Dynamic Modelling and Stability in Social Security Schemes

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy by

HUMBERTO GODÍNEZ OLIVARES

October, 2016

A Carolina y Nicolás

Intelligence is the ability to adapt to change.

Stephen Hawking

Acknowledgments

I would like to thank all those people and institutions who made this thesis possible and an unforgettable experience for me.

Foremost, I would like to express my gratitude to my supervisor Dr. Carmen Boado-Penas for her continuous support of my PhD studies and research, for her patience, motivation, enthusiasm, and immense knowledge. Her guidance helped me in all the time of research and writing of this thesis. Carmen, thanks for supporting me in all my decisions! I could not have imagined having a better advisor, mentor and friend for my PhD studies ¡¡ Thanks !!

Besides my supervisor, I would like to thank the rest of my thesis committee Prof. Brendan McCabe and Prof. Pierre Devolder for their encouragement, insightful comments, advice and suggestions.

Thanks to the Institute for Financial and Actuarial Mathematics at The University of Liverpool, in particular to Dr. Corina Constantinescu for her selfless support. To all the anonymous referees and those who provided comments during the conferences and seminars where the papers from this thesis were presented.

I am also thankful for the financial support received from Petroleos Mexicanos, the National Council on Science and Technology, the IPE Pensions Scholarship Fund, the Mexican Ministry of Education and the European Union's Seventh Framework Programme for research, technological development and demonstration under grant agreement No. 318984-RARE.

All my colleagues on the PhD programme, especially of those in Room 310 (Jiajie, Moyu, Renchao, and Vasilis) + Dimitirs (a.k.a Pedro), my friends of L17 (Adriana, Alfredo, Lulú), my Rockers' (a special mention to Serch and Claus for all the love given to us and the home feeling whenever we were in need), to my *Familia Olivares* for always being there. To Gloria and Evaristo for your love and support. To Arturo for being the first to visit me and for your sour personality. To Alex, Betito, Ile, Ines, Mau and Val for the great times we shared and the amazing *ficha* time! and thanks to Eduardo and Heidi for the amazing summer in 2015.

I would like to give special thanks to my Mom, Dad, Pily and Beky. *Mamá, Papá, Pily y Beky gracias por toda la confianza incondicional, apoyo, y amor. Gracias por siempre creer en mi. Los amo.*

Finally, my deepest sense of gratitude to the most important people in my life, my beloveds wife and son, Carolina and Nicolás. Caro, thanks to your love, sleepless nights, encouragement, and continuous support I am here. This thesis would not have been possible without you. I will never forget your words "it is going to be worth it" and it was. I love you. Nicolás, thanks because this last year has been the best of my life thanks to your laughs, smiles, milestones and love! Thanks both for filling my life with love and happiness. I love you.

Abstract

Since social security involves several individual parameters, in recent years considerable attention has been focused on the impact these parameters on pension and social security systems. The literature on pensions has long been highlighting concerns that public Pay-As-You-Go (PAYGO) pension systems will turn out to be unsustainable in the long run and are a concern for most countries around the world, from the industrialised nations to the developing countries. The common trend in responses to what is a pensions crisis is a wave of parametric pension adjustments during the last years. These parametric reforms include, among others, changes in the contribution ceilings, increases in the retirement age, reductions in the indexation of pensions or even carrying out a structural reform from a Defined Benefit pension system to a Notional Defined Contribution (NDC).

Following this process of reforming the pension system, this thesis is focused on the most important innovation in public pension schemes over the past years, first on Actuarial Balancing Mechanism (ABM) in PAYGO and second in some aspects of the Notional Defined Contributions, both in a deterministic framework. The ABM mechanism, that uses non-linear optimization models, identifies and applies an optimal path of these variables into a PAYGO system and absorbs fluctuations in longevity, fertility rates, life expectancy or any other events in a pension system. For the NDC, the Survivor Dividend (SD), also called inheritance gains, kept by most NDCs is analysed under different assumptions to calculate the maximum mortality decrease a scheme can cover if the SD is not distributed and whether the SD is a potential solution to cover the longevity.

The research has considerable potential impact. It addresses a clear need in political, business, economic and societal contexts. This project also bridges the gap between academics and policy makers for better pension's public policies under alternative financial and economic scenarios. As a result, it will allow to design and assess the path of reforms in a more efficient manner. Further development will include a stochastic framework, considering stochastic dynamic programming, robustness, sensitivity analysis and error bounds.

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Introduction

The public pension systems are usually financed on a Pay-As-You-Go (PAYGO) basis where pensions for retirees are paid by the contributions of the working-age population. A successfully PAYGO system require a balance between the expenditure on pensions and the income from contributions made by the active workers, that is known as intergenerational solidarity[1], and long-term sustainability. Recently, the evolution and forecast of the relevant parameters involved in the provision of pension schemes are major topics of on-going discussion for the policy and decision-making committees in many countries, because ageing populations are exerting significant pressures on the public finances of virtually all countries.

Birth rates have dramatically decreased and with continuous improvements in life expectancy due to improved health care and medical innovations, pensions are paid over a longer time horizon, which causes great difficulties when the system does not expect such improvements and this will raise serious concerns for the sustainability of the PAYGO pension systems. The Congress of the United States (2005) projected that, in a few decades, their Social Security system will be paying out more in benefits than it collects in taxes. This imbalance will be growing and is expected to remain. In Europe, European Commission (2012b) (White Paper) and European Commission (2010) (Green Paper), shows a clear increase in life expectancy at birth for males and females of 7.9 and 6.5 years respectively from 2010 to 2060. Additionally, the European population aged 60+ is estimated to become almost twice as high in 2060 as it was in the early 2000s. Moreover, in 2012, pension expenditure represented more than 10% of GDP and this is forecast to rise to 12.5% in 2060 in the EU. Furthermore, the recent global financial crisis and a severe world-wide public debt crisis have exerted additional stress on pension systems.

[1] Haberman & Zimbidis (2002b), define intergenerational solidarity as the willingness of different groups of people to participate in a common pool, sharing actual experience, including any losses emerging. In the context of public pension systems, the concept refers to both young and old generations.

The common trend in responses to what is a pensions crisis is a wave of parametric pension adjustments during the last years in countries like France, Greece, Hungary, Japan, Romania and Spain[2]. These parametric reforms include, among others, changes in the contribution ceilings, increases in the retirement age, reductions in the indexation of pensions or even carrying out a structural reform from a Defined Benefit PAYGO pension system to a Notional Defined Contribution (NDC) or even to a funding pension schemes. In Latin America, for most of their history, social protection systems have been based on a Pay-As-You-Go finance system (Rofman, Apella, & Vezza (2015)). However, the pressure in their finance forced them to implement several reforms. Since the 1980's, most of the region countries made structural reforms replacing totally or partially their PAYGO system with programs containing a fully funded component of individually capitalized accounts. As a result, a reduction of expected benefits and a transfer of financial market and volatility risk from the state to the individual happened.

Those reforms need to reach the objective of better managed and designed pensions system. A useful way of reaching these objectives is to provide better information on the status and financial development of a public pension system by means of an actuarial balance[3].

Even though the information provided by the actuarial balance can help to take better and more informed decisions, some of the countries decide not to use it, i.e., USA, or even take some parametric reforms via emergency modifications in legislation without compiling any balance and knowing the real situation of the pension system. Following this suddenly changes in legislations, Vidal-Meliá, Boado-Penas, & Settergren (2009) and Vidal-Meliá, Boado-Penas, & Settergren (2010) propose an *Automatic Balance Mechanism* (ABM) that the authors define as a set of pre-determined measures established by law to be applied immediately as required according to the solvency or sustainability indicator[4]. Its purpose, through successive application, is to re-establish the

[2]See Whitehouse (2009a), Whitehouse (2009b), OECD (2012).

[3]For interested readers see Settergren (2008) for Sweden and The 2015 Annual report of the Board of Trustees of the Federal Old-Age and Survivors Insurance and Federal Disability Insurance Trust Funds (OASDI) (2015) for the USA

[4]According Knell, Kohler-Toglhofer, & Prammer (2006), the term sustainability has many definitions, though it almost always refers to the fiscal policies of a government, the public sector or the pension system. One of the most widely accepted definitions in the area of pensions is that of a position where there is no need to increase the contribution rate in

financial equilibrium of PAYGO pension systems with the aim of making those systems viable without the repeated intervention of the legislators. They argue that there are three reasons for introducing an ABM method, first, to adapt the system to changes in socioeconomic and demographic conditions; secondly, to create a credible institutional framework in the sense that promises of pension payments are kept; and finally, to minimise the use of the pension system as an electoral weapon. As examined by Turner (2009) at least 12 countries have indexation of benefits linked to life expectancy or any other kind of automatic adjustment (For more information on ABM's see, for instance, Turner (2007), Turner (2009), Vidal-Meliá et al. (2009), OECD (2012), and OECD (2014)).

Another major change is what is known as Notional Defined Contribution pension schemes, first developed about 20 years ago in countries such as Italy, Latvia, Poland and Sweden. Other countries, Holzmann, Palmer, & Robalino (2012), including Egypt, China and Greece are seriously considering the introduction of NDCs. The NDC model retains Pay-As-You-Go state financing but mimics a privately funded DC plan. Workers continue to pay for today's pensioners but their contributions are also credited to notional accounts, which get a rate of return broadly linked to earnings growth. When they retire their pension benefits are based on the notional capital they have accumulated, which is turned into annuities through a formula based on life expectancy at their retirement age (London (2013)).

However in notional accounts, the return that contributions earn is calculated on the basis of a macroeconomic index that tries to reflect the financial health of the system (i.e. changes in salaries, GDP growth) and not market returns. This return is called the notional interest rate. When the individual retires, the accumulated contributions, or notional account, are converted into a life annuity according to standard actuarial practice. Therefore, the amount of the initial pension depends on the mortality of the retiring cohort, potential future pension indexation and the technical interest rate used to discount the cash flows.

Following this process of reforming the pension system, this thesis is focused on the most important innovation in public pension schemes over the past years, first on Actuarial Balancing Mechanism in PAYGO and second in some

the future. On the other hand, the concept of solvency refers to the ability of a pension scheme's assets to meet the scheme's liabilities indicator.

aspects of the Notional Defined Contributions, both in a deterministic framework. The aim is to twofold: to design Automatic Balancing Mechanisms to restore the liquidity and sustainability of Pay-As-You-Go pension system based on minimizing changes in the main variables, such as the contribution rate, normal retirement age and indexation of pensions and to analyse the Survivor Dividend, also called inheritance gains, kept by most NDCs can be used to cover an unexpected longevity increase.

Given the above objectives, the project will be divided in four different chapters. The first chapter analyse to what extent the Survivor Dividend kept by most NDCs can be used to cover an unexpected longevity increase. Formulas are developed under different assumptions (constant or according to Lee-Carter mortality improvements) to calculate the maximum mortality decrease a scheme can cover if the Survivor Dividend is not distributed. We also apply the formulas using Polish, Latvian and Swedish life tables and show that the non-distribution of the Survivor Dividend is a potential solution to cover the longevity risk of NDCs. The second chapter designs optimal strategies, also known as Automatic Balancing Mechanisms, using non-linear optimisation techniques to guarantee the required level of liquidity in Pay-As-You-Go pension systems through changes in the key variables of the system, such as, the contribution rate, retirement age and/or indexation of pensions. In particular, this method aims to keep the future percentage of growth rates of the projections for the contribution rate, the normal retirement age and the indexation of pensions at a minimum level using a logarithm function. The model is studied for two interesting cases, the symmetric and the asymmetric. Under the symmetric design, our model determines whether the contribution rate and the age of normal retirement (and indexation of pensions) are reduced (increased) when the buffer fund has a surplus or increased (decreased) in periods of deficit. In the absence of a symmetric design any surplus is kept by the system, so the key variables can only increase or maintain the same level. The logarithmic cost criterion try to minimise the percentage of changes in the variables over time instead of trying to stabilise them around an initial set value, as in the case of using a quadratic function. However, the choice of a logarithmic function present some weaknesses for the symmetric design as the percentage of change could the positive or negative. The optimisation (minimisation) problem will try to push the variables in lower bounds of the constraints setting the fund in negative values. The aim of third chapter is to design an Actuarial Balancing Mechanism to restore the sustainability of a Pay-As-You-Go pension system. The model is adapted to different population structures. Expected and unexpected shocks are analysed in order to see how a buffer fund absorbs these effects. In the fourth chapter an alternative reform of the Spanish contributory retirement pension system is designed based on optimal strategies to restore liquidity through changes in the key variables of the system (the contribution rate, retirement age and/or indexation of pensions) and at the same time an assessment of the reformed Spanish system it is done with the focus on its liquidity in the long run. Furthermore, our ABMs can be easily generalised including more parameters and is applicable to any PAYGO pension system.

However, in light of real-world pension systems with complex dynamics, the introduction of a stochastic framework is necessary. Some useful techniques for this purpose are stochastic dynamic programming, robustness, sensitivity analysis and error bounds . These techniques are used for modelling optimisation problems that involve uncertainty. The models presented in the thesis are deterministic optimisation problems since we are assuming the parameters are known. However in real world problems some parameters are unknown or random.

The impact of the thesis is to bring forth an improved understanding of the complex issues around making public policies related to pensions systems using different financial and economic scenarios. It addresses a clear need in political, business, economic and societal contexts. This project also bridges the gap between academics and policy makers for better pension's public policies under alternative financial and economic scenarios. As a result, it will allow to design and assess the path of reforms in a more efficient manner.

1

Longevity Risk in Notional Defined Contribution Pension Schemes: A Solution

S. Arnold (-Gaille), M. del C. Boado-Penas and H. Godínez Olivares[1]

Notional defined contribution pension schemes (NDCs) aim at reproducing the logic of a financial defined contribution plan under a pay-as-you-go framework. Of particular interest is how the accumulated capital of a deceased person is used when the death occurs prior to retirement. While in most countries this accumulated capital (called survivor dividend, SD) is kept by the scheme, in Sweden it is distributed among the same cohort survivors. This chapter aims to analyse to what extent the SD kept by most NDCs can be used to cover an unexpected longevity increase. We develop formulas under different assumptions (constant or according to Lee-Carter mortality improvements) to calculate the maximum mortality decrease a scheme can cover if the SD is not distributed. We also apply the formulas using Polish, Latvian and Swedish life tables and show that the non-distribution of the SD is a potential solution to cover the longevity risk of NDCs.

Keywords: notional defined contribution; longevity risk; pay-as-you-go; public pensions; Lee-Carter Model; retirement.

1.1 Introduction

Retirement systems across the world are undergoing major reforms in order to adapt to continuously changing economic and demographic factors. Among these major changes is what is known as notional defined contribution pension

[1] This chapter is based on the paper 'Longevity Risk in Notional Defined Contribution Pension Schemes: A Solution' by Arnold (-Gaille), Boado-Penas, & Godínez-Olivares (2016). The Geneva Papers on Risk and Insurance - Issues and Practice. 41 (1), 24-52.

schemes (NDCs), first developed about 20 years ago in countries such as Italy, Latvia, Poland and Sweden. Other countries, Holzmann et al. (2012), including Egypt, China and Greece are seriously considering the introduction of NDCs.[2]

These pension schemes are ruled by the same principle: they attempt to reproduce the logic of a financial defined contribution pension plan within a pay-as-you-go framework. Pension contributions throughout the participant's working life are accumulated in a fictive or notional account (World Bank (2005); The Swedish Pension System (2015)). However in notional accounts, the return that contributions earn is calculated on the basis of a macroeconomic index that tries to reflect the financial health of the system (i.e. changes in salaries, GDP growth) and not market returns. This return is called the notional interest rate. When the individual retires, the accumulated contributions, or notional account, are converted into a life annuity according to standard actuarial practice. Therefore, the amount of the initial pension depends on the mortality of the retiring cohort, potential future pension indexation and the technical interest rate used to discount the cash flows.

Currently, most NDC schemes are accumulating a capital that is not necessarily used to finance pension benefits. Indeed, when a death occurs prior to the retirement age, the accumulated capital of the deceased person is kept by the scheme.[3] The scheme then accumulates some reserves, with no clear purpose. For instance, in Italy, Latvia and Poland, this money is not used and implicitly becomes a component of general public revenues. Specifically, in Poland and Latvia, these revenues provide the required funding for some un-

[2]Interested readers in NDCs can consult, for example, Lindbeck & Persson (2003); Williamson (2004); Börsch-Supan (2006); Holzmann & Palmer (2006); Vidal-Meliá, Domínguez-Fabián, & Devesa-Carpio (2006); Auerbach & Lee (2009); Vidal-Meliá et al. (2010); Whitehouse (2010); Auerbach & Lee (2011); Chłoń-Domińczak, Franco, & Palmer (2012)

[3]See Holzmann & Palmer (2006) and Chłoń-Domińczak et al. (2012) for particularities of NDCs in different countries.

funded insurance commitments, for example, legacy costs from the old pension system.

Sweden, The Swedish Pension System (2015), is the only country that distributes the accumulated capital of the deceased person among the survivors of the same birth cohort.[4] In Sweden, this capital is known as inheritance gains, but it can also be called the "survivor dividend".[5] The notional account of each surviving individual increases then faster and thus the pension benefits paid by the system from the retirement age are higher. In Sweden, the system is then actuarially fair: the present value of the total contributions is equivalent to the present value of the total benefits.

As for any pension schemes, life insurance companies and many other institutions, NDC schemes are facing challenges due to an increasing longevity. The longevity risk is often described as having two components, the individual and the aggregate. The first, called idiosyncratic risk, *refers to random fluctuations around an expected value and is due to uncertainty in an individual's lifetime. Insurance companies reduce this risk by insuring a sufficiently large number of persons*. The second component is a systematic risk, that is, *a systematic deviation from the expected lifetime due to unexpected mortality improvements*, Gaille (2012). See also Stallard (2006). Therefore, in the case of a life annuity with guaranteed benefits, individuals would tend to systematically outlive company savings (also called mathematical reserves). To manage this systematic risk, the life insurance business is currently either (1) transferring the risk to another party or (2) retaining the risk. Risk transfer can be performed

[4]The survivor dividend is only distributed to surviving contributors and therefore works differently from the lottery in the historical tontines where, as investors die, their shares are forfeited, with the entire fund going to the last surviving investor. For more details on tontines, see Forman & Sabin (2015).

[5]Boado-Penas & Vidal-Meliá (2014). For an extended version that includes population growth, see Vidal-Meliá, Boado-Penas, & Navarro-Cabo (2015).

through reinsurance or securitisation.[6] However, Maurer et al. (2013); Olivieri & Pitacco (2008),reinsurance is usually expensive and reinsurers are not willing to insure systematic risks regarding the overall insurance/reinsurance market. Capital market instruments may then help in transferring the risk through, for example, longevity bonds or mortality swaps, whose payments are linked to some measure of longevity in some reference population. Currently, however, the longevity-linked product market is underdeveloped, Blake & Borrows (2001); Cowley & Cummins (2005); Maurer et al. (2013), and the age structure of the reference population does not necessarily reflect the age structure of the insured population, Hári, De Waegenaere, Melenberg, & Nijman (2008). It is therefore questionable whether insurers would be able to purchase the required number of these products and, even if they could, the systematic risk would not be perfectly hedged. Many life insurers are then compelled to self-insure their systematic risk. To do so, some insurers use natural hedging, either through a second business line reciprocally affected by mortality changes or by offering products combining both life and death benefits. Since the effectiveness of natural hedging has been questioned, Gründl, Post, & Schulze (2006); Maurer et al. (2013); Olivieri & Pitacco (2008), many insurers are reduced to setting aside an appropriate level of funding in order to meet unexpected mortality improvements, usually termed a buffer fund or solvency margin (Hári et al. (2008); Olivieri & Pitacco (2008)). The annuity prices, Maurer et al. (2013), are then not actuarially fair but include premiums that protect the insurer against adverse mortality developments with a specified degree of confidence.

In practice, NDC pension schemes are sufficiently large not to be significantly affected by the idiosyncratic part of the longevity risk. However, like life in-

[6]Naturally, an insurance company can avoid any systematic risk by adjusting the benefits to unanticipated mortality developments (see e.g. Maurer, Mitchell, Rogalla, & Kartashov (2013) and references therein) and thus providing a nonguaranteed life annuity.

surers, they should take into account the systematic risk if they wish to be sustainable in the long run. Since pension schemes usually use official period life tables[7] to calculate the amount of the initial pension, potential future aggregate longevity improvements (the systematic risk) are ignored, which can be dangerous for the financial sustainability of the scheme. Of all the methods dealing with systematic longevity risk proposed in the literature and used by the private insurance market, only conservative mortality assumptions can be applied in NDC schemes, resulting in the inclusion of a longevity loading in the contributions paid to the system, or passing the remaining risk on to the government. In this line, Alho, Bravo, & Palmer (2013) proposed a method to allocate the systematic cohort risk to the government in a transparent way by creating an NDC bond.[8] Knell (2012) was also interested in the impact of increasing life expectancy on NDC pension schemes. He showed how the retirement age and the notional interest rate need to be adjusted for the system to be balanced. However, he assumed that all members of a generation reach their cohort life expectancy and thus, no death occurs prior to the life expectancy age. Under such a strong assumption, the survivor dividend is ignored.

In this chapter, we are interested in analysing if the amount of the survivor dividend is sufficiently large to cover the aggregate longevity risk faced by NDC pension schemes. The survivor dividend can be seen as the buffer fund or solvency margin of life insurers, in the sense that the contributions paid by the annuitants are not actuarially fair but contain a premium for unexpected mortality improvements. However, three important differences appear

[7]For example, in Italy, period tables are used with a revision every three years (Belloni & Maccheroni (2013)), while in Sweden annual period tables are used. See Börsch-Supan (2006) for more details.

[8]For a definition of NDC bonds, see Palmer (2013).

between the two systems: (1) in life insurance business, the additional premium is clearly meant to finance longevity improvements, while in NDC schemes, the additional contribution is kept by the scheme, with no clear purpose; (2) life insurers are fully funded and thus a long-time horizon perspective is required, while NDC pension schemes are under a pay-as-you-go system and thus not only the long-term sustainability but also the short-term liquidity of the system is required; (3) life insurers fix their buffer fund in order to reduce their underfunding probability to some pre-specified level (e.g. 2.5 per cent), while in the current analysis, the survivor dividend is given and we are rather trying to assess what mortality reduction can be financed with it. Therefore, we first show that if the survivor dividend is not distributed, it can be used to finance a decrease in mortality. The maximum mortality decrease (assumed constant over ages) that a scheme can cover is developed with an explicit formula. In a second stage, the formula is adapted in order to include future mortality decreases according to the Lee-Carter model, Lee & Carter (1992), an internationally recognised mortality forecasting method with the property of constant relative speed of mortality decline what allows us to simplify our formulas. We use the life tables from three NDC countries, namely Sweden, Poland and Latvia, and different assumptions regarding changes in salaries[9] and indexation of pensions to illustrate our model. The main conclusions are that the survivor dividend can finance a mortality decrease of around 30 per cent or, in some cases, an increase in the life expectancy at age 65 of more than 3 years.

The chapter is structured as follows. The next section introduces the concept of the survivor dividend in the context of NDC pension schemes. In the subse-

[9] We only consider positive changes in our numerical example. Therefore different levels of salary growth and pension indexation are discussed.

quent section, we develop an explicit formula to calculate the maximum mortality decrease that the scheme can cover. The mortality decrease is assumed first to be constant over ages and second, to follow the Lee-Carter model. In the penultimate section, we provide a numerical illustration of the effects of the non-distribution of the survivor dividend under a generic NDC scheme using mortality tables from Latvia, Poland and Sweden. The final section concludes the chapter and presents possible directions for future research.

1.2 The survivor dividend in NDC pension schemes

The survivor dividend, at a specific age, is defined as the portion of the account balances of contributors resulting from the distribution of the account balances of participants who do not survive to retirement age. In current NDC pension schemes, Sweden is the only country that includes the survivor dividend in the calculation of the initial pension. The formulas developed in the next section and the resulting applications introduced in the penultimate section are based on a model developed by Boado-Penas & Vidal-Meliá (2014). Therefore, this section summarises their key results that will be needed throughout this chapter. These include the mathematical formulas used to calculate the initial pension by first considering the survivor dividend and second, ignoring the dividend. We also introduce the following important result from Boado-Penas & Vidal-Meliá (2014): if the survivor dividend is included in the accumulated capital used to calculate the amount of the initial pension, the contribution rate paid by the contributors (the credited contribution rate) and the contribution rate that makes the system balanced (the balanced contribution rate) coincide. On the other hand, when the survivor dividend is not distributed, the credited contribution rate is higher than the balanced rate and the system would continuously accumulate financial reserves. Following the above model, we assume that: (1) the credited contribution rate θ_c is fixed over time; (2) the contribution base[10] grows at an annual rate of g; (3) the pension benefits are indexed at an annual rate of λ; (4) the contributions and benefits are payable yearly in advance; and (5) the population remains constant over time.[11]

1.2.1 Initial pension, excluding the survivor dividend

If the accumulated survivor dividend is not included in the calculation of the initial pension in year t for an individual reaching the retirement age, then the amount of the initial pension, denoted $P^{nd}_{(x_e+A,t)}$, can be calculated as

$$P^{nd}_{(x_e+A,t)} = \frac{K^{nd}_{(x_e+A,t)}}{\ddot{a}^{\lambda}_{x_e+A,g}}, \tag{1.1}$$

where

$$K^{nd}_{(x_e+A,t)} = \theta_c \sum_{k=0}^{A-1} y_{(x_e+k,-A+k+t)}(1+g)^{(A-k)} = \theta_c \sum_{k=0}^{A-1} y_{(x_e+k,t)} \tag{1.2}$$

and

x_e is the entry age in the system, that is, the first age at which a contribution is paid;

A is the number of years during which contributions are paid and thus $x_e + A$ represents the retirement age;

$K^{nd}_{(x,t)}$ is the accumulated notional capital at time t for an individual age x, without taking into account the survivor dividend;

[10] The terms contribution base and salary are used as synonyms.
[11] Additional details of the model are described by Boado-Penas & Vidal-Meliá (2014).

$\ddot{a}^{\lambda}_{x,g} = \sum_{k=0}^{\infty} \left(\frac{1+\lambda}{1+g}\right)^k {}_kp_x$ is a whole life annuity-due indexed at rate λ with a technical interest rate equal to g, for an individual age x;

${}_kp_x$ is the probability for an individual age x to survive up to age $x+k$ and by convention ${}_1p_x = p_x$;

$y_{(x,t)}$ is the average contribution base at time t for an individual age x, with $y_{(x,1)}(1+g)^{t-1} = y_{(x,t)}$.

1.2.2 Initial pension considering the survivor dividend

If the accumulated survivor dividend is included in the calculation of the initial pension for any individual who reaches the retirement age, then the amount of the initial pension at time t, denoted $P_{(x_e+A,t)}$, is calculated as

$$P_{(x_e+A,t)} = \frac{K_{(x_e+A,t)}}{\ddot{a}^{\lambda}_{x_e+A,g}} = \frac{K^{nd}_{(x_e+A,t)} + D^{ac}_{(x_e+A,t)}}{\ddot{a}^{\lambda}_{x_e+A,g}} \tag{1.3}$$

with

$$\begin{aligned} K_{(x_e+A,t)} &= \frac{\theta_c \sum_{k=0}^{A-1} l_{(x_e+k,-A+k+t)} y_{(x_e+k,-A+k+t)} (1+g)^{(A-k)}}{l_{(x_e+A,t)}} \\ &= \frac{\theta_c \sum_{k=0}^{A-1} l_{(x_e+k,t)} y_{(x_e+k,t)}}{l_{(x_e+A,t)}} \end{aligned} \tag{1.4}$$

where

$K_{x,t}$ is the accumulated notional capital at time t for an individual age x, including the survivor dividend;

$D^{ac}_{(x,t)}$ is the accumulated survivor dividend at time t and age x;

$l_{(x,t)}$ is the number of individuals alive at time t and age x, assuming no growth of the population, that is, $l_{(x,t)} = l_{(x,1)}$.

1.2.3 Balanced system contribution rate, including the survivor dividend

According to Boado-Penas & Vidal-Meliá (2014), the contribution rate paid by the contributors θ_c (credited contribution rate) and the contribution rate that makes the system balanced θ^* are equal if the survivor dividend is included in the calculation of the pension. Indeed, we have

$$P_{(x_e+A,t)} \sum_{k=0}^{\infty} l_{(x_e+A+k,t)} \left[\frac{1+\lambda}{1+g}\right]^k = \theta^* \sum_{k=0}^{A-1} l_{(x_e+k,t)} y_{(x_e+k,t)} \tag{1.5}$$

Substituting the expression of $P_{(x_e+A,t)}$ in Formula 1.5, it follows that

$$\begin{aligned} \frac{\theta_c \sum_{k=0}^{A-1} l_{(x_e+k,t)} y_{(x_e+k,t)}}{l_{(x_e+A,t)} \ddot{a}^{\lambda}_{x_e+A,g}} l_{(x_e+A,t)} \ddot{a}^{\lambda}_{x_e+A,g} = & \quad \theta^* \sum_{k=0}^{A-1} l_{(x_e+k,t)} y_{(x_e+k,t)} \\ \implies & \theta_c = \theta^*. \end{aligned} \tag{1.6}$$

The total value of the contributions is then equal to the total value of the benefits, and the system does not accumulate any reserves.

1.2.4 Balanced system contribution rate, excluding the survivor dividend

If the survivor dividend is not distributed, then the contribution rate that makes the system balanced, θ^{nd*}, can be calculated as

$$P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \ddot{a}^{\lambda}_{x_e+A,g} = \theta^{nd*} \sum_{k=0}^{A-1} l_{(x_e+k,t)} y_{(x_e+k,t)}. \tag{1.7}$$

Substituting the expression for $P^{nd}_{(x_e+A,t)}$, we see that

$$\theta_c = \theta^{nd*} \frac{K_{(x_e+A,t)}}{K^{nd}_{(x_e+A,t)}} \implies \theta_c \geq \theta^{nd*}. \tag{1.8}$$

Consequently, the system continuously accumulates financial reserves. Indeed, by paying the contribution rate θ_c when only a contribution θ^{nd*} is required for a balanced system, savings are made. However, an assumption implicitly made is that mortality rates are not changing over time. As it is known that longevity is increasing over time, the following section analyses under which conditions the accumulated reserves due to the non-distribution of the survivor dividend can be used to cover future mortality improvements.

1.3 Explicit formulas

As demonstrated in the previous section, NDC pension schemes that do not distribute the survivor dividend accumulate some reserves. These reserves can be used to face some unexpected risks, such as an unexpected decrease of mortality. Two different scenarios are analysed. First, we assume that the contribution base is not growing ($g = 0$) and the pension benefits remain constant ($\lambda = 0$). Second, we allow for a constant growth of salaries ($g \neq 0$) and a constant indexation of pensions ($\lambda \neq 0$).

1.3.1 No growth of salaries and no indexation of pensions

This section studies three different approaches for a mortality decrease (or equivalently, a survival increase). The first one considers a constant increase in the survival rate over ages, that is, the probability of surviving at each age increases by the same proportion a. We then look for the maximum increase in survival, or the maximal value of a , the system can afford. Since a constant increase in the survival rates over ages may not be reasonable compared to past mortality improvements, the second approach uses the observed pattern in the rate of change of past mortality rates. We allow for different mortality improvements by age, while imposing at the same time a constant structure between mortality rates, that is, the structure found through the Lee-Carter model (Lee & Carter (1992)). The maximum mortality decrease the system can cover under the restriction imposed by the Lee-Carter model is thus developed. Finally, a much simpler approach based on improvements in life expectancy is proposed.

Case 1: Constant improvement in the survival rates

In order to find the maximal survival increase an NDC scheme can afford, we first need to find the amount the system can use to cover such longevity increases. The total expenditure on pensions at time t, E_t^{nd}, when the survivor dividend is not included in the accumulated notional capital of the individuals can be written as

$$
\begin{aligned}
E_t^{nd} &= \sum_{k=0}^{\infty} P^{nd}_{(x_e+A+k,t)} l_{(x_e+A+k,t)} \\
&= \sum_{k=0}^{\infty} P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)\,k}p_{x_e+A} \\
&= P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} (1 + e_{x_e+A}) \qquad (1.9)
\end{aligned}
$$

since there is no pension indexation, no growth of salaries and $l_{(x_e+A+k,t)} = l_{(x_e+A,t)\,k}p_{x_e+A}$. Note that e_{x_e+A} denotes the life expectancy at retirement age.

However, the contributions paid to the system could cover pension benefits increased by the survivor dividend. Therefore, E_t, the total expenditure on pensions that the scheme can theoretically cover when the survivor dividend is included, is

$$\begin{aligned} E_t &= \sum_{k=0}^{\infty} P_{(x_e+A+k,t)} l_{(x_e+A+k,t)} \\ &= P_{(x_e+A,t)} l_{(x_e+A,t)} (1 + e_{x_e+A}) \\ &= P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} (1 + e_{x_e+A}) + D^{ac}_{(x_e+A,t)} l_{(x_e+A,t)} \end{aligned} \tag{1.10}$$

since $P_{(x_e+A,t)} = \left[P^{nd}_{(x_e+A,t)} + D^{ac}_{(x_e+A,t)}/(1 + e_{x_e+A})\right]$. The NDC scheme saves then an amount of

$$E_t - E^{nd}_t = D^{ac}_{(x_e+A,t)} l_{(x_e+A,t)} \tag{1.11}$$

when the pension $P^{nd}_{(x_e+A,t)}$ is paid instead of the pension $P_{(x_e+A,t)}$. Formula 1.11 represents the amount available to finance an unexpected increase in longevity.

We now consider an unexpected increase in the survival rates and its impact on the NDC expenditure. An increase in the survival rates of a per cent at all ages from the retirement age would result in pension expenditure E^{nd*}_t

$$\begin{aligned} E^{nd*}_t &= P_{(x_e+A,t)} l_{(x_e+A,t)} \sum_{k=0}^{\infty} (1 + a)^k {}_k p_{x_e+A} \\ &= P_{(x_e+A,t)} l_{(x_e+A,t)} \sum_{k=0}^{\infty} \frac{1}{(1 + b)^k} {}_k p_{x_e+A} \\ &= P_{(x_e+A,t)} l_{(x_e+A,t)} \ddot{a}_{x_e+A,b} \end{aligned} \tag{1.12}$$

with $b = (1/(1+a)) - 1 < 0$ and $\ddot{a}_{x_e+A,b}$ representing a life annuity-due evaluated with a negative interest rate of b. The expenditure increases then by an amount of

$$E_t^{nd*} - E_t^{nd} = P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \left[\ddot{a}_{x_e+A,b} - (1 + e_{x_e+A})\right]. \tag{1.13}$$

In order to find the maximal mortality decrease the system can afford, we need to solve with respect to b the following inequality:

$$P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \left[\ddot{a}_{x_e+A,b} - (1 + e_{x_e+A})\right] \leq D^{ac}_{(x_e+A,t)} l_{(x_e+A,t)}$$

which leads to

$$\ddot{a}_{x_e+A,b} - (1 + e_{x_e+A}) \leq \frac{D^{ac}_{(x_e+A,t)}}{P^{nd}_{(x_e+A,t)}} \tag{1.14}$$

This inequality can be solved numerically by several environments for mathematical computing.

Formula 1.14 reveals that an increase in the survivor dividend would allow the system to afford a more important mortality decrease, which is intuitive. At the same time, if the initial amount of the pension, $P^{nd}_{(x_e+A,t)}$, is higher and thus closer to $P_{(x_e+A,t)}$, $D^{ac}_{(x_e+A,t)}$ would consequently be lower and the scheme would less easily cope with a longevity increase.

Case 2: Mortality decrease according to the Lee-Carter model

The Lee-Carter model (Lee & Carter (1992)), has become a standard in mortality modelling. It decomposes the logarithm of the force of mortality, that is the instantaneous rate of mortality at a certain age x, in two components, one capturing the age pattern of average mortality rates and the other a common time trend with differential impacts by age,

$$ln\big(\mu_{(x,t)}\big) = \alpha_x + \beta_x \kappa_t + \epsilon_{(x,t)} \tag{1.15}$$

where

$\mu_{(x,t)}$ is the force of mortality at age x, in year t;

α_x is the mean value over time, at age x, of the logarithm of the force of mortality;

β_x is the relative speed of mortality change at age x. It reflects the impact of the time trend represented by κ_t on age, that is the higher its absolute value, the greater the impact of mortality changes over time on age x;

κ_t is the mortality rate trend over time;

$\epsilon_{(x,t)}$ is the historical influence not captured by the model or, equivalently, error with mean zero and variance σ^2 (homoscedasticity).

The parameters of the model can be estimated numerically through maximum likelihood estimation, assuming that the number of deaths follows a Poisson distribution. Additional details on the model properties and estimation procedure are provided by Delwarde & Denuit (2006) or Gaille (2012). The model was successfully applied in many countries[12] and owes its success mainly to its simplicity. Parameters are easily interpretable and forecasts are easily performed. At the same time, the model presents a few weaknesses, such as the lack of smoothness of the mortality rates across ages.[13] The model has also a

[12]For example, Tuljapurkar, Li, & Boe (2000), Wilmoth (1996) and Lundström & Qvist (2004).

[13]The pros and cons of the Lee-Carter model have been discussed in several papers; see, for example, Booth & Tickle (2008).

very interesting property, extremely relevant in our analysis. It assumes that the relative speed of mortality decline at different ages, β_x, is constant over time. Therefore, once we know the general mortality decline over time, we multiply it by β_x in order to find the mortality decline for age x, and thus we have different decreases for different ages. This property allows us to define elegant formulas. The first step is then to fit the Lee-Carter model to past mortality rates in order to have an estimation of the β_x parameters. In a second stage, we need to determine the maximal general mortality decline, represented by the maximal decrease of κ_t, an NDC scheme can afford, using the β_x parameters found in the first stage. Formally, denote p_x as the current survival rates at age x and p_x^* as the survival rates following a mortality decrease according to the Lee-Carter model. Under constant age specific mortality rates within bands of age and time, we can show that

$$p_x^* = [p_x]^{(\hat{B}_x)^\gamma} \tag{1.16}$$

with $\hat{B}_x = e^{\hat{\beta}_x}$ and γ is defined as the general mortality decrease, that is, the decrease of κ_t. The hat on the parameters indicates estimated values.

Using the same procedure described through Formulas 1.12-1.14 in the subsection "Case 1: Constant improvement in the survival rates" with a survival increase represented by Formula 1.16, we can show that the maximal general mortality decrease the system can afford is given by the highest value of γ respecting

$$\Big[\sum_{k=0}^{\infty}\prod_{j=0}^{k}\left[p_{x_e+A+j}\right]^{(\hat{B}_{x_e+A+j})^\gamma} - e_{x_e+A}\Big] \leq \frac{D^{ac}_{(x_e+A,t)}}{P^{nd}_{(x_e+A,t)}}. \tag{1.17}$$

Details are provided in Appendix A.

Even if this inequality can easily be solved by several environments for statistical computing, Formula 1.17 can be simplified if we assume that the mortality decrease is relatively small. Under such an assumption, we can show that Formula 1.17 simplifies to

$$\ddot{a}_{x_e+A,LC(\gamma)} - (1 + e_{x_e+A}) \leq \frac{D^{ac}_{(x_e+A,t)}}{P^{nd}_{(x_e+A,t)}}, \tag{1.18}$$

where the subscript $LC(\gamma)$ indicates that the present value is computed with an interest rate of $-1 < \zeta_x^y = [1/(e^{(-\hat{\mu_x}\hat{\beta_x})})]^\gamma - 1 < 0$ (See Appendix B for details), that is, the interest rate is changing over time according to the β_x parameters of the Lee-Carter model and a general mortality decrease of γ. In order to find the maximal mortality decrease the system can afford, we now need to solve Formula 1.18 with respect to γ. It is interesting to note that the maximum mortality decrease that the system can afford is found in a relatively similar way when survival rates increase by a constant proportion over all ages (section "Case 1: Constant improvement in the survival rates", Formula 1.14) and when the survival increase is related to the Lee-Carter model (Formula 1.18). The difference between the two inequalities lies in the discounting interest rate that is used. In both cases, the interest rates are negative. However, in Formula 1.14, the interest rate is constant over time, while in Formula 1.18, it changes.

Case 3: Life expectancy improvement at the retirement age

In the preceding sections, Formulas 1.14, 1.17 and 1.18 give a precise description of the longevity improvements through modified survival or mortality rates an NDC scheme can cover, and which, to be solved, require some environments for statistical computing. When there is no indexation of pensions and no salary growth, it is however possible to get a quick idea of the longevity increase magnitude (in years) that an NDC scheme can cover. Indeed, an NDC scheme pays at retirement age an initial pension amount of $P^{nd}_{(x_e+A,t)}$, while it could theoretically pay an amount of $P_{(x_e+A,t)}$, corresponding to a life expectancy of e_{x_e+A}. If mortality decreases, the life expectancy will increase to, for example, $e^{"}_{x_e+A}$. We define Δe as the increase in life expectancy

$$\Delta e = e^{"}_{x_e+A} - e_{x_e+A},$$

with

$$e^{"}_{x_e+A} > e_{x_e+A},$$

and $P^{"}_{(x_e+A,t)}$ as the pension benefit paid at the retirement age corresponding to a life expectancy of $e^{"}_{x_e+A}$ and including the survivor dividend. Since this result only holds if there is no indexation of pensions and no salary growth, Formula 1.3 simplifies and we have

$$P^{"}_{(x_e+A,t)} = \frac{K^{nd}_{(x_e+A,t)} + D^{ac}_{(x_e+A,t)}}{e^{"}_{x_e+A}} < P_{(x_e+A,t)}.$$

As we are interested in finding the conditions under which $P^{"}_{(x_e+A,t)} = P^{nd}_{(x_e+A,t)}$, we have

$$\frac{K^{nd}_{(x_e+A,t)} + D^{ac}_{(x_e+A,t)}}{e^{"}_{x_e+A}} = \frac{K^{nd}_{(x_e+A,t)}}{1 + e_{x_e+A}}. \tag{1.19}$$

Using $e^{"}_{x_e+A} = \Delta e + e_{x_e+A}$ and solving Formula 1.19, we find

$$\Delta e = \frac{(1 + e_{x_e+A}) D^{ac}_{(x_e+A,t)}}{K^{nd}_{(x_e+A,t)}}. \tag{1.20}$$

Formula 1.20 represents the maximal increase in life expectancy the system can finance. Naturally, a different mortality pattern can reproduce the same life expectancy. Therefore, different age-specific mortality decreases can explain an increase of Δe, with important consequences for the age structure of the population. However, Formula 1.20 provides a straightforward way to get a first estimation of the longevity risk that the scheme can cover.

1.3.2 With growth of salaries and indexation of pensions

Case 4: Constant improvement in survival rates

As mentioned in the section "The survivor dividend in NDC pension schemes", Formulas 1.5–1.7, the total expenditure on pensions when the survivor dividend is excluded or included are, respectively

$$E^{nd}_t = P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \ddot{a}^{\lambda}_{x_e+A,g}, \tag{1.21}$$

$$\begin{aligned} E_t &= P_{(x_e+A,t)} l_{(x_e+A,t)} \ddot{a}^{\lambda}_{x_e+A,g} \\ &= \left[P^{nd}_{(x_e+A,t)} + \frac{D^{ac}_{(x_e+A,t)}}{\ddot{a}^{\lambda}_{x_e+A,g}} \right] l_{(x_e+A,t)} \ddot{a}^{\lambda}_{x_e+A,g}. \end{aligned} \tag{1.22}$$

Therefore, the amount that the system saves if the pension $P^{nd}_{(x_e+A,t)}$ is paid instead of $P_{(x_e+A,t)}$ is equal to

$$E_t - E^{nd}_t = D^{ac}_{(x_e+A,t)} l_{(x_e+A,t)}. \tag{1.23}$$

Formula 1.23 represents the amount available to finance an unexpected increase in longevity. It is interesting to note that this formula is identical to Formula 1.11. Salary growth and pension indexation seem to have no direct impact on the amount available for an unexpected increase in longevity. While this is true for pension indexation, the growth of salaries g is directly affecting the survivor dividend, $D^{ac}_{(x_e+A,t)}$. This result is reasonable since Formula 1.23 represents a saved capital accumulated before any pension benefits are being paid.

If the survival rates increase by a per cent at all ages from retirement, the total expenditure on pensions E_t^{nd*} becomes

$$\begin{aligned} E_t^{nd*} &= P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \sum_{k=0}^{\infty} \left[\frac{1+\lambda}{1+g}\right] (1+a)^k {}_k p_{x_e+A} \\ &= P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \sum_{k=0}^{\infty} \left[\frac{1+\lambda^*}{1+g}\right]^k {}_k p_{x_e+A} \\ &= P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \ddot{a}^{\lambda^*}_{x_e+A,g} \end{aligned} \tag{1.24}$$

with $\lambda^* = (1+\lambda)(1+a) - 1$. The increase in survival acts as an increase in the pension indexation rate. The pension expenditure increases then by an amount of

$$E_t^{nd*} - E_t^{nd} = P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \left[\ddot{a}^{\lambda^*}_{x_e+A,g} - \ddot{a}^{\lambda}_{x_e+A,g}\right], \tag{1.25}$$

and the maximal mortality decrease the system can afford is found by solving with respect to λ^* the following inequality:

$$\left[\ddot{a}^{\lambda^*}_{x_e+A,g} - \ddot{a}^{\lambda}_{x_e+A,g}\right] \leq \frac{D^{ac}_{(x_e+A,t)}}{P^{nd}_{(x_e+A,t)}}. \tag{1.26}$$

Case 5: Mortality decrease according to the Lee-Carter model

The introduction of growth of salaries and indexation of pensions do not lead to any particular difficulties, and the development introduced in the section "Case 2: Mortality decrease according to the Lee-Carter model" is used with no further adaptation. Formula 1.17 becomes

$$\left[\sum_{k=0}^{\infty} \left(\frac{1+\lambda}{1+g}\right)^{k+1} \prod_{j=0}^{k} \left[p_{x_e+A+j}\right]^{(\hat{B}_{x_e+A+j})^{\gamma}} - \ddot{a}^{\lambda}_{x_e+A,g}\right] \leq \frac{D^{ac}_{(x_e+A,t)}}{P^{nd}_{(x_e+A,t)}} - 1, \tag{1.27}$$

while the approximation 1.18 becomes

$$\ddot{a}^{\lambda}_{x_e+A,LC(g,\gamma)} - \ddot{a}^{\lambda}_{x_e+A,g} \leq \frac{D^{ac}_{(x_e+A,t)}}{P^{nd}_{(x_e+A,t)}}, \tag{1.28}$$

with the maximal general mortality decrease the system can afford represented by the highest value of γ (see Appendix C for details). The subscript $LC(g,\gamma)$ indicates that the present value is computed with an interest rate of $\psi^{\gamma}_{x} = (1+g)(1+\zeta^{\gamma}_{x}) - 1 = (1+g)(1/(e^{(-\hat{\mu_x}\hat{\beta_x})}))^{\gamma}$, that is, an interest rate changing over time according to the β_x parameters of the Lee-Carter model, a general mortality decrease of γ and a rate of growth of the salaries of g. Formula 1.28 extends Formula 1.18 to a more general framework, since salary and pension growth are now included. In both equations, the discounting rate of interest changes over time. However, the discounting rate of interest ψ^{γ}_{x} not necessarily negative, contrary to the discounting rate ζ^{γ}_{x}. Indeed, ψ^{γ}_{x} includes the growth of the salaries g, which is normally positive. The growth of salaries can then counterbalance the effect of ζ_x^γ.

1.4 Numerical illustration

This section presents the effects of the non-distribution of the survivor dividend under a generic NDC scheme using the official period life tables from Latvia, Poland and Sweden. Only the observed mortality rates in these countries are used and not the population structures by age, which are kept similar for the three countries in order to allow some comparisons of the results.

We first present the results, assuming no growth of the salaries and no indexation of pensions, based on

- a constant increase in the survival probability and the equivalent maximum mortality decrease that an NDC pension system can cover, as developed in the subsection "Case 1: Constant improvement in the survival rates";
- mortality decreases according to the Lee-Carter model presented in the subsection "Case 2: Mortality decrease according to the Lee-Carter model";
- the increase in life expectancy at retirement that the scheme can cover if the survivor dividend is not distributed following the methodology presented in the subsection "Case 3: Life expectancy improvement at the retirement age".

Second, we present the results, assuming growth of salaries and indexation of pensions, based on

- a constant increase in the survival probability (from the subsection "Case 4: Constant improvement in survival rates");
- a mortality decrease following the Lee-Carter model (from the subsection "Case 5: Mortality decrease according to the Lee-Carter model").

Finally, a retrospective analysis is performed where we analyse our generic NDC scheme to see if it could have covered past mortality decreases observed since 1980.

The life tables are obtained from the Human Mortality Database (2014). They contain demographic information, including the number of people alive, the number of deaths and life expectancy for various countries. For our case studies, we use the latest available period life tables by single age for each country.[14] Mortality rates are computed as the ratio between the number of registered deaths and the number of people alive at age x. The life expectancy at birth equals 73.72 in Latvia, 75.71 in Poland and 81.76 years in Sweden.

The salary structure is obtained from the European Union statistical office.[15] The same structure with a mean monthly salary of $2,752$[16] at the beginning of the analysis is used for the three NDC countries in order to make the results comparable.

The model described in the sections "The survivor dividend in NDC pension schemes" and "Explicit formulas" is applied, using the following assumptions: (1) the entry age[17] into the system is fixed at 25; (2) the retirement age of

[14] The latest available life table for Latvia and Sweden is from 2011 and for Poland from 2009.

[15] Eurostat Database (2014).

[16] Mean monthly earnings of industry, construction and services of 25 EU Member States.

[17] The conclusions are similar with an entry age of 30 or 35, since mortality rates are extremely low at these ages. For instance, the mortality decrease Latvia can cover would be affected by less than 1 per cent with an entry age of 30 and less than 2 per cent with an entry age of 35.

the participants is 65; (3) the individuals pay continuously a contribution of 16 per cent of their contribution bases until they reach retirement age; and (4) the highest age an individual can survive is 110. Therefore, an individual stays in the NDC system for a maximum of 85 years.

1.4.1 No growth of the salaries and no indexation of pensions

Application 1: Constant improvement in the survival rates

Table 1.1 shows the main results under the mortality scenarios of Latvia, Poland and Sweden. The difference between the total expenditure on pension when the survivor dividend is not included, E_t^{nd}, and the expenditure after the inclusion of the dividend, E_t, represents the amount available to finance an unexpected increase in longevity according to Formula 1.11. By using Latvian mortality tables, we find that the system would accumulate a reserve of 286 million if the dividends were not distributed. This reserve could then finance a constant increase over ages in the survival probability of 1.85 per cent or equivalently, a decrease in mortality of 35.32 per cent. Finally, by applying Formula 1.20 we find that the maximal increase in life expectancy at retirement the system can finance is $\Delta e = 3.65$. The discrepancies in values between the columns of Table 1.1 are caused only by the different mortality tables used. Sweden has the highest life expectancy of the three NDC countries studied. As fewer deaths occur prior to retirement, its accumulated dividend is lower. Sweden can then cover a smaller survival improvement: if the dividends were not distributed, the system could finance an increase in the survival probability of only 0.59 per cent or equivalently, a decrease in mortality rates of 17.24 per cent.

Table 1.1: Results with no growth of the salaries and no indexation of pensions.

	Latvia	Poland	Sweden
E_t^{nd} (in mill.)	1,324	1,395	1,598
E_t (in mill.)	1,610	1,642	1,710
Increase in life expectancy Δe	3.65	3.12	1.42
Increase in survival probability a	1.85%	1.53%	0.59%
Maximum mortality decrease	35.32%	31.86%	17.24%
Life expectancy after retirement	16.39	17.10	19.84
Negative interest rate b	-1.81%	-1.50%	-0.58%

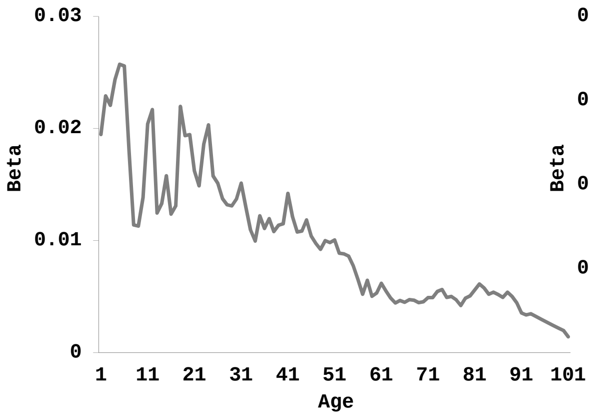
Application 2: Mortality decrease according to the Lee-Carter model

In order to fit the Lee-Carter model to past mortality rates, data over a long time horizon is required. Such data is obtained from the Human Mortality Database (2014). While data for Poland and Sweden are known to be reliable at least for the last 50 years (Fihel & Jasilionis (2011); Glei, Lundström, & Wilmoth (2012)), some cautions are required for Latvia, Jasilionis (2013), since their data presents some concerns during the period of Soviet rule (1940-1989). Therefore, the Lee-Carter model is estimated on a time horizon of 20 years for Latvia (percentage of explained variance of 64.50 per cent). With respect to Poland and Sweden, the Lee-Carter model is estimated twice, once on a 20-year horizon and once on a 50 year period. We chose the model with the highest explained variance, resulting in a fitting period of 20 years for Poland (percentage of explained variance of 89.30 per cent) and 50 years for Sweden (percentage of explained variance of 83.40 per cent). The resulting estimated β_x parameters are shown in Figure 1.1.

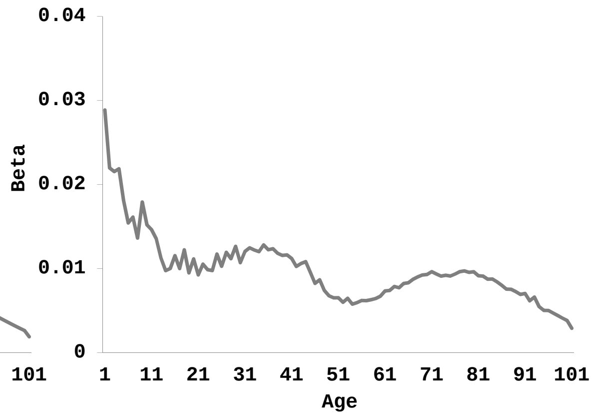
As shown in Figure 1.1, the values of the β_x parameter decrease with age.[18] The mortality decline has then been more important at young ages over the

[18] The Lee-Carter model is estimated over ages 0-100, since data for older ages fluctuates too much to provide a reliable fit.

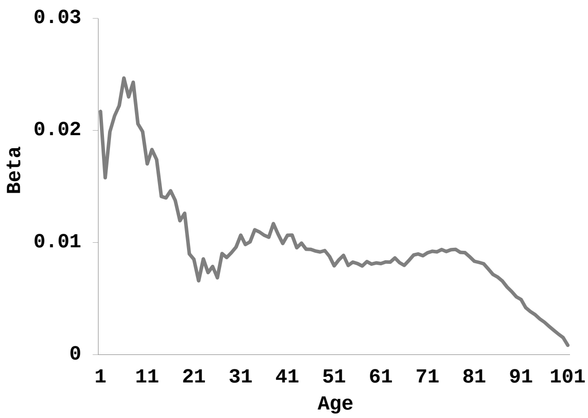
Figure 1.1: β_x parameters of the Lee-Carter model in Latvia, Poland and Sweden.



(a) Latvia



(b) Poland



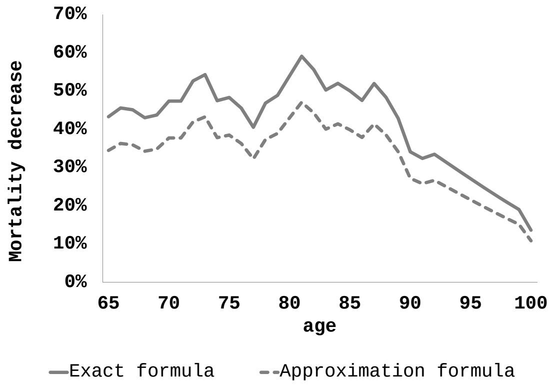
(c) Sweden

past few decades. In our analysis, we only need the values from the age of retirement onwards.

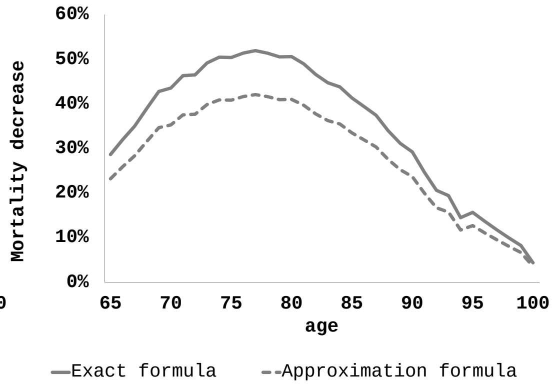
Once the β_x parameters are found, the maximal general mortality decrease that the system can cover, γ, is estimated, first with Formula 1.17 and second with Formula 1.18. Figure 1.2 shows the age-specific mortality decrease that the system can cover by multiplying the resulting value of γ with the estimated

β_x parameter. The solid line represents the exact formula (Formula 1.17) and the dashed line the approximation (Formula 1.18).

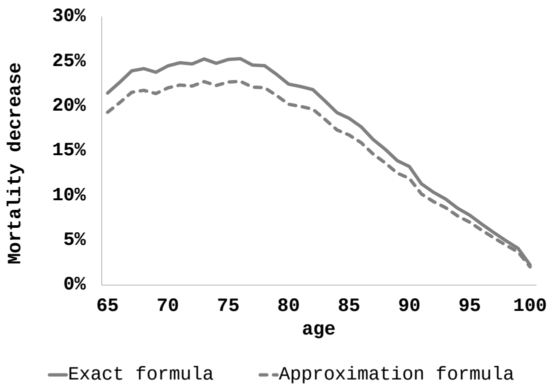
Figure 1.2: Age-specific mortality decrease according to the Lee-Carter model for Latvia, Poland and Sweden.



(a) Latvia



(b) Poland



(c) Sweden

Three aspects should be highlighted with respect to Figure 1.2. First, instead of a constant mortality decrease of 35.32, 31.86 and 17.24 per cent in Latvia, Poland and Sweden respectively (Table 1.1), the decline of mortality changes with age. The highest decline in mortality occurs for ages between 75 and 80 and takes an average value of 49.6 per cent in Latvia, 47.8 per cent in Poland and 24.9 per cent in Sweden. The lowest decline in mortality is found at ages 95-100, with values of 21.5, 10.6 and 5.3 per cent in Latvia, Poland and Sweden respectively. This structure of mortality decrease is in line with past studies[19] where the highest mortality decline from the retirement age was also observed to be around age 70. Second, it is interesting to note that, even if the values resulting from the approximation are lower than within the exact calculation, the results are relatively close to each other. Indeed, the structure of the mortality decrease over ages is similar. Third, the increase in life expectancy Δe and the total expenditure on pensions with, E_t, and without, E_t^{nd}, the survivor dividend remain equal to the ones calculated under the assumption of a constant mortality decrease over ages (Table 1.1). The mortality decrease by age is then different between the section "Application 1: Constant improvement in the survival rates" and "Application 2: Mortality decrease according to the Lee-Carter model", but the general impact remains the same.

1.4.2 With growth of salaries and indexation of pensions

Application 3: Constant improvement in survival rates

This subsection presents the results with growth of salaries ($g \neq 0$) and indexation of pensions ($\lambda \neq 0$) described in the section "Case 4: Constant improvement in survival rates". Tables 1.2-1.4 introduce the results under different levels of salary growth and indexation of pensions for the three mortality tables analysed.[20] As explained in the section "Case 4: Constant improvement in survival rates", salary growth has a direct impact on the survivor dividend:

[19]See D'Amato, Piscopo, & Russolollo (2011).

[20]It is assumed that the growth of salaries is at least equal to the growth of pensions, since increases in indexation higher than increases in growth of salaries pose fiscal problems to the governments. Usually, indexations of pensions are linked to price inflation, where wages commonly increase faster than prices (see Whitehouse (2009b)).

the higher the salary growth, the higher the accumulated survivor dividend at retirement age. Therefore, we observe in Table 1.2(a) that, for a fixed pension indexation, the affordable increase in survival probability is higher when the growth of the salary increases. The same observation holds for Table 1.2(b)-(d). The Latvian NDC scheme can afford a maximal decrease inmortality of 43 per cent and an increase in life expectancy of 4.24 years when $g = 3$ per cent and $\lambda = 0.5$ per cent. Tables 1.3 and 1.4 present the results with Polish and Swedish life tables. The trends are similar to the ones observed in Latvia, even if the levels are different. For example, Table 1.3(d) indicates that the maximal Polish mortality decrease the system can cover is 39.1 per cent when $g = 3$ per cent and $\lambda = 0.5$ per cent, while the minimal decrease is of 31.9 per cent when $g = \lambda$. In terms of life expectancy, the increase varies from 3.12 to 3.62 (Table 1.3(c)). With Swedish life tables, the maximal affordable mortality decrease varies from 17.2 to 21.9 per cent (Table 1.4(d)), while the corresponding increase in the life expectancy goes from 1.42 to 1.64 (Table 1.4(c)).

Table 1.2: Results under different levels of salary growth g and indexation of pensions λ -Latvia.

(a) Increase in survival probability a

g \ λ	**0.5%**	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
0.5%	1.85%					
1.0%	1.89%	1.85%				
1.5%	1.94%	1.89%	1.85%			
2.0%	2.00%	1.94%	1.89%	1.85%		
2.5%	2.05%	2.00%	1.94%	1.89%	1.85%	
3.0%	2.10%	2.05%	1.99%	1.94%	1.89%	1.85%

(b) Modified indexation rate λ^*

0.5%	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
2.35%					
2.40%	2.86%				
2.45%	2.91%	3.37%			
2.51%	2.96%	3.42%	3.88%		
2.56%	3.02%	3.47%	3.93%	4.39%	
2.61%	3.07%	3.52%	3.98%	4.44%	4.90%

(c) Increase in life expectancy

g \ λ	**0.5%**	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
0.5%	3.65					
1.0%	3.76	3.65				
1.5%	3.87	3.76	3.65			
2.0%	3.99	3.87	3.76	3.65		
2.5%	4.11	3.99	3.87	3.76	3.65	
3.0%	4.24	4.11	3.99	3.87	3.76	3.65

(d) Decrease in mortality

0.5%	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
35.3%					
36.7%	35.3%				
38.2%	36.7%	35.3%			
39.8%	38.2%	36.7%	35.3%		
41.3%	39.7%	38.2%	36.7%	35.3%	
43.0%	41.3%	39.7%	38.2%	36.7%	35.3%

Table 1.3: Results under different levels of salary growth g and indexation of pensions λ -Poland.

(a) Increase in survival probability a

g \ λ	**0.5%**	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
0.5%	1.53%					
1.0%	1.57%	1.53%				
1.5%	1.61%	1.57%	1.53%			
2.0%	1.65%	1.61%	1.57%	1.53%		
2.5%	1.69%	1.65%	1.61%	1.57%	1.53%	
3.0%	1.75%	1.69%	1.65%	1.61%	1.57%	1.53%

(b) Modified indexation rate λ^*

0.5%	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
2.03%					
2.07%	2.54%				
2.12%	2.58%	3.05%			
2.16%	2.62%	3.09%	3.56%		
2.20%	2.67%	3.13%	3.60%	4.06%	
2.25%	2.71%	3.17%	3.64%	4.11%	4.57%

(c) Increase in life expectancy

g \ λ	**0.5%**	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
0.5%	3.12					
1.0%	3.22	3.12				
1.5%	3.31	3.22	3.12			
2.0%	3.41	3.31	3.21	3.12		
2.5%	3.51	3.41	3.31	3.21	3.12	
3.0%	3.62	3.51	3.41	3.31	3.21	3.12

(d) Decrease in mortality

0.5%	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
31.9%					
33.2%	31.9%				
34.6%	33.2%	31.9%			
36.1%	34.6%	33.2%	31.9%		
37.6%	36.1%	34.6%	33.2%	31.9%	
39.1%	37.6%	36.0%	34.6%	33.2%	31.9%

Table 1.4: Results under different levels of salary growth g and indexation of pensions λ -Sweden.

(a) Increase in survival probability a

g \ λ	**0.5%**	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
0.5%	0.59%					
1.0%	0.60%	0.59%				
1.5%	0.62%	0.60%	0.59%			
2.0%	0.64%	0.62%	0.60%	0.59%		
2.5%	0.65%	0.64%	0.62%	0.60%	0.59%	
3.0%	0.67%	0.65%	0.64%	0.62%	0.60%	0.59%

(b) Modified indexation rate λ^*

0.5%	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
1.09%					
1.11%	1.59%				
1.12%	1.61%	2.09%			
1.14%	1.62%	2.11%	2.60%		
1.16%	1.64%	2.13%	2.61%	3.10%	
1.17%	1.66%	2.14%	2.63%	3.12%	3.60%

(c) Increase in life expectancy

g \ λ	**0.5%**	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
0.5%	1.42					
1.0%	1.46	1.42				
1.5%	1.50	1.46	1.42			
2.0%	1.55	1.50	1.46	1.42		
2.5%	1.59	1.55	1.50	1.46	1.42	
3.0%	1.64	1.59	1.54	1.50	1.46	1.42

(d) Decrease in mortality

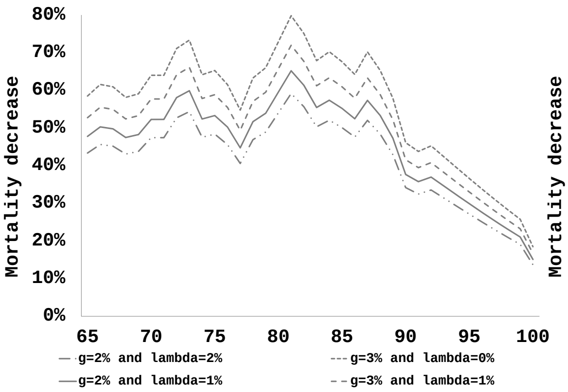
0.5%	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**
17.2%					
18.1%	17.2%				
19.0%	18.1%	17.2%			
19.9%	19.0%	18.1%	17.2%		
20.9%	19.9%	19.0%	18.1%	17.2%	
21.9%	20.9%	19.9%	19.0%	18.1%	17.2%

Application 4: Mortality decrease according to the Lee-Carter model

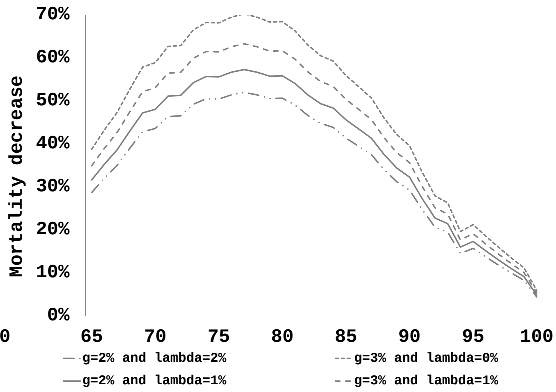
Formulas 1.27-1.28 developed in the section "Case 5: Mortality decrease according to the Lee-Carter model" are applied with the three analysed life tables. We choose to analyse four different scenarios with respect to salary growth (g) and indexation of pensions λ: (i) $g = 2$ per cent and $\lambda = 1$ per cent (solid line);(ii) $g = 3$ per cent and $\lambda = 1$ per cent (dashed line); (iii) $g = 3$ per cent and $\lambda = 0$ per cent (pointed line); and (iv) $g = 2$ per cent and $\lambda = 2$ per cent (dashed and points line).

Figure 1.3 shows the maximal mortality decrease according to Formula 1.27 (section "Case 5: Mortality decrease according to the Lee-Carter model"), that is, the exact mortality decrease. The values presented in Figure 1.3 result from the multiplication of the general mortality decrease γ by the β_x parameters found in the section "Application 2: Mortality decrease according to the Lee-Carter model". Naturally, the patterns are similar to the ones described in the section "Application 2: Mortality decrease according to the Lee-Carter model", while the levels change due to the different values of pension and salary growth. A higher salary growth g, and a lower pension indexation, λ, allow the system to cover a more important mortality decrease. Finally, the same remark holds as with no pension and salary growth (section "Application 2: Mortality decrease according to the Lee-Carter model"): the increase in life expectancy, Δe, the total expenditure on pensions with, E_t, and without the survivor dividend, E_t^{nd},remain equal to the ones calculated under the assumption of a constant mortality decrease over ages (Tables 1.2-1.4).

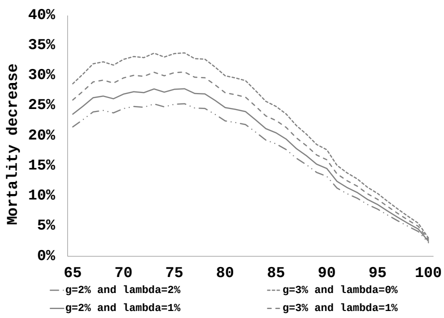
Figure 1.3: Age-specific mortality decrease according to the Lee-Carter model for Latvia, Poland and Sweden, with growth of salaries and pension indexation.



(a) Latvia



(b) Poland



(c) Sweden

Retrospective analysis

In the preceding sections, we were interested in finding the maximal mortality decrease (or survival increase) the system could theoretically face, under different assumptions. In this last part of our analysis, we consider whether the accumulated survivor dividend of the generic NDC scheme we modelled could have covered the effective mortality decline observed from 1980 to the year of the most recent available life tables, that is, 2009 in Poland and 2011 in Latvia and Sweden. The salary structure is kept constant over time for comparability reasons. The total expenditure on pensions, when the survivor dividend is not included, is computed using two different mortality tables in each country. The first one refers to the total expenditure on pensions when the latest available mortality table is used, denoted $E_{2011}^{nd(2011)}$ in Latvia and Sweden and $E_{2009}^{nd(2009)}$ in Poland. The second one refers to the total expenditure on pensions when the 1980 mortality table is used, denoted $E_{2011}^{nd(1980)}$ in Latvia and Sweden and $E_{2009}^{nd(1980)}$ in Poland. The difference between the two provides the additional cost that the system will need to cover as a result of past mortality improvements if the NDC scheme uses the 1980 period life tables to determine the pension benefits instead of using the most recently available life tables. If this additional cost is smaller than the amount corresponding to the survivor dividend accumulated in 2011 (2009 for Poland), then the scheme is able to cover the increase in longevity. Table 1.5 presents the results for the three analysed life tables. In the three cases, we conclude that the survivor dividend could have covered past increases in life expectancy, since the increase in costs is smaller than the accumulated reserves, that is, $E_t^{nd(t)} - E_t^{nd(1980)} < D_{(65,t)}^{ac(1980)}$, with $t = 2009, 2011$.

Table 1.5: Retrospective analysis.

	Latvia[21]	**Poland**	**Sweden**
E_t^{nd} (in mill.)	1,324	1,395	1,598
$E_t^{nd(1980)}$ (in mill.)	1,217	1,279	1,453
$E_t^{nd} - E_t^{nd(1980)}$ (in mill.)	107	116	145
Survivor dividend (in mill.), using 1980 tables: $D_{(65,t)}^{ac(1980)}$	310	290	203
Is $E_t^{nd} - E_t^{nd(1980)} < D_{(65,t)}^{ac(1980)}$?	Yes	Yes	Yes

[21] t=2009 for Poland and t=2011 for Latvia and Sweden.

1.5 Conclusions

In the present chapter we show that the survivor dividend can cover an unexpected increase in longevity, reflected either by a decrease in mortality rates, an increase in survival probabilities or an increase in life expectancy at the retirement age. We found the mathematical formulas for the maximal mortality improvement an NDC scheme can cover under different scenarios, and applied them to three different case studies.

We first assume a constant mortality decrease over ages and we find that the maximum decrease that can be financed is 35.3 per cent with Latvian, 31.9 per cent with Polish and 17.2 per cent with Swedish mortality tables. Second, we derive age-specific mortality decreases the system can finance under the Lee-Carter model, resulting in a mortality decline as small as 2.3 per cent for ages around 100 in Sweden and as large as 60 per cent for ages around 80 in Latvia. Third, the impacts of different salary and pension growths are analysed. As expected, an increased salary growth and a reduced pension indexation result in more important mortality improvements that an NDC scheme can cover. Finally, the retrospective analysis demonstrates that mortality improvements over the last 30 years could have been financed through the survivor dividend. Indeed, in the three mortality scenarios presented, the amount of the accumulated survivor dividend exceeds by far the amount required to finance mortality improvements observed from 1980 to 2011 (Latvia, Sweden) or 1980 to 2009 (Poland). The survivor dividend proves then to be an interesting solution for NDC schemes to finance future longevity improvements, provided that this dividend is not distributed as in Sweden.

This chapter shows that the reserves that an NDC scheme accumulates when it does not distribute the survivor dividend are much larger than the required amount to finance the longevity risk. Therefore, part of the accumulated reserve could be used to finance some other benefits or unexpected risks. For instance, the survivor dividend could cover the introduction of a minimum pension providing a minimum standard of living for the pensioners, in addition to covering future longevity improvements. It would be also of interest to determine if the dividend could act as an automatic balancing mechanism[22] to reestablish the liquidity and/or the sustainability in pay-as-you-go pension systems. Distributing or not the survivor dividend becomes a very important decision that any NDC countries should seriously consider, especially the countries currently considering the introduction of NDC pension schemes such as Egypt, China and Greece.

To conclude, this chapter highlights reserves that several NDC schemes are currently accumulating and not using. These reserves can be used in different ways, one of them discussed here being the financing of future mortality improvements. It is important that governments of countries using NDC schemes discuss, decide and reveal for which purpose these accumulates reserves are meant to be used.

[22] According to Vidal-Meliá et al. (2009), an automatic balancing mechanism is a set of predetermined measures established by law to be applied immediately according to a solvency or sustainability indicator.

2

How to finance pensions: Optimal strategies for pay-as-you-go pension systems

Godínez Olivares, H., Boado-Penas, M.C., and Pantelous, A.A. [1]

The aim of this chapter is to design optimal strategies using non-linear dynamic programming to guarantee the required level of liquidity in pay-as-you-go pension systems through changes in the key variables of the system, such as, the contribution rate, retirement age and/or indexation of pensions. These strategies, also known as Automatic Balancing Mechanisms (ABMs), calculate the optimal path of these variables over time, managing fluctuations in longevity, fertility rates, salary growth or any other kind of uncertainty faced by the pension scheme without repeated legislative intervention. A numerical application of our model, that uses the projection of the population structure of two representative countries, illustrates the main findings of the chapter.

Keywords: financial equilibrium, optimisation, pay-as-you-go, public pensions, risk.

2.1 Introduction

The evolution and forecast of the relevant parameters involved in the provision of pension schemes are major topics of on-going discussion for the policy and decision-making committees in many EU countries. Their public pension systems are usually financed on a Pay-As-You-Go (PAYGO) basis where pensions

[1] This chapter is based on the paper 'How to finance pensions: Optimal strategies for pay-as-you-go pension systems' by Godínez-Olivares, Boado-Penas, & Pantelous (2016). Journal of Forecasting. 35(1), 13-33.

for retirees are paid by the contributions of the working-age population. It is well understood that PAYGO systems require a balance between the benefits paid to the pensioners and the contributions made by the active workers, what is also referred as *intergenerational solidarity*[2]. A successfully PAYGO system needs to reflect intergenerational solidarity and long-term sustainability.

The decline in fertility rates, the increase in longevity and the current forecasts for the ageing of the baby-boom generation will contribute to a substantial increase in the old-age dependency ratio, and this will raise serious concerns for the sustainability of the PAYGO pension systems. With continuous improvements in life expectancy, pensions are paid over a longer time horizon, which causes great difficulties when the system does not expect such improvements. The European Commission (2012b), shows a clear increase in life expectancy at birth for males and females of 7.9 and 6.5 years respectively from 2010 to 2060. Furthermore, the European population aged 60+ is increasing annually by two million and is estimated to become almost twice as high in 2060 as it was in the late 1990s and early 2000s. Moreover, in 2012, pension expenditure represented a very large and rising share of public expenditure: more than 10 per cent of GDP and this is forecast to rise to 12.5 per cent in 2060 in the EU as a whole. Thus Milevsky (2010) considers a PAYGO system as a high-risk funding method where the sponsors provide benefits to retirees when they are due and payable, and a real fund is never accumulated. Brown (1992) states that countries should develop an acceptable wealth transfer equilibrium given their ageing population and increase the contributions to maintain balance. According to Valdés-Prieto (2006) many European PAYGO pension systems

[2] Haberman & Zimbidis (2002b), define intergenerational solidarity as the willingness of different groups of people to participate in a common pool, sharing actual experience, including any losses emerging. In the context of public pension systems, the concept refers to both young and old generations.

tend to require periodic adjustments to maintain the long-term sustainability, mainly because of demographic and economic uncertainty. Furthermore, García-Ferrer & Del Hoyo (1991) states that population predictions have frequently failed to give proper guidance on the inherent degree of uncertainty and consequently demographic forecasts have quickly become obsolete (see also Lee & Shripad (1998a) and Lee & Shripad (1998b)). In fact, the common trend in responses to what is a pensions crisis is a wave of parametric pension adjustments in Europe, for example in France, Greece, Hungary, Romania and Spain[3]. These parametric reforms usually include, among others, changes in the contribution ceilings, increases in the retirement age or changes in the indexation of pensions.

Following this process of reforms, Vidal-Meliá et al. (2009) and Vidal-Meliá et al. (2010) propose an *Automatic Balance Mechanism*[4] (ABM) defined as a set of predetermined measures established by law to be applied immediately as required according to an indicator that reflects the financial health of the system. Its purpose, through successive application, is to re-establish the financial equilibrium of PAYGO pension systems with the aim of making those systems viable without the repeated intervention of the legislators. They argue that there are three reasons for introducing an ABM method, first, to adapt the system to changes in socio-economic and demographic conditions; secondly, to create a credible institutional framework in the sense that promises of pension payments are kept; and finally, to minimise the use of the pension system as an electoral weapon. As examined by Turner (2009) at least 12 countries have

[3]See Whitehouse (2009a,b) and OECD (2012).
[4]See Appendix D.

indexation of benefits linked to life expectancy or any other kind of automatic adjustment[5].

It is important to highlight that three papers, Haberman & Zimbidis (2002b), Pantelous & Zimbidis (2008) and Gannon, Legros, & Touzé (2013), propose parametric reforms in the PAYGO pension systems introducing the concept of a liquidity or contingency fund in order to absorb unexpected events that might affect their liquidity. As emphasized by Pantelous & Zimbidis (2008), this non-zero contingency fund is acting as a buffer, fluctuating deliberately in the short run and absorbing partially or completely the uncertainty in mortality, fertility rates or other events. Similarly, Gannon et al. (2013) define this buffer fund as the inter-temporal budget balance of the pension system that brings promised future expenditures in line with expected future revenues. Siebert (2005) states that at any given moment the system adjusts its benefits in accordance with the inter-temporal budget constraint.

The aim of this chapter is to design a new automatic balancing mechanism (for the very first time, according to the authors' knowledge) to restore liquidity into the system under a non-linear framework using different population projections (Spain and Japan). This ABM is the result of minimising a chosen logarithmic function and simultaneously calculating the optimal path for the contribution rate, the retirement age and the indexation of pensions for the PAYGO system. As a result, the derived contingency fund is able to absorb fluctuations in longevity, fertility rates, salary growth or any other events. Our method aims to keep the future growth rates of the projections for the contribution rate, the normal retirement age and the indexation of pensions at a minimum level and provide the system with a long-term sustainability. Fur-

[5] For more information on ABM's see, for instance, Turner (2009), Vidal-Meliá et al. (2009), OECD (2012) and Appendix D.

thermore, our ABM model can be easily generalised including more parameters and is applicable to any PAYGO pension system.

Following this introduction, the chapter is organised as follows. The next section describes the mathematical preliminaries together with the numerical optimisation and briefly discusses the main notation and definitions. Moreover, it gives details about our model and the projections of the parameters involved. Section 3 shows a representative and practical application for two different structures of population and suggests how an ABM model should be designed for both symmetric and asymmetric cases. Section 4 provides concluding comments of the chapter. The present chapter has two appendices, first the main automatic balancing mechanisms of some countries are described (Appendix D); second the gradient method is explained in more detail (Appendix E).

2.2 Model formulation for a PAYGO pension system

This section presents the optimisation model together with the optimal paths of the parameters involved in the PAYGO system such as the contribution rate, the age of normal retirement and the indexation of pensions. The parameters of the dynamic model are the buffer fund, also called the contingency fund,$\{F\}_{n\in\mathbb{N}}$, that represents the *state equation* or *state trajectory* and the decision variables: contribution rate$\{c\}_{n\in\mathbb{N}}$, age of retirement $\{x^{(r)}\}_{n\in\mathbb{N}}$ and the indexation of pensions $\{\lambda\}_{n\in\mathbb{N}}$. The model uses an objective function and minimises the growth rates of the decision variables, as described in eq.2.1.

In general the decision variables are represented by $\{d\}_n = \{d_n^0, d_n^1, ..., d_n^k\}$ while specifically in our model $d_n^j = (c_n^j, x_n^{(r)j}, \lambda_n^j)$ minimises $f_n(d_n^j, n)$ in eq. 2.1. The variable F_n fluctuates deliberately to absorb changes in fertility, mortality projections and any other events that might affect the liquidity indicator in the pension system. Consequently, the aim of the forecast variables is to act as an ABM, re-establishing the liquidity of the PAYGO pension system over time according to pre-established rules without the intervention of the state.

2.2.1 Mathematical preliminaries

This subsection presents the mathematical notation and terminology for a better understanding of the optimal path of the decision variables for our PAYGO model. Following Bertsekas (1999), our mathematical optimisation model is represented by a decisions set $\{d\}_n$ and a cost function f that maps elements D into real numbers. The set D consists of the available decisions d and the cost $f(d)$ is a scalar measure of undesirability of choosing decisions d. In general, the aim is to find an optimal $d^* \in D$ such that $f(d^*) \leq f(d)$, $\forall d \in D$. In our problem, $f_n(d_n^j, n)$ is defined by eq.(2.1) and D is the set of all the real values that the projected variables could take. Some basic definitions associated with an optimisation non-linear problem in discrete time are introduced below.[6]

Definition 2.1. *A time set N is a subgroup of $(\Re, +)$. For each set Q and interval I, the set of all maps from I into Q is denoted by $Q^I = \{q | q : I \rightarrow Q\}$.*

In our case, N_+ is the set of non-negative elements.

Definition 2.2. *A system $\sum = (N, D, U, f, F)$ consists of a: i) A time set N; ii) A non-empty set D called the decision space of $\sum$; iii) A non-empty set U called the constraints of $\sum$; iv) A map $f : D_n \rightarrow D$ called the transition map of $\sum$, which is defined on a subset D_n of $\{(n, \sigma, x, u) | \sigma, n \in N, \sigma \leq n, x \in X, u \in U^{[\sigma,n)}\}$ such that the non-triviality, restriction, semi-group and identity holds; and v) A state trajectory $\{F\}_{n\in\mathbb{N}}$.*

[6]See Sontag (1998), Bertsekas (1999), Bertsekas (2005) and Bradley (1997).

The *cost* or *objective* function f is defined as $f_n(d_n^j, n)$ where d_n^j is the permissible decisions that is chosen from the set D and n is the step of the process with $n \in N$. In our model, D consists of the values that the projected decision variables can take every year n. Bradley (1997) states that the next state depends entirely on the current state of the process and the current decision made. As noted by Bertsekas (1999), $d_n^j = (d_n^0, d_n^1, ..., d_n^k)$ is the *decision trajectory* and the equation $d_{n+1} = F_n(d_n, n)$ is called the *state equation* or *state trajectory*. The set U_n specify constraints on the decision vectors. The system constraints U could be, $g_j(D) \leq 0; j = 1, 2, ...m$; non-linear constraint and $h_k(D) = 0; k = 1, 2, ..., l$; linear constraint equations. In our model, the constraints are defined by the minimum and maximum values of the projected decision variables and rates of change in each step/year. In other words, we need to find a sequence $d_n^j = (d_n^0, d_n^1, ..., d_n^k)$ which

$$min f_n(d_n^j, n)$$

$$s.t.: \begin{cases} d_{n+1} = F_n(d_n^j, n), n = 1, 2, ..., \omega; \\ d_0 : given; \\ d_n^j = (c_n^j, x_n^{(r)j}, \lambda_n^j) \in \Re^n, j = 1, 2, ..., \omega; \\ h_k(U) = 0; k = 1, 2, ..., l; \\ g_j(U) \leq 0; j = 1, 2, ...m; \end{cases}$$

Following general ideas about optimal non-linear optimisation in discrete time, the *Gradient method* is used in the chapter. The gradient method is a well-developed method that works with inequality constraints. These inequalities are modified to be equalities using a linear slack variable (see Venkataraman (2009)). The new equality constraints are expanded into Taylor series. Then with these linear equations, the constraint equations are used to reduce the number of independent variables. The modified problem minimises $f(D)$ subject to $h_k(D) = 0; k = 1, 2, ..., l$ and $g_j(D) + q_{n+j} = 0; j = 1, 2, ...m$, $d_n^{min} \leq q_i \leq d_n^{max}, i = 1, 2, ..., v$ and $d_{p+j} \geq 0; j = 1, 2, ...m$ with $D = [d_1, d_2, ..., d_N]$. See Appendix E for more details about the gradient method.

2.2.2 PAYGO Model

Haberman & Zimbidis (2002b) propose two different models[7] to identify the optimal path for the contribution rate and retirement age only. The functional objective that determines the smoothness of the path includes weight changes for the two variables. These weights reflect the expectations of the participants in the pension system as well as the underlying demographic trends. The functional objective has a quadratic form that minimises the distance between the projected variables and their pre-defined values. Using standard linearisation procedures the authors obtained a closed solution for the optimal path of the projections. Both models try to keep the variables at the same level over time. Similarly, Gannon et al. (2013) propose a model where the objective function is quadratic and the system is adjusted with the introduction of two coefficients which modify the present and future payroll taxes and pension expenditure. However, they do not specify the values of the key variables that are involved in a pension system (contribution rate, age of retirement and indexation of pensions).

Following Haberman & Zimbidis (2002b), Pantelous & Zimbidis (2008) construct a discrete time stochastic model including several variables such as different investment strategies, contribution rates, retirement ages and levels of

[7] The first deterministic model is based on a continuous framework while the second one is a stochastic model in discrete time as, in practice, it is impermissible to make changes to the variables in continuous time.

pension benefits. Finally, the authors allow the fund to take positive or negative values. Again, the functional objective has a quadratic form and the optimal path of the control variables is assumed to be stabilised around the initial pre-established value. Both Haberman & Zimbidis (2002b) and Pantelous & Zimbidis (2008) did not try to solve the non-linear problem directly, but a linearisation method has been suggested instead.

In our approach, we suggest a different way to forecast the contribution rate and the age of normal retirement. In particular, the problem is settled in a non-linear framework and the optimal path for the decision variables are developed. Additionally, the indexation of pensions is introduced as a third decision variable. Another advantage comparing with previous papers is that the functional objective that is used for the determination of the optimal, smooth path for the projected variables is defined to minimise the percentage of the changes over time using a logarithm function.[8] The logarithmic function is widely used in economics because its elasticity can easily be found. In our case, the logarithmic cost criterion measures the percentage of changes in particular variables over time. So we are introducing the idea of minimising the change rate of the decision variables involved in the projections (contribution rate, normal retirement age and indexation of pensions) instead of trying to stabilise them around an initial set value, as in the case of previous models using a quadratic function.

[8] We use the logarithmic function due to its nature of being strictly monotonic increasing and concave in the entire domain. More details about interesting properties and comments for this type of function in economics can be found in Luenberger (1992) and Kübler (2010).

Analytically, the non-linear function to minimise is given by:

$$f(d_n, n) = min_{c_n, x_n^{(r)}, \lambda_n} \Big[\sum_{n=1}^{N} \Big[\theta_1 log\Big(\frac{c_{n+1}}{c_n}\Big) + \epsilon_1\theta_2 log\Big(\frac{x_{n+1}^{(r)}}{x_n^{(r)}}\Big) + \epsilon_2\theta_3 log\Big(\frac{\lambda_{n+1}}{\lambda_n}\Big)\Big]\Big] \quad (2.1)$$

where

$c_n, x_n^{(r)}, \lambda_n$ are the projected variables during year n.

θ_1, θ_2, and θ_3 are the weights that measure the impact that occurs when the projected variables c_n, $x_n^{(r)}$ and λ_n change over time, where $\theta_1 + \theta_2 + \theta_3 = 1$[9].

ϵ_1 and ϵ_2 are coefficients that have been introduced in the growth rates of the age of normal retirement and in the indexation of pensions to deal with the metric problems that are present due to the nature of the logarithmic function, as c_n, $x_n^{(r)}$ and λ_n have different units and constraints[10], i.e.

$$\epsilon_1 = \frac{log(c_\Delta)}{log(x_\Delta^{(r)})};$$
$$\epsilon_2 = \frac{log(c_\Delta)}{log(\lambda_{1\Delta})}\mathbb{1}_{\{\frac{\lambda_{n+1}}{\lambda_n} \geq 0\}} + \frac{log(c_\Delta)}{log(\lambda_{2\Delta})}\mathbb{1}_{\{\frac{\lambda_{n+1}}{\lambda_n} \leq 0\}},$$

where, $\mathbb{1}_{\{j \in \omega\}}$ is the characteristic or indicator function. It takes the value of 1 for all values that satisfied the condition $j \in \omega$ and 0 in other case.

The non-linear model is derived from the basic equation in a defined benefit (DB) PAYGO system and, similarly to Haberman & Zimbidis (2002b), we

[9] As noted by Haberman & Zimbidis (2002b) the weights θ are key parameters which reflect the expectations of all participants in the pension system as well as the underlying demographic trends. The parameters would be obtained after research and negotiations with all the parties involved in the public pension system.

[10] c_Δ, $x_\Delta^{(r)}$ and $\lambda_{1\Delta}$ are the maximum increase that are allowed (and $\lambda_{2\Delta}$ decrease for λ_n) per year in the contribution rate c_n, the age of normal retirement $x_n^{(r)}$ and the indexation of pensions λ_n.

introduce a contingency fund into the basic equation. The dynamics of the fund (state equation $F_n(d_n, n)$) can be expressed as:

$$F_n = (1 + J_n)F_{n-1} + c_n W_n(n, g, x_n^{(r)}) - B_n(n, g, x_n^{(r)}, \lambda_n), \tag{2.2}$$

where J_n is the growth risk-free rate (investments in T-bills) of the fund during year n; $c_n, x_n^{(r)}, \lambda_n$ are the projected variables during year n; W_n is the total contribution base paid at n; B_n is the total expenditure on pensions at n that depends on the growth of salaries, g, the retirement age $x_n^{(r)}$ and the indexation of pensions λ_n.[11]

Remark 2.1. *Extensive discussion about and properties of the contingency fund can be found in Haberman & Zimbidis (2002b). In theory, upper and lower bounds can be imposed in the contingency fund. Setting upper bound could make sense due to the restriction of risk-free investments of the fund whereas lower bound might be set to prevent high levels of deficit. In this chapter, negative values of the fund are acceptable since our aim is to get a smooth transition of the contribution rate, age of retirement and indexation of pensions to make the model applicable.*[12] *Negative values of the fund could be covered by government debt, however, non-smooth jumps in the key parameters are not easy to justify in any pension parametric reform.*

The model is studied for two interesting cases, the symmetric and the asymmetric. Palmer (2013) states that under a symmetric ABM, any surplus that might arise would be automatically distributed. In the absence of a symmetric

[11] The indexation is a key parameter in a pension system and it refers to the revalorisation of the pensions in payment; according to Whitehouse, D'Addio, Chomik, & Reilly (2009) most OECD countries adjust pensions in payment to reflect changes in costs and standards of living. The procedures for such adjustment play an essential role in providing income security in retirement.

[12] Upper and lower bounds were imposed as the 10 per cent of the GDP for each country.

ABM an undistributed surplus is maintained. Furthermore, Alho et al. (2013) state that the balancing mechanism can be symmetric, adjusting for both positive and negative deviations from the financial health indicator, or asymmetric, as in the Swedish system, which balances only if the solvency ratio is less than 1, i.e. the amount of assets is less than the amount of liabilities.[13] Under the *symmetric* design, our model determines whether the contribution rate and the age of normal retirement (and indexation of pensions) are reduced when the buffer fund has a surplus (deficit) or increased (decreased) in periods of deficit (surplus).

At the end of the period of study, the value of the fund is enforced to be equal to zero, i.e. $F_n = 0$, if $n = N$, as a constraint in the optimisation under the symmetric case to avoid that the system accumulates reserves at all times. Under the *asymmetric* case, changes in the projected variables only occur when a deficit in the fund arises, thus the terminal condition is $F_n \geq 0$, if $n = N$.

The constraints, $h_k(D), g_j(D)$ for the values of the forecast variables are as follows: for the contribution rate, c_n, the age of normal retirement, $x_n^{(r)}$, and the indexation of pensions, λ_n, we impose upper and lower bounds, $c_{max}, x_{max}^{(r)}, \lambda_{max} \in \mathbb{R}_{\geq 0}$ and $c_{min}, x_{min}^{(r)} \in \mathbb{R}_{\geq 0}$, $\lambda_{min} \in \mathbb{R}$ respectively. Finally, a smooth path for the projected variables is obtained when the following constraints are imposed: $c_{1\Delta} \leq \dfrac{c_{n+1}}{c_n} \leq c_{2\Delta}$; $x_{1\Delta}^{(r)} \leq \dfrac{x_{n+1}^{(r)}}{x_n^{(r)}} \leq x_{2\Delta}^{(r)}$; $\lambda_{1\Delta} \leq \dfrac{\lambda_{n+1}}{\lambda_n} \leq \lambda_{2\Delta}$; where $c_{1\Delta}, c_{2\Delta}, x_{1\Delta}^{(r)}, x_{2\Delta}^{(r)}, \lambda_{1\Delta}, \lambda_{2\Delta} \in \Re$.

[13] Under the particular case of the Swedish pensions system Alho et al. (2013) argue that with symmetric balancing, systematic overestimation of life expectancy transfers money from the accounts of the cohorts, whose pensions were underestimated, to other cohorts. In contrast, in the asymmetric scenario, if the system were otherwise in exact equilibrium, systematic underestimation of life expectancy for pensioner cohorts causes the ratio of assets to liabilities to fall below unity, causing a downward adjustment of the accounts of all workers and pensioners, thereby spreading out the consequence to all workers and pensioners in the insurance pool.

The solution of the optimisation problem is studied in a deterministic, discrete time framework and derived numerically.[14]

Thus, eventually, we need to numerically solve the problem (2.1):

$$min_{c_n,x_n^{(r)},\lambda_n} \left[\sum_{n=1}^{N}\left[\theta_1 log\left(\frac{c_{n+1}}{c_n}\right)+\epsilon_1\theta_2 log\left(\frac{x_{n+1}^{(r)}}{x_n^{(r)}}\right)+\epsilon_2\theta_3 log\left(\frac{\lambda_{n+1}}{\lambda_n}\right)\right]\right]$$

$$s.t. = \begin{cases} F_n = (1+J_n)F_{n-1} + c_nW_n(n,g,x_n^{(r)}) - B_n(n,g,x_n^{(r)},\lambda_n); \\ c_{min} \leq c_n \leq c_{max}; x_{min}^{(r)} \leq x_n^{(r)} \leq x_{max}^{(r)}; \\ \lambda_{min} \leq \lambda_n \leq \lambda_{max}; \\ c_{1\Delta} \leq \frac{c_{n+1}}{c_n} \leq c_{2\Delta}; x_{1\Delta}^{(r)} \leq \frac{x_{n+1}^{(r)}}{x_n^{(r)}} \leq x_{2\Delta}^{(r)}; \\ \lambda_{1\Delta} \leq \frac{\lambda_{n+1}}{\lambda_n} \leq \lambda_{2\Delta}; \\ \theta_1+\theta_2+\theta_3 = 1; \\ F_N \geq 0; \\ \epsilon_1 = \frac{log(c_\Delta)}{log(x_\Delta^{(r)})}; \\ \epsilon_2 = \frac{log(c_\Delta)}{log(\lambda_{1\Delta})}\mathbb{1}_{\{\frac{\lambda_{n+1}}{\lambda_n}\geq 0\}} + \frac{log(c_\Delta)}{log(\lambda_{2\Delta})}\mathbb{1}_{\{\frac{\lambda_{n+1}}{\lambda_n}\leq 0\}} \end{cases} \tag{2.3}$$

2.3 Numerical application

This section presents the forecasts for the contribution rate, the normal retirement age and the indexation of pensions for a generic PAYGO pension system in a dynamic non-linear framework. Our goal is to show the numerical

[14]Due to the nature of the non-linear dynamical system and the forecasts of the three main variables we have $3(N-1)$ parameters for the optimisation problem. To enable numerical computation methods (LaValle (2006)), a family of trajectories need to be specified in terms of a parameter space and the optimisation can be viewed as an incremental search in the parameter space while all constraints are being satisfied.

application of our model using the population structure of two representative countries: Spain and Japan. The projected data from Spain is obtained from the National Statistics Institute of Spain. Japan's data is provided by the National Institute of Population and Social Security Research.

These countries present a very different age structure. Japan represents a mature-aged population and Spain an ageing one. For comparison purposes the two populations have been normalised. Moreover, the normalised structures of these populations are very helpful in investigating different scenarios and easily extrapolating useful conclusions for other countries with similar population characteristics.

The dynamics behind the total contribution base, W_n, and the total expenditure on pensions, B_n, are complex as they depend on variables such as the growth of salaries, g, the normal retirement age, $x_n^{(r)}$, and the indexation of pensions, λ_n.

The total contribution base, W_n, is modelled at time $n = 1$ as:

$$W_1 = \left(\sum_{x=x_e}^{x_1^{(r)}-1} l_{x,1} * wage(x) \right),$$

where, x_e is the entry age into the labour market.

$l_{x,1}$ is the number of people alive at age x at time 1. We assume that $l_{x,n}$ is distributed uniformly over the year.

$wage(x)$ is the average wage[15] for individuals who are x years old.

For $n > 1$, we have:

[15] For simplicity, the terms wage, salary and contribution base are considered to be synonyms.

$$W_n = \left(\sum_{x=x_e}^{\lfloor x_n^{(r)} \rfloor - 1} (l_{x,n}) * wage(x) * (1+g)^n \right) + (x_n^{(r)} mod \lfloor x_n^{(r)} \rfloor) l_{\lfloor x_n^{(r)} \rfloor, n} * wage(\lfloor x_n^{(r)} \rfloor)(1+g)^n \quad (2.4)$$

Where, $\lfloor x_n^{(r)} \rfloor$ is the floor function, i.e., $\lfloor x_n^{(r)} \rfloor$ maps a real number to the largest previous integer number and $x_n^{(r)} mod \lfloor x_n^{(r)} \rfloor$ is the modulus operation that finds the remainder of the division $x_n^{(r)} / \lfloor x_n^{(r)} \rfloor$.

Modelling $B(n)$ is a more complicated task as indexation of pensions, λ_n is dynamic over time. The expenditure on pensions at year n=1 can be expressed as:

$$B_1 = P_{x_1^{(r)},1} l_{x_1^{(r)},1} + P_{x_1^{(r)}+1,1} l_{x_1^{(r)}+1,1} + P_{x_1^{(r)}+2,1} l_{x_1^{(r)}+2,1} + ... = \sum_{x=x_1^{(r)}}^{\omega} P_{x,1} l_{x,1},$$

where, $P_{x_1^{(r)},1}$ is the average pension at time 1 for pensioners aged $x_1^{(r)}$ and ω is the last age to which a person can survive.

For $n > 1$, the total expenditure on pensions is forecast as follows:

$$B_n = \left(1 - (x_n^{(r)} mod \lfloor x_n^{(r)} \rfloor) l_{\lfloor x_n^{(r)} \rfloor, n} \right) * P_{\lfloor x_n^{(r)} \rfloor, n} + \sum_{x=\lceil x_n^{(r)} \rceil}^{\omega} P_{x,n} l_{x,n} \quad (2.5)$$

Where, $\lceil x_n^{(r)} \rceil$ is the ceiling function, i.e., $\lceil x_n^{(r)} \rceil$ maps a real number to the smallest next integer number, and $P_{x,n}$ can be written as:

$$P_{x,n} = P_{x-1,n-1} * (1 + \lambda_{n-1}),$$

with $P_{x_n^{(r)},n} = P_{x_n^{(r)},n} \mathbb{1}_{\{x_n^{(r)},n=1\}} + P_{x_n^{(r)},n} * (1+g)^t \mathbb{1}_{\{x_n^{(r)},n>1\}}$.

2.3.1 Data

The entry age into the labour market is assumed to be 20, i.e, $x_e = 20$. Under the projection of the Spanish population structure the average age in year 0 is 48.9 and for Japanese structure is 52.7 and the median 52. In the last year of study[16] (n=20) the average age increases by 6.2 years for Spain and 4.8 years for Japan. Optimal non-linear optimisation has been applied using the same level of projective salaries and pensions over time in order to make the results comparable.

Since the forecasts are realised for the three decision variables, the minimum value for the contribution rate, age of normal retirement and indexation of pensions are given respectively by 15 per cent, 65 and -2 per cent and the upper values are 20 per cent, 68 and 2 per cent respectively. It is also assumed that the change in the contribution rate varies between 0.3 per cent and 0.5 per cent, the age of normal retirement between 1.5 and 3 months and the indexation of pensions between -2 per cent and 2 per cent. These values are in line with the most important reforms in the 34 OECD member countries between January 2009 and September 2013 (see OECD (2014)).[17] The accumulated

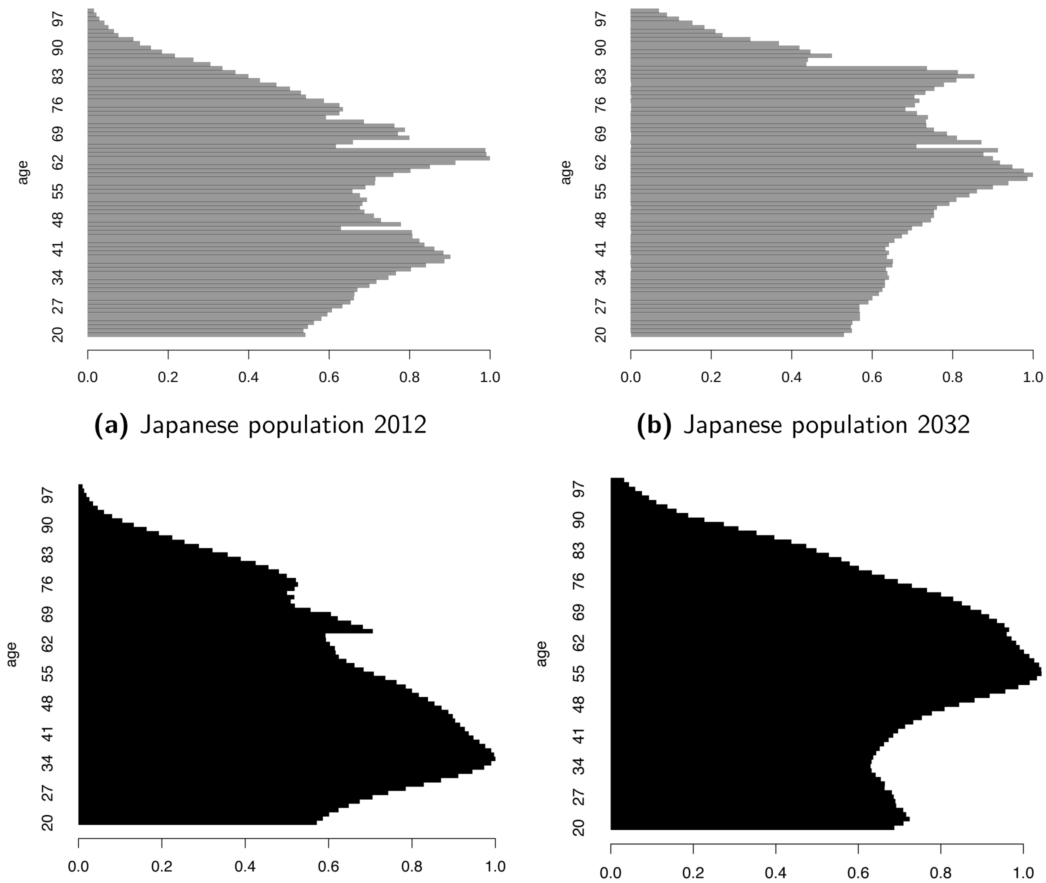
[16]When we apply the model in a longer horizon, 40 years, we have a similar convergence in the values of the projected variables with a difference in how the fund accumulate the surpluses.

[17]Some indexation rules move towards less generous benefits, for example, the indexation of pensions in Austria, Greece, Portugal and Slovenia have frozen automatic adjustments for all but the lowest earners. In Spain, the retirement age is being raised gradually from 65 to 67 years in 2027. In France the contribution rate for the civil servants will increase 0.27 per cent per year during the next years.

fund increases at an annual rate of 3 per cent[18] while the annual salary growth is equal to 2.5 per cent.

Figure 2.1 shows the two different structures of population that we are using in this numerical example. The pyramid represents the normalised mature population of Japan (grey colour) versus the ageing population of Spain (light colour) to highlight the differences.

Figure 2.1: Projection of the Spanish population versus Japanese population structure.



(a) Japanese population 2012

(b) Japanese population 2032

(c) Spanish population 2012

(d) Spanish population 2032

[18] An average value for the past 20 years of 1 year T-bills according to the Federal Reserve of the United States of America. A sensitivity analysis of this value is provided in section 3.3.

Japan's population has two peaks in 2012 at ages 64-66 and 38-41 corresponding to the first baby boom of 1947-1949 and second baby boom 1971-1974 respectively. By 2032 population forecasts are worse showing two peaks at ages 83-85 and 59-61.

For Spain's population the peak is at ages 34-36 in 2012 corresponding to the creation of the Social Security and the demographic boom in the 1960s and early 1970s. By 2032 the population trend continues and the peak is at age 53-55, corresponding to the same population that in 2012.

Under the Spanish population structure, Figure 2.1c, there are 2.96 contributors who finance each pensioner, compared with 2.29 under Japan population, Figure 2.1a. This ratio is known as the age-dependency[19] ratio and it worsens over time, reaching values of 1.97 and 1.56, Figure 2.1d and 2.1b respectively, in 20 years. No unemployment is considered in our analysis.

Figure 2.2: Salary and pension structure.

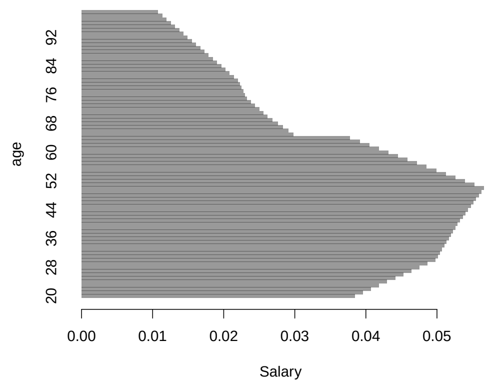


Figure 2.2 shows the salary and pension structure at the beginning of our study. The wage structure shows a concave function, Katz & Autor (1999), Becker (1962), Becker (1993), Ben-Porath (1967), Mincer (1974), provide a coherent explanation of relatively timeless qualitative features of the wage structure that have been found in almost every country: higher earnings for

[19]This ratio measures the number of elderly people relative to those of working age.

more-educated workers and upward sloping and concave age earnings profiles. This concave function in Figure 2.2 shows the peak of earnings at age 52 after that the salary function starts to decrease with age. We use the same salary and pension structure for both population structures which allows us to compare the results. The initial pension is assumed to be 60 per cent[20] of the contribution base in a particular year.

2.3.2 Results

One of the main characteristics of this model is that it allows the system to accumulate surpluses. In this sense, we could design both an asymmetric and a symmetric design of the pension system.[21]

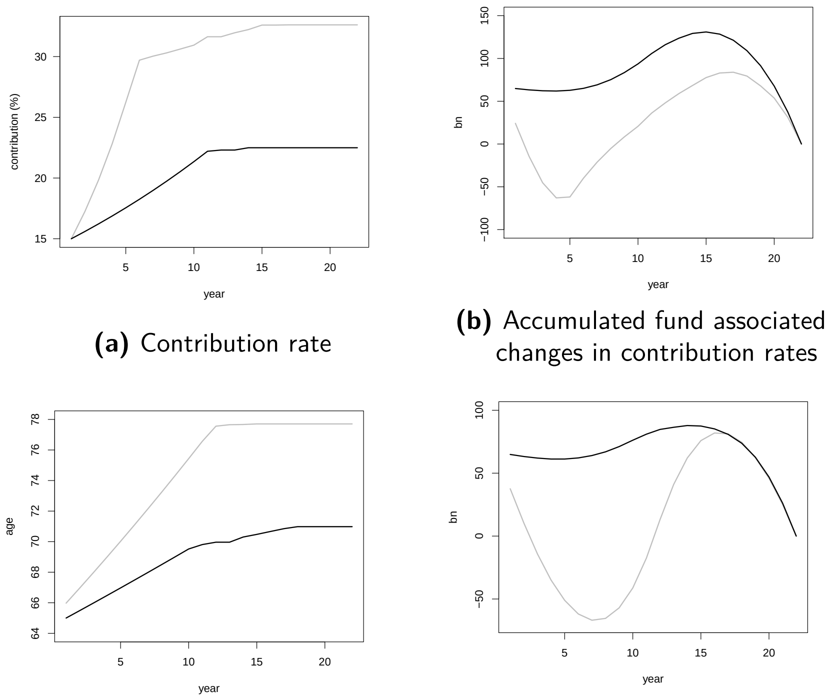
Figure 2.3 shows the optimal forecasting strategy when only one decision variable is involved.[22] For the ageing structure, we can see that if we designed an ABM to restore the liquidity into the system with only one decision variable, the age of normal retirement, Figure 2.3c (black line), would need to increase to 71 years whereas the contribution rate, Figure 2.3a, would increase to 22.5 per cent. For the mature aged population (grey line), the results are even worse. The age of normal retirement (Figure 2.3c) and contribution rate (Figure 2.3a) needs to increase to 78 years and 32 per cent respectively. At the same time the fund remains negative for the first 9 years but the system keeps balanced afterwards.

[20] This is the average initial pension for a median earner in the EU and very similar to the median earner of the OECD countries that is 57.9 per cent according with the OECD (see OECD (2014)).

[21] In some cases both the asymmetric and symmetric designs provide the same results, see for example figure 2.4a and 2.4b.

[22] We set one of the parameters $\theta = 1$ and the other two equal to zero depending on which variable we projected.

Figure 2.3: Results when only one decision variable is forecasted under the Spanish (black line) and Japanese (grey line) structures.



(a) Contribution rate

(b) Accumulated fund associated to changes in contribution rates

(c) Age of retirement

(d) Accumulated fund associated to changes in retirement age

If the indexation of pensions is forecast as the only decision variable no solutions are found for $F_N \geq 0$. However, if we decrease the limit of the lower bound, we find that the indexation would need to decrease for the mature population until -14.6 per cent and for the ageing population to -5.0 per cent to set $F_N \geq 0$, which makes it very inapplicable in practice.

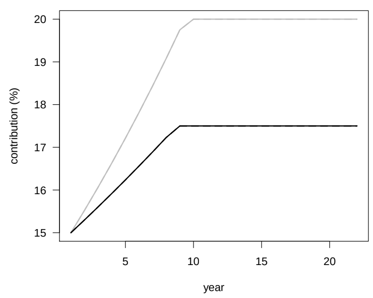
Figure 2.4 shows the results when we project the three variables simultaneously with the same weight in the parameters.[23] Figure 2.4a shows that the adjustment in the contribution rate for the ageing population structure increases from 15.0 per cent to 17.5 per cent after 9 years, remaining unchanged after that under both the symmetric and asymmetric designs. At the same

[23]We set $\theta_j = 1/3$ for $j = 1, 2, 3$.

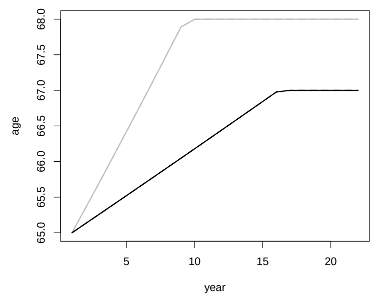
time, Figure 2.4b, the age of normal retirement increases from 65 to 67 over a period of 17 years.

The indexation of pensions under the symmetric scenario, Figure 2.4c, decreases at the beginning of the period and then increases after ten years in order to redistribute the surpluses that have been accumulated during the period. It can be shown that the optimal path of the indexation of pensions stabilises around the value of 0.77 per cent after ten years. However, under the asymmetric design, the indexation decreases and stabilizes around the value of 0.72 per cent.

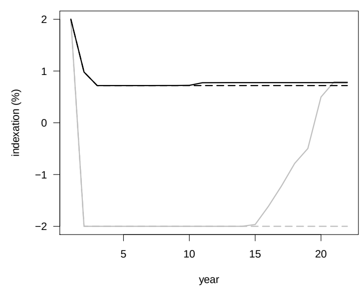
Figure 2.4: Results when the three decision variables are forecasted simultaneously for the symmetric (solid line) and asymmetric (dashed line) with Spanish (black line) and Japanese (grey line) population structure.



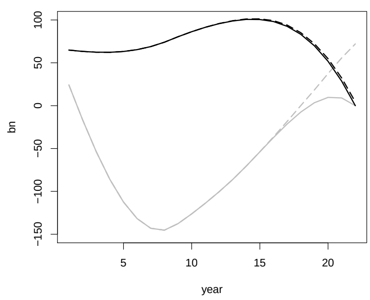
(a) Contribution rate



(b) Age of retirement



(c) Indexation of pensions



(d) Accumulated fund

As can be shown in Figure 2.4d, unsurprisingly, the fund presents an accumulation period during the first 15 years and a de-accumulation period over the latter years. The main reason for this is to absorb fluctuations in the population structure owing mainly to changes in longevity and fertility rates. As expected, under the asymmetric design, the amount of the accumulated fund is higher.

Since the Japanese population structure is “older”, projections of the decision variables take higher values. For example, under the symmetric design (grey solid line) the contribution rate needs to increase from 15 to 20 per cent, Figure 2.4a, the age of normal retirement from 65 to 68, Figure 2.4b, and the growth of pensions, Figure 2.4c, needs to decrease by an annual 2 per cent over almost a 15 year period in order to restore the liquidity into the system. The accumulated fund, Figure 2.4d, is in deficit for the first 17 years. The difference with the asymmetric design (grey dashed line) is that the amount of the pension decreases by 2 per cent over all the period and consequently the fund accumulates a higher amount in the last years of the study.

The results indicate that the projected decision variables take higher values (or lower values for the indexation of pensions) under the mature aged population to make the system balance. Also, the values of the projected decision variables are lower (or higher values for the indexation of pensions) than in the case where only one decision variable is involved.

2.3.3 Sensitivity analysis

This section first adds expected and unexpected shocks into our analysis in order to see how the fund absorbs these effects. Second, a sensitivity analysis of different levels of investment rate (J), growth rate of salaries (g) and different percentages of the contribution base to calculate the initial pension are analysed. From now on, the three decision variables (contribution rate, c_n, age of normal retirement, $x_n^{(r)}$, and indexation of pensions, λ_n) are projected.

Expected shocks

In the case of expected shocks, we assume that these are known at the start of the study. As an example, for the ageing population structure, we consider a specific cohort with a lower life expectancy,[24] which translates into fewer pensioners at the end of our study period. Figure 2.5 shows how, under the symmetric case, the surplus resulting from a lower expenditure on pensions is redistributed through a decrease in the contribution rate (Figure 2.5a) and age of normal retirement (Figure 2.5b), and an increase in the indexation of pensions (Figure 2.5c). However, under the asymmetric design, no surpluses are re-distributed and as a result of this, the value of the accumulated fund is higher. Also, the contribution rate and the age of normal retirement remain at their maximum value and the indexation of pension at the minimum.

Table 2.1 shows, for the ageing population structure, the difference between the forecast of the decision variables with and without shocks for the symmetric scenario using the idea of the specific cohort with a lower life expectancy. The age of normal retirement needs to increase 0.862 years under symmetric design at the end of the study compared with the scenario without shocks. As expected, the value of the accumulated fund at the end of the study is equal in the presence of a shock. That is because, under the symmetric scenario, the model redistributes any surpluses that may arise.

Table 2.2 shows for the ageing population structure the difference between the forecast without and with shocks for the asymmetric scenario using the idea of a cohort with a lower life expectancy. As can be seen, the difference occurs in the indexation of pensions and in the age of retirement after year 12. The

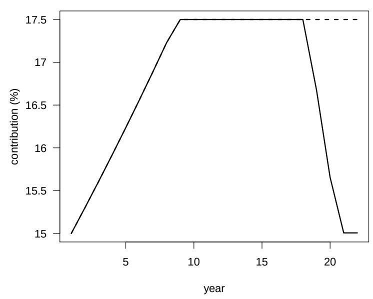
[24]We consider that a cohort has two years less life expectancy than the others.

Table 2.1: Difference between forecasts of the decision variables (contribution rate c_n, age of normal retirement $x_n^{(r)}$ and indexation of pensions λ_n) with shock of a specific cohort with a lower life expectancy and without shock under the ageing population for the symmetric case.

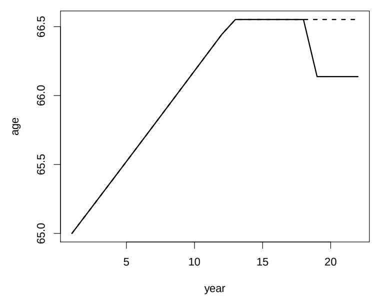
Year	c_n	$x_n^{(r)}$	λ_n	**F**	**Year**	c_n	$x_n^{(r)}$	λ_n	**F**
2012	0	0	0	0	2023	0	0.1%	-0.7%	23.0
2013	0	0	-0.3%	0.2	2024	0	2.5%	-0.7%	28.3
2014	0	0	-0.5%	0.6	2025	0	15.8%	-0.8%	35.4
2015	0	0	-0.5%	1.5	2026	0	29.1%	-0.9%	44.5
2016	0	0	-0.6%	2.6	2027	0	42.5%	-1.0%	55.7
2017	0	0	-0.5%	4.2	2028	0	44.8%	-1.0%	68.7
2018	0	0	-0.5%	6.1	2029	0	44.8%	-1.1%	83.3
2019	0	0	-0.6%	8.5	2030	0.8%	86.2%	-1.2%	41.4
2020	0	0	-0.6%	11.3	2031	1.8%	86.2%	-1.2%	2.3
2021	0	0	-0.7%	14.7	2032	2.5%	86.2%	-1.2%	-36.3
2022	0	0	-0.6%	18.5	2033	2.5%	86.2%	-1.2%	0

Note: The value of the Fund (F) is in thousands.

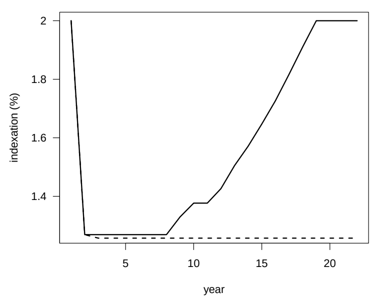
Figure 2.5: Expected shock for the symmetric (solid line) and asymmetric (dashed line): cohort with lower life expectancy.



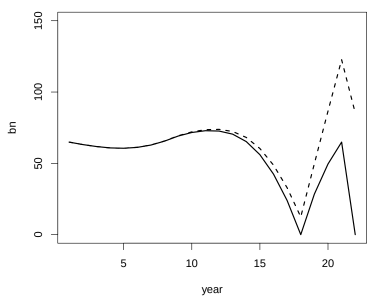
(a) Contribution rate



(b) Age of retirement



(c) Indexation of pensions



(d) Accumulated fund

value of the fund is lower in the presence of shocks as that amount is used to absorb the changes of the system.

Table 2.2: Difference between forecasts of the decision variables (contribution rate c_n, age of normal retirement $x_n^{(r)}$ and indexation of pensions λ_n) with shock of a specific cohort with a lower life expectancy and without shock under the ageing population for the asymmetric case.

Year	c_n	$x_n^{(r)}$	λ_n	**F**	**Year**	c_n	$x_n^{(r)}$	λ_n	**F**
2012	0	0	0	0	2023	0	0.1%	-0.5%	21.8
2013	0	0	-0.3%	0.2	2024	0	2.5%	-0.5%	26.5
2014	0	0	-0.5%	0.6	2025	0	1.58%	-0.5%	32.5
2015	0	0	-0.5%	1.5	2026	0	29.1%	-0.5%	40.1
2016	0	0	-0.5%	2.6	2027	0	4.25%	-0.5%	49.5
2017	0	0	-0.5%	4.1	2028	0	44.8%	-0.5%	59.7
2018	0	0	-0.5%	6.1	2029	0	44.8%	-0.5%	70.9
2019	0	0	-0.5%	8.4	2030	0	44.8%	-0.5%	20.07
2020	0	0	-0.5%	11.1	2031	0	44.8%	-0.5%	-34.8
2021	0	0	-0.5%	14.3	2032	0	44.8%	-0.5%	-94.0
2022	0	0	-0.5%	17.8	2033	0	44.8%	-0.5%	-85.3

Note: The value of the Fund (F) is in thousands.

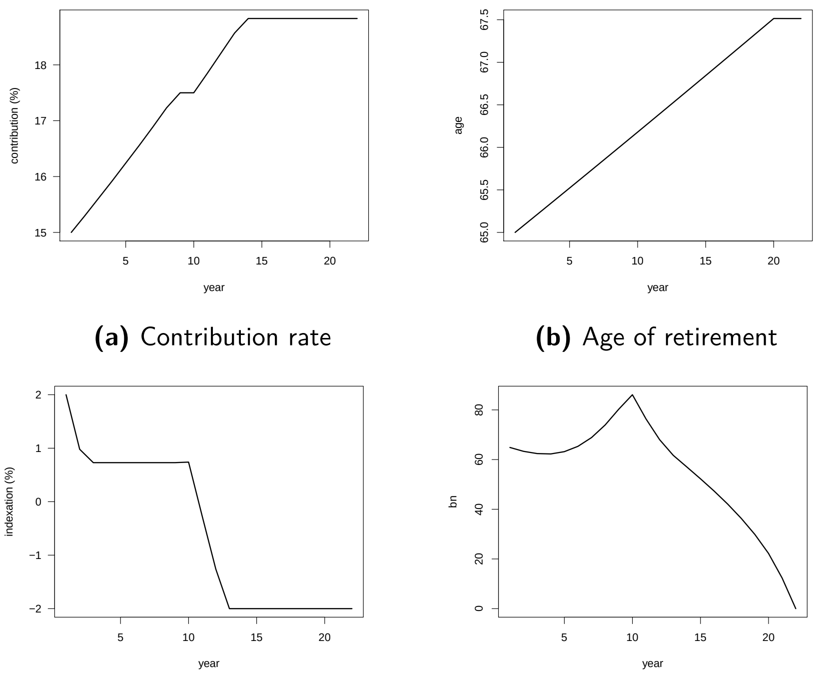
Unexpected shocks

We assume that the total number of contributors suddenly decreases[25] (we suppose a constant 20 per cent less contributors) after year 10. In this case, the model re-calculates the optimal path of the decision variables from year 10 to the end of the study. As we can see in Figure 2.6, the contribution rate increases after being stable for 2 years whereas the indexation of pensions decreases from 0.74 to -2.0 per cent and the age of normal retirement increases

[25] The sudden decrease of the number of contributors is more common than we thought and can occur in financial crisis, war periods, and pandemic diseases, among others. For example, Osterholm (2005) state that if we translate the rate of death associated with the 1918 influenza virus to that in the current population, there could be 1.7 million deaths in the United States and 180 million to 360 million deaths globally. Furthermore, according with Duguay (2006) despite what most people believe, young adults were the worst affected group. In Spain, owing to the financial crisis the unemployment rate increases from 8 per cent in 2006 to around 25 per cent in 2012.

to of 67.5. In this case no surpluses arise that can be redistributed, and therefore both symmetric and asymmetric cases provide the same forecasts for the decision variables. Figure 2.6d shows how the fund absorbs this kind of shock.

Figure 2.6: Unexpected shock: number of contributors suddenly decreases.



(a) Contribution rate

(b) Age of retirement

(c) Indexation of pensions

(d) Accumulated fund

Different levels of investment rate (J) for the buffer fund

Figure 2.7 show the results of the difference of the values for the decision variables between the investment rate in the buffer fund of 3 per cent compared with two different levels, 1 per cent (solid) and 5 per cent (dashes and dots) under symmetric scenario (black line) whereas Figure 2.8 show the results for the asymmetric design (grey line) for the ageing population when we forecast simultaneously the three decision variables, i.e. the results of $c_n(J = 3\%) -$

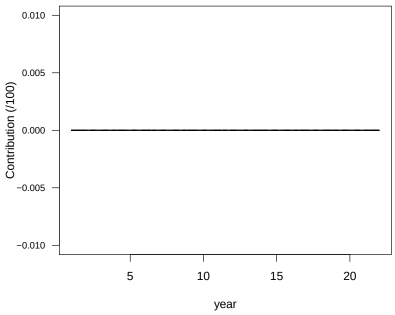
$c_n(J=q), x_n^{(r)}(J=3\%) - x_n^{(r)}(J=q), \lambda_n(J=3\%) - \lambda_n(J=q)$, with $q=1$ per cent and 5 per cent.

Figures 2.7a and 2.8a show the difference in level of the contribution rate. As can be seen, the difference is zero in all cases, so the investment rate does not affect the contribution rate when the three decision variables are simultaneously projected. However, the indexation and age of retirement are affected but in a very small amount. For the age of normal retirement (Figures 2.7b and 2.8b) the changes are between -0.0006 and 0.00005 and for the indexation of pensions (Figures 2.7c and 2.8c) are between -0.0045 and 0.003 under the symmetric and asymmetric design. So we can say that the results of the three decision variables under two different levels of investment rate remain almost the same than with an investment rate of 3 per cent. The fund (Figures 2.7d and 2.8d) presents a small change in the long-term comparing with the previous results, when the investment rate is 5 per cent the fund tends to accumulates more and vice versa, however, for the asymmetric scenario (Figure 2.8d) of the investment rate of 1 per cent the fund at the end accumulate more money than the fund under 3 per cent, this is due to the fact that the indexation and retirement age need to increase more in the first years and under the asymmetric design the system cannot redistribute the surpluses obtained for this faster increase.

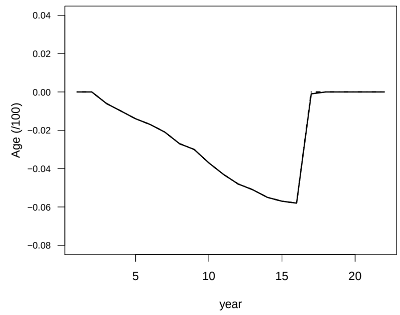
Different levels of growth of salaries (g)

This subsection considers different levels of growth of salaries (g) equal to 1.5 and 3.5 per cent respectively (Figure 2.9-symmetric design- and Figure 2.10-asymmetric design-) and compare the results with the growth of salaries at 2.5 per cent.

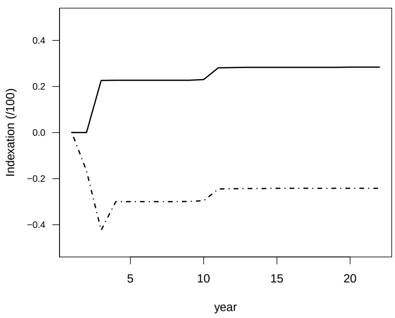
Figure 2.7: Differences of the values of the decision variables. Base scenario (J=3%) versus an investment rate of 1% (solid) and 5% (dots and dashes) under symmetric (black) design.



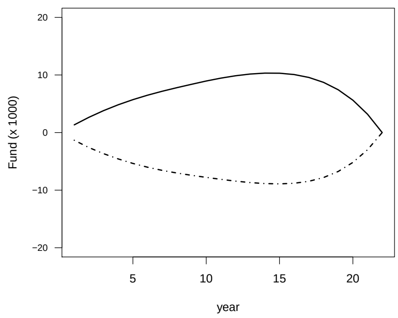
(a) Contribution rate



(b) Age of retirement



(c) Indexation of pensions

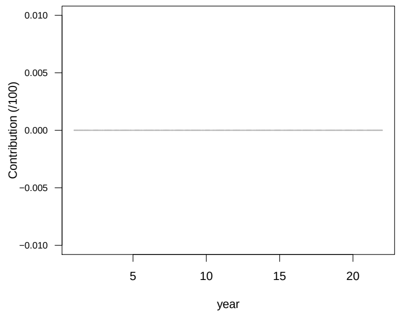


(d) Fund

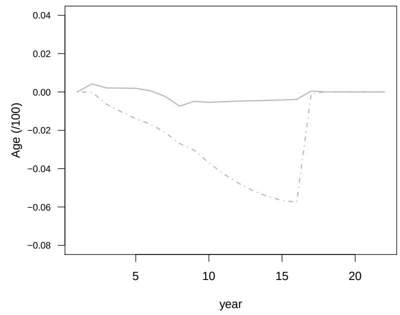
The results show $c_n(g = 2.5\%) - c_n(g = q)$, $x_n^{(r)}(g = 2.5\%) - x_n^{(r)}(g = q)$, $\lambda_n(g = 2.5\%) - \lambda_n(g = q)$, with $q = 1.5$ per cent and 3.5 per cent. The contribution rate and age of retirement stabilise at the same level, no change is observed. As in the case of the investment rate sensitivity analysis, the contribution rate (Figures 2.9a and 2.10a) does not suffer any change. The age of normal retirement (Figures 2.9b and 2.10b) has very minor modifications that are in the range of -0.00055 and 0.0001.

The indexation of pensions (Figures 2.9c and 2.10c) is the variable that is more affected but again at minimum levels. The difference from the case of the growth of salaries of 2.5 per cent goes from -0.01 and 0.015. However, the fund (Figures 2.9d and 2.10d) converges almost to the same at the end of the study, showing differences that are nearly zero except for the asymmetric case

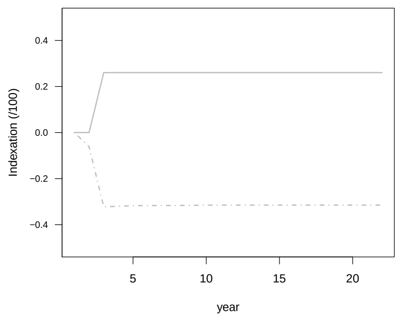
Figure 2.8: Differences of the values of the decision variables. Base scenario ($J = 3\%$) versus an investment rate of 1% (solid) and 5% (dots and dashes) under asymmetric (grey) design.



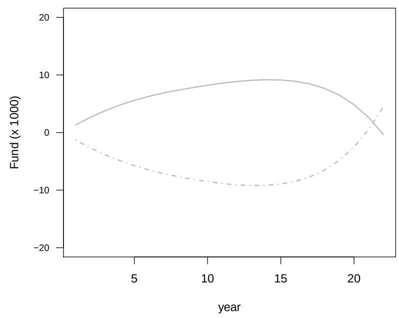
(a) Contribution rate



(b) Age of retirement



(c) Indexation of pensions



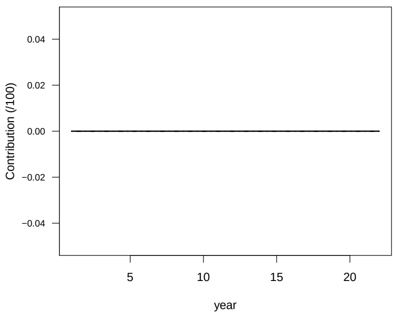
(d) Fund

(Figure 2.10d) under the growth rate of 1.5 per cent that accumulates more surpluses owing to a faster decrease of the indexation of pensions comparing with the growth of salaries if 2.5 per cent.

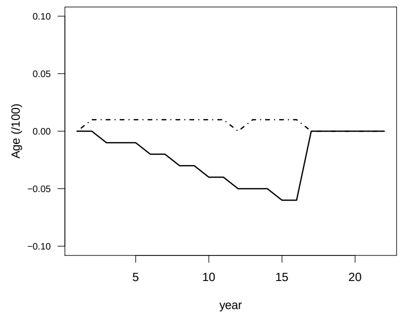
Different levels of contribution base

In this subsection we analyse different percentage levels of the average contribution base to calculate the initial pension. Instead of having a 60 per cent of the contribution base as the initial pension we set it at 50 per cent or 70 per cent. The results are shown in Figure 2.11 for the symmetric design and Figure 2.12 for the asymmetric. For the contribution rate (Figures 2.11a and 2.12a) the change occurs when the percentage of the contribution base is set at 50 per cent. The contribution level under the symmetric and asymmetric scenario

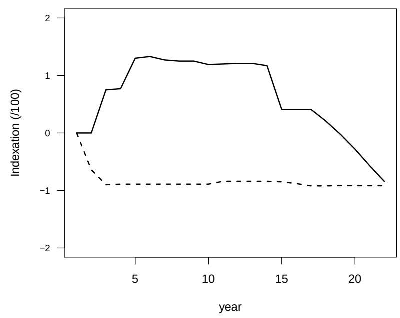
Figure 2.9: Differences of the values of the decision variables. Base scenario (g=2.5%) versus a salary growth rate of 1.5% (solid) and 3.5% (dots and dashes) under symmetric (black) design.



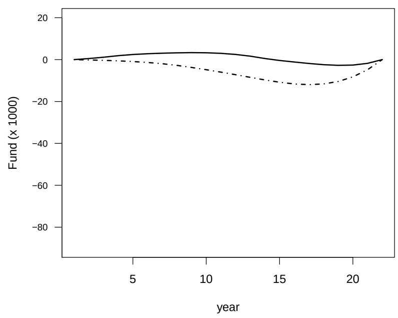
(a) Contribution rate



(b) Age of retirement



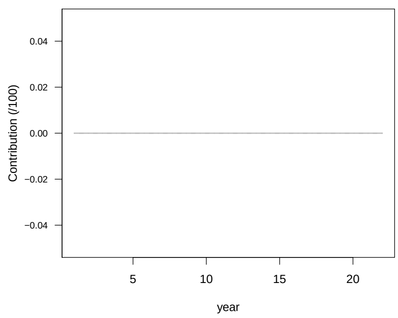
(c) Indexation of pensions



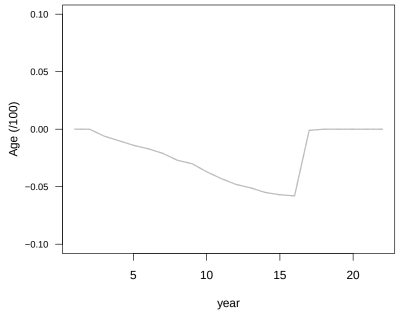
(d) Fund

is lower comparing with the initial pension of 60 per cent of the contribution base by no more than 0.02. The age of retirement suffers modification under the asymmetric scenario at an initial pension of 70 per cent of the contribution base, the values increase slowly than the values under the initial pension of 60 per cent owing to a faster decrease of the indexation of pension (Figure 2.11c and 2.12c). However, the age of retirement under the initial pension of 70 per cent converges to the same value as under an initial pension of 60 per cent. Again, under both designs, symmetric (Figures 2.11c and 2.12c) and asymmetric, minimum modifications occur in the indexation of pensions, the differences are situated between -0.015 and 0.030 with respect an initial pension of 60 per cent. The surpluses or deficits of the fund are consistent with what we expected. Under an initial pension of 50 per cent of the average

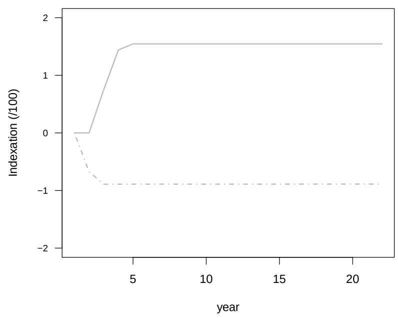
Figure 2.10: Differences of the values of the decision variables. Base scenario ($g = 2.5\%$) versus a salary growth rate of 1.5% (solid) and 3.5% (dots and dashes) under asymmetric (grey) design.



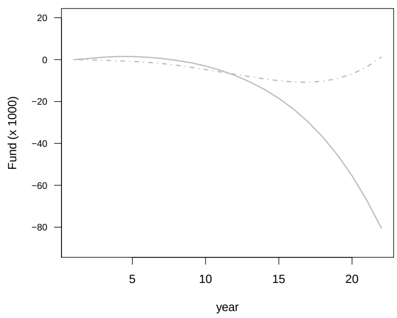
(a) Contribution rate



(b) Age of retirement



(c) Indexation of pensions



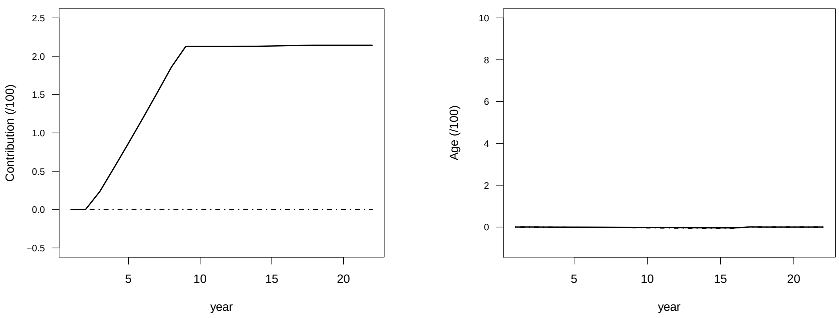
(d) Fund

contribution base the fund accumulates less money than under the 60 per cent of initial pension and under 70 per cent accumulates more because the system has higher payments of benefits and so the fund needs to have more surpluses to cover them.

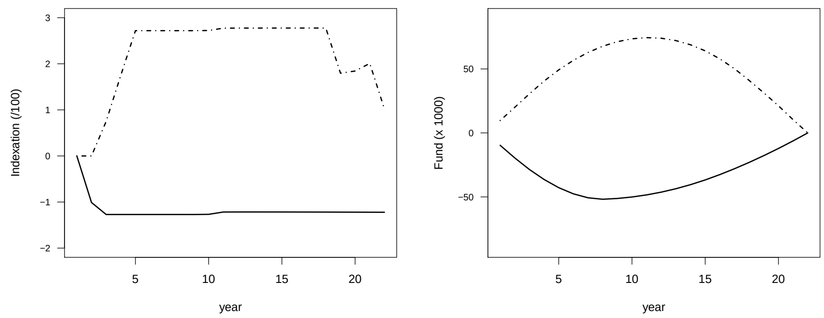
2.4 Conclusions

In social security reforms there is no only one way to proceed and each country has developed its own model. However, it is clear that any solution chosen should meet at least two requirements: financial and social viability (Sales-Sarrapy, Solis-Soberon, & Villagomez-Amezcua (1998)). The aim of this proposed automatic balancing mechanism is to re-establish the liquidity

Figure 2.11: Differences of the values for the decision variables. Base scenario (initial pension of 60% of the average contribution base) versus a 50% (solid) and 70% (dots and dashes) under symmetric (black) design.



(a) Contribution rate **(b)** Age of retirement

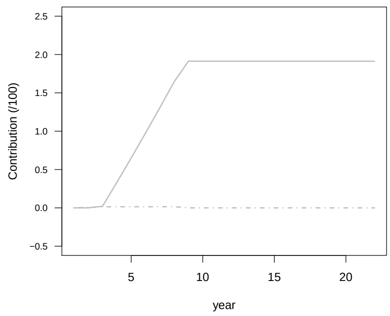


(c) Indexation of pensions **(d)** Fund

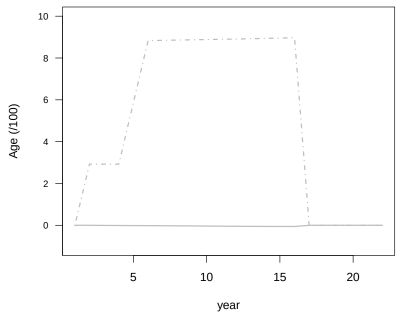
of PAYGO pension systems without the repeated intervention of the legislators. This chapter extends the ideas first proposed by Haberman & Zimbidis (2002b) and Pantelous & Zimbidis (2008) and develops a model for the optimal non-linear problem in a pension scheme including the indexation of the pensions as an additional estimated variable.

The model presented in this chapter could be an alternative to the traditional parametric reforms of the PAYGO systems around the world. We obtained, for two different population structures (an ageing and a mature), an optimal level of the contingency fund given an optimal growth of the contribution rate, age of normal retirement and indexation of pensions based on an annual interest rate of 3 per cent. We show that the contribution rate, age of normal

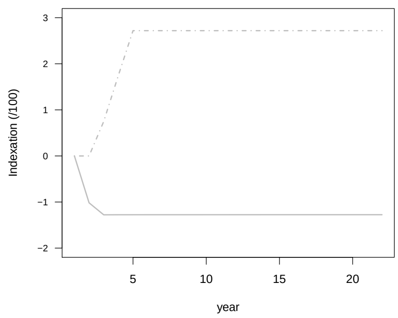
Figure 2.12: Differences of the values for the decision variables. Base scenario (initial pension of 60% of the average contribution base) versus a 50% (solid) and 70% (dots and dashes) under asymmetric (grey) design.



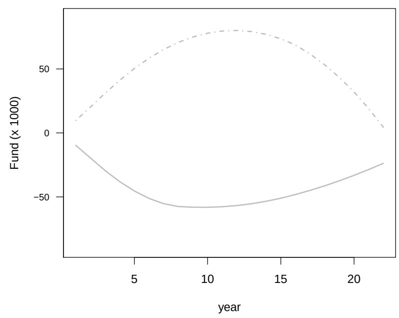
(a) Contribution rate



(b) Age of retirement



(c) Indexation of pensions



(d) Fund

retirement and indexation stabilises at the end of the period of analysis. Furthermore, we show under different investment and salary growth rates and percentages of the average contribution base to calculate the initial pension, that the contribution rate, age of retirement and indexation of pensions suffer very minor modification. We can conclude that the model presented converges to similar values at the end of the study under different scenarios in our sensitivity analysis. The results are completely consistent with the theoretical analysis that we could expect. The contingency fund absorbs the fluctuation of the demographic patterns and the economic variables involved. In contrast with a quadratic objective, the introduction of the logarithmic function as the objective function allows us to analyse and choose between two designs: the symmetric and the asymmetric one. With the model presented we can conclude that an ABM not necessarily needs only to increase the key variables in a pension system, but also under the symmetric scenario, the fund could redistribute the surpluses to the active workers and pensioners relaxing the political difficulties that the legislators could face when implementing the reform. On the other hand, the asymmetric scenario allows the government to accumulate financial reserves in case of any unexpected shocks.

When we first establish an ABM of this type, we need to set up the number of projection variables to be included. The main advantages of a mechanism of this type is to guide the system back onto the road to long-term liquidity and at the same time to automate the measures to be taken, isolating them from the political arena, avoiding any delay and lack of time perspective. However, this ABM also allows some flexibility in the sense that the number of variables to be projected can be changed to adapt the system to a specific situation. At the same time, it is also possible to impose more restrictions to the model to keep, for example, the normal retirement age or the contribution rate constant during some years, making the ABM more applicable in practice.

Planning a smooth sequence of change in the key variables of a pension plan requires involving several aspects at the same time. Nevertheless, the model presented in this chapter involves most of them, an inter-temporal budget pension constraint (the contingency fund); an objective function that allows us to forecast the contribution rate, age of retirement and indexation of pensions without the restrictions that a quadratic loss function has; the forecast of the population; the flexibility of the policy maker to choose how many parameters want to include; and finally, the recalibration of the parameters with the ABM.

3

Optimal strategies for pay-as-you-go pension finance: A sustainability framework

Godínez Olivares, H., Boado-Penas, M.C., and Haberman, S. [1]

The aim of this chapter is to design an automatic balancing mechanism to restore the sustainability of a pay-as-you-go (PAYGO) pension system based on changes in its main variables, such as the contribution rate, normal retirement age and indexation of pensions. Using nonlinear optimisation, this mechanism, identifies and applies an optimal path of these variables to a PAYGO system in the long run and absorbs fluctuations in longevity, fertility rates, salary growth or any other events in a pension system.

Keywords: optimisation; pay-as-you-go; public pensions; risk; sustainability.

3.1 Introduction

Public pension systems are usually financed on a pay-as-you-go (PAYGO) basis where pensions for retirees are paid by the contributions of the working-age population. A successful PAYGO system requires a balance between the expenditure on pensions and the income from contributions made by the active workers over time, usually termed inter-generational solidarity.[2]

[1]This chapter is based on the paper 'Optimal strategies for pay-as-you-go pension finance: A sustainability framework' by Godínez-Olivares, Boado-Penas, & Haberman (2016). Insurance: Mathematics and Economics. 69, 117-126.

[2]Haberman & Zimbidis (2002b) define inter-generational solidarity as the willingness of different groups of people, in this case young and old generations, to participate in a common pool sharing actual experience, including any losses emerging.

Birth rates have dramatically decreased and with continuous improvements in life expectancy, due to improved health care and medical innovations OECD (2015), pensions are paid over a longer time horizon. This causes great difficulties for pension finances and raises serious concerns about the sustainability of the pay-as-you-go pension systems.

In the U.S., The 2015 Annual report of the Board of Trustees of the Federal Old-Age and Survivors Insurance and Federal Disability Insurance Trust Funds (OASDI) (2015) forecasts that the net present value of expenditure on pensions exceeds the net present value from contributions through the period 2015-2089 and that there is a 50% probability of trust fund depletion by the end of 2034 under intermediate assumptions. In Europe, The European Commission (2012b) (White Paper) and The European Commission (2010) (Green Paper) shows a clear increase in life expectancy at birth for males and females of 7.9 and 6.5 years respectively from 2010 to 2060 and that this will cause an increase of the pension expenditure from 10% to 12.5% of GDP in 2060.[3] Furthermore, the increase in unemployment rates, resulting from the recent global crisis, have exerted additional stress on pension systems.

The common trend resulting from these events, that negatively affect the financial health of the systems, is a wave of parametric, or even structural reforms, by changing the formula to calculate the initial pension from a Defined Benefit (DB) PAYGO to a Notional Defined Contribution (NDC), with the aim of reducing the expenditure on pensions.[4]

[3]In both papers, the age of retirement, contribution rate and indexation of pensions are fixed according to current legislation in 2010 and 2012 for the Green and White paper respectively.

[4]See Whitehouse et al. (2009), Whitehouse (2009a), OECD (2012). See Holzmann et al. (2012) for more details on NDCs.

In this respect, Boado-Penas, Valdés-Prieto, & Vidal-Meliá (2008) warn that some politicians, researchers and public opinion mistakenly consider the annual cash-flow deficit or surplus to be an indicator of the pay-as-you-go system's solvency or sustainability[5]; i.e. they confuse an annual liquidity indicator with a sustainability indicator. An actuarial balance needs to be compiled in order to assess whether or not a system is sustainable in the long run. The actuarial balance, according to Barr & Diamond (2009), is necessary to have the "big picture" of the whole system looking for explicit or implicit assets and not only focusing on the future liabilities of the pay-as-you-go system. The actuarial balance, Boado-Penas & Vidal-Meliá (2013), also supplies a positive incentive to improve financial management by eliminating or at least minimising the traditional mismatch between relatively short-vision of both politician and voters -often only four years- and the indefinite time horizon of the system itself.

Even though the information provided by the actuarial balance can help to take better and more informed decisions, some countries have decided not to use it, i.e., U.S. Other countries, without compiling any balance, have adopted some parametric reforms via emergency modifications in legislation. In this respect, Vidal-Meliá et al. (2009) and Vidal-Meliá et al. (2010) define *Automatic Balance Mechanisms* (ABMs) as a set of pre-determined measures established by law to be applied immediately as required according to a sustainability indicator or any other indicator that reflects the financial health of the system. Its purpose, through successive application, is to re-establish the financial equilibrium of pay-as-you-go pension systems with the aim of making those sys-

[5]According to Knell et al. (2006), the term sustainability has many definitions, though it almost always refers to the fiscal policies of a government, the public sector or the pension system. On the other hand, the concept of solvency, mostly applicable in the insurance sector, refers to the ability of a pension scheme's assets to meet the scheme's liabilities indicator. Henceforth we will use the term sustainability.

tems viable without the repeated intervention of the legislators. According to D'Addio & Whitehouse (2012) three main automatic mechanisms can be considered for changing pension values. First, adjustments can be made in benefit levels to reflect changes in life expectancy; second, adjustments can be made through revaluation of earlier years' contribution bases and third, adjustments may occur through the indexation of pension payments. In practice, OECD (2012), Turner (2007) and Turner (2009) state that at least 12 countries link the indexation of pensions to life expectancy or some other type of indicator.

It is important to highlight that Haberman & Zimbidis (2002b) and Godínez-Olivares, Boado-Penas, & Pantelous (2016), propose parametric reforms for restoring the liquidity in the PAYGO pension schemes using a buffer fund. In particular, Haberman & Zimbidis (2002b), using standard linearisation procedures, obtain a closed solution for the optimal paths of the contribution rate and retirement age while trying to keep these variables at the same level over time. Godínez-Olivares, Boado-Penas, & Pantelous (2016) propose an Automatic Balance Mechanism to restore the liquidity of the system at the end of a 20-year horizon. In this case, the authors use a logarithmic function to minimise the growth rate of the contribution rate, retirement age and indexation on pensions.

In this chapter, for the very first time according to the authors' knowledge, two different automatic balancing mechanisms are developed using nonlinear optimisation techniques to restore the sustainability over a 75-year time horizon into a DB-pay-as-you-go pension system while keeping the system liquid at all times. The functional objective function is set to guarantee that the net present value of the income from contributions is sufficient to cover the pension expenditure in the long run.

Following this introduction, the next section of the chapter describes the methodology to compile the actuarial balance with the aim of assessing the sustainability of the system. The third section describes the optimisation techniques in a pay-as-you-go pension system and the mathematical preliminaries together with the main notation and definitions. Section 3.4 describes the two Automatic Balance Mechanisms proposed to guarantee the sustainability into a PAYGO pension scheme. Section 3.5 shows a representative application given a population structure and suggests how an ABM should be designed for both symmetric and asymmetric cases. Section 3.6 concludes and makes suggestions for further research.

3.2 The actuarial balance (AB)

The paper by Plamondon et al. (2002) is a first attempt to conceptualise the actuarial balance of the PAYGO system defined as the difference between an income rate and a cost rate computed over various periods. In our case, this is related to the Sustainability mechanism defined in section 3.4.1. In the same line, Boado-Penas & Vidal-Meliá (2013)[6] and Billig & Ménard (2013) describe the different types of actuarial balance for the pay-as-you-go pension systems.

According to Boado-Penas & Vidal-Meliá (2013), the main methodology used to compile the actuarial balance in nonfinancial Defined Benefit systems could be described as an aggregate accounting projection model that compares the net present value (NPV) of the expenditure on pensions and the income from contributions, in a long time horizon.

[6]This paper also includes the methodology of the actuarial balance sheet, in the accounting sense of the term, that is used in the Swedish notional pension system.

This kind of actuarial balance uses a forecast demographic scenario to determine the future evolution of the number of contributors and pensioners according to the rules of the pension system. The macroeconomic scenario that determines the amounts of future contributions and pensions is exogenous.

In practice, the actuarial balance (AB) at time zero is defined as the difference between the NPV of future contributions valued at zero and the NPV of future pension benefits valued at zero.

In practice, allowing for particular differences between countries, actuarial balances are compiled, on a regular basis, in countries such as U.S, Japan or Canada, amongst others. The actuarial balance of the U.S. social security programs (The 2015 Annual report of the Board of Trustees of the Federal Old-Age and Survivors Insurance and Federal Disability Insurance Trust Funds (OASDI) (2015)), is aimed at measuring the system's financial solvency over a 75 year time horizon. In Japan, the Actuarial Affairs Division of the Ministry of Health, Labour and Welfare (2009) compiles an actuarial balance at least every five years with a 95-year time[7] that includes an automatic balancing mechanism to make the system sustainable in such horizon of time.[8] In Canada, actuarial valuation reports on the Canada Pension Plan (CPP) are prepared by the Office of the Chief Actuary (2012) every three years. These reports determine a minimum contribution rate and show projections of the Plan's contributions, expenditures and assets for the next 75 years.[9]

[7]The U.S and Japanese actuarial balances include the financial assets (buffer fund) at the beginning of the valuation period and also the level of the fund that reaches a magnitude of one-year's expenditure at the end of the time horizon.

[8]A modified indexation is applied to both the revaluation of the contribution bases and pensions in payment. It takes into account both improvements in life expectancy and population increases (or decreases).

[9]Each actuarial valuation report is based on a number of best-estimate assumptions that reflect the judgement of the Chief Actuary of the CPP as to future demographic and economic conditions.

After compiling an actuarial balance, the next section presents the nonlinear optimisation model for two different types of Automatic Balance Mechanisms, to restore the sustainability into the system, involving changes in the contribution rate, retirement age and indexation on pensions.

3.3 Optimisation techniques in a PAYGO pension system

This section presents the optimisation[10] techniques used to calculate the optimal and smooth paths of the variables involved in the PAYGO system such as the contribution rate, the age of normal retirement and the indexation of pensions. We extend the optimisation model developed by Godínez-Olivares, Boado-Penas, & Pantelous (2016) which focused only on the liquidity of the system in a 20-year time horizon. In this chapter, we are particularly interested in finding the optimal paths of the key variables that guarantee not only the short run equilibrium (liquidity in the system) but also the long-run sustainability of the pension finance.

In a general nonlinear optimisation problem (NLO)[11] the functions and parameters involved are:

a The decision variables represented by $\{d\}_n = \{d_n^0, d_n^1, ..., d_n^v\} \in D$; where $D \in \Re^n$ is the decision space, n is the step of the process with $n \in \mathbb{N}$ and v is the number of variables included in d. Specifically in this model

[10] Optimisation techniques are extensively used in economics in the intertemporal consumption context (see for example, Obstfeld & Rogoff (1996), Wilcoxen (1989), Walde (2010)). However, there are not many application on pensions systems (see for example, Devolder, Bosch Princep, & Dominguez Fabian (2002), Feldstein & Liebman (2010), Haberman & Vigna (2002a), Roberts (2002) and Sayan & Kiraci (2001)).

[11] For more details see Bertsekas (1999) and Bertsekas (2005).

$d_n^j = (c_n^j, x_n^{(r)j}, \lambda_n^j)$ where $\{c\}_{n\in\mathbb{N}}$ is the contribution rate, $\{x^{(r)}\}_{n\in\mathbb{N}}$ is the age of retirement, $\{\lambda\}_{n\in\mathbb{N}}$ is the indexation of pensions; and n is the year;

b A function $f_n(d_n^j, n)$, that is called, the objective function of the NLO. In our case, $f_n(d_n^j, n)$ is expressed by a minimisation function;

c The set of feasible solutions $\mathcal{F} = \{d_n^j \in D | h_k(d_n^j) = 0, k = 1, ...l$ and $g_j(D) \leq 0, j = 1, 2, ...m\}$; where h_k for $k = 1, ...l$ represent l linear constraints and g_j for $j = 1, ..., m$ represent m nonlinear constraints. In our model, following the ideas introduced by Godínez-Olivares, Boado-Penas, & Pantelous (2016), the constraints are defined by the upper and lower bounds and change of rates of the key variables at time n and also by the liquidity restrictions, and;

d If d^* minimises (or maximises) f, then d^* is called an optimal solution of the NLO. If $\mathcal{F} = \emptyset$ then the NLO is not feasible, i.e., there are no d^* that minimises (or maximises) the objective function.

Following Godínez-Olivares, Boado-Penas, & Pantelous (2016), we applied the Generalized Reduced Gradient (GRG) algorithm to find the optimal path of the variables. This iterative method allows us to find the optimal solution of the NLO. The GRG is the extension of the gradient method to constrained and bounded optimisation problems. It was first developed by Abadie & Carpenter (1969) and is still today one of the most powerful nonlinear optimisation algorithms.[12]

In practice, building the model (modelling phase) is as important and sometimes more difficult than the solving phase (Nemirovski (1999)). The modelling

[12]See, for example, Kallrath (1999), Bazaraa, Sheraldi, & Shetly (1993) and Wright (1996).

phase, i.e. the formulation of the Automatic Balance Mechanism, is discussed in the following section.

3.4 Automatic Balancing Mechanism (ABMs) to restore sustainability

This section presents two different designs of Automatic Balance Mechanisms to restore both the liquidity and sustainability of a pay-as-you-go pension system to find the optimal path for the decision variables -contribution rate, age of normal retirement and indexation of pensions- in a PAYGO pension scheme.

The first Automatic Balance Mechanism (*Sustainability ABM*) restores the system's sustainability measured via the actuarial balance as the difference between the net present value of the future income from contributions and the expenditure on pensions in the long run. The second Automatic Balance Mechanism (*Sustainability ABM with fund*) takes into account the accumulated value of the financial assets (buffer fund) emerging from the difference between the income from contributions and expenditure on pensions in each year.

The next subsections describe how the two different Automatic Balance Mechanisms are built.

3.4.1 Sustainability ABM

The optimisation function, $f_n(d_n^j, n)$, to minimise for the *Sustainability ABM* for a time horizon N, subject to some constraints, is as follows:

$$
min_{c_n x_n, \lambda_n} \sum_{n=0}^{N} \frac{c_n W_n(g_n, x_n^{(r)})}{(1+\delta)^n} - \sum_{n=0}^{N} \frac{B_n(g_n, x_n^{(r)}, \lambda_n)}{(1+\delta)^n}
$$

$$
s.t. = \begin{cases} c_{min} \leq c_n \leq c_{max}; x_{min}^{(r)} \leq x_n^{(r)} \leq x_{max}^{(r)}; \\ \lambda_{min} \leq \lambda_n \leq \lambda_{max}; \\ c_{1\Delta} \leq \dfrac{c_{n+1}}{c_n} \leq c_{2\Delta}; x_{1\Delta}^{(r)} \leq \dfrac{x_{n+1}^{(r)}}{x_n^{(r)}} \leq x_{2\Delta}^{(r)}; \\ \lambda_{1\Delta} \leq \dfrac{\lambda_{n+1}}{\lambda_n} \leq \lambda_{2\Delta}; \\ \dfrac{c_n W_n(g_n, x_n^{(r)})}{B_n(g_n, x_n^{(r)}, \lambda_n)} \geq 1; \end{cases} \tag{3.1}
$$

where:

c_n is the contribution rate during year n; $W_n(g_n, x_n^{(r)})$ is the total contribution base[13] paid at n that depends on the growth of salaries g_n and the retirement age at n, $x_n^{(r)}$; B_n is the total expenditure on pensions at n that depends on g_n, $x_n^{(r)}$, and the indexation of pensions at n, λ_n; $\delta > 0$ is the discount rate.

$c_{min}, x_{min}^{(r)}, \lambda_{min} \in \Re$ and $c_{max}, x_{max}^{(r)}, \lambda_{max} \in \Re$ are lower and upper bounds of the decision variables respectively. These bounds are set in order to avoid possible unrealistic changes in the key variables of the pension system.

Smooth constraints are also necessary to prevent jumps in path of the contribution rate, age of retirement and indexation of pensions. Mathematically, smooth constraints are set as: $c_{1\Delta} \leq \frac{c_{n+1}}{c_n} \leq c_{2\Delta}; x_{1\Delta}^{(r)} \leq \frac{x_{n+1}^{(r)}}{x_n^{(r)}} \leq x_{2\Delta}^{(r)}; \lambda_{1\Delta} \leq \frac{\lambda_{n+1}}{\lambda_n} \leq \lambda_{2\Delta}$
where $c_{1\Delta}, c_{2\Delta}, x_{1\Delta}^{(r)}, x_{2\Delta}^{(r)}, \lambda_{1\Delta}, \lambda_{2\Delta} \in \Re$.

Finally, the liquidity restriction $c_n W_n(g_n, x_n^{(r)}) \geq B_n(g_n, x_n^{(r)}, \lambda_n)$ for all n is imposed to ensure that the income from contributions at n covers the annual expenditure on pensions. Consequently, the liquidity indicator that measures

[13]The contribution base of the system is defined as the total wages of the working population.

the liquidity of the system, taking only into account the income from contributions and the expenditure of pensions is defined as:

$$L_n = \frac{c_n W_n(g_n, x_n^{(r)})}{B_n(g_n, x_n^{(r)}, \lambda_n)} \tag{3.2}$$

Since $\delta > 0$ and thanks to the liquidity restriction, the optimal solution for this Automatic Balance Mechanism built by equation 3.1 should be zero.[14] However, it is well-known that in practice it is not possible to have an exact equilibrium for the valuation problem, and that is the reason for minimising the difference between the income from contributions and the expenditure on pensions.

3.4.2 An alternative design for the Sustainability ABM (*Sustainability ABM with fund*)

The amount of the buffer fund, F_n, or even the returns that it generates, might have a positive effect on a pension system facing unexpected changes in the demographic or economic projections. For example, the returns generated by the buffer fund in Spain covered the shortfall in contributions during 2010 (of State for Social Security. (2011)). With this in mind, the *Sustainability ABM with fund*, that includes the accumulation of the fund over time, can be expressed as:

$$F_n = (1 + J_n)F_{n-1} + c_n W_n(g_n, x_n^{(r)}) - B_n(g_n, x_n^{(r)}, \lambda_n), \tag{3.3}$$

[14] Under the current adverse demographic scenario, it will be necessary to make adjustments in the key variables in order to ensure sustainability of the system in the long run. A value higher than zero would mean that changes to the key variables would be even harsher to the individual members of the pension system. On the other hand, a value lower than zero would require assumptions to be made with respect to the interest rate and that are outside of the scope of this chapter.

where J_n is the return[15] of the fund during year n. The variable F_n fluctuates deliberately to absorb changes in fertility, mortality projections and any other events that might affect the liquidity and sustainability indicator in the pension system.

In contrast to the Automatic Balance Mechanism built by equation 3.1, the *Sustainability ABM with fund* is focused on the minimisation of the present value of the buffer fund. The objective function to minimise, subject to some constraints, is as follows:

$$min_{c_n x_n, \lambda_n} \sum_{n=0}^{N} \frac{F_n(c_n, g_n, x_n^{(r)}, \lambda_n, J_n)}{(1+\delta)^n}$$

$$s.t. = \begin{cases} c_{min} \leq c_n \leq c_{max}; x_{min}^{(r)} \leq x_n^{(r)} \leq x_{max}^{(r)}; \\ \lambda_{min} \leq \lambda_n \leq \lambda_{max}; \\ c_{1\Delta} \leq \dfrac{c_{n+1}}{c_n} \leq c_{2\Delta}; x_{1\Delta}^{(r)} \leq \dfrac{x_{n+1}^{(r)}}{x_n^{(r)}} \leq x_{2\Delta}^{(r)}; \\ \lambda_{1\Delta} \leq \dfrac{\lambda_{n+1}}{\lambda_n} \leq \lambda_{2\Delta}; \\ F_n \geq 0 \end{cases} \tag{3.4}$$

The same constraints for the contribution rate, the age of retirement and the indexation of pensions as in the *Sustainability ABM* (eq. 3.1) are imposed. However, the liquidity constraint is now set as $F_n \geq 0$, for all n, to ensure liquidity in the system.

Under the *Sustainability ABM with fund*, the natural liquidity indicator that emerges from the objective function and takes into account the accumulated value of the buffer fund is expressed as follows:

[15] According to Yermo (2008) one of the most remarkable aspects of the regulatory environment of the reserve funds surveyed is that with the exception of Ireland, Japan, Korea, and Sweden, there are no major investment limitations. The study of the different investment strategies is out of the scope of this chapter, therefore our analysis assumes a risk-free investment rate as in the case of the United States, Belgium and Spain.

$$Lf_n = \frac{(1 + J_n)F_{n-1} + c_n W_n(g_n, x_n^{(r)})}{B_n(g_n, x_n^{(r)}, \lambda_n)} \tag{3.5}$$

The same liquidity indicator as for the *Sustainability ABM* (without the inclusion of the buffer fund eq. 3.2), can be calculated but, under the *Sustainability ABM with fund* model, we cannot guarantee that its value would be higher than or equal to one.

3.4.3 Symmetric and asymmetric designs for the *Sustainability ABM* and the alternative design (*Sustainability ABM with fund*)

Most of parametric reforms are based on the general view that expenditure on pensions is increasing, based on an increase in life expectancy, and/or on the fact that the amount of the buffer fund is decreasing. In this respect, Automatic Balance Mechanisms are designed to face adverse demographic and economic changes (asymmetric ABM), but these could also be designed to benefit the system's participants in good times.

Palmer (2013) states that under a symmetric Automatic Balance Mechanism, any surplus (defined as the difference between the income from contributions and the expenditure on pensions at n) that might arise would be automatically distributed. In the absence of a symmetric ABM, an undistributed surplus is maintained in the system. Furthermore, Alho et al. (2013) state that the balancing mechanism can be symmetric, adjusting for both positive and negative deviations from the financial health indicator. Analytically, for the asymmetric case the change in the contribution rate and age of normal retirement are enforced to be strictly greater or equal to zero (strictly lower or equal to zero for the indexation of pensions).

In our model, under the *symmetric* design, our ABMs determine whether the contribution rate, the age of normal retirement (and indexation of pensions) are reduced (increased) when the system has a surplus and increased (decreased) in periods of deficit. Analytically, relative changes in the contribution rate and age of normal retirement are enforced to be strictly greater or equal to one (strictly lower or equal to one for the indexation of pensions) under the asymmetric design. In contrast, under the *symmetric* case, the rate of change in the variables could be positive or negative.

3.5 Numerical Example

This section presents a numerical example using the ABMs defined by the optimisation problems 3.1 and 3.4 developed in Section 3.4. First, the main data and assumptions are presented, secondly the results are discussed and finally this section provides a sensitivity analysis of the results.

3.5.1 Data

- We use the demographic structure of the European population (Figure 3.1a) from 2013 to 2087 obtained from Eurostat.[16] It can be seen that the peak of population in 2013 (Figure 3.1a, grey pyramid) is at ages 40-50 in 2013, corresponding to the demographic boom in the 1960s and early 1970s. By 2087, there are no clear peaks in the population. Also, the age-dependency ratio[17], decreases over time, with 3.63 contributor financing

[16]Data obtained from Eurostat Database, http://ec.europa.eu/eurostat, accessed 1 March 2015.

[17]This ratio measures the number of elderly people relative to those of working age. It is calculated as the number of contributors divided by the number of pensioners.

one pensioner in 2013 to 1.94 in 2087 as shown in Figure 3.1b. For comparison purposes the demographic structure has been normalised.[18]

- The European salary and pension structures over time have been used.[19]

- The salaries ($g_n = g$) are assumed to increase at an annual constant rate of 2.5%[20] while the buffer fund is assumed to increase at an annual rate (J_n) of 3%[21].

- The initial pension is set at 55% of final salary[22] and $P_{x,n}$ can be written as:

$$P_{x,n} = P_{x-1,n-1} * (1 + \lambda_{n-1}),$$

  where $P_{x_n^{(r)},n} = P_{x_n^{(r)},n} \mathbb{1}_{\{x_n^{(r)},n=1\}} + P_{x_n^{(r)},n} * (1+g)^n \mathbb{1}_{\{x_n^{(r)},n>1\}}$.

- The lower bounds for the contribution rate, age of normal retirement and indexation of pensions are given respectively by 15%, 65 and -5%; the upper bounds are 20%, 70[23] and 5% respectively.

- For smooth changes, it is also assumed that the change in the contribution rate varies between 0.3% and 0.7%, the age of normal retirement between 1.5 and 4 months and the indexation of pensions between -0.5% and 0.5%. These values are in line with the most important reforms in

[18] The normalisation rescales data values of the population per age to a new range of values between zero and one.

[19] Data obtained from Eurostat Database, http://ec.europa.eu/eurostat accessed 1 March 2015. The salary structure, shown in Figure 3.1c, has been normalised to allow comparisons.

[20] Average growth of the hourly labour costs in Europe from 2004 to 2014. Data obtained from Eurostat Database, http://ec.europa.eu/eurostat, accessed 1 March 2015.

[21] This value, which is used for the *Sustainability ABM with fund*, is in line with the average value of the Euribor rates during the last 15 years.

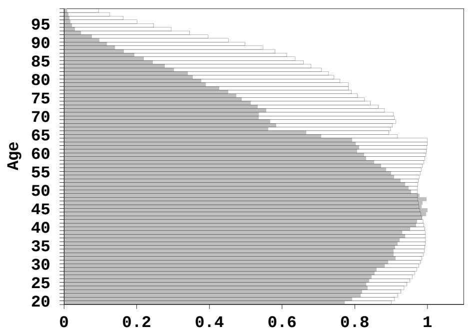
[22] According to Creighton (2014), this level is in line with the average replacement rate which measures the pension as a percentage of a worker's pre-retirement income.

[23] As shown in the results, the optimal solutions would never reach this upper bound. Therefore, setting up a higher value would not modify our results.

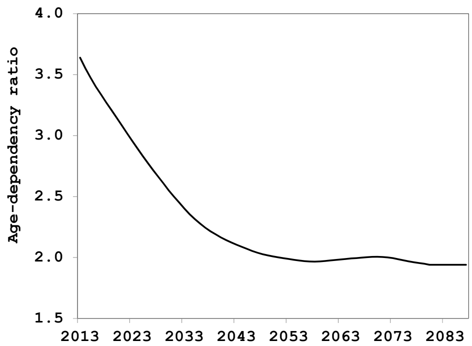
the 34 OECD member countries[24] between January 2009 and September 2013.

- No unemployment is considered in our analysis.

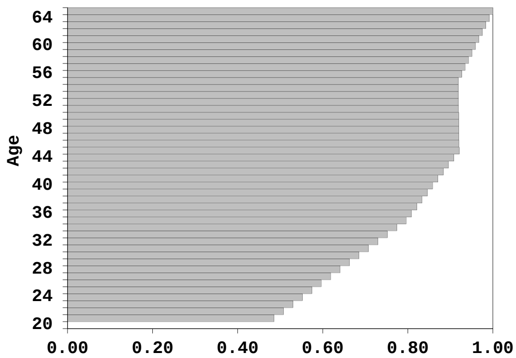
Figure 3.1: Normalised European population structure in 2013 (grey) and 2087 (transparent), evolution of the age-dependency ratio, and salary structure in 2013.



(a) European population 2013 and 2087



(b) Age-dependency ratio 2013 - 2087



(c) Salary structure European population 2013

[24]See OECD (2014).

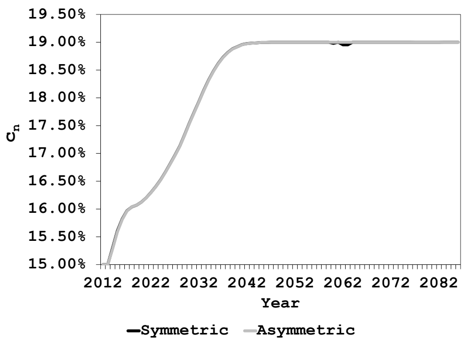
3.5.2 Results

Figure 3.2 shows the optimal paths[25] for the *Sustainability ABM* when the three decision variables (contribution rate, retirement age and indexation of pensions) are modified simultaneously under both the asymmetric (base case) and symmetric cases. Under both cases, the contribution rate (Figure 3.2a) stabilises at 19%, and only from 2060 to 2064 is the contribution rate lower under the symmetric design. At the same time, the age of normal retirement (Figure 3.2b) stabilises at 67.5 under both scenarios, although for the symmetric case the retirement age is lower than under the asymmetric design from 2055 to 2077. The indexation of pensions (Figure 3.2c) stabilises at -1%[26] at the end of the period of analysis. The liquidity indicator, defined as the ratio of the income from contributions to the expenditure on pensions at every year, stabilises around one. The difference between the symmetric and asymmetric cases is explained by the population structure: the age-dependency ratio (Figure 3.1b) shows a small blip in the projected ratio between 2058 and 2069, i.e. there are more active workers to finance the pensioners.

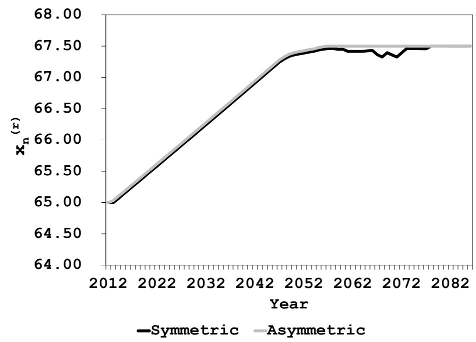
[25]For both ABMs the optimal solutions found are local minimums, however, as the initial values for the decision variables are close to the real values, the local minimums are the best solutions.

[26]When a lower bound of the indexation of pensions is set to zero percent, the *Sustainability ABM* stabilises at 21% for the contribution rate, 68.44 for the retirement age and 0% for the indexation of pensions; the *Sustainability ABM with fund*, stabilises at 20.45%, 68 and 0% respectively. In practice, some PAYGO pension systems (i.e. Sweden (The Swedish Pension System (2015))) have already adopted a negative indexation to make its pension system sustainable in the long run. For this reason, we allow for a negative indexation (with a lower bound of -5%) to keep a more general setting.

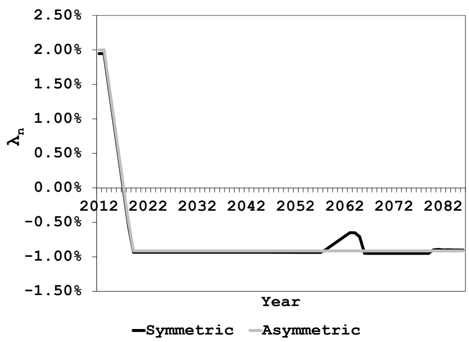
Figure 3.2: Results of *Sustainability ABM* when the three variables are projected simultaneously for the symmetric (black line) and asymmetric (grey line) with European population structure.



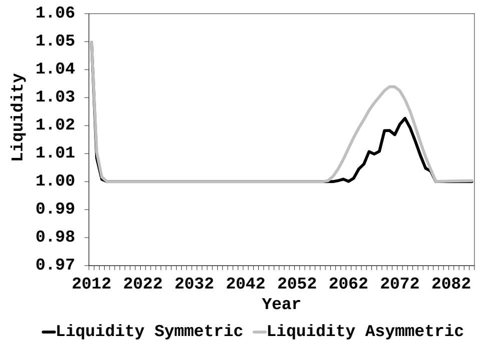
(a) Contribution rate



(b) Age of retirement



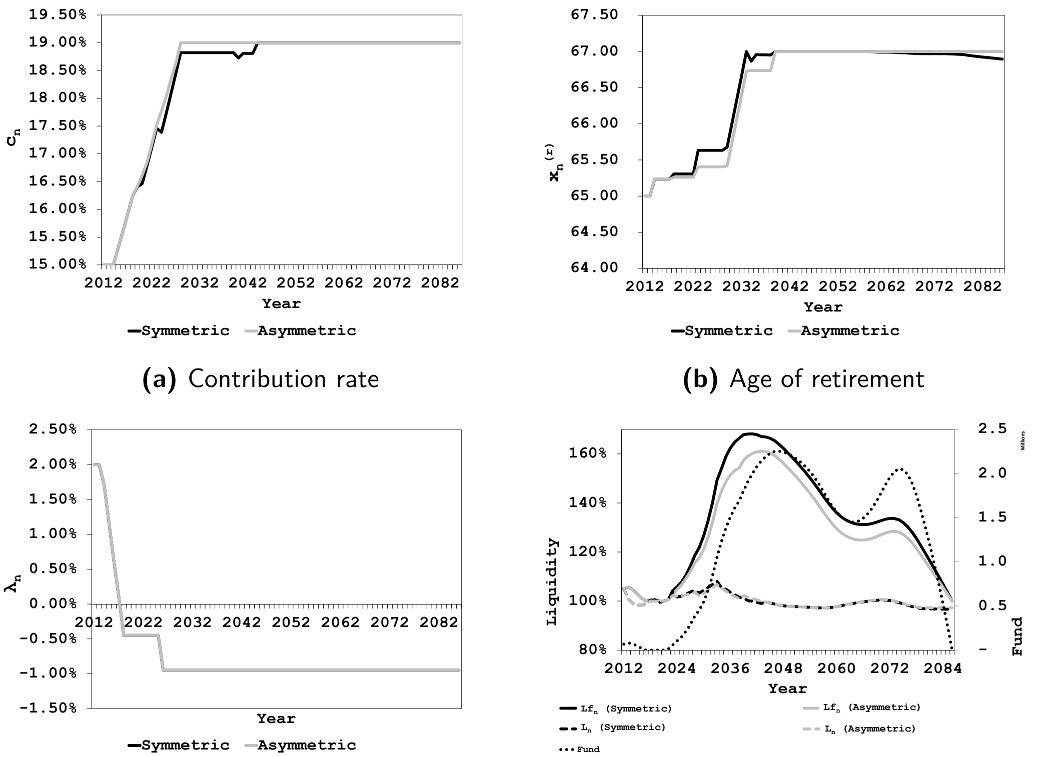
(c) Indexation of pensions



(d) Liquidity indicator (L_n)

Figure 3.3 shows the results of the *Sustainability ABM with fund*, which is the modified Automatic Balance Mechanism that takes into account the amount of financial assets of the system. Under this design, the optimal paths of the contribution rate, age of normal retirement (and indexation of pensions) are expected to take lower (higher) values than for the *Sustainability ABM* design. Figure 3.3 shows that, under the symmetric design (black line), the contribution rate (Figure 3.3a) needs to increase to 19%, as in the *Sustainability ABM*. However, the age of normal retirement increases to 66.8 at the end of the study, 0.7 less than under the *Sustainability ABM* design. The indexation of pensions takes the same path as for the *Sustainability ABM* design.

Figure 3.3: Results of the *Sustainability ABM with fund* when the three variables are projected simultaneously - for the symmetric (black line) and asymmetric scenario (grey line) with European population structure.



(a) Contribution rate **(b)** Age of retirement

(c) Indexation of pensions **(d)** Liquidity indicator

Figure 3.3d shows the liquidity indicator without buffer fund defined by eq. (3.2) (dashed line) and including the amount of the buffer fund defined by eq. (3.5) (solid line) under both the symmetric (black line) and asymmetric (grey line) designs. The liquidity indicator including buffer fund remains always greater than 1, that is, the system always has enough money to cover the expenditure on pensions in every year. However, if we do not take into account the buffer fund (dashed line), the system shows a permanent deficit from year 2040 onwards.

The Automatic Balance Mechanism could also be designed around having only one modified variable rather than having three variables that are modified simultaneously. For the *Sustainability ABM*, if the contribution rate is the only decision variable, it would need to increase to 31.43%, whereas if the age of retirement is the only variable it would increase to 78.04 years. If the indexation of pensions is set as the only decision variable, the benefits would decrease at an annual rate of 5.15%.

3.5.3 Sensitivity analysis

This sub-section performs a sensitivity analysis for different levels of salaries growth (g), percentages of the contribution base to calculate the initial pension, and age-dependency ratios in order to see how the main variables of the system react to these changes. From now on, only the *Sustainability ABM* under an asymmetric design is analysed.[27]

Different levels of growth of salaries (g)

This subsection analyses two different levels of salary growth (g). When the growth of salaries is 0.5%, the contribution rate stabilises at 19.7% (Figure 3.4a), that is 0.7% more than for the base scenario, whereas the retirement age needs to increase to 68.5 (Figure 3.4b), one year more than the base scenario, and the indexation of pension stabilises at -2.1% (Figure 3.4c), that is 1.1% less than the base scenario.

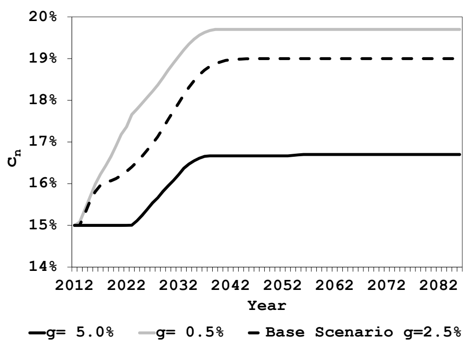
If the growth of salaries is set at 5%, the contribution rate stabilises at 16.7%, 2.3% lower than the base scenario with a salary growth of 2.5%, the age of

[27]The results for the *Sustainability ABM with fund* are not presented since the conclusions drawn are very similar to the *Sustainability ABM*.

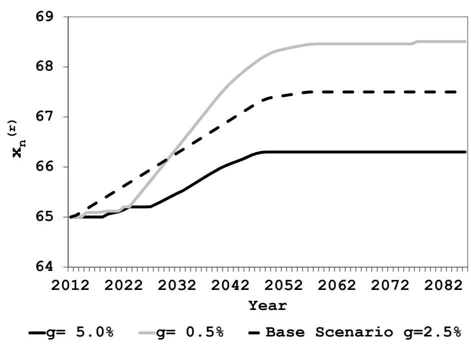
retirement stabilises at 66.3, that is 1.2 years below the base scenario, whereas the indexation of pensions stabilises at -0.4%, that is 0.6% more than the base scenario.

The liquidity of the system (Figure 3.4d) follows a similar path for the different growth of salary analysed. The increase from 2058 to 2070 corresponds to the increase in the age-dependency ratio shown in the demographic projections. As the liquidity of the system is always greater than 1, it is possible to conclude that the objective of the Automatic Balance Mechanism is achieved.

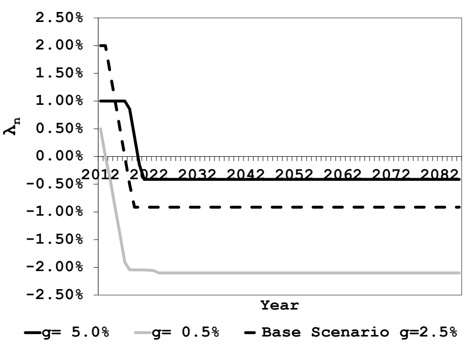
Figure 3.4: Results of the *Sustainability ABM* under different levels of growth of salaries: 0.5% (grey), 5% (black) and base scenario (dashed).



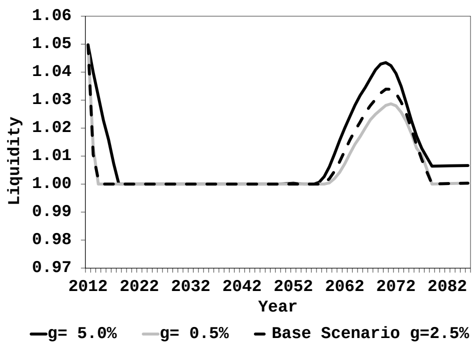
(a) Contribution rate



(b) Age of retirement



(c) Indexation of pensions



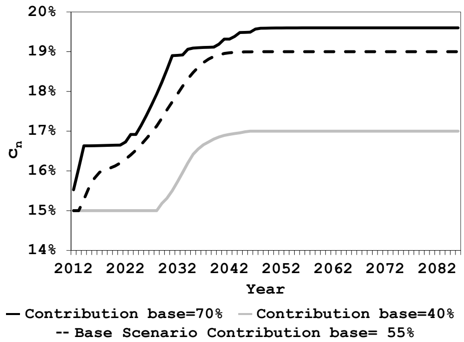
(d) Liquidity indicator (L_n)

Different percentage levels of the initial pension

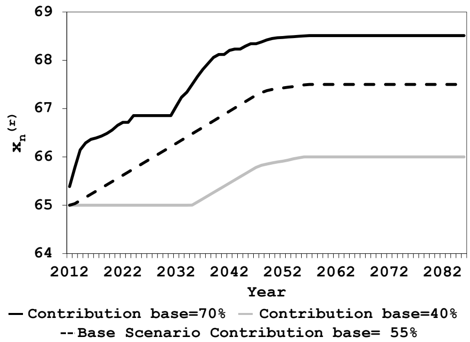
This subsection analyses different percentage levels of initial pension, specifically the initial pension is set at 40% or 70% of the contribution base. The contribution rate (Figure 3.5a) stabilises at 17% and 19.6% respectively, instead of 19% for the base case scenario. The age of normal retirement stabilises at age 66 and 68.5 respectively, while it stabilises at 67 under the base scenario.

The value of the indexation of pensions stabilises at -0.45% and -1.75% for a 40% and 70% of the contribution base respectively. The liquidity of the system is maintained around one for the whole of the period analysed; thus, the target of the Automatic Balance Mechanism is achieved with these levels of contribution base.

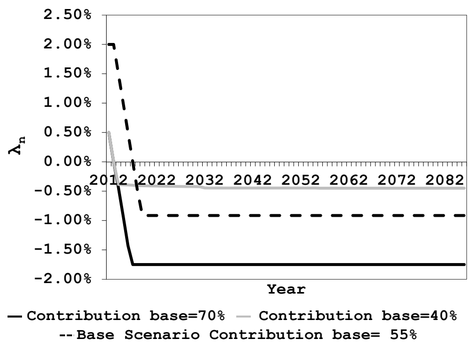
Figure 3.5: Results of the *Sustainability ABM* under different levels of initial pension: 40% (grey), 70% (black) and base scenario (dashed).



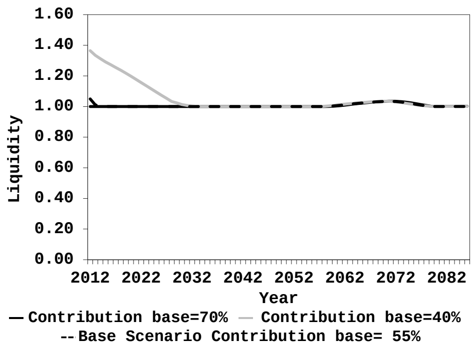
(a) Contribution rate



(b) Age of retirement



(c) Indexation of pensions



(d) Liquidity indicator (L_n)

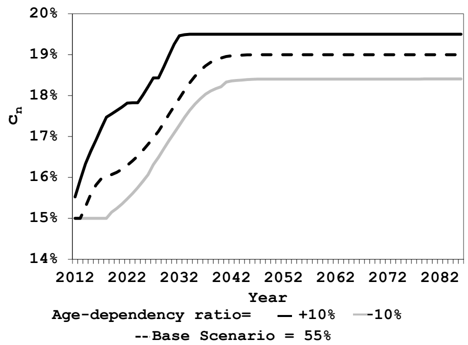
Different levels of age-dependency ratio

A lower (higher) level of the age-dependency ratio translates into a smaller or greater number of pensioners in our study.[28] When the age-dependency ratio (grey line) is 10% lower (Figure 3.6a), the contribution rate stabilises below the base case scenario by 0.6%, the age of retirement (Figure 3.6b) by 0.5 years and the indexation of pensions above by 0.41%. Whereas, when the age-dependency ratio increases by 10% , the contribution rate and the age of

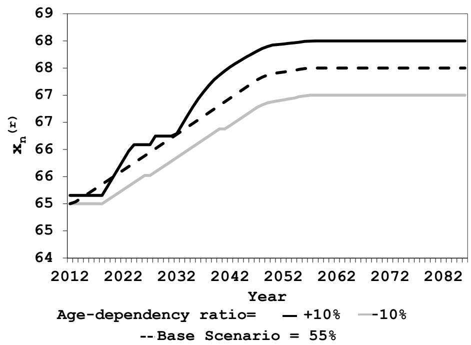
[28]The constant increase or decrease in the age-dependency ratio affects all ages by the same percentage.

retirement are higher by 0.5% and 0.5 years respectively with respect to the base scenario. The indexation of pensions (Figure 3.6c) is lower by 0.20%. In this case, the liquidity of the system (Figure 3.6d) is obtained and, again, from 2058 to 2069 the system shows a surplus owing to the given demographic structure during these years.

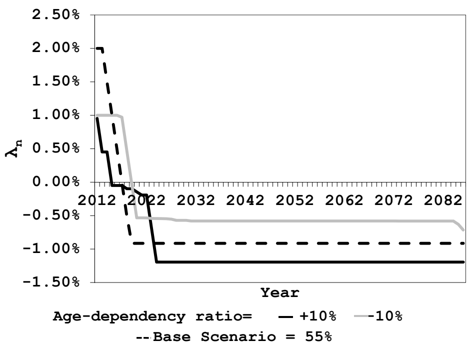
Figure 3.6: Results of the *Sustainability ABM* under different levels of age-dependency ratio: -10% (grey), +10% (black) and base scenario (dashed).



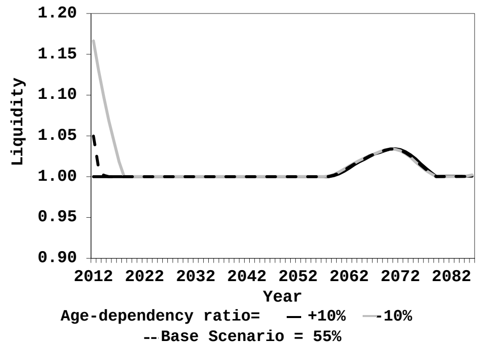
(a) Contribution rate



(b) Age of retirement



(c) Indexation of pensions



(d) Liquidity indicator (L_n)

This sub-section has shown how the three sensitivities presented (different levels of growth of salaries (g), percentage of the levels of the initial pension and levels of age-dependency ratio) are all in the expected direction. It is also remarkable that, even in the most optimistic scenario, the key variables will need to be adjusted to guarantee the sustainability of the system in the long run.

3.6 Conclusions

Restoring the long-term sustainability of a Pay-As-You-Go pension system is on the agenda for most governments. The expenditure on pensions is increasing in line with the increase in longevity, the forecasts for the ageing of the baby-boom generation and any other random events that negatively affect to the financial health of the system. At the same time, the income from contributions does not increase at the same rate mainly due to the decline in the fertility rates.

Social security decisions to restore the sustainability usually involve political risk in the sense that the horizon of the policymakers is less than that of the system itself. This chapter aims to provide a solution to restore the long-term sustainability of a pay-as-you-go system using automatic balancing mechanisms, isolating the measures to be taken from the political arena. Using optimisation techniques, the Automatic Balance Mechanisms presented in this chapter propose a theoretical framework to calculate the optimal path of the contribution rate, age of retirement and indexation of pensions.

Some politicians, researchers and sections of public opinion mistakenly consider the annual cash-flow deficit or surplus, that is the liquidity indicator, to be an indicator of the PAYGO system's sustainability. Therefore, the two Automatic Balance Mechanisms presented in this chapter focus on restoring the sustainability of the pension scheme while keeping the system liquid at all times. The Automatic Balance Mechanisms built are based on minimising the present value of future cash flows (income from contributions and expenditure on pensions) including or not the accumulated value of the buffer fund of the system.

The Automatic Balance Mechanisms presented in this chapter could be implemented as an alternative to the parametric pension reforms of the pay-as-you-go systems around the world. Parametric reforms are usually taken via emergency modifications in legislation with the aim of re-establishing the financial equilibrium in the system. Our proposed ABM not only focuses on the short term liquidity of the system but also on its sustainability in the long run. The main drawback of our ABMs is that the projections of demographic, economic and financial variables might be far away from the real values.

The chapter also discusses two different designs that could be implemented, the symmetric or asymmetric design. Under the *Sustainability ABM*, no surpluses are accumulated, whereas it is possible to build an alternative Automatic Balance Mechanism which includes a buffer fund, and accumulates financial assets (*Sustainability ABM with fund*). For both *Sustainability ABM* and *Sustainability ABM with fund* mechanisms, the contribution rate and indexation of pension stabilise at the same level, 19% and -1% respectively, whereas the age of normal retirement, when the buffer fund is included (*Sustainability ABM with fund*), stabilises 0.7 years below the *Sustainability ABM* design, i.e. 67.5 and 66.8 respectively.

This chapter shows that the sustainability of a generic pension system using European population structure and salary profile is not guaranteed in the long run, even in the most optimistic scenario with a lower dependency ratio or an increase in the total contribution base.

Finally, based on the model presented in this chapter, at least two important directions for future research can be identified. First, it would be interesting to apply Automatic Balance Mechanisms to different countries which have recently carried out parametric reforms and assess whether these reforms have any mathematical basis in the long run, in terms of our optimisation framework. Another direction would be to make the model more representative of the real world by means of a periodic recalibration of the optimal path of the key variables in line with current economic and demographic projections.

4

An alternative pension reform for Spain based on optimisation techniques

H. Godínez Olivares and M. del C. Boado-Penas[1]

The aim of this chapter is to twofold: to design an alternative reform of the Spanish contributory retirement pension system based on optimal strategies to restore liquidity through changes in the key variables of the system (the contribution rate, retirement age and/or indexation of pensions) and at the same time to assess the reformed Spanish system with the focus on its liquidity in the long run. These optimal strategies, which we call Automatic Balancing Mechanisms (ABMs), calculate the optimal path of these variables over time and absorb fluctuations in longevity, fertility rates, life expectancy, salary growth or any other kind of uncertainty faced by the pension system.

Keywords: financial equilibrium, pay-as-you-go, public pensions, optimisation, reform, risk, Spain.

4.1 Introduction

Public pension systems are usually financed on a Pay-As-You-Go (PAYGO) basis where pensions for retirees are paid by the contributions of the working-age population. It is well understood that this method of financing requires a balance between the benefits paid to the pensioners and the contributions made by the active workers.

The decline in fertility rates, the increase in longevity and the current forecasts for the ageing of the baby-boom generation all point to a substantial increase

[1]This chapter is based on the paper 'An alternative pension reform for Spain based on optimisation techniques' by Godínez-Olivares & Boado-Penas (2015). Anales del Instituto de Actuarios Españoles, 3a época, (21) 107-134.

in the age dependency ratio, and this will raise serious concerns for the sustainability of PAYGO pension systems. This is a worldwide problem, and many European countries have already carried out some parametric reforms of their systems in the sense that promises of payment may be reasonably respected. For example in Italy the normal retirement age has increased from 60 to 65 for men and from 55 to 60 for women while the indexation of pension payments has been modified in order to reduce their rate of increase. Germany has also introduced penalties for early retirement.[2] As stated in Holzmann, Orenstein, & Rutkowski (2003), countries such as Sweden, Poland, Italy and Latvia have undertaken some structural reforms by changing the formula to calculate their initial pension from a Defined Benefit (DB) to a Notional Defined-Contribution (NDC) pension system.[3]

In the meantime, some countries, such as, Sweden, Germany and Japan, have decided to set up an Automatic Balancing Mechanism (ABM). This has been defined as a set of predetermined measures established by law to be applied immediately as required according to an indicator that reflects the financial health of the system (Vidal-Meliá et al. (2009, 2010)). Its purpose, through successive application, is to re-establish the financial equilibrium of PAYGO pension systems without the repeated intervention of the legislator. Turner (2009) found that at least 12 countries have the indexation of benefits linked to life expectancy or some other kind of automatic adjustment.[4]

Spain is not an exception to the pension crisis and concerns about the financial sustainability of the pension system have driven reforms enacted in recent

[2] Detailed pension reforms for European countries can be found in Whiteford & Whitehouse (2006).

[3] According to Holzmann et al. (2003), many countries, including Egypt, China and Greece, are seriously considering the introduction of NDCs.

[4] For more information on ABMs see, for instance, Turner (2007, 2009), Vidal-Meliá et al. (2009) and OECD (2012).

years. In 2060, there will be fewer than two people of active-age per retiree compared to more than three currently while life expectancy at the retirement age will have increased from 16.5 to 20.8. There are many debates regarding the effectiveness of the Spanish reforms and lot of researchers, Vidal-Meliá (2014), De la Fuente & Doménech (2013), state that the 2011 reform that introduced parametric changes and the commitment to a sustainability factor by 2027 is not sufficient. According to Vidal-Meliá (2014) the 2011 reform preserves the structure of the pre-reform formula for calculating the initial retirement pension and reproduces most of its main flaws. Responding to the strong pressures from the European Commission, the Spanish Committee of Experts designed and introduced a new indexation on pensions (annual revaluation factor), to be implemented from the beginning of 2014, and sustainability factor[5] (intergenerational equity factor) for the pension system to be implemented in 2019 with the aim of reducing spending on pensions.

To our knowledge, the Spanish pension system reform does not have any mathematical basis other than trying to adapt the system to the demographic scenario, without any proof, based on the general view that indicators, such as the old-age dependency ratio, expenditure on pensions or life expectancy, will rise over time. With this in mind, the aim of this chapter is to twofold. First, it seeks to design an alternative reform of the Spanish pension system, different to the ones proposed in Act 27/2011 and Act 23/2013, based on optimal strategies, that not only adapts the system to economic and demographic changes with a mathematical basis but also restores liquidity to the Spanish pension system through changes in the contribution rate, retirement age and/or indexation of pensions. Secondly, it assesses the reformed Spanish system (Act

[5]For more information on the new indexation on pensions and sustainability factor see Act 23/2013.

27/2011) with the focus on its financial equilibrium in the long run. These optimal strategies, which we call Automatic Balancing Mechanisms[6], calculate the optimal path of these variables over time and absorb fluctuations in longevity, fertility rates, life expectancy, salary growth or any other kind of uncertainty faced by the pension system.

The chapter is organised as follows. The next section presents some data on the Spanish pension system. The third section describes the main notation, definitions and the numerical optimisation method. The fourth section shows the practical application of this method to the Spanish pension system and suggests how an ABM should be designed, for both symmetric and asymmetric cases, to restore liquidity to the system. This section also provides a sensitivity analysis of the results. The fifth section focuses on the long run and provides an analysis over a 75-year horizon for the Spanish pension system. The final section concludes. Two appendices present the expressions to calculate the income from contributions and the expenditure on pensions, and; provide a detailed description of the formulae to calculate the monthly pension benefit before and after the Spanish pension reform.

4.2 Brief description of the Spanish pension system

This chapter is focused on the reform of the contributory public social insurance programs run by the State and in particular to the old-age (also called retirement) contingency.[7] The retirement contingency is financed by contributions from employees and employers.

[6]See Appendix D for characteristics of some countries with ABMs in place.

[7]These programs award benefits to compensate for income no longer earned due to sickness, accident, family care responsibilities, unemployment, disability, old age or death.

The key features of the reform are: a) raising the statutory retirement age from 65 to 67 over the period 2013-2027, b) lengthening the period used to calculate the base pension[8] from 15 to 25 years over the period 2013-2022, c) increasing the number of contributory years for a full pension from 35 to 37[9] and d) changing the adjustment factors for anticipating the retirement age and the incentives for remaining in the labour market.[10]

At the end of 2013, a reform was passed that establishes a new revaluation index and regulates the sustainability factor that complements the parametric changes. Under this reform, from 2014, pensions will be adjusted according to the performance of variables, such as revenue, expenditure and the number of pensions. This will replace the former system, in force since 1997, which linked pensions to the rate of change of the CPI. Moreover, from 2019, starting pensions will be automatically linked to the increase in life expectancy (the sustainability factor[11]).

Table 4.1 presents the main data on the retirement contingency during the last six years. Consequences of the reform are difficult to value at this first stage given that the reform will be applied gradually until 2027. It can be seen that the actual number of contributors has been decreasing while the number of pensioners increases. Thus the number of contributors funding a retirement pension falls from 3.87 to 2.98 over the last six years. At the same time the monthly benefit and the life expectancy at retirement age increase and this worsens the sustainability of the pension system.

[8] Also called regulating base. For more details of the calculation see Appendix G.

[9] Other features regarding the early retirement and voluntary work extension can be consulted in Act27/2011.

[10] For particularities or further explanation regarding the reform, please see the Appendix G and Act27/2011.

[11] Examination of the sustainability factor (see Act 23/2013) is beyond the scope of this chapter whose main aim is to shed some light on how variables such as the contribution

Table 4.1: Some data for the Spanish Pension System (old-age contingency).

Data \Years	2008	2010	2012	2014
Number of contributors (in thousands)	19,129.81	17,667.47	16,845.86	16,553.79
Number of retirement pensions (in thousands)	4,936.84	5,123.88	5,330.20	5,558.96
Contributors/retirement pension ratio	3.87	3.45	3.16	2.98
Life expectancy at age 65	20	20.6	20.5	20.98
Monthly Average old-age pension	810.48	883.27	946.32	999.77

According to Act27/2011, the aim of the reform is to meet the enormous challenge posed by an ageing population and to partially correct the imbalance between contributions made and pensions received. This chapter assesses whether the main changes in the reform Act27/2011 are justified from a mathematical point of view with the aim to restore the liquidity into the system in the long run. The next section designs automatic balancing mechanisms to provide an alternative reform and assess the current reform of the Spanish pension system on a theoretical basis.

4.3 Liquidity ABMs

In this section two ABMs are built to analyse and restore the liquidity of the Spanish contributory retirement pension system, which is financed by the PAYGO method. These ABMs are based on the work of Godínez-Olivares, Boado-Penas, & Pantelous (2016), in which the objective is to minimise

rate, retirement age and indexation on pensions should be modified to re-establish liquidity in the system in the long run.

changes over time in the main variables, such as the contribution rate, normal retirement age and indexation of pensions. This mechanism, which uses an intertemporal optimisation model, identifies and applies an optimal path for the contribution rate, normal retirement age and indexation of pensions to a PAYGO system.

The first ABM does not include the accumulation or deficit of the buffer fund, while the second includes the accumulation or deficit of the fund over the time. The aim of both ABMs is to absorb fluctuations in longevity, fertility rates, life expectancy, salary growth or any other random events in a pension system.

The main purposes of the ABM, through successive applications, are to re-establish the financial equilibrium of PAYGO adapting the system to changes in socio-economic and demographic conditions; to create a credible institutional framework to increase the likelihood that promises of pension payments will be respected; and to minimise the use of the pension system as an electoral tool.[12]

The functional objective is a logarithmic function which is set to minimise the percentage of the changes over time of the key variables. The logarithmic function is widely used in statistical forecasting (Nau (2015)). Its great advantage is that small changes in the natural log of a variable are directly interpretable as percentage changes, to a very close approximation, and because changes in the natural logarithm are (almost) equal to percentage changes in the original series, it follows that the slope of a trend line fitted to logged data is equal to the average percentage growth in the original series. Godínez-Olivares, Boado-Penas, & Pantelous (2016) introduced the idea of minimising the rate change

[12]Nevertheless the system still is subject to a kind of political risk in the sense that the planning horizon of politicians is much shorter than the horizon of the pension system.

of the decision variables involved in the projections of the contribution rate, normal retirement age and indexation of pensions.

The optimisation problem is defined by the objective function:

$$f(d_n,n) = min_{c_n,x_n^{(r)},\lambda_n}\Big[\sum_{n=1}^{N}\Big[\theta_1 log\Big(\frac{c_{n+1}}{c_n}\Big)+ \\ \epsilon_1\theta_2 log\Big(\frac{x_{n+1}^{(r)}}{x_n^{(r)}}\Big)+\epsilon_2\theta_3 log\Big(\frac{\lambda_{n+1}}{\lambda_n}\Big)\Big]\Big] \quad (4.1)$$

where c_n is the contribution rate; $x_n^{(r)}$, is the age of normal retirement; and, λ_n, is the indexation of pensions. θ_i , $i = 1, 2, 3$ are the weights which measure the impact that occurs when the key variables are projected and $\theta_1 + \theta_2 + \theta_3 = 1$. These weights reflect the interest of the policy maker, since, a value of 0 in one parameter means that no change is allowed in this particular variable.

A metric problem is present due to the nature of the logarithmic function, as c_n, $x_n^{(r)}$ and λ_n have different units and constraints. To deal with this problem, ϵ_i, $i = 1, 2$ have been introduced in the growth rates of the age of normal retirement and in the indexation of pensions;

$$\epsilon_1 = \frac{log(c_\Delta)}{log(x_\Delta^{(r)})};$$
$$\epsilon_2 = \frac{log(c_\Delta)}{log(\lambda_{1\Delta})}\mathbb{1}_{\{\frac{\lambda_{n+1}}{\lambda_n}\geq 0\}} + \frac{log(c_\Delta)}{log(\lambda_{2\Delta})}\mathbb{1}_{\{\frac{\lambda_{n+1}}{\lambda_n}\leq 0\}},$$

where c_{Δ}, $x_{\Delta}^{(r)}$ and $\lambda_{1\Delta}$ are the maximum increase that are allowed (and $\lambda_{1\Delta}$ for λ_n) per year in the contribution rate c_n, the age of normal retirement $x_n^{(r)}$ and the indexation of pensions λ_n; and $\mathbb{1}_{\{j \in \omega\}}$ is the indicator function.[13]

The system's liquidity is measured in two ways; first, without the inclusion of the buffer fund, that is, only the income from contributions and the expenditure of pensions are taking into account; and secondly, the accumulation of the buffer fund over time and the actuarial projections of the income from contributions and the pensions to be paid are included in the liquidity of the system. The following subsections describe the constraints set in our optimisation model to build the ABMs.

4.3.1 Liquidity ABM without buffer fund

In order built a realistic problem, the following constraints are imposed on the Liquidity ABM without buffer fund; for the contribution rate, c_n, the age of normal retirement, $x_n^{(r)}$, and the indexation of pensions, λ_n, upper ($c_{max}, x_{max}^{(r)}, \lambda_{max} \in \Re$) and lower ($c_{min}, x_{min}^{(r)}, \lambda_{min} \in \Re$) bounds are imposed. These bounds are set in order to avoid possible unrealistic increases in the key variables of the pension system. Another constraint to be considered is the smooth constraint. This constraint is necessary to prevent jumps in the contribution rate, age of retirement and indexation of pensions. Mathematically, the smooth constraint is set as:

$$c_{1\Delta} \leq \frac{c_{n+1}}{c_n} \leq c_{2\Delta};\ x_{1\Delta}^{(r)} \leq \frac{x_{n+1}^{(r)}}{x_n^{(r)}} \leq x_{2\Delta}^{(r)};\ \lambda_{1\Delta} \leq \frac{\lambda_{n+1}}{\lambda_n} \leq \lambda_{2\Delta};$$

where $c_{1\Delta}, c_{2\Delta}, x_{1\Delta}^{(r)}, x_{2\Delta}^{(r)}, \lambda_{1\Delta}, \lambda_{2\Delta} \in \Re$.

[13] The indicator function takes the value of 1 if the condition $j \in \omega$ is satisfied and 0 in all other cases.

Finally, the liquidity restriction is imposed as: $c_nW_n(n, g, x_n^{(r)}) - B_n(n, g, x_n^{(r)}, \lambda_n) \geq 0$. W_n is the total contribution base[14] paid at n an B_n is the total expenditure on pensions at n that depend on the growth of salaries, g, the retirement age $x_n^{(r)}$, and the indexation of pensions λ_n. The liquidity restriction helps to impose maximum levels of deficit or surplus into the system, and setting this restriction greater or equal to zero means that no deficit is allowed in the pension system.

The optimisation function to minimise is as follows:

$$f(d_n, n) = min_{c_n, x_n^{(r)}, \lambda_n} \Big[\sum_{n=1}^{N} \Big[\theta_1 log\Big(\frac{c_{n+1}}{c_n}\Big) + \epsilon_1\theta_2 log\Big(\frac{x_{n+1}^{(r)}}{x_n^{(r)}}\Big) + \epsilon_2\theta_3 log\Big(\frac{\lambda_{n+1}}{\lambda_n}\Big)\Big]\Big] \tag{4.2}$$

$$s.t. = \begin{cases} c_nW_n(n, g, x_n^{(r)}) - B_n(n, g, x_n^{(r)}, \lambda_n) \geq 0; \\ c_{min} \leq c_n \leq c_{max}; x_{min}^{(r)} \leq x_n^{(r)} \leq x_{max}^{(r)}; \\ \lambda_{min} \leq \lambda_n \leq \lambda_{max}; \\ c_{1\Delta} \leq \dfrac{c_{n+1}}{c_n} \leq c_{2\Delta}; x_{1\Delta}^{(r)} \leq \dfrac{x_{n+1}^{(r)}}{x_n^{(r)}} \leq x_{2\Delta}^{(r)}; \\ \lambda_{1\Delta} \leq \dfrac{\lambda_{n+1}}{\lambda_n} \leq \lambda_{2\Delta}; \\ \theta_1 + \theta_2 + \theta_3 = 1; \\ \epsilon_1 = \dfrac{log(c_\Delta)}{log(x_\Delta^{(r)})}; \\ \epsilon_2 = \dfrac{log(c_\Delta)}{log(\lambda_{1\Delta})} \mathbb{1}_{\{\frac{\lambda_{n+1}}{\lambda_n} \geq 0\}} + \dfrac{log(c_\Delta)}{log(\lambda_{2\Delta})} \mathbb{1}_{\{\frac{\lambda_{n+1}}{\lambda_n} \leq 0\}} \end{cases}$$

[14]See Appendix F for details of calculations for the total contribution base and expenditure on pensions.

4.3.2 Liquidity ABM with buffer fund

In this subsection a contingency fund is introduced into the minimisation function. It is important to highlight that four papers, Godínez-Olivares, Boado-Penas, & Pantelous (2016), Haberman & Zimbidis (2002b), Pantelous & Zimbidis (2008), and Gannon et al. (2013), propose parametric reforms in the PAYGO pension systems introducing the concept of a liquidity or contingency fund in order to absorb unexpected events that might affect their liquidity. Gannon, Legros and Touzé (2013) define this buffer fund as the intertemporal budget balance of the pension system that brings promised future expenditures in line with expected future revenues. For example, the interest generated by the buffer fund in Spain covered the shortfall in contributions during 2010 (Vidal-Meliá (2014)).

The liquidity ABM with the buffer fund calculates the optimal path for the contribution rate, age of normal retirement and indexation of pensions building an optimisation problem that is solved by numerical analysis (Godínez-Olivares, Boado-Penas, & Pantelous (2016)) including the dynamics of the fund, F_n, that can be expressed as:

$$F_n = (1 + J_n)F_{n-1} + c_n W_n(n, g, x_n^{(r)}) - B_n(n, g, x_n^{(r)}, \lambda_n), \tag{4.3}$$

where J_n is the growth risk-free rate of the fund during year n; $c_n, x_n^{(r)}, \lambda_n$ are the projected variables during year n; W_n is the total contribution base paid at n; B_n is the total expenditure on pensions at n. The derived contingency fund is able to absorb fluctuations in longevity, fertility rates, salary growth or any other events.

In this ABM, the liquidity restriction is changed to, $F_n \geq 0$, that is, $(1 + J_n)F_{n-1} + c_n W_n(n, g, x_n^{(r)}) \geq B_n(n, g, x_n^{(r)}, \lambda_n)$. The other restrictions set for the ABM without buffer fund remains the same.

The optimisation problem to minimise in the ABM including the buffer fund is defined by:

$$f(d_n, n) = min_{c_n, x_n^{(r)}, \lambda_n} \Big[\sum_{n=1}^{N} \Big[\theta_1 log\Big(\frac{c_{n+1}}{c_n}\Big) + \epsilon_1 \theta_2 log\Big(\frac{x_{n+1}^{(r)}}{x_n^{(r)}}\Big) + \epsilon_2 \theta_3 log\Big(\frac{\lambda_{n+1}}{\lambda_n}\Big)\Big]\Big] \quad (4.4)$$

$$s.t. = \begin{cases} c_{min} \leq c_n \leq c_{max}; x_{min}^{(r)} \leq x_n^{(r)} \leq x_{max}^{(r)}; \\ \lambda_{min} \leq \lambda_n \leq \lambda_{max}; \\ c_{1\Delta} \leq \dfrac{c_{n+1}}{c_n} \leq c_{2\Delta}; x_{1\Delta}^{(r)} \leq \dfrac{x_{n+1}^{(r)}}{x_n^{(r)}} \leq x_{2\Delta}^{(r)}; \\ \lambda_{1\Delta} \leq \dfrac{\lambda_{n+1}}{\lambda_n} \leq \lambda_{2\Delta}; \\ \theta_1 + \theta_2 + \theta_3 = 1; \\ \epsilon_1 = \dfrac{log(c_\Delta)}{log(x_\Delta^{(r)})}; \\ \epsilon_2 = \dfrac{log(c_\Delta)}{log(\lambda_{1\Delta})} \mathbb{1}_{\{\frac{\lambda_{n+1}}{\lambda_n} \geq 0\}} + \dfrac{log(c_\Delta)}{log(\lambda_{2\Delta})} \mathbb{1}_{\{\frac{\lambda_{n+1}}{\lambda_n} \leq 0\}}; \\ F_N \geq 0; \end{cases}$$

The next subsections describe two indicators to measure the liquidity of the system according to the inclusion or exclusion of the buffer fund.

4.3.3 Liquidity indicators

Two indicators emerge from our model to measure the liquidity of the system. The first one, which does not include the buffer fund, is defined by:

$$I_{NoBF} = \frac{c_n W_n(n, g, x_n^{(r)})}{B_n(n, g, x_n^{(r)}, \lambda_n)} \tag{4.5}$$

The second includes the buffer fund as follows:

$$I_{BF} = \frac{(1 + J_n)F_{n-1} + c_n W_n(n, g, x_n^{(r)})}{B_n(n, g, x_n^{(r)}, \lambda_n)} \tag{4.6}$$

Both indicators measure the liquidity of the system, however, in the first one, only the income from contributions and the expenditure of pensions are included, whereas, in the second one, the accumulated buffer fund is added to the indicator.

4.3.4 Symmetric and asymmetric designs

Our ABM also allows us to design symmetric and asymmetric cases under both ABMs. Palmer (2013) states that under a symmetric ABM, any surplus that might arise would be automatically distributed. In the absence of a symmetric ABM, an undistributed surplus will be maintained. Under the symmetric design, the ABMs determine whether the contribution rate, the age of normal retirement and (indexation of pensions) are reduced (increased) when the system has a surplus or increased (decreased) in periods of deficit (Godínez-Olivares, Boado-Penas, & Pantelous (2016)). Analytically, for the asymmetric case the change in the contribution rate and age of normal retirement are enforced to be strictly greater or equal to zero (strictly lower or equal to zero for the indexation of pensions). In other words, under the symmetric case, the change in the variables could be positive or negative.

The following section shows the application of the ABMs to the Spanish pension system.

4.4 ABMs for the Spanish pension system

This section presents different designs of ABMs for the Spanish pension system. First, the main data and assumptions are presented, secondly the results are discussed and finally this section provides a sensitivity analysis of the results.

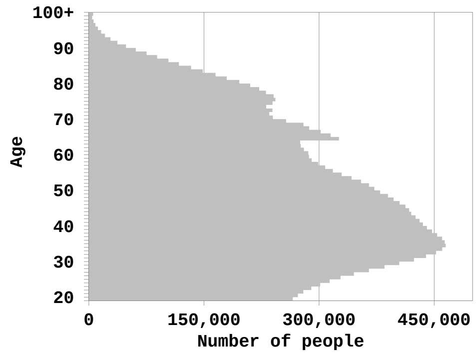
4.4.1 Main data and assumptions

Demographic data

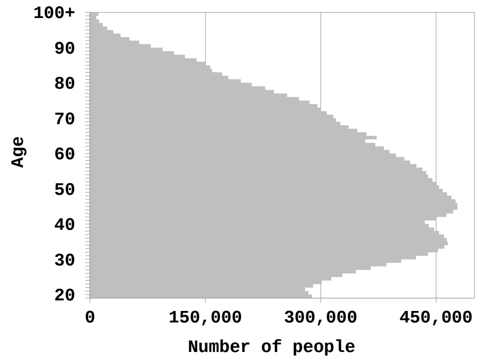
According to the National Institute of Statistics (Instituto Nacional de Estadística (INE)[15]) population growth will progressively decrease over the next few decades. If current demographic trends continue, Spain will lose a more than million people (2.2%) in the next 15 years and more than 5.6 million (12.1%) in the next 50 years, continuing the negative trend that began in 2012, hence the Spanish population will be reduced to 45.8 million in 2024 and to 40.9 million in 2064. The reduction is mainly due to the progressive increase in deaths and low fertility rates. Figure 4.1 shows the population structure for years 2014, 2024 and 2034. It can be seen that the largest group in 2014 is aged 35-39 while the most representative group in 2014 will be the one aged 45-49.

[15]Instituto Nacional de Estadística, from http://www.ine.es, accessed 1 June 2015.

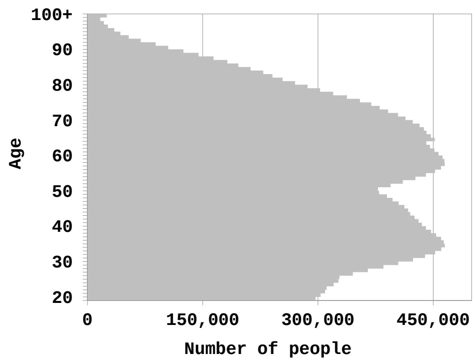
Figure 4.1: Spanish population structure.



(a) Spanish population 2014



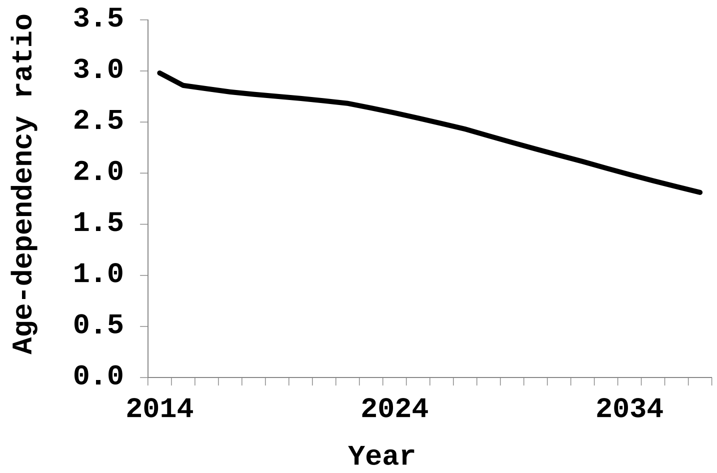
(b) Spanish population 2024



(c) Spanish population 2034

The increase in life expectancy also changes the structure of the population. In fact, life expectancy at birth will be around 84.0 years for men and 88.7 for women in 2030, 4 and 3 years more respectively from current values. Despite the decrease in the number of population and an increase in life expectancy, the number of deaths would continue to grow as a result of population ageing. In the period 2012-2032 more than six million deaths are expected, around 7% higher than those observed in the previous 15 years (1999-2012), which results in a decrease in the value of the old-age dependency ratio (Figure 4.2). Specifically, in 20 years there will be more than 11 million people over 64 years in Spain, that is an increase of 34%.

Figure 4.2: Projected old-age dependency ratio.



Other main data and assumptions

To set the minimisation problem, some assumptions are needed:

- The entry age into the labour market, x_e, is set in 20[16].
- At the beginning of the study, the decision variable contribution rate is assumed to be 18%, that is the weight that represents the expenditure on pensions over the total contributory expenditure.
- At the beginning of the study, the decision variable retirement age is assumed to be 65, that is the legal retirement age before the reform.
- At the beginning of the study, the decision variable indexation on pensions is assumed to be 1%. Although pensions are usually linked to the change of CPI, we want to see how this variable reacts with the economic and demographic scenario.

[16]The entry age into the system does not have real implications in our model since we match the number of contributors and pensioners through the dependency ratio.

- The upper and lower bounds of the contribution rate, age of normal retirement and indexation of pensions are 23%, 69 and 2% and 18%, 65 and -2% respectively.[17]

- In order to have a smooth transition, it is also assumed that the change over time[18] in the contribution rate varies between 0.3% and 0.5%, the age of normal retirement between 1.5 and 3 months and the indexation of pensions between -2% and 2%.

- The buffer fund is assumed to increase at an annual rate[19] of 3% while the annual salary growth[20] is 2%.

- The salary structure is assumed to be the mean annual earnings in Spain of each age group[21].

- The average net replacement rate considered in our analysis is kept[22] at 75%. This value is higher than the replacement rate of 68.8% in the OECD-34. As a result of setting the replacement rate at 75%, the ratio of the average pension to the average salary in 2014 becomes 60% which is in line with real data[23].

[17] Negative indexation would be a very unpopular and an unlikely measure to be taken in Spain. However some PAYGO pension systems like Sweden has already adopted a negative indexation for some years to make its pension system solvent. For this reason, we would keep a more general setting in our model allowing for negative indexation with a minimum value of -2%.

[18] These values are in line with the most important reforms in the 34 OECD member countries between January 2009 and September 2013 (see OECD (2014)).

[19] Historical average of the Euribor in the last 15 years.

[20] We assume the expected growth of the GDP in the next 20 years according to European Commission (2012a).

[21] Eurostat Database, from http://ec.europa.eu/eurostat accessed 1 June 2015.

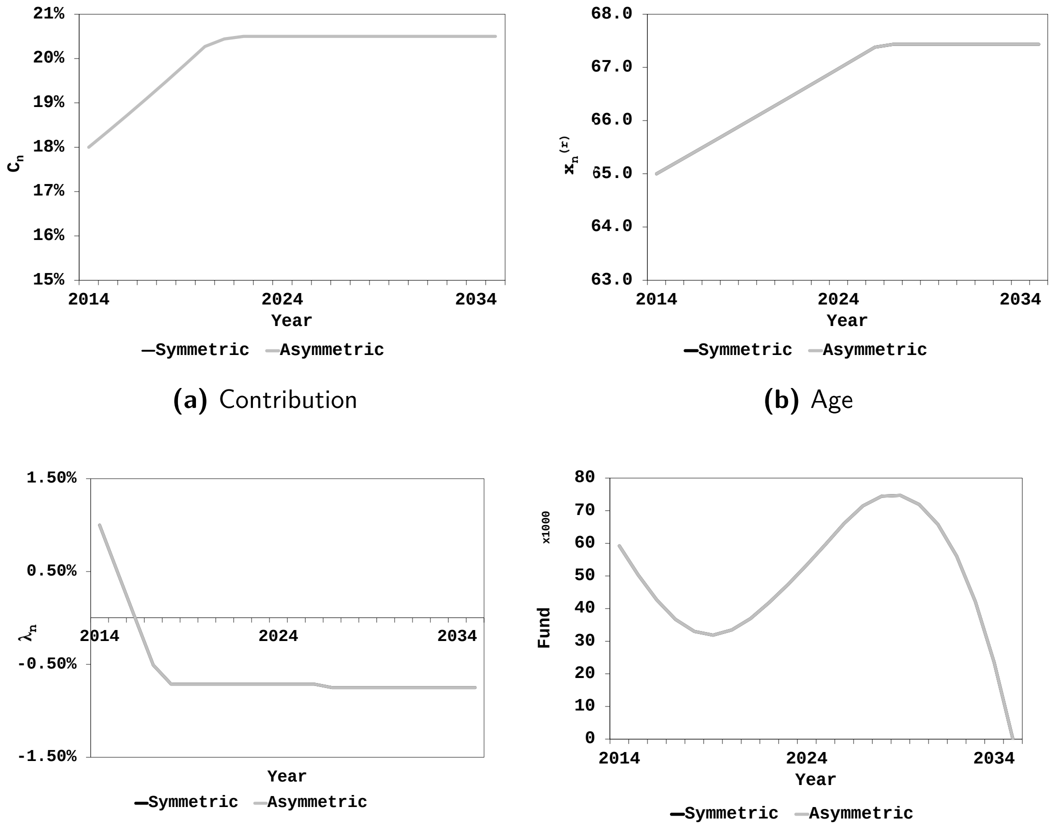
[22] Sensitivity analysis is provided regarding this variable.

[23] According to the OECD (2011) the Spanish pension reform on a full career worker reduces the replacement rate to 73.9% (on the OECD's standard assumptions of 2.5% price inflation and 2% real earnings growth).

4.4.2 Results

Figure 4.3 presents the optimal path for the contribution rate, age of normal retirement and the indexation of pensions when the buffer fund is included. This ABM, in contrast with the ABM without a buffer fund, accumulates the surpluses of the system. In both symmetric and asymmetric scenarios the optimal path of the decision variables is the same and the effect of the inclusion of the fund is seen, particularly, in the levels of the contribution rate, Figure 4.3a, and age of normal retirement, Figure 4.3b. The contribution rate stabilises at 20.5% and the age of normal retirement at 67.4. The indexation of pensions, Figure 4.3c, shows minor changes; with the inclusion of the buffer fund it stabilises at -0.75%. The dynamics of the fund are shown in Figure 4.3d. The accumulation period decreases over the last 5 years of the analysis, leaving the fund in levels near to zero as imposed in the restrictions.

Figure 4.3: Liquidity ABM with buffer fund (Black line-Symmetric. Grey line -Asymmetric.



(a) Contribution

(b) Age

(c) Indexation of pensions

(d) Fund

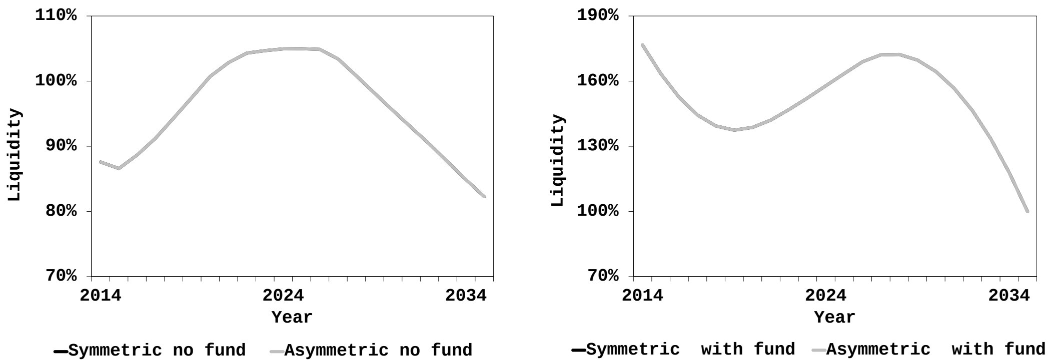
To illustrate how the inclusion of the buffer fund contributes to the liquidity of the system, two liquidity indicators for the ABM with buffer fund[24] are shown in Figure 4.4[25]. The first one, Figure 4.4a, represent the ratio (equation (4.5)) between the income from contributions and the expenditure of pensions, it is seen that the indicator, from year 2020 -2028, is below 1, so, the income from contribution does not cover the amount of pensions to be paid during 2014-2020 and 2029-2034. However, when the fund is included in the ratio

[24]For the ABM without buffer fund is not possible to calculate these indicators, because the optimisation problem does not take into account the accumulation of the surpluses.

[25]As can be seen, the optimal path is exactly the same for both scenarios.

(equation (4.6)), Figure 4.4b, the ratio is always above 1. So, the inclusion of the buffer fund is contributing to the liquidity of the system.[26]

Figure 4.4: Liquidity Indicators for the ABM with buffer fund (Black line-Symmetric. Grey line- Asymmetric).



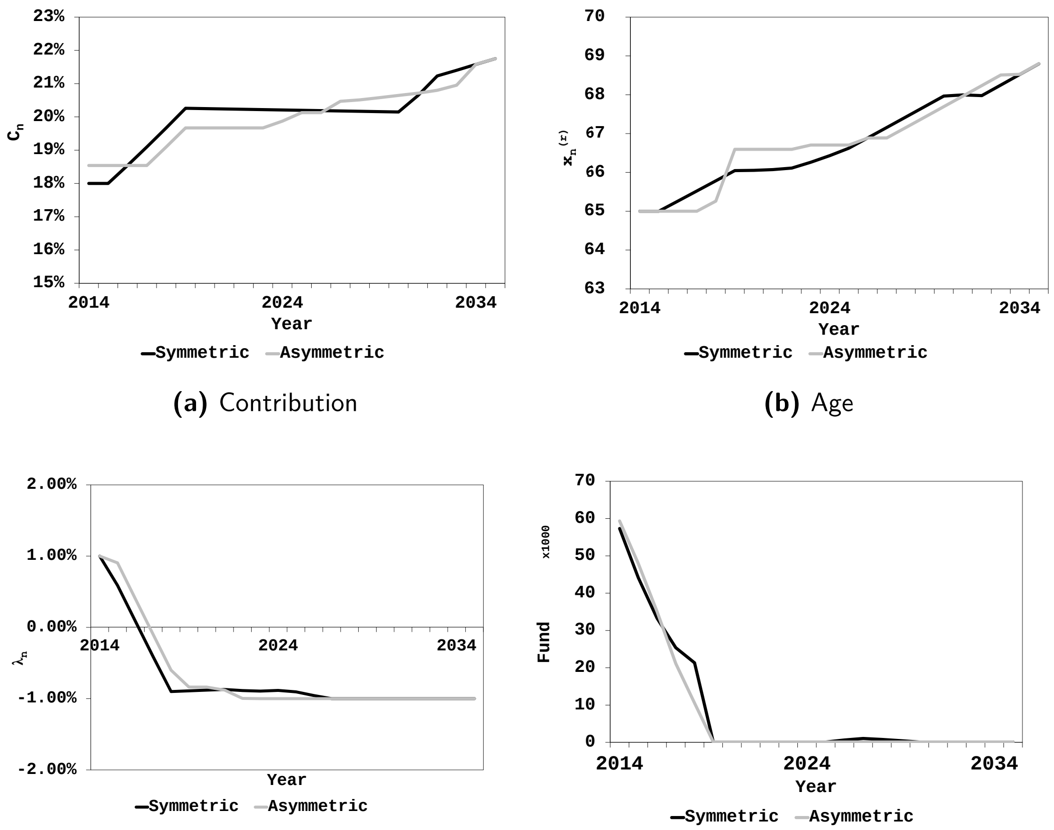
(a) Liquidity Indicator without buffer fund (Contributions/Benefits)

(b) Liquidity Indicator with buffer fund (Fund/Benefits)

If we apply the liquidity ABM without including the buffer fund, the projections of the contribution rate, age of normal retirement and indexation of pensions for Spain can be seen in Figure 4.5. In both scenarios, the symmetric and asymmetric, the contribution rate, Figure 4.5a, increases from 18% to 21.75%. The age of normal retirement, Figure 4.5b, increases to 68.8; 1.25% and 1.4 years more respectively compared to the ABM with the buffer fund. The indexation of pensions, Figure 4.5c, shows a small decrease compared to the previous ABM, in both the symmetric and asymmetric design, down to -1% by 2034. Figure 4.5d, shows that the liquidity indicator is always greater than 1, so, the expenditure of pensions is always covered by the income from contributions, as set in the liquidity restriction.

[26]Given that the optimal path for the decision variables is the same, results under symmetric and asymmetric scenarios are identical.

Figure 4.5: Liquidity ABM without buffer fund (Black line-Symmetric. Grey line -Asymmetric).



(a) Contribution

(b) Age

(c) Indexation of pensions

(d) Fund

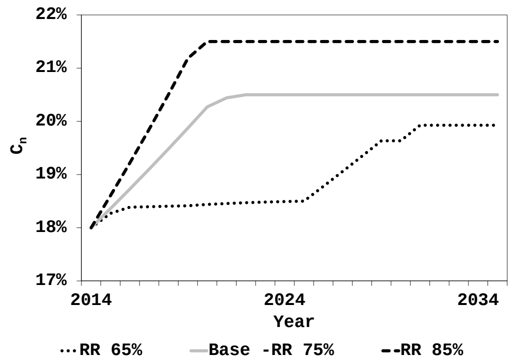
4.4.3 Sensitivity analysis

This section presents, for the ABM with buffer fund under the asymmetric scenario, the results of increasing or decreasing the replacement rate (RR); increasing or decreasing the old-age dependency ratio; and, the results when only one variable is projected and the other two are fixed in the current values.

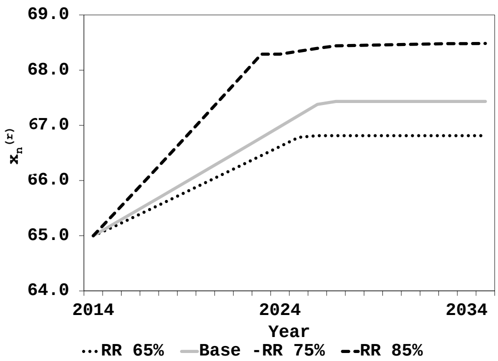
In Figure 4.6, two levels of replacement rates are analysed. First, we suppose a replacement rate of 65% (black points line). Under this scenario, the contribution rate increases only by 1.9% to stabilise at 19.% (0.51% less than in the base scenario of a replacement rate of 75%); the age of normal retirement increases to 66.8 (0.6 years less than in the base scenario) and the indexation of pensions decreases to -0.55%, that is 0.2% less compared with a replacement rate of 75%. When the replacement rate is set 85%, the contribution rate stabilises at 21.50%, the age of retirement at 68.5 and the indexation of pensions at -1.0% (that is, 1% and 1.1 years more, and 0.25% less indexation of pensions comparing with the scenario under the replacement rate of 75%).

Something interesting in this analysis is the dynamic of the buffer fund. For the replacement rate of 85%, the buffer fund has surpluses at the end of the analysis; however, as we are imposing an asymmetric scenario, the surpluses cannot be redistributed by the system.

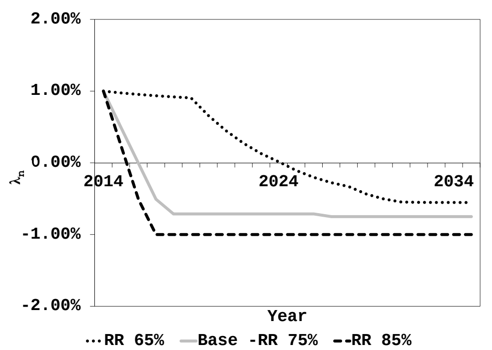
Figure 4.6: Liquidity ABM with buffer fund (Black dashed line - RR 85%. Grey line- RR 75% -base scenario-. Black points line - RR 65%).



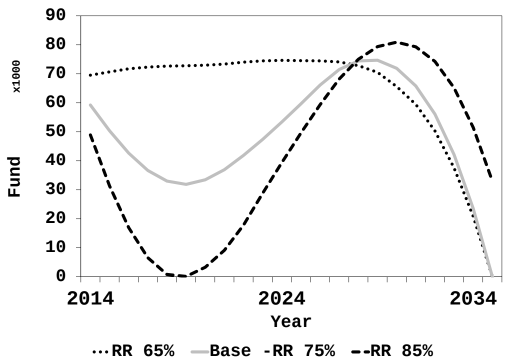
(a) Contribution



(b) Age



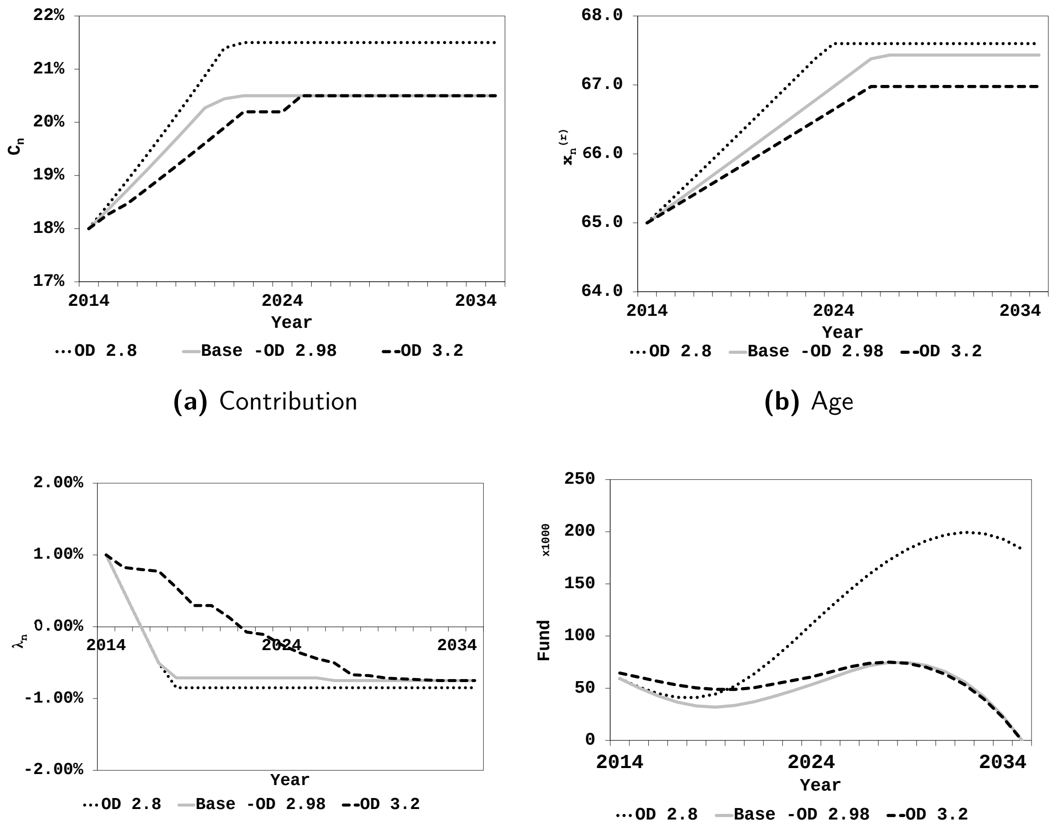
(c) Indexation of pensions



(d) Fund

Figure 4.7 shows the results when different levels of the old-age dependency ratio (OD) are taken into account. First, a decrease of the ratio from 2.98 to 2.8 is analysed, then an increase to 3.2 is analysed. In the first case, the contribution rate and indexation of pension stabilises at the same level as in the base scenario, but the age of retirement stabilises at 67 instead at 67.5. That is, basically, because the age of retirement affect both, pensioners and contributors. In the second case, when the old-dependency ratio starts at 3.2, the contribution rate increase to 21.5%, the age to 67.6 and the indexation of pensions -0.85%. Again, the asymmetric design is leaving the fund with surpluses at the end of the study when the old-dependency ratio is 2.8 in the first year of analysis.

Figure 4.7: Liquidity ABM with buffer fund (Black dashed line- OD 2.8. Grey line- OD 2.98 -base scenario-. Black points line - OD 3.2).



(a) Contribution

(b) Age

(c) Indexation of pensions

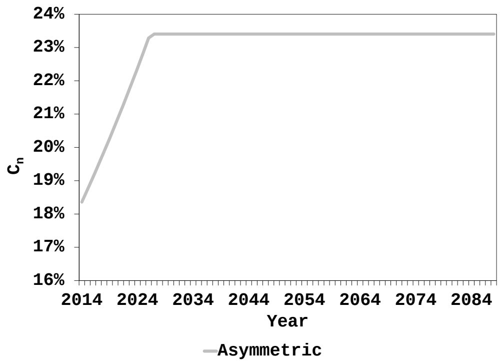
(d) Fund

The ABMs proposed have the ability to modify only one variable, instead of modifying the three of them simultaneously. Therefore, we analyse the case when only one variable is studied. If we set the age of retirement and indexation of pensions at 65 and 1% respectively, the contribution rate would need to increase to 30%. If the age of retirement is the only player, having set the contribution rate and indexation of pensions at values 18% and 1% respectively, it would need to increase to 72.3. On the contrary, if the indexation of pensions is the only decision variable, the pension benefits would need to decrease by -10.3% every year.

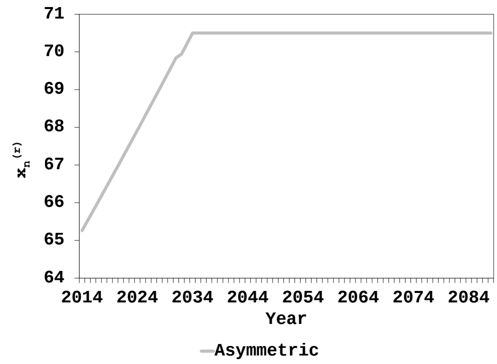
4.5 Long run analysis of the Spanish pension system

In contrast, with the previous section, the model calculates the optimal path of the contribution rate, age of normal retirement, and indexation of pensions over a 75 year time horizon. Figure 4.8a, presents an increase of 2.9% in the contribution rate comparing with the 20-year time horizon and the value stabilises at 23.4%. The age of retirement, Figure 4.8b, stabilises at 70.4, that is three years more than for the 20 years time horizon. The indexation of pensions, Figure 4.8c, now decreases to -1.5%, that is, double the number compared to the ABM over the 20 year horizon. Figure 4.8d, shows the dynamics of the buffer fund, which shows an accumulation period of almost 40 years after which it decreases reaching the value of zero at the end of the 75 years.

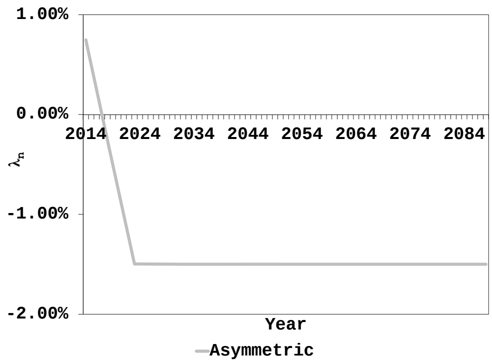
Figure 4.8: Liquidity ABM with buffer fund (Grey line Asymmetric) in the long run.



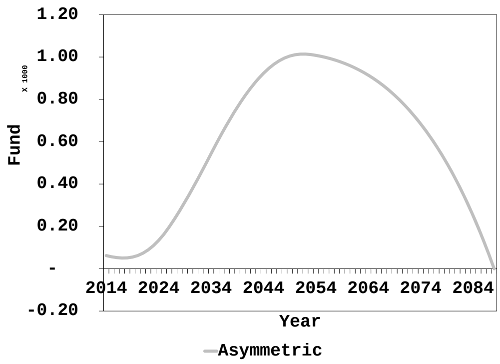
(a) Contribution



(b) Age



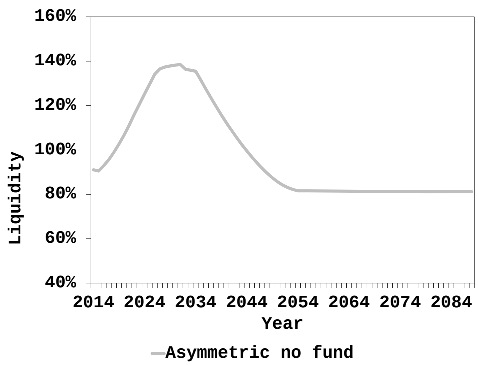
(c) Indexation of pensions



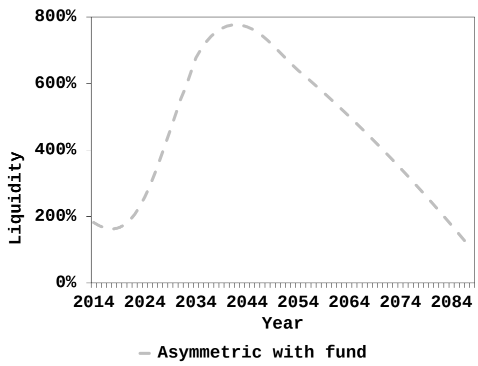
(d) Fund

Figure 4.9 shows the liquidity indicators defined in equation (4.5) (Figure 4.9a), and equation (4.6) (Figure 4.9b). The liquidity indicator without the buffer fund, shows a deficit in the system from 2045 onwards, in contrast to the indicator with the buffer fund, which does not present any deficit in the 75 years.

Figure 4.9: Liquidity Indicators for the ABM with buffer fund (Grey line- Asymmetric) in the long run.



(a) Liquidity Indicator without buffer fund (Contributions/Benefits)



(b) Liquidity Indicator with buffer fund (Fund/Benefits)

Finally, if only one variable is projected at a time, the contribution rate needs to increase to 47% when the age of retirement and indexation of pensions are fixed to 65 and 1% respectively; the age of retirement needs to increase from 65 to 79.9 if the contribution rate and indexation of pensions are fixed to 18% and 1% respectively; and the indexation of pension needs to decrease to -9.2% when the contribution rate and age of normal retirement are fixed to 18% and 65.

4.6 Conclusions

Restoring the liquidity of a PAYGO pension system is on the agenda for most governments and Spain is no exception. Even after the 2011 reform, the pension system's sustainability in the long run has been widely questioned. This chapter contributes to the debate on the policies needed to strengthen the pension system and proposes alternative reforms based on optimisation techniques to re-establish the liquidity of the Spanish PAYGO pension system without the repeated intervention of the legislator. The aim of these alternative reforms, also called Automatic Balancing Mechanisms (ABMs), is to guide the system back onto the road to long-term liquidity and at the same time to automate the measures to be taken, isolating them from the political arena, avoiding any delay and lack of time perspective.

The ABMs can be designed in different ways considering or not in our analysis the amount of the accumulated buffer fund. We show that for the Spanish pension system the contribution rate stabilises at 20.5% and the age of normal retirement and indexation of pensions at 67.5 and -0.75% respectively after 20 years. As expected, if the amount of the buffer fund is not included in our analysis, the values for the contribution rate, age of normal retirement and indexation of pensions are higher (lower for the indexation of pensions) and stabilise at 21.75%, 68.8 and -1% respectively.

The ABMs proposed have the ability to modify only one variable, instead of modifying the three of them simultaneously. Therefore, if the contribution rate is the only player into the system, it would need to increase to 30%, whereas if the age of retirement is the only variable it would need to increase to 72.3, questioning then the measures taken by Act27/2011. On the contrary, if the indexation of pensions is set as the only decision variable, the pension benefits must decrease by -10.3% every year.

Following some of the ideas of the current Swedish ABM we design a symmetric ABM that determines whether the contribution rate, the age of normal retirement (and indexation of pensions) are reduced (increased) when the system has a surplus or increased (decreased) in periods of deficit. Although there are some differences in the values of the variables under the symmetric and asymmetric design during the first years, at the end of the analysis both ABMs reach similar values for the contribution rate, retirement age and indexation of pensions.

Some countries like Japan or the USA are concerned about the sustainability of their system in a 95 or 75-time horizon. The results for the Spanish pension system if a 75-year time horizon is considered are quite alarming. If the three decision variables are modified simultaneously, the contribution rate stabilises at 23.4, whereas the retirement age needs to increase to 70.4 and the indexation on pension decreases -1.5% every year.

This chapter questions the adequacy of the current reforms and considers that the sustainability of the Spanish pension system is not guaranteed in a long run, unless the structure of population changes dramatically with an increase in the number of contributors and a decrease in the number of pensioners.

5
Conclusions

In social security reforms there is no only one way to proceed and each country has developed its own model. However, it is clear that any solution chosen should meet at least two requirements: financial and social viability (Sales-Sarrapy et al. (1998)). In this thesis, the survivor dividend of the Notional Defined Contributions is analysed and Automatic Balancing Mechanism are developed.

In the first chapter we show that the survivor dividend can cover an unexpected increase in longevity, reflected either by a decrease in mortality rates, an increase in survival probabilities or an increase in life expectancy at the retirement age. We found the mathematical formulas for the maximal mortality improvement an NDC scheme can cover under different scenarios, and applied them to three different case studies.

Then in the second chapter, a new automatic balancing mechanism to restore liquidity into the system under a nonlinear framework was designed. This ABM is the result of minimising a chosen logarithmic function and simultaneously calculating the optimal path for the contribution rate, the retirement age and the indexation of pensions for the PAYGO system. This method aims to keep the future changes of the projections for the contribution rate, the normal retirement age and the indexation of pensions at a minimum level and provide liquidity into the system.

In the third chapter, the previous model is extended. Two new ABMs are developed. The first ABM is the result of minimising the difference in present value between spending on pensions and income from contributions for the next 75 years taking only into account the initial level of financial reserves (buffer fund) and no other deficit or surplus during the analysis. The second ABM design takes into account the buffer fund. This non-zero contingency fund is acting as a buffer, fluctuating deliberately in the short run and absorbing partially or completely the uncertainty in mortality, fertility rates or other events. The buffer fund could have a positively effect in a pension system. If the fund is big enough the interests generated even in risk-free assets could be high to face unexpected changes in the demographic or economic projections. This ABM will restore the sustainability of a PAYGO minimising the present value of the buffer fund over the next 75 years.

The fourth chapter questions the adequacy of the current reforms and considers that the sustainability of the Spanish pension system is not guaranteed in a long run, unless the structure of population changes dramatically with an increase in the number of contributors and a decrease in the number of pensioners.

The automatic balancing mechanisms presented in this thesis could be an alternative to the traditional parametric reforms of the PAYGO systems around the world. We show that the contribution rate, age of normal retirement and indexation of pensions stabilise at the end of the period of analysis and converge to similar values under different scenarios in our sensitivity analysis. The contingency fund absorbs the fluctuation of the demographic patterns and the economic variables involved.

Two designs are analysed: the symmetric and the asymmetric one. With the models presented we can conclude that an ABM not necessarily needs only to increase the key variables in a pension system, but also under the symmetric scenario, the fund could redistribute the surpluses to the active workers and pensioners relaxing the political difficulties that the legislators could face when implementing the reform. On the other hand, the asymmetric scenario allows the government to accumulate financial reserves in case of any unexpected shocks.

When we first establish an ABM of this type, we need to set up the number of projection variables to be included. The main advantages of a mechanism of this type is to guide the system back onto the road to long-term solvency and liquidity and at the same time to automate the measures to be taken, isolating them from the political arena, avoiding any delay and lack of time perspective. Another feature of our model, is that the ABMs also allows some flexibility in the sense that the number of variables to be projected can be changed to adapt the system to a specific situation. At the same time, it is also possible to impose more restrictions to the model to keep, for example, the normal retirement age or the contribution rate constant during some years, making the ABM more applicable in practice.

In the same line, further research could be done by seeking to assess whether a sustainability factor linked to life expectancy is sufficient to guarantee the financial stability in the pension system and, considering this sustainability factor, designs different optimal strategies, that involve variables such as the contribution rate, age of retirement and indexation on pensions, to restore the long-term financial equilibrium of the system. Extension to the model are possible by including different contingencies in our model, i.e, disability, long term care and by developing a dynamic optimisation method.

This thesis analysed the Survivor Dividend of a NDC and developed Automatic Balancing Mechanisms in a DB PAYGO pension system in a deterministic framework. Further development will include the introduction of a stochastic framework and techniques such as stochastic dynamic programming, robustness, sensitivity analysis and error bounds.

Planning a smooth sequence of change in the key variables of a pension system requires involving several aspects at the same time. Nevertheless, the ABMs presented in this thesis involves most of them, an intertemporal budget pension constraint (the contingency fund); an objective function that allows us to forecast the contribution rate, age of retirement and indexation of pensions without the restrictions that a quadratic loss function has; the flexibility of the policy maker to choose how many parameters want to include; and finally, the possibility of recalibration of the parameters involved the automatic balancing mechanism.

A

Mortality decrease according to the Lee-Carter model

We prove Formulas 1.16 and 1.17 from the section “Case 2: Mortality decrease according to the Lee-Carter model”. By assuming that mortality rates are observed up to year T, we have

$$\begin{aligned} ln\big(\hat{\mu}_{(x,T+1)}\big) &= \hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_{T+1} \\ &= ln\big(\hat{\mu}_{(x,T)}\big) + \hat{\beta}_x\big(\kappa_{T+1} - \hat{\kappa}_T\big) \\ &= ln\big(\hat{\mu}_{(x,T)}\big) + \hat{\beta}_x \gamma, \end{aligned}$$

where γ is defined as the general mortality decrease that happens between year T and $T+1$, that is, a measure of mortality improvement. The hat on the parameters indicates estimated values. More generally, we have

$$ln\big(\hat{\mu}_x^*\big) = ln\big(\hat{\mu}_x\big) + \hat{\beta}_x \gamma,$$

with $\hat{\mu}_x^*$ representing the force of mortality following a general mortality decrease of γ, and $\hat{\mu}_x$ the current mortality. Instead of comparing mortality rates between two different years, T and $T+1$, we compare two mortality scenarios. Alternatively, we can write

$$\hat{\mu}_x^* = \hat{\mu}_x\big(\hat{B}_x\big)^{\gamma}, \tag{A.1}$$

with $\hat{B}_x = e^{\hat{\beta}_x}$. By assuming constant age-specific mortality rates within bands of age and time, the following relation holds

$$\mu_x = -ln\big(p_x\big),$$

with

$$\mu_{x+\delta} = \mu_x, 0 \leq \delta < 1.$$

Using this assumption, Formula A.1 becomes

$$p_x^* = [p_x]^{(\hat{B}_x)^\gamma}.$$

The total expenditure on pensions following a mortality decrease according to the Lee–Carter model then becomes

$$E_t^{nd^{LC}} = P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \Big[1 + \sum_{k=0}^{\infty} \prod_{j=0}^{k} \big[p_{x_e+A+j}\big]^{(\hat{B}_{x_e+A+j})^\gamma}\Big] \tag{A.2}$$

and the total increase in the expenditure the NDC scheme will need to cover is

$$E_t^{nd^{LC}} - E_t^{nd} = P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \Big[\sum_{k=0}^{\infty} \prod_{j=0}^{k} \big[p_{x_e+A+j}\big]^{(\hat{B}_{x_e+A+j})^\gamma} - e_{x_e+A}\Big]. \tag{A.3}$$

The maximal general mortality decrease the system can afford is given by the highest value of γ respecting

$$\Big[\sum_{k=0}^{\infty} \prod_{j=0}^{k} \big[p_{x_e+A+j}\big]^{(\hat{B}_{x_e+A+j})^\gamma} - e_{x_e+A}\Big] \leq \frac{D^{ac}_{(x_e+A,t)}}{P^{nd}_{(x_e+A,t)}}. \tag{A.4}$$

B

Approximation of the mortality decrease according to the Lee-Carter model

We prove Formula 1.18 from the section "Case 2: Mortality decrease according to the Lee-Carter model" by using the following approximation:

$$ln(\mu_x^*) - ln(\mu_x) \approx \frac{\mu_x^* - \mu_x}{\mu_x},$$

which is valid if μ_x and μ_x^* are close enough to each other. This can be easily proven using Taylor series. Under such assumption, $\hat{\beta}_x\gamma$ represents the percentage of mortality decreaseat age x. It follows that

$$\hat{\mu}_x^* \approx \hat{\mu}_x[1 + \hat{\beta}_x\gamma].$$

By assuming a constant age-specific mortality rates within bands of age and time, the survival probability, p_x^*, corresponding to a decrease of mortality following the Lee-Carter model becomes

$$\begin{aligned} p_x^* &\approx e^{(-\hat{\mu}_x(1+\hat{\beta}_x\gamma))} \\ &\approx p_x(M_x)^{\gamma}, \end{aligned}$$

with $(M_x) = e^{(-\hat{\mu}_x \hat{\beta}_x)}$.

The total expenditure on pensions becomes

$$\begin{aligned}
\tilde{\mathrm{E}}_t^{nd^{LC}} &= P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \Big[1 + \sum_{k=1}^{\infty} \Big({}_k p_{x_e+a} \prod_{j=0}^{k-1} (M_{x_e+A+j})^{\gamma} \Big) \Big] \\
&= P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \Big[1 + \sum_{k=1}^{\infty} \Big({}_k p_{x_e+a} \prod_{j=0}^{k-1} \frac{1}{1+\zeta^{\gamma}_{x_e+A+j}} \Big) \Big], \qquad \text{(B.1)}
\end{aligned}$$

with $(M_x)^{\gamma} = (1/(1+\zeta^{\gamma}_x))$. Since $(M_x)^{\gamma} > 1$ for all ages (a decrease in mortality is equivalent to an increase in survival rate), $1 < \zeta^{\gamma}_x < 0$. Formula B.1 is similar to Formula 1.12. The difference between the two equations lies in the discounting interest rate that is used. In both cases, the interest rates are negative. However, in Formula 1.12 the interest rate is constant over time, while in Formula B.1 it changes. Formula B.1 can then be written as

$$\tilde{\mathrm{E}}_t^{nd^{LC}} = P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \ddot{\mathrm{a}}_{x_e+A,LC(\gamma)}$$

where the subscript $LC(\gamma)$ indicates that the present value is computed with an interest rate changing over time according to the β_x parameters of the Lee-Carter model and a general mortality decrease of γ. It follows that the total expenditure increases by

$$\tilde{\mathrm{E}}_t^{nd^{LC}} - E^{nd}_t = P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} [\ddot{\mathrm{a}}_{x_e+A,LC(\gamma)} - (1 + e_{x_e+A})].$$

In order to find the maximal mortality decrease the system can afford, we need to solve with respect to γ the following inequality:

$$\ddot{\mathrm{a}}_{x_e+A,LC(\gamma)} - (1 + e_{x_e+A}) \leq \frac{D^{ac}_{(x_e+A,t)}}{P^{nd}_{(x_e+A,t)}}. \qquad \text{(B.2)}$$

C

Mortality decrease according to the Lee-Carter model, with growth of the salaries and indexation of pensions

With salary and pension growth, Formulas A.2 and A.3 respectively become

$$E_t^{nd^{LC}} = P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \Big[1 + \sum_{k=0}^{\infty} \Big(\frac{1+\lambda}{1+g} \Big)^{k+1} \prod_{j=0}^{k} \big[p_{x_e+A+j} \big]^{(\hat{B}_{x_e+A+j})^{\gamma}} \Big] \quad \text{(C.1)}$$

$$E_t^{nd^{LC}} - E_t^{nd} =$$

$$P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \Big[1 + \sum_{k=0}^{\infty} \Big(\frac{1+\lambda}{1+g} \Big)^{k+1} \prod_{j=0}^{k} \big[p_{x_e+A+j} \big]^{(\hat{B}_{x_e+A+j})^{\gamma}} - \ddot{a}^{\lambda}_{x_e+A,g} \Big]. \quad \text{(C.2)}$$

The maximal general mortality decrease the system can afford is represented by the highest value of γ respecting

$$\Big[\sum_{k=0}^{\infty} \Big(\frac{1+\lambda}{1+g} \Big)^{k+1} \prod_{j=0}^{k} \big[p_{x_e+A+j} \big]^{(\hat{B}_{x_e+A+j})^{\gamma}} - \ddot{a}^{\lambda}_{x_e+A,g} \Big] \leq \frac{D^{ac}_{(x_e+A,t)}}{P^{nd}_{(x_e+A,t)}} - 1. \quad \text{(C.3)}$$

Finally, the approximation developed in Appendix B can also be applied in a model including salary and pension growth. Formula (C.1) becomes then

$$\begin{aligned}\tilde{\mathrm{E}}_t^{nd^{LC}} &= P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \Big[1+ \\ &\sum_{k=1}^{\infty} \Big(\frac{1+\lambda}{1+g}\Big)^k \Big({}_kp_{x_e+a} \prod_{j=0}^{k-1} \Big(\frac{1}{1+\zeta^{\gamma}_{x_e+A+j}}\Big)\Big)\Big] \\ &= P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \Big[1+ \sum_{k=1}^{\infty} \Big({}_kp_{x_e+a} \prod_{j=0}^{k-1} \Big(\frac{1+\lambda}{1+\psi^{\gamma}_{x_e+A+j}}\Big)\Big)\Big] \qquad \text{(C.4)}\end{aligned}$$

with $\psi^{\gamma}_x = (1+g)(1+\zeta^{\gamma}_x) - 1 = (1+g)(1/e^{(-\hat{\mu}_x\hat{\beta}_x)})^{\gamma} - 1$. Formula C.4 extends Formula B.1 to a more general framework, since salary and pension growth are now included. In both equations, the discounting rate of interest changes over time. However, the discounting rate of interest ψ^{γ}_x is not necessarily negative, contrary to the discounting rate ζ^{γ}_x. Indeed, ψ^{γ}_x includes the growth of the salaries, g, which is usually positive. The growth of the salaries can then counterbalance the effect of ζ^{γ}_x. Formula C.4 can then be written as

$$\tilde{\mathrm{E}}_t^{nd^{LC}} = P^{nd}_{(x_e+A,t)} l_{(x_e+A,t)} \ddot{\mathrm{a}}^{\lambda}_{x_e+A,LC(g,\gamma)}, \qquad \text{(C.5)}$$

where the subscript $LC(g,\gamma)$ indicates that the present value is computed with an interest rate changing over time according to the β_x parameters of the Lee-Carter model, a general mortality decrease of γ and a rate of growth of the salaries of g. In order to find the maximal mortality decrease the system can afford, we need to solve with respect to γ the following inequality:

$$\ddot{\mathrm{a}}^{\gamma}_{x_e+A,LC(g,\gamma)} - \ddot{\mathrm{a}}^{\gamma}_{x_e+A,g} \leq \frac{D^{ac}_{(x_e+A,t)}}{P^{nd}_{(x_e+A,t)}}. \qquad \text{(C.6)}$$

D

Automatic balancing mechanisms

This section shows some of the main countries that have an ABM in place. Many countries have introduced automatic mechanisms, that mainly affect the pension level, to take into account changes in life expectancy. Other ABMs are linked to other indicators, such as sustainability or solvency indicator, that reflect the financial health of the system.

For a deep understanding see Vidal-Meliá et al. (2009), Fall & Bloch (2014), Turner (2007), Turner (2009), OECD (2012) and OECD (2014). The list is sorted alphabetically.

Table D.1: Automatic balancing mechanisms

Country	**Type**	**Main Characteristic**
Canada	DB	Actuarial projections are carried every 3 years for a time horizon of 75 years, if the plan is not financially sustainable a (semi-automatic) mechanism will be triggered by increasing the contribution rate by the amount necessary to cover 50% of the deficit, and the benefits are "frozen" for three years.
Finland	DB	The life expectancy coefficient automatically adjusts the amount of pensions in payment as longevity increases (or decreases).
France	DB & points	France's automatic adjustment mechanism operates maintaining a constant ratio between the duration of activity and the expected duration of retirement. However, the government has the right not to make these adjustments if labour market conditions do not support the extra years of work.
Germany	DB	The formula for revaluing pensions includes a sustainability factor that takes into account the system's rate of dependence.
Italy	NDC	Italy uses a transformation coefficient. This coefficient is reviewed every three years in line with changes in mortality rates at different ages up to 2019 and every two years after that date. However, the adjustment is not completely automatic, because it requires legislative approval.

Table D.1 – Automatic balancing mechanisms... continued

Country	**Type**	**Main Characteristic**
Japan	DB	A modified indexation it is applied to both the revaluation of the contribution bases and the revaluation of pensions in payment. It takes into account both improvements in life expectancy and population increases (or decreases). It could be modified every five years, due to the actuarial report.
Latvia	NDC	Latvia uses unisex life expectancy at retirement age to convert the NDC account balance to an annuity. It bases life expectancy on projected cohort life tables, which are adjusted annually.
Norway	NDC	The system has unisex life-expectancy indexing of benefits at retirement. The life expectancy divisors are determined for each cohort, based mainly on remaining life expectancy.
Poland	NDC	Poland uses an annuity divisor which is revised annually. It is based on average life expectancy at retirement age.
Portugal	DB	The pension reform includes an indexation to its social security benefits taking into account improvements in life expectancy. The reduction of benefits is based directly on the percentage change in life expectancy.
Sweden	NDC	The ABM is triggered if a solvency ratio is less than one and it consists basically of reducing the growth in pension liability, that is, the pension in paymen and the contributors' notional accumulated capital.

E
The Gradient Method

Following the general ideas about optimal non-linear optimisation in discrete time, the **Gradient method** is used. This is an active set method that works with inequality constraints. Then, the inequalities are modified to equality using a linear slack variable (see Venkataraman (2009)). Finally, the model and constraint equations are expanded in a Taylor series, and only the first order terms will be retained. Then, with these linear equations, the constraint equations could be used to reduce the number of independent variables. The modified problem is given by minimizing $f(D)$ subject to $h_k(D) = 0; k = 1, 2, ..., l$ and $F_j(D) + q_{n+j} = 0; j = 1, 2, ...m$, $d_n^{min} \leq q_i \leq d_n^{max}, i = 1, 2, ..., v$ and $d_{p+j} \geq 0; j = 1, 2, ...m$ with $D = [d_1, d_2, ..., d_N]$.

The variable [D] is partitioned in [Z] and [Y], an independent and dependent set respectively:

$$
\begin{aligned}
[D] =& [d_1, d_2, ..., d_v, d_{p+1}, d_{p+2}, ..., d_{p+m}]^T; \\
[D] =& [Y, Z]^T = [z_1, z_2, ..., z_{n-l}, y_1, y_2, y_{l+m}]^T.
\end{aligned}
$$

Finally, the equality constraint is $[H] = [h, g]^T$.

Definition E.1. *A **Generalized Gradient** problem with two variables is to minimize $f(Z, Y)$ s.t. $H(Z, Y) = 0$ and $z_i^{min} \leq z_i \leq z_i^{max}, i = 1, 2, ..., n - l$ and $y_t^{min} \leq y_t \leq y_t^{max}$, $t = 1, 2, ..., m + l$, with the linear sub-problem:*

$$\begin{aligned}
&Min\ \tilde{f}(\triangle Z, \triangle Y) = f(Q^i) + \nabla_z f(Q^i)^T \triangle Z + \nabla_y f(Q^i)^T \triangle Y \\
&\quad s.t.\ \ \tilde{H}_k(\triangle Z, \triangle Y) = H_k(Q^i) + \nabla_z H_k(Q^i)^T \triangle Z + \nabla_y H_k(Q^i)^T \triangle Y = 0,
\end{aligned}$$

with $\triangle Z, \triangle Y$ constrained.

In our problem

$$f(Z,Y) = f(d_n, n) = \sum_{n=1}^{N}\left[\theta_1 log\left(\frac{c_{n+1}}{c_n}\right) + \epsilon_1\theta_2 log\left(\frac{x^{(r)}_{n+1}}{x^{(r)}_n}\right) + \epsilon_2\theta_3 log\left(\frac{\lambda_{n+1}}{\lambda_n}\right)\right],$$

with $d_n = [x^{(r)}_n, c_n, \lambda_n]$.

$H(Z,Y) = 0$ is $F_n, (d_n, n) = 0$ and $c_{max} \leq c_n \leq c_{min}, x^{(r)}_{max} \leq x^{(r)}_n \leq x^{(r)}_{min}, \lambda_{max} \leq \lambda_n \leq \lambda_{min}; n = 1, 2, ..., r - l$ and $c_{(1)max} \leq c_{(1)n} \leq c_{(1)min}, x^{(r)}_{(1)max} \leq x^{(r)}_{(1)n} \leq x^{(r)}_{(1)min}, \lambda_{(1)max} \leq \lambda_{(1)n} \leq \lambda_{(1)min}; n = 1, 2, ..., m + l$ with the linear sub-problem as defined above $F_N \geq 0$ is transformed in $F_N + q_{n+j} = 0; j = 1, 2, ...m$

Finally, with the inequalities constraints for the increase or decrease of the projected variables, $\frac{c_{n+1}}{c_n} \leq c_\Delta; \frac{x^{(r)}_{n+1}}{x^{(r)}_n} \leq x_\Delta$; we do the same as in the fund's constraint adding a variable $y(i)_j;\ i = 1, 2, 3; j = 1, 2, ...m$ such that $\frac{c_{n+1}}{c_n} + y(1)_j = c_\Delta;\ \frac{x^{(r)}_{n+1}}{x^{(r)}_n} + y(2)_j = x_\Delta$;

Bertsekas demonstrates that feasible direction methods generate sequences of feasible points $\{d^k\}$ by searching along descents directions.

Definition E.2. *Given a feasible vector d, a* ***feasible direction*** *at d is a vector $d \neq 0$ such that $d + \alpha d$ is feasible for all $\alpha > 0$ that are sufficiently small.*

Lemma E.1. *Stationarity of Limit Points for Feasible Direction Methods. (Bertsekas (1999)) Let* $\{x_k\}$ *be a sequence generated by the feasible direction method* $x_{k+1} = x_k + \alpha_k d_k$. *Assume that* $\{d_k\}$ *is gradient related and that* α^k *is chosen by the limited minimization rule . Then every limit point of* $\{x^k\}$ *is a stationary point.*

Assuming that $H(X^i) = 0$ we can write

$$[\nabla_z H_1^T; \nabla_z H_2^T; \ldots; \nabla_z H_{l+m}^T]^T \triangle Z + [\nabla_y H_1^T; \nabla_y H_2^T; \ldots; \nabla_y H_{l+m}^T]^T \triangle Y =$$

$$[A] \triangle Z + [B] \triangle Y = 0 \; with \; \triangle Y = -[B]^{-1}[A] \triangle Z.$$

Now, doing substitution in the objective function:

$$\tilde{f}(\triangle Z) = f(Q^i) + \nabla_z f(Q^i)^T \triangle Z - \nabla_y f(Q^i)^T [B]^{-1}[A] \triangle Z$$

$$= f(Q^i) + [\nabla_z f(Q^i)^T - \nabla_y f(Q^i)^T [B]^{-1}[A]] \triangle Z$$

$$= f(Q^i) + [\nabla_z f(Q^i) - ([B]^{-1}[A])^T \nabla_y f(Q^i)]^T \triangle Z$$

$$= f(Q^i) + [G_R]^T \triangle Z,$$

where $[G_R]$ is the reduced gradient of the function $f(Q)$ and provides the search direction.

In general, we can not guarantee that the iterations converge to a global minimum unless the objection function obeys significant restrictions. For f convex we have the following theorem; see Bertsekas (1999) for the proof.

Theorem E.1. *Optimality Condition*

a if d^* *is a local minimum of* f *over* D *then* $\bigtriangledown f(d*)'(d - d^*) \geq 0, \forall d \in D.$

b *if f is convex over D, then the condition of part (a) is also sufficient for d^* to minimize over X.*

As we have a constraint problem to solve, we need to find an optimal vector d^* in the closed convex set D; see also Bertsekas (1999).

Theorem E.2. *(Projection Theorem): Let D be a non-empty, closed, and convex subset of $\Re^n$*

a *For every $z \in \Re^n$, there exists a unique $d^* \in Q$ that minimizes $||z - d||$ over all $d \in D$. This vector is called the* ***projection*** *of z on D and is denoted by $[z]^+$*

b *Given some $z \in \Re^n$, a vector $d^* \in Q$ is equal to the projection $[z]^+$ if and only if*

$$(z - d^*)'(d - d^*) \leq 0, \forall d \in D.$$

c *The mapping $f : \Re^n D$ defined by $f(d) = [z]^+$ is continuous and non-expansive, that is,*

$$||[d]^+ - [y]^+|| \leq ||d - y||, \forall d, y \in D.$$

d *In the case where D is a subspace, a vector $d^* \in D$ is equal to the projection $[z]^+$ if and only if $z - d^*$ is orthogonal to d, that is*

$$(z - d^*)'q = 0, \forall d \in D.$$

As in our case, the function $f(d_n, n)$ in equation 2.3 is convex and the set of the constraints D is close, non-empty and convex, so we can find d^* to minimize $f(d_n, n)$

Algorithm: The Gradient Method

Here, two necessary steps are described in a symbolic way.

Computing step size α

Step 1:	Find $G_R, S = -G_R$; and choose α.
Step 2:	j=1.
	$\triangle Z = \alpha S; Z = Z^i + \triangle Z;$
	$\triangle Y^j = -[B]^{-1}[A] \triangle Z;$
Step 3:	$Y^{j+1} = Y^j + \triangle Y^j$.
	$Q^{j+1} = [Z; Y^{j+1}]^T$,
	if $[H(Q^{j+1})] = 0$; Stop. Converged.
	Else $j = j + 1$.
	$\triangle Y^j = [B]^{-1}[-H(Q^{j+1})]$.
	Go to step 3.
Step 4:	Do steps 1-3 for 2 values of $\alpha[\alpha_1, \alpha_2]$ and calculate $f(\alpha)$.
	With $\alpha_0 = 0$.
	Use quadratic interpolation to obtain α^*.
	Do steps 1-3 for α^* to end the iteration.

The main Algorithm

Step 1:	Choose Q_1, N_s (number of iterations).
	Choose ζ (for convergence and stopping).
	Set p=1 (iteration counter).
Step 2:	Identify [Z], [Y].
	Calculate [A], [B].
	Calculate $[G_R]$.
	Calculate optimum stepsize α^*.
	Calculate Q_{p+1}.
Step 3:	**Convergence for SQP:**
	if h_k =0, for k=1,2,...l;
	if g_j= 0, for j=1,2,...m;
	if KT condition are satisfied.
	Converged.
	Stop.
	Stopping Criteria:
	$\triangle Q = Q_{p+1} - Q_p$,
	if $\triangle Q^T \triangle Q \leq \zeta$. Stop,
	if $p = N_s$ Stop (maximum iterations reached).
	Continue.
	p = p+1.
	Go to step 2.

F

Expressions for income from contributions and expenditure on pensions

The total contribution base for year 1 is modelled as a function of the individuals' average wage, $wage(x)$ at age x, the number of people alive, $l_{x,1}$ (uniformly distributed over the year), at age x at the first year of study (that is, at time 1) and the entry age into the labour market, x_e. Thus, at time $n = 1$, the total contribution base, W_1, is modelled as:

$$W_1 = \left(\sum_{x=x_e}^{x_1^{(r)}-1} l_{x,1} * wage(x) \right),$$

For $n > 1$, the total contribution base, W_n, depends additionally on variables such as the growth of salaries, g and the normal retirement age, $x_n^{(r)}$. Where the floor function is $\lfloor x_n^{(r)} \rfloor$; i.e., it maps a real number to the largest previous integer number and $mod \lfloor x_n^{(r)} \rfloor$ is the modulus operation that finds the remainder of the division $x_n^{(r)} / \lfloor x_n^{(r)} \rfloor$. Therefore, the expression of the total contribution base for n greater than 1 is:

$$W_n = \left(\sum_{x=x_e}^{\lfloor x_n^{(r)} \rfloor -1} (l_{x,n}) * wage(x) * (1+g)^n \right) + (x_n^{(r)} mod \lfloor x_n^{(r)} \rfloor) l_{\lfloor x_n^{(r)} \rfloor, n} * wage(\lfloor x_n^{(r)} \rfloor)(1+g)^n \quad \text{(F.1)}$$

The dynamics of $B_n > 1$ could be written as:

$$B_n = \left(1 - (x_n^{(r)} mod \lfloor x_n^{(r)} \rfloor) l_{\lfloor x_n^{(r)} \rfloor, n}\right) * P_{\lfloor x_n^{(r)} \rfloor, n} + \sum_{x=\lceil x_n^{(r)} \rceil}^{\omega} P_{x,n} l_{x,n} \tag{F.2}$$

where $P_{x,n} = P_{x-1,n-1} * (1 + \lambda_{n-1})$, $\lceil x_n^{(r)} \rceil$ is the ceiling function, i.e., $\lceil x_n^{(r)} \rceil$ maps a real number to the smallest next integer number, $l_{x,n}$, is the number of people alive at age x in time n; w the last age to which a person can survive, and λ is the indexation of pensions that is dynamic over time and the expenditure on pensions at year n=1 is equal to:

$$B_1 = P_{x_1^{(r)},1} l_{x_1^{(r)},1} + P_{x_1^{(r)}+1,1} l_{x_1^{(r)}+1,1} + P_{x_1^{(r)}+2,1} l_{x_1^{(r)}+2,1} + ... = \sum_{x=x_1^{(r)}}^{\omega} P_{x,1} l_{x,1},$$

G

The Spanish pension reform

In Spain, in order to be eligible to receive a retirement pension it is necessary to contribute for at least 15 years, including at least two of them in the last 15 years prior your retirement. The figure G.1 shows the main differences in the expression to calculate the amount of the monthly initial pension, pre-reform (P_{x_r}) and post-reform ($P^{`}_{x_r}$) where:

$wage_i$: is the contribution base of one individual in month i. $x_{(r)}$: is the retirement age. S_1: is the monthly contribution base in month 1, that is, the month before retirement. S_i: is the monthly contribution base in month i. $\%_{x_{(r)},C}$: is the percentage to be applied to the base[1] pension before the reform according to the retirement age, $x_{(r)}$, and number of contributory years, C. $(\%_{x_{(r)},C})^{`}$: is the percentage to be applied to the base pension after the reform according to the retirement age, $x_{(r)}$, and number of contributory years, C. $\%_C$: is the percentage to be applied to the base pension before the reform according to the total number of years of contributions (C). $(\%_C)^{`}$: is the percentage to be applied to the base pension after the reform according to the total number of years of contributions (C).

The resulting pension should be higher than the minimum pension per year and is capped if its value exceeded the amount of the maximum pension.

[1]The table shows a very simplified expression of this value without considering the particularities of different groups.

It is worth highlighting that before the reform, one individual who contributes 35 years and retires at the age 65 has a full amount of pension equal to the base pension. Once that the reform is totally implemented in 2027 the individual should retire at the age 65 after 38.5 years of contributions or at the age of 67 after 37 contributory year to have full pension.

Figure G.1: Expressions to calculate the pension benefits before and after the reform based on Act27/2011.

Pre-Reform	Post-Reform
$P_{x_r} = \%_C \%_{x_r,C} \overbrace{\frac{\sum_{i=1}^{24} wage_i + \sum_{i=25}^{180} wage_i \frac{RPI_{25}}{RPI_i}}{210}}^{\text{Base Pension}}$ With	$P'_{x_r} = (\%_C)'(\%_{x_r,C})' \overbrace{\frac{\sum_{i=1}^{24} wage_i + \sum_{i=25}^{300} wage_i \frac{RPI_{25}}{RPI_i}}{350}}^{\text{Base Pension}}$ With
$\%_{x_r,C} = \begin{cases} 100\% - (65 - x_r)\cdot 8\%, & 60 \le x_r < 65 \text{ and } 15 \le C < 30 \\ 100\% - (65 - x_r)\cdot 7.5\%, & 60 \le x_r < 65 \text{ and } 30 \le C < 35 \\ 100\% - (65 - x_r)\cdot 7\%, & 60 \le x_r < 65 \text{ and } 35 \le C \le 37 \\ 100\% - (65 - x_r)\cdot 6.5\%, & 60 \le x_r < 65 \text{ and } 38 \le C \le 39 \\ 100\% - (65 - x_r)\cdot 6\%, & 60 \le x_r < 65 \text{ and } C \ge 40 \\ 100\% - (x_r - 65)\cdot 2\%, & x_r \ge 65 \text{ and } 30 \le C < 40 \\ 100\% - (x_r - 65)\cdot 3\%, & x_r \ge 65 \text{ and } C \ge 40 \end{cases}$ And	$(\%_{x_r,C})' = \begin{cases} \overbrace{\begin{cases} 100\% - (67 - x_r)\cdot 8\%, & 65 \le x_r < 67 \text{ and } 35 \le C < 38.5 \\ 100\% - (65 - x_r)\cdot 7.5\%, & 63 \le x_r < 65 \text{ and } 38.5 \le C < 41.5 \\ 100\% - (65 - x_r)\cdot 7\%, & 63 \le x_r < 65 \text{ and } 41.5 \le C \le 44.5 \\ 100\% - (65 - x_r)\cdot 6.5\%, & 63 \le x_r < 65 \text{ and } C \ge 44.5 \end{cases}}^{\text{Worker voluntarily gives up her job}} \\ \overbrace{\begin{cases} 100\% - (x_r - 67)\cdot 2\%, & x_r \ge 67 \text{ and } 15 \le C < 25 \\ 100\% - (x_r - 67)\cdot 2.75\%, & x_r \ge 67 \text{ and } 25 \le C < 37 \\ 100\% - (x_r - 67)\cdot 4\%, & x_r \ge 67 \text{ and } C \ge 37 \\ 100\% - (x_r - 65)\cdot 4\%, & x_r \ge 65 \text{ and } C \ge 38.5 \end{cases}}^{\text{Worker voluntarily prolongs her working life}} \end{cases}$ And
$\%_C = \begin{cases} 0, & C < 15 \\ 50\% + (C - 15)\cdot 3\%, & 15 \le C \le 25 \\ 80\% + (C - 25)\cdot 2\%, & 25 < C \le 35 \\ 100\%, & C > 35 \end{cases}$	$(\%_C)' = \begin{cases} 0, & C < 15 \\ 50\% + 2.28\%(C - 15), & 15 \le C < 35.67 \\ 80\% + 2\%(C - 35.67), & 35.67 \le C < 37 \\ 100\%, & C > 37 \end{cases}$

References

Abadie, J., & Carpenter, J. (1969). Practical methods of optimization. In R. Fletcher (Ed.), (chap. Generalization of the Wolfe Reduced Gradient Method to the Case of nonlinear constraints.). New York: Academic Press.

Actuarial Affairs Division of the Ministry of Health, Labour and Welfare. (2009). *Financial report on the public pension system fy 2009.* Tokyo..

Alho, J., Bravo, J., & Palmer, E. (2013). Non-financial defined contribution pension schemes in a changing pension world, volume 2: Gender, politics and financial stability. In R. Holzmann, E. Palmer, & D. Robalino (Eds.), (pp. 395–436). Washington, DC: World Bank Publications.

Arnold (-Gaille), S., Boado-Penas, M., & Godínez-Olivares, H. (2016). Longevity risk in notional defined contribution pension schemes: A solution. *The Geneva Papers on Risk and Insurance—Issues and Practice*, *41*(1), 24–52.

Auerbach, A., & Lee, R. (2009). Social security policy in a changing environment. In J. Brown, J. Liebman, & D. Wise (Eds.), (pp. 43–72). Chicago: University of Chicago Press.

Auerbach, A., & Lee, R. (2011). Welfare and generational equity in sustainable unfunded pension systems. *Journal of Public Economics*, *95*(1–2), 16–27.

Barr, N., & Diamond, P. (2009). *Pension reform: A short guide.* New York: Oxford University Press.

Bazaraa, M., Sheraldi, H., & Shetly, C. (1993). *Nonlinear programming* (2nd ed.). Wiley.

Becker, G. (1962). Investment in human capital: a theoretical analysis. *Journal of Public Economics*, *70*(1), 9–49.

Becker, G. (1993). *Human capital* (3rd ed.). Chicago: University of Chicago Press.

Belloni, M., & Maccheroni, C. (2013). Actuarial fairness when longevity increases: An evaluation of the italian pension system. *The Geneva Papers on Risk and Insurance—Issues and Practice*, *38*(4), 638–674.

Ben-Porath, Y. (1967). The production of human capital and the life cycle of earnings. *Journal of Political Economy*, *75*(1), 352–365.

Bertsekas, D. (1999). *Nonlinear programming* (2nd ed.). Nashua: Athena Scientific.

Bertsekas, D. (2005). *Dynamic programming and optimal control* (3rd ed., Vol. 1). Nashua: Athena Scientific.

Billig, A., & Ménard, J. (2013). Actuarial balance sheets as a tool to assess the sustainability of social security pension systems. *International Social Security Review*, *66*(2), 31-52.

Blake, D., & Borrows, W. (2001). Survivor bonds: Helping to hedge mortality risk. *The Journal of Risk and Insurance*, *68*(2), 339–348.

Boado-Penas, M., Valdés-Prieto, S., & Vidal-Meliá, C. (2008). An actuarial balance sheet for pay-as-you-go finance: Solvency indicators for spain and sweden. *Fiscal Studies*, *29*(1), 89–134.

Boado-Penas, M., & Vidal-Meliá, C. (2013). Nonfinancial defined contribution pension schemes in a changing pension world: Vol.2, gender, politics, and financial stability. In R. Holzmann, E. Palmer, & D. Robalino (Eds.), (pp. 443–484). Washington, DC: World Bank.

Boado-Penas, M., & Vidal-Meliá, C. (2014). Nonfinancial defined contribution pension schemes: Is a survivor dividend necessary to make the system balanced? *Applied Economics Letters*, *21*(4), 242–247.

Booth, H., & Tickle, L. (2008). Mortality modelling and forecasting: A review of methods. *Annals of Actuarial Science*, *3*(1–2), 3–43.

Börsch-Supan, A. (2006). Pension reform: Issues and prospects for non-financial defined contribution (ndc) schemes. In R. Holzmann & E. Palmer (Eds.), (pp. 35–55). Washington, DC: World Bank Publications.

Bradley, S. (1997). *Applied mathematical programming* (3rd ed.). Boston: Addison-Wesley.

Brown, R. (1992). Pay-as-you-go funding stability: an age of eligibility model. *Transactions of the International Congress of Actuaries*, *4*, 35–56.

Chłoń-Domińczak, A., Franco, D., & Palmer, E. (2012). Non-financial defined contribution pension schemes in a changing pension world, volume 1: Progress, lessons, and implementation. In R. Holzmann, E. Palmer, & D. Robalino (Eds.), (pp. 31–84). Washington, DC: World Bank Publications.

Congress of the United States. (2005). *Quantifying uncertainty in the analysis of long-term social security projections.* USA.

Cowley, A., & Cummins, J. (2005). Securitization of life insurance assets and liabilities. *The Journal of Risk and Insurance*, *72*(2), 193–226.

Creighton, H. (2014). *Europe's ageing demography.* UK.

D'Addio, A. C., & Whitehouse, E. (2012). Towards financial sustainability of pension systems: The role of automatic-adjustment mechanisms in oecd and eu countries. *Bundesamt für Sozialversicherungen. Forschungsbericht Nr. 8/12.*

D'Amato, V., Piscopo, G., & Russololllo, M. (2011). The mortality of the italian population: Smoothing techniques on the lee–carter model. *The Annals of Applied Statistics*, *5*(2A), 705–724.

De la Fuente, A., & Doménech, R. (2013). The financial impact of spanish pension reform: A quick estimate. *Journal of Pension Economics and Finance*, *12*(1), 111–137.

Delwarde, A., & Denuit, M. (2006). Construction de tables de mortalité périodiques et prospectives [construction of period and prospective life tables]. *Economica.*

Devolder, P., Bosch Princep, M., & Dominguez Fabian, I. (2002). Stochastic optimal control of annuity contracts. *Insurance: Mathematics and Economics*, *33*(2), 227–238.

Duguay, M. (2006). *Preparing for a pandemic: avian flu and your pension plans.* Stockholm. Retrieved 06.05.2015, from `http://mmc.dppl.com/knowledgecenter/avian_flu_and_pension_plans.pdf`

European Commission. (2010). *Green paper. towards adequate, sustainable and safe european pension systems.* Belgium.

European Commission. (2012a). *The 2012 ageing report: Economic and budgetary projections for the 27 eu member states (2010-2060).* Brussels. (European Economy 2–2012)

European Commission. (2012b). *White paper. an agenda for adequate, safe and sustainable pensions.* Belgium.

Eurostat Database. (2014). *Mean annual earnings by sex, age and economic activity.* Retrieved 11.08.2014, from `http://ec.europa.eu/eurostat/data/database`

Fall, F., & Bloch, D. (2014). Overcoming vulnerabilities of pension systems. *OECD Economics Department Working Papers 1133.*

Feldstein, M., & Liebman, J. (2010). Handbook of public economics, volume 4. In A. Auerbach & M. Feldstein. (Eds.), (chap. Social security). New York: Elsevier Science B.V.

Fihel, A., & Jasilionis, D. (2011). *About mortality data for Poland, Human Mortality Database, Berkeley /Rostock: University of California/Max Planck Institute for Demographic Research.* Retrieved 22.09.2014, from `www.mortality.org`

Forman, J., & Sabin, M. (2015). Tontine pensions. *University of Pennsylvania Law Review*, *163*(3), 775–831.

Gaille, S. (2012). Forecasting mortality: When academia meets practice. *European Actuarial Journal*, *2*(1), 49–76.

Gannon, F., Legros, F., & Touzé, V. (2013). *Automatic adjustment mechanisms and budget balancing of pensions schemes.* (Working paper, OFCE)

García-Ferrer, A., & Del Hoyo, J. (1991). Analysis and prediction of the population in Spain: 1910–2000. *Journal of Forecasting*, *10*, 347–369.

Glei, D., Lundström, H., & Wilmoth, J. (2012). *About mortality data for sweden, human mortality database, berkeley /rostock: University of california/max planck institute for demographic research.* Retrieved 22.09.2014, from `www.mortality.org`

Godínez-Olivares, H., & Boado-Penas, M. (2015). An alternative pension reform for spain based on optimisation techniques. *Anales del Instituto de Actuarios Españoles*, *3era*(21), 107–134.

Godínez-Olivares, H., Boado-Penas, M., & Haberman, S. (2016). Optimal strategies for pay-as-you-go pension finance: A sustainability framework. *Insurance: Mathematics and Economics*, *69*, 117–126.

Godínez-Olivares, H., Boado-Penas, M., & Pantelous, A. (2016). How to finance pensions: Optimal strategies for pay-as-you-go pension systems. *Journal of Forecasting*, *35*(1), 13–33.

Gründl, H., Post, T., & Schulze, R. (2006). To hedge or not to hedge: Managing demographic risk in life insurance companies. *The Journal of Risk and Insurance*, *73*(1), 19–41.

Haberman, S., & Vigna, E. (2002a). Optimal investment strategies and risk measures in defined contribution pension schemes. *Insurance: Mathematics and Economics*, *31*(1), 35–69.

Haberman, S., & Zimbidis, A. (2002b). An investigation of the pay-as-you-go financing method using a contingency fund and optimal control techniques. *North American Actuarial Journal*, *6*, 60–75.

Hári, N., De Waegenaere, A., Melenberg, B., & Nijman, T. (2008). Longevity risk in portfolios of pension annuities. *Insurance: Mathematics and Economics*, *42*(2), 505–519.

Holzmann, R., Orenstein, M. A., & Rutkowski, M. (2003). *Pension reform in europe: process and progress.* Washington, DC: World Bank Publications.

Holzmann, R., & Palmer, E. (2006). *Pension reform: Issues and prospects for notional defined contribution (ndc) schemes.* Washington, DC: World Bank Publications.

Holzmann, R., Palmer, E., & Robalino, D. (2012). *Non-financial defined contribution pension schemes in a changing pension world, volume 1: Progress, lessons, and implementation.* Washington, DC: World Bank Publications.

Human Mortality Database. (2014). *University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany).* Retrieved 11.08.2014, from `www.mortality.org`

Jasilionis, D. (2013). *About mortality data for latvia, human mortality database, berkeley /rostock: University of california/max planck institute for demographic research.* Retrieved 22.09.2014, from `www.mortality.org`

Kallrath, J. (1999). *Nonlinear optimization: Models, algorithms, tools and support.* (AIRO News, IV(n.2), 5-10)

Katz, L., & Autor, D. (1999). Handbook of labor economics. In O. Ashenfelter & D. Card (Eds.), (Vol. 3A, pp. 1463–1555). Amsterdam: Elsevier.

Knell, M. (2012). Increasing life expectancy and pay-as-you-go pension systems. *ÖNB Working Paper 179, Wien: Oesterreichische Nationalbank.*

Knell, M., Kohler-Toglhofer, W., & Prammer, D. (2006). The austrian pension system -how recent reforms have changed fiscal sustainability and pension benefits. *Monetary Policy & the Economy*, *Q2*(6), 69–93.

Kübler, B. (2010). *Risk classification by means of clustering.* (Peter Lang: Frankfurt am Main)

LaValle, S. (2006). *Planning algorithms.* Cambridge: Cambridge University Press.

Lee, R., & Carter, L. (1992). Modeling and forecasting u.s. mortality. *Journal of the American Statistical Association*, *87*(419), 659–671.

Lee, R., & Shripad, T. (1998a). Frontiers in the economics of aging. In D. Wise (Ed.), (pp. 393–428). Chicago: University of Chicago Press.

Lee, R., & Shripad, T. (1998b). Uncertain demographic futures and social security finances. *American Economic Review*, *88*(2), 237–241.

Lindbeck, A., & Persson, M. (2003). The gains from pension reform. *Journal of Economic Literature*, *41*(1), 74–112.

London, P. (2013). *Reforming public pensions. the notional solution.* Retrieved 27.07.2016, from `http://http://www.economist.com/blogs/freeexchange/2013/01/reforming-public-pensions`

Luenberger, D. (1992). Benefit functions and duality. *Journal of Mathematical Economics*, *21*(5), 461–481.

Lundström, H., & Qvist, J. (2004). Mortality forecasting and trend shifts: An application of the lee–carter model to swedish mortality data. *International Statistical Review*, *72*(1), 37–50.

Maurer, R., Mitchell, O., Rogalla, R., & Kartashov, V. (2013). Lifecycle portfolio choice with systematic longevity risk and variable investment-linked deferred annuities. *The Journal of Risk and Insurance*, *80*(3), 649–676.

Milevsky, M. (2010). *The calculus of retirement income: Financial models for pension annuities and life insurance.* Cambridge: Cambridge University Press.

Mincer, J. (1974). *Schooling, experience, and earnings.* New York: NBER.

Nau, R. (2015). *Statistical forecasting: notes on regression and time series analysis.* Duke University. Retrieved 16.06.2015, from `http://people.duke.edu/~rnau/411home.htm`

Nemirovski, A. (1999). *Optimization ii. numerical methods for nonlinear continuous optimization.* New York: Spring.

Obstfeld, M., & Rogoff, K. (1996). *Foundations of international macroeconomics.* Cambdrige, Massachussetts: The MIT press.

OECD. (2011). *Pensions at a glance 2011: Retirement income systems in oecd and g20 countries* (Tech. Rep.). Paris: OECD Publishing.

OECD. (2012). Putting pensions on auto-pilot: automatic-adjustment mechanisms and financial sustainability of retirement-income systems. In *Oecd pensions outlook 2012.* Paris: OECD Publishing.

OECD. (2014). *Pensions at a glance 2013: Oecd and g20 indicators.* Paris.

OECD. (2015). *Health at a glance 2015: Oecd indicators.* Paris.

Office of the Chief Actuary. (2012). *Actuarial report (26th) on the canada pension plan.* Canada.

of State for Social Security., S. (2011). *Social security reserve fund. changes and actions in 2010 and status as at 31 december 2010. report to parliament.* Spain. Retrieved 05.05.2016, from `http://www.seg-social.es/Internet_1/Estadistica/FondodeReservadelaS48074/Informes/InfoAnuEE2k10/InfCortGen2k10/index.htm`

Olivieri, A., & Pitacco, E. (2008). Solvency requirements for life annuity: Some comparisons. *Giornale dell'Istituto Italiano degli Attuari*, *71*(1/2), 59–82.

Osterholm, M. (2005). Preparing for the next pandemic. *New England Journal of Medicine*, *352*, 1839–1842.

Palmer, E. (2013). Non-financial defined contribution pension schemes in a changing pension world, volume 2: Gender, politics and financial stability. In R. Holzmann, E. Palmer, & D. Robalino (Eds.), (pp. 309–341). Washington, DC: World Bank Publications.

Pantelous, A., & Zimbidis, A. (2008). Dynamic reforming of a quasi pay-as-you-go social security system within a discrete stochastic multidimensional framework using optimal control methods. *Applicationes Mathematicae*, *35*, 121–144.

Plamondon, P., A., D., Binet, G., Cichon, M., McGillivray, W., Bedard, M., & H. Perez-Montas, H. (2002). Actuarial practice in social security. quantitative methods in social protection series. *Geneva. International Social Security Association and International Labour Office.*

Roberts, M. (2002). *Can pay-as-you-go pensions raise the capital stock?* University of Nottingham.

Rofman, R., Apella, I., & Vezza, E. (2015). Beyond contributory pensions. fourteen experiences with coverage expansion in latin america. In R. Rofman, I. Apella, & E. Vezza (Eds.), (pp. 1–44). Washington, DC: World Bank Publications.

Sales-Sarrapy, C., Solis-Soberon, F., & Villagomez-Amezcua, A. (1998). Privatizing social security. In M. Feldstein (Ed.), (pp. 135–175). Chicago: University of Chicago Press.

Sayan, S., & Kiraci, A. (2001). Identification of parametric policy options for rehabilitating a pay-as-you-go based pension system: an optimization analysis for turkey. *Applied Economics Letters*, *8*(2), 89–93.

Settergren, O. (2008). *Defining meaningful information to assess the financial status of social security pension schemes* (Tech. Rep.). Limassol, Cyprus: Paper presented at the Technical Seminar for Social Security Actuaries and Statisticians. October 30-31.

Siebert, H. (2005). *The german economy: Beyond the social market.* Princeton: Princeton University Press.

Sontag, E. (1998). *Mathematical control theory: Deterministic finite dimensional systems* (2nd ed.). New York: Springer.

Stallard, E. (2006). Demographic issues in longevity risk analysis. *The Journal of Risk and Insurance*, *73*(4), 575–609.

The 2015 Annual report of the Board of Trustees of the Federal Old-Age and Survivors Insurance and Federal Disability Insurance Trust Funds (OASDI). (2015). *Annual report* (Tech. Rep.). Washington, DC: U.S. Government Printing Office.

The Swedish Pension System. (2015). *Orange report: Annual report of the swedish pension system 2014.* Stockholm.

Tuljapurkar, S., Li, N., & Boe, C. (2000). A universal pattern of mortality decline in the G7 countries. *Nature*, *405*(6788), 789–792.

Turner, J. (2007). *Autopilot: self-adjusting mechanisms for sustainable retirement systems.* Schaumburg.

Turner, J. (2009). *Social security financing: automatic adjustment to restore solvency.* Washington, DC.

Valdés-Prieto, S. (2006). Restructuring retirement risks. In D. Blitzstein, O. Mitchell, & S. Utkus (Eds.), (pp. 226–242). Philadelphia: Oxford University Press for the Pension Research Council.

Venkataraman, P. (2009). *Applied optimisation with matlab programming* (2nd ed.). Hoboken: Wiley.

Vidal-Meliá, C. (2014). An assessment of the 2011 spanish pension reform using the swedish system as a benchmark. *Journal of Pension Economics and Finance*, *13*(3), 297–333.

Vidal-Meliá, C., Boado-Penas, M., & Navarro-Cabo, F. (2015). Notional defined contribution pension systems: Why does only sweden distribute the survivor dividend? *Journal of Economic Policy Reform*, *in press*.

Vidal-Meliá, C., Boado-Penas, M., & Settergren, O. (2009). Automatic balance mechanisms in pay-as-you-go pension systems. *The Geneva Papers on Risk and Insurance - Issues and Practice*, *34*(2), 287–317.

Vidal-Meliá, C., Boado-Penas, M., & Settergren, O. (2010). Pension fund risk management. financial and actuarial modelling. In M. Micocci, G. Gregoriou, & G. Masala (Eds.), (pp. 419–472). New York: Chapman & Hall/CRC.

Vidal-Meliá, C., Domínguez-Fabián, I., & Devesa-Carpio, J. (2006). Subjective economic risk to beneficiaries in notional defined contribution accounts (ndcs). *The Journal of Risk and Insurance*, *73*(3), 489–515.

Walde, K. (2010). *Applied intertemporal optimization*. Gutenberg Press.

Whiteford, P., & Whitehouse, E. (2006). Pension challenges and pension reforms in oecd countries. *Oxford review of economic policy*, *22*(1), 78–94.

Whitehouse, E. (2009a). Pensions during the crisis: impact on retirement income systems and policy responses. *The Geneva Papers on Risk and Insurance—Issues and Practice*, *34*, 536–547.

Whitehouse, E. (2009b). Pensions, purchasing-power risk, inflation and indexation. *OECD Social, Employment and Migration working papers No. 77*.

Whitehouse, E. (2010). Decomposing notional defined-contribution pensions: Experience of oecd countries' reforms. *OECD Social, Employment and Migration working papers No. 109*.

Whitehouse, E., D'Addio, A., Chomik, R., & Reilly, A. (2009). Two decades of pension reform: what has been achieved and what remains to be done? *The Geneva Papers on Risk and Insurance—Issues and Practice*, *34*, 515–535.

Wilcoxen, P. (1989). *Intertemporal optimization in general equilibrium: A practical introduction.* Australia.

Williamson, J. (2004). Assessing the pension reform potential of a notional defined contribution pillar. *International Social Security Review*, *57*(1), 47–64.

Wilmoth, J. (1996). Health and mortality among elderly populations. In G. Caselli & A. Lopez (Eds.), (pp. 266–287). Oxford/New York: Clarendon Press/Oxford University Press.

World Bank. (2005). *Notional accounts: Notional defined contribution plans as a pension reform strategy. world bank pension reform primer 33382.* Washington, DC.

Wright, S. (1996). *Primal-dual interior-point methods.* Philadelphia.

Yermo, J. (2008). *Governance and investment of public pension reserve funds in selected oecd countries* (Tech. Rep.). Paris: OECD Working Papers on Insurance and Private Pensions, No. 15. OECD Publishing.