2} - 32\,\mathrm{H}_{4,0} \\
& - 32\,\mathrm{H}_{4,1} - 3296/3\,\mathrm{H}_{-3,-1,0} + 1248\,\mathrm{H}_{-3,0,0} - 928\,\mathrm{H}_{-2,-2,0} - 4880/3\,\mathrm{H}_{-2,-1,0} \\
& - 7936/3\,\mathrm{H}_{-2,-1,2} + 3168\,\mathrm{H}_{-2,0,0} + 640/3\,\mathrm{H}_{-2,2,0} + 224\,\mathrm{H}_{-2,2,1} - 800\,\mathrm{H}_{-1,-3,0} \\
& - 2408/3\,\mathrm{H}_{-1,-2,0} - 2560\,\mathrm{H}_{-1,-2,2} - 4960\,\mathrm{H}_{-1,-1,-1}\,\zeta_2 - 18080/3\,\mathrm{H}_{-1,-1,2} \\
& - 3040\,\mathrm{H}_{-1,-1,3} + 2048/3\,\mathrm{H}_{-1,2,0} + 896\,\mathrm{H}_{-1,2,1} + 160\,\mathrm{H}_{-1,3,0} + 160\,\mathrm{H}_{-1,3,1} \\
& - 64\,\mathrm{H}_{1,-2,0} + 1808/5\,\mathrm{H}_{1,0,0} + 184\,\mathrm{H}_{1,1,0} + 192\,\mathrm{H}_{1,1,1} - 32/3\,\mathrm{H}_{1,1,2} \\
& + 64/3\,\mathrm{H}_{1,2,0} - 320\,\mathrm{H}_{2,-2,0} - 256/3\,\mathrm{H}_{2,0,0} - 256/3\,\mathrm{H}_{2,1,0} - 96\,\mathrm{H}_{2,1,1} \\
& + 32/3\,\mathrm{H}_{3,0,0} - 32/3\,\mathrm{H}_{3,1,0} + 1248\,\mathrm{H}_{-2,-1,-1,0} - 8336/3\,\mathrm{H}_{-2,-1,0,0} \\
& + 1456\,\mathrm{H}_{-2,0,0,0} + 960\,\mathrm{H}_{-1,-2,-1,0} - 2400\,\mathrm{H}_{-1,-2,0,0} + 960\,\mathrm{H}_{-1,-1,-2,0} \\
& + 1424\,\mathrm{H}_{-1,-1,-1,0} + 4480\,\mathrm{H}_{-1,-1,-1,2} - 13064/3\,\mathrm{H}_{-1,-1,0,0} - 320\,\mathrm{H}_{-1,-1,2,0} \\
& - 320\,\mathrm{H}_{-1,-1,2,1} + 2380\,\mathrm{H}_{-1,0,0,0} - 320\,\mathrm{H}_{-1,2,0,0} - 1184\,\mathrm{H}_{0,0,0,0} - 64/3\,\mathrm{H}_{1,0,0,0} \\
& - 32/3\,\mathrm{H}_{1,1,0,0} + 32/3\,\mathrm{H}_{1,1,1,0} + 160\,\mathrm{H}_{2,0,0,0} - 320\,\mathrm{H}_{2,1,0,0} - 960\,\mathrm{H}_{-1,-1,-1,-1,0} \\
& + 4000\,\mathrm{H}_{-1,-1,-1,0,0} - 1920\,\mathrm{H}_{-1,-1,0,0,0} + 800\,\mathrm{H}_{-1,0,0,0,0} - 136\,\mathrm{H}_{0,0,0,0,0} \\
& - 4/3\,(189 + 1864\,\zeta_2)\,\mathrm{H}_{-2,0} + 40/3\,(209 - 168\,\zeta_2)\,\mathrm{H}_{-1,0,0} - 8/15\,(569 \\
& + 1050\,\zeta_2)\,\mathrm{H}_{2,0} + 8\,(607 + 20\,\zeta_2)\,\mathrm{H}_{-1,2} - 8/15\,(799 + 150\,\zeta_2)\,\mathrm{H}_{2,1} + 8/15\,(971 \\
& + 7200\,\zeta_2)\,\mathrm{H}_{-1,-1,0} + 8/15\,(1623 + 490\,\zeta_2)\,\mathrm{H}_{1,0} + 8/15\,(2283 + 370\,\zeta_2)\,\mathrm{H}_{1,1} \\
& - 4/15\,(16091 + 1240\,\zeta_2)\,\mathrm{H}_3 + 2/75\,(17569 + 11800\,\zeta_3 + 21440\,\zeta_2)\,\mathrm{H}_1 \\
& - 2/15\,(18767 - 780\,\zeta_2)\,\mathrm{H}_{0,0,0} + 2/75\,(32084 + 101450\,\zeta_3 + 210835\,\zeta_2 \\
& + 2660\,\zeta_2^2)\,\mathrm{H}_0 + 4/225\,(46153 - 94500\,\zeta_3 - 309150\,\zeta_2)\,\mathrm{H}_{-1,0} + 2/75\,(47983 \\
& + 3000\,\zeta_3 - 31050\,\zeta_2)\,\mathrm{H}_2 + 2/225\,(63894 - 112350\,\zeta_5 + 5400\,\zeta_4 + 574155\,\zeta_3 \\
& - 98758\,\zeta_2 - 87000\,\zeta_2\,\zeta_3 + 17775\,\zeta_2^2) + 2/225\,(103519 + 36600\,\zeta_3 \\
& + 252300\,\zeta_2)\,\mathrm{H}_{0,0} - 8\,(347\,\zeta_3 + 541\,\zeta_2)\,\mathrm{H}_{-2} + 8/3\,(1530\,\zeta_3 + 2527\,\zeta_2)\,\mathrm{H}_{-1,-1} \\
& - 4/15\,(21720\,\zeta_3 + 17239\,\zeta_2 - 1770\,\zeta_2^2)\,\mathrm{H}_{-1} \Big\} + p_{qq}(-x) \Big\{ 5600/3\,\mathrm{H}_{-3}\,\zeta_2 \\
& - 52\,\mathrm{H}_4 + 560\,\mathrm{H}_5 - 512\,\mathrm{H}_{-4,0} + 112/3\,\mathrm{H}_{-3,0} - 1472\,\mathrm{H}_{-3,2} - 3024\,\mathrm{H}_{-2,-1}\,\zeta_2 \\
& - 240\,\mathrm{H}_{-2,2} - 8048/3\,\mathrm{H}_{-2,3} - 3008\,\mathrm{H}_{-1,-2}\,\zeta_2 - 400\,\mathrm{H}_{-1,3} - 2096\,\mathrm{H}_{-1,4} \\
& + 8\,\mathrm{H}_{2,2} - 96\,\mathrm{H}_{2,3} + 84\,\mathrm{H}_{3,0} + 128\,\mathrm{H}_{3,1} + 416/3\,\mathrm{H}_{3,2} + 976/3\,\mathrm{H}_{4,0} + 432\,\mathrm{H}_{4,1} \\
& + 2368/3\,\mathrm{H}_{-3,-1,0} - 4480/3\,\mathrm{H}_{-3,0,0} + 640\,\mathrm{H}_{-2,-2,0} + 112\,\mathrm{H}_{-2,-1,0}
\end{aligned}
$$

$$+ 2624\,\mathrm{H}_{-2,-1,2} - 16\,\mathrm{H}_{-2,0,0} - 768\,\mathrm{H}_{-2,2,0} - 2912/3\,\mathrm{H}_{-2,2,1} + 2624/3\,\mathrm{H}_{-1,-3,0}$$
$$+ 120\,\mathrm{H}_{-1,-2,0} + 2624\,\mathrm{H}_{-1,-2,2} + 4032\,\mathrm{H}_{-1,-1,-1}\,\zeta_2 + 384\,\mathrm{H}_{-1,-1,2}$$
$$+ 12448/3\,\mathrm{H}_{-1,-1,3} - 192\,\mathrm{H}_{-1,2,0} - 256\,\mathrm{H}_{-1,2,1} - 288\,\mathrm{H}_{-1,2,2} - 2800/3\,\mathrm{H}_{-1,3,0}$$
$$- 3616/3\,\mathrm{H}_{-1,3,1} + 64/3\,\mathrm{H}_{2,-2,0} - 8\,\mathrm{H}_{2,1,0} - 32/3\,\mathrm{H}_{2,1,2} + 64/3\,\mathrm{H}_{2,2,0}$$
$$+ 512/3\,\mathrm{H}_{3,0,0} + 96\,\mathrm{H}_{3,1,0} + 96\,\mathrm{H}_{3,1,1} - 800\,\mathrm{H}_{-2,-1,-1,0} + 2512\,\mathrm{H}_{-2,-1,0,0}$$
$$- 1856\,\mathrm{H}_{-2,0,0,0} - 768\,\mathrm{H}_{-1,-2,-1,0} + 7520/3\,\mathrm{H}_{-1,-2,0,0} - 736\,\mathrm{H}_{-1,-1,-2,0}$$
$$- 288\,\mathrm{H}_{-1,-1,-1,0} - 3648\,\mathrm{H}_{-1,-1,-1,2} + 240\,\mathrm{H}_{-1,-1,0,0} + 3968/3\,\mathrm{H}_{-1,-1,2,0}$$
$$+ 1664\,\mathrm{H}_{-1,-1,2,1} + 36\,\mathrm{H}_{-1,0,0,0} - 1216/3\,\mathrm{H}_{-1,2,0,0} - 544/3\,\mathrm{H}_{-1,2,1,0}$$
$$- 192\,\mathrm{H}_{-1,2,1,1} - 596/3\,\mathrm{H}_{0,0,0,0} - 64\,\mathrm{H}_{2,0,0,0} - 64\,\mathrm{H}_{2,1,0,0} + 32/3\,\mathrm{H}_{2,1,1,0}$$
$$+ 768\,\mathrm{H}_{-1,-1,-1,-1,0} - 3424\,\mathrm{H}_{-1,-1,-1,0,0} + 8720/3\,\mathrm{H}_{-1,-1,0,0,0}$$
$$- 1264\,\mathrm{H}_{-1,0,0,0,0} + 272\,\mathrm{H}_{0,0,0,0,0} - 8\,(4 - 310\,\zeta_3 - 53\,\zeta_2)\,\mathrm{H}_{-1,0} + 16/3\,(6$$
$$+ 43\,\zeta_2)\,\mathrm{H}_{2,1} + 16/3\,(6 + 53\,\zeta_2)\,\mathrm{H}_{2,0} - 32\,(8 + 149\,\zeta_2)\,\mathrm{H}_{-1,-1,0} - 4/3\,(24$$
$$- 308\,\zeta_3 - 183\,\zeta_2)\,\mathrm{H}_2 - 16/3\,(31 + 41\,\zeta_2)\,\mathrm{H}_{-1,2} - 4/3\,(54 + 572\,\zeta_3$$
$$- 107\,\zeta_2)\,\mathrm{H}_{0,0} + 8/3\,(61 + 164\,\zeta_2)\,\mathrm{H}_3 + 8/3\,(79 + 894\,\zeta_2)\,\mathrm{H}_{-1,0,0} + 8/3\,(85$$
$$+ 1184\,\zeta_2)\,\mathrm{H}_{-2,0} - 2/3\,(239 + 904\,\zeta_2)\,\mathrm{H}_{0,0,0} - 2/15\,(1205 - 360\,\zeta_4 - 10\,\zeta_3$$
$$+ 555\,\zeta_2 + 916\,\zeta_2^2)\,\mathrm{H}_0 - 4/15\,(2480\,\zeta_5 + 75\,\zeta_3 - 60\,\zeta_2 - 3660\,\zeta_2\,\zeta_3 + 408\,\zeta_2^2)$$
$$- 48\,(85\,\zeta_3 + 11\,\zeta_2)\,\mathrm{H}_{-1,-1} + 8/3\,(1100\,\zeta_3 + 111\,\zeta_2)\,\mathrm{H}_{-2} + 4/15\,(2055\,\zeta_3$$
$$+ 140\,\zeta_2 - 476\,\zeta_2^2)\,\mathrm{H}_{-1} \Big\} + 1200\,\mathrm{H}_{-3}\,\zeta_2 + 7264/3\,\mathrm{H}_4 - 288\,\mathrm{H}_{-4,0}$$
$$- 1424\,\mathrm{H}_{-3,0} - 672\,\mathrm{H}_{-3,2} - 3744\,\mathrm{H}_{-2,-1}\,\zeta_2 - 5232\,\mathrm{H}_{-2,2} - 2016\,\mathrm{H}_{-2,3}$$
$$- 3648\,\mathrm{H}_{-1,-2}\,\zeta_2 - 7696\,\mathrm{H}_{-1,3} - 2112\,\mathrm{H}_{-1,4} - 512/3\,\mathrm{H}_{1,3} + 288\,\mathrm{H}_{2,2}$$
$$- 384\,\mathrm{H}_{2,3} + 2704/3\,\mathrm{H}_{3,0} + 3616/3\,\mathrm{H}_{3,1} + 1056\,\mathrm{H}_{-3,-1,0} - 1056\,\mathrm{H}_{-3,0,0}$$
$$+ 960\,\mathrm{H}_{-2,-2,0} + 5648/3\,\mathrm{H}_{-2,-1,0} + 3072\,\mathrm{H}_{-2,-1,2} - 13040/3\,\mathrm{H}_{-2,0,0}$$
$$- 192\,\mathrm{H}_{-2,2,0} - 192\,\mathrm{H}_{-2,2,1} + 960\,\mathrm{H}_{-1,-3,0} + 3808/3\,\mathrm{H}_{-1,-2,0} + 3072\,\mathrm{H}_{-1,-2,2}$$
$$+ 5952\,\mathrm{H}_{-1,-1,-1}\,\zeta_2 + 29792/3\,\mathrm{H}_{-1,-1,2} + 3648\,\mathrm{H}_{-1,-1,3} - 4096/3\,\mathrm{H}_{-1,2,0}$$
$$- 1792\,\mathrm{H}_{-1,2,1} - 192\,\mathrm{H}_{-1,3,0} - 192\,\mathrm{H}_{-1,3,1} + 2676\,\mathrm{H}_{0,0,0} + 640/3\,\mathrm{H}_{1,-2,0}$$
$$+ 64/3\,\mathrm{H}_{1,0,0} + 384\,\mathrm{H}_{2,-2,0} + 224\,\mathrm{H}_{2,0,0} + 544/3\,\mathrm{H}_{2,1,0} + 192\,\mathrm{H}_{2,1,1}$$
$$- 1344\,\mathrm{H}_{-2,-1,-1,0} + 2976\,\mathrm{H}_{-2,-1,0,0} - 1344\,\mathrm{H}_{-2,0,0,0} - 1152\,\mathrm{H}_{-1,-2,-1,0}$$
$$+ 2880\,\mathrm{H}_{-1,-2,0,0} - 1152\,\mathrm{H}_{-1,-1,-2,0} - 6016/3\,\mathrm{H}_{-1,-1,-1,0} - 5376\,\mathrm{H}_{-1,-1,-1,2}$$
$$+ 22192/3\,\mathrm{H}_{-1,-1,0,0} + 384\,\mathrm{H}_{-1,-1,2,0} + 384\,\mathrm{H}_{-1,-1,2,1} - 13184/3\,\mathrm{H}_{-1,0,0,0}$$
$$+ 384\,\mathrm{H}_{-1,2,0,0} + 4016/3\,\mathrm{H}_{0,0,0,0} - 320/3\,\mathrm{H}_{1,0,0,0} + 512/3\,\mathrm{H}_{1,1,0,0} - 192\,\mathrm{H}_{2,0,0,0}$$
$$+ 384\,\mathrm{H}_{2,1,0,0} + 1152\,\mathrm{H}_{-1,-1,-1,-1,0} - 4800\,\mathrm{H}_{-1,-1,-1,0,0} + 2304\,\mathrm{H}_{-1,-1,0,0,0}$$
$$- 960\,\mathrm{H}_{-1,0,0,0,0} + 64\,(1 + \zeta_2)\,\mathrm{H}_{1,1} + 64/3\,(3 + 16\,\zeta_2)\,\mathrm{H}_{1,0} + 32/3\,(15 - 3\,\zeta_3$$
$$+ 64\,\zeta_2)\,\mathrm{H}_1 + 16/5\,(17 + 840\,\zeta_2)\,\mathrm{H}_{-2,0} + 16/5\,(353 + 30\,\zeta_2)\,\mathrm{H}_{2,1} - 32/15\,(373$$

$$+ 2160\,\zeta_2)\,\mathrm{H}_{-1,-1,0} - 16\,(487 + 12\,\zeta_2)\,\mathrm{H}_{-1,2} + 8/15\,(1483 + 1260\,\zeta_2)\,\mathrm{H}_{2,0}$$
$$- 8/3\,(1649 - 1008\,\zeta_2)\,\mathrm{H}_{-1,0,0} + 8/3\,(1789 + 126\,\zeta_2)\,\mathrm{H}_3 - 16/225\,(3290$$
$$- 21600\,\zeta_5 + 77625\,\zeta_3 + 631\,\zeta_2 - 10125\,\zeta_2\,\zeta_3 + 4695\,\zeta_2^2) - 4/75\,(6793$$
$$+ 1800\,\zeta_3 - 26450\,\zeta_2)\,\mathrm{H}_2 - 8/225\,(27083 - 56700\,\zeta_3 - 268200\,\zeta_2)\,\mathrm{H}_{-1,0}$$
$$- 4/225\,(32249 + 5400\,\zeta_3 + 159000\,\zeta_2)\,\mathrm{H}_{0,0} - 2/225\,(42037 + 344400\,\zeta_3$$
$$+ 692400\,\zeta_2 + 2700\,\zeta_2^2)\,\mathrm{H}_0 + 40/3\,(234\,\zeta_3 + 463\,\zeta_2)\,\mathrm{H}_{-2} - 32/3\,(459\,\zeta_3$$
$$+ 1025\,\zeta_2)\,\mathrm{H}_{-1,-1} + 8/15\,(17945\,\zeta_3 + 13864\,\zeta_2 - 1062\,\zeta_2^2)\,\mathrm{H}_{-1}$$
$$+ 2/3\,\left(g_2^E(x) - g_2^O(x)\right) - 46/3\,\left(g_1^E(x) - g_1^O(x)\right) - 4/3\,\zeta_3\,\delta(1-x)\Big)$$
$$+ C_F\,(C_A - 2\,C_F)\,C_A\Big(x^2\Big\{ - 1408\,\mathrm{H}_{-2}\,\zeta_2 - 1104/5\,\mathrm{H}_2 - 208\,\mathrm{H}_3 - 800\,\mathrm{H}_4$$
$$+ 256\,\mathrm{H}_{-3,0} - 384\,\mathrm{H}_{-2,0} + 1408\,\mathrm{H}_{-2,2} + 1408\,\mathrm{H}_{-1,-1}\,\zeta_2 - 176\,\mathrm{H}_{-1,2}$$
$$+ 704\,\mathrm{H}_{-1,3} + 224\,\mathrm{H}_{1,0}\,\zeta_2 - 96\,\mathrm{H}_{1,3} + 704\,\mathrm{H}_{-2,0,0} + 384\,\mathrm{H}_{-1,-1,0}$$
$$- 1408\,\mathrm{H}_{-1,-1,2} - 984\,\mathrm{H}_{-1,0,0} + 3192/5\,\mathrm{H}_{0,0,0} + 256\,\mathrm{H}_{1,-2,0} + 96\,\mathrm{H}_{2,0,0}$$
$$- 704\,\mathrm{H}_{-1,-1,0,0} + 128\,\mathrm{H}_{-1,0,0,0} - 256\,\mathrm{H}_{0,0,0,0} - 128\,\mathrm{H}_{1,0,0,0} + 96\,\mathrm{H}_{1,1,0,0}$$
$$- 416/3\,(7 + 6\,\zeta_2)\,\mathrm{H}_{-1,0} - 8/15\,(132 + 135\,\zeta_3 + 2291\,\zeta_2 - 24\,\zeta_2^2)$$
$$+ 16/15\,(853 + 990\,\zeta_2)\,\mathrm{H}_{0,0} - 8/25\,(2769 - 3100\,\zeta_3 - 855\,\zeta_2)\,\mathrm{H}_0 - 8\,(8\,\zeta_3$$
$$- 37\,\zeta_2)\,\mathrm{H}_1 - 16\,(66\,\zeta_3 - 23\,\zeta_2)\,\mathrm{H}_{-1}\Big\} + x^3\Big\{384\,\mathrm{H}_{-2}\,\zeta_2 + 456/5\,\mathrm{H}_3$$
$$+ 1008/5\,\mathrm{H}_4 - 192/5\,\mathrm{H}_{-3,0} + 944/5\,\mathrm{H}_{-2,0} - 384\,\mathrm{H}_{-2,2} - 768\,\mathrm{H}_{-1,-1}\,\zeta_2$$
$$- 912/5\,\mathrm{H}_{-1,2} - 384\,\mathrm{H}_{-1,3} - 192\,\mathrm{H}_{-2,0,0} + 224/5\,\mathrm{H}_{-1,-1,0} + 768\,\mathrm{H}_{-1,-1,2}$$
$$+ 1544/5\,\mathrm{H}_{-1,0,0} - 772/5\,\mathrm{H}_{0,0,0} - 48/5\,\mathrm{H}_{2,0,0} + 384\,\mathrm{H}_{-1,-1,0,0}$$
$$- 192/5\,\mathrm{H}_{-1,0,0,0} + 192/5\,\mathrm{H}_{0,0,0,0} - 12/25\,(623 + 500\,\zeta_2)\,\mathrm{H}_{0,0} + 8/25\,(1649$$
$$+ 1320\,\zeta_2)\,\mathrm{H}_{-1,0} + 64/5\,(45\,\zeta_3 + 16\,\zeta_2)\,\mathrm{H}_{-1} - 4\,(60\,\zeta_3 + 37\,\zeta_2)\,\mathrm{H}_0$$
$$- 4/25\,(995\,\zeta_3 - 1869\,\zeta_2 - 72\,\zeta_2^2)\Big\} + \left(1/x^2 + 9\,x^3\right)\Big\{ - 352/15\,\mathrm{H}_{-2,0}$$
$$+ 128/3\,\mathrm{H}_{-1,-1}\,\zeta_2 + 152/15\,\mathrm{H}_{-1,2} + 64/3\,\mathrm{H}_{-1,3} - 16/5\,\mathrm{H}_{1,0}\,\zeta_2 + 16/15\,\mathrm{H}_{1,3}$$
$$- 112/45\,\mathrm{H}_{-1,-1,0} - 128/3\,\mathrm{H}_{-1,-1,2} - 772/45\,\mathrm{H}_{-1,0,0} - 64/15\,\mathrm{H}_{1,-2,0}$$
$$- 64/3\,\mathrm{H}_{-1,-1,0,0} + 32/15\,\mathrm{H}_{-1,0,0,0} + 32/15\,\mathrm{H}_{1,0,0,0} - 16/15\,\mathrm{H}_{1,1,0,0}$$
$$- 4/225\,(1429 + 1320\,\zeta_2)\,\mathrm{H}_{-1,0} + 4/15\,(20\,\zeta_3 - 19\,\zeta_2)\,\mathrm{H}_1 - 32/45\,(45\,\zeta_3$$
$$+ 16\,\zeta_2)\,\mathrm{H}_{-1}\Big\} + \left(1/x + 9\,x^2\right)\Big\{ - 128/3\,\mathrm{H}_{-1}\,\zeta_2 - 184/15\,\mathrm{H}_1 + 184/15\,\mathrm{H}_2$$
$$- 112/5\,\mathrm{H}_3 + 64/15\,\mathrm{H}_{-2,0} - 472/9\,\mathrm{H}_{-1,0} + 128/3\,\mathrm{H}_{-1,2} + 964/45\,\mathrm{H}_{0,0}$$
$$+ 64/3\,\mathrm{H}_{-1,0,0} - 64/15\,\mathrm{H}_{0,0,0} + 16/15\,\mathrm{H}_{1,0,0} + 16/225\,(386 + 375\,\zeta_2)\,\mathrm{H}_0$$
$$- 4/225\,(2864 - 1500\,\zeta_3 + 575\,\zeta_2)\Big\} + (1-x)\Big\{136/3\,\mathrm{H}_{-3}\,\zeta_2 + 2048/3\,\mathrm{H}_4$$
$$+ 16/3\,\mathrm{H}_5 - 16\,\mathrm{H}_{-4,0} + 768\,\mathrm{H}_{-3,0} - 32/3\,\mathrm{H}_{-3,2} - 4000/3\,\mathrm{H}_{-2,-1}\,\zeta_2$$

$$
\begin{aligned}
&- 1504\,\mathrm{H}_{-2,2} - 1976/3\,\mathrm{H}_{-2,3} - 1280\,\mathrm{H}_{-1,-2}\,\zeta_2 - 22744/9\,\mathrm{H}_{-1,2} \\
&- 6016/3\,\mathrm{H}_{-1,3} - 800\,\mathrm{H}_{-1,4} - 128/3\,\mathrm{H}_{1,2} + 304/3\,\mathrm{H}_{1,3} + 2272/9\,\mathrm{H}_{2,1} \\
&+ 32\,\mathrm{H}_{2,2} - 160\,\mathrm{H}_{2,3} + 616/3\,\mathrm{H}_{3,0} + 320\,\mathrm{H}_{3,1} - 16/3\,\mathrm{H}_{3,2} + 208/3\,\mathrm{H}_{-3,-1,0} \\
&- 56\,\mathrm{H}_{-3,0,0} + 112/3\,\mathrm{H}_{-2,-2,0} - 2336/3\,\mathrm{H}_{-2,-1,0} + 3920/3\,\mathrm{H}_{-2,-1,2} \\
&+ 832/3\,\mathrm{H}_{-2,0,0} + 16/3\,\mathrm{H}_{-2,2,0} + 320\,\mathrm{H}_{-1,-3,0} - 1040\,\mathrm{H}_{-1,-2,0} + 1280\,\mathrm{H}_{-1,-2,2} \\
&+ 2560\,\mathrm{H}_{-1,-1,-1}\,\zeta_2 + 8960/3\,\mathrm{H}_{-1,-1,2} + 1280\,\mathrm{H}_{-1,-1,3} - 896/3\,\mathrm{H}_{-1,2,0} \\
&- 1280/3\,\mathrm{H}_{-1,2,1} + 64\,\mathrm{H}_{1,-2,0} - 1472/15\,\mathrm{H}_{1,0,0} + 128/3\,\mathrm{H}_{1,1,0} + 32/3\,\mathrm{H}_{1,1,2} \\
&- 64/3\,\mathrm{H}_{1,2,0} + 320\,\mathrm{H}_{2,-2,0} + 64/3\,\mathrm{H}_{2,0,0} - 32\,\mathrm{H}_{2,1,0} + 16/3\,\mathrm{H}_{3,1,0} \\
&- 160/3\,\mathrm{H}_{-2,-1,-1,0} + 2152/3\,\mathrm{H}_{-2,-1,0,0} - 592/3\,\mathrm{H}_{-2,0,0,0} + 640\,\mathrm{H}_{-1,-2,0,0} \\
&+ 2416/3\,\mathrm{H}_{-1,-1,-1,0} - 2560\,\mathrm{H}_{-1,-1,-1,2} - 56\,\mathrm{H}_{-1,-1,0,0} + 8\,\mathrm{H}_{-1,0,0,0} \\
&+ 160\,\mathrm{H}_{-1,2,0,0} + 644/3\,\mathrm{H}_{0,0,0,0} + 16\,\mathrm{H}_{1,1,0,0} - 32/3\,\mathrm{H}_{1,1,1,0} - 160\,\mathrm{H}_{2,0,0,0} \\
&+ 160\,\mathrm{H}_{2,1,0,0} - 1280\,\mathrm{H}_{-1,-1,-1,0,0} + 320\,\mathrm{H}_{-1,-1,0,0,0} - 320\,\mathrm{H}_{-1,0,0,0,0} \\
&+ 16\,\mathrm{H}_{0,0,0,0,0} + 160\,(1 + 2\,\zeta_2)\,\mathrm{H}_{2,0} - 64/9\,(71 + 18\,\zeta_2)\,\mathrm{H}_{1,1} - 8/3\,(91 \\
&+ 58\,\zeta_2)\,\mathrm{H}_{1,0} + 32/9\,(158 + 315\,\zeta_2)\,\mathrm{H}_{-1,0,0} + 64/15\,(418 + 195\,\zeta_2)\,\mathrm{H}_{-2,0} \\
&+ 4/45\,(491 - 90\,\zeta_2)\,\mathrm{H}_{0,0,0} - 4/27\,(626 + 1404\,\zeta_3 + 2799\,\zeta_2)\,\mathrm{H}_{1} - 8/9\,(2891 \\
&+ 1800\,\zeta_2)\,\mathrm{H}_{-1,-1,0} - 8/27\,(2920 - 1269\,\zeta_2)\,\mathrm{H}_{2} + 8/15\,(3869 + 10\,\zeta_2)\,\mathrm{H}_{3} \\
&- 4/15\,(7844 + 190\,\zeta_3 + 3135\,\zeta_2)\,\mathrm{H}_{0,0} + 4/135\,(54673 + 32400\,\zeta_3 \\
&+ 78120\,\zeta_2)\,\mathrm{H}_{-1,0} + 2/2025\,(217793 + 318600\,\zeta_5 - 48600\,\zeta_4 - 1320750\,\zeta_3 \\
&+ 1831500\,\zeta_2 + 396900\,\zeta_2\,\zeta_3 + 174825\,\zeta_2^2) - 2/2025\,(1192459 + 507600\,\zeta_3 \\
&+ 2489850\,\zeta_2 - 16740\,\zeta_2^2)\,\mathrm{H}_{0} - 8\,(240\,\zeta_3 + 323\,\zeta_2)\,\mathrm{H}_{-1,-1} + 8/3\,(379\,\zeta_3 \\
&+ 418\,\zeta_2)\,\mathrm{H}_{-2} + 4/9\,(4686\,\zeta_3 + 2795\,\zeta_2 + 72\,\zeta_2^2)\,\mathrm{H}_{-1} \Big\} + p_{qq}(-x) \Big\{ \\
&- 2048/3\,\mathrm{H}_{-3}\,\zeta_2 + 1648/9\,\mathrm{H}_{4} - 160\,\mathrm{H}_{5} + 160/3\,\mathrm{H}_{-4,0} - 3920/9\,\mathrm{H}_{-3,0} \\
&+ 704\,\mathrm{H}_{-3,2} + 1312\,\mathrm{H}_{-2,-1}\,\zeta_2 - 352/9\,\mathrm{H}_{-2,2} + 3472/3\,\mathrm{H}_{-2,3} + 1296\,\mathrm{H}_{-1,-2}\,\zeta_2 \\
&- 352/9\,\mathrm{H}_{-1,3} + 768\,\mathrm{H}_{-1,4} - 512/3\,\mathrm{H}_{2,0}\,\zeta_2 - 128\,\mathrm{H}_{2,1}\,\zeta_2 + 224/3\,\mathrm{H}_{2,3} \\
&- 352/9\,\mathrm{H}_{3,1} - 64/3\,\mathrm{H}_{3,2} - 64\,\mathrm{H}_{4,0} - 320/3\,\mathrm{H}_{4,1} + 128/3\,\mathrm{H}_{-3,-1,0} \\
&+ 1120/3\,\mathrm{H}_{-3,0,0} + 64/3\,\mathrm{H}_{-2,-2,0} + 3872/9\,\mathrm{H}_{-2,-1,0} - 1312\,\mathrm{H}_{-2,-1,2} \\
&- 6128/9\,\mathrm{H}_{-2,0,0} + 928/3\,\mathrm{H}_{-2,2,0} + 1280/3\,\mathrm{H}_{-2,2,1} - 128/3\,\mathrm{H}_{-1,-3,0} \\
&+ 3872/9\,\mathrm{H}_{-1,-2,0} - 1312\,\mathrm{H}_{-1,-2,2} - 1920\,\mathrm{H}_{-1,-1,-1}\,\zeta_2 - 1984\,\mathrm{H}_{-1,-1,3} \\
&+ 704/9\,\mathrm{H}_{-1,2,1} + 160/3\,\mathrm{H}_{-1,2,2} + 784/3\,\mathrm{H}_{-1,3,0} + 1280/3\,\mathrm{H}_{-1,3,1} \\
&- 64/3\,\mathrm{H}_{2,-2,0} + 16\,\mathrm{H}_{2,0,0} + 32/3\,\mathrm{H}_{2,1,2} - 64/3\,\mathrm{H}_{2,2,0} - 160/3\,\mathrm{H}_{3,0,0} \\
&+ 64/3\,\mathrm{H}_{3,1,0} - 2128/3\,\mathrm{H}_{-2,-1,0,0} + 1424/3\,\mathrm{H}_{-2,0,0,0} - 32\,\mathrm{H}_{-1,-2,-1,0} \\
&- 704\,\mathrm{H}_{-1,-2,0,0} - 64\,\mathrm{H}_{-1,-1,-2,0} - 3520/9\,\mathrm{H}_{-1,-1,-1,0} + 1920\,\mathrm{H}_{-1,-1,-1,2}
\end{aligned}
$$

$$+ 5984/9\,\mathrm{H}_{-1,-1,0,0} - 1792/3\,\mathrm{H}_{-1,-1,2,0} - 2560/3\,\mathrm{H}_{-1,-1,2,1}$$
$$- 5912/9\,\mathrm{H}_{-1,0,0,0} + 448/3\,\mathrm{H}_{-1,2,0,0} - 160/3\,\mathrm{H}_{-1,2,1,0} + 2972/9\,\mathrm{H}_{0,0,0,0}$$
$$+ 128/3\,\mathrm{H}_{2,0,0,0} + 128/3\,\mathrm{H}_{2,1,0,0} - 32/3\,\mathrm{H}_{2,1,1,0} + 1088\,\mathrm{H}_{-1,-1,-1,0,0}$$
$$- 2432/3\,\mathrm{H}_{-1,-1,0,0,0} + 928/3\,\mathrm{H}_{-1,0,0,0,0} - 64\,\mathrm{H}_{0,0,0,0,0} + 16/27\,(17$$
$$- 396\,\zeta_2)\,\mathrm{H}_3 - 32/27\,(17 - 144\,\zeta_2)\,\mathrm{H}_{-1,2} + 16/3\,(21 - 46\,\zeta_3 - 27\,\zeta_2)\,\mathrm{H}_2$$
$$+ 64/9\,(67 + 312\,\zeta_2)\,\mathrm{H}_{-1,-1,0} + 16/27\,(463 + 288\,\zeta_2)\,\mathrm{H}_{0,0,0} - 16/27\,(679$$
$$+ 2277\,\zeta_2)\,\mathrm{H}_{-2,0} - 16/27\,(973 + 1548\,\zeta_2)\,\mathrm{H}_{-1,0,0} - 8/81\,(2129 + 8964\,\zeta_3$$
$$- 1305\,\zeta_2)\,\mathrm{H}_{-1,0} + 4/81\,(6125 + 3888\,\zeta_3 - 5229\,\zeta_2)\,\mathrm{H}_{0,0} + 2/135\,(17715$$
$$- 3240\,\zeta_4 - 16620\,\zeta_3 - 5440\,\zeta_2 + 4176\,\zeta_2^2)\,\mathrm{H}_0 + 2/81\,(6372\,\zeta_5 - 10260\,\zeta_3$$
$$- 8794\,\zeta_2 - 14040\,\zeta_2\,\zeta_3 + 1107\,\zeta_2^2) + 32/9\,(543\,\zeta_3 - 55\,\zeta_2)\,\mathrm{H}_{-1,-1}$$
$$- 16/9\,(687\,\zeta_3 - 143\,\zeta_2)\,\mathrm{H}_{-2} + 8/135\,(2640\,\zeta_3 + 4360\,\zeta_2 + 423\,\zeta_2^2)\,\mathrm{H}_{-1}\Big\}$$
$$- 2176\,\mathrm{H}_3 - 2896/3\,\mathrm{H}_4 - 2480/3\,\mathrm{H}_{-3,0} + 1536\,\mathrm{H}_{-2,-1}\,\zeta_2 + 7280/3\,\mathrm{H}_{-2,2}$$
$$+ 768\,\mathrm{H}_{-2,3} + 1536\,\mathrm{H}_{-1,-2}\,\zeta_2 + 35024/9\,\mathrm{H}_{-1,2} + 10816/3\,\mathrm{H}_{-1,3} + 960\,\mathrm{H}_{-1,4}$$
$$- 192\,\mathrm{H}_{1,0}\,\zeta_2 + 256/3\,\mathrm{H}_{1,3} - 4544/9\,\mathrm{H}_{2,1} - 160/3\,\mathrm{H}_{2,2} + 192\,\mathrm{H}_{2,3} - 808/3\,\mathrm{H}_{3,0}$$
$$- 1280/3\,\mathrm{H}_{3,1} + 3248/3\,\mathrm{H}_{-2,-1,0} - 1536\,\mathrm{H}_{-2,-1,2} + 64/3\,\mathrm{H}_{-2,0,0}$$
$$- 384\,\mathrm{H}_{-1,-3,0} + 3760/3\,\mathrm{H}_{-1,-2,0} - 1536\,\mathrm{H}_{-1,-2,2} - 3072\,\mathrm{H}_{-1,-1,-1}\,\zeta_2$$
$$- 15488/3\,\mathrm{H}_{-1,-1,2} - 1536\,\mathrm{H}_{-1,-1,3} + 1792/3\,\mathrm{H}_{-1,2,0} + 2560/3\,\mathrm{H}_{-1,2,1}$$
$$+ 136/45\,\mathrm{H}_{0,0,0} - 640/3\,\mathrm{H}_{1,-2,0} - 32/3\,\mathrm{H}_{1,0,0} - 384\,\mathrm{H}_{2,-2,0} - 176/3\,\mathrm{H}_{2,0,0}$$
$$+ 160/3\,\mathrm{H}_{2,1,0} - 768\,\mathrm{H}_{-2,-1,0,0} + 192\,\mathrm{H}_{-2,0,0,0} - 768\,\mathrm{H}_{-1,-2,0,0}$$
$$- 2768/3\,\mathrm{H}_{-1,-1,-1,0} + 3072\,\mathrm{H}_{-1,-1,-1,2} - 792\,\mathrm{H}_{-1,-1,0,0} + 1184/3\,\mathrm{H}_{-1,0,0,0}$$
$$- 192\,\mathrm{H}_{-1,2,0,0} - 1120/3\,\mathrm{H}_{0,0,0,0} + 320/3\,\mathrm{H}_{1,0,0,0} - 256/3\,\mathrm{H}_{1,1,0,0} + 192\,\mathrm{H}_{2,0,0,0}$$
$$- 192\,\mathrm{H}_{2,1,0,0} + 1536\,\mathrm{H}_{-1,-1,-1,0,0} - 384\,\mathrm{H}_{-1,-1,0,0,0} + 384\,\mathrm{H}_{-1,0,0,0,0}$$
$$+ 16/3\,(23 + 2\,\zeta_3 - 46\,\zeta_2)\,\mathrm{H}_1 + 148/5\,(69 + 40\,\zeta_2)\,\mathrm{H}_{0,0} - 8/3\,(107$$
$$+ 144\,\zeta_2)\,\mathrm{H}_{2,0} + 128/9\,(277 + 135\,\zeta_2)\,\mathrm{H}_{-1,-1,0} - 40/9\,(607 + 216\,\zeta_2)\,\mathrm{H}_{-2,0}$$
$$- 8/9\,(1237 + 1512\,\zeta_2)\,\mathrm{H}_{-1,0,0} + 8/27\,(1781 - 2070\,\zeta_2)\,\mathrm{H}_2 + 4/45\,(6520$$
$$- 3780\,\zeta_5 + 12005\,\zeta_3 - 17351\,\zeta_2 - 4860\,\zeta_2\,\zeta_3 - 1788\,\zeta_2^2) - 8/135\,(51538$$
$$+ 19440\,\zeta_3 + 70065\,\zeta_2)\,\mathrm{H}_{-1,0} + 4/2025\,(630679 + 390150\,\zeta_3 + 1303920\,\zeta_2)\,\mathrm{H}_0$$
$$- 8/3\,(432\,\zeta_3 + 707\,\zeta_2)\,\mathrm{H}_{-2} + 8/3\,(864\,\zeta_3 + 1763\,\zeta_2)\,\mathrm{H}_{-1,-1} - 8/45\,(21945\,\zeta_3$$
$$+ 10810\,\zeta_2 + 216\,\zeta_2^2)\,\mathrm{H}_{-1} - 1/3\,\left(g_2^E(x) - g_2^O(x)\right) + 23/3\,\left(g_1^E(x) - g_1^O(x)\right)$$
$$+ 2/3\,\zeta_3\,\delta(1-x)\Big)$$
$$+ C_F\,(C_A - 2\,C_F)\,n_f\Big(x^2\Big\{ - 64\,\mathrm{H}_{-1}\,\zeta_2 + 128\,\mathrm{H}_{-2,0} + 1168/15\,\mathrm{H}_{-1,0}$$
$$- 2032/15\,\mathrm{H}_{0,0} - 128\,\mathrm{H}_{-1,-1,0} + 192\,\mathrm{H}_{-1,0,0} - 192\,\mathrm{H}_{0,0,0} + 16/15\,(12 + 120\,\zeta_3$$

$$+ 199\,\zeta_2) + 16/25\,(139 + 100\,\zeta_2)\,\mathrm{H}_0 \Big\} + x^3 \Big\{ 192/5\,\mathrm{H}_{-1}\,\zeta_2 - 96/5\,\mathrm{H}_0\,\zeta_2$$
$$- 384/5\,\mathrm{H}_{-2,0} - 2224/25\,\mathrm{H}_{-1,0} + 1272/25\,\mathrm{H}_{0,0} + 384/5\,\mathrm{H}_{-1,-1,0}$$
$$- 576/5\,\mathrm{H}_{-1,0,0} + 288/5\,\mathrm{H}_{0,0,0} - 24/25\,(40\,\zeta_3 + 53\,\zeta_2) \Big\} + \big(1/x^2 + 9\,x^3\big) \Big\{$$
$$- 32/15\,\mathrm{H}_{-1}\,\zeta_2 + 64/15\,\mathrm{H}_{-2,0} + 952/225\,\mathrm{H}_{-1,0} - 64/15\,\mathrm{H}_{-1,-1,0}$$
$$+ 32/5\,\mathrm{H}_{-1,0,0} \Big\} + \big(1/x + 9\,x^2\big) \Big\{ - 472/225\,\mathrm{H}_0 + 224/15\,\mathrm{H}_{-1,0} - 32/5\,\mathrm{H}_{0,0}$$
$$+ 8/225\,(179 + 60\,\zeta_2) \Big\} + (1-x) \Big\{ 320/3\,\mathrm{H}_{-2}\,\zeta_2 - 976/27\,\mathrm{H}_1 + 656/27\,\mathrm{H}_2$$
$$+ 32/3\,\mathrm{H}_3 - 160\,\mathrm{H}_{-3,0} - 2336/9\,\mathrm{H}_{-2,0} - 320/3\,\mathrm{H}_{-1,-1}\,\zeta_2 - 128/9\,\mathrm{H}_{-1,2}$$
$$+ 5192/15\,\mathrm{H}_{0,0} + 128/9\,\mathrm{H}_{1,1} - 64/9\,\mathrm{H}_{2,1} + 640/3\,\mathrm{H}_{-2,-1,0} - 640/3\,\mathrm{H}_{-2,0,0}$$
$$+ 640/3\,\mathrm{H}_{-1,-2,0} + 1280/3\,\mathrm{H}_{-1,-1,0} - 3008/9\,\mathrm{H}_{-1,0,0} + 120\,\mathrm{H}_{0,0,0}$$
$$- 640/3\,\mathrm{H}_{-1,-1,-1,0} + 320\,\mathrm{H}_{-1,-1,0,0} - 160\,\mathrm{H}_{-1,0,0,0} - 8/135\,(5969$$
$$- 900\,\zeta_2)\,\mathrm{H}_{-1,0} - 8/2025\,(30059 + 55800\,\zeta_3 + 58680\,\zeta_2 + 8100\,\zeta_2^2)$$
$$+ 8/2025\,(30157 - 13500\,\zeta_3 - 7425\,\zeta_2)\,\mathrm{H}_0 + 64/9\,(15\,\zeta_3 + 32\,\zeta_2)\,\mathrm{H}_{-1} \Big\}$$
$$+ p_{qq}(-x) \Big\{ - 416/9\,\mathrm{H}_{-2}\,\zeta_2 - 320/27\,\mathrm{H}_3 - 64/9\,\mathrm{H}_4 + 800/9\,\mathrm{H}_{-3,0}$$
$$+ 1600/27\,\mathrm{H}_{-2,0} + 64/9\,\mathrm{H}_{-2,2} + 320/9\,\mathrm{H}_{-1,-1}\,\zeta_2 + 640/27\,\mathrm{H}_{-1,2} + 64/9\,\mathrm{H}_{-1,3}$$
$$+ 64/9\,\mathrm{H}_{3,1} - 704/9\,\mathrm{H}_{-2,-1,0} + 1088/9\,\mathrm{H}_{-2,0,0} - 704/9\,\mathrm{H}_{-1,-2,0}$$
$$- 640/9\,\mathrm{H}_{-1,-1,0} + 2560/27\,\mathrm{H}_{-1,0,0} - 128/9\,\mathrm{H}_{-1,2,1} - 664/27\,\mathrm{H}_{0,0,0}$$
$$+ 640/9\,\mathrm{H}_{-1,-1,-1,0} - 1088/9\,\mathrm{H}_{-1,-1,0,0} + 1088/9\,\mathrm{H}_{-1,0,0,0} - 368/9\,\mathrm{H}_{0,0,0,0}$$
$$+ 32/81\,(83 - 63\,\zeta_2)\,\mathrm{H}_{-1,0} - 8/27\,(87 - 66\,\zeta_3 - 23\,\zeta_2)\,\mathrm{H}_0 - 16/81\,(191$$
$$- 45\,\zeta_2)\,\mathrm{H}_{0,0} - 64/27\,(12\,\zeta_3 + 25\,\zeta_2)\,\mathrm{H}_{-1} + 16/405\,(1350\,\zeta_3 + 415\,\zeta_2$$
$$+ 513\,\zeta_2^2) \Big\} - 128\,\mathrm{H}_{-2}\,\zeta_2 - 1312/27\,\mathrm{H}_2 - 64/3\,\mathrm{H}_3 + 192\,\mathrm{H}_{-3,0}$$
$$+ 1280/3\,\mathrm{H}_{-2,0} + 128\,\mathrm{H}_{-1,-1}\,\zeta_2 + 256/9\,\mathrm{H}_{-1,2} - 17912/45\,\mathrm{H}_{0,0} + 128/9\,\mathrm{H}_{2,1}$$
$$- 256\,\mathrm{H}_{-2,-1,0} + 256\,\mathrm{H}_{-2,0,0} - 256\,\mathrm{H}_{-1,-2,0} - 5632/9\,\mathrm{H}_{-1,-1,0}$$
$$+ 1664/3\,\mathrm{H}_{-1,0,0} - 496/3\,\mathrm{H}_{0,0,0} + 256\,\mathrm{H}_{-1,-1,-1,0} - 384\,\mathrm{H}_{-1,-1,0,0}$$
$$+ 192\,\mathrm{H}_{-1,0,0,0} - 8/45\,(430 - 1600\,\zeta_3 - 1433\,\zeta_2 - 216\,\zeta_2^2) + 16/135\,(5039$$
$$- 540\,\zeta_2)\,\mathrm{H}_{-1,0} - 16/2025\,(28942 - 8100\,\zeta_3 - 6075\,\zeta_2)\,\mathrm{H}_0 - 128/3\,(3\,\zeta_3$$
$$+ 8\,\zeta_2)\,\mathrm{H}_{-1} \Big), \tag{A.45}$$

$$\delta c^{(3)}_{L,ns}(x) =$$
$$C_F\,(C_A - 2\,C_F)\,C_F \Big(x^{-2} \Big\{ 384/5\,\mathrm{H}_{-2}\,\zeta_2 + 256\,\mathrm{H}_{-1}\,\zeta_3 + 19072/75\,\mathrm{H}_{-1}\,\zeta_2$$

$$
\begin{aligned}
&- 64/5\,\mathrm{H}_{1}\,\zeta_3\; + 6784/75\,\mathrm{H}_{1}\,\zeta_2\; + 128/5\,\mathrm{H}_{2}\,\zeta_2\; - 512/15\,\mathrm{H}_{-2,0}\; - 256/5\,\mathrm{H}_{-2,2}\\
&- 4672/15\,\mathrm{H}_{-1,-1}\,\zeta_2\; - 25936/225\,\mathrm{H}_{-1,0}\; + 704/3\,\mathrm{H}_{-1,0}\,\zeta_2\; - 13568/75\,\mathrm{H}_{-1,2}\\
&- 640/3\,\mathrm{H}_{-1,3}\; + 448/15\,\mathrm{H}_{1,0}\,\zeta_2\; + 64/5\,\mathrm{H}_{1,1}\,\zeta_2\; - 128/15\,\mathrm{H}_{1,3}\; + 256/5\,\mathrm{H}_{-2,-1,0}\\
&- 256/5\,\mathrm{H}_{-2,0,0}\; + 128/5\,\mathrm{H}_{-1,-2,0}\; + 11008/75\,\mathrm{H}_{-1,-1,0}\; + 896/3\,\mathrm{H}_{-1,-1,2}\\
&- 14848/75\,\mathrm{H}_{-1,0,0}\; - 128/5\,\mathrm{H}_{-1,2,0}\; - 128/5\,\mathrm{H}_{-1,2,1}\; + 256/15\,\mathrm{H}_{1,-2,0}\\
&- 128/5\,\mathrm{H}_{-1,-1,-1,0}\; + 640/3\,\mathrm{H}_{-1,-1,0,0}\; - 256/3\,\mathrm{H}_{-1,0,0,0}\; - 128/15\,\mathrm{H}_{1,0,0,0}\\
&+ 128/15\,\mathrm{H}_{1,1,0,0}\Big\} + x^{-1}\Big\{16688/225\; - 1216/5\,\zeta_3\; - 13696/75\,\zeta_2\\
&- 192\,\mathrm{H}_{-2}\,\zeta_2\; - 672\,\mathrm{H}_{-1}\,\zeta_3\; - 4448/15\,\mathrm{H}_{-1}\,\zeta_2\; - 12848/225\,\mathrm{H}_{0}\; - 3968/15\,\mathrm{H}_{0}\,\zeta_2\\
&- 1536/25\,\mathrm{H}_{1}\; + 64/3\,\mathrm{H}_{1}\,\zeta_3\; - 4432/15\,\mathrm{H}_{1}\,\zeta_2\; + 256/25\,\mathrm{H}_{2}\; - 64\,\mathrm{H}_{2}\,\zeta_2\\
&+ 3328/15\,\mathrm{H}_{3}\; - 128/3\,\mathrm{H}_{-2,0}\; + 128\,\mathrm{H}_{-2,2}\; + 2464/3\,\mathrm{H}_{-1,-1}\,\zeta_2\; - 5248/75\,\mathrm{H}_{-1,0}\\
&- 1888/3\,\mathrm{H}_{-1,0}\,\zeta_2\; + 800/3\,\mathrm{H}_{-1,2}\; + 1664/3\,\mathrm{H}_{-1,3}\; + 11648/75\,\mathrm{H}_{0,0}\\
&- 128/5\,\mathrm{H}_{1,0}\; - 416/3\,\mathrm{H}_{1,0}\,\zeta_2\; - 128/5\,\mathrm{H}_{1,1}\; - 32\,\mathrm{H}_{1,1}\,\zeta_2\; + 64\,\mathrm{H}_{1,3}\; + 128/5\,\mathrm{H}_{2,0}\\
&+ 128/5\,\mathrm{H}_{2,1}\; - 128\,\mathrm{H}_{-2,-1,0}\; + 128\,\mathrm{H}_{-2,0,0}\; - 64\,\mathrm{H}_{-1,-2,0}\; - 896/15\,\mathrm{H}_{-1,-1,0}\\
&- 2368/3\,\mathrm{H}_{-1,-1,2}\; + 112\,\mathrm{H}_{-1,0,0}\; + 64\,\mathrm{H}_{-1,2,0}\; + 64\,\mathrm{H}_{-1,2,1}\; + 1408/15\,\mathrm{H}_{0,0,0}\\
&- 256/3\,\mathrm{H}_{1,-2,0}\; - 128/15\,\mathrm{H}_{1,0,0}\; + 64\,\mathrm{H}_{-1,-1,-1,0}\; - 1664/3\,\mathrm{H}_{-1,-1,0,0}\\
&+ 704/3\,\mathrm{H}_{-1,0,0,0}\; + 128/3\,\mathrm{H}_{1,0,0,0}\; - 64\,\mathrm{H}_{1,1,0,0}\Big\} + x\Big\{46136/75\; - 3256/15\,\zeta_2^2\\
&+ 1024\,\zeta_5\; - 15112/5\,\zeta_3\; - 155656/225\,\zeta_2\; + 480\,\zeta_2\,\zeta_3\; + 800\,\mathrm{H}_{-3}\,\zeta_2\\
&+ 2080\,\mathrm{H}_{-2}\,\zeta_3\; + 7664/3\,\mathrm{H}_{-2}\,\zeta_2\; - 1888/5\,\mathrm{H}_{-1}\,\zeta_2^2\; + 3472\,\mathrm{H}_{-1}\,\zeta_3\\
&+ 15008/5\,\mathrm{H}_{-1}\,\zeta_2\; - 54904/25\,\mathrm{H}_{0}\; - 16\,\mathrm{H}_{0}\,\zeta_2^2\; - 880\,\mathrm{H}_{0}\,\zeta_3\; - 14112/5\,\mathrm{H}_{0}\,\zeta_2\\
&- 89792/75\,\mathrm{H}_{1}\; + 32\,\mathrm{H}_{1}\,\zeta_3\; - 2864/15\,\mathrm{H}_{1}\,\zeta_2\; + 43456/75\,\mathrm{H}_{2}\; - 64\,\mathrm{H}_{2}\,\zeta_3\\
&+ 752\,\mathrm{H}_{2}\,\zeta_2\; + 10752/5\,\mathrm{H}_{3}\; + 224\,\mathrm{H}_{3}\,\zeta_2\; + 1664/3\,\mathrm{H}_{4}\; - 192\,\mathrm{H}_{-4,0}\\
&- 1760/3\,\mathrm{H}_{-3,0}\; - 448\,\mathrm{H}_{-3,2}\; - 2496\,\mathrm{H}_{-2,-1}\,\zeta_2\; - 848/3\,\mathrm{H}_{-2,0}\; + 1792\,\mathrm{H}_{-2,0}\,\zeta_2\\
&- 6176/3\,\mathrm{H}_{-2,2}\; - 1344\,\mathrm{H}_{-2,3}\; - 2432\,\mathrm{H}_{-1,-2}\,\zeta_2\; - 3264\,\mathrm{H}_{-1,-1}\,\zeta_3\\
&- 12992/3\,\mathrm{H}_{-1,-1}\,\zeta_2\; - 44008/225\,\mathrm{H}_{-1,0}\; + 1344\,\mathrm{H}_{-1,0}\,\zeta_3\; + 8864/3\,\mathrm{H}_{-1,0}\,\zeta_2\\
&- 9184/3\,\mathrm{H}_{-1,2}\; - 128\,\mathrm{H}_{-1,2}\,\zeta_2\; - 6688/3\,\mathrm{H}_{-1,3}\; - 1408\,\mathrm{H}_{-1,4}\\
&+ 131608/225\,\mathrm{H}_{0,0}\; - 64\,\mathrm{H}_{0,0}\,\zeta_3\; - 1984/3\,\mathrm{H}_{0,0}\,\zeta_2\; - 512/5\,\mathrm{H}_{1,0}\; - 320/3\,\mathrm{H}_{1,0}\,\zeta_2\\
&- 512/5\,\mathrm{H}_{1,1}\; + 256/3\,\mathrm{H}_{1,3}\; + 896/5\,\mathrm{H}_{2,0}\; + 448\,\mathrm{H}_{2,0}\,\zeta_2\; + 896/5\,\mathrm{H}_{2,1}\; + 64\,\mathrm{H}_{2,1}\,\zeta_2\\
&- 256\,\mathrm{H}_{2,3}\; + 64\,\mathrm{H}_{3,0}\; + 64\,\mathrm{H}_{3,1}\; + 704\,\mathrm{H}_{-3,-1,0}\; - 704\,\mathrm{H}_{-3,0,0}\; + 640\,\mathrm{H}_{-2,-2,0}\\
&+ 992\,\mathrm{H}_{-2,-1,0}\; + 2048\,\mathrm{H}_{-2,-1,2}\; - 4960/3\,\mathrm{H}_{-2,0,0}\; - 128\,\mathrm{H}_{-2,2,0}\; - 128\,\mathrm{H}_{-2,2,1}\\
&+ 640\,\mathrm{H}_{-1,-3,0}\; + 576\,\mathrm{H}_{-1,-2,0}\; + 2048\,\mathrm{H}_{-1,-2,2}\; + 3968\,\mathrm{H}_{-1,-1,-1}\,\zeta_2\\
&- 1792/15\,\mathrm{H}_{-1,-1,0}\; - 3072\,\mathrm{H}_{-1,-1,0}\,\zeta_2\; + 11456/3\,\mathrm{H}_{-1,-1,2}\; + 2432\,\mathrm{H}_{-1,-1,3}\\
&- 5744/3\,\mathrm{H}_{-1,0,0}\; + 1792\,\mathrm{H}_{-1,0,0}\,\zeta_2\; - 128\,\mathrm{H}_{-1,2,0}\; - 128\,\mathrm{H}_{-1,2,1}\; - 128\,\mathrm{H}_{-1,3,0}
\end{aligned}
$$

$$- 128\,\mathrm{H}_{-1,3,1} + 18896/15\,\mathrm{H}_{0,0,0} - 128/3\,\mathrm{H}_{1,-2,0} + 128/5\,\mathrm{H}_{1,0,0} + 256\,\mathrm{H}_{2,-2,0}$$
$$- 448/3\,\mathrm{H}_{2,0,0} - 896\,\mathrm{H}_{-2,-1,-1,0} + 1984\,\mathrm{H}_{-2,-1,0,0} - 896\,\mathrm{H}_{-2,0,0,0}$$
$$- 768\,\mathrm{H}_{-1,-2,-1,0} + 1920\,\mathrm{H}_{-1,-2,0,0} - 768\,\mathrm{H}_{-1,-1,-2,0} - 1024\,\mathrm{H}_{-1,-1,-1,0}$$
$$- 3584\,\mathrm{H}_{-1,-1,-1,2} + 8032/3\,\mathrm{H}_{-1,-1,0,0} + 256\,\mathrm{H}_{-1,-1,2,0} + 256\,\mathrm{H}_{-1,-1,2,1}$$
$$- 3232/3\,\mathrm{H}_{-1,0,0,0} + 256\,\mathrm{H}_{-1,2,0,0} + 656/3\,\mathrm{H}_{0,0,0,0} + 64/3\,\mathrm{H}_{1,0,0,0}$$
$$- 256/3\,\mathrm{H}_{1,1,0,0} - 128\,\mathrm{H}_{2,0,0,0} + 256\,\mathrm{H}_{2,1,0,0} + 768\,\mathrm{H}_{-1,-1,-1,-1,0}$$
$$- 3200\,\mathrm{H}_{-1,-1,-1,0,0} + 1536\,\mathrm{H}_{-1,-1,0,0,0} - 640\,\mathrm{H}_{-1,0,0,0,0} \Big\} + x^2 \Big\{ 3608/25$$
$$+ 5152/15\,\zeta_2^2 - 24352/15\,\zeta_3 + 12352/75\,\zeta_2 + 4928/3\,\mathrm{H}_{-2}\,\zeta_2 + 1344\,\mathrm{H}_{-1}\,\zeta_3$$
$$+ 4224/5\,\mathrm{H}_{-1}\,\zeta_2 + 1144/25\,\mathrm{H}_0 - 3904/3\,\mathrm{H}_0\,\zeta_3 - 24032/15\,\mathrm{H}_0\,\zeta_2$$
$$- 2464/25\,\mathrm{H}_1 + 128/3\,\mathrm{H}_1\,\zeta_3 - 9248/15\,\mathrm{H}_1\,\zeta_2 - 3424/25\,\mathrm{H}_2 - 64\,\mathrm{H}_2\,\zeta_2$$
$$+ 13312/15\,\mathrm{H}_3 + 3712/3\,\mathrm{H}_4 - 896/3\,\mathrm{H}_{-3,0} - 3328/15\,\mathrm{H}_{-2,0} - 4736/3\,\mathrm{H}_{-2,2}$$
$$- 4928/3\,\mathrm{H}_{-1,-1}\,\zeta_2 + 19072/75\,\mathrm{H}_{-1,0} + 3776/3\,\mathrm{H}_{-1,0}\,\zeta_2 - 2240/3\,\mathrm{H}_{-1,2}$$
$$- 3328/3\,\mathrm{H}_{-1,3} - 10432/75\,\mathrm{H}_{0,0} - 1536\,\mathrm{H}_{0,0}\,\zeta_2 - 192/5\,\mathrm{H}_{1,0} - 832/3\,\mathrm{H}_{1,0}\,\zeta_2$$
$$- 192/5\,\mathrm{H}_{1,1} - 64\,\mathrm{H}_{1,1}\,\zeta_2 + 128\,\mathrm{H}_{1,3} - 192/5\,\mathrm{H}_{2,0} - 192/5\,\mathrm{H}_{2,1} + 128\,\mathrm{H}_{3,0}$$
$$+ 128\,\mathrm{H}_{3,1} + 128\,\mathrm{H}_{-2,-1,0} - 3328/3\,\mathrm{H}_{-2,0,0} + 128\,\mathrm{H}_{-1,-2,0} + 2944/15\,\mathrm{H}_{-1,-1,0}$$
$$+ 4736/3\,\mathrm{H}_{-1,-1,2} - 1184/3\,\mathrm{H}_{-1,0,0} - 128\,\mathrm{H}_{-1,2,0} - 128\,\mathrm{H}_{-1,2,1}$$
$$+ 8992/15\,\mathrm{H}_{0,0,0} - 512/3\,\mathrm{H}_{1,-2,0} - 64/5\,\mathrm{H}_{1,0,0} - 128\,\mathrm{H}_{2,0,0} - 128\,\mathrm{H}_{-1,-1,-1,0}$$
$$+ 3328/3\,\mathrm{H}_{-1,-1,0,0} - 1408/3\,\mathrm{H}_{-1,0,0,0} + 1664/3\,\mathrm{H}_{0,0,0,0} + 256/3\,\mathrm{H}_{1,0,0,0}$$
$$- 128\,\mathrm{H}_{1,1,0,0} \Big\} + x^3 \Big\{ - 2576/25\,\zeta_2^2 + 3216/5\,\zeta_3 + 2696/25\,\zeta_2$$
$$- 2336/5\,\mathrm{H}_{-2}\,\zeta_2 - 384\,\mathrm{H}_{-1}\,\zeta_3 - 9776/25\,\mathrm{H}_{-1}\,\zeta_2 + 1824/5\,\mathrm{H}_0\,\zeta_3$$
$$+ 13248/25\,\mathrm{H}_0\,\zeta_2 - 96/5\,\mathrm{H}_1\,\zeta_3 + 3472/25\,\mathrm{H}_1\,\zeta_2 + 96/5\,\mathrm{H}_2\,\zeta_2 - 6944/25\,\mathrm{H}_3$$
$$- 1664/5\,\mathrm{H}_4 + 64\,\mathrm{H}_{-3,0} + 5664/25\,\mathrm{H}_{-2,0} + 448\,\mathrm{H}_{-2,2} + 2336/5\,\mathrm{H}_{-1,-1}\,\zeta_2$$
$$+ 2696/25\,\mathrm{H}_{-1,0} - 352\,\mathrm{H}_{-1,0}\,\zeta_2 + 6944/25\,\mathrm{H}_{-1,2} + 320\,\mathrm{H}_{-1,3} - 2696/25\,\mathrm{H}_{0,0}$$
$$+ 1984/5\,\mathrm{H}_{0,0}\,\zeta_2 + 224/5\,\mathrm{H}_{1,0}\,\zeta_2 + 96/5\,\mathrm{H}_{1,1}\,\zeta_2 - 64/5\,\mathrm{H}_{1,3} - 192/5\,\mathrm{H}_{3,0}$$
$$- 192/5\,\mathrm{H}_{3,1} - 192/5\,\mathrm{H}_{-2,-1,0} + 320\,\mathrm{H}_{-2,0,0} - 192/5\,\mathrm{H}_{-1,-2,0}$$
$$- 5664/25\,\mathrm{H}_{-1,-1,0} - 448\,\mathrm{H}_{-1,-1,2} + 7584/25\,\mathrm{H}_{-1,0,0} + 192/5\,\mathrm{H}_{-1,2,0}$$
$$+ 192/5\,\mathrm{H}_{-1,2,1} - 7584/25\,\mathrm{H}_{0,0,0} + 128/5\,\mathrm{H}_{1,-2,0} + 64/5\,\mathrm{H}_{2,0,0}$$
$$+ 192/5\,\mathrm{H}_{-1,-1,-1,0} - 320\,\mathrm{H}_{-1,-1,0,0} + 128\,\mathrm{H}_{-1,0,0,0} - 704/5\,\mathrm{H}_{0,0,0,0}$$
$$- 64/5\,\mathrm{H}_{1,0,0,0} + 64/5\,\mathrm{H}_{1,1,0,0} \Big\} - 187568/225 + 8176/15\,\zeta_3 - 22736/75\,\zeta_2$$
$$+ 544\,\mathrm{H}_{-2}\,\zeta_2 + 1632\,\mathrm{H}_{-1}\,\zeta_3 + 18224/15\,\mathrm{H}_{-1}\,\zeta_2 + 2024/45\,\mathrm{H}_0 + 2432/5\,\mathrm{H}_0\,\zeta_2$$
$$+ 101792/75\,\mathrm{H}_1 - 64\,\mathrm{H}_1\,\zeta_3 + 4368/5\,\mathrm{H}_1\,\zeta_2 + 54496/75\,\mathrm{H}_2 + 160\,\mathrm{H}_2\,\zeta_2$$
$$- 5056/15\,\mathrm{H}_3 - 128\,\mathrm{H}_{-3,0} + 128/5\,\mathrm{H}_{-2,0} - 320\,\mathrm{H}_{-2,2} - 1984\,\mathrm{H}_{-1,-1}\,\zeta_2$$

$$+ 42544/75\, \mathrm{H}_{-1,0} + 1536\, \mathrm{H}_{-1,0}\, \zeta_2 - 3424/3\, \mathrm{H}_{-1,2} - 1216\, \mathrm{H}_{-1,3}$$
$$- 9424/75\, \mathrm{H}_{0,0} + 832/5\, \mathrm{H}_{1,0} + 448\, \mathrm{H}_{1,0}\, \zeta_2 + 832/5\, \mathrm{H}_{1,1} + 64\, \mathrm{H}_{1,1}\, \zeta_2 - 256\, \mathrm{H}_{1,3}$$
$$- 64/5\, \mathrm{H}_{2,0} - 64/5\, \mathrm{H}_{2,1} + 448\, \mathrm{H}_{-2,-1,0} - 512\, \mathrm{H}_{-2,0,0} + 384\, \mathrm{H}_{-1,-2,0}$$
$$+ 736/5\, \mathrm{H}_{-1,-1,0} + 1792\, \mathrm{H}_{-1,-1,2} - 2240/3\, \mathrm{H}_{-1,0,0} - 128\, \mathrm{H}_{-1,2,0} - 128\, \mathrm{H}_{-1,2,1}$$
$$- 3056/15\, \mathrm{H}_{0,0,0} + 256\, \mathrm{H}_{1,-2,0} - 64/15\, \mathrm{H}_{1,0,0} - 384\, \mathrm{H}_{-1,-1,-1,0}$$
$$+ 1600\, \mathrm{H}_{-1,-1,0,0} - 768\, \mathrm{H}_{-1,0,0,0} - 128\, \mathrm{H}_{1,0,0,0} + 256\, \mathrm{H}_{1,1,0,0}$$
$$- 16\, \left(g_1^E(x) - g_1^O(x)\right) \Bigg)$$
$$+ {\color{blue} C_F\, (C_A - 2\, C_F)\, C_A} \Bigg(+ x^{-2} \Bigg\{ - 128\, \mathrm{H}_{-1}\, \zeta_3 - 2048/45\, \mathrm{H}_{-1}\, \zeta_2 + 64/3\, \mathrm{H}_1\, \zeta_3$$
$$- 304/15\, \mathrm{H}_1\, \zeta_2 - 1408/15\, \mathrm{H}_{-2,0} + 512/3\, \mathrm{H}_{-1,-1}\, \zeta_2 - 28144/225\, \mathrm{H}_{-1,0}$$
$$- 1408/15\, \mathrm{H}_{-1,0}\, \zeta_2 + 608/15\, \mathrm{H}_{-1,2} + 256/3\, \mathrm{H}_{-1,3} - 64/5\, \mathrm{H}_{1,0}\, \zeta_2 + 64/15\, \mathrm{H}_{1,3}$$
$$- 448/45\, \mathrm{H}_{-1,-1,0} - 512/3\, \mathrm{H}_{-1,-1,2} - 3088/45\, \mathrm{H}_{-1,0,0} - 256/15\, \mathrm{H}_{1,-2,0}$$
$$- 256/3\, \mathrm{H}_{-1,-1,0,0} + 128/15\, \mathrm{H}_{-1,0,0,0} + 128/15\, \mathrm{H}_{1,0,0,0} - 64/15\, \mathrm{H}_{1,1,0,0} \Bigg\}$$
$$+ x^{-1} \Bigg\{ - 51104/225 + 320/3\, \zeta_3 - 368/9\, \zeta_2 + 352\, \mathrm{H}_{-1}\, \zeta_3 - 112/3\, \mathrm{H}_{-1}\, \zeta_2$$
$$+ 29984/225\, \mathrm{H}_0 + 320/3\, \mathrm{H}_0\, \zeta_2 - 736/15\, \mathrm{H}_1 - 64/3\, \mathrm{H}_1\, \zeta_3 + 296/3\, \mathrm{H}_1\, \zeta_2$$
$$+ 736/15\, \mathrm{H}_2 - 448/5\, \mathrm{H}_3 + 3776/15\, \mathrm{H}_{-2,0} - 1408/3\, \mathrm{H}_{-1,-1}\, \zeta_2 + 3784/9\, \mathrm{H}_{-1,0}$$
$$+ 832/3\, \mathrm{H}_{-1,0}\, \zeta_2 - 80/3\, \mathrm{H}_{-1,2} - 704/3\, \mathrm{H}_{-1,3} + 3856/45\, \mathrm{H}_{0,0} + 224/3\, \mathrm{H}_{1,0}\, \zeta_2$$
$$- 32\, \mathrm{H}_{1,3} - 128\, \mathrm{H}_{-1,-1,0} + 1408/3\, \mathrm{H}_{-1,-1,2} + 856/3\, \mathrm{H}_{-1,0,0} - 256/15\, \mathrm{H}_{0,0,0}$$
$$+ 256/3\, \mathrm{H}_{1,-2,0} + 64/15\, \mathrm{H}_{1,0,0} + 704/3\, \mathrm{H}_{-1,-1,0,0} - 128/3\, \mathrm{H}_{-1,0,0,0}$$
$$- 128/3\, \mathrm{H}_{1,0,0,0} + 32\, \mathrm{H}_{1,1,0,0} \Bigg\} + x \Bigg\{ 36064/225 - 704/15\, \zeta_2^2 - 224\, \zeta_5$$
$$+ 3736/9\, \zeta_3 - 7240/9\, \zeta_2 - 288\, \zeta_2\, \zeta_3 - 768\, \mathrm{H}_{-2}\, \zeta_3 - 576\, \mathrm{H}_{-2}\, \zeta_2$$
$$- 128/5\, \mathrm{H}_{-1}\, \zeta_2^2 - 3104/3\, \mathrm{H}_{-1}\, \zeta_3 - 2032/3\, \mathrm{H}_{-1}\, \zeta_2 + 233168/225\, \mathrm{H}_0$$
$$+ 160/3\, \mathrm{H}_0\, \zeta_3 + 9592/9\, \mathrm{H}_0\, \zeta_2 + 1184/3\, \mathrm{H}_1 + 32/3\, \mathrm{H}_1\, \zeta_3 + 320/3\, \mathrm{H}_1\, \zeta_2$$
$$+ 224/3\, \mathrm{H}_2 - 352\, \mathrm{H}_2\, \zeta_2 - 12688/15\, \mathrm{H}_3 - 192\, \mathrm{H}_4 - 544\, \mathrm{H}_{-3,0} + 1024\, \mathrm{H}_{-2,-1}\, \zeta_2$$
$$- 49504/45\, \mathrm{H}_{-2,0} - 640\, \mathrm{H}_{-2,0}\, \zeta_2 + 2816/3\, \mathrm{H}_{-2,2} + 512\, \mathrm{H}_{-2,3} + 1024\, \mathrm{H}_{-1,-2}\, \zeta_2$$
$$+ 1536\, \mathrm{H}_{-1,-1}\, \zeta_3 + 4480/3\, \mathrm{H}_{-1,-1}\, \zeta_2 - 16832/15\, \mathrm{H}_{-1,0} - 768\, \mathrm{H}_{-1,0}\, \zeta_3$$
$$- 1120\, \mathrm{H}_{-1,0}\, \zeta_2 + 1504\, \mathrm{H}_{-1,2} + 2752/3\, \mathrm{H}_{-1,3} + 640\, \mathrm{H}_{-1,4} + 64952/45\, \mathrm{H}_{0,0}$$
$$+ 784/3\, \mathrm{H}_{0,0}\, \zeta_2 + 64\, \mathrm{H}_{1,0}\, \zeta_2 - 128/3\, \mathrm{H}_{1,3} - 256\, \mathrm{H}_{2,0}\, \zeta_2 + 128\, \mathrm{H}_{2,3}$$
$$+ 2176/3\, \mathrm{H}_{-2,-1,0} - 1024\, \mathrm{H}_{-2,-1,2} - 1088/3\, \mathrm{H}_{-2,0,0} - 256\, \mathrm{H}_{-1,-3,0}$$
$$+ 2432/3\, \mathrm{H}_{-1,-2,0} - 1024\, \mathrm{H}_{-1,-2,2} - 2048\, \mathrm{H}_{-1,-1,-1}\, \zeta_2 + 4960/3\, \mathrm{H}_{-1,-1,0}$$
$$+ 1280\, \mathrm{H}_{-1,-1,0}\, \zeta_2 - 5504/3\, \mathrm{H}_{-1,-1,2} - 1024\, \mathrm{H}_{-1,-1,3} - 544\, \mathrm{H}_{-1,0,0}$$
$$- 896\, \mathrm{H}_{-1,0,0}\, \zeta_2 + 664/45\, \mathrm{H}_{0,0,0} + 128/3\, \mathrm{H}_{1,-2,0} - 64/5\, \mathrm{H}_{1,0,0} - 256\, \mathrm{H}_{2,-2,0}$$

$$+ 224/3\, \mathrm{H}_{2,0,0} - 512\, \mathrm{H}_{-2,-1,0,0} + 128\, \mathrm{H}_{-2,0,0,0} - 512\, \mathrm{H}_{-1,-2,0,0}$$
$$- 2048/3\, \mathrm{H}_{-1,-1,-1,0} + 2048\, \mathrm{H}_{-1,-1,-1,2} + 1088/3\, \mathrm{H}_{-1,-1,0,0} - 800/3\, \mathrm{H}_{-1,0,0,0}$$
$$- 128\, \mathrm{H}_{-1,2,0,0} - 208/3\, \mathrm{H}_{0,0,0,0} - 64/3\, \mathrm{H}_{1,0,0,0} + 128/3\, \mathrm{H}_{1,1,0,0} + 128\, \mathrm{H}_{2,0,0,0}$$
$$- 128\, \mathrm{H}_{2,1,0,0} + 1024\, \mathrm{H}_{-1,-1,-1,0,0} - 256\, \mathrm{H}_{-1,-1,0,0,0} + 256\, \mathrm{H}_{-1,0,0,0,0} \Big\} + x^2 \Big\{$$
$$- 26432/75 + 128/15\, \zeta_2^2 + 112\, \zeta_3 - 39416/45\, \zeta_2 - 2816/3\, \mathrm{H}_{-2}\, \zeta_2$$
$$- 704\, \mathrm{H}_{-1}\, \zeta_3 - 32/3\, \mathrm{H}_{-1}\, \zeta_2 - 31952/75\, \mathrm{H}_0 + 1984/3\, \mathrm{H}_0\, \zeta_3 + 1712/5\, \mathrm{H}_0\, \zeta_2$$
$$- 368/5\, \mathrm{H}_1 - 128/3\, \mathrm{H}_1\, \zeta_3 + 592/3\, \mathrm{H}_1\, \zeta_2 - 368/5\, \mathrm{H}_2 - 4096/15\, \mathrm{H}_3$$
$$- 1600/3\, \mathrm{H}_4 + 512/3\, \mathrm{H}_{-3,0} - 1152/5\, \mathrm{H}_{-2,0} + 2816/3\, \mathrm{H}_{-2,2}$$
$$+ 2816/3\, \mathrm{H}_{-1,-1}\, \zeta_2 - 8656/9\, \mathrm{H}_{-1,0} - 1664/3\, \mathrm{H}_{-1,0}\, \zeta_2 + 416/3\, \mathrm{H}_{-1,2}$$
$$+ 1408/3\, \mathrm{H}_{-1,3} + 6616/9\, \mathrm{H}_{0,0} + 704\, \mathrm{H}_{0,0}\, \zeta_2 + 448/3\, \mathrm{H}_{1,0}\, \zeta_2 - 64\, \mathrm{H}_{1,3}$$
$$+ 1408/3\, \mathrm{H}_{-2,0,0} + 256\, \mathrm{H}_{-1,-1,0} - 2816/3\, \mathrm{H}_{-1,-1,2} - 528\, \mathrm{H}_{-1,0,0} + 400\, \mathrm{H}_{0,0,0}$$
$$+ 512/3\, \mathrm{H}_{1,-2,0} + 32/5\, \mathrm{H}_{1,0,0} + 64\, \mathrm{H}_{2,0,0} - 1408/3\, \mathrm{H}_{-1,-1,0,0} + 256/3\, \mathrm{H}_{-1,0,0,0}$$
$$- 512/3\, \mathrm{H}_{0,0,0,0} - 256/3\, \mathrm{H}_{1,0,0,0} + 64\, \mathrm{H}_{1,1,0,0} \Big\} + x^3 \Big\{ 192/25\, \zeta_2^2 - 1592/15\, \zeta_3$$
$$+ 4984/25\, \zeta_2 + 256\, \mathrm{H}_{-2}\, \zeta_2 + 192\, \mathrm{H}_{-1}\, \zeta_3 + 1024/15\, \mathrm{H}_{-1}\, \zeta_2 - 160\, \mathrm{H}_0\, \zeta_3$$
$$- 296/3\, \mathrm{H}_0\, \zeta_2 + 32\, \mathrm{H}_1\, \zeta_3 - 152/5\, \mathrm{H}_1\, \zeta_2 + 304/5\, \mathrm{H}_3 + 672/5\, \mathrm{H}_4 - 128/5\, \mathrm{H}_{-3,0}$$
$$- 224/15\, \mathrm{H}_{-2,0} - 256\, \mathrm{H}_{-2,2} - 256\, \mathrm{H}_{-1,-1}\, \zeta_2 + 4984/25\, \mathrm{H}_{-1,0} + 704/5\, \mathrm{H}_{-1,0}\, \zeta_2$$
$$- 304/5\, \mathrm{H}_{-1,2} - 128\, \mathrm{H}_{-1,3} - 4984/25\, \mathrm{H}_{0,0} - 160\, \mathrm{H}_{0,0}\, \zeta_2 - 96/5\, \mathrm{H}_{1,0}\, \zeta_2$$
$$+ 32/5\, \mathrm{H}_{1,3} - 128\, \mathrm{H}_{-2,0,0} + 224/15\, \mathrm{H}_{-1,-1,0} + 256\, \mathrm{H}_{-1,-1,2} + 1544/15\, \mathrm{H}_{-1,0,0}$$
$$- 1544/15\, \mathrm{H}_{0,0,0} - 128/5\, \mathrm{H}_{1,-2,0} - 32/5\, \mathrm{H}_{2,0,0} + 128\, \mathrm{H}_{-1,-1,0,0}$$
$$- 64/5\, \mathrm{H}_{-1,0,0,0} + 128/5\, \mathrm{H}_{0,0,0,0} + 64/5\, \mathrm{H}_{1,0,0,0} - 32/5\, \mathrm{H}_{1,1,0,0} \Big\} + 94336/225$$
$$- 736/3\, \zeta_3 + 24824/45\, \zeta_2 - 768\, \mathrm{H}_{-1}\, \zeta_3 - 128\, \mathrm{H}_{-1}\, \zeta_2 - 29152/225\, \mathrm{H}_0$$
$$- 3248/15\, \mathrm{H}_0\, \zeta_2 - 272\, \mathrm{H}_1 - 352\, \mathrm{H}_1\, \zeta_2 - 1040/3\, \mathrm{H}_2 + 896/5\, \mathrm{H}_3$$
$$- 7712/15\, \mathrm{H}_{-2,0} + 1024\, \mathrm{H}_{-1,-1}\, \zeta_2 - 42496/45\, \mathrm{H}_{-1,0} - 640\, \mathrm{H}_{-1,0}\, \zeta_2$$
$$+ 1024/3\, \mathrm{H}_{-1,2} + 512\, \mathrm{H}_{-1,3} - 2488/9\, \mathrm{H}_{0,0} - 256\, \mathrm{H}_{1,0}\, \zeta_2 + 128\, \mathrm{H}_{1,3}$$
$$+ 1280/3\, \mathrm{H}_{-1,-1,0} - 1024\, \mathrm{H}_{-1,-1,2} - 512\, \mathrm{H}_{-1,0,0} + 112/3\, \mathrm{H}_{0,0,0} - 256\, \mathrm{H}_{1,-2,0}$$
$$+ 32/15\, \mathrm{H}_{1,0,0} - 512\, \mathrm{H}_{-1,-1,0,0} + 128\, \mathrm{H}_{-1,0,0,0} + 128\, \mathrm{H}_{1,0,0,0} - 128\, \mathrm{H}_{1,1,0,0}$$
$$+ 8\, \left(g_1^E(x) - g_1^O(x)\right) \Big)$$
$$+ C_F\, (C_A - 2\, C_F)\, n_f \Big(+ x^{-2} \Big\{ - 128/15\, \mathrm{H}_{-1}\, \zeta_2 + 256/15\, \mathrm{H}_{-2,0} + 4768/225\, \mathrm{H}_{-1,0}$$
$$- 256/15\, \mathrm{H}_{-1,-1,0} + 128/5\, \mathrm{H}_{-1,0,0} \Big\} + x^{-1} \Big\{ 6688/225 + 128/15\, \zeta_2$$
$$+ 64/3\, \mathrm{H}_{-1}\, \zeta_2 - 2848/225\, \mathrm{H}_0 - 128/3\, \mathrm{H}_{-2,0} - 2512/45\, \mathrm{H}_{-1,0} - 128/5\, \mathrm{H}_{0,0}$$

$$+ 128/3\,\mathrm{H}_{-1,-1,0} - 64\,\mathrm{H}_{-1,0,0}\Big\} + x\Big\{13472/225 + 128/5\,\zeta_2^2 + 1408/9\,\zeta_3$$
$$+ 6608/45\,\zeta_2 - 256/3\,\mathrm{H}_{-2}\,\zeta_2 - 256/3\,\mathrm{H}_{-1}\,\zeta_3 - 1408/9\,\mathrm{H}_{-1}\,\zeta_2$$
$$- 28256/225\,\mathrm{H}_0 + 128/3\,\mathrm{H}_0\,\zeta_3 + 32/3\,\mathrm{H}_0\,\zeta_2 + 128\,\mathrm{H}_{-3,0} + 1792/9\,\mathrm{H}_{-2,0}$$
$$+ 256/3\,\mathrm{H}_{-1,-1}\,\zeta_2 + 3872/15\,\mathrm{H}_{-1,0} - 128/3\,\mathrm{H}_{-1,0}\,\zeta_2 - 11984/45\,\mathrm{H}_{0,0}$$
$$- 512/3\,\mathrm{H}_{-2,-1,0} + 512/3\,\mathrm{H}_{-2,0,0} - 512/3\,\mathrm{H}_{-1,-2,0} - 2816/9\,\mathrm{H}_{-1,-1,0}$$
$$+ 2176/9\,\mathrm{H}_{-1,0,0} - 160/3\,\mathrm{H}_{0,0,0} + 512/3\,\mathrm{H}_{-1,-1,-1,0} - 256\,\mathrm{H}_{-1,-1,0,0}$$
$$+ 128\,\mathrm{H}_{-1,0,0,0}\Big\} + x^2\Big\{1168/25 + 256/3\,\zeta_3 + 6944/45\,\zeta_2 - 128/3\,\mathrm{H}_{-1}\,\zeta_2$$
$$+ 1168/25\,\mathrm{H}_0 + 128/3\,\mathrm{H}_0\,\zeta_2 + 256/3\,\mathrm{H}_{-2,0} + 6368/45\,\mathrm{H}_{-1,0} - 5792/45\,\mathrm{H}_{0,0}$$
$$- 256/3\,\mathrm{H}_{-1,-1,0} + 128\,\mathrm{H}_{-1,0,0} - 128\,\mathrm{H}_{0,0,0}\Big\} + x^3\Big\{ - 128/5\,\zeta_3 - 848/25\,\zeta_2$$
$$+ 64/5\,\mathrm{H}_{-1}\,\zeta_2 - 64/5\,\mathrm{H}_0\,\zeta_2 - 128/5\,\mathrm{H}_{-2,0} - 848/25\,\mathrm{H}_{-1,0} + 848/25\,\mathrm{H}_{0,0}$$
$$+ 128/5\,\mathrm{H}_{-1,-1,0} - 192/5\,\mathrm{H}_{-1,0,0} + 192/5\,\mathrm{H}_{0,0,0}\Big\} - 3408/25 - 544/15\,\zeta_2$$
$$- 128/3\,\mathrm{H}_{-1}\,\zeta_2 - 7216/225\,\mathrm{H}_0 + 256/3\,\mathrm{H}_{-2,0} + 6656/45\,\mathrm{H}_{-1,0} + 224/5\,\mathrm{H}_{0,0}$$
$$- 256/3\,\mathrm{H}_{-1,-1,0} + 128\,\mathrm{H}_{-1,0,0}\Big), \qquad \text{(A.46)}$$

$$\delta c_{3,ns}^{(3)}(x) =$$
$$+ C_F\,(C_A - 2\,C_F)\,C_F\Big((1+x)^{-1}\Big\{11200/3\,\mathrm{H}_{-3}\,\zeta_2 - 104\,\mathrm{H}_4 + 1120\,\mathrm{H}_5 - 1024\,\mathrm{H}_{-4,0}$$
$$+ 224/3\,\mathrm{H}_{-3,0} - 2944\,\mathrm{H}_{-3,2} - 6048\,\mathrm{H}_{-2,-1}\,\zeta_2 - 480\,\mathrm{H}_{-2,2} - 16096/3\,\mathrm{H}_{-2,3}$$
$$- 6016\,\mathrm{H}_{-1,-2}\,\zeta_2 - 800\,\mathrm{H}_{-1,3} - 4192\,\mathrm{H}_{-1,4} + 16\,\mathrm{H}_{2,2} - 192\,\mathrm{H}_{2,3} + 168\,\mathrm{H}_{3,0}$$
$$+ 256\,\mathrm{H}_{3,1} + 832/3\,\mathrm{H}_{3,2} + 1952/3\,\mathrm{H}_{4,0} + 864\,\mathrm{H}_{4,1} + 4736/3\,\mathrm{H}_{-3,-1,0}$$
$$- 8960/3\,\mathrm{H}_{-3,0,0} + 1280\,\mathrm{H}_{-2,-2,0} + 224\,\mathrm{H}_{-2,-1,0} + 5248\,\mathrm{H}_{-2,-1,2} - 32\,\mathrm{H}_{-2,0,0}$$
$$- 1536\,\mathrm{H}_{-2,2,0} - 5824/3\,\mathrm{H}_{-2,2,1} + 5248/3\,\mathrm{H}_{-1,-3,0} + 240\,\mathrm{H}_{-1,-2,0}$$
$$+ 5248\,\mathrm{H}_{-1,-2,2} + 8064\,\mathrm{H}_{-1,-1,-1}\,\zeta_2 + 768\,\mathrm{H}_{-1,-1,2} + 24896/3\,\mathrm{H}_{-1,-1,3}$$
$$- 384\,\mathrm{H}_{-1,2,0} - 512\,\mathrm{H}_{-1,2,1} - 576\,\mathrm{H}_{-1,2,2} - 5600/3\,\mathrm{H}_{-1,3,0} - 7232/3\,\mathrm{H}_{-1,3,1}$$
$$+ 128/3\,\mathrm{H}_{2,-2,0} - 16\,\mathrm{H}_{2,1,0} - 64/3\,\mathrm{H}_{2,1,2} + 128/3\,\mathrm{H}_{2,2,0} + 1024/3\,\mathrm{H}_{3,0,0}$$
$$+ 192\,\mathrm{H}_{3,1,0} + 192\,\mathrm{H}_{3,1,1} - 1600\,\mathrm{H}_{-2,-1,-1,0} + 5024\,\mathrm{H}_{-2,-1,0,0} - 3712\,\mathrm{H}_{-2,0,0,0}$$
$$- 1536\,\mathrm{H}_{-1,-2,-1,0} + 15040/3\,\mathrm{H}_{-1,-2,0,0} - 1472\,\mathrm{H}_{-1,-1,-2,0} - 576\,\mathrm{H}_{-1,-1,-1,0}$$
$$- 7296\,\mathrm{H}_{-1,-1,-1,2} + 480\,\mathrm{H}_{-1,-1,0,0} + 7936/3\,\mathrm{H}_{-1,-1,2,0} + 3328\,\mathrm{H}_{-1,-1,2,1}$$
$$+ 72\,\mathrm{H}_{-1,0,0,0} - 2432/3\,\mathrm{H}_{-1,2,0,0} - 1088/3\,\mathrm{H}_{-1,2,1,0} - 384\,\mathrm{H}_{-1,2,1,1}$$
$$- 1192/3\,\mathrm{H}_{0,0,0,0} - 128\,\mathrm{H}_{2,0,0,0} - 128\,\mathrm{H}_{2,1,0,0} + 64/3\,\mathrm{H}_{2,1,1,0}$$
$$+ 1536\,\mathrm{H}_{-1,-1,-1,-1,0} - 6848\,\mathrm{H}_{-1,-1,-1,0,0} + 17440/3\,\mathrm{H}_{-1,-1,0,0,0}$$
$$- 2528\,\mathrm{H}_{-1,0,0,0,0} + 544\,\mathrm{H}_{0,0,0,0,0} - 16\,(4 - 310\,\zeta_3 - 53\,\zeta_2)\,\mathrm{H}_{-1,0} + 32/3\,(6$$

$$+ 43\,\zeta_2)\,\mathrm{H}_{2,1} + 32/3\,(6 + 53\,\zeta_2)\,\mathrm{H}_{2,0} - 64\,(8 + 149\,\zeta_2)\,\mathrm{H}_{-1,-1,0} - 8/3\,(24$$
$$- 308\,\zeta_3 - 183\,\zeta_2)\,\mathrm{H}_2 - 32/3\,(31 + 41\,\zeta_2)\,\mathrm{H}_{-1,2} - 8/3\,(54 + 572\,\zeta_3$$
$$- 107\,\zeta_2)\,\mathrm{H}_{0,0} + 16/3\,(61 + 164\,\zeta_2)\,\mathrm{H}_3 + 16/3\,(79 + 894\,\zeta_2)\,\mathrm{H}_{-1,0,0} + 16/3\,(85$$
$$+ 1184\,\zeta_2)\,\mathrm{H}_{-2,0} - 4/3\,(239 + 904\,\zeta_2)\,\mathrm{H}_{0,0,0} - 4/15\,(1225 - 360\,\zeta_4 - 10\,\zeta_3$$
$$+ 555\,\zeta_2 + 916\,\zeta_2^2)\,\mathrm{H}_0 - 8/15\,(2480\,\zeta_5 + 75\,\zeta_3 - 60\,\zeta_2 - 3660\,\zeta_2\,\zeta_3 + 408\,\zeta_2^2)$$
$$- 96\,(85\,\zeta_3 + 11\,\zeta_2)\,\mathrm{H}_{-1,-1} + 16/3\,(1100\,\zeta_3 + 111\,\zeta_2)\,\mathrm{H}_{-2} + 8/15\,(2055\,\zeta_3$$
$$+ 140\,\zeta_2 - 476\,\zeta_2^2)\,\mathrm{H}_{-1}\Big\} + (1/x + x^2)\Big\{ - 432\,\mathrm{H}_{-1,-1}\,\zeta_2 + 336\,\mathrm{H}_{-1,0}\,\zeta_2$$
$$- 560/3\,\mathrm{H}_{-1,2} - 288\,\mathrm{H}_{-1,3} + 32\,\mathrm{H}_{-1,-2,0} + 2224/9\,\mathrm{H}_{-1,-1,0} + 416\,\mathrm{H}_{-1,-1,2}$$
$$- 1952/9\,\mathrm{H}_{-1,0,0} - 32\,\mathrm{H}_{-1,2,0} - 32\,\mathrm{H}_{-1,2,1} - 32\,\mathrm{H}_{-1,-1,-1,0} + 288\,\mathrm{H}_{-1,-1,0,0}$$
$$- 128\,\mathrm{H}_{-1,0,0,0} + 8/9\,(396\,\zeta_3 + 349\,\zeta_2)\,\mathrm{H}_{-1}\Big\} + (1/x - x^2)\Big\{176/3\,\mathrm{H}_{1,0}\,\zeta_2$$
$$+ 16\,\mathrm{H}_{1,1}\,\zeta_2 - 32/3\,\mathrm{H}_{1,3} + 64\,\mathrm{H}_{1,-2,0} - 32\,\mathrm{H}_{1,0,0,0} + 32/3\,\mathrm{H}_{1,1,0,0} + 56/3\,(2\,\zeta_3$$
$$+ 5\,\zeta_2)\,\mathrm{H}_1\Big\} + (1 - x)\Big\{ - 5728/3\,\mathrm{H}_{-3}\,\zeta_2 - 2476/3\,\mathrm{H}_4 - 1928/3\,\mathrm{H}_5$$
$$+ 2176/3\,\mathrm{H}_{-4,0} + 1040/3\,\mathrm{H}_{-3,0} + 4480/3\,\mathrm{H}_{-3,2} + 7648/3\,\mathrm{H}_{-2,-1}\,\zeta_2$$
$$+ 336\,\mathrm{H}_{-2,2} + 2528\,\mathrm{H}_{-2,3} + 2400\,\mathrm{H}_{-1,-2}\,\zeta_2 + 4624/3\,\mathrm{H}_{-1,3} + 1744\,\mathrm{H}_{-1,4}$$
$$+ 856/3\,\mathrm{H}_{1,2} + 64\,\mathrm{H}_{1,3} - 472/3\,\mathrm{H}_{2,2} + 32\,\mathrm{H}_{2,3} - 1912/3\,\mathrm{H}_{3,0} - 2624/3\,\mathrm{H}_{3,1}$$
$$- 128\,\mathrm{H}_{3,2} - 1072/3\,\mathrm{H}_{4,0} - 464\,\mathrm{H}_{4,1} - 832\,\mathrm{H}_{-3,-1,0} + 5056/3\,\mathrm{H}_{-3,0,0}$$
$$- 608\,\mathrm{H}_{-2,-2,0} - 1904/3\,\mathrm{H}_{-2,-1,0} - 6592/3\,\mathrm{H}_{-2,-1,2} + 2192/3\,\mathrm{H}_{-2,0,0}$$
$$+ 2368/3\,\mathrm{H}_{-2,2,0} + 3008/3\,\mathrm{H}_{-2,2,1} - 2144/3\,\mathrm{H}_{-1,-3,0} - 560/3\,\mathrm{H}_{-1,-2,0}$$
$$- 2112\,\mathrm{H}_{-1,-2,2} - 3040\,\mathrm{H}_{-1,-1,-1}\,\zeta_2 - 576\,\mathrm{H}_{-1,-1,2} - 10624/3\,\mathrm{H}_{-1,-1,3}$$
$$+ 2240/3\,\mathrm{H}_{-1,2,0} + 1024\,\mathrm{H}_{-1,2,1} + 288\,\mathrm{H}_{-1,2,2} + 2704/3\,\mathrm{H}_{-1,3,0}$$
$$+ 3520/3\,\mathrm{H}_{-1,3,1} - 704/3\,\mathrm{H}_{1,-2,0} + 592/3\,\mathrm{H}_{1,0,0} + 184\,\mathrm{H}_{1,1,0} + 192\,\mathrm{H}_{1,1,1}$$
$$- 32/3\,\mathrm{H}_{1,1,2} + 64/3\,\mathrm{H}_{1,2,0} + 128/3\,\mathrm{H}_{2,-2,0} - 992/3\,\mathrm{H}_{2,0,0} - 232/3\,\mathrm{H}_{2,1,0}$$
$$- 96\,\mathrm{H}_{2,1,1} + 32/3\,\mathrm{H}_{2,1,2} - 64/3\,\mathrm{H}_{2,2,0} - 160\,\mathrm{H}_{3,0,0} - 320/3\,\mathrm{H}_{3,1,0} - 96\,\mathrm{H}_{3,1,1}$$
$$+ 704\,\mathrm{H}_{-2,-1,-1,0} - 6944/3\,\mathrm{H}_{-2,-1,0,0} + 1968\,\mathrm{H}_{-2,0,0,0} + 576\,\mathrm{H}_{-1,-2,-1,0}$$
$$- 6080/3\,\mathrm{H}_{-1,-2,0,0} + 544\,\mathrm{H}_{-1,-1,-2,0} + 368\,\mathrm{H}_{-1,-1,-1,0} + 2752\,\mathrm{H}_{-1,-1,-1,2}$$
$$- 2248/3\,\mathrm{H}_{-1,-1,0,0} - 3776/3\,\mathrm{H}_{-1,-1,2,0} - 1600\,\mathrm{H}_{-1,-1,2,1} + 2552/3\,\mathrm{H}_{-1,0,0,0}$$
$$+ 1408/3\,\mathrm{H}_{-1,2,0,0} + 544/3\,\mathrm{H}_{-1,2,1,0} + 192\,\mathrm{H}_{-1,2,1,1} - 1604/3\,\mathrm{H}_{0,0,0,0}$$
$$+ 64\,\mathrm{H}_{1,0,0,0} - 224\,\mathrm{H}_{1,1,0,0} + 32/3\,\mathrm{H}_{1,1,1,0} + 32\,\mathrm{H}_{2,0,0,0} + 128\,\mathrm{H}_{2,1,0,0}$$
$$- 32/3\,\mathrm{H}_{2,1,1,0} - 576\,\mathrm{H}_{-1,-1,-1,-1,0} + 2624\,\mathrm{H}_{-1,-1,-1,0,0} - 7568/3\,\mathrm{H}_{-1,-1,0,0,0}$$
$$+ 1104\,\mathrm{H}_{-1,0,0,0,0} - 408\,\mathrm{H}_{0,0,0,0,0} - 8/3\,(11 - 1500\,\zeta_2)\,\mathrm{H}_{-1,-1,0} + 16/3\,(31$$
$$- 363\,\zeta_2)\,\mathrm{H}_{-1,0,0} - 8/3\,(37 + 64\,\zeta_2)\,\mathrm{H}_{2,0} - 8/3\,(83 + 80\,\zeta_2)\,\mathrm{H}_{2,1} - 8/3\,(163$$
$$- 265\,\zeta_2)\,\mathrm{H}_{0,0,0} + 8/3\,(169 + 70\,\zeta_2)\,\mathrm{H}_{-1,2} + 8/3\,(255 - 26\,\zeta_2)\,\mathrm{H}_{1,0} - 4/3\,(359$$

$$+ 2216\,\zeta_2)\,\mathrm{H}_{-2,0} + 8/3\,(387 + 62\,\zeta_2)\,\mathrm{H}_{1,1} - 4/3\,(617 + 324\,\zeta_2)\,\mathrm{H}_3 - 2/3\,(743$$
$$- 568\,\zeta_3 + 60\,\zeta_2)\,\mathrm{H}_1 + 4/9\,(1531 - 4824\,\zeta_3 - 3432\,\zeta_2)\,\mathrm{H}_{-1,0} + 2/3\,(2431$$
$$- 640\,\zeta_3 + 60\,\zeta_2)\,\mathrm{H}_2 + 2/9\,(4379 + 4464\,\zeta_3 + 3594\,\zeta_2)\,\mathrm{H}_{0,0} - 2/45\,(23995$$
$$+ 1080\,\zeta_4 - 27900\,\zeta_3 - 18990\,\zeta_2 - 3804\,\zeta_2^2)\,\mathrm{H}_0 + 2/45\,(34520 + 26970\,\zeta_5$$
$$+ 1080\,\zeta_4 + 17205\,\zeta_3 - 25900\,\zeta_2 - 23160\,\zeta_2\,\zeta_3 - 1329\,\zeta_2^2) + 8\,(408\,\zeta_3$$
$$+ 95\,\zeta_2)\,\mathrm{H}_{-1,-1} - 8/3\,(971\,\zeta_3 + 245\,\zeta_2)\,\mathrm{H}_{-2} - 4/15\,(4485\,\zeta_3 + 1745\,\zeta_2$$
$$- 122\,\zeta_2^2)\,\mathrm{H}_{-1}\Big\} - 400\,\mathrm{H}_{-3}\,\zeta_2 + 96\,\mathrm{H}_{-4,0} + 224\,\mathrm{H}_{-3,2} + 1248\,\mathrm{H}_{-2,-1}\,\zeta_2$$
$$+ 672\,\mathrm{H}_{-2,3} + 1216\,\mathrm{H}_{-1,-2}\,\zeta_2 - 6704/3\,\mathrm{H}_{-1,3} + 704\,\mathrm{H}_{-1,4} + 288\,\mathrm{H}_{2,2}$$
$$+ 128\,\mathrm{H}_{2,3} - 352\,\mathrm{H}_{-3,-1,0} + 352\,\mathrm{H}_{-3,0,0} - 320\,\mathrm{H}_{-2,-2,0} - 1024\,\mathrm{H}_{-2,-1,2}$$
$$+ 64\,\mathrm{H}_{-2,2,0} + 64\,\mathrm{H}_{-2,2,1} - 320\,\mathrm{H}_{-1,-3,0} - 32/3\,\mathrm{H}_{-1,-2,0} - 1024\,\mathrm{H}_{-1,-2,2}$$
$$- 1984\,\mathrm{H}_{-1,-1,-1}\,\zeta_2 + 288\,\mathrm{H}_{-1,-1,2} - 1216\,\mathrm{H}_{-1,-1,3} - 3328/3\,\mathrm{H}_{-1,2,0}$$
$$- 1536\,\mathrm{H}_{-1,2,1} + 64\,\mathrm{H}_{-1,3,0} + 64\,\mathrm{H}_{-1,3,1} - 128\,\mathrm{H}_{2,-2,0} + 544/3\,\mathrm{H}_{2,1,0}$$
$$+ 192\,\mathrm{H}_{2,1,1} + 448\,\mathrm{H}_{-2,-1,-1,0} - 992\,\mathrm{H}_{-2,-1,0,0} + 448\,\mathrm{H}_{-2,0,0,0}$$
$$+ 384\,\mathrm{H}_{-1,-2,-1,0} - 960\,\mathrm{H}_{-1,-2,0,0} + 384\,\mathrm{H}_{-1,-1,-2,0} + 128/3\,\mathrm{H}_{-1,-1,-1,0}$$
$$+ 1792\,\mathrm{H}_{-1,-1,-1,2} + 784\,\mathrm{H}_{-1,-1,0,0} - 128\,\mathrm{H}_{-1,-1,2,0} - 128\,\mathrm{H}_{-1,-1,2,1}$$
$$- 5152/3\,\mathrm{H}_{-1,0,0,0} - 128\,\mathrm{H}_{-1,2,0,0} + 64\,\mathrm{H}_{2,0,0,0} - 128\,\mathrm{H}_{2,1,0,0}$$
$$- 384\,\mathrm{H}_{-1,-1,-1,-1,0} + 1600\,\mathrm{H}_{-1,-1,-1,0,0} - 768\,\mathrm{H}_{-1,-1,0,0,0} + 320\,\mathrm{H}_{-1,0,0,0,0}$$
$$+ 16/3\,(12\,x^{-1} + 65 + 6\,x^2)\,\mathrm{H}_{-2,-1,0} - 16/3\,(12\,x^{-1} + 157 + 78\,x^2)\,\mathrm{H}_{-2,2}$$
$$- 16/3\,(12\,x^{-1} + 211 + 54\,x^2)\,\mathrm{H}_{-2,0,0} - 4/9\,(231\,x^{-1} + 2948 + 93\,x^2 + 1512\,\zeta_3$$
$$- 4872\,\zeta_2)\,\mathrm{H}_{-1,0} + 32\,(17 + 48\,\zeta_2)\,\mathrm{H}_{-1,-1,0} + 32/3\,(41 - x^2)\,\mathrm{H}_{2,0,0} + 16\,(53$$
$$- 2\,\zeta_2)\,\mathrm{H}_{2,1} + 16\,(61 + 10\,x^2)\,\mathrm{H}_{0,0,0,0} - 16/3\,(85 + 18\,x^2)\,\mathrm{H}_{-3,0} - 8\,(93$$
$$+ 112\,\zeta_2)\,\mathrm{H}_{-1,0,0} + 32/3\,(107 + 3\,x^2)\,\mathrm{H}_{3,1} - 16/3\,(107 - 12\,\zeta_2)\,\mathrm{H}_{-1,2}$$
$$+ 32/3\,(151 + 28\,x^2)\,\mathrm{H}_4 + 16/3\,(157 + 6\,x^2)\,\mathrm{H}_{3,0} - 16/9\,(159 + 139\,x^2$$
$$+ 504\,\zeta_2)\,\mathrm{H}_{-2,0} + 8/3\,(191 - 84\,\zeta_2)\,\mathrm{H}_{2,0} + 8/3\,(435 + 70\,x^2 - 42\,\zeta_2)\,\mathrm{H}_3$$
$$- 4/3\,(1033 - 24\,\zeta_3 - 24\,\zeta_2\,x^{-1} + 238\,\zeta_2 + 12\,\zeta_2\,x^2)\,\mathrm{H}_2 - 4/9\,(1169 - 93\,x^2$$
$$- 72\,\zeta_3 + 4008\,\zeta_2 + 888\,\zeta_2\,x^2)\,\mathrm{H}_{0,0} + 4/9\,(1785 + 488\,x^2)\,\mathrm{H}_{0,0,0} + 2/9\,(9043$$
$$- 8016\,\zeta_3 - 1752\,\zeta_3\,x^2 - 6384\,\zeta_2 - 1816\,\zeta_2\,x^2 + 36\,\zeta_2^2)\,\mathrm{H}_0 - 4/45\,(5760\,\zeta_5$$
$$+ 14400\,\zeta_3 + 5930\,\zeta_3\,x^2 - 7780\,\zeta_2 + 465\,\zeta_2\,x^2 + 2700\,\zeta_2\,\zeta_3 - 2988\,\zeta_2^2$$
$$- 582\,\zeta_2^2\,x^2) + 32/3\,(153\,\zeta_3 - 25\,\zeta_2)\,\mathrm{H}_{-1,-1} - 8/3\,(390\,\zeta_3 - 36\,\zeta_2\,x^{-1} - 379\,\zeta_2$$
$$- 162\,\zeta_2\,x^2)\,\mathrm{H}_{-2} + 8/15\,(2105\,\zeta_3 + 1580\,\zeta_2 + 354\,\zeta_2^2)\,\mathrm{H}_{-1}$$
$$+ \big(g_1^E(x) + g_2^E(x) - g_1^O(x) - g_2^O(x)\big)\,\Big\{ - 2/3 \Big\} + \delta(1-x)\,\Big\{ 4/3\,\zeta_3 \Big\}\Big)$$
$$+ C_F\,(C_A - 2\,C_F)\,C_A\Big((1+x)^{-1}\Big\{ - 4096/3\,\mathrm{H}_{-3}\,\zeta_2 + 3296/9\,\mathrm{H}_4 - 320\,\mathrm{H}_5$$

$$
\begin{aligned}
&+ 320/3\,\mathrm{H}_{-4,0} - 7840/9\,\mathrm{H}_{-3,0} + 1408\,\mathrm{H}_{-3,2} + 2624\,\mathrm{H}_{-2,-1}\,\zeta_2 - 704/9\,\mathrm{H}_{-2,2} \\
&+ 6944/3\,\mathrm{H}_{-2,3} + 2592\,\mathrm{H}_{-1,-2}\,\zeta_2 - 704/9\,\mathrm{H}_{-1,3} + 1536\,\mathrm{H}_{-1,4} - 1024/3\,\mathrm{H}_{2,0}\,\zeta_2 \\
&- 256\,\mathrm{H}_{2,1}\,\zeta_2 + 448/3\,\mathrm{H}_{2,3} - 704/9\,\mathrm{H}_{3,1} - 128/3\,\mathrm{H}_{3,2} - 128\,\mathrm{H}_{4,0} - 640/3\,\mathrm{H}_{4,1} \\
&+ 256/3\,\mathrm{H}_{-3,-1,0} + 2240/3\,\mathrm{H}_{-3,0,0} + 128/3\,\mathrm{H}_{-2,-2,0} + 7744/9\,\mathrm{H}_{-2,-1,0} \\
&- 2624\,\mathrm{H}_{-2,-1,2} - 12256/9\,\mathrm{H}_{-2,0,0} + 1856/3\,\mathrm{H}_{-2,2,0} + 2560/3\,\mathrm{H}_{-2,2,1} \\
&- 256/3\,\mathrm{H}_{-1,-3,0} + 7744/9\,\mathrm{H}_{-1,-2,0} - 2624\,\mathrm{H}_{-1,-2,2} - 3840\,\mathrm{H}_{-1,-1,-1}\,\zeta_2 \\
&- 3968\,\mathrm{H}_{-1,-1,3} + 1408/9\,\mathrm{H}_{-1,2,1} + 320/3\,\mathrm{H}_{-1,2,2} + 1568/3\,\mathrm{H}_{-1,3,0} \\
&+ 2560/3\,\mathrm{H}_{-1,3,1} - 128/3\,\mathrm{H}_{2,-2,0} + 32\,\mathrm{H}_{2,0,0} + 64/3\,\mathrm{H}_{2,1,2} - 128/3\,\mathrm{H}_{2,2,0} \\
&- 320/3\,\mathrm{H}_{3,0,0} + 128/3\,\mathrm{H}_{3,1,0} - 4256/3\,\mathrm{H}_{-2,-1,0,0} + 2848/3\,\mathrm{H}_{-2,0,0,0} \\
&- 64\,\mathrm{H}_{-1,-2,-1,0} - 1408\,\mathrm{H}_{-1,-2,0,0} - 128\,\mathrm{H}_{-1,-1,-2,0} - 7040/9\,\mathrm{H}_{-1,-1,-1,0} \\
&+ 3840\,\mathrm{H}_{-1,-1,-1,2} + 11968/9\,\mathrm{H}_{-1,-1,0,0} - 3584/3\,\mathrm{H}_{-1,-1,2,0} \\
&- 5120/3\,\mathrm{H}_{-1,-1,2,1} - 11824/9\,\mathrm{H}_{-1,0,0,0} + 896/3\,\mathrm{H}_{-1,2,0,0} - 320/3\,\mathrm{H}_{-1,2,1,0} \\
&+ 5944/9\,\mathrm{H}_{0,0,0,0} + 256/3\,\mathrm{H}_{2,0,0,0} + 256/3\,\mathrm{H}_{2,1,0,0} - 64/3\,\mathrm{H}_{2,1,1,0} \\
&+ 2176\,\mathrm{H}_{-1,-1,-1,0,0} - 4864/3\,\mathrm{H}_{-1,-1,0,0,0} + 1856/3\,\mathrm{H}_{-1,0,0,0,0} - 128\,\mathrm{H}_{0,0,0,0,0} \\
&+ 32/27\,(17 - 396\,\zeta_2)\,\mathrm{H}_3 - 64/27\,(17 - 144\,\zeta_2)\,\mathrm{H}_{-1,2} + 32/3\,(21 - 46\,\zeta_3 \\
&- 27\,\zeta_2)\,\mathrm{H}_2 + 128/9\,(67 + 312\,\zeta_2)\,\mathrm{H}_{-1,-1,0} + 32/27\,(463 + 288\,\zeta_2)\,\mathrm{H}_{0,0,0} \\
&- 32/27\,(679 + 2277\,\zeta_2)\,\mathrm{H}_{-2,0} - 32/27\,(973 + 1548\,\zeta_2)\,\mathrm{H}_{-1,0,0} - 16/81\,(2129 \\
&+ 8964\,\zeta_3 - 1305\,\zeta_2)\,\mathrm{H}_{-1,0} + 8/81\,(6125 + 3888\,\zeta_3 - 5229\,\zeta_2)\,\mathrm{H}_{0,0} \\
&+ 4/135\,(17805 - 3240\,\zeta_4 - 16620\,\zeta_3 - 5440\,\zeta_2 + 4176\,\zeta_2^2)\,\mathrm{H}_0 \\
&+ 4/81\,(6372\,\zeta_5 - 10260\,\zeta_3 - 8794\,\zeta_2 - 14040\,\zeta_2\,\zeta_3 + 1107\,\zeta_2^2) + 64/9\,(543\,\zeta_3 \\
&- 55\,\zeta_2)\,\mathrm{H}_{-1,-1} - 32/9\,(687\,\zeta_3 - 143\,\zeta_2)\,\mathrm{H}_{-2} + 16/135\,(2640\,\zeta_3 + 4360\,\zeta_2 \\
&+ 423\,\zeta_2^2)\,\mathrm{H}_{-1} \Big\} + (1/x + x^2)\Big\{ 256\,\mathrm{H}_{-1,-1}\,\zeta_2 - 160\,\mathrm{H}_{-1,0}\,\zeta_2 + 152/3\,\mathrm{H}_{-1,2} \\
&+ 128\,\mathrm{H}_{-1,3} - 80/9\,\mathrm{H}_{-1,-1,0} - 256\,\mathrm{H}_{-1,-1,2} - 788/9\,\mathrm{H}_{-1,0,0} - 128\,\mathrm{H}_{-1,-1,0,0} \\
&+ 32\,\mathrm{H}_{-1,0,0,0} - 16/9\,(108\,\zeta_3 + 31\,\zeta_2)\,\mathrm{H}_{-1} \Big\} + (1/x - x^2)\Big\{ - 112/3\,\mathrm{H}_{1,0}\,\zeta_2 \\
&+ 16/3\,\mathrm{H}_{1,3} - 64\,\mathrm{H}_{1,-2,0} + 32\,\mathrm{H}_{1,0,0,0} - 16/3\,\mathrm{H}_{1,1,0,0} - 4/3\,(20\,\zeta_3 + 19\,\zeta_2)\,\mathrm{H}_1 \Big\} \\
&+ (1-x)\Big\{ 728\,\mathrm{H}_{-3}\,\zeta_2 + 464/9\,\mathrm{H}_4 + 496/3\,\mathrm{H}_5 - 208/3\,\mathrm{H}_{-4,0} + 3776/9\,\mathrm{H}_{-3,0} \\
&- 2144/3\,\mathrm{H}_{-3,2} - 3328/3\,\mathrm{H}_{-2,-1}\,\zeta_2 + 2176/9\,\mathrm{H}_{-2,2} - 1048\,\mathrm{H}_{-2,3} \\
&- 1040\,\mathrm{H}_{-1,-2}\,\zeta_2 - 4832/9\,\mathrm{H}_{-1,3} - 608\,\mathrm{H}_{-1,4} - 128/3\,\mathrm{H}_{1,2} - 16/3\,\mathrm{H}_{1,3} \\
&+ 32\,\mathrm{H}_{2,2} - 128/3\,\mathrm{H}_{2,3} + 616/3\,\mathrm{H}_{3,0} + 3232/9\,\mathrm{H}_{3,1} + 16\,\mathrm{H}_{3,2} + 64\,\mathrm{H}_{4,0} \\
&+ 320/3\,\mathrm{H}_{4,1} + 80/3\,\mathrm{H}_{-3,-1,0} - 1288/3\,\mathrm{H}_{-3,0,0} + 16\,\mathrm{H}_{-2,-2,0} - 992/9\,\mathrm{H}_{-2,-1,0} \\
&+ 3248/3\,\mathrm{H}_{-2,-1,2} + 5024/9\,\mathrm{H}_{-2,0,0} - 304\,\mathrm{H}_{-2,2,0} - 1280/3\,\mathrm{H}_{-2,2,1}
\end{aligned}
$$

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\begin{aligned}
&- 64/3\,\mathrm{H}_{-1,-3,0} - 2576/9\,\mathrm{H}_{-1,-2,0} + 1056\,\mathrm{H}_{-1,-2,2} + 1408\,\mathrm{H}_{-1,-1,-1}\,\zeta_2 \\
&+ 128\,\mathrm{H}_{-1,-1,2} + 1728\,\mathrm{H}_{-1,-1,3} - 896/3\,\mathrm{H}_{-1,2,0} - 4544/9\,\mathrm{H}_{-1,2,1} \\
&- 160/3\,\mathrm{H}_{-1,2,2} - 784/3\,\mathrm{H}_{-1,3,0} - 1280/3\,\mathrm{H}_{-1,3,1} + 704/3\,\mathrm{H}_{1,-2,0} - 16\,\mathrm{H}_{1,0,0} \\
&+ 128/3\,\mathrm{H}_{1,1,0} + 32/3\,\mathrm{H}_{1,1,2} - 64/3\,\mathrm{H}_{1,2,0} - 128/3\,\mathrm{H}_{2,-2,0} + 128\,\mathrm{H}_{2,0,0} \\
&- 32\,\mathrm{H}_{2,1,0} - 32/3\,\mathrm{H}_{2,1,2} + 64/3\,\mathrm{H}_{2,2,0} + 160/3\,\mathrm{H}_{3,0,0} - 16\,\mathrm{H}_{3,1,0} \\
&- 160/3\,\mathrm{H}_{-2,-1,-1,0} + 1976/3\,\mathrm{H}_{-2,-1,0,0} - 480\,\mathrm{H}_{-2,0,0,0} + 32\,\mathrm{H}_{-1,-2,-1,0} \\
&+ 576\,\mathrm{H}_{-1,-2,0,0} + 64\,\mathrm{H}_{-1,-1,-2,0} + 2128/9\,\mathrm{H}_{-1,-1,-1,0} - 1408\,\mathrm{H}_{-1,-1,-1,2} \\
&- 2360/9\,\mathrm{H}_{-1,-1,0,0} + 1792/3\,\mathrm{H}_{-1,-1,2,0} + 2560/3\,\mathrm{H}_{-1,-1,2,1} \\
&+ 2720/9\,\mathrm{H}_{-1,0,0,0} - 544/3\,\mathrm{H}_{-1,2,0,0} + 160/3\,\mathrm{H}_{-1,2,1,0} - 2600/9\,\mathrm{H}_{0,0,0,0} \\
&- 256/3\,\mathrm{H}_{1,0,0,0} + 368/3\,\mathrm{H}_{1,1,0,0} - 32/3\,\mathrm{H}_{1,1,1,0} - 32/3\,\mathrm{H}_{2,0,0,0} - 224/3\,\mathrm{H}_{2,1,0,0} \\
&+ 32/3\,\mathrm{H}_{2,1,1,0} - 832\,\mathrm{H}_{-1,-1,-1,0,0} + 2240/3\,\mathrm{H}_{-1,-1,0,0,0} - 736/3\,\mathrm{H}_{-1,0,0,0,0} \\
&+ 80\,\mathrm{H}_{0,0,0,0,0} + 160/3\,(3 + 2\,\zeta_2)\,\mathrm{H}_{2,0} - 56/3\,(13 - 2\,\zeta_2)\,\mathrm{H}_{1,0} - 64/9\,(71 \\
&+ 18\,\zeta_2)\,\mathrm{H}_{1,1} + 32/9\,(71 + 36\,\zeta_2)\,\mathrm{H}_{2,1} - 8/27\,(235 + 576\,\zeta_2)\,\mathrm{H}_{-1,2} - 8/9\,(461 \\
&+ 2136\,\zeta_2)\,\mathrm{H}_{-1,-1,0} - 28/81\,(605 + 702\,\zeta_3 - 126\,\zeta_2)\,\mathrm{H}_{0,0} - 4/27\,(644 \\
&+ 1332\,\zeta_3 + 891\,\zeta_2)\,\mathrm{H}_{1} + 16/27\,(793 + 405\,\zeta_2)\,\mathrm{H}_{3} + 16/27\,(961 \\
&+ 1170\,\zeta_2)\,\mathrm{H}_{-1,0,0} - 8/27\,(1250 + 603\,\zeta_2)\,\mathrm{H}_{0,0,0} + 16/27\,(1270 \\
&+ 2061\,\zeta_2)\,\mathrm{H}_{-2,0} - 8/81\,(1822 - 486\,\zeta_4 + 153\,\zeta_3 + 3072\,\zeta_2 + 459\,\zeta_2^2)\,\mathrm{H}_{0} \\
&- 8/27\,(2227 - 828\,\zeta_3 + 27\,\zeta_2)\,\mathrm{H}_{2} + 4/81\,(9655 + 14040\,\zeta_3 + 8406\,\zeta_2)\,\mathrm{H}_{-1,0} \\
&- 2/405\,(185225 + 36180\,\zeta_5 + 9720\,\zeta_4 + 42840\,\zeta_3 - 114350\,\zeta_2 - 62100\,\zeta_2\,\zeta_3 \\
&- 13446\,\zeta_2^2) + 8/9\,(1215\,\zeta_3 - 334\,\zeta_2)\,\mathrm{H}_{-2} - 8/9\,(1740\,\zeta_3 + 11\,\zeta_2)\,\mathrm{H}_{-1,-1} \\
&+ 4/135\,(8850\,\zeta_3 - 4565\,\zeta_2 - 1062\,\zeta_2^2)\,\mathrm{H}_{-1} \Big\} - 512\,\mathrm{H}_{-2,-1}\,\zeta_2 - 256\,\mathrm{H}_{-2,3} \\
&- 512\,\mathrm{H}_{-1,-2}\,\zeta_2 + 1616/9\,\mathrm{H}_{-1,2} + 1152\,\mathrm{H}_{-1,3} - 320\,\mathrm{H}_{-1,4} - 4544/9\,\mathrm{H}_{2,1} \\
&- 160/3\,\mathrm{H}_{2,2} - 64\,\mathrm{H}_{2,3} - 808/3\,\mathrm{H}_{3,0} - 1280/3\,\mathrm{H}_{3,1} - 1168/3\,\mathrm{H}_{-2,-1,0} \\
&+ 512\,\mathrm{H}_{-2,-1,2} + 128\,\mathrm{H}_{-1,-3,0} - 304\,\mathrm{H}_{-1,-2,0} + 512\,\mathrm{H}_{-1,-2,2} \\
&+ 1024\,\mathrm{H}_{-1,-1,-1}\,\zeta_2 - 256\,\mathrm{H}_{-1,-1,2} + 512\,\mathrm{H}_{-1,-1,3} + 1792/3\,\mathrm{H}_{-1,2,0} \\
&+ 2560/3\,\mathrm{H}_{-1,2,1} + 128\,\mathrm{H}_{2,-2,0} + 160/3\,\mathrm{H}_{2,1,0} + 256\,\mathrm{H}_{-2,-1,0,0} - 64\,\mathrm{H}_{-2,0,0,0} \\
&+ 256\,\mathrm{H}_{-1,-2,0,0} + 944/3\,\mathrm{H}_{-1,-1,-1,0} - 1024\,\mathrm{H}_{-1,-1,-1,2} - 2504/3\,\mathrm{H}_{-1,-1,0,0} \\
&+ 2176/3\,\mathrm{H}_{-1,0,0,0} + 64\,\mathrm{H}_{-1,2,0,0} - 64\,\mathrm{H}_{2,0,0,0} + 64\,\mathrm{H}_{2,1,0,0} - 512\,\mathrm{H}_{-1,-1,-1,0,0} \\
&+ 128\,\mathrm{H}_{-1,-1,0,0,0} - 128\,\mathrm{H}_{-1,0,0,0,0} - 8/9\,(132\,x^{-1} + 113 - 10\,x^2 \\
&- 360\,\zeta_2)\,\mathrm{H}_{-2,0} - 4/27\,(1317\,x^{-1} + 3544 + 1371\,x^2 - 2592\,\zeta_3 + 7362\,\zeta_2)\,\mathrm{H}_{-1,0} \\
&- 320/3\,(1 + 6\,\zeta_2)\,\mathrm{H}_{-1,-1,0} + 56/9\,(1 + 72\,\zeta_2)\,\mathrm{H}_{-1,0,0} - 32\,(7 + 2\,x^2)\,\mathrm{H}_{0,0,0,0} \\
&+ 16\,(29 + 16\,x^2)\,\mathrm{H}_{-2,2} - 16/3\,(31 - x^2)\,\mathrm{H}_{2,0,0} - 4/9\,(47 - 457\,x^2 - 1464\,\zeta_2
\end{aligned}
$$

$$- 444\,\zeta_2\,x^2)\,\mathrm{H}_{0,0} + 16/3\,(61 + 12\,x^2)\,\mathrm{H}_{-3,0} + 32/3\,(67 + 12\,x^2)\,\mathrm{H}_{-2,0,0}$$
$$- 8/3\,(107 - 48\,\zeta_2)\,\mathrm{H}_{2,0} - 16/3\,(109 + 25\,x^2)\,\mathrm{H}_4 - 8/3\,(216 + 19\,x^2)\,\mathrm{H}_3$$
$$+ 4/9\,(624 + 197\,x^2)\,\mathrm{H}_{0,0,0} + 8/27\,(1061 + 306\,\zeta_2)\,\mathrm{H}_2 - 4/81\,(16481$$
$$- 14742\,\zeta_3 - 4428\,\zeta_3\,x^2 - 9450\,\zeta_2 - 1629\,\zeta_2\,x^2)\,\mathrm{H}_0 + 4/45\,(1260\,\zeta_5 + 5405\,\zeta_3$$
$$+ 955\,\zeta_3\,x^2 - 6445\,\zeta_2 - 2285\,\zeta_2\,x^2 + 1620\,\zeta_2\,\zeta_3 - 396\,\zeta_2^2 + 312\,\zeta_2^2\,x^2)$$
$$+ 8/3\,(144\,\zeta_3 - 247\,\zeta_2 - 96\,\zeta_2\,x^2)\,\mathrm{H}_{-2} - 8/3\,(288\,\zeta_3 - 155\,\zeta_2)\,\mathrm{H}_{-1,-1}$$
$$- 8/45\,(4725\,\zeta_3 + 1310\,\zeta_2 - 72\,\zeta_2^2)\,\mathrm{H}_{-1} + \big(g_1^E(x) + g_2^E(x) - g_1^O(x) - g_2^O(x)\big)\,\Big\{$$
$$1/3\Big\} + \delta(1-x)\Big\{ - 2/3\,\zeta_3\Big\}\Big)$$
$$+ C_F\,(C_A - 2\,C_F)\,n_f\Big(+ (1+x)^{-1}\Big\{ - 832/9\,\mathrm{H}_{-2}\,\zeta_2 - 640/27\,\mathrm{H}_3 - 128/9\,\mathrm{H}_4$$
$$+ 1600/9\,\mathrm{H}_{-3,0} + 3200/27\,\mathrm{H}_{-2,0} + 128/9\,\mathrm{H}_{-2,2} + 640/9\,\mathrm{H}_{-1,-1}\,\zeta_2$$
$$+ 1280/27\,\mathrm{H}_{-1,2} + 128/9\,\mathrm{H}_{-1,3} + 128/9\,\mathrm{H}_{3,1} - 1408/9\,\mathrm{H}_{-2,-1,0}$$
$$+ 2176/9\,\mathrm{H}_{-2,0,0} - 1408/9\,\mathrm{H}_{-1,-2,0} - 1280/9\,\mathrm{H}_{-1,-1,0} + 5120/27\,\mathrm{H}_{-1,0,0}$$
$$- 256/9\,\mathrm{H}_{-1,2,1} - 1328/27\,\mathrm{H}_{0,0,0} + 1280/9\,\mathrm{H}_{-1,-1,-1,0} - 2176/9\,\mathrm{H}_{-1,-1,0,0}$$
$$+ 2176/9\,\mathrm{H}_{-1,0,0,0} - 736/9\,\mathrm{H}_{0,0,0,0} + 64/81\,(83 - 63\,\zeta_2)\,\mathrm{H}_{-1,0} - 16/27\,(87$$
$$- 66\,\zeta_3 - 23\,\zeta_2)\,\mathrm{H}_0 - 32/81\,(191 - 45\,\zeta_2)\,\mathrm{H}_{0,0} - 128/27\,(12\,\zeta_3 + 25\,\zeta_2)\,\mathrm{H}_{-1}$$
$$+ 32/405\,(1350\,\zeta_3 + 415\,\zeta_2 + 513\,\zeta_2^2)\Big\} + (1/x + x^2)\Big\{ - 32/3\,\mathrm{H}_{-1}\,\zeta_2$$
$$+ 64/3\,\mathrm{H}_{-2,0} + 376/9\,\mathrm{H}_{-1,0} - 64/3\,\mathrm{H}_{-1,-1,0} + 32\,\mathrm{H}_{-1,0,0}\Big\} + (1-x)\Big\{$$
$$224/9\,\mathrm{H}_{-2}\,\zeta_2 - 976/27\,\mathrm{H}_1 + 656/27\,\mathrm{H}_2 + 608/27\,\mathrm{H}_3 + 64/9\,\mathrm{H}_4 - 512/9\,\mathrm{H}_{-3,0}$$
$$- 1120/27\,\mathrm{H}_{-2,0} - 64/9\,\mathrm{H}_{-2,2} - 128/9\,\mathrm{H}_{-1,-1}\,\zeta_2 - 1024/27\,\mathrm{H}_{-1,2}$$
$$- 64/9\,\mathrm{H}_{-1,3} + 128/9\,\mathrm{H}_{1,1} - 64/9\,\mathrm{H}_{2,1} - 64/9\,\mathrm{H}_{3,1} + 320/9\,\mathrm{H}_{-2,-1,0}$$
$$- 704/9\,\mathrm{H}_{-2,0,0} + 320/9\,\mathrm{H}_{-1,-2,0} + 256/9\,\mathrm{H}_{-1,-1,0} - 2944/27\,\mathrm{H}_{-1,0,0}$$
$$+ 128/9\,\mathrm{H}_{-1,2,1} + 1456/27\,\mathrm{H}_{0,0,0} - 256/9\,\mathrm{H}_{-1,-1,-1,0} + 512/9\,\mathrm{H}_{-1,-1,0,0}$$
$$- 800/9\,\mathrm{H}_{-1,0,0,0} + 368/9\,\mathrm{H}_{0,0,0,0} + 16/81\,(119 - 45\,\zeta_2)\,\mathrm{H}_{0,0} - 8/81\,(341$$
$$+ 90\,\zeta_3 + 96\,\zeta_2)\,\mathrm{H}_0 - 8/81\,(857 - 144\,\zeta_2)\,\mathrm{H}_{-1,0} + 8/405\,(3350 - 900\,\zeta_3$$
$$- 2315\,\zeta_2 - 702\,\zeta_2^2) + 64/27\,(3\,\zeta_3 + 22\,\zeta_2)\,\mathrm{H}_{-1}\Big\} + 128/3\,\mathrm{H}_{-2}\,\zeta_2$$
$$- 1312/27\,\mathrm{H}_2 - 64/3\,\mathrm{H}_3 - 64\,\mathrm{H}_{-3,0} - 128/9\,\mathrm{H}_{-2,0} - 128/3\,\mathrm{H}_{-1,-1}\,\zeta_2$$
$$+ 256/9\,\mathrm{H}_{-1,2} + 128/9\,\mathrm{H}_{2,1} + 256/3\,\mathrm{H}_{-2,-1,0} - 256/3\,\mathrm{H}_{-2,0,0}$$
$$+ 256/3\,\mathrm{H}_{-1,-2,0} + 256/3\,\mathrm{H}_{-1,-1,0} + 256/9\,\mathrm{H}_{-1,0,0} - 256/3\,\mathrm{H}_{-1,-1,-1,0}$$
$$+ 128\,\mathrm{H}_{-1,-1,0,0} - 64\,\mathrm{H}_{-1,0,0,0} - 16/3\,(19 + 6\,x^2)\,\mathrm{H}_{0,0,0} - 8/9\,(63 + 47\,x^2)\,\mathrm{H}_{0,0}$$
$$+ 16/27\,(175 + 36\,\zeta_2)\,\mathrm{H}_{-1,0} + 16/81\,(458 - 108\,\zeta_3 + 135\,\zeta_2 + 54\,\zeta_2\,x^2)\,\mathrm{H}_0$$

$$+ 128/9\,(3\,\zeta_3 + \zeta_2)\,\mathrm{H}_{-1} - 8/45\,(160\,\zeta_3 - 120\,\zeta_3\,x^2 - 565\,\zeta_2 - 235\,\zeta_2\,x^2 + 72\,\zeta_2^2)\Big)\,. \tag{A.47}$$

A.8 Input Quantities for Small-x Resummation of Structure Functions

Here we give the input used for the structure function resummations, in addition to the functions already given in Eqs. (4.14) and (4.15).

A.8.1 Non-Singlet Input: $\hat{F}_{L,ns}$ and $\hat{F}_{3,ns}$

The input quantities for the resummation of $\hat{F}_{L,ns}$ and $\hat{F}_{3,ns}$ read:

$$\begin{aligned}
c_{L,ns}^{(1,0)} &= +\,4 - 4N + 4N^2\\
c_{L,ns}^{(1,1)} &= +\,4 - [4 - 4\zeta_2]N\\
c_{L,ns}^{(1,2)} &= +\,[8 - 2\zeta_2]\\
c_{L,ns}^{(2,0)} &= +\,8C_F N^{-2} + (12C_F - 4\beta_0)N^{-1} + \Big(\frac{40}{3}C_A + \frac{38}{3}\beta_0 - [74 + 8\zeta_2]C_F\Big)\\
c_{L,ns}^{(2,1)} &= -\,8C_F N^{-3} - (4C_F - 4\beta_0)N^{-2} - \Big(\frac{40}{3}C_A + \frac{50}{3}\beta_0 - [70 + 20\zeta_2]C_F\Big)N^{-1}\\
c_{L,ns}^{(3,0)} &= +\,40C_F^2 N^{-4} + (64C_F^2 - 36\beta_0 C_F)N^{-3} + \Big(\frac{112}{3}\beta_0 C_F - 120C_A^2\zeta_2 + 8\beta_0^2\\
&\quad - [168 + 416\zeta_2]C_F^2 + \Big[\frac{200}{3} + 384\zeta_2\Big]C_A C_F\Big)N^{-2}
\end{aligned} \tag{A.48}$$

$$\begin{aligned}
c_{3,ns}^{(1,0),-} &= +\,2N^{-2} + N^{-1} - [7 + 2\zeta_2]\\
c_{3,ns}^{(1,1),-} &= -\,2N^{-3} - N^{-2} + [1 + 3\zeta_2]N^{-1}\\
c_{3,ns}^{(1,2),-} &= +\,2N^{-4} + N^{-3} - [1 + 3\zeta_2]N^{-2}\\
c_{3,ns}^{(2,0),-} &= +\,10C_F N^{-4} + (10C_F - 5\beta_0)N^{-3} + (10C_A + 10\beta_0 - [33 + 24\zeta_2]C_F)N^{-2}\\
c_{3,ns}^{(2,1),-} &= -\,26C_F N^{-5} - (26C_F - 13\beta_0)N^{-4} - \Big(\frac{70}{3}C_A + \frac{68}{3}\beta_0 - [71 + 68\zeta_2]C_F\Big)N^{-3}\\
c_{3,ns}^{(3,0),-} &= +\,60C_F^2 N^{-6} + (90C_F^2 - \frac{182}{3}\beta_0 C_F)N^{-5} + \Big(\frac{143}{3}\beta_0 C_F - 120C_A^2\zeta_2 + \frac{46}{3}\beta_0^2\\
&\quad - [142 + 524\zeta_2]C_F^2 + \Big[\frac{260}{3} + 384\zeta_2\Big]C_A C_F\Big)N^{-4}
\end{aligned} \tag{A.49}$$

An overall factor of C_F has been omitted.

A.8.2 Singlet Input: $\hat{F}_{2,q}$, $\hat{F}_{2,g}$, $\hat{F}_{L,q}$, $\hat{F}_{L,g}$, $\hat{F}_{\phi,q}$ and $\hat{F}_{\phi,g}$

Here we show the input for the resummation of the singlet structure functions. In all cases we state beneath the functions if an overall colour factor has been omitted in the typesetting. The functions labelled ps should be added to the corresponding non-singlet parts of Eq. (4.14), Eq. (4.15), Eq. (A.48) to form the full singlet quantity.

$$\gamma^{(1)}_{qq,ps} = + 8N^{-3} + 4N^{-2} + 8N^{-1}$$

$$\gamma^{(2)}_{qq,ps} = + (64C_F - 64C_A)N^{-5} + \Big(\frac{16}{3}n_f + 24C_F - \frac{232}{3}C_A\Big)N^{-4} - \Big(\frac{232}{9}n_f + \Big[\frac{404}{9} + 8\zeta_2\Big]C_A - [160 - 96\zeta_2]C_F\Big)N^{-3} \qquad \text{(A.50)}$$

Here an overall factor of $C_F n_f$ has been omitted.

$$\gamma^{(0)}_{qg} = - 2N^{-1} + 2 - 3N$$

$$\gamma^{(1)}_{qg} = - (4C_F - 8C_A)N^{-3} + (6C_F + 4C_A)N^{-2} + (8C_A - [28 - 8\zeta_2]C_F)N^{-1}$$

$$\gamma^{(2)}_{qg} = + (32C_F n_f - 16C_F^2 + 32C_A C_F - 64C_A^2)N^{-5} - \Big(\frac{152}{3}C_F n_f - 12C_F^2 + \frac{16}{3}C_A n_f - \frac{44}{3}C_A C_F + \frac{56}{3}C_A^2\Big)N^{-4} + \Big(\frac{1370}{9}C_F n_f - \frac{64}{9}C_A n_f - [89 - 56\zeta_2]C_F^2 + \Big[\frac{1171}{9} - 96\zeta_2\Big]C_A C_F - \Big[\frac{1724}{9} - 12\zeta_2\Big]C_A^2\Big)N^{-3} \qquad \text{(A.51)}$$

Here an overall factor of n_f has been omitted.

$$\gamma^{(0)}_{gq} = + 4N^{-1} + 2 + 6N$$

$$\gamma^{(1)}_{gq} = + (8C_F - 16C_A)N^{-3} + (8C_F - 16C_A)N^{-2} - \Big(\frac{128}{9}n_f + 14C_F - \Big[\frac{332}{9} - 16\zeta_2\Big]C_A\Big)N^{-1}$$

$$\gamma^{(2)}_{gq} = - (64C_F n_f - 32C_F^2 + 64C_A C_F - 128C_A^2)N^{-5} + \Big(\frac{16}{3}C_F n_f + 48C_F^2 + \frac{32}{3}C_A n_f - \frac{376}{3}C_A C_F + \frac{400}{3}C_A^2\Big)N^{-4} - \Big(\frac{2380}{9}C_F n_f - \frac{992}{9}C_A n_f - \frac{2446}{9}C_A C_F - \Big[\frac{280}{9} + 104\zeta_2\Big]C_A^2 + [42 + 48\zeta_2]C_F^2\Big)N^{-3} \qquad \text{(A.52)}$$

Here and overall factor of C_F has been omitted.

$$\gamma^{(0)}_{gg} = + 4C_A N^{-1} + \Big(\frac{2}{3}n_f - \frac{5}{3}C_A\Big) + ([7 + 4\zeta_2]C_A)N$$

$$\gamma^{(1)}_{gg} = + (8C_F n_f - 16C_A^2)N^{-3} - \Big(12C_F n_f + \frac{8}{3}C_A n_f + \frac{4}{3}C_A^2\Big)N^{-2} + \Big(32C_F n_f - \frac{76}{9}C_A n_f - \Big[\frac{74}{9} + 16\zeta_2\Big]C_A^2\Big)N^{-1}$$

$$\gamma^{(2)}_{gg} = + (32C_F^2 n_f - 128C_A C_F n_f + 128C_A^3)N^{-5} - \Big(\frac{16}{3}C_F n_f^2 + 24C_F^2 n_f - \frac{232}{3}C_A C_F n_f - 32C_A^2 n_f - 16C_A^3\Big)N^{-4} - \Big(\frac{184}{9}C_F n_f^2 - \frac{16}{9}C_A n_f^2 - \Big[\frac{208}{3} + 24\zeta_2\Big]C_A^2 n_f - [120 - 32\zeta_2]C_F^2 n_f - \Big[\frac{2612}{9} + 160\zeta_2\Big]C_A^3 + \Big[\frac{3548}{9} + 96\zeta_2\Big]C_A C_F n_f\Big)N^{-3} \qquad \text{(A.53)}$$

$$
\begin{aligned}
c_{2,ps}^{(2,0)} &= -\,20N^{-4} - 2N^{-3} - [56 - 16\zeta_2]N^{-2} \\
c_{2,ps}^{(2,1)} &= +\,52N^{-5} + 2N^{-4} + [160 - 56\zeta_2]N^{-3} \\
c_{2,ps}^{(3,0)} &= -\,(240C_F - 240C_A)N^{-6} - \Big(\frac{368}{9}n_f + \frac{440}{3}C_F - \frac{3416}{9}C_A\Big)N^{-5} + \Big(\frac{1784}{27}n_f \\
&\quad - \Big[572 - \frac{1328}{3}\zeta_2\Big]C_F + \Big[\frac{16984}{27} - \frac{320}{3}\zeta_2\Big]C_A\Big)N^{-4} \qquad \text{(A.54)}
\end{aligned}
$$

Here an overall factor of $C_F n_f$ has been omitted.

$$
\begin{aligned}
c_{2,g}^{(1,0)} &= +\,2N^{-2} - 2N^{-1} + [6 - 2\zeta_2] \\
c_{2,g}^{(1,1)} &= -\,2N^{-3} + 2N^{-2} - [6 - 3\zeta_2]N^{-1} \\
c_{2,g}^{(1,2)} &= +\,2N^{-4} - 2N^{-3} + [6 - 3\zeta_2]N^{-2} \\
c_{2,g}^{(2,0)} &= +\,(10C_F - 20C_A)N^{-4} - (3C_F + 2C_A)N^{-3} + ([16 - 16\zeta_2]C_F \\
&\quad - [58 - 8\zeta_2]C_A)N^{-2} \\
c_{2,g}^{(2,1)} &= -\,(26C_F - 52C_A)N^{-5} + (3C_F + 2C_A)N^{-4} - ([20 - 44\zeta_2]C_F \\
&\quad - [166 - 32\zeta_2]C_A)N^{-3} \\
c_{2,g}^{(3,0)} &= -\,(120C_F n_f - 60C_F^2 + 120C_A C_F - 240C_A^2)N^{-6} + \Big(\frac{1636}{9}C_F n_f + \frac{44}{3}C_F^2 \\
&\quad - \frac{8}{9}C_A n_f - \frac{1636}{9}C_A C_F + \frac{1436}{9}C_A^2\Big)N^{-5} + \Big(\frac{532}{27}C_A n_f + \Big[\frac{178}{3} - \frac{524}{3}\zeta_2\Big]C_F^2 \\
&\quad - \Big[\frac{4589}{27} - \frac{656}{3}\zeta_2\Big]C_A C_F - \Big[\frac{17782}{27} - 88\zeta_2\Big]C_F n_f + \Big[\frac{27338}{27} - 56\zeta_2\Big]C_A^2\Big)N^{-4} \\
&\qquad \text{(A.55)}
\end{aligned}
$$

Here an overall factor of n_f has been omitted.

$$
\begin{aligned}
c_{\phi,q}^{(1,0)} &= -\,4N^{-2} - 4N^{-1} + [5 + 4\zeta_2] \\
c_{\phi,q}^{(1,1)} &= +\,4N^{-3} + 4N^{-2} + [1 - 6\zeta_2]N^{-1} \\
c_{\phi,q}^{(1,2)} &= -\,4N^{-4} - 4N^{-3} - [1 - 6\zeta_2]N^{-2} \\
c_{\phi,q}^{(2,0)} &= -\,(20C_F - 40C_A)N^{-4} + (12n_f + [16 - 16\zeta_2]C_A + [21 + 32\zeta_2]C_F)N^{-2} \\
&\quad - \Big(\frac{32}{3}n_f + 28C_F - \frac{344}{3}C_A\Big)N^{-3} \\
c_{\phi,q}^{(2,1)} &= +\,(52C_F - 104C_A)N^{-5} + (32n_f + 76C_F - 328C_A)N^{-4} - \Big(\frac{196}{9}n_f \\
&\quad + [25 + 104\zeta_2]C_F + \Big[\frac{1196}{9} - 80\zeta_2\Big]C_A\Big)N^{-3} \\
c_{\phi,q}^{(3,0)} &= +\,(240C_F n_f - 120C_F^2 + 240C_A C_F - 480C_A^2)N^{-6} - \Big(\frac{440}{9}C_F n_f + 224C_F^2 \\
&\quad - \frac{1072}{9}C_A n_f - \frac{8960}{9}C_A C_F + \frac{13960}{9}C_A^2\Big)N^{-5} - \Big(32n_f^2 - \frac{6592}{27}C_A n_f \\
&\quad - \Big[\frac{2338}{27} - \frac{1120}{3}\zeta_2\Big]C_A C_F - \Big[\frac{308}{3} + 328\zeta_2\Big]C_F^2 - \Big[\frac{17456}{27} - 176\zeta_2\Big]C_F n_f \\
&\quad + \Big[\frac{69928}{27} - \frac{208}{3}\zeta_2\Big]C_A^2\Big)N^{-4} \qquad \text{(A.56)}
\end{aligned}
$$

Here an overall factor of C_F has been omitted.

$$
\begin{aligned}
c_{\phi,g}^{(1,0)} &= -\,4C_AN^{-2} + \Big(\frac{2}{3}n_f - \frac{23}{3}C_A\Big)N^{-1} - \Big(\frac{16}{9}n_f - \Big[\frac{118}{9} + 4\zeta_2\Big]C_A\Big)\\
c_{\phi,g}^{(1,1)} &= +\,4C_AN^{-3} - \Big(\frac{2}{3}n_f - \frac{23}{3}C_A\Big)N^{-2} + \Big(\frac{16}{9}n_f - \Big[\frac{64}{9} + 6\zeta_2\Big]C_A\Big)N^{-1}\\
c_{\phi,g}^{(1,2)} &= -\,4C_AN^{-4} + \Big(\frac{2}{3}n_f - \frac{23}{3}C_A\Big)N^{-3} - \Big(\frac{16}{9}n_f - \Big[\frac{64}{9} + 6\zeta_2\Big]C_A\Big)N^{-2}\\
c_{\phi,g}^{(2,0)} &= -\,(20C_Fn_f - 40C_A^2)N^{-4} + (14C_Fn_f - 4C_An_f + 78C_A^2)N^{-3} + \Big(\frac{8}{9}n_f^2\\
&\quad - \frac{22}{9}C_An_f + \frac{833}{9}C_A^2 - [34 - 16\zeta_2]C_Fn_f\Big)N^{-2}\\
c_{\phi,g}^{(2,1)} &= +\,(52C_Fn_f - 104C_A^2)N^{-5} - \Big(30C_Fn_f - \frac{44}{3}C_An_f + \frac{698}{3}C_A^2\Big)N^{-4} - \Big(\frac{8}{3}n_f^2\\
&\quad - \frac{142}{9}C_An_f - [94 - 56\zeta_2]C_Fn_f + \Big[\frac{2857}{9} - 32\zeta_2\Big]C_A^2\Big)N^{-3}\\
c_{\phi,g}^{(3,0)} &= -\,(120C_F^2n_f - 480C_AC_Fn_f + 480C_A^3)N^{-6} - \Big(\frac{536}{9}C_Fn_f^2 - \frac{44}{3}C_F^2n_f\\
&\quad - \frac{3140}{9}C_AC_Fn_f - \frac{352}{9}C_A^2n_f + \frac{10000}{9}C_A^3\Big)N^{-5} + \Big(\frac{3508}{27}C_Fn_f^2 - \frac{328}{27}C_An_f^2\\
&\quad + \Big[\frac{560}{27} - 48\zeta_2\Big]C_A^2n_f - \Big[162 - \frac{616}{3}\zeta_2\Big]C_F^2n_f + \Big[\frac{16622}{27} - \frac{256}{3}\zeta_2\Big]C_AC_Fn_f\\
&\quad - \Big[\frac{59902}{27} + 224\zeta_2\Big]C_A^3\Big)N^{-4}
\end{aligned} \tag{A.57}
$$

$$
\begin{aligned}
c_{L,ps}^{(2,0)} &= -\,16N^{-2} + [16 + 16\zeta_2]\\
c_{L,ps}^{(2,1)} &= +\,16N^{-3} - 32N^{-2} + [72 - 40\zeta_2]N^{-1}\\
c_{L,ps}^{(3,0)} &= -\,(160C_F - 160C_A)N^{-4} - \Big(\frac{64}{3}n_f + 16C_F - \frac{496}{3}C_A\Big)N^{-3} + \Big(\frac{512}{9}n_f\\
&\quad - [80 - 256\zeta_2]C_F + \Big[\frac{400}{9} - 112\zeta_2\Big]C_A\Big)N^{-2}
\end{aligned} \tag{A.58}
$$

Here an overall factor of C_Fn_f has been omitted.

$$
\begin{aligned}
c_{L,g}^{(1,0)} &= +\,4 - 6N + 7N^2\\
c_{L,g}^{(1,1)} &= +\,8 - [12 - 4\zeta_2]N\\
c_{L,g}^{(1,2)} &= +\,[16 - 2\zeta_2]\\
c_{L,g}^{(2,0)} &= +\,(8C_F - 16C_A)N^{-2} - 8C_FN^{-1} - ([4 + 8\zeta_2]C_F - [16 + 16\zeta_2]C_A)\\
c_{L,g}^{(2,1)} &= -\,(8C_F - 16C_A)N^{-3} + (16C_F - 32C_A)N^{-2} - ([12 - 20\zeta_2]C_F\\
&\quad - [72 - 40\zeta_2]C_A)N^{-1}\\
c_{L,g}^{(3,0)} &= -\,(80C_Fn_f - 40C_F^2 + 80C_AC_F - 160C_A^2)N^{-4} + \Big(\frac{464}{3}C_Fn_f - 20C_F^2\\
&\quad + \frac{16}{3}C_An_f - \frac{152}{3}C_AC_F + \frac{56}{3}C_A^2\Big)N^{-3} + \Big(\frac{80}{9}C_An_f - [16 + 96\zeta_2]C_F^2\\
&\quad + \Big[\frac{308}{9} + 144\zeta_2\Big]C_AC_F - \Big[\frac{3416}{9} - 64\zeta_2\Big]C_Fn_f + \Big[\frac{3640}{9} - 120\zeta_2\Big]C_A^2\Big)N^{-2}
\end{aligned} \tag{A.59}
$$

Here an overall factor of n_f has been omitted.

A.9 a_s^5 Predictions from the Small-x Resummation of DIS Structure Functions

We present here the explicit a_s^5 predictions of the all-a_s-order expressions computed in Chapter 4. For the non-singlet anomalous dimension,

$$\begin{aligned}\gamma_{ns}^{(4),+}(N) = &-448C_F^5N^{-9} + \Big(-1120C_F^5 + 560\beta_0C_F^4\Big)N^{-8} + \Big(3600C_A^2C_F^3\zeta_2 \\ &- 640/3\beta_0C_F^4 - 240\beta_0^2C_F^3 + [-1280 + 11840\zeta_2]C_F^5 \\ &+ [-3200/3 - 11520\zeta_2]C_AC_F^4\Big)N^{-7} + \mathcal{O}(N^{-6}).\end{aligned} \tag{A.60}$$

For the non-singlet coefficient functions,

$$\begin{aligned}c_{2,ns}^{(5)}(N) = &+2652C_F^5N^{-10} + \Big(8418C_F^5 - 17012/3\beta_0C_F^4\Big)N^{-9} + \Big(-15840C_A^2C_F^3\zeta_2 \\ &- 23546/3\beta_0C_F^4 + 14363/3\beta_0^2C_F^3 + [6040 + 50688\zeta_2]C_AC_F^4 \\ &+ [6438 - 56508\zeta_2]C_F^5\Big)N^{-8} + \mathcal{O}(N^{-7})\end{aligned} \tag{A.61}$$

$$\begin{aligned}c_{L,ns}^{(5)}(N) = &+1560C_F^5N^{-8} + \Big(3736C_F^5 - 8920/3\beta_0C_F^4\Big)N^{-7} + \Big(-10800C_A^2C_F^3\zeta_2 \\ &- 1504\beta_0C_F^4 + 6574/3\beta_0^2C_F^3 - [2064 + 35648\zeta_2]C_F^5 \\ &+ [11120/3 + 34560\zeta_2]C_AC_F^4\Big)N^{-6} + \mathcal{O}(N^{-5})\end{aligned} \tag{A.62}$$

$$\begin{aligned}c_{3,ns}^{(5),-}(N) = &+2652C_F^5N^{-10} + \Big(6630C_F^5 - 17012/3\beta_0C_F^4\Big)N^{-9} + \Big(-15840C_A^2C_F^3\zeta_2 \\ &- 11374/3\beta_0C_F^4 + 14363/3\beta_0^2C_F^3 + [66 - 56508\zeta_2]C_F^5 \\ &+ [6040 + 50688\zeta_2]C_AC_F^4\Big)N^{-8} + \mathcal{O}(N^{-7})\end{aligned} \tag{A.63}$$

For the *singlet* splitting functions,

$$\begin{aligned}\gamma_{qq}^{(4)}(N) \;=\; &\gamma_{ns}^{(4),+}(N) + n_f\,C_F\Big\{N^{-9}\Big(-7168\,C_A^3 + 7168\,C_F\,C_A^2 - 5376\,C_F^2\,C_A \\ &+ 3584\,C_F^3 + 7168\,n_f\,C_F\,C_A - 5376\,n_f\,C_F^2\Big) + N^{-8}\Big(-7936\,C_A^3 \\ &+ \frac{38720}{3}\,C_F\,C_A^2 - \frac{41984}{3}\,C_F^2\,C_A + \frac{12272}{3}\,C_F^3 - \frac{1792}{3}\,n_f\,C_A^2 \\ &+ 1088\,n_f\,C_F\,C_A + \frac{6656}{3}\,n_f\,C_F^2 - \frac{896}{3}\,n_f^2\,C_F\Big) + N^{-7}\Big(-\frac{256}{9}\,n_f^2\,C_A \\ &+ \frac{20480}{9}\,n_f^2\,C_F - \frac{32}{3}\,[442 + 105\,\zeta_2]\,n_f\,C_A^2 - \frac{32}{9}\,[7054 + 243\,\zeta_2]\,C_A^3 \\ &+ \frac{4}{3}\,[9109 - 19668\,\zeta_2]\,C_F^3 - \frac{4}{9}\,[9211 - 72108\,\zeta_2]\,C_F^2\,C_A \\ &+ \frac{16}{9}\,[16829 + 1602\,\zeta_2]\,n_f\,C_F\,C_A + \frac{8}{9}\,[24337 - 22320\,\zeta_2]\,C_F\,C_A^2 \\ &- \frac{8}{9}\,[33715 - 9216\,\zeta_2]\,n_f\,C_F^2\Big)\Big\} + \mathcal{O}(N^{-6}),\end{aligned} \tag{A.64}$$

$$
\begin{aligned}
\gamma_{qg}^{(4)}(N) \;=\; & n_f \Bigg\{ N^{-9} \Big(-7168\, C_A^4 + 3584\, C_F\, C_A^3 - 1792\, C_F^2\, C_A^2 + 896\, C_F^3\, C_A - 448\, C_F^4 \\
& + 10752\, n_f\, C_F\, C_A^2 - 7168\, n_f\, C_F^2\, C_A + 2688\, n_f\, C_F^3 - 1792\, n_f^2\, C_F^2 \Big) \\
& + N^{-8} \Big(-\frac{4096}{3}\, C_A^4 + 2368\, C_F\, C_A^3 - \frac{7840}{3}\, C_F^2\, C_A^2 + \frac{6064}{3}\, C_F^3\, C_A \\
& - \frac{584}{3}\, C_F^4 - 1792\, n_f\, C_A^3 - 4736\, n_f\, C_F\, C_A^2 + \frac{6272}{3}\, n_f\, C_F^2\, C_A - \frac{7424}{3}\, n_f\, C_F^3 \\
& + \frac{1792}{3}\, n_f^2\, C_F\, C_A + \frac{11648}{3}\, n_f^2\, C_F^2 \Big) + N^{-7} \Big(-128\, n_f^2\, C_A^2 \\
& + \frac{52672}{27}\, n_f^2\, C_F\, C_A - \frac{427424}{27}\, n_f^2\, C_F^2 + \frac{128}{9}\, n_f^3\, C_F \\
& - \frac{2}{3}\, [2915 - 13216\, \zeta_2]\, C_F^4 - \frac{16}{27}\, [5216 + 3375\, \zeta_2]\, n_f\, C_A^3 \\
& + \frac{2}{27}\, [29293 - 244260\, \zeta_2]\, C_F^3\, C_A - \frac{16}{27}\, [59326 + 8199\, \zeta_2]\, C_A^4 \\
& - \frac{4}{27}\, [73415 - 115992\, \zeta_2]\, C_F^2\, C_A^2 + \frac{8}{27}\, [81626 - 37539\, \zeta_2]\, C_F\, C_A^3 \\
& + \frac{4}{27}\, [114685 - 57816\, \zeta_2]\, n_f\, C_F^3 - \frac{8}{27}\, [118813 - 41067\, \zeta_2]\, n_f\, C_F^2\, C_A \\
& + \frac{8}{27}\, [181400 + 18351\, \zeta_2]\, n_f\, C_F\, C_A^2 \Big) \Bigg\} + \mathcal{O}(N^{-6})\,, \qquad \text{(A.65)}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{gq}^{(4)}(N) \;=\; & C_F \Bigg\{ N^{-9} \Big(14336\, C_A^4 - 7168\, C_F\, C_A^3 + 3584\, C_F^2\, C_A^2 - 1792\, C_F^3\, C_A + 896\, C_F^4 \\
& - 21504\, n_f\, C_F\, C_A^2 + 14336\, n_f\, C_F^2\, C_A - 5376\, n_f\, C_F^3 + 3584\, n_f^2\, C_F^2 \Big) \\
& + N^{-8} \Big(16128\, C_A^4 - \frac{43904}{3}\, C_F\, C_A^3 + \frac{31808}{3}\, C_F^2\, C_A^2 - 6944\, C_F^3\, C_A \\
& + 2240\, C_F^4 + 3584\, n_f\, C_A^3 - \frac{46592}{3}\, n_f\, C_F\, C_A^2 + \frac{51968}{3}\, n_f\, C_F^2\, C_A \\
& - 4928\, n_f\, C_F^3 - \frac{3584}{3}\, n_f^2\, C_F\, C_A - \frac{7168}{3}\, n_f^2\, C_F^2 \Big) + N^{-7} \Big(256\, n_f^2\, C_A^2 \\
& - \frac{318208}{27}\, n_f^2\, C_F\, C_A + \frac{750464}{27}\, n_f^2\, C_F^2 - \frac{256}{9}\, n_f^3\, C_F \\
& + \frac{112}{3}\, [42 - 437\, \zeta_2]\, C_F^4 + \frac{112}{27}\, [191 + 1017\, \zeta_2]\, C_F\, C_A^3 \\
& - \frac{64}{27}\, [8005 - 5517\, \zeta_2]\, n_f\, C_F^3 + \frac{8}{27}\, [13313 + 104940\, \zeta_2]\, C_F^3\, C_A \\
& + \frac{32}{27}\, [14392 + 3375\, \zeta_2]\, n_f\, C_A^3 - \frac{8}{27}\, [17711 + 77652\, \zeta_2]\, C_F^2\, C_A^2 \\
& + \frac{32}{27}\, [37616 + 17019\, \zeta_2]\, C_A^4 + \frac{16}{27}\, [63557 - 20547\, \zeta_2]\, n_f\, C_F^2\, C_A \\
& - \frac{16}{27}\, [149746 + 32031\, \zeta_2]\, n_f\, C_F\, C_A^2 \Big) \Bigg\} + \mathcal{O}(N^{-6})\,, \qquad \text{(A.66)}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{gg}^{(4)}(N) \;=\; & N^{-9}\Big(14336\,C_A^5 \;-28672\,n_f\,C_F\,C_A^3 \;+10752\,n_f\,C_F^2\,C_A^2 \;-3584\,n_f\,C_F^3\,C_A \\
& +896\,n_f\,C_F^4 \;+10752\,n_f^2\,C_F^2\,C_A \;-3584\,n_f^2\,C_F^3\Big) \;+N^{-8}\Big(\frac{8960}{3}\,C_A^5 \\
& +\frac{17920}{3}\,n_f\,C_A^4 \;+\frac{14848}{3}\,n_f\,C_F\,C_A^3 \;+\frac{15040}{3}\,n_f\,C_F^2\,C_A^2 \;-\frac{9536}{3}\,n_f\,C_F^3\,C_A \\
& +\frac{1168}{3}\,n_f\,C_F^4 \;-5376\,n_f^2\,C_F\,C_A^2 \;-10048\,n_f^2\,C_F^2\,C_A \;+\frac{11264}{3}\,n_f^2\,C_F^3 \\
& +\frac{896}{3}\,n_f^3\,C_F^2\Big) \;+N^{-7}\Big(\frac{2560}{3}\,n_f^2\,C_A^3 \;-256\,n_f^3\,C_F\,C_A \;+\frac{14080}{9}\,n_f^3\,C_F^2 \\
& +\frac{640}{9}\,[164+81\,\zeta_2]\,n_f\,C_A^4 \;+\frac{640}{9}\,[907+396\,\zeta_2]\,C_A^5 \\
& +\frac{4}{3}\,[2171-7692\,\zeta_2]\,n_f\,C_F^4 \;-\frac{32}{9}\,[3274+495\,\zeta_2]\,n_f^2\,C_F\,C_A^2 \\
& -\frac{224}{9}\,[5438+1431\,\zeta_2]\,n_f\,C_F\,C_A^3 \;-\frac{4}{9}\,[18349-39132\,\zeta_2]\,n_f\,C_F^3\,C_A \\
& -\frac{8}{9}\,[26605-5184\,\zeta_2]\,n_f^2\,C_F^3 \;+\frac{16}{9}\,[38371+3978\,\zeta_2]\,n_f^2\,C_F^2\,C_A \\
& +\frac{8}{9}\,[43463-14940\,\zeta_2]\,n_f\,C_F^2\,C_A^2\Big) \;+\mathcal{O}(N^{-6})\,, \qquad \text{(A.67)}
\end{aligned}
$$

and for the singlet coefficient functions,

$$
\begin{aligned}
c_{2,q}^{(5)}(N) \;=\; & c_{2,ns}^{(5)}(N)+n_f\,C_F\,\Big\{N^{-10}\Big(42432\,C_A^3 \;-42432\,C_F\,C_A^2 \;+31824\,C_F^2\,C_A \\
& -21216\,C_F^3 \;-42432\,n_f\,C_F\,C_A \;+31824\,n_f\,C_F^2\Big) \;+N^{-9}\Big(\frac{5366608}{45}\,C_A^3 \\
& -\frac{7102528}{45}\,C_F\,C_A^2 \;+\frac{2208812}{15}\,C_F^2\,C_A \;-\frac{511648}{15}\,C_F^3 \;-\frac{81248}{9}\,n_f\,C_A^2 \\
& -\frac{1361056}{45}\,n_f\,C_F\,C_A \;-\frac{243376}{15}\,n_f\,C_F^2 \;+\frac{72448}{9}\,n_f^2\,C_F\Big) \\
& +N^{-8}\Big(\frac{17696}{9}\,n_f^2\,C_A \;-\frac{2472352}{135}\,n_f^2\,C_F \;-\frac{4}{5}\,[74593-180392\,\zeta_2]\,C_F^3 \\
& -\frac{16}{135}\,[102961-37125\,\zeta_2]\,n_f\,C_A^2 \;-\frac{16}{45}\,[429100-30021\,\zeta_2]\,n_f\,C_F\,C_A \\
& +\frac{16}{135}\,[1390214-523683\,\zeta_2]\,n_f\,C_F^2 \;+\frac{16}{135}\,[3063709-69039\,\zeta_2]\,C_A^3 \\
& -\frac{16}{135}\,[3126887-924570\,\zeta_2]\,C_F\,C_A^2 \\
& +\frac{2}{135}\,[7465355-11586096\,\zeta_2]\,C_F^2\,C_A\Big)\Big\} +\mathcal{O}(N^{-7})\,, \qquad \text{(A.68)}
\end{aligned}
$$

$$
\begin{aligned}
c_{2,g}^{(5)}(N) \;=\; & n_f\,\Big\{N^{-10}\Big(42432\,C_A^4 \;-21216\,C_F\,C_A^3 \;+10608\,C_F^2\,C_A^2 \;-5304\,C_F^3\,C_A \\
& +2652\,C_F^4 \;-63648\,n_f\,C_F\,C_A^2 \;+42432\,n_f\,C_F^2\,C_A \;-15912\,n_f\,C_F^3 \\
& +10608\,n_f^2\,C_F^2\Big) \;+N^{-9}\Big(\frac{3616288}{45}\,C_A^4 \;-\frac{2668904}{45}\,C_F\,C_A^3 \\
& +\frac{1762432}{45}\,C_F^2\,C_A^2 \;-\frac{1089206}{45}\,C_F^3\,C_A \;+\frac{50356}{15}\,C_F^4 \;-\frac{17600}{9}\,n_f\,C_A^3 \\
& -\frac{1868624}{45}\,n_f\,C_F\,C_A^2 \;+\frac{1566016}{45}\,n_f\,C_F^2\,C_A \;+\frac{334184}{45}\,n_f\,C_F^3 \\
& +\frac{81248}{9}\,n_f^2\,C_F\,C_A \;-\frac{1239104}{45}\,n_f^2\,C_F^2\Big) \;+N^{-8}\Big(\frac{12464}{27}\,n_f^2\,C_A^2 \\
& -\frac{1105856}{135}\,n_f^2\,C_F\,C_A \;-\frac{8848}{9}\,n_f^3\,C_F \;+\frac{16}{135}\,[39757+61020\,\zeta_2]\,n_f\,C_A^3 \\
& +\frac{1}{15}\,[59357-673108\,\zeta_2]\,C_F^4 \;-\frac{4}{135}\,[365911-3205476\,\zeta_2]\,C_F^3\,C_A \\
& -\frac{4}{15}\,[648293-174498\,\zeta_2]\,C_F\,C_A^3 \;+\frac{2}{45}\,[1977587-1966200\,\zeta_2]\,C_F^2\,C_A^2 \\
& +\frac{8}{135}\,[2407760-1256427\,\zeta_2]\,n_f\,C_F^2\,C_A \\
& +\frac{4}{135}\,[3500111-241380\,\zeta_2]\,n_f^2\,C_F^2 \\
& -\frac{2}{135}\,[4630465-3452868\,\zeta_2]\,n_f\,C_F^3 \;+\frac{4}{135}\,[11350279+666720\,\zeta_2]\,C_A^4 \\
& -\frac{4}{135}\,[12031717+12510\,\zeta_2]\,n_f\,C_F\,C_A^2\Big)\Big\} +\mathcal{O}(N^{-7})\,, \qquad \text{(A.69)}
\end{aligned}
$$

$$
\begin{aligned}
c^{(5)}_{L,q}(N) \;=\;& c^{(5)}_{L,ns}(N) + n_f\, C_F \Big\{ N^{-8} \Big(24960\, C_A^3 \, - 24960\, C_F\, C_A^2 \, + 18720\, C_F^2\, C_A \\
& - 12480\, C_F^3 \, - 24960\, n_f\, C_F\, C_A \, + 18720\, n_f\, C_F^2 \Big) \, + N^{-7} \Big(\frac{436192}{9}\, C_A^3 \\
& - \frac{602368}{9}\, C_F\, C_A^2 \, + \frac{198248}{3}\, C_F^2\, C_A \, - \frac{33904}{3}\, C_F^3 \, - \frac{30400}{9}\, n_f\, C_A^2 \\
& - \frac{27328}{9}\, n_f\, C_F\, C_A \, - \frac{56272}{3}\, n_f\, C_F^2 \, + \frac{33920}{9}\, n_f^2\, C_F \Big) \, + N^{-6} \Big(\frac{6208}{9}\, n_f^2\, C_A \\
& - \frac{305536}{27}\, n_f^2\, C_F \, + \frac{160}{27}\, [278 + 513\,\zeta_2]\, n_f\, C_A^2 \, - \frac{56}{3}\, [359 - 4436\,\zeta_2]\, C_F^3 \\
& - \frac{32}{9}\, [18590 - 2271\,\zeta_2]\, n_f\, C_F\, C_A \, - \frac{8}{27}\, [52207 + 341604\,\zeta_2]\, C_F^2\, C_A \\
& - \frac{16}{27}\, [180227 - 109944\,\zeta_2]\, C_F\, C_A^2 \, + \frac{16}{27}\, [218827 - 20610\,\zeta_2]\, C_A^3 \\
& + \frac{8}{27}\, [278473 - 110304\,\zeta_2]\, n_f\, C_F^2 \Big) \Big\} + \mathcal{O}(N^{-5})\,, \qquad \text{(A.70)}
\end{aligned}
$$

$$
\begin{aligned}
c^{(5)}_{L,g}(N) \;=\;& n_f \Big\{ N^{-8} \Big(24960\, C_A^4 \, - 12480\, C_F\, C_A^3 \, + 6240\, C_F^2\, C_A^2 \, - 3120\, C_F^3\, C_A \\
& + 1560\, C_F^4 \, - 37440\, n_f\, C_F\, C_A^2 \, + 24960\, n_f\, C_F^2\, C_A \, - 9360\, n_f\, C_F^3 \\
& + 6240\, n_f^2\, C_F^2 \Big) \, + N^{-7} \Big(\frac{230272}{9}\, C_A^4 \, - \frac{178352}{9}\, C_F\, C_A^3 \, + \frac{139120}{9}\, C_F^2\, C_A^2 \\
& - \frac{93860}{9}\, C_F^3\, C_A \, + \frac{2140}{3}\, C_F^4 \, + \frac{7040}{9}\, n_f\, C_A^3 \, + \frac{22528}{9}\, n_f\, C_F\, C_A^2 \\
& - \frac{1856}{9}\, n_f\, C_F^2\, C_A \, + \frac{96152}{9}\, n_f\, C_F^3 \, + \frac{30400}{9}\, n_f^2\, C_F\, C_A \, - \frac{164144}{9}\, n_f^2\, C_F^2 \Big) \\
& + N^{-6} \Big(\frac{3424}{27}\, n_f^2\, C_A^2 \, - \frac{200672}{27}\, n_f^2\, C_F\, C_A \, - \frac{3104}{9}\, n_f^3\, C_F \\
& - \frac{2}{3}\, [1261 + 42056\,\zeta_2]\, C_F^4 \, + \frac{32}{27}\, [4967 + 4212\,\zeta_2]\, n_f\, C_A^3 \\
& + 4\, [5905 - 13404\,\zeta_2]\, C_F^2\, C_A^2 \, + \frac{16}{27}\, [11939 + 102024\,\zeta_2]\, C_F^3\, C_A \\
& - \frac{8}{3}\, [22553 - 9334\,\zeta_2]\, C_F\, C_A^3 \, - \frac{20}{27}\, [46153 - 35208\,\zeta_2]\, n_f\, C_F^3 \\
& + \frac{16}{27}\, [101897 - 7344\,\zeta_2]\, n_f^2\, C_F^2 \, + \frac{8}{27}\, [191519 - 130122\,\zeta_2]\, n_f\, C_F^2\, C_A \\
& + \frac{8}{27}\, [458999 + 12744\,\zeta_2]\, C_A^4 \, - \frac{8}{27}\, [542135 - 18342\,\zeta_2]\, n_f\, C_F\, C_A^2 \Big) \Big\} \\
& + \mathcal{O}(N^{-5})\,. \qquad \text{(A.71)}
\end{aligned}
$$

A.10 The Small-x Behaviour of the Fourth-Order QCD Splitting Functions at Large-n_f

In Chapter 5 we computed fourth-order contributions to the n_f^3 terms of the singlet anomalous dimensions and the n_f^2 terms of the non-singlet anomalous dimensions. For future reference, the leading small-x behaviour of the associated splitting functions is presented here.

$$
P^{\pm}_{ns}\big|_{n_f^3} = \ln^3 x \Big(-\frac{8}{81}\, C_F \Big) \, + \ln^2 x \Big(-\frac{88}{81}\, C_F \Big) \, + \ln x \Big(-\frac{64}{27}\, C_F \Big)\,, \qquad \text{(A.72)}
$$

$$
\begin{aligned}
P^{+}_{ns}\big|_{n_f^2} =\;& \ln^4 x \Big(\frac{4}{9}\, C_F^2 \Big) \, + \ln^3 x \Big(\frac{152}{27}\, C_F^2 \, + \frac{44}{27}\, C_F\, C_A \Big) \\
& + \ln^2 x \Big(\frac{16}{81}\, [134 + 9\,\zeta_2]\, C_F^2 \, + \frac{4}{27}\, [161 - 36\,\zeta_2]\, C_F\, C_A \Big) \\
& + \ln x \Big(+ \frac{8}{81}\, [967 + 72\,\zeta_2]\, C_F^2 \, + \frac{1}{81}\, [7561 - 2736\,\zeta_2 + 864\,\zeta_3]\, C_F\, C_A \Big)\,,
\end{aligned}
$$
(A.73)

$$P^{-}_{ns}\big|_{n_f^2} = \ln^4 x \left(\frac{4}{9}\, C_F\, C_A \,- \frac{4}{9}\, C_F^2 \right) \,+ \ln^3 x \left(\frac{692}{81}\, C_F\, C_A \,- \frac{664}{81}\, C_F^2 \right)$$
$$+ \ln^2 x \left(\frac{4}{81}\, [1081 - 36\,\zeta_2]\, C_F\, C_A \,- \frac{16}{27}\, [55 + 9\,\zeta_2]\, C_F^2 \right)$$
$$+ \ln x \left(\frac{1}{27}\, [4131 - 304\,\zeta_2 + 384\,\zeta_3]\, C_F\, C_A \,- \frac{8}{81}\, [241 + 384\,\zeta_2 + 72\,\zeta_3]\, C_F^2 \right), \tag{A.74}$$

$$P_{qq,ps}\big|_{n_f^3} = \frac{1}{x} \left(\frac{64}{27}\, [1 - 6\,\zeta_3]\, C_F \right) \,+ \ln^4 x \left(-\frac{4}{27}\, C_F \right) \,+ \ln^3 x \left(-\frac{232}{81}\, C_F \right)$$
$$+ \ln^2 x \left(-\frac{16}{81}\, [73 + 18\,\zeta_2]\, C_F \right) \,+ \ln x \left(-\frac{32}{81}\, [59 + 87\,\zeta_2 + 36\,\zeta_3]\, C_F \right), \tag{A.75}$$

$$P_{qg}\big|_{n_f^3} = \frac{1}{x} \left(\frac{256}{729}\, [17 - 54\,\zeta_3]\, C_F \,- \frac{64}{729}\, [7 + 54\,\zeta_3]\, C_A \right) \,+ \ln^4 x \left(\frac{278}{81}\, C_F \,- \frac{4}{27}\, C_A \right)$$
$$+ \ln^3 x \left(\frac{232}{243}\, C_A \,+ \frac{20}{243}\, [193 + 72\,\zeta_2]\, C_F \right)$$
$$+ \ln^2 x \left(\frac{4}{27}\, [835 + 180\,\zeta_2 + 72\,\zeta_3]\, C_F \,- \frac{2}{243}\, [277 - 576\,\zeta_2]\, C_A \right)$$
$$+ \ln x \left(\frac{4}{243}\, [643 + 543\,\zeta_2 - 288\,\zeta_3]\, C_A \right.$$
$$\left. + \frac{32}{243}\, [1988 + 1137\,\zeta_2 + 180\,\zeta_3 - 108\,\zeta_4]\, C_F \right), \tag{A.76}$$

$$P_{gq}\big|_{n_f^3} = \frac{1}{x} \left(-\frac{128}{81}\, [1 - 6\,\zeta_3]\, C_F \right), \tag{A.77}$$

$$P_{gg}\big|_{n_f^3} = \frac{1}{x} \left(\frac{32}{243}\, [5 + 18\,\zeta_3]\, C_A - \frac{64}{243}\, [17 - 54\,\zeta_3]\, C_F \right) \,+ \ln^4 x \left(-\frac{4}{27}\, C_F \right)$$
$$+ \ln^3 x \left(\frac{184}{81}\, C_F \,- \frac{16}{81}\, C_A \right) \,+ \ln^2 x \left(\frac{152}{81}\, C_A - \frac{32}{81}\, [35 - 9\,\zeta_2]\, C_F \right)$$
$$+ \ln x \left(\frac{16}{81}\, [179 - 138\,\zeta_2 + 144\,\zeta_3]\, C_F \,- \frac{4}{81}\, [115 - 48\,\zeta_2]\, C_A \right). \tag{A.78}$$

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