

# MODAL ANALYSIS OF MULTI-DEGREE-OF-FREEDOM SYSTEMS WITH SINGULAR MATRICES - ANALYTICAL DYNAMICS APPROACH

Athanasios A. Pantelous<sup>1</sup> and Antonina Pirrotta\*<sup>2</sup>, M. ASCE

## ABSTRACT

Complex mechanical (e.g. multi-body) systems with different types of constraints are generally performed through analytical dynamics methods. In some cases, however, it is possible that the (augmented) mass and/or stiffness matrices may derive to be singular, consequently the modal analysis, which is used extensively in the classical dynamics literature, fails. In this paper, if the uniqueness condition is satisfied by the constraints, a properly modified modal analysis is elucidated into analytical dynamics leading to the evaluation of the natural frequencies in a simple and straightforward way. Under that framework, advances of both classical and analytical dynamics are taken into consideration for evaluating the structural response.

**Keywords:** Modal Analysis; Analytical Dynamics; Constrained Mechanical Systems; Singular Matrices.

## INTRODUCTION

In analytical dynamics, one of the most fundamental and consequently, well studied problem for more than 200 years ago is the determination of equations of motion for constrained mechanical systems (Pars 1979; Roberts and Spanos 2003; Ardema 2005). The pioneering works of (Lagrange 1787) and (Gauss 1829) have inspired and influenced many other researchers. Thus, for the formulation of the equations of motion, at the beginning of 90's, an alternative and very interesting

---

<sup>1</sup>Reader, Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool, L69 7ZL, UK. E-mail: a.pantelous@liverpool.ac.uk.

<sup>2</sup>Associate Professor, Dipartimento di Ingegneria Civile, Ambientale, Aerospaziale, dei Materiali (DICAM), Italy Honorary Professor, Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool, L69 7ZL. (corresponding author). E-mail: antonina.pirrotta@unipa.it, pirrotta@liverpool.ac.uk.

approach has been proposed (Udwadia and Kalaba 1992). With their seminal work, additional constraint forces have been introduced and eventually, the equations of constrained mechanical system have been augmented. Under this new framework, it derives that the explicit computation of constraint forces is not always an easy task to perform, especially in complex cases, such as for multi-body systems (Laulusa and Bauchau 2007; De Falco et al. 2009; Schutte and Udwadia 2011; Mariti et al. 2011; Garcia de Jalón and Guetiérrez-López 2013; Fragkoulis et al. 2015; Fragkoulis et al. 2016).

In the present paper, our attention focuses on a rather recent approach for the formulation of equations of motion of constrained systems, which has been proposed and studied thoughtfully in a series of papers (Udwadia and Kalaba 1992; Udwadia et al. 1997; Udwadia and Kalaba 2001; Udwadia and Kalaba 2002; Udwadia and Kalaba 2007; Udwadia and Schutte 2010; Udwadia and Di Massa 2011; Udwadia and Wanichanon 2012; Udwadia and Wanichanon 2013). Particularly, under our framework, by adapting the technique introduced in (Udwadia and Phohomsiri 2006) for the formulation of equations of motion in cases where the mass matrix can be singular, an alternative approach is proposed for the results presented therein which is related to the modal analysis. In more details, given a *linear* mechanical system subject to a number of *linear* constraints, unavoidably additional constraint forces have to be introduced in the system in order to guarantee that the imposed constraints are always satisfied. A workaround for this situation is to set up the equations of motion neglecting the dependence between generalized coordinates imposed by the constraints and then apply a methodology based on the Moore-Penrose (pseudo) inverse matrix theory (Greville 1960; Campbell and Meyer 1979; Ben-Israel and Greville 2003) to incorporate the constraints in the modified equations of motion. On the formation of the unconstrained equations of motion, the mass matrix of the system may be singular. This might be either due to the dependence between the generalized coordinates chosen to describe the system or occasions where it is possible to assign null mass to a body whose inertia is negligible. Note that some of the structural systems considered herein are related to the so-called *singular* systems described, in general, by a set of differential-algebraic equations (Kalogeropoulos and Pantelous 2008; Gashi and Pantelous 2013;

Kalogeropoulos et al. 2014; Gashi and Pantelous 2015).

The main advantage of the approach proposed in (Udwadia and Phohomsiri 2006) is that it allows us to model easily complex mechanical (e.g. multi-body) systems by decomposing them into a collection of independently modeled subsystems, whose equations of motion can be easily formulated. It is only at the second stage of this approach where the constraints are taken into account, and lead to modified equations of motion, regardless of whether a singular mass matrix has encountered in the original equations or not (Garcia de Jalón and Guetiérrez-López 2013; Antoniou et al. 2016). A second important advantage is that it provides an explicit formula for the acceleration, without engaging any auxiliary variables such as the Lagrange multipliers (Schiehlen 1984; Pradhan et al. 1997). It should be noticed that the method is applicable to systems subject to holonomic and non-holonomic constraints or their combination, as well as systems where the constraint forces may or may not be ideal.

For engineers, although reaching the solution is an important task (Antoniou et al. 2016), it is much more significant to know the natural structural frequencies to predict detrimental dynamic effects. Just think of the *resonance* phenomenon that occurs when the natural *structural* frequencies are very close to the *excitation* frequencies; especially, for design control devices (Di Matteo et al. 2014a; Di Matteo et al. 2014b), i.e., useful for mitigation of vibrations like tuned mass dampers or tuned liquid column damper that are tuned to the natural frequency of the system to be controlled. More generally in engineering applications, it is of fundamental importance to know the values of natural frequencies. Actually, this is the reason for the wide use of modal analysis, when the original system response is obtained through a superposition of modal responses shaped by the *mode shapes* itself, and they are as many as the frequencies of the system. But, looking at the fundamental matrices of the augmented system or whenever a system is characterised by a singular mass matrix, then the classical modal analysis may not be applicable in the current form.

In the present paper, if the uniqueness condition, which is shown in (Udwadia and Phohomsiri 2006), is satisfied by the constraints, a proper modified modal analysis is elucidated, valid for these augmented systems or singular mass matrix systems instead, leading to the evaluation of the

*natural* frequencies as first step in a simple and straightforward way as is derived clearly in the following sections. Indeed, the proposed formulation fits ideally to the case of linear time invariant (LTI) underdamped mechanical systems subject to linear constraints.

## STATE-VARIABLE FORMULATION BASED ON THE MOORE-PENROSE THEORY

Considering a structural system for evaluating the dynamic response, dynamics equilibrium equations may be referred to the minimum set of coordinates, however, for complex systems as the multi-body ones, writing the equation of motion using the minimum set of coordinates is a hard task (Bae and Haug 1987; Featherstone 1987; Critchley and Anderson 2003; De Falco et al. 2009). As regards, choosing redundant set of coordinates, it makes easier the way of writing dynamics equilibrium equations. In this context, the set of equations is in an algebraic-differential form and composed of a lot of equations but with a simple algebraic structure. By using analytical dynamics tool, the solution provides not only information about the motion, but also on the forces of constraint. What makes the difference is the possibility to have singular mass matrices so that the classical modal analysis is not more applicable. Hereafter a section dedicated to the solution procedure for such a system used in the literature (Udwadia and Phohomsiri 2006). In this regard, consider an  $l$ -DOF system of the form

$$\begin{aligned} \mathbf{M}_u \ddot{\mathbf{u}}(t) + \mathbf{C}_u \dot{\mathbf{u}}(t) + \mathbf{K}_u \mathbf{u}(t) &= \mathbf{f}_u(t), \\ \mathbf{u}(0) &= \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0, \end{aligned} \tag{1}$$

being  $\mathbf{u}$  the  $l$ -vector of the coordinates, and  $\mathbf{f}_u(t)$  the  $l$ -vector of external forces.  $\mathbf{M}_u$ ,  $\mathbf{C}_u$ ,  $\mathbf{K}_u$  are the mass, damping and stiffness ( $l \times l$ ) matrices, respectively, corresponding to the system Eq. (1). Further, consider that the above system is subjected to  $m$ -constraints as

$$\mathbf{A}(\mathbf{u}, \dot{\mathbf{u}}, t) \ddot{\mathbf{u}} = \mathbf{b}(\mathbf{u}, \dot{\mathbf{u}}, t), \tag{2}$$

being  $\mathbf{A}$  an  $(m \times l)$  matrix and  $\mathbf{b}$  an  $m$ -vector. To simplify the procedure, assuming,  $\mathbf{b}(\mathbf{u}, \dot{\mathbf{u}}, t) =$

$-\mathbf{E}\dot{\mathbf{u}} - \mathbf{L}\mathbf{u} + \mathbf{F}$ , Eq. (2) may be rewritten as

$$\mathbf{A}(\mathbf{u}, \dot{\mathbf{u}}, t) \ddot{\mathbf{u}} = -\mathbf{E}\dot{\mathbf{u}} - \mathbf{L}\mathbf{u} + \mathbf{F}. \quad (3)$$

Next, combining Eq. (1) with Eq. (3), the system is written in the form

$$\begin{aligned} \bar{\mathbf{M}}_u \ddot{\mathbf{u}}(t) + \bar{\mathbf{C}}_u \dot{\mathbf{u}}(t) + \bar{\mathbf{K}}_u \mathbf{u}(t) &= \bar{\mathbf{f}}_u(t), \\ \mathbf{u}(0) &= \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0, \end{aligned} \quad (4)$$

with

$$\bar{\mathbf{M}}_u = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^+ \mathbf{A}) \mathbf{M}_u \\ \mathbf{A} \end{bmatrix}, \quad (5)$$

$$\bar{\mathbf{C}}_u = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^+ \mathbf{A}) \mathbf{C}_u \\ \mathbf{E} \end{bmatrix}, \quad (6)$$

$$\bar{\mathbf{K}}_u = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^+ \mathbf{A}) \mathbf{K}_u \\ \mathbf{L} \end{bmatrix}, \quad (7)$$

$$\bar{\mathbf{f}}_u = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^+ \mathbf{A}) \mathbf{f}_u \\ \mathbf{F} \end{bmatrix}, \quad (8)$$

and  $\mathbf{A}^+ (l \times m)$  is the so called Moore-Penrose inverse of  $\mathbf{A}$ .

For such a system using the analytical dynamics approach, the acceleration response is evaluated by

$$\ddot{\mathbf{u}}(t) = \bar{\mathbf{M}}_u^+ [-\bar{\mathbf{C}}_u \dot{\mathbf{u}}(t) - \bar{\mathbf{K}}_u \mathbf{u}(t) + \bar{\mathbf{f}}_u(t)] + [\mathbf{I} - \bar{\mathbf{M}}_u^+ \bar{\mathbf{M}}_u] \mathbf{q}(t), \quad (9)$$

where  $\bar{\mathbf{M}}_u^+ (l \times (l + m))$  is the Moore-Penrose inverse of  $\bar{\mathbf{M}}_u$ , and  $\mathbf{q}(t)$  is an arbitrary vector involved in the definition of the Moore-Penrose inverse matrix, that does not contribute when the  $((l + m) \times l)$  matrix  $\bar{\mathbf{M}}_u$  has full rank  $l$ , returning, in this case, a unique response solution (Udwadia and Phohomsiri 2006). It should be mentioned here that a simple, general, and explicit form of

equations of motion for general constrained mechanical systems by fully preserving the physical meaning of the systems without using the generalized Moore-Penrose (MP) inverse of the matrix  $A$  for the determination of the unconstrained auxiliary system appears in (Udwadia and Wanichanon 2012; Udwadia and Wanichanon 2013). Instead, the transpose of  $A$  is just used to describe the unconstrained auxiliary system; further discussion is omitted as it is out of the scope of the paper.

Furthermore, it is worth stressing that even when the solution is unique and is carried out through the above procedure, all dynamics features remain hidden. To highlight these characteristics a proper modal analysis has been proposed recovering all physical meaning, as detailed in the following section.

## PROPOSED MODAL ANALYSIS FRAMED INTO ANALYTICAL DYNAMICS

Dealing with systems referred to redundant coordinates or with those having singular mass matrices, the general approach framed into analytical dynamics furnishes the final solution in efficient and elegant way, although the mass matrix is singular. Just due to this singularity effect, modal analysis is not applicable, let alone that now the relevant matrices are rectangular. However engineers cannot overlook an approach rich of physical meanings (Udwadia and Wanichanon 2012; Udwadia and Wanichanon 2013).

To aim at this hereafter a proper modal analysis is proposed that solves out the differential system of equations referred to redundant variables or system with singular mass matrix, decoupling the system itself and returning the main dynamics characteristics as frequency and mode shape. The main idea is to evaluate the eigenvalues  $\bar{\omega}_j^2$  and eigenvectors  $\bar{\phi}_j$  ( $j = 1, 2 \dots l$ ) of the following matrix

$$[\bar{\mathbf{M}}_u^+ \bar{\mathbf{K}}_u] . \quad (10)$$

Then, considering the modal matrix  $\bar{\Phi}$  ( $l \times l$ ) containing the eigenvectors  $\bar{\phi}_j$  as columns, the fun-

damental following relationships hold true

$$\begin{aligned}\mathbf{I} &= \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{M}}_u \bar{\Phi} = \text{diag} \{1\}, \\ \bar{\Omega} &= \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{K}}_u \bar{\Phi} = \text{diag} \{\bar{\omega}_j^2\}, \\ \bar{\Lambda} &= \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{C}}_u \bar{\Phi} = \text{diag} \{2\bar{\zeta}_j \bar{\omega}_j\},\end{aligned}\tag{11}$$

where  $\mathbf{I} (l \times l)$  is the identity matrix while  $\bar{\Omega} (l \times l)$  and  $\bar{\Lambda} (l \times l)$  are diagonal matrices, and  $\bar{\omega}_j$  and  $\bar{\zeta}_j$  are respectively, undamped natural circular frequencies and values of the damping ratio of the system. Moreover, introducing the following modal transformation

$$\mathbf{u}(t) = \bar{\Phi} \mathbf{p}(t),\tag{12}$$

into the Eq. (4), it leads to

$$\bar{\mathbf{M}}_u \bar{\Phi} \ddot{\mathbf{p}}(t) + \bar{\mathbf{C}}_u \bar{\Phi} \dot{\mathbf{p}}(t) + \bar{\mathbf{K}}_u \bar{\Phi} \mathbf{p}(t) = \bar{\mathbf{f}}_u(t).\tag{13}$$

Then pre-multiplying by  $[\bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+]$ , the original system (13) is transformed into

$$\bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{M}}_u \bar{\Phi} \ddot{\mathbf{p}}(t) + \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{C}}_u \bar{\Phi} \dot{\mathbf{p}}(t) + \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{K}}_u \bar{\Phi} \mathbf{p}(t) = \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{f}}_u(t).\tag{14}$$

Next, by considering the relations Eq. (11), it is decoupled in the form

$$\begin{aligned}\ddot{\mathbf{p}}(t) + \bar{\Lambda} \dot{\mathbf{p}}(t) + \bar{\Omega} \mathbf{p}(t) &= \bar{\mathbf{f}}(t), \\ \mathbf{p}(0) &= \bar{\Phi}^{-1} \mathbf{u}_0, \quad \dot{\mathbf{p}}(0) = \bar{\Phi}^{-1} \dot{\mathbf{u}}_0,\end{aligned}\tag{15}$$

being  $\bar{\mathbf{f}}(t) = \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{f}}_u(t)$ . Then, the system response may be evaluated as a superposition of modal responses  $\mathbf{p}(t)$  as

$$u_i(t) = \sum_{j=1}^l \bar{\phi}_{ij} p_j(t), \quad i = 1, 2 \dots l,\tag{16}$$

where  $p_j(t)$  is the solution response of the following uncoupled  $j^{th}$  equation of the system in Eq.

(15)

$$\ddot{p}_j(t) + 2\bar{\zeta}_j \bar{\omega}_j \dot{p}_j(t) + \bar{\omega}_j^2 p_j(t) = \bar{f}_j(t). \quad (17)$$

It is worth underscoring that the main goal of this procedure that is to return the physical meaning of frequency and mode shape is achieved. Specifically, for redundant systems, since the package of  $l$  undamped natural frequencies  $\bar{\omega}_j(t)$  contains the undamped natural frequencies (say  $n$  values with  $n < l$ ) of the system referred to  $n$  strictly variables together with null frequencies pertaining rigid motions, that are expected, since redundant variables are present. As regards mode shapes, they are provided by the eigenvectors correspondent to the eigenvalues  $\bar{\omega}_j^2(t)$ . In this direction, let us proceed a little further in order to avoid to calculate analytically the  $\bar{\mathbf{M}}_u^+$ , for the evaluation of natural frequencies

$$\begin{aligned} \det(\bar{\mathbf{M}}_u^+ \bar{\mathbf{K}}_u - \bar{\omega} I_l) &= \det\left((\bar{\mathbf{M}}_u^T \bar{\mathbf{M}}_u)^{-1} \bar{\mathbf{M}}_u^T \bar{\mathbf{K}}_u - \bar{\omega} I_l\right) = \\ \det\left((\bar{\mathbf{M}}_u^T \bar{\mathbf{M}}_u)^{-1}\right) \det(\bar{\mathbf{M}}_u^T \bar{\mathbf{K}}_u - \bar{\omega} \bar{\mathbf{M}}_u^T \bar{\mathbf{M}}_u) &= 0 \Leftrightarrow \\ \det(\bar{\mathbf{M}}_u^T (\bar{\mathbf{K}}_u - \bar{\omega} \bar{\mathbf{M}}_u)) &= 0. \end{aligned} \quad (18)$$

Let  $\begin{pmatrix} l+m \\ m \end{pmatrix}$  be the possible  $l \times l$ - submatrices of  $\bar{\mathbf{M}}_u^T$  and  $(\bar{\mathbf{K}}_u - \bar{\omega} \bar{\mathbf{M}}_u)$ . Let  $I_{l+m,l}$  denote the set of  $l$ -element subsets of  $[l+m] = \{1, 2, \dots, l+m\}$ . For each subset  $S \in I_{l+m,l}$ , we can uniquely write  $S = \{n_1, n_2, \dots, n_l\}$ , where  $1 \leq n_1 < n_2 < \dots < n_l \leq l+m$ . Let  $\bar{\mathbf{M}}_{u,S}^T$  be the  $l \times l$  matrix formed from  $\bar{\mathbf{M}}_u^T$  by keeping only the rows with row index in  $S$ , and removing the rest. Thus, the row  $i$  of  $\bar{\mathbf{M}}_{u,S}^T$  is equal to the row  $n_i$  of  $\bar{\mathbf{M}}_u^T$ .

Then, by applying the Cauchy-Binet theorem (Gohberg et al. 1986), since  $\bar{\mathbf{M}}_u^T$  is a  $l \times (l+m)$  matrix and  $(\bar{\mathbf{K}}_u - \bar{\omega} \bar{\mathbf{M}}_u)$  is a  $(l+m) \times l$  matrix, then we obtain that

$$\det(\bar{\mathbf{M}}_u^T (\bar{\mathbf{K}}_u - \bar{\omega} \bar{\mathbf{M}}_u)) = \sum_S \det(\bar{\mathbf{M}}_{u,S}^T) \det((\bar{\mathbf{K}}_u - \bar{\omega} \bar{\mathbf{M}}_u)_S). \quad (19)$$

Additionally, we have that

$$\bar{\mathbf{K}}_u - \bar{\omega} \bar{\mathbf{M}}_u = \begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+ \mathbf{A}) (\mathbf{K}_u - \bar{\omega} \mathbf{M}_u) \\ -\bar{\omega} \mathbf{A} \end{bmatrix}, \quad (20)$$

and  $\text{rank}(K) = \text{rank}(M) = \text{rank}(\mathbf{K}_u) = \text{rank}(\mathbf{M}_u) = r$ . We are interested in calculating

$$\det((\bar{\mathbf{K}}_u - \bar{\omega} \bar{\mathbf{M}}_u)_S) = \det\left(\begin{bmatrix} (\mathbf{I}_l - \mathbf{A}^+ \mathbf{A}) (\mathbf{K}_u - \bar{\omega} \mathbf{M}_u) \\ -\bar{\omega} \mathbf{A} \end{bmatrix}_S\right) = 0, \text{ for } S \in \binom{l+m}{l}. \quad (21)$$

Actually, after some algebraic calculations, it can be seen that the only determinant among a choice of  $\binom{l+m}{l}$  candidates which does not contain linear dependent rows is the following one

$$\det\left(\begin{bmatrix} [(\mathbf{I}_l - \mathbf{A}^+ \mathbf{A}) (\mathbf{K}_u - \bar{\omega} \mathbf{M}_u)]_{r \times l} \\ -\bar{\omega} \mathbf{A}_{m \times l} \end{bmatrix}\right) = 0, \quad (22)$$

where the matrix

$$[(\mathbf{I}_l - \mathbf{A}^+ \mathbf{A}) (\mathbf{K}_u - \bar{\omega} \mathbf{M}_u)]_{r \times l} \quad (23)$$

contains  $r$  independent rows from

$$(\mathbf{I}_l - \mathbf{A}^+ \mathbf{A}) (\mathbf{K}_u - \bar{\omega} \mathbf{M}_u). \quad (24)$$

What is more, it can be seen that the following  $l \times l$  matrix

$$\begin{bmatrix} [(\mathbf{I}_l - \mathbf{A}^+ \mathbf{A}) (\mathbf{K}_u - \bar{\omega} \mathbf{M}_u)]_{r \times l} \\ -\bar{\omega} \mathbf{A}_{m \times l} \end{bmatrix} \quad (25)$$

is invertible, i.e., full rank

$$\text{rank} \begin{bmatrix} [(\mathbf{I}_l - \mathbf{A}^+ \mathbf{A}) (\mathbf{K}_u - \bar{\omega} \mathbf{M}_u)]_{(l-m) \times l} \\ -\bar{\omega} \mathbf{A}_{m \times l} \end{bmatrix} = l = r + m. \quad (26)$$

Thus, we can conclude that  $\bar{\omega}$  is eigenvalue of  $(\bar{\mathbf{K}}_u - \bar{\omega}_j^2 \bar{\mathbf{M}}_u) \bar{\phi}_j = \mathbf{0}$ , if  $\bar{\omega}$  is eigenvalue of

$$\begin{bmatrix} [(\mathbf{I}_l - \mathbf{A}^+ \mathbf{A}) (\mathbf{K}_u - \bar{\omega} \mathbf{M}_u)]_{(l-m) \times l} \\ -\bar{\omega} \mathbf{A}_{m \times l} \end{bmatrix} \bar{\phi}_j = \mathbf{0}. \quad (27)$$

To better understanding this statement it follows a simple but vivid example which is solved through the proposed modal analysis.

## NUMERICAL EXAMPLE

In this section a numerical example is provided to show how simple it is to perform the proposed modal analysis for systems with singular matrices (Udwadia and Phohomsiri 2006; Fragkoulis et al. 2016).

### 2-DOF Underdamped Linear Structural System

Considering the system composed of two masses  $m_1$  and  $m_2$  depicted in Fig. 1, where the first mass  $m_1$  is connected to the ground and to the second mass  $m_2$  through a linear spring in parallel with a linear damper of coefficients  $k_1, C_1$  and  $k_2, C_2$ , respectively.

Selecting  $m_1 = m_2 = 1$ ,  $C_1 = C_2 = 0.1$  and  $K_1 = K_2 = 1$ , it leads to the following two values of undamped natural frequencies:  $\omega_1 = \sqrt{0.38} = 0.616$ ,  $\omega_2 = \sqrt{2.62} = 1.618$ .

Further, selecting the following general assigned conditions  $x_1(0) = 1$ ,  $x_2(0) = -1$ ,  $\dot{x}_1(0) = 2$ ,  $\dot{x}_2(0) = 0$ ,  $f_1(t) = 1$ ,  $f_2(t) = 10 \sin(10t)$ , both time history-displacements  $x_1(t)$  and  $x_2(t)$  are depicted in Fig. 2.

Now, consider the same system as above modeled as a multi-body one composed of two separate subsystems as shown in Fig. 3. The matrix form equilibrium Eqs. (1) are particularized as

$$\mathbf{M}_u = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & m_2 \\ 0 & m_2 & m_2 \end{bmatrix}, \mathbf{C}_u = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_2 \end{bmatrix}, \mathbf{K}_u = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_2 \end{bmatrix}, \quad (28)$$

$$\mathbf{u} = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}, \quad (29)$$

while the assigned conditions are related to those of the original system as

$$\mathbf{u}_0 = \begin{bmatrix} x_1(0) \\ x_1(0) \\ x_2(0) - x_1(0) \end{bmatrix}, \dot{\mathbf{u}}_0 = \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_1(0) \\ \dot{x}_2(0) - \dot{x}_1(0) \end{bmatrix}, \mathbf{f}_u = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_2(t) \end{bmatrix}. \quad (30)$$

It is worth stressing that the relations Eq. (30), between restricted variable system  $x_j(t)$  and the redundant ones  $u_i(t)$ , come out from a clear view of the main system from a physical standpoint. The latter statement emerges absolutely necessary to obtain the restricted variable responses depicted in Fig. 2 from analytical dynamics, although generally this step is ignored in the literature!

Next, consider that the above system is subjected to the following constraints as  $u_2(t) = u_1(t) + l_{1,0} + d = \bar{u}_1(t) + d$ , where  $l_{1,0}$  the unstretched length of the first spring.

Differentiating twice the constraint equation, it is possible to particularize Eq. (3) in the form

$$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (31)$$

Considering the same numerical values as before it leads to

$$\bar{\mathbf{M}}_u = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \bar{\mathbf{C}}_u = \begin{bmatrix} 0.05 & 0 & 0 \\ 0.05 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \end{bmatrix}, \bar{\mathbf{K}}_u = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad (32)$$

$$\bar{\mathbf{f}}_u = \begin{bmatrix} 0.5 + 5\sin(10t) \\ 0.5 + 5\sin(10t) \\ 10\sin(10t) \\ 0 \end{bmatrix}. \quad (33)$$

Based now on Eq. (27), since we have that  $r = 2$ ,  $m = 1$  and  $l = 3$ , we take that

$$[(\mathbf{I}_l - \mathbf{A}^+ \mathbf{A})(\mathbf{K}_u - \bar{\omega} \mathbf{M}_u)]_{2 \times 3} = \begin{bmatrix} \frac{1}{2}(k_1 - \bar{\omega} m_1) & -\frac{1}{2}\bar{\omega} m_2 & -\frac{1}{2}\bar{\omega} m_2 \\ 0 & -\frac{1}{2}\bar{\omega} m_2 & k_2 - \frac{1}{2}\bar{\omega} m_2 \end{bmatrix}, \quad (34)$$

and

$$[-\bar{\omega} \mathbf{A}]_{1 \times 3} = \begin{bmatrix} -\bar{\omega} & \bar{\omega} & 0 \end{bmatrix}. \quad (35)$$

Then,

$$\det \left( \begin{bmatrix} [(\mathbf{I}_l - \mathbf{A}^+ \mathbf{A})(\mathbf{K}_u - \bar{\omega} \mathbf{M}_u)]_{2 \times 3} \\ -\bar{\omega} \mathbf{A}_{1 \times 3} \end{bmatrix} \right) = \det \left( \begin{bmatrix} \frac{1}{2}(k_1 - \bar{\omega} m_1) & -\frac{1}{2}\bar{\omega} m_2 & -\frac{1}{2}\bar{\omega} m_2 \\ 0 & -\frac{1}{2}\bar{\omega} m_2 & k_2 - \frac{1}{2}\bar{\omega} m_2 \\ -\bar{\omega} & \bar{\omega} & 0 \end{bmatrix} \right) = 0. \quad (36)$$

After some algebraic calculations, the following cubic polynomial is derived, i.e.,

$$\frac{1}{2}\bar{\omega} (\bar{\omega}^2 - a\bar{\omega} + b\bar{\omega}) = 0, \quad (37)$$

where, again,  $a = \frac{k_1 m_1 + k_2 m_2 + k_2 m_1}{m_1 m_2}$  and  $b = \frac{k_1 k_2}{m_1 m_2}$ . It is worth underscoring that using this procedure the main frequencies of the original system are recovered, i.e.,  $\tilde{\omega} = \bar{\omega}$ , together with the null frequency that stresses a rigid motion as expected.

Finally replacing the numbers, the eigenvalues  $\bar{\omega}_j^2 = (0, 0.38, 2.62)$  and eigenvectors  $\bar{\phi}_j$  ( $j = 1, 2, 3$ ) of the Eq. (27) have been evaluated.

Introducing these matrices into the system Eq. (4), the response  $\mathbf{u}(t)$  is evaluated and depicted in Fig. 4. Notice that the first two components  $u_1(t)$  and  $u_2(t)$  coincide one another and with  $x_1(t)$  of the original system as expected by a physical point of view. To recover  $x_2(t)$  it needs summing the  $u_2(t)$  and  $u_3(t)$  time histories as shown in Fig. 4.

In particular, considering the modal matrix  $\bar{\Phi}$  ( $3 \times 3$ ) containing the eigenvectors  $\bar{\phi}_j$  as columns,

$$\bar{\Phi} = \begin{bmatrix} 0 & 1.618 & -0.618 \\ 1 & 1.618 & -0.618 \\ 0 & 1 & 1 \end{bmatrix}. \quad (38)$$

the fundamental following relationships hold true

$$\bar{\Omega} = \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{K}}_u \bar{\Phi} = \text{diag} \{0, 0.38, 2.62\}, \quad (39)$$

$$\bar{\Lambda} = \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{C}}_u \bar{\Phi} = \text{diag} \{0, 0.038, 0.262\}. \quad (40)$$

Further, solving the system in Eq. (15), the modal responses  $p_j(t)$  ( $j = 1, 2, 3$ ) (depicted in Fig. 5) are obtained useful for applying the modal transformation  $\mathbf{u}(t) = \bar{\Phi} \mathbf{p}(t)$  that returns the structural response absolutely equal to responses depicted in Fig. 4.

## CONCLUDING REMARKS

The governing equation of motion of complex underdamped mechanical systems (e.g. multi-body systems) are easily formulated decomposing them into a collection of independently modeled subsystems. The solution, getting using analytical dynamics tool, provides not only information

about the motion, but also on the forces of constraint. However proceeding in this way, it is possible to have singular mass matrices so that classical modal analysis is not any more applicable. But in structural design, it is of fundamental importance to know the values of natural frequencies that is the reason for the wide use of modal analysis performing the system response through a superposition of modal responses shaped by the mode shapes itself, as many as the frequencies of the system. In the present paper, if the uniqueness condition is satisfied by the constraints, a proper modified modal analysis is introduced, valid for these systems having singular matrices, leading to the evaluation of the natural frequencies as first step in a simple and straightforward way. Indeed, the proposed formulation fits ideally to the case of linear time invariant underdamped mechanical systems subject to linear constraints. Finally, it should be emphasised that the validity of the resulting methodology we have taken advantage of the fact that the  $\bar{M}_u^+$  is not needed to be calculated analytically. Although the reported example deals with systems with redundant coordinates, the authors underscore the validity of the proposed procedure for systems having singular mass matrix as well. However, due to space limitations, further discussion is omitted.

## ACKNOWLEDGEMENT

We are extremely grateful to the anonymous two reviewers and the associate editor who have afforded us considerable assistance in enhancing both the quality of the findings and the clarity of their presentation. We would like to gratefully acknowledge the support of this work by the EPSRC and ESRC Centre for Doctoral Training on Quantification and Management of Risk & Uncertainty in Complex Systems & Environments (EP/L015927/1).

## REFERENCES

- Antoniou, E. N., Pantelous, A. A., Kougioumtzoglou, I. A., and Pirrotta, A. "Response determination of linear dynamical systems with singular matrices: A polynomial matrix theory approach." *Appl. Math. Model.* (forthcoming).
- Ardema, M. D. (2005). *Analytical dynamics: Theory and applications*. Science & Business Media, Springer-Verlag, New York, USA.

- Bae, D. S., and Haug, E. J. (1987). "A recursive formulation for constraint mechanical system dynamics: Part II Closed loop systems." *Mech. Struct. Mach.*, 15(4), 481-506.
- Ben-Israel, A., and Greville, T. N. E. (2003). *Generalized Inverses: Theory and applications, 2nd Edition*, Springer-Verlag, New York, USA.
- Campbell, S. L., and Meyer, Jr C. D. (1979). *Generalized Inverses of Linear Transformations*, Dover Publications, New York, USA.
- Critchley, J. H., and Anderson, K. S. (2003). "A generalized recursive coordinate reduction method for multi-body dynamic systems." *Int. J. Multiscale Comp. Eng.*, 1(2-3), 181-200.
- De Falco, D., Pennestrì, E., and Vita, L. (2009). "Investigation of the Influence of Pseudoinverse Matrix Calculations on Multibody Dynamics Simulations by Means of the Udwadia-Kalaba Formulation." *J. Aerosp. Eng.*, 22(4), 365-372.
- Di Matteo, A., Lo Iacono, F., Navarra, G., and Pirrotta, A. (2014a). "Direct evaluation of the equivalent linear damping for TLCD systems in random vibration for pre-design purposes." *Int. J. Non Linear Mech.*, 63, 19-30.
- Di Matteo, A., Lo Iacono, F., Navarra, G., and Pirrotta, A. (2014b). "Experimental validation of a direct pre-design formula for TLCD." *Eng. Struct.*, 75, 528-538.
- Featherstone, R. (1987). *Robot Dynamics Algorithms*, Kluwer, New York, USA.
- Fragkoulis, V. C., Kougioumtzoglou, I. A., Pantelous, A. A., and Pirrotta, A. (2015). "Higher order matrix differential equations with singular coefficient matrices." In *Proc. of the Inter. Confer. on Numeric. Anal. and Applied Maths.* (ICNAAM-2014) (Vol. 1648, p. 340002). AIP Publishing.
- Fragkoulis, V. C., Kougioumtzoglou, I. A., and Pantelous, A. A. (2016). "Linear random vibration of structural systems with singular matrices." *ASCE J. Eng. Mech.*, 142(2), 10.1061/(ASCE)EM.1943-7889.0001000, 04015081.
- Garcia de Jalón J. and Guetiérrez-López, M. D. (2013). "Multibody dynamics with redundant constraints and singular mass matrix: existence, uniqueness, and determination of solutions for accelerations and constraint forces." *Multibody Syst. Dyn.* 30(3), 311-341.
- Gashi, B. and Pantelous, A. A. (2013). "Linear backward stochastic differential equations of de-

- scriptor type: regular systems.” *Stoch. Anal. Appl.*, 31(1), 142-166.
- Gashi, B. and Pantelous, A. A. (2015). ”Linear backward stochastic differential systems of descriptor type with structure and applications to engineering.” *Prob. Eng. Mech.*, 40, 1-11.
- Gauss, C. F. (1829). ”Uber ein neues allgemeines Grundgesetz der Mechanik”. *J Reine. Angew. Math.*, 4, 232-235.
- Gohberg, I., Lancaster, P., and Rodman, L. (1986). ”*Invariant subspaces of matrices with applications (Classics in Applied Mathematics)*” (Vol. 51). Society for Industrial and Applied Mathematics (SIAM), 2nd Revised Edition, USA.
- Greville, T. N. E. (1960). ”Some Applications of the Pseudoinverse of a Matrix.” *SIAM Rev.*, 2, 15-22.
- Kalogeropoulos, G. I. and Pantelous, A. A. (2008). ”On Generalized Regular Stochastic Differential Delay Systems with Time Invariant Coefficients.” *Stoch. Anal. Appl.*, 26(5), 1076-1094.
- Kalogeropoulos, G. I., Karageorgos, A. D., and Pantelous, A. A. (2014). ”On the solution of higher order linear homogeneous complex  $\sigma - \alpha$  descriptor matrix differential systems of Apostol-Kolodner type.” *J. Frank. Inst.*, 351(3), 1756-1777.
- Lagrange, J. L. (1787). ”*Mecanique analytique*”. Paris: Mme Ve Courcier, France.
- Laulusa, A., and Bauchau, O. A. (2007). ”Review of classical approaches for constraint enforcement in multibody systems.” *ASME. J. Comput. Nonlinear Dynam.*, 3(1):011004-011004-8. doi:10.1115/1.2803257.
- Mariti, L., Belfiore, N. P., Pennestrì, E., and Valentini, P. P. (2011), ”Comparison of solution strategies for multibody dynamics equations.” *Int. J. Numer. Meth. Eng.*, 88(7), 637-656.
- Pars, L. A. (1979). *A treatise on analytical dynamics*, CT: Ox Bow Press, Woodridge, USA.
- Pradhan, S., Modi, V. J., and Misra, A. K. (1997). ”Order N formulation for flexible multi-body systems in tree topology: Lagrangian approach.” *J. Guid. Control Dyn.*, 20(4), 665-672.
- Roberts, J. B., and Spanos, P. D. (2003). *Random Vibration and Statistical Linearization*, Dover Publications, New York, USA.
- Schiehlen, W. O. (1984). ”Dynamics of complex multibody systems.” *SM Arch.*, 9: 159-195.

- Schutte, A. D., and Udwadia, F. E. (2011). "New approach to the modeling of complex multi-body dynamical systems." *ASME J. Appl. Mech.*, 78(2), 021018.
- Udwadia, F. E., and Di Massa, G. (2011). "Sphere rolling on a moving surface: Application of the fundamental equation of constrained motion." *Simul. Model. Pract. Th.*, 19(4), 1118-1138.
- Udwadia, F. E., and Kalaba, R. E. (1992). "A new perspective on constrained motion." *Proc. R. Soc. A* 439, 407-410.
- Udwadia, F. E., Kalaba, R. E. and Hee-Chang, E. (1997). "Equation of motion for constrained mechanical systems and the extended D'Alembert's principle." *Q. Appl. Math.* LV, 321-331.
- Udwadia, F. E., and Kalaba, R. E. (2001). "Explicit equations of motion for mechanical systems with nonideal constraints." *ASME J. Appl. Mech.* 68, 462-467.
- Udwadia, F. E., and Kalaba, R. E. (2002). "On the foundations of analytical dynamics." *Int. J. Nonlin. Mech.* 37, 1079-1090.
- Udwadia, F. E., and Kalaba, R. E. (2007). *Analytical Dynamics. A New Approach*, Cambridge University Press, New York, USA.
- Udwadia, F. E., and Phohomsiri, P. (2006). "Explicit equations of motion for constrained mechanical systems with singular mass matrices and applications to multi-body dynamics." *Proc. R. Soc. A*, 462, 2097-2117.
- Udwadia, F. E., and Schutte, A. D. (2010). "Equations of motion for general constrained systems in Lagrangian mechanics.", *Acta Mech.*, 213(1), 111-129.
- Udwadia, F. E., and Wanichanon T. (2012). "Explicit Equations of Motion of Constrained Systems with Applications to Multi-body Dynamics.", Springer New York, 315-348.
- Udwadia, F. E., and Wanichanon, T. (2013). "On general nonlinear constrained mechanical systems.", *Numer. Algebra, Contr. Optim.*, 3(3), 425-443.

# List of Figures

1	A two degree-of-freedom linear structural system . . . . .	19
2	Time history-displacement: $x_1(t), x_2(t)$ . . . . .	20
3	Modeling of the system shown in Fig. 1 using more than two coordinates . . . . .	21
4	Time history response: $u_1(t), u_2(t), u_3(t), u_2(t) + u_3(t)$ . . . . .	22
5	Time history modal response: $p_1(t), p_2(t), p_3(t)$ . . . . .	23

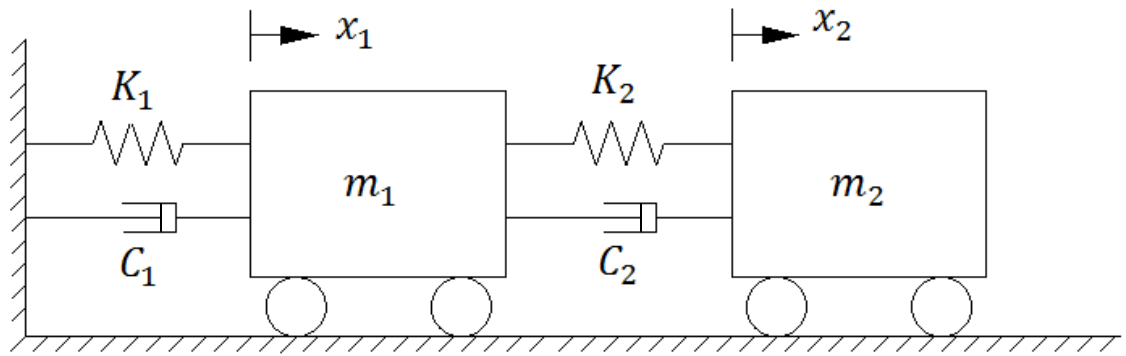


FIG. 1: A two degree-of-freedom linear structural system

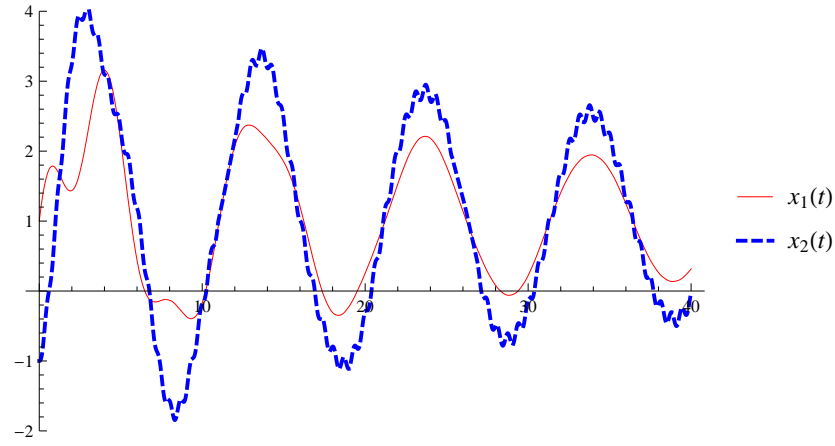


FIG. 2: Time history-displacement:  $x_1(t)$ ,  $x_2(t)$

Sub-system 1

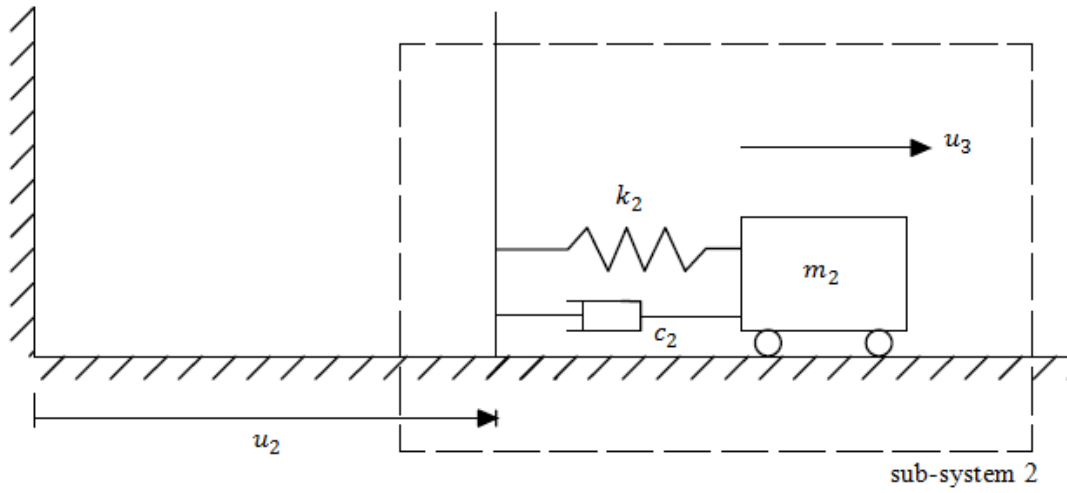
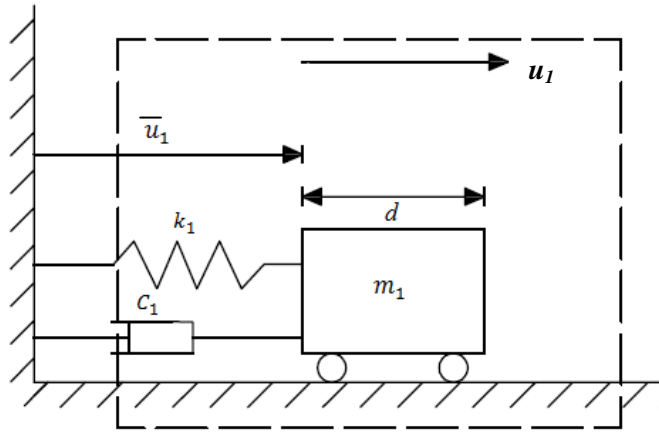


FIG. 3: Modeling of the system shown in Fig. 1 using more than two coordinates

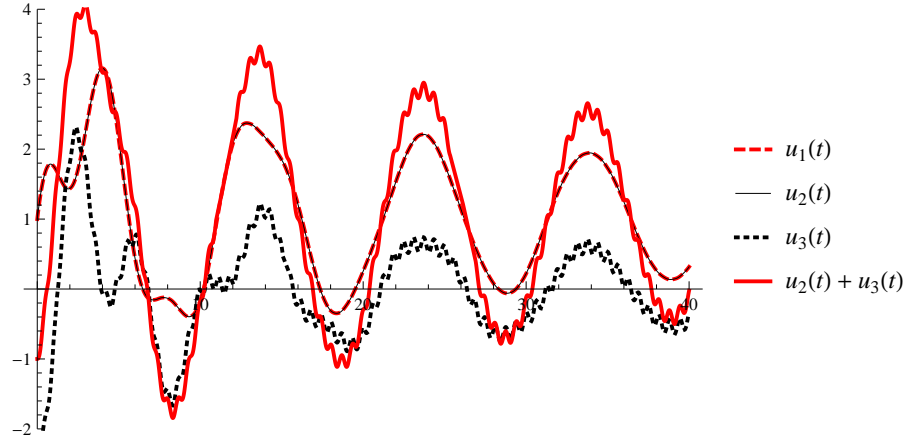


FIG. 4: Time history response:  $u_1(t)$ ,  $u_2(t)$ ,  $u_3(t)$ ,  $u_2(t) + u_3(t)$

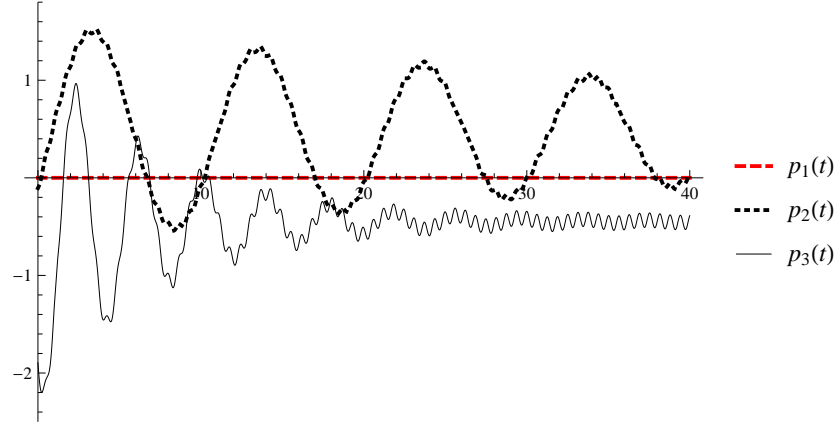


FIG. 5: Time history modal response:  $p_1(t)$ ,  $p_2(t)$ ,  $p_3(t)$