Counting components of an integral lamination

S. Öykü Yurttaş* and Toby Hall†

Abstract

We present an efficient algorithm for calculating the number of components of an integral lamination on an n-punctured disk, given its Dynnikov coordinates. The algorithm requires $O(n^2M)$ arithmetic operations, where M is the sum of the absolute values of the Dynnikov coordinates.

1 Introduction

Systems of mutually disjoint essential simple closed curves have played a central rôle in the study of mapping class groups of surfaces since the work of Dehn. Such systems are usually described combinatorially using techniques such as train tracks or the Dehn–Thurston coordinate system [10]. Given such a combinatorial description, it can be difficult to determine even elementary properties of the system, such as the number of curves which it contains.

In the case where the surface is an n-punctured disk D_n, a particularly beautiful description of such systems of curves – or *integral laminations* – is given by the Dynnikov coordinate system [4], which provides an explicit bijection from the set of integral laminations on D_n to $\mathbb{Z}^{2n-4} \setminus \{0\}$.

In the case $n = 3$, the Dynnikov coordinates of an integral lamination consist of a pair of integers, and the number of connected components of the lamination is the greatest common divisor of these integers. No analogous formula is known when $n \geq 4$.

In this paper we describe an algorithm for calculating the number of components of an integral lamination from its Dynnikov coordinates. The algorithm proceeds by the repeated application of three moves, each of which simplifies the lamination and either leaves the number of components unchanged, or reduces it by a known amount. The algorithm can be seen as complementary to that of Dynnikov and Wiest [5], which works with *interval identification systems*: combinatorial descriptions of a rather different nature, ideally suited to their goal of comparing algebraic and geometric notions of braid complexity.

2010 *Mathematics subject classification.* 57M50, 57N05, 20F36

*Dicle University, Science Faculty, Mathematics Department, 21280, Diyarbakır, Turkey, e-mail: saadet.yurttas@dicle.edu.tr

†Department of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL, UK, e-mail: T.Hall@liverpool.ac.uk

The three moves are described, and their properties are analysed, in Lemmas 6, 7, and 8, before the algorithm itself (Algorithm 9) is presented. In order to ease implementation, the formal descriptions of the moves and the algorithm are entirely in terms of Dynnikov coordinates rather than topological properties of the corresponding laminations. This method of presentation also makes it straightforward to analyse the complexity of the algorithm (Lemma 10): calculating the number of components of an integral lamination on the n-punctured disk requires $O(n^2M)$ arithmetic operations, where M is the sum of the absolute values of the Dynnikov coordinates. Here an *arithmetic operation* means adding, subtracting, comparing, taking the maximum, or taking the minimum of two integers, each of size $O(n^2M^2)$.

We are grateful to the referee for pointing out that this resolves, for surfaces of genus zero, a long-standing conjecture regarding the existence of a polynomial algorithm to decide whether an integral lamination, specified in terms of a coordinate system, is connected or not (see also Problem 2 of [9]); and that the fact that earlier approaches to the problem based on Dehn–Thurston or train track coordinates were unsuccessful indicates the utility of Dynnikov coordinates. Other positive results in this direction include the work of Haas and Susskind [6] on genus 2 surfaces.

The algorithm has been implemented as part of the second author's program `Dynn`, available at `http://pcwww.liv.ac.uk/maths/tobyhall/software/`. In addition to having good theoretical complexity, the algorithm is efficient in practice. Calculating the number of components of 10000 integral laminations on D_{10} with randomly generated Dynnikov coordinates between -10 and 10, -1000 and 1000, and -100000 and 100000 took an average of 0.000089, 0.00033, and 0.00099 seconds per lamination on a standard notebook PC with an Intel i5 processor. On D_{100}, the corresponding times were 0.0013, 0.010, and 0.058 seconds per lamination.

2 Preliminaries

2.1 Integral laminations on the punctured disk

Let $n \geq 3$, and let D_n be a standard model of the n-punctured disk in the plane, with the punctures arranged along the horizontal diameter. A simple closed curve in D_n is *inessential* if it bounds an unpunctured disk, a once-punctured disk, or an n-punctured disk, and is *essential* otherwise.

An *integral lamination* $\mathcal{L}$ in D_n is a non-empty union of pairwise disjoint unoriented essential simple closed curves in D_n, up to isotopy. We write $\mathcal{L}_n$ for the set of integral laminations on D_n.

Given an integral lamination $\mathcal{L}$, we write $X(\mathcal{L}) \geq 1$ for the number of components of a representative of $\mathcal{L}$. The aim of this paper is to describe an algorithm for calculating $X(\mathcal{L})$ from the *Dynnikov coordinates* of $\mathcal{L}$.

2.2 The Dynnikov coordinate system

The Dynnikov coordinate system [4] provides, for each $n \geq 3$, a bijection $\rho\colon \mathcal{L}_n \to \mathbb{Z}^{2n-4} \setminus \{0\}$, which we now define.

Construct *Dynnikov arcs* α_i $(1 \leq i \leq 2n-4)$ and β_i $(1 \leq i \leq n-1)$ in D_n as depicted in Figure 1. Given $\mathcal{L} \in \mathcal{L}_n$, let L be a representative of $\mathcal{L}$ which intersects each of these arcs minimally (such an L is called a *minimal representative* of $\mathcal{L}$). Write α_i (respectively β_i) for the number of intersections of L with the arc α_i (respectively the arc β_i). This overload of notation will not give rise to any ambiguity, since it will always be stated explicitly when the symbols α_i and β_i refer to arcs rather than to integers.

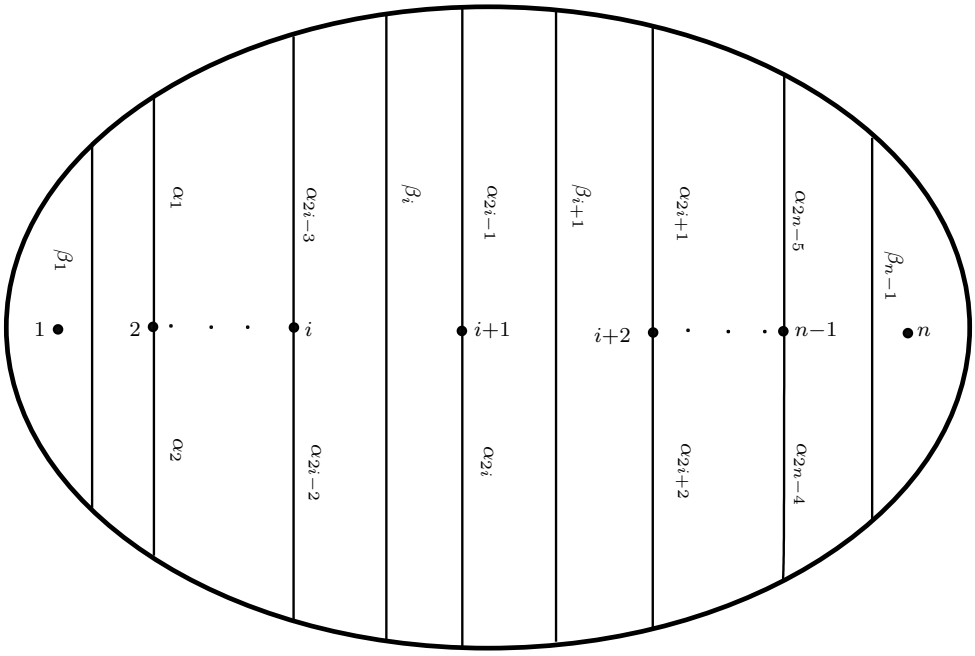

Figure 1: The arcs α_i and β_i

The *Dynnikov coordinate function* $\rho\colon \mathcal{L}_n \to \mathbb{Z}^{2n-4} \setminus \{0\}$ is defined by

$$\rho(\mathcal{L}) = (a;\, b) = (a_1, \ldots, a_{n-2};\, b_1, \ldots, b_{n-2}),$$

where

$$a_i = \frac{\alpha_{2i} - \alpha_{2i-1}}{2} \qquad \text{and} \qquad b_i = \frac{\beta_i - \beta_{i+1}}{2} \tag{1}$$

for $1 \leq i \leq n-2$.

The intersection numbers α_i and β_i (and hence the integral lamination $\mathcal{L}$) can be recovered from the Dynnikov coordinates $(a;\, b)$ using the following formulae (see for

example [7]):

$$\beta_i = 2 \max_{1 \le k \le n-2} \left(|a_k| + b_k^+ + \sum_{j=1}^{k-1} b_j \right) - 2 \sum_{j=1}^{i-1} b_j, \tag{2}$$

$$\alpha_i = \begin{cases} (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \ge 0, \\ (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{1+\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \le 0. \end{cases} \tag{3}$$

Here x^+ denotes $\max(x, 0)$, and $\lceil x \rceil$ denotes the smallest integer which is not less than x.

We next mention some relevant facts about the interpretation of the Dynnikov coordinates. Let $1 \le i \le n-2$, and let Δ_i denote the subset of D_n bounded by the arcs β_i and β_{i+1}. Let L be a minimal representative of $\mathcal{L}$, and consider the connected components of $L \cap \Delta_i$. By minimality, each such component is of one of four types:

- A *right loop* component, which has both endpoints on the arc β_i and intersects both of the arcs α_{2i-1} and α_{2i};
- A *left loop* component, which has both endpoints on the arc β_{i+1} and intersects both of the arcs α_{2i-1} and α_{2i};
- An *above* component, which has one endpoint on each of the arcs β_i and β_{i+1}, and intersects the arc α_{2i-1} but not the arc α_{2i}; or
- A *below* component, which has one endpoint on each of the arcs β_i and β_{i+1}, and intersects the arc α_{2i} but not the arc α_{2i-1}.

Clearly there cannot be both left loop and right loop components. It follows immediately from (1) that there are $|b_i|$ loop components, which are left loops if $b_i < 0$, and right loops if $b_i > 0$; and that $2a_i$ is the difference between the number of below components and the number of above components.

Now suppose that $2 \le i \le n-2$, and that $b_{i-1} < 0$ and $b_i > 0$, so that Δ_{i-1} contains left loop components and Δ_i contains right loop components. Lemma 2 below says that these loop components join up to give simple closed curves provided that $a_{i-1} = a_i$.

Definition 1 (Elementary curve)**.** An essential simple closed curve in D_n is *elementary* (about punctures i and $i+1$) if it is isotopic to a simple closed curve which is contained in $\Delta_{i-1} \cup \Delta_i$ for some $1 \le i \le n-1$. (Here Δ_0 is the subset of D_n containing puncture 1 and bounded by β_1; and Δ_{n-1} is the subset of D_n containing puncture n and bounded by β_{n-1}.)

Lemma 2. *Let L be a minimal representative of an integral lamination $\mathcal{L} \in \mathcal{L}_n$ with Dynnikov coordinates $(a; b)$. Let $2 \le i \le n-2$, and suppose that $b_{i-1} < 0$, $b_i > 0$, and $a_{i-1} = a_i$.*

Then L has $\min(-b_{i-1},\, b_i)$ components which are elementary curves about punctures i and $i+1$.

Proof. We suppose without loss of generality that $b_i \geq -b_{i-1}$ (otherwise reflect in the arc β_i) and that $a_{i-1} = a_i \geq 0$ (otherwise reflect in the horizontal diameter of the disk).

Let there be $A \geq 0$ above components of $L \cap \Delta_i$. Then the endpoints of components of $L \cap \Delta_i$ on β_i, ordered from top to bottom, come: A from above components; $2b_i$ from b_i right loop components; and $A + 2a_i$ from below components.

Since these endpoints must agree with those of components of $L \cap \Delta_{i-1}$ on β_i, there are also $2(A + a_i + b_i) = 2(A + a_{i-1} + b_i)$ of these, which, again from top to bottom, come: $A + b_i + b_{i-1}$ from above components; $-2b_{i-1}$ from $-b_{i-1}$ left loop components; and $A + b_i + b_{i-1} + 2a_i$ from below components.

Therefore the jth outermost right loop component in Δ_i joins the endpoint in position $A + j$ to that in position $A + 2b_i + 1 - j$ $(1 \leq j \leq b_i)$; and the kth outermost left loop component in Δ_{i-1} joins the endpoint in position $A + b_i + b_{i-1} + k$ to that in position $A + b_i - b_{i-1} + 1 - k$ $(1 \leq k \leq -b_{i-1})$. It follows that the kth outermost left loop component shares its endpoints with the $(b_i + b_{i-1} + k)^{\text{th}}$ outermost right loop component, yielding $-b_{i-1}$ elementary curve components as required. □

2.3 The action of the braid group

The mapping class group $\mathrm{MCG}(D_n)$ of D_n is isomorphic to the n-braid group B_n modulo its center [1], so that elements of $\mathrm{MCG}(D_n)$ can be represented in terms of the Artin braid generators σ_i $(1 \leq i \leq n-1)$. In this paper we adopt the convention of Birman's book [2], that σ_i exchanges punctures i and $i+1$ in the *counterclockwise* direction.

The action of $\mathrm{MCG}(D_n)$ on $\mathcal{L}_n$ can be calculated using the *update rules* of Theorem 3 below (see for example [4, 8, 3, 7, 11]), which describe how Dynnikov coordinates transform under the action of the Artin generators and their inverses. In this theorem statement we again use the notation x^+ to denote $\max(x, 0)$.

Theorem 3 (Update rules)**.** *Let $\mathcal{L} \in \mathcal{L}_n$ have Dynnikov coordinates $(a;\, b)$, and let $1 \leq i \leq n-1$.*

(a) Let $(a';\, b')$ be the Dynnikov coordinates of the integral lamination $\sigma_i(\mathcal{L})$. Then $a'_j = a_j$ and $b'_j = b_j$ for all $j \notin \{i-1, i\}$, and

- *If $i = 1$ then*

$$\begin{aligned} a'_1 &= b_1 - \max(0,\ b_1^+ - a_1), \\ b'_1 &= b_1^+ - a_1. \end{aligned} \tag{4}$$

- *If* $2 \le i \le n-2$ *then*

$$\begin{aligned} a'_{i-1} &= \max(a_{i-1} + b^+_{i-1},\ a_i + b_{i-1}), \\ a'_i &= b_i - \max(-a_{i-1},\ b^+_i - a_i), \\ b'_{i-1} &= a_i + b_{i-1} + b_i - \max(a_{i-1} + b^+_{i-1} + b^+_i,\ a_i + b_{i-1}), \\ b'_i &= \max(a_{i-1} + b^+_{i-1} + b^+_i,\ a_i + b_{i-1}) - a_i. \end{aligned} \tag{5}$$

- *If* $i = n-1$ *then*

$$\begin{aligned} a'_{n-2} &= \max(a_{n-2} + b^+_{n-2},\ b_{n-2}), \\ b'_{n-2} &= b_{n-2} - a_{n-2} - b^+_{n-2}. \end{aligned} \tag{6}$$

(b) Let $(a''; b'')$ *be the Dynnikov coordinates of the integral lamination* $\sigma_i^{-1}(\mathcal{L})$*. Then* $a''_j = a_j$ *and* $b''_j = b_j$ *for all* $j \notin \{i-1, i\}$*, and*

- *If* $i = 1$ *then*

$$\begin{aligned} a''_1 &= \max(0,\ a_1 + b^+_1) - b_1, \\ b''_1 &= a_1 + b^+_1. \end{aligned} \tag{7}$$

- *If* $2 \le i \le n-2$ *then*

$$\begin{aligned} a''_{i-1} &= \min(a_{i-1} - b^+_{i-1},\ a_i - b_{i-1}), \\ a''_i &= \max(a_{i-1},\ a_i + b^+_i) - b_i, \\ b''_{i-1} &= -a_i + b_{i-1} + b_i - \max(-a_{i-1} + b^+_{i-1} + b^+_i,\ -a_i + b_{i-1}), \\ b''_i &= a_i + \max(b^+_{i-1} + b^+_i - a_{i-1},\ b_{i-1} - a_i). \end{aligned} \tag{8}$$

- *If* $i = n-1$ *then*

$$\begin{aligned} a''_{n-2} &= \min(a_{n-2} - b^+_{n-2},\ -b_{n-2}), \\ b''_{n-2} &= a_{n-2} + b_{n-2} - b^+_{n-2}. \end{aligned} \tag{9}$$

3 The algorithm

Given $(a; b) \in \mathbb{Z}^{2n-4} \setminus \{0\}$, we write $X(a; b) := X(\mathcal{L})$, where $\mathcal{L} = \rho^{-1}(a; b) \in \mathcal{L}_n$. Algorithm 9 below computes $X(a; b)$ from $(a; b)$.

3.1 The case $n = 3$

The following result is an expression of the well-known fact that the braid group B_3 acts on $\mathcal{L}_3$ by Euclid's algorithm.

Lemma 4. *Let* $(a_1;\, b_1) \in \mathbb{Z}^2 \setminus \{0\}$. *Then* $X(a_1;\, b_1) = \gcd(a_1,\, b_1)$.

Proof. It can easily be seen from Theorem 3 that the action of the Artin generators of B_3 on $\mathcal{L}_3$, when expressed in Dynnikov coordinates, preserves $\gcd(a_1,\, b_1)$.

Let L be a minimal representative of $\mathcal{L}$, and let C be any component of L. Since C is essential, it bounds a disk containing 2 of the 3 punctures, and hence there is a braid $\sigma \in B_3$ such that $\sigma(C)$ is an elementary curve about punctures 1 and 2. Since the components of $\sigma(L)$ are disjoint, it consists of $X(a_1;\, b_1)$ elementary curves about these punctures, and hence $\rho(\sigma(L)) = (0;\, X(a_1;\, b_1))$. Therefore $X(a_1;\, b_1) = \gcd(0,\, X(a_1;\, b_1)) = \gcd(a_1,\, b_1)$ as required. □

3.2 Extended Dynnikov coordinates

The first step of Algorithm 9 is to add two "dummy" punctures, one to the left and one to the right of the existing punctures.

The motivation for this is that one of the moves of the algorithm fills in a puncture (see Section 3.4.1). If this were done without the dummy punctures, it could result in boundary-parallel components, and therefore take us out of the realm of integral laminations. While it would be possible to calculate, and compensate for, the number of such boundary-parallel components, this would involve an inversion of Dynnikov coordinates using (2) and (3) each time that a puncture is filled in, and would therefore decrease the efficiency of the algorithm.

An additional benefit of the dummy punctures is to simplify the statement of the algorithm. Once the dummy punctures have been added, the braid group B_n acts on the central n punctures of an $(n+2)$-punctured disk, so that the update rules are always given by (5) and (8), avoiding the need for separate end cases (4), (6), (7), and (9).

Introducing the dummy punctures involves the extension of Figure 1 to include additional punctures labelled 0 and $n+1$, and additional arcs β_0 (between punctures 0 and 1); α_{-1} and α_0 (each with an endpoint on puncture 1); β_n (between punctures n and $n+1$); and α_{2n-3} and α_{2n-2} (each with an endpoint on puncture n). Additional coordinates a_0, b_0, a_{n-1} and b_{n-1} can then be defined using (1).

To describe an integral lamination $\mathcal{L} \in \mathcal{L}_n$ in these extended coordinates, observe that we have $\alpha_{-1} = \alpha_0$, $\alpha_{2n-3} = \alpha_{2n-2}$, and $\beta_0 = \beta_n = 0$, so that

$$\begin{aligned} a_0 &= a_{n-1} = 0, \\ b_0 &= -\max_{1 \le k \le n-2} \left(|a_k| + b_k^+ + \sum_{j=1}^{k-1} b_j \right), \quad \text{and} \\ b_{n-1} &= -b_0 - \sum_{j=1}^{n-2} b_j, \end{aligned} \tag{10}$$

using $b_0 = (\beta_0 - \beta_1)/2 = -\beta_1/2$, $b_{n-1} = (\beta_{n-1} - \beta_n)/2 = \beta_{n-1}/2$, and (2).

As mentioned above, we will always consider the action of the braid group B_n on integral laminations on this $(n+2)$-punctured disk, using (5) and (8) for $1 \leq i \leq n-1$.

Definition 5 (Central lamination)**.** An integral lamination $\mathcal{L}$ on the $(n+2)$-punctured disk is said to be *central* if it satisfies $\beta_0 = \beta_n = 0$.

We will see that the algorithm moves all preserve the property of centrality, so that we will have

$$a_0 = a_{n-1} = 0, \quad b_0 \leq 0, \quad \text{and} \quad b_{n-1} \geq 0$$

throughout.

3.3 The complexity function

Given Dynnikov coordinates $(a;\, b) = (a_0, \ldots, a_{n-1};\, b_0, \ldots, b_{n-1}) \in \mathbb{Z}^{2n} \setminus \{0\}$ of a central lamination, we write $n(a;\, b) = n$, and define $i(a;\, b)$ by

a) $i(a;\, b) = 0$ if $b_i = 0$ for any i, and otherwise

b) $i(a;\, b)$ is the smallest $i \in \{1, \ldots, n-1\}$ with $b_i > 0$.

Note that such a smallest i must exist in case b), since $b_{n-1} > 0$; and that $b_{i-1} < 0$ since $b_0 < 0$.

Progress through the algorithm is measured by decrease, in the lexicographic order, of the *complexity*

$$\mathcal{C}(a;\, b) = \left(n(a;\, b),\ \sum_{i=0}^{n(a;\, b)-1} |b_i|,\ \ i(a;\, b) \right) \in \mathbb{N}^3.$$

3.4 The moves

In this section we describe and analyse each of the three moves of the algorithm: Filling in a puncture; Erasing elementary components; and Untwisting. While the interpretation of each of these moves is explained briefly at the start, and is clarified in the proofs of the relevant lemmas, the formal descriptions of the moves and the statements of their properties are given entirely in terms of Dynnikov coordinates, with the intention of making it easier for a reader to implement them.

Examples of the application of each of these moves can be found in the extended example of Section 3.7.

3.4.1 Filling in a puncture

This move is applied when some $b_i = 0$, so that a minimal representative of the lamination has no loops about puncture $i+1$. The minimal representative therefore remains minimal when the puncture is filled in.

Lemma 6. *Let $(a;\, b) \in \mathbb{Z}^{2n}$ be the Dynnikov coordinates of a central integral lamination $\mathcal{L}$, with $n > 3$.*

Suppose that $b_i = 0$ for some i. Let $(a';\, b') \in \mathbb{Z}^{2n-2}$ be obtained from $(a;\, b)$ by erasing the coordinates a_i and b_i (so that $a'_j = a_j$ and $b'_j = b_j$ for $j < i$, while $a'_j = a_{j+1}$ and $b'_j = b_{j+1}$ for $j \geq i$).

Then $(a';\, b')$ are the Dynnikov coordinates of a central integral lamination with $X(a';\, b') = X(a;\, b)$ and $\mathcal{C}(a';\, b') < \mathcal{C}(a;\, b)$.

Proof. Let $L \subset D_{n+2}$ be a minimal representative of $\mathcal{L}$. We write L' for the same disjoint union of simple closed curves, regarded as a subset of the $(n+1)$-punctured disk D_{n+1} obtained by filling in puncture $i+1$. Take Dynnikov arcs in D_{n+1} given by $\alpha'_j = \alpha_j$ for $-1 \leq j \leq 2i-2$; $\alpha'_j = \alpha_{j+2}$ for $2i-1 \leq j \leq 2n-4$; $\beta'_j = \beta_j$ for $0 \leq j \leq i$; and $\beta'_j = \beta_{j+1}$ for $i+1 \leq j \leq n-1$. We shall show that

a) Each component of L' is essential in D_{n+1}, so that L' is a representative of an integral lamination $\mathcal{L}'$;

b) L' is a minimal representative of $\mathcal{L}'$; and

c) $\mathcal{L}'$ is central.

It follows from b) that the Dynnikov coordinates $(a';\, b')$ of $\mathcal{L}'$ are as given in the statement of the lemma. Since $X(a;\, b)$ and $X(a';\, b')$ are both equal to the number of components of L, and $n(a';\, b') = n(a;\, b) - 1$, this will establish the result.

a) Since $\mathcal{L}$ is central, L' does not intersect the arcs $\beta'_0 = \beta_0$ or $\beta'_{n-1} = \beta_n$, and hence has no components which bound an $(n+1)$-punctured disk. Since every component of L bounds a disk containing at least two punctures, no component of L' can bound an unpunctured disk; and if there were a component bounding a once-punctured disk, it would coincide with a component of L bounding a disk containing puncture $i+1$ and one other puncture. If this puncture were to the right (respectively left) of puncture $i+1$, then L would have a left (respectively right) loop component in Δ_i, and we would have $b_i < 0$ (respectively $b_i > 0$). Since $b_i = 0$, this is impossible.

b) Suppose for a contradiction that L' is not a minimal representative of $\mathcal{L}'$. Then there is a Dynnikov arc γ' in D_{n+1} and a component J of $L' \setminus \gamma'$ which is an arc whose union with the segment of γ' bounded by its endpoints forms a simple closed

curve C bounding an unpunctured disk D. Since L is a minimal representative of $\mathcal{L}$ and γ' is a Dynnikov arc in D_{n+2}, the disk D must contain the filled in puncture $i+1$. Again, since L is minimal, if γ' is to the right (respectively left) of puncture $i+1$, then J does not intersect the Dynnikov arc β_i (respectively β_{i+1}); hence L has a left (respectively right) loop component in Δ_i, so that $b_i < 0$ (respectively $b_i > 0$). This is the required contradiction.

c) We have $\beta'_0 = \beta_0 = 0$ and $\beta'_{n-1} = \beta_n = 0$ (or, if $i = n-1$, then $\beta'_{n-1} = \beta_{n-1} = 0$, since $b_{n-1} = 0$ and $\beta_n = 0$), so $\mathcal{L}'$ is central.

□

3.4.2 Erasing elementary components

This move is applied when a minimal representative of the lamination contains components which are elementary curves. We erase these elementary components, thereby simplifying the lamination.

Lemma 7. *Let $(a;\, b) \in \mathbb{Z}^{2n}$ be the Dynnikov coordinates of a central integral lamination $\mathcal{L}$ with $n > 3$.*

Suppose that $b_j \neq 0$ for all j, so that $i = i(a;\, b) > 0$ and $M = \min(-b_{i-1},\, b_i) > 0$; and that $a_{i-1} = a_i$. Let $(a';\, b') \in \mathbb{Z}^{2n}$ be defined by $a' = a$; $b'_{i-1} = b_{i-1} + M$; $b'_i = b_i - M$; and $b'_j = b_j$ for all $j \neq i-1,\, i$.

Then $(a';\, b')$ are the Dynnikov coordinates of a central integral lamination with $X(a';\, b') = X(a;\, b) - M$ and $\mathcal{C}(a';\, b') < \mathcal{C}(a;\, b)$.

Proof. Let L be a minimal representative of $\mathcal{L}$. By Lemma 2, L has M components which are elementary curves about punctures i and $i+1$. Moreover, L has other components besides these, since all of the b_j are non-zero (an integral lamination consisting entirely of elementary curves about these punctures would have $b_j = 0$ for all $j \neq i-1,\, i$). Therefore the union of simple closed curves L' obtained from L by erasing these elementary curves is a minimal representative of a central integral lamination $\mathcal{L}'$.

Erasing the elementary curves reduces the number of intersections with the arcs α_{2i-3}, α_{2i-2}, α_{2i-1}, and α_{2i} by M; and the number of intersections with the arc β_i by $2M$. Therefore $\mathcal{L}'$ has Dynnikov coordinates $(a';\, b')$ as given in the statement of the lemma.

Clearly $X(a';\, b') = X(a;\, b) - M$; and $n(a';\, b') = n(a;\, b)$, while $\sum_{i=0}^{n-1} |b'_i| = \sum_{i=0}^{n-1} |b_i| - 2M$, so that $\mathcal{C}(a', b') < \mathcal{C}(a, b)$. □

3.4.3 Untwisting

This move is applied when there are two consecutive punctures, with a left loop about the left puncture and a right loop about the right puncture, but no elementary curves

about these two punctures. Applying an appropriate braid generator simplifies the lamination.

Lemma 8. *Let $(a;\, b) \in \mathbb{Z}^{2n}$ be the Dynnikov coordinates of a central integral lamination $\mathcal{L}$ with $n > 3$.*

Suppose that $b_j \neq 0$ for all j, so that $i = i(a;\, b) > 0$; and that $a_{i-1} \neq a_i$. Let $(a';\, b') \in \mathbb{Z}^{2n}$ be defined by $a'_j = a_j$ and $b'_j = b_j$ for all $j \neq i-1,\, i$, and:

Case Ia): $0 < a_{i-1} - a_i$ **and** $b_i - b_{i-1} \leq a_{i-1} - a_i$**.**

$$\begin{aligned} a'_{i-1} &= a_i - b_{i-1}, \\ a'_i &= a_{i-1} - b_i, \\ b'_{i-1} &= b_i, \\ b'_i &= b_{i-1}. \end{aligned}$$

Case Ib): $0 < a_{i-1} - a_i < b_i - b_{i-1}$**.**

$$\begin{aligned} a'_{i-1} &= \min(a_i - b_{i-1},\ a_{i-1}), \\ a'_i &= \max(a_{i-1} - b_i,\ a_i), \\ b'_{i-1} &= b_{i-1} + (a_{i-1} - a_i), \\ b'_i &= b_i - (a_{i-1} - a_i). \end{aligned}$$

Case IIa): $0 < a_i - a_{i-1}$ **and** $b_i - b_{i-1} \leq a_i - a_{i-1}$**.**

$$\begin{aligned} a'_{i-1} &= a_i + b_{i-1}, \\ a'_i &= a_{i-1} + b_i, \\ b'_{i-1} &= b_i, \\ b'_i &= b_{i-1}. \end{aligned}$$

Case IIb): $0 < a_i - a_{i-1} < b_i - b_{i-1}$**.**

$$\begin{aligned} a'_{i-1} &= \max(a_{i-1},\ a_i + b_{i-1}), \\ a'_i &= \min(a_i,\ a_{i-1} + b_i), \\ b'_{i-1} &= b_{i-1} + (a_i - a_{i-1}), \\ b'_i &= b_i - (a_i - a_{i-1}). \end{aligned}$$

Then $(a';\, b')$ are the Dynnikov coordinates of a central integral lamination with $X(a';\, b') = X(a;\, b)$ and $\mathcal{C}(a';\, b') < \mathcal{C}(a;\, b)$.

Proof. The coordinates $(a'; b')$ in the statement of the lemma are obtained from $(a; b)$ by applying the braid generator σ_i^{-1} (case I) or σ_i (case II), using the inequalities pertaining to each case to resolve some of the maxima and minima in (5) and (8). Therefore $(a'; b')$ are the Dynnikov coordinates of a central integral lamination with $X(a'; b') = X(a; b)$, and it only remains to show that the complexity has decreased.

In all cases we have $n(a'; b') = n(a; b)$. In cases Ia) and IIa) we also have $\sum_{i=0}^{n-1} |b'_i| = \sum_{i=0}^{n-1} |b_i|$: however $i(a'; b') = i(a; b) - 1$, since the first positive component of b' is b'_{i-1}. Therefore $\mathcal{C}(a'; b') < \mathcal{C}(a; b)$.

In case Ib) we have that $b_{i-1} < 0$, $b_i > 0$, and $0 < a_{i-1} - a_i < b_i - b_{i-1}$, and we proceed by considering cases.

a) If $a_{i-1} - a_i \leq \min(-b_{i-1}, b_i)$ then

$$\begin{aligned} |b'_{i-1}| + |b'_i| &= (-b_{i-1} - (a_{i-1} - a_i)) + (b_i - (a_{i-1} - a_i)) \\ &= |b_{i-1}| + |b_i| - 2(a_{i-1} - a_i) < |b_{i-1}| + |b_i|. \end{aligned}$$

b) If $-b_{i-1} < a_{i-1} - a_i \leq b_i$ then

$$\begin{aligned} |b'_{i-1}| + |b'_i| &= (b_{i-1} + (a_{i-1} - a_i)) + (b_i - (a_{i-1} - a_i)) \\ &= b_{i-1} + b_i < |b_{i-1}| + |b_i|. \end{aligned}$$

c) If $b_i < a_{i-1} - a_i \leq -b_{i-1}$ then

$$\begin{aligned} |b'_{i-1}| + |b'_i| &= (-b_{i-1} - (a_{i-1} - a_i)) + (-b_i + (a_{i-1} - a_i)) \\ &= -b_{i-1} - b_i < |b_{i-1}| + |b_i|. \end{aligned}$$

d) If $\max(-b_{i-1}, b_i) < a_{i-1} - a_i$ then

$$\begin{aligned} |b'_{i-1}| + |b'_i| &= (b_{i-1} + (a_{i-1} - a_i)) + (-b_i + (a_{i-1} - a_i)) \\ &< b_{i-1} - b_i + 2(b_i - b_{i-1}) = b_i - b_{i-1} = |b_{i-1}| + |b_i|. \end{aligned}$$

Therefore $\mathcal{C}(a'; b') < \mathcal{C}(a; b)$. A similar argument applies in case IIb). □

3.5 Statement of the algorithm

Algorithm 9 below computes the number of components of an integral lamination $\mathcal{L} \in \mathcal{L}_n$. We assume that $n > 3$, since otherwise the number of components is given by Lemma 4. The algorithm works with a pair $((a; b), Y)$, where $(a; b)$ are extended Dynnikov coordinates and Y is a non-negative integer which counts the number of elementary curve components which have been erased: the quantity $X(a; b) + Y$ remains constant throughout.

Algorithm 9. Let $(a;\, b) \in \mathbb{Z}^{2n-4}$ be the Dynnikov coordinates of an integral lamination $\mathcal{L} \in \mathcal{L}_n$, with $n > 3$.

Step 1 Replace $(a;\, b)$ with $(a;\, b) \in \mathbb{Z}^{2n}$ given by (10). Set $Y = 0$ and input the pair $((a;\, b),\, Y)$ to Step 2.

Step 2 **If** $b_i = 0$ for some i, then let $(a';\, b')$ be given by *Filling in a puncture* (Lemma 6). If $n(a';\, b') = 3$ then input $((a';\, b'),\, Y)$ to Step 5: if $n(a';\, b') > 3$ then input $((a';\, b'),\, Y)$ to Step 2.

Otherwise, input $((a;\, b),\, Y)$ to Step 3.

Step 3 Let $i = i(a;\, b)$. **If** $a_{i-1} = a_i$, then let $(a';\, b')$ be given by *Erasing elementary components* (Lemma 7), and input $((a';\, b'),\, Y + \min(-b_{i-1},\, b_i))$ to Step 2.

Otherwise, input $((a;\, b),\, Y)$ to Step 4.

Step 4 Let $(a';\, b')$ be given by *Untwisting* (Lemma 8). Input $((a';\, b'),\, Y)$ to Step 3 in cases Ia) and IIa) of the Lemma, or to Step 2 in cases Ib) and IIb).

Step 5 Since $n(a;\, b) = 3$, we have $(a;\, b) = (0, a_1, 0;\, b_0, b_1, b_2)$. The number of components of the original integral lamination is given by

$$\gcd(a_1, b_1) + Y + \min(-b_0,\, b_2,\, -|a_1| - b_0 - b_1^+).$$

Proof. It is immediate from Lemmas 6, 7, and 8 that the quantity $X(a\,;b) + Y$ remains constant throughout the algorithm, and that $\mathcal{C}(a\,;b)$ decreases each time one of the moves is applied. The algorithm therefore terminates (i.e. reaches Step 5), since there are no infinite strictly decreasing sequences in $\mathbb{N}^3$; and the number of components of the starting integral lamination is equal to $X(a;\, b) + Y$, where $(a;\, b) = (0, a_1, 0;\, b_0, b_1, b_2)$ and Y are the inputs to Step 5. It therefore only remains to show that

$$X(0, a_1, 0;\, b_0, b_1, b_2) = \gcd(a_1, b_1) + \min(-b_0,\, b_2,\, -|a_1| - b_0 - b_1^+).$$

Now $(0, a_1, 0;\, b_0, b_1, b_2)$ are the Dynnikov coordinates of a central lamination on D_5. Let L be a minimal representative. We shall show that the number Z of components of L which bound a disk containing the three central punctures is given by $Z = \min(-b_0,\, b_2,\, -|a_1| - b_0 - b_1^+)$. This will complete the proof, since erasing these components and filling in the two end punctures yields a representative of an integral lamination on D_3 with Dynnikov coordinates $(a_1;\, b_1)$, which has $\gcd(a_1,\, b_1)$ components by Lemma 4.

Now Z is the minimum of the number of left loop components in Δ_0; the number of right loop components in Δ_2; the number of above components in Δ_1; and the number of below components in Δ_1. The first two of these numbers are $-b_0$ and b_2; and the third and fourth are $\alpha_1 - |b_1|$ and $\alpha_2 - |b_1|$.

Suppose first that $b_1 \geq 0$. Since the lamination is central, we have $\beta_0 = 0$, and hence $\beta_1 = -2b_0$ by (1). Then (3) gives that $\alpha_1 = -a_1 - b_0$ and $\alpha_2 = a_1 - b_0$. Therefore

$$Z = \min(-b_0,\, b_2,\, -a_1 - b_0 - b_1,\, a_1 - b_0 - b_1) = \min(-b_0,\, b_2,\, -|a_1| - b_0 - b_1^+)$$

as required. A similar argument applies in the case $b_1 \leq 0$, when $\alpha_1 = -a_1 - b_0 - b_1$ and $\alpha_2 = a_1 - b_0 - b_1$. □

3.6 Complexity of the algorithm

In this section we analyse the complexity of the algorithm, when applied to an integral lamination $\mathcal{L} \in \mathcal{L}_n$ with Dynnikov coordinates $(a;\, b) \in \mathbb{Z}^{2n-4}$.

Write $M = \sum_{i=1}^{n-2}(|a_i| + |b_i|)$. By an *arithmetic operation* we mean adding, subtracting, comparing, taking the maximum, or taking the minimum of two integers. As we will see, these integers have absolute value $O(n^2M^2)$ thoughout the algorithm, so that the cost of each arithmetic operation is logarithmic in n and M.

Steps 1 and 5 are each carried out only once in the algorithm. Step 1 involves $O(n^2)$ arithmetic operations, while Step 5 involves $O(\log(n^2M^2))$ arithmetic operations (to calculate the greatest common divisor). Observe that the Dynnikov coordinates $(a';\, b')$ produced by Step 1 satisfy

$$M' := \sum_{i=0}^{n-1}(|a_i'| + |b_i'|) \leq M + 2 \max_{1 \leq k \leq n-2} \left(|a_k| + b_k^+ + \sum_{j=1}^{k-1} b_j \right) + \sum_{j=1}^{n-2} |b_j| = O(M).$$

Now consider the main body of the algorithm, consisting of Steps 2, 3, and 4. This can naturally be regarded as a loop: at each iteration, $O(n)$ arithmetic operations are carried out to scan the b coordinates and identify whether some $b_i = 0$; and, if not, to find $i = i(a;\, b)$ and to determine whether or not $a_{i-1} = a_i$. According to the results of these tests, one of the three moves *Filling in a puncture*, *Erasing elementary components*, or *Untwisting* is carried out.

Each of these three moves involves $O(1)$ arithmetic operations; and none of them increases $\sum_{i=1}^{n-1} |b_i|$.

Filling in a puncture is carried out $O(n)$ times during the course of the algorithm.

Erasing elementary components strictly decreases $\sum_{i=1}^{n-1} |b_i|$, and so is carried out $O(M)$ times during the course of the algorithm.

Untwisting strictly decreases $\sum_{i=1}^{n-1} |b_i|$ in cases Ib) and IIb), so these cases are carried out $O(M)$ times during the course of the algorithm. Cases Ia) and IIa) leave $\sum_{i=1}^{n-1} |b_i|$ constant. However, since they decrease $i(a;\, b)$ by exactly 1, these cases are repeated $O(n)$ times before either *Erasing elementary components* or Case Ib) or IIb) is applied. Moreover, no scanning of the b coordinates is necessary between successive applications of these cases.

The main body of the algorithm therefore involves $O(n^2M)$ arithmetic operations. Observing that none of the moves increases $\max_{i=0}^{n-1} |a_i|$ by more than $\sum_{i=1}^{n-1} |b_i|$ (which remains $\leq M$ throughout the algorithm), we see that the maximum size of the integers involved in arithmetic operations is $O(n^2M^2)$.

We therefore have the following result:

Lemma 10. *Let $(a;\, b)$ be the Dynnikov coordinates of an integral lamination $\mathcal{L} \in \mathcal{L}_n$ with $n > 3$, and write $M = \sum_{i=1}^{n-2}(|a_i| + |b_i|)$. Then applying Algorithm 9 to $\mathcal{L}$ requires $O(n^2M)$ arithmetic operations, each carried out on a pair of integers of sizes $O(n^2M^2)$.*

3.7 An example

In this section we use Algorithm 9 to compute the number of components of the integral lamination $\mathcal{L} \in \mathcal{L}_6$ with Dynnikov coordinates $\rho(\mathcal{L}) = (-1,\, -2,\, -2,\, 1\,;\, -1,\, 2,\, -2,\, 2)$. The successive moves are illustrated in Figure 2.

1. *Extend coordinates*: apply (10) to replace the coordinates with

$$(a;\, b) = (0,\, -1,\, -2,\, -2,\, 1,\, 0\,;\, -3,\, -1,\, 2,\, -2,\, 2,\, 2),$$

   and input $((a;\, b),\, 0)$ to the main algorithm (Step 2).

2. *Untwisting*: we have $b_i \neq 0$ for all i, so we proceed to Step 3. The first positive b coordinate is b_2, so we have $i(a;\, b) = 2$. Since $a_1 \neq a_2$ we proceed to Step 4.

   Since $a_1 - a_2 = 1 > 0$ and $b_2 - b_1 = 3 > a_1 - a_2$ we are in Case Ib). We get

$$(a;\, b) = (0,\, -1,\, -2,\, -2,\, 1,\, 0\,;\, -3,\, 0,\, 1,\, -2,\, 2,\, 2),$$

   and input $((a;\, b),\, 0)$ to Step 2.

3. *Fill in puncture 2*: since $b_1 = 0$ we fill in puncture 2. We get

$$(a;\, b) = (0,\, -2,\, -2,\, 1,\, 0\,;\, -3,\, 1,\, -2,\, 2,\, 2),$$

   and input $((a;\, b),\, 0)$ to Step 2.

4. *Untwisting*: we have $b_i \neq 0$ for all i, so we proceed to Step 3. The first positive b coordinate is b_1, so we have $i(a;\, b) = 1$. Since $a_0 \neq a_1$ we proceed to Step 4.

   Since $a_0 - a_1 = 2 > 0$ and $b_1 - b_0 = 4 > a_0 - a_1$ we are again in Case Ib). We get

$$(a;\, b) = (0,\, -1,\, -2,\, 1,\, 0\,;\, -1,\, -1,\, -2,\, 2,\, 2),$$

   and input $((a;\, b),\, 0)$ to Step 2.

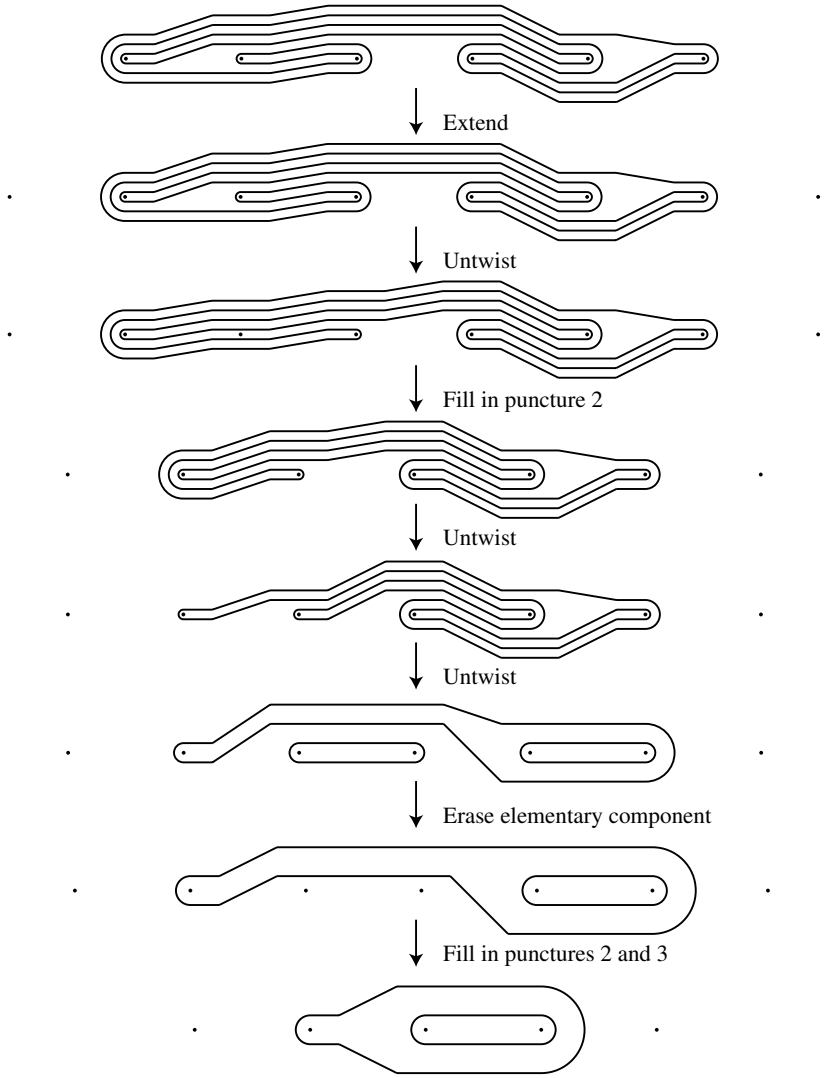

Figure 2: An example of the algorithm

5. *Untwisting*: we have $b_i \neq 0$ for all i, so we proceed to Step 3. The first positive b coordinate is b_3, so we have $i(a;\, b) = 3$. Since $a_2 \neq a_3$, we proceed to Step 4.

   Since $a_3 - a_2 = 3 > 0$ and $b_3 - b_2 = 4 > a_3 - a_2$ we are in Case IIb). We get

$$(a;\, b) = (0,\, -1,\, -1,\, 0,\, 0\,;\; -1,\, -1,\, 1,\, -1,\, 2).$$

6. *Erasing an elementary component*: we have $b_i \neq 0$ for all i, so we proceed to Step 3. The first positive b coordinate is b_2, so we have $i(a;\, b) = 2$. Since $a_1 = a_2$, we can erase elementary components about punctures 2 and 3. Since $\min(-b_1, b_2) = 1$, there is 1 such component. We get

$$(a;\, b) = (0,\, -1,\, -1,\, 0,\, 0\,;\; -1,\, 0,\, 0,\, -1,\, 2),$$

and input $((a;\, b),\, 1)$ to Step 2.

7. *Fill in punctures 2 and 3*: since $b_1 = b_2 = 0$ we fill in punctures 2 and 3 (following the algorithm strictly, we first fill in puncture 2, and then fill in the new puncture 2, which is the previous puncture 3). We get

$$(a;\, b) = (0,\, 0,\, 0\,;\ -1,\, -1,\, 2).$$

   Since $n(a;\, b) = 3$, we input $((a;\, b),\, 1)$ to Step 5.

8. *Number of components*: we have $Y = 1$; $\gcd(a_1, b_1) = \gcd(0, -1) = 1$; and $\min(-b_0,\, b_2,\, -|a_1| - b_0 - b_1^+) = \min(1,\, 2,\, 1) = 1$ (corresponding to the fact that our final lamination has one component bounding a disk containing the three central punctures).

   Therefore $X(-1,\, -2,\, -1,\, 0\,;\ -1,\, 2,\, 1,\, -1) = 3$.

References

[1] E. Artin, *Theory of braids*, Ann. of Math. (2) **48** (1947), 101–126.

[2] J. S. Birman, *Braids, links, and mapping class groups*, Princeton University Press, Princeton, N.J., 1974, Annals of Mathematics Studies, No. 82.

[3] P. Dehornoy, I. Dynnikov, D. Rolfsen, and B. Wiest, *Ordering braids*, Mathematical Surveys and Monographs, vol. 148, American Mathematical Society, Providence, RI, 2008.

[4] I. Dynnikov, *On a Yang-Baxter mapping and the Dehornoy ordering*, Uspekhi Mat. Nauk **57** (2002), no. 3(345), 151–152.

[5] I. Dynnikov and B. Wiest, *On the complexity of braids*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 4, 801–840.

[6] A. Haas and P. Susskind, *The connectivity of multicurves determined by integral weight train tracks*, Trans. Amer. Math. Soc. **329** (1992), no. 2, 637–652. MR 1028309

[7] T. Hall and S. Ö. Yurttaş, *On the topological entropy of families of braids*, Topology Appl. **156** (2009), no. 8, 1554–1564.

[8] J-O. Moussafir, *On computing the entropy of braids*, Funct. Anal. Other Math. **1** (2006), no. 1, 37–46.

[9] R. C. Penner, *Probing mapping class groups using arcs*, Problems on mapping class groups and related topics, Proc. Sympos. Pure Math., vol. 74, Amer. Math. Soc., Providence, RI, 2006, pp. 97–114. MR 2264535

[10] R. C. Penner and J. L. Harer, *Combinatorics of train tracks*, Annals of Mathematics Studies, vol. 125, Princeton University Press, Princeton, NJ, 1992.

[11] S. Ö. Yurttaş, *Geometric intersection of curves on punctured disks*, Journal of the Mathematical Society of Japan **65** (2013), no. 4, 1554–1564.