# Cut-and-Project Tilings Constructed From Crystallographic Tilings

Thesis Submitted In Accordance With The Requirements Of
The University of Liverpool For The Degree Of Doctor In Philosophy

by

Hawazin Daif Allah Alzahrani

Faculty of Engineering and Physical Sciences
Department Of Mathematical Sciences

October 5, 2016

# The University of Liverpool

**Hawazin Daif Allah Alzahrani**
**Doctor of Philosophy**
**Mathematical Sciences**
**Cut-and-Project Tilings Constructed From Crystallographic Tilings**
**March, 2016**

# Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

# Copyright Statement

# Contents

# List of Figures

# Acknowledgments

First, and above all, I praise God the almighty for providing me with this opportunity and granting me the capability to proceed, and getting me through various difficulties. I have experienced Your guidance day by day. You are the one who helped me finish my degree, and I will keep trusting You in my future.

This thesis appears in its current form due to the assistance and guidance of several people. I would therefore like to offer my sincere thanks to all of them:

First and foremost, I want to thank my supervisor Dr.Thomas Eckl. It has been an honor to be his first PhD student. He has taught me, both consciously and unconsciously, how good work is done. I appreciate his contributions with regard to time, ideas, and support which made my PhD experience productive and stimulating. I am also thankful for the excellent example he has set as a successful mathematician.

For this PhD thesis, I would like to thank my reading committee members Dr. Alan Haynes and Dr. Oleg Karpenkov for their time, critical review, and being part of this thesis. Thank you.

I would especially like to thank the mathematicians, researchers, and the staff in the mathematics department at Liverpool University. Each one of you have been there to help and support me whenever I needed.

I will never forget my colleagues and friends in the department and those who have lived with me here in the UK. A warm thank you for their support and help throughout the six years that I have been here.

I would also like to thank my family for their love and encouragement. My sisters and brothers, thank you for pushing me forward to finish this dissertation and keeping in touch even when we were far from each other. Thank you.

For my parents who raised me with a love of science and supported me in all my pursuits. Your prayers were what sustained me thus far. Mum and Dad, you have always lavished love on me without expecting reciprocation. I have learned a lot from my parents, from their success and hard times. Yet, I have always respected them most for not giving up. I dedicate this thesis to them.

And most of all, to my loving, supportive, encouraging, and patient husband Naif, whose faithful support during the final stages of this PhD is thoroughly appreciated. He has been an exceptional asset in my life, both personally and professionally. He has taught me to always push the boundaries no matter what the limit. Without his support, I probably would not have made it this far. I love you.

To my beloved little princesses, my daughters, Alhanouf and Asayel: thanks for being with me during my study here in UK, and making my stay here wonderful. I would like to express my thanks to both of you for being such good girls always cheering me up. You are my everything.

Hawazin Alzahrani

2016

# Abstract

Aperiodic "quasi-crystallographic" tilings were first constructed around 1970, most prominently by R.Penrose, and gained interest throughout science when the so-called quasi-crystals were discovered in the mid 80s (and won D. Shechtman a Nobel prize in 2011). Aperiodic tilings turned out to be a useful model for these new molecular structures, in particular explaining their formerly unknown symmetries.

An important method to construct such aperiodic tilings is the method of canonical projection from higher dimensional lattices. Lattices are the orbits of special type of crystallographic groups. For example, Penrose tilings can be obtained from a lattice tiling of $\mathbb{E}^5$, by the cut-and project method.

The main and final aim of the research project will be to develop a mathematical theory of crystallographic tilings and generalize the method of canonical projection to other crystallographic groups than lattices.

Using this method one can hope to construct (interesting), completely new types of aperiodic tilings.

# Chapter 0

# Introduction

## 0.1 Objective of Research

There has been plenty of work done with regard to crystallography. After the discovery of quasi-crystals in the 1980s, crystallographers became interested in other substances found in nature. New areas had been opened in mathematics. Modern crystallographic groups and tiling theory improved and developed from different directions and perspectives.

Penrose tiling is a good example allowing one to think and build upon one's knowledge. In the mid of 70s, Roger Penrose discovered aperiodic tiles, which gained popular attention [22]. Penrose tilings displayed a remarkable rotational symmetry that: every pattern in the tiling appears to rotated by 36 degrees, and in the same manner.

The second direction we should consider is the discovery of quasi-crystals. In the 1980s, Shechtman discovered a new class of solids called quasi-crystals. Quasi-crystals have a structures that are ordered but not periodic. A quasi-crystal pattern can occupy all space, but lacks translational symmetry. These patterns have sharp diffraction patterns and some of them have 8 or 10 folds; (see [29] for details).

Immediately after the discovery of quasi-crystals, scientists noticed that quasi-crystals can be modeled by aperiodic tilings and other cut-and-project tilings; (see the presentation of history in[26]).

Today, after several decades, many kinds of aperiodic crystals have been discovered and many of their properties and structures are known; (see [29] and [26]). These tilings, in turn, can be generated in a number of ways including matching rules, cut-and-project methods, and substitutions.

Tilings were then studied from a dynamical system point of view. Tilings of $\mathbb{E}^n$ generate dynamical systems by moving them around with translations (or more general isometries) of $\mathbb{E}^n$, and closing the resulting orbit with respect to a distance function defined between arbitrary tilings. This began in the 1980s, when people studied substitution tilings in one and two dimensions and then moved to higher dimensional substitution. See [7], [30], [27], and [20] for more details.

In the beginning, the procedure we intend to follow was:
Pick out tiles and project them. This technique works in Penrose tiling, but we have seen lots of problems. For instance, looking at these techniques [10], we see that they have a monograph on cut-and-project (set of points) that are more specific than Delone sets, because simplicity is important. This is followed by an immediate specialization in lattices and projecting those sets of points. A remark that is always made is that we can obtain the tilings and the Voronoi-cell tilings from the set of points, but this was never studied. There are difficulties on this basis; one of these difficulties is that the automorphism group may gets larger (see Chapter 5).

The main question of the thesis is:

**Does the cut-and-project method still work when we replace lattices with more general "crystallographic" tilings, and what are the properties of the resulting cut-and- project tilings?**

The first problem was that surprisingly enough, no notion of "crystallographic tiling" in the following very natural sense seems to exist in the literature.

**Definition 0.1.1** [=Definition 5.1.2]
An isometrically simple tiling $T \subset \mathbb{E}^n$ is crystallographic if its automorphism group $Aut(T)$ in $Isom(\mathbb{E}^n)$ is crystallographic.

Then the automorphism group of a crystallographic tiling should completely determines the equivalence type. However, a reasonable classification theory is not possible with respect to the standard equivalence relation (Translational-Mutual Local Derivability; for short MLD) in [26]. We will see in several examples that two tilings are MLD, but do not have the same automorphism group. To resolve this issue, we introduce **isometrical**-MLD by imposing the taxicab metric $d_O$ on $Isom(\mathbb{E}^n)$ (as in section 1.2), then replacing translations by isometries in the original definition, as we will see in Chapter 4.

**Definition 0.1.2** [=Definition 4.2.9]
Let $T$ and $T'$ two tilings of $\mathbb{E}^n$. Then $T'$ is called *$\gamma$-locally derivable* from $T$ if there exists a radius $R$, such that for $x \in \mathbb{E}^n$ and $\phi \in Isom(\mathbb{E}^n)$, we derive the following:

$$[T]_{B_R(x)} = [\phi(T)]_{B_R(x)} \text{ implies } [T']_{\{\gamma(x)\}} = [\gamma\phi\gamma^{-1}T']_{\{\gamma(x)\}}.$$

If $T$ is $\gamma^{-1}$-locally derivable from $T'$, and $T'$ is $\gamma$-locally derivable from $T$, then $T$ and $T'$ are called *isometrical-MLD*.
Informally this means that the properties of $T'$ at each point $\gamma(x) \in \mathbb{E}^n$ are determined by the properties of $T$ in a ball of some given radius $R$ around $x$.

This definition allows to prove strong results:

**Theorem 0.1.3** [=Theorem 5.2.1]
*For any two crystallographic tilings $T, T'$ of $\mathbb{E}^n$; $T'$ is $\gamma$-LD from $T$, if and only if*

$$\gamma \; Aut(T) \; \gamma^{-1} \subset Aut(T').$$

As a consequence of Theorem 0.1.3, we are able to prove:

**Theorem 0.1.4** [=Theorem 5.2.3 ]
*Two crystallographic tilings are MLD if and only if their automorphism groups are conjugated by an isomorphism.*

Vice versa, there are not more crystallographic groups than crystallographic tilings:

**Theorem 0.1.5** [=Theorem 5.3.1 ]
*For every crystallographic group $\Gamma \subset Isom(\mathbb{E}^n)$, there exists a simple tiling $T$ with $Aut(T) = \Gamma$.*

Up till now, we have constructed crystallographic tilings, which were obviously a generalization of lattice tiling.

A well-known method to construct aperiodic tilings (for example, the Penrose tiling) was the cut-and-project method. This method starts with a lattice in a high-dimensional space, wherein a subset of the tiling is selected and projected to a lower dimensional space.

We can generalize the cut-and-project construction from lattices to these more general crystallographic tilings. The ingredients of the construction are as follows:

A *point set data* $\{(X_i, t_i)_i\}$ of a tiling $T$ is a finite set of points $X_i$ for each prototile $t_i$ that is invariant under the isometry group of $t_i$; (see Definition 6.1.2).

**Theorem 0.1.6** [=Proposition 6.1.3 ]
*Given a point set data* $\{(X_i, t_i)_i\}$ *the point set* $X_T = \bigcup_{t\in T, \gamma(t_i)=t} \gamma(X_i)$ *is a Delone set.*

Here is a brief description of cut-and-project data for the Euclidean space $\mathbb{E}^n$:
Let $E \subset \mathbb{E}^n$ be an $m$-dimensional projection subspace, $E^\perp \subset \mathbb{E}^n$ be the orthogonal complement, and fix a *window* $K \subset E^\perp$. Then the projection to $E$ for the intersection of $X_T$ with the cylinder $K \times E$, yields $X_{T'}$ (see Section 6.2).

**Theorem 0.1.7** [=Theorem 6.2.11]
$X_{T'}$ *is a Delone set.*

**Theorem 0.1.8** [=Theorem 6.2.12 ]
*The Voronoi-cell tiling* $VT(X_{T'})$ *associated to the Delone set* $X_{T'}$ *is simple tiling.*

Note that we first tried a different procedure to construct a cut-and-project tiling: We picked certain faces of tiles and projected them. This technique works in the case of Penrose tilings but in more general settings, lots of problems occurred: In particular, the projected tiles intersect each other or they leave gaps on the projection subspace.

In monographs like [10] on the cut-and-project method, only Delone sets are projected, and immediately specialized to orbits of lattices. Usually a remark is added that tilings can be obtained from the point sets by using e.g. Voronoi-cell decomposition, but the properties of the resulting tilings are rarely studied, let alone proven. For example, the phenomena that Voronoi-cell tilings can have larger automorphism groups than the Delone sets from which they are constructed (see Section 6.1), is never discussed.

We conclude this section by highlighting some open questions about tilings constructed by the cut-and-project method from crystallographic tilings:

(i) What are the hulls of the cut-and-project tilings?
One possibility could be that it is (a quotient of) the hull of the projected crystallographic tiling, since moving around this tiling together with window $K$ and projection subspace $E$ will move around the cut-and-project tiling on $E$.

(ii) How does the orbit of the cut-and-project tiling lie inside the hull?
If the orbit is dense but nowhere open, the cut-and-project tiling could be a candidate for an aperiodic tiling.

(iii) Do we obtain MLD cut-and-project tilings when we change to crystallographic tilings with the same automorphism group, respectively choose different point set data?

## 0.2 Thesis Outline

The thesis structure is as follows:

Chapter 1 collects definitions, notations, concepts, constructions and techniques mainly from Euclidean geometry, group theory and topology. In this chapter, we look at the groups $Trans(\mathbb{E}^n)$, $\mathbb{E}^n$ and $Isom(\mathbb{E}^n)$ in the Euclidean space, and we define several topologies on them which turned out to be the same [17] ,[14]. Bieberbach's Theorem is also presented [9] as it is an important tool for our work in the following chapters. We also introduce the Taxicab metric and discuss its properties. Finally, in this chapter, we define and discuss first properties of Delone set, convex hulls and polytopes.

In Chapter 2, we define tilings in general, followed by simple tiling. Voronoi-cell tiling is constructed from a crystallographic group by Voronoi-cell decomposition of orbits. Then, we prove that the constructed Voronoi-cell tiling is simple.
It is important to know that (in this chapter and the following ones), even if results and examples look familiar, it is difficult to find proofs and constructions in the literature, so we provide or expand them ourselves.

Within Chapter 3, we discuss the metric on the tiling space $\Omega_T$. Important to mention that this metric is defined using isometries, not just in the case of translation as in [26]. Tiling spaces and its hull were also discussed.

Chapter 4 is dedicated to studying in details the equivalence relation between tilings, and also between tiling spaces. We define isometrically mutual local derivability and discuss its properties in tilings and tiling spaces.

In Chapter 5, we introduce the definition of crystallographic tiling. Then, we give an example by calculating the automorphism group of $T$ to demonstrate whether $Aut(T)$ is crystallographic or not. A relation between two crystallographic tilings is discovered wherein two crystallographic tilings are MLD if and only if their automorphism groups are conjugated by an isomorphism. Finally, we construct a crystallographic tiling by proving that, for every crystallographic group, $\Gamma \subset Isom(\mathbb{E}^n)$, there exists a simple tiling $T$ with $Aut(T) = \Gamma$.

Within Chapter 6, we describe a more general construction of cut-and-project tilings from an arbitrary given crystallographic tiling T not only lattices, and given cut-and-project data (that is, projection subspace and window), in detail.

# Chapter 1

# Mathematical Preliminaries

In this chapter, we will illustrate basic definitions and notations which will provide the background for our work in the following chapters. This has been constructed through several chapters from [8], [9], [23], [16], and [21].

To understand the context of this work, it was important to devote a great deal of time and effort to acquire a solid foundation in crystallographic groups. This was achieved through a combination of background reading and work [31],[28], [2], and[32].

## 1.1 Crystallography

In crystallography, a crystal should have a bounded pattern that is repeated until it fills up space. This repeating cell can be described as a covering domain. Our main sources for the theoretical material presented in this section are [9], [33] and [3].

Crystallography is no more than the study of certain permutation groups on points of Euclidean space. We begin by investigating the permutations of $\mathbb{R}^n$: translations and isometries in general.

### 1.1.1 Affine Space and Euclidean Space

An affine space is a geometric structure that generalizes the properties of Euclidean space independent of the concepts of distance. Also, it has no distinguished point that serves as an origin.

**Notation 1.1.1** (i) $\mathbb{E}^n = \{(x_1, x_2, ..., x_n) \mid \quad x_1, ...., x_n \in \mathbb{R}\}$ is a set of points with no additional structure.

(ii) $Perm(\mathbb{E}^n)$ is the group of $1-1$ maps/permutations, with group composition given by the composition of maps.

(iii) $Trans(\mathbb{E}^n)$ is the group of translations which is a subgroup of $Perm(\mathbb{E}^n)$: For each $(\tau_1, ..., \tau_n) \in \mathbb{R}^n$, there is a translation $\tau : \mathbb{E}^n \to \mathbb{E}^n$ such that $\tau : (x_1, ..., x_n) \to (x_1 + \tau_1, ..., x_n + \tau_n)$.

Notice that $\tau$ is a $1-1$ map, as the inverse of $\tau$ is given by $(-\tau_1, ..., -\tau_n)$.

**Fact 1.1.2** $Trans(\mathbb{E}^n)$ is an $n$-dimensional real vector space, together with two binary operations (addition and scalar multiplication), which are defined as follows:

- $s + \tau = (s_1 + \tau_1, ..., s_n + \tau_n)$ ; $s, \tau \in Trans(\mathbb{E}^n)$.
- $\lambda\tau = (\lambda\tau_1, ..., \lambda\tau_n)$ ; $\lambda \in \mathbb{R}$ and $\tau \in Trans(\mathbb{E}^n)$.

**Definition 1.1.3** *Affine space* is defined as the set $\mathbb{E}^n$, together with its group of translations, $Trans(\mathbb{E}^n)$.

This description of space does not specify a basis. It does not even pick out an origin. We can clarify this situation through the following construction:

**Construction 1.1.4** Fixing a particular point $O \in \mathbb{E}^n$ via the evaluation map $ev_O : Trans(\mathbb{E}^n) \to \mathbb{E}^n$, given by $\tau \to \tau(O)$, so $0 \to O$. Consequently, this map endows $\mathbb{E}^n$ with an origin $O$, symbolized by $\mathbb{E}^n_O$. Addition and scalar multiplication in $\mathbb{E}^n_O$ are derived from the evaluation map $ev_O$.

Take two points $O$ and $O'$ in $\mathbb{E}^n$. $\mathbb{E}^n_O$ is a vector space obtained from $\mathbb{E}^n$ by fixing the point $O$ via the evaluation map $ev_O$, which is given by $\tau \to \tau(O)$. A similar process is undertaken for $\mathbb{E}^n_{O'}$. Then, we can obtain an isomorphism between $\mathbb{E}^n_O$ and $\mathbb{E}^n_{O'}$ by:

$$\mathbb{E}^n_O \xleftarrow{\cong} Trans(E^n) \xrightarrow{\cong} \mathbb{E}^n_{O'}$$

Here, if $\tau \in Trans(\mathbb{E}^n)$, we can describe the map $\mathbb{E}^n_O \to \mathbb{E}^n_{O'}$ as sending $\tau(O) \to \tau(O')$. Let $\overrightarrow{OO'} \in Trans(\mathbb{E}^n)$ denote the unique translation sending $O$ to $O'$. Then,

$$\overrightarrow{OO'}\big(\tau(O)\big) = (\overrightarrow{OO'} + \tau)(O) = (\tau + \overrightarrow{OO'})(O) = \tau(O').$$

This means that $\overrightarrow{OO'} : \mathbb{E}^n_O \to \mathbb{E}^n_{O'}$ is a vector space isomorphism.

In particular,

$$\overrightarrow{OO'}(P +_O Q) = \overrightarrow{OO'}(P) +_{O'} \overrightarrow{OO'}(Q)$$

and

$$\overrightarrow{OO'}(\lambda \cdot_O P) = \lambda \cdot_{O'} \overrightarrow{OO'}(P)$$

where $\lambda \in \mathbb{R}$ and $P, Q \in \mathbb{E}^n$.

**Definition 1.1.5** If $f \in Aff(\mathbb{E}^n)$, the map $ad\ f : Trans(\mathbb{E}^n) \to Trans(\mathbb{E}^n)$ is defined as

$$(ad\ f)(\tau) = f\tau f^{-1}\ ; \tau \in Trans(\mathbb{E}^n)\,.$$

Notice that $ad\ f(\tau)$ is really a map $Trans(\mathbb{E}^n) \to Trans(\mathbb{E}^n)$. This follows easily from $Trans(\mathbb{E}^n) \lhd Aff(\mathbb{E}^n)$; see [9] for more details.

**Notation 1.1.6**

- For $s, \tau \in Trans(\mathbb{E}^n)$; $\langle s, \tau\rangle = (s_1, ..., s_n)\cdot(\tau_1, ..., \tau_n) = \sum_{i=1}^n s_i\tau_i$ we equip $Trans(\mathbb{E}^n)$ with an inner product $\langle\cdot,\cdot\rangle$. We can induce an inner product $\langle\cdot,\cdot\rangle_O$ on every $\mathbb{E}^n_O$ via the evaluation map $ev_O$.
- For $\tau \in Trans(\mathbb{E}^n)$, we define the norm on $\tau$ by:

$$\parallel \tau \parallel = \sqrt{\langle\tau,\tau\rangle}.$$

- Similarly as in $Trans(\mathbb{E}^n)$, the norm on $\mathbb{E}^n_O$ induced by $ev_O : Trans(\mathbb{E}^n) \to \mathbb{E}^n$.

Now, we can introduce distance on $\mathbb{E}^n$ using the norms above:
For $P, Q \in \mathbb{E}^n$, we define the distance by

$$d(P,Q) = \parallel P - Q \parallel\,.$$

Notice that, the norm on $\mathbb{E}^n_O$, and the norm topology on $\mathbb{E}^n_{O'}$ induce the same distance on $\mathbb{E}^n_O$ and $\mathbb{E}^n_{O'}$:

It suffices to show that for $P, Q \in \mathbb{E}^n$ $\parallel P -_O Q \parallel_O = \parallel P -_{O'} Q \parallel_{O'}$:
Suppose $\tau \in Trans(\mathbb{E}^n)$ $s.t.$ $\tau(O) = O'$. Then,
$\parallel R \parallel_O = \parallel \tau(R) \parallel_{O'} = \parallel \tau(O) - O + R \parallel_{O'}$, hence

$$\begin{aligned} \parallel P -_O Q \parallel_O &= \parallel (P - O) - (Q - O) + O \parallel_O \\ &= \parallel P - Q + O \parallel_O \\ &= \parallel \tau(O) - O + P - Q + O \parallel_{O'} \\ &= \parallel P - Q + O' \parallel_{O'} \\ &= \parallel P -_{O'} Q \parallel_{O'}\,. \end{aligned} \tag{1.1}$$

From now on, the distance on $\mathbb{E}^n$ agrees with the norm on each $\mathbb{E}^n_O$. In particular, for two points $P, Q \in \mathbb{E}^n$, we define $\| P - Q \| := \| P -_O Q \|$, independently of $O$.

**Remark 1.1.7** The orthogonal group of $Trans(\mathbb{E}^n)$, expressed as $O(Trans(\mathbb{E}^n))$, is the group of all linear maps $\theta : Trans(\mathbb{E}^n) \to Trans(\mathbb{E}^n)$, which preserve the inner product, that is $\langle \theta(\tau), \theta(p) \rangle = \langle \tau, p \rangle \ ; \ \tau, p \in Trans(\mathbb{E}^n)$.

**Notation 1.1.8** $Isom(\mathbb{E}^n)$ denotes the group of all isometries on $\mathbb{E}^n$, that are the distance-preserving affine maps on $\mathbb{E}^n$.
Note that $Isom(\mathbb{E}^n) \cong Trans(\mathbb{E}^n) \rtimes O(\mathbb{E}^n_O)$, for any point $O \in \mathbb{E}^n$.

### 1.1.2 The topological spaces $Trans(\mathbb{E}^n), \mathbb{E}^n, \mathbb{E}^n_O$ and $Isom(\mathbb{E}^n)$

Working through different chapters in[21], [32] and [13], we introduced different topologies on $Trans(\mathbb{E}^n), \mathbb{E}^n, \mathbb{E}^n_O$ and $Isom(\mathbb{E}^n)$ by describing open and closed sets, and converging sequences on the topologies of these spaces, and defining its basis.

Even if theses topologies turn out to be the same, it is useful to have them available. One reason for this is that it will be easier to view and check results and proofs.

#### The norm topology on $Trans(\mathbb{E}^n)$

The vector space $Trans(\mathbb{E}^n)$ has a norm topology or strong topology, derived from its inner product. See Notation 1.1.6.

**Definition 1.1.9** For $\tau \in Trans(\mathbb{E}^n)$, we define the norm of $\tau$ by:

$$\| \tau \| = \sqrt{\langle \tau, \tau \rangle}.$$

**Definition 1.1.10** For the normed space $(Trans(\mathbb{E}^n), \|\|)$, we define the following:

(i) Given $\tau \in Trans(\mathbb{E}^n)$, the *open ball* centered at $\tau$ with radius $\epsilon > 0$ is the set:
$$B(\tau, \epsilon) = \{\tau' \in Trans(\mathbb{E}^n) \quad | \; \| \tau - \tau' \| < \epsilon\}.$$

(ii) $U$ is *open set* in $Trans(\mathbb{E}^n)$ if:

$$\forall \tau \in U \quad \exists \epsilon > 0 \quad s.t. \quad B(\tau, \epsilon) \subseteq U.$$

(iii) $W$ is closed $\Longleftrightarrow W^c$ is open,
where $W^c$ is the complement of $W$, $W^c = Trans(\mathbb{E}^n) \setminus W$.

The collection $\tau$ of open sets in $Trans(\mathbb{E}^n)$ forms a topology on $Trans(\mathbb{E}^n)$, called the *norm topology*.

**Definition 1.1.11** A sequence $(\tau_n) \in Trans(\mathbb{E}^n)$ converges in norm topology to $\tau \in Trans(\mathbb{E}^n)$, expressed by $\tau_n \longrightarrow_n \tau$ if:

$$\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} \quad s.t. \quad \| \tau_n - \tau \| < \epsilon \;\; \forall n \geq N \; .$$

Similarly in $\mathbb{E}^n_O$ for any origin $O \in \mathbb{E}^n$.

**The norm topology on $\mathbb{E}^n$**

Open and close sets in $\mathbb{E}^n$ are defined in the same way as in Definition 1.1.10 with respect to the norm on $\mathbb{E}^n_O$ induced by $ev_O$. This does not depend on the choice of $O$.

The norm topology on $\mathbb{E}^n_O$, $O$-topology agrees with the norm topology on $\mathbb{E}^n_{O'}$, $O'$-topology. In other words, the norm topologies on $\mathbb{E}^n_O$ and $\mathbb{E}^n_{O'}$ define the same topology on $\mathbb{E}^n$.

**The topology of pointwise convergence on $Trans(\mathbb{E}^n)$**

**Definition 1.1.12** Considering $\tau \in Trans(\mathbb{E}^n)$ as a map $\tau : \mathbb{E}^n \to \mathbb{E}^n$, we say that a sequence $(\tau_n)$ *converges pointwise* to $\tau$, as expressed by $\tau_n \longrightarrow_{pc} \tau$ if

$$\forall \epsilon > 0 \; \forall P \in \mathbb{E}^n \; \exists N \in \mathbb{N} \quad s.t. \quad \| \tau_n(P) - \tau(P) \| < \epsilon \; \forall n \geq N \; .$$

Closed sets in pointwise topology of $Trans(\mathbb{E}^n)$ are defined using sequences.

**Definition 1.1.13** For pointwise topology of $Trans(\mathbb{E}^n)$, we define:

(i) A set $F$ in $Trans(\mathbb{E}^n)$ is *closed* if:

$$\forall \tau_n \longrightarrow_{pc} \tau \;\; ; \;\; (\tau_n) \subset F \implies \tau \in F \; .$$

(ii) A set $U$ is *open* $\Longleftrightarrow U^c$ is closed.

**Topology of uniform convergence on $Trans(\mathbb{E}^n)$**

**Definition 1.1.14** Considering $\tau \in Trans(\mathbb{E}^n)$ as a map $\tau : \mathbb{E}^n \to \mathbb{E}^n$, we say that a sequence $(\tau_n)$ in $Trans(\mathbb{E}^n)$ converges *uniformly* to $\tau$, as expressed by $\tau_n \longrightarrow_{uc} \tau$ if:

$$\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} \;\; s.t. \;\; \| \tau_n(P) - \tau(P) \| < \epsilon \;\; \forall n \geq N, \;\; \forall P \in E^n.$$

**Theorem 1.1.15** *The topologies of pointwise convergence and uniform convergence of $Trans(\mathbb{E}^n)$ agree with the norm topology of $Trans(\mathbb{E}^n)$.*

Proof We aim to show that the following are equivalent:

(i) $\tau_n \longrightarrow_n \tau$.

(ii) $\tau_n \longrightarrow_{uc} \tau$.

(iii) $\tau_n \longrightarrow_{pc} \tau$.

(i) $\Longrightarrow$ (ii) If $\tau_n \longrightarrow_n \tau$ in $Trans(\mathbb{E}^n)$, then $\tau_n(O) \longrightarrow \tau(O)$ in $\mathbb{E}^n_O$. Since $s(P) = s(O) + P$ for any translation $s \in Trans(\mathbb{E}^n)$ and any $P \in \mathbb{E}^n_O$, we see that:

$$\| \tau_n(P) - \tau(P) \|_O = \| \tau_n(O) + P - \tau(O) - P \|_O = \| \tau_n(O) - \tau(O) \|_O .$$

Consequently, $\tau_n \longrightarrow_{uc} \tau$ in the topology of uniform convergence for $E^n_O$.
(ii) $\Longrightarrow$ (iii) If $\tau_n \longrightarrow_{uc} \tau$, then $\tau_n(P) \longrightarrow \tau(P)$ for any point $P \in \mathbb{E}^n_O$, implies $\tau_n \longrightarrow_{pc} \tau$ in $Trans(\mathbb{E}^n)$.
(iii) $\Longrightarrow$ (i) If $\tau_n \longrightarrow_{pc} \tau$, then $\tau_n(O) \longrightarrow \tau(O)$ in $\mathbb{E}^n_O$. So, $\tau_n \longrightarrow_n \tau$, as $ev_O$ preserves the norms on $Trans(\mathbb{E}^n)$ and $\mathbb{E}^n_O$.
Hence, the topologies of pointwise convergence and uniform convergence of $Trans(\mathbb{E}^n)$ agree with the norm topology of $Trans(\mathbb{E}^n)$. □

Henceforth, we do not distinguish between norm topology, the topology of pointwise convergence, and the topology of uniform convergence on $Trans(\mathbb{E}^n)$. Also, we simply write $\tau_n \longrightarrow \tau$ for a sequence $\tau_n$ converging to $\tau$ in $Trans(\mathbb{E}^n)$.

**The topology on $O(\mathbb{E}^n_O)$**

The topology on $O(\mathbb{E}^n_O)$ induced by the *operator norm* on $O(\mathbb{E}^n_O) \subset End(\mathbb{E}^n_O)$. That is $\| a \|_{op} = \sup\{\frac{\|a(w)\|}{\|w\|} \mid w \neq 0\}$ ; $a \in O(\mathbb{E}^n_O)$ and $w \in \mathbb{E}^n_O$. Notice that open and closed sets are defined in the same way as in Definition 1.1.10 (with respect to operator norm).

**The topologies on $Isom(\mathbb{E}^n)$**

The topology on $Isom(\mathbb{E}^n)$ is the product topology on the set $Trans(\mathbb{E}^n) \rtimes O(\mathbb{E}^n_O)$ having as basis the collection of all open sets of the form $U \times V$, where $U$ is an open subset of $Trans(\mathbb{E}^n)$ as in Definition 1.1.13, and $V$ is an open subset of $O(\mathbb{E}^n_O)$ as in Definition 1.1.10.

We also can define the topology of pointwise convergence on $Isom(\mathbb{E}^n)$ in the same way as in the section about (the topology of pointwise convergence on $Trans(\mathbb{E}^n)$).

**Lemma 1.1.16** *The topology of pointwise convergence on $Isom(\mathbb{E}^n)$ coincides with the product topology on $Trans(\mathbb{E}^n) \rtimes O(\mathbb{E}^n_O)$.*

In particular, the product topology on $Isom(\mathbb{E}^n)$ is induced by $Trans(\mathbb{E}^n) \rtimes O(\mathbb{E}^n_O)$ and $Trans(\mathbb{E}^n) \rtimes O(\mathbb{E}^n_{O'})$ for different $O, O' \in \mathbb{E}^n$.

### 1.1.3 Crystallographic Groups and Bieberbach's Theorem

We will now introduce crystallographic groups. The following definition agrees with our understanding of crystals.

**Definition 1.1.17** A subgroup $\Gamma$ of $Isom(\mathbb{E}^n)$ is called *crystallographic group* if it is discrete and $Isom(\mathbb{E}^n)/\Gamma$ is compact.

Notice that a subgroup $\Gamma$ of $Isom(\mathbb{E}^n)$ is called a *discrete subgroup*, if for a sequence $(y_n) \subset \Gamma$ and $y \in \Gamma$ such that $y_n \longrightarrow y$, the sequence of $y_n$ is eventually constant.
From "crystals", point of view, "discrete" should mean that the crystal pattern never bunches up, that is, there are no accumulation points in the orbit of a point.

**Remark 1.1.18** It is enough to test that $(y_n)$ is eventually constant for $y = id_{\mathbb{E}^n}$, the identity map in $Isom(\mathbb{E}^n)$.

**Example 1.1.19** Crystallographic groups in two dimensions are called wallpaper groups, whereas crystallographic groups in three dimensions are called space groups.

The bridge between geometry and algebra is found in the following remarkable theorem as stated by Bieberbach.

**Theorem 1.1.20** [[9]; Bieberbach's Theorem, pp.532]
*Let $\Gamma$ be a crystallographic group of $Isom(\mathbb{E}^n)$. Then, $\Gamma$ satisfies the following two conditions:*

**(1)** *$\Gamma \cap Trans(\mathbb{E}^n)$ is a finitely generated abelian group of rank $n = dim(\mathbb{E}^n)$, which spans $Trans(\mathbb{E}^n)$ as a vector space.*

**(2)** *ad $\Gamma \cong \Gamma/\Gamma \cap Trans(\mathbb{E}^n)$; the point group of $\Gamma$ is finite (see Definition 1.1.5 for the meaning of ad).*

By adding some translations to $\Gamma$ we obtain a crystallographic group, the so-called *symmorph* of $\Gamma$, which is the product of its translations and orthogonal maps making up the point group of $\Gamma$:

**Proposition 1.1.21** [[9], p.534-535]
*For every crystallographic group $\Gamma \subset Isom(\mathbb{E}^n)$ there exists a crystallographic group $\Gamma^*$ such that*

*(i)* *$\Gamma^* \cap Trans(\mathbb{E}^n)$ is a lattice of full rank containing $\Gamma \cap Trans(\mathbb{E}^n)$.*

*(ii)* *the point group of $\Gamma^*$ is isomorphic to the point group of $\Gamma$, and*

*(iii)* *there is a point $O \in \mathbb{E}^n$ and a subgroup $G \subset \Gamma^* \cap O(\mathbb{E}^n_O)$ isomorphic to the point group of $\Gamma^*$ such that*

$$\Gamma^* \cong (\Gamma^* \cap Trans(\mathbb{E}^n)) \rtimes G \subset Trans(\mathbb{E}^n) \rtimes O(\mathbb{E}^n_O) = Isom(\mathbb{E}^n)\,.$$

### 1.1.4 Crystallographic Restriction for Crystallographic Groups

The cornerstone of the theory of crystallographic groups is the Crystallographic Restriction Theorem. See Chapter 3.2 in [19] for more details.

A crystallographic group $\Gamma \subset Isom(\mathbb{E}^n)$ can only contain finite cyclic groups $C_k$ where the order $k$ is an element of a finite set of integers only depending on $n$. If $n = 2, 3$ then $k \in \{1, 2, 3, 4, 6\}$.

**Theorem 1.1.22** [[19], Crystallographic Restriction Theorem, p.49-53]
*Let $\Gamma \subset Isom(\mathbb{E}^n)$ be a crystallographic group, and let $R \in O(\mathbb{E}^n_O)$ be an orthogonal map fixing the origin $O \in \mathbb{E}^n$. Then $R \in \Gamma$ implies that the characteristic polynomial $p(\lambda) = det(R - \lambda 1_{\mathbb{E}^n_O})$ has integers coefficient only, that is $p(\lambda) \in \mathbb{Z}[\lambda]$.*

This theorem will not be used later on. On the other hand, it shows that the cut-and-project method may produce tilings or point sets in $\mathbb{E}^n$ that have symmetries allowed by the Crystallographic Restriction Theorem only in $\mathbb{E}^N$ for $N > n$.

**Theorem 1.1.23** [[19], Theorem3.2, page 52]
*Consider a locally finite planar point set with n-fold symmetry that is constructed from a lattice in $\mathbb{R}^d$ by a symmetry preserving (partial) projection. Then, d is bounded from below by a number only depending on n.*

The cut-and-project tilings from crystallographic tilings discussed in Chapter 6 should allow to further decrease the lower bound in this theorem. This is left to further investigations.

## 1.2 Taxicab Metric On $Isom(\mathbb{E}^n)$

Before defining a metric on tilings, we introduce a metric $d_O$ on the isometries $Isom(\mathbb{E}^n) \subset Trans(\mathbb{E}^n) \rtimes End(\mathbb{E}^n_O)$ for each center $O \in \mathbb{E}^n$. We can choose the taxicab metric:

$$d_O(\phi, \psi) = \|\tau - \sigma\|_{Eucl} + \|b - a\|_{op} \ ,$$

derived from the Euclidean metric on $Trans(\mathbb{E}^n)$ and the operator norm on $End(\mathbb{E}^n_O)$; where $\phi, \psi \in Isom(\mathbb{E}^n)$ such that $\phi = \tau a, \psi = \sigma b$.

Some properties of $\| \cdot \|_{op}$ and $d_O$ are presented in the following two lemmas:

**Lemma 1.2.1** *(i)* $\|a\|_{op} = 1$ *;* $a \in O(\mathbb{E}^n_O)$.

*(ii)* $\|a \cdot b\|_{op} \leq \|a\|_{op} \cdot \|b\|_{op}$*;* $a, b \in End(\mathbb{E}^n_O)$*, with equality if* $a \in O(\mathbb{E}^n_O)$.

*(iii)* $\|aba^{-1} - id_{\mathbb{E}^n}\|_{op} = \|b - id_{\mathbb{E}^n}\|_{op}$ *;* $a, b \in O(\mathbb{E}^n_O)$.

*(iv)* $\|ab - id_{\mathbb{E}^n}\|_{op} \leq \|a - id_{\mathbb{E}^n}\|_{op} + \|b - id_{\mathbb{E}^n}\|_{op}$ *;* $a, b \in O(\mathbb{E}^n_O)$.

*(v)* $a \cdot \sigma = \overrightarrow{Oa\big(\sigma(O)\big)} \cdot a$ *;* $a \in O(\mathbb{E}^n_o)$ *and* $\sigma \in Trans(\mathbb{E}^n)$*. Here,* $\overrightarrow{Oa\big(\sigma(O)\big)}$ *is the translation from* $O$ *to* $a\big(\sigma(O)\big)$.

Proof (i) By definition of the operator norm and the orthogonal group, as $a \in O(\mathbb{E}^n_O)$ preserves distances.

(ii) By definition, we also see that $\|a \cdot b\|_{op} \leq \|a\|_{op} \cdot \|b\|_{op}$ if $a, b \in End(\mathbb{E}^n_O)$, and $\|a \cdot b\|_{op} = \|a\|_{op} \cdot \|b\|_{op}$ if furthermore $a \in O(\mathbb{E}^n_O)$.

(iii) Notice that,

$$\begin{aligned} \|aba^{-1} - id_{\mathbb{E}^n}\|_{op} &= \|aba^{-1} - a \cdot id_{\mathbb{E}^n} \cdot a^{-1}\|_{op} \\ &= \|a\|_{op} \cdot \|b - id_{\mathbb{E}^n}\|_{op} \cdot \|a^{-1}\|_{op} \\ &= \|b - id_{\mathbb{E}^n}\|_{op} \ ; \quad \text{as } a \in O(\mathbb{E}^n_O). \end{aligned} \tag{1.2}$$

(iv) Notice that,

$$\begin{aligned} \|ab - id_{\mathbb{E}^n}\|_{op} &= \|ab - a + a - id_{\mathbb{E}^n}\|_{op} \\ &\leq \|a\|_{op} \cdot \|b - id_{\mathbb{E}^n}\|_{op} + \|a - id_{\mathbb{E}^n}\|_{op} \\ &\leq \|a - id_{\mathbb{E}^n}\|_{op} + \|b - id_{\mathbb{E}^n}\|_{op} \ ; \quad \text{as } a \in O(\mathbb{E}^n_O). \end{aligned} \tag{1.3}$$

(v) Choose $p \in \mathbb{E}^n$. Let $A$ be the linear map in $GL(\mathbb{R}^n)$, describing $a$ as a linear map with center $O$; we have:

$$\begin{aligned}
(a \cdot \sigma)(p) = a(p+\sigma) &= A(p+\sigma-O)+O \\
&= A(p-O)+A\cdot\sigma+O \\
&= A(p-O)+O+A\cdot\sigma \\
&= a(p)+A\cdot\sigma \\
&= A\cdot\sigma+a(p) \\
&= A\big(\sigma(O)-O\big)+a(p) \\
&= \big(A\cdot\big(\sigma(O)-O\big)+O-O\big)\big(a(p)\big) \\
&= \overrightarrow{Oa\big(\sigma(O)\big)}\big(a(p)\big).
\end{aligned} \tag{1.4}$$

Hence, $a \cdot \sigma = \overrightarrow{Oa\big(\sigma(O)\big)} \cdot a$ . □

**Lemma 1.2.2** *(i)* $d_O(a\cdot\sigma, id_{\mathbb{E}^n}) = d_O(\sigma\cdot a, id_{\mathbb{E}^n})$.

*(ii)* $d_O(\chi^{-1}, id_{\mathbb{E}^n}) = d_O(\chi, id_{\mathbb{E}^n})$.

*(iii)* $d_O(\chi\cdot\psi, id_{\mathbb{E}^n}) \leq d_O(\chi, id_{\mathbb{E}^n}) + d_O(\psi, id_{\mathbb{E}^n})$.

*(iv)* $d_O(\sigma,\tau) = \|\sigma-\tau\|_{Eucl}$ ; $\sigma,\tau \in Trans(\mathbb{E}^n)$.

*(v)* $d_{\sigma(O)}(\chi\phi\chi^{-1}, \chi\psi\chi^{-1}) = d_O(\phi,\psi)$; $\chi,\phi,\psi \in Isom(\mathbb{E}^n)$ .

*(vi)* $d_O(\chi\psi\chi^{-1}, id_{\mathbb{E}^n}) \leq d_O(\psi, id_{\mathbb{E}^n})\big(1+d_O(\chi, id_{\mathbb{E}^n})\big)$.

*(vii)* $d_{\sigma(O)}(\phi,\psi) \leq (1+\|\sigma\|_{Eucl})d_O(\phi,\psi)$ *for all* $\sigma \in Trans(\mathbb{E}^n), \phi,\psi \in Isom(\mathbb{E}^n)$.

Proof (i) Notice that,

$$\begin{aligned}
d_O(a\cdot\sigma, id_{\mathbb{E}^n}) &= d_O(\overrightarrow{Oa\big(\sigma(O)\big)}\cdot a, id_{\mathbb{E}^n});\ \text{by Lemma 1.2.1 (v)} \\
&= \|a\big(\sigma(O)\big)-O\|_{Eucl}+\|a\|_{op} \\
&= \|\sigma\|_{Eucl}+\|a\|_{op};\ \text{since } a \in O(\mathbb{E}^n_O) \\
&= d_O(\sigma\cdot a, id_{\mathbb{E}^n}).
\end{aligned} \tag{1.5}$$

(ii) Take $\chi = \sigma \cdot a$, then $\chi^{-1} = a^{-1}\sigma^{-1} = \overrightarrow{Oa^{-1}\big(\sigma^{-1}(O)\big)} \cdot a^{-1}$. Then:

$$\begin{aligned} d_O(\chi^{-1}, id_{\mathbb{E}^n}) &= d_O(\overrightarrow{Oa^{-1}\big(\sigma^{-1}(O)\big)} \cdot a^{-1}, id_{\mathbb{E}^n}) \\ &= \|a^{-1}\big(\sigma^{-1}(O)\big)\|_{Eucl} + \|a^{-1} - id_{\mathbb{E}^n}\|_{op} \\ &= \|\sigma^{-1}\|_{Eucl} + \|a^{-1} - a^{-1} \cdot a\|_{op} \\ &= \|-\sigma\|_{Eucl} + \|a^{-1}\|_{op} \cdot \|id_{\mathbb{E}^n} - a\|_{op}; \text{by Lemma 1.2.1 (ii)} \\ &= \|\sigma\|_{Eucl} + \|a - id_{\mathbb{E}^n}\|_{op} \\ &= d_O(\chi, id_{\mathbb{E}^n}). \end{aligned} \tag{1.6}$$

(iii) Take $\chi = \sigma \cdot a$ and $\psi = \tau \cdot b$. Then:

$$\begin{aligned} d_O(\chi \cdot \psi, id_{\mathbb{E}^n}) &= d_O(\sigma\overrightarrow{Oa\big(\tau(O)\big)} \cdot ab, id_{\mathbb{E}^n}) \\ &= \|\sigma + \overrightarrow{Oa\big(\tau(O)\big)}\|_{Eucl} + \|a \cdot b - id_{\mathbb{E}^n}\|_{op} \\ &\leq \|\sigma\|_{Eucl} + \|\tau\|_{Eucl} + \|a - id_{\mathbb{E}^n}\|_{op} + \|b - id_{\mathbb{E}^n}\|_{op}; \\ &\quad \text{by Lemma 1.2.1 (iv)} \\ &= d_O(\sigma a, id_{\mathbb{E}^n}) + d_O(\tau b, id_{\mathbb{E}^n}) \\ &= d_O(\chi, id_{\mathbb{E}^n}) + d_O(\psi, id_{\mathbb{E}^n}). \end{aligned} \tag{1.7}$$

(iv) Obvious by definition of $d_O$.

(v) Take $\chi = \sigma a, \phi = \tau b$ and $\psi = \mu c$, such that $a, b, c \in O(\mathbb{E}^n_O)$ and $\sigma, \tau, \mu \in Trans(\mathbb{E}^n)$. Notice that

$$\begin{aligned} d_{\sigma(O)}(\sigma\phi\sigma^{-1}, \sigma\psi\sigma^{-1}) &= d_{\sigma(O)}(\sigma\tau b\sigma^{-1}, \sigma\mu c\sigma^{-1}) \\ &= \|\sigma\tau\sigma^{-1} - \sigma\mu\sigma^{-1}\|_{Eucl} + \|c - b\|_{op} \\ &= \|\tau - \mu\|_{Eucl} + \|c - b\|_{op} \\ &= d_O(\phi, \psi). \end{aligned} \tag{1.8}$$

Then:

$$\begin{aligned}
d_{\sigma(O)}(\chi\phi\chi^{-1}, \chi\psi\chi^{-1}) &= d_{\sigma(O)}(\sigma a\phi a^{-1}\sigma^{-1}, \sigma a\psi a^{-1}\sigma^{-1}) \\
&= d_O(a\phi a^{-1}, a\psi a^{-1});\ \text{by (1.8)} \\
&= d_O(a\tau b a^{-1}, a\mu c a^{-1}) \\
&= d_O(\overrightarrow{Oa\tau(O)}aba^{-1}, \overrightarrow{Oa\mu(O)}aca^{-1}) \\
&= \|\overrightarrow{Oa\tau(O)} - \overrightarrow{Oa\mu(O)}\|_{Eucl} + \|aba^{-1} - aca^{-1}\|_{op} \\
&= \|\tau - \mu\|_{Eucl} + \|b - c\|_{op} \\
&= d_O(\phi, \psi).
\end{aligned} \tag{1.9}$$

(vi) Take $\chi = \sigma a$ and $\psi = \tau b$; then:

$$\begin{aligned}
\chi\psi\chi^{-1} &= \sigma \cdot a \cdot \tau \cdot b \cdot a^{-1} \cdot \sigma^{-1} \\
&= \sigma \cdot \overrightarrow{Oa(\tau(O))} \cdot aba^{-1} \cdot \sigma^{-1} \\
&= \sigma \cdot \overrightarrow{Oa(\tau(O))} \cdot \overrightarrow{Oaba^{-1}(\sigma^{-1}(O))} \cdot aba^{-1}.
\end{aligned} \tag{1.10}$$

Now, if we calculate the metric $d_O$, we get:

$$\begin{aligned}
d_O(\chi\psi\chi^{-1}, id_{\mathbb{E}^n}) &= \|\sigma + \overrightarrow{Oa(\tau(O))} + \overrightarrow{Oaba^{-1}(\sigma^{-1}(O))}\|_{Eucl} + \|aba^{-1} - id_{\mathbb{E}^n}\|_{op} \\
&\leq \|\tau\| + \|\sigma + \overrightarrow{Oaba^{-1}\big(\sigma^{-1}(O)\big)}\|_{Eucl} + \|b - id_{\mathbb{E}^n}\|_{op} \\
&= d_O(\psi, id_{\mathbb{E}^n}) + \|\sigma + \overrightarrow{Oaba^{-1}\big(\sigma^{-1}(O)\big)}\| \\
&= d_O(\psi, id_{\mathbb{E}^n}) + \|\overrightarrow{Oaba^{-1}\big(\sigma(O)\big)} - \overrightarrow{O\sigma(O)}\| \\
&= d_O(\psi, id_{\mathbb{E}^n}) + \|aba^{-1}\big(\sigma(O)\big) - \sigma(O)\| \\
&= d_O(\psi, id_{\mathbb{E}^n}) + \|\big(aba^{-1} - id_{\mathbb{E}^n}\big)\big(\sigma(O)\big)\| \\
&\leq d_O(\psi, id_{\mathbb{E}^n}) + \|aba^{-1} - id_{\mathbb{E}^n}\|_{op} \cdot \|\sigma\| \\
&= d_O(\psi, id_{\mathbb{E}^n}) + \|b - id_{\mathbb{E}^n}\|_{op} \cdot \|\sigma\|.
\end{aligned} \tag{1.11}$$

Therefore,

$$d_O(\chi\psi\chi^{-1}, id_{\mathbb{E}^n}) \leq d_O(\psi, id_{\mathbb{E}^n})\big(1 + d_O(\chi, id_{\mathbb{E}^n})\big).$$

(vii) By (v) we know $d_{\sigma(O)}(\phi, \psi) = d_O(\sigma^{-1}\phi\sigma, \sigma^{-1}\psi\sigma)$, hence with $\phi = \tau_1\beta_1$, $\psi = \tau_2\beta_2$, where $\tau_1, \tau_2 \in Trans(\mathbb{E}^n)$ and $\beta_1, \beta_2 \in O(\mathbb{E}^n_O)$,

$$
\begin{aligned}
d_O(\sigma^{-1}\tau_1\beta_1\sigma, \sigma^{-1}\tau_2\beta_2\sigma) &= d_O\big(\sigma^{-1}\tau_1\overrightarrow{O\beta_1(\sigma(O))}\beta_1, \sigma^{-1}\tau_2\overrightarrow{O\beta_2(\sigma(O))}\beta_2\big) \\
&\leq \|\tau_1 - \tau_2\|_{Eucl} + \|\overrightarrow{O\beta_1(\sigma(O))} - \overrightarrow{O\beta_2(\sigma(O))}\|_{Eucl} \\
&+ \|\beta_1, \beta_2\|_{op} \\
&\leq \|\tau_1 - \tau_2\|_{Eucl} + \|\beta_1 - \beta_2\|_{op} \cdot \|\sigma\|_{Eucl} + \|\beta_1 - \beta_2\|_{op} \\
&\leq (1 + \|\sigma\|_{Eucl}) d_O(\phi, \psi).
\end{aligned} \tag{1.12}
$$

Hence,

$$d_{\sigma(O)}(\phi, \psi) \leq (1 + \|\sigma\|_{Eucl}) d_O(\phi, \psi).$$

□

**Proposition 1.2.3** *The metric $d_O$ makes $Isom(\mathbb{E}^n)$ into a metric space.*

PROOF Clearly, by definition, $d_O$ is symmetric, positive, and $d_O(\tau a, \tau a) = 0$. We aim to prove that $d_O(\tau a, \tau'' a'') \leq d_O(\tau a, \tau' a') + d_O(\tau' a', \tau'' a'')$. This follows from the triangle inequalities for $\| \cdot \|_{Eucl}$ and $\| \cdot \|_{op}$:

$$
\begin{aligned}
d_O(\tau a, \tau'' a'') &= \|\tau - \tau''\|_{Eucl} + \|a - a''\|_{op} \\
&\leq \|\tau - \tau'\|_{Eucl} + \|a - a'\|_{op} + \|\tau' - \tau''\|_{Eucl} + \|a' - a''\|_{op} \\
&= d_O(\tau a, \tau' a') + d_O(\tau' a', \tau'' a'').
\end{aligned} \tag{1.13}
$$

Therefore, the triangle inequality holds for this metric. Hence, the metric $d_O$ on $Isom(\mathbb{E}^n)$ is a metric space. □

The topology on $Isom(\mathbb{E}^n)$ induced by the metric $d_O$ coincides with the topology introduced in Section 1.1.2.

## 1.3 Delone Sets, Convex Hulls and Polytopes

### 1.3.1 Delone sets

There is a good way to construct simple tilings from point sets $D \subset \mathbb{E}^n$ with certain properties, as we will see later through, the so-called Voronoi-cell tilings.

**Definition 1.3.1** A set $X$ in $\mathbb{E}^n$ is *relatively dense* if it has a finite covering radius. The *covering radius* of $X$ is the infimum of radii $r$, such that every point of $\mathbb{E}^n$ is within distance $r$ of at least one point in X; that is, it is the smallest radius, such that closed balls of that radius centered at the points of X have $\mathbb{E}^n$ as their union:

$$\text{covering radius} = \inf\{r : \bigcup_{x\in X} B_r(x) = \mathbb{E}^n\}.$$

**Definition 1.3.2** A set $X$ in $\mathbb{E}^n$ is *uniformly discrete* if it has a nonzero packing radius. The *packing radius* of $X$ is half that of the infimum of distances between distinct members of $X$:

$$\text{packing radius} = \frac{1}{2}\inf\{d(x, x') : x \neq x' \in X\}.$$

Based on the definitions above, we can define Delone sets as follows:

**Definition 1.3.3** A *Delone* set is a set $X$ in $\mathbb{E}^n$ that is both uniformly discrete and relatively dense, i.e., if there are numbers $R > r > 0$, such that each ball of radius r contains at most one point of $X$, and every ball of radius R contains at least one point of $X$.

### 1.3.2 Convex Hulls and Polytopes

The convex hull of a set $X$ of points in $\mathbb{R}^n$ is the smallest convex set that contains $X$.

**Definition 1.3.4** A *convex polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points $\{p_1, ..., p_k\} \subset \mathbb{R}^n$.

**Remark 1.3.5** : The convex hull of points $\{p_1, ..., p_k\} \subset \mathbb{R}^n$ is defined as the set of points $\{p = \sum_{i=1}^k t_i p_i \mid \sum_{i=1}^k t_i = 1\} \subseteq \mathbb{R}^n$, which we denote by $\langle p_1, ..., p_k \rangle$.

- We say that the points $p_1, ..., p_k$ *minimally generate* the convex hull $\langle p_1, ..., p_k \rangle$ if the convex hull of a proper subset of $\{p_1, ..., p_k\}$ is also a proper subset of $\langle p_1, ..., p_k \rangle$.

- In general, $D \subset \mathbb{R}^n$ is called a *convex set* if for all $p, q \in D$ and for all $t \in [0,1]$, we have $tp + (1-t)q \in D$.

**Lemma 1.3.6** *The convex hull is convex.*

PROOF Take the convex hull $P = \{\sum_{i=1}^n t_i p_i \mid \sum_{i=1}^n t_i = 1\}$ and take $Q, R \in P$ such that:

$$Q = \sum_{i=1}^n a_i p_i \quad ; \sum_{i=1}^n a_i = 1$$

and

$$R = \sum_{i=1}^n b_i p_i \quad ; \sum_{i=1}^n b_i = 1.$$

We aim to show that the line connecting $Q, R$ is contained in $P$, i.e., we must show that

$$(1-t)Q + tR \in P \ ; 0 \leq t \leq 1.$$

Now:

$$\begin{aligned} (1-t)Q + tR &= (1-t)\sum_{i=1}^n a_i p_i + t\sum_{i=1}^n b_i p_i \\ &= \sum_{i=1}^n \big((1-t)a_i + tb_i\big)p_i \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} \sum_{i=1}^n \big((1-t)a_i + tb_i\big) &= (1-t)\sum_{i=1}^n a_i + t\sum_{i=1}^n b_i \\ &= 1 - t + t \\ &= 1. \end{aligned} \tag{1.15}$$

Hence, $(1-t)Q + tR \in P$, and so, $P$ is convex. □

**Definition 1.3.7** An *open half-space* [*closed half-space*] is defined as the following set:
$\{x \in \mathbb{R}^d \mid \langle x, y\rangle > \alpha\}$ [respectively $\{x \in \mathbb{R}^d \mid \langle x, y\rangle \geq \alpha\}$] for suitable $y \in \mathbb{R}^d, y \neq 0, \alpha \in \mathbb{R}$ .

Let $K$ be a subset of $\mathbb{R}^d$. We say that a hyperplane $H = \{x \in \mathbb{R}^d \mid \langle x, u\rangle = \alpha\}$ cuts $K$ if the two open half-spaces determined by $H$ contain points of $K$, i.e., there exists $x_1, x_2 \in K$ such that $\langle x_1, u\rangle < \alpha$ and $\langle x_2, u\rangle > \alpha$.

**Definition 1.3.8** We say that $H$ is a *supporting hyperplane* of $K$ if $H$ does not cut $K$, but $H \cap \overline{K} \neq \emptyset$; where $\overline{K}$ is the closure of $K$ in $\mathbb{R}^n$.

**Definition 1.3.9** Suppose $K$ is a convex subset of $\mathbb{R}^d$. A set $F \subset K$ is a *face* of $K$, if either $F = \emptyset$ or $F = K$, or if there exists a supporting hyperplane $H$ of $K$, such that $F = K \cap H$. These faces are called *proper faces.*

**Definition 1.3.10** We say that a maximal proper face of $K$ is a *facet* of $K$.

**Lemma 1.3.11** *Let $P$ be a convex polytope, and let $Aut(P) \subset Isom(\mathbb{E}^n)$ be the group of isometries $\gamma \in Isom(\mathbb{E}^n)$ such that $\gamma(P) = P$. Then there is a finite set $K \subset P$ such that $\gamma(K) = K$ for all $\gamma \in Aut(P)$.*

Proof First, we have to show that, $Aut(P)$ is finite. Notice that convex polytopes are the convex hull of a finite set of extremal vertices. From this we can conclude that $Aut(P)$ is contained in the permutation group of extremal vertices. Since the permutation group is finite, this means that the symmetry group of a convex polytope is finite. Consequently, we can take one point $p \in P$ and the orbit $Aut(P) \cdot p$ under the isometry group. As $Aut(P)$ is finite, the orbit is also finite, and by construction, it is clear that the orbit is invariant under the isometry group of $P$. □

**Theorem 1.3.12** [[11], Theorem 1, page 31-32]
*Each polytope $K \subset \mathbb{R}^d$ is the intersection of a finite family of closed half-spaces; the smallest such family consists of those closed half-spaces containing $K$, whose boundaries are the affine hulls of the facets of $K$.*

**Definition 1.3.13** Let $t$ be a convex polytope in $\mathbb{R}^n$; particularly, let $t$ be the convex hull of the points $p_1, ..., p_k \in \mathbb{R}^n$. Assume that the points $p_1, ..., p_k$ minimally generate $t$. Then:

(i) The points $p_1, ..., p_k$ are called the *vertices* of $t$.

(ii) The convex hull $E_{ij}$ of two points $p_i, p_j$ (that is, the connecting line segment) is called an *edge* of $t$ if $E_{ij} \subset \partial t$.

(iii) The convex hull $F_{i_1,...,i_e} = \langle p_{i_1}, ..., p_{i_e} \rangle$ is called an $m$*-face* if $F_{i_1,...,i_e} \subset \partial t$, $F_{i_1,...,i_e}$ has dimension $m$, and an $(m-1)$-face is called a *facet.*

# Chapter 2

# Simple Tilings In General

First and foremost, we will discuss certain kind of tilings (simple tilings). Then, we will construct periodic tilings from crystallographic groups (by Voronoi-cell decomposition).
To this end, I worked through [26] with my supervisor along with several chapters from [12],[4] and [25].

## 2.1 Tilings

A tiling is a subdivision of $\mathbb{E}^n$ into areas $\{t_i\}_{i\in I}$ called tiles. More precisely:

**Definition 2.1.1** A set of convex polytopes $\{t_i\}_{i\in I}$ in $\mathbb{E}^n$ is called a *tiling* of $\mathbb{E}^n$ if:

(i) $\bigcup t_i = \mathbb{E}^n$ .

(ii) $t_i \cap t_j \subseteq \partial t_i \cap \partial t_j \ \ \forall i \neq j$.

**Example 2.1.2** Tilings constructed by thick rhombs and thin rhombs prototiles which were discovered by Penrose are called Penrose tilings, see Figure 2.1. It is clear that Figure 2.1 is a tiling as all conditions in Definition 2.1.1 are satisfied.

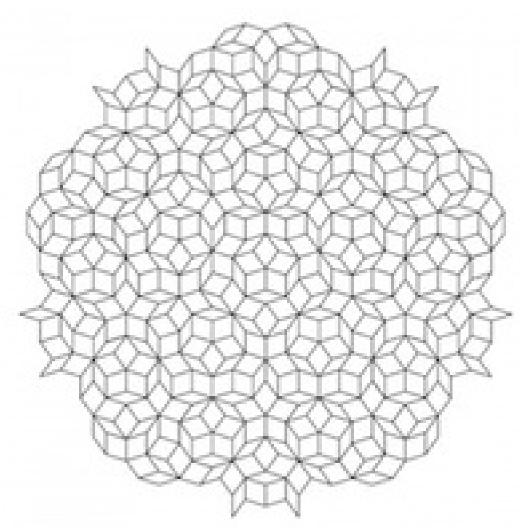


Figure 2.1: Penrose tiling by thick and thin rhombs "prototiles"; [15].

Note that, all tilings with "thick rhomb" and "thin rhomb" should meet with the same marking edges to obtain Penrose tilings, as in Figure 2.2. The thick rhombs has angles of 72, 108, 72, and 108 degrees, whereas the thin rhombs has four corners with angles of 36, 144, 36, and 144 degrees. Notice, patches can be only joined if the arrows coincide.

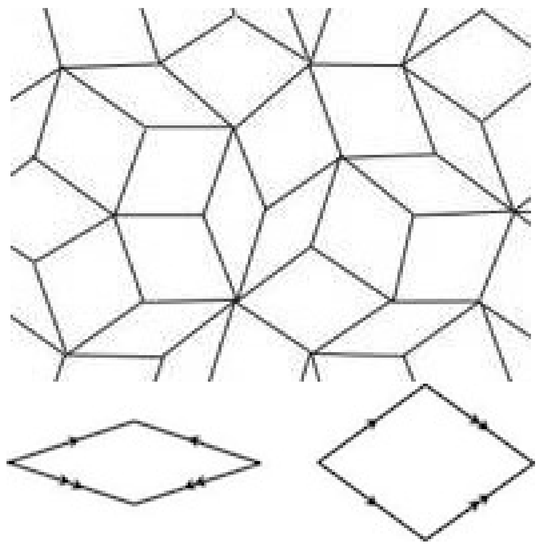


Figure 2.2: Thick and thin rhombs matching rules; [18].

In Definition 2.1.1; we restrict our choice of tiles to be convex polytopes. It is possible to construct tilings with non-convex, non-polytopes tiles. It is believed that we can turn such tilings into (in whatever sense) equivalent tilings with convex polytopes as tiles, using for example the Voronoi-cell decomposition. But this is rarely proven in the literature.

**Definition 2.1.3** A *patch* of tiles in a giving tiling $T$ is a number of tiles of the tiling $T$, whose union is a bounded subset of $\mathbb{R}^n$.

**Notation 2.1.4** (i) If $A$ is a bounded subset of $\mathbb{R}^n$, then $[T]_A$ denotes the patch of a tiling $T$ consisting of all tiles that intersect $A$.

(ii) If $T$ is a tiling, then $T + r$ is the tiling with all tiles in $T$ shifted by $r$, where $r \in \mathbb{R}^n$. More generally, if $\phi$ is an isometry of $\mathbb{E}^n$, then $\phi(T)$ is the tiling, with all tiles in $T$ mapped by $\phi$.

**Definition 2.1.5** Let $P$ be a set of points $p_1, p_2, ...$ in $\mathbb{E}^n$. The *Voronoi-cells* of $P$ subdivide $\mathbb{E}^n$ into cells, such that a point q lies in the cell $V_p(P)$, corresponding to $p_i \in P$ iff $||q - p_i|| \leq ||q - p_j||$ for each $p_i \in P$ , $i \neq j$.

**Construction 2.1.6** Given a Delone set $D$ with covering radius $R$ and packing radius $r$, the Voronoi-cells

$$V_p(D) = \{x \in \mathbb{E}^n : \ ||x - p|| \leq ||x - q|| \ \text{ for all } q \in D\}$$

satisfy the following properties:

(i) $V_p(D)$ is a closed convex polytope containing $B(p, r)$ and contained in $B(p, R)$.

(ii) Two Voronoi-cells intersect only at their boundaries.

Therefore, the union $\{V_p(D) : p \in D\}$ of Voronoi-cells is a tiling in the sense of Definition 2.1.1. It is called the Voronoi-cell tiling associated to $D$ and denoted by $VT(D)$.

## 2.2 Simple Tilings

The main reason for introducing simple tilings is that the more general definition of tilings in Definition 2.1.1 allows for strange shapes and arrangements.

**Definition 2.2.1** An *(isometrically) simple tiling* of $\mathbb{R}^n$ is a tiling in which:

(i) There is only a finite number of tile types called *prototiles*, up to isometries. That is, there is a finite subset $\{t_1, ..., t_r\} \subset T$, such that every tile $t \in T$ is obtained from one of the $t_1, ..., t_r$ say $t_i$, by an isometry, and $t$ can not be obtained from $t_j; j \neq i$ by an isometry.

(ii) Tiles meet full-facet to full-facet. This means that, for all tiles $t_i, t_j$ in the tiling $T$, the intersection $t_i \cap t_j$ is either empty, or both a face of $t_i$ and a face of $t_j$. In particular, if the dimension of $t_i \cap t_j$ is $n-1$, then $t_i \cap t_j$ is both a facet of $t_i$ and a facet of $t_j$.

**Example 2.2.2** The tiling in Figure 2.3 is not simple; since the second condition (ii) of Definition 2.2.1 is false. The picture shows that for some $t_i, t_j \in T$ the intersection $t_i \cap t_j$ is neither a facet of $t_i$ nor $t_j$.

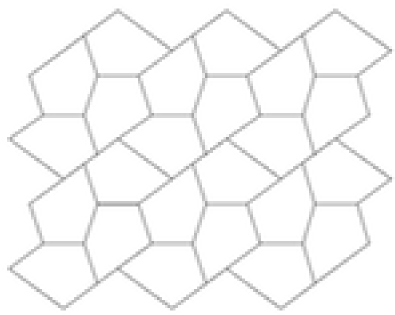


Figure 2.3: Non-simple pentagon tiling; [6].

**Remark 2.2.3** Originally, simple tilings were defined as built up from a finite number of prototiles up to translations. The class of isometrically (or rotationally) simple tiling is larger:
An example for this is pinwheel tiling, which is constructed as the substitution tiling induced by Conway's triangle decomposition of a rectangular triangle with side lengths 1 and 2 at a right angle, as shown in Figure 2.4:

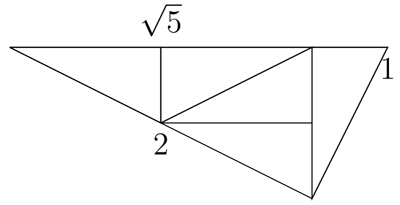



Figure 2.4: Pinwheel Tiling (Conway triangle decomposition into homothetic smaller triangles).

There were extensively studied by Radin [24] who showed that the triangular tiles point to an infinite number of directions. Hence, they can not be obtained as translates of a finite number of prototypes.

**Remark 2.2.4** To distinguish between translationally and isometrically simple tilings, we add "translationally" and "isometrically".
When speaking of simple tilings we always mean "isometrically simple tiling".

Usually, Voronoi-cell tiling constructed from a Delone set $VT(D)$ is **not** simple. On the other hand, under some conditions, this is the case. To find these conditions, we need the following proposition as a preparation.

**Proposition 2.2.5** *For a Delone set $X \subset \mathbb{E}^n$, there exists a radius $R$ such that: for all $x \in X$, $B_R(x)$ contains the Voronoi-cell of $x$.*

Proof As a Delone set, $X \subset \mathbb{E}^n$ is relatively dense, that is

$$\exists r \; \forall y \in \mathbb{E}^n \;:\; B_r(y) \cap X \neq \emptyset.$$

Let $y_1, ..., y_n, y_{n+1}$ be the vertices of a regular $n$-simplex $S$ in $\mathbb{E}^n$ barycentered in a point $x_\circ \in X$, such that $d(y_i, x_\circ) = R' \gg r$ for $i = 1, ..., n, n+1$. This means in particular that the difference vectors $y_1 - x_\circ, ..., y_n - x_\circ$ are linearly independent and $y_{n+1} - x_\circ = -\sum_{i=1}^{n}(y_i - x_\circ)$.
Relative density implies:

$$\exists x_i \in B_r(y_i) \cap X \;; i = 1, ..., n, n+1.$$

Now, by continuity and homogeneity of the linear independence condition, the difference vectors $x_1 - x_\circ, ..., x_n - x_\circ$ are still linearly independent and $x_{n+1} - x_\circ = -\sum_{i=1}^n a_i(x_i - x_\circ)$ with $a_i > 0$, $i = 1, ..., n$.
Consequently, there is an affine-linear transformation $\alpha$ of $\mathbb{E}^n$ continuously dependent on the $x_1, ..., x_{n+1}$, such that:

$$\alpha(x_\circ) = (0, ..., 0) \ , \ \alpha(x_1) = (1, 0, ..., 0) \ , \ \alpha(x_n) = (0, ..., 0, 1) \quad \text{and}$$

$$\alpha(x_{n+1}) = \alpha\big(-\sum_{i=1}^n a_i(x_i - x_\circ)\big) = -(a_1, ..., a_n).$$

Then, the Voronoi-cell $V_{x_\circ}(\{x_\circ, x_1, ..., x_{n+1}\})$ is given as the intersection of

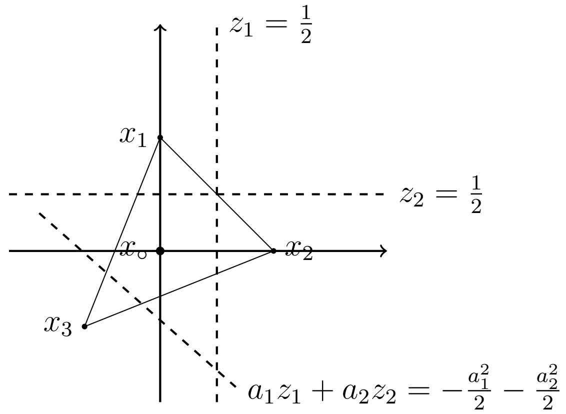



Figure 2.5: Voronoi-cell of simplicial point configuration in $\mathbb{E}^2$.

the half-spaces:

$$\{z_i \le \frac{1}{2} \ ; i = 1, ..., n\}$$

and

$$\{\sum_{i=1}^n a_i z_i \ge -\sum_{i=1}^n \frac{a_i^2}{2}\} \ ;$$

where $z_1, ..., z_n$ are coordinates of $\mathbb{E}^n_O$.
This shows that the Voronoi-cell $V_{x_\circ}$ is compact and continuously dependent on $x_1, ..., x_{n+1}$.
Since the diameter $D(x_1, ..., x_{n+1})$ of $V_{x_\circ}(\{x_\circ, x_1, ..., x_{n+1}\})$ also depends continuously on $x_1, ..., x_{n+1}$, we can find an $R > 0$ such that:
For points $x \in X$ with $\|x - x_\circ\| > 2 \cdot R$ , the half-space $\{y \in \mathbb{E}^n \ : \ \|y - x_\circ\| \le \|y - x\|\}$ contains $V_{x_\circ}(\{x_\circ, x_1, ..., x_{n+1}\})$; hence, it

contains $V_{x_\circ}(X)$.
Note that the construction of this radius $R$ only depends on the relative position of $x_1, ..., x_{n+1}$ to $x_\circ$, and since we have chosen $y_i$ as the vertices of a regular simplex centered in $x_\circ$, we can choose the same $R$ for all $x_\circ \in X$. This implies that $R$ satisfies the assertion of the proposition.

□

**Proposition 2.2.6** *The Delone set $X$ has simple Voronoi-cell tiling $VT(D)$ if for $R \gg 0$ there are only finitely many point set configurations $X \cap B_R(x)$ for all points $x \in X$, up to isometries.*

PROOF Immediate consequence of Proposition 2.2.5 and Construction 2.1.6.

□

## 2.3 Voronoi-cell Tilings from Crystallographic Groups

Recall that $\Gamma \subset Isom(\mathbb{E}^n)$ is a crystallographic group if $\Gamma$ is discrete and $Isom(\mathbb{E}^n)/\Gamma$ is compact. Then, upon picking a point $p \in \mathbb{E}^n$:
$\Gamma(p) = \{\gamma(p) : \gamma \in \Gamma\} \subset \mathbb{E}^n$ is the $\Gamma$-orbit of $p$.

**Theorem 2.3.1** *The Voronoi-cell tiling $VT(\Gamma(p))$ is a simple tiling, and $\Gamma(p)$ is a Delone set.*

PROOF First, we aim to show that $VT(\Gamma(p))$ has a finite number of tiles (up to translation). Notice that the connection between Voronoi-cells $V_p(\Gamma(p))$ and $V_{\gamma(p)}(\Gamma(p))$ is given by the equality $V_{\gamma(p)}(\Gamma(p)) = \gamma(V_p)(\Gamma(p))$, because

$$||\gamma(p) - \gamma(q)|| = ||p - q||.$$

As every $\gamma \in \Gamma$ can be written as a product of the form

$$\gamma = t \cdot \gamma_p \quad ; t \in T^n, \gamma_p \in \Gamma_p,$$

where $\Gamma_p$ is the point group of $\Gamma$ centered in $p$ (see Theorem 1.1.20), and $T^n$ is a lattice of full rank in $Trans(\mathbb{E}^n)$ containing $\Gamma \cap Trans(\mathbb{E}^n)$ (see Proposition 1.1.21 in Section 1.1.3). Therefore, every $V_{\gamma(p)}(\Gamma(p))$ is a translation of $\gamma_p(V_p(\Gamma(p)))$, i.e.,

$$V_{\gamma(p)}(\Gamma(p)) = \gamma(V_p(\Gamma(p))) = t \cdot \gamma_p(V_p(\Gamma(p))).$$

Hence, we only have a finite number of tiles if we allow translations, as $\gamma_p(V_p(\Gamma(p)))$ contains only finitely many possible tiles because $\Gamma_p$ is finite.

Second, we aim to show that, the Voronoi-cell $V_p(\Gamma(p))$ is a convex polytope, i.e., it is a convex hull of finitely many points $p_i$ or equivalently, it is the intersection of a finite number of half-spaces, according to Theorem 1.3.12. We know that:

$$\begin{aligned} V_p(\Gamma(p)) &= \{x \in \mathbb{E}^n \ : ||x-p|| \leq ||x-q|| \quad \forall q \in \Gamma(p)\} \\ &= \bigcap_{q \in \Gamma(p)} \{x \in \mathbb{E}^n \ : ||x-p|| \leq ||x-q||\} \end{aligned} \tag{2.1}$$

where $H_{p,q} := \{x \in \mathbb{E}^n \ : ||x-p|| \leq ||x-q||\}$ is a half space given by the equation $(q-p) \cdot x \leq \frac{1}{2}(q-p) \cdot (p+q)$.
Suppose $t_1, ..., t_n$ are the generators of the lattice $T \subset \Gamma$.
**Claim:** Only finitely many translations $t$ in the lattice $T \subset \Gamma$ are needed, such that the half spaces $H_{p,t\gamma(p)}$ cut out the Voronoi-cell $V_p(\Gamma(p))$, where $\gamma$ runs through $\Gamma_p$.
**Proof of the claim:** We know that the Voronoi-cell $V_p(\Gamma(p))$ is contained in the intersection of the half spaces $H_{p,\pm t_i(p)}$, and hence, $V_p(\Gamma(p))$ is compact. Therefore, there is a maximal distance $K$ such that:

$$||x-p|| < K \ \forall x \in V_p(\Gamma(p)).$$

Since $\Gamma_p$ is finite, there exists $M : ||p-\gamma(p)|| < M$ for all $\gamma \in \Gamma_p$. Now, if $||t|| \to \infty$, then $||p - t\gamma(p)|| \to \infty$; also notice that:

$$\begin{aligned} ||p - t\gamma(p)|| &= ||p - \gamma(p) - t|| \\ &\geq |\|p-\gamma(p)\| - \|t\|| \end{aligned} \tag{2.2}$$

Hence,

$$\begin{aligned} \frac{1}{2}||p - t\gamma(p)|| &\geq \frac{1}{2}|\|p-\gamma(p)\| - \|t\|| \\ &\geq \frac{1}{2}||t|| - \frac{1}{2}M > K \end{aligned} \tag{2.3}$$

for $||t|| > M + 2K$. This means that, the half space $H_{p,t\gamma(p)}$ contains $V_p(\Gamma(p))$ in its interior, and hence, is not necessary to cut out $V_p(\Gamma(p))$. The claim follows because there are only finitley many $t$ in the lattice $T$ of $\Gamma$ with $||t|| \leq M + 2K$.
Finally, we aim to show that the tiles of $VT(\Gamma(p))$ meets full facet to full facet. Let $L_{p,q} := H_{p,q} \cap H_{q,p}$ be a hyperplane cutting out a facet $V_p(\Gamma(p)) \cap L_{p,q}$ of $V_p$. Notice that $L_{p,q} \neq L_{p,q'}$, implies $dim(L_{p,q} \cap L_{p,q'}) = n-2$. Therefore, a facet of $V_p(\Gamma(p))$ is cut out by exactly one $L_{p,q} = L_{q,p}$. Hence,

$$\begin{aligned} x \in V_p(\Gamma(p)) \cap L_{p,q} &\Longleftrightarrow \|x-p\| = \|x-q\| \leq \|x-q'\| \quad \forall q' \in \Gamma - \{p,q\} \\ &\Longleftrightarrow x \in V_q(\Gamma(q)) \cap L_{p,q}. \end{aligned} \tag{2.4}$$

Therefore, $V_p(\Gamma(p)) \cap L_{p,q} = V_q(\Gamma(q)) \cap L_{p,q}$ and $V_p(\Gamma(p)), V_q(\Gamma(q))$ meet full facet to full facet. Hence, $VT(\Gamma(p))$ is a simple tiling.

We have already shown that $V_{\gamma(p)}(\Gamma(p)) = \gamma(V_p(\Gamma(p)))$. Hence there exist $r > 0$ such that $B_r(\gamma(p)) \subset V_{\gamma(p)}(\Gamma(p))$ and $R > 0$ such that $B_R(\gamma(p)) \supset V_{\gamma(p)}(\Gamma(p))$, for all $\gamma \in \Gamma$. Then $r$ is a packing radius and $R$ a covering radius of $\Gamma(p)$, and we conclude that $\Gamma(p)$ is a Delone set. □

# Chapter 3

# Metrics on Simple Tiling Spaces

From each simple tiling $T$, we will construct a space $\Omega_T$ of tilings and study the topology of $\Omega_T$. The first step is to define a metric on the space $\Omega$ of all simple tilings.

## 3.1 The Tiling Metric $d$

For two tilings $T$,$T'$ of $\mathbb{E}^n$, we say that $T, T'$ are $\epsilon$-close if they agree on a ball of radius $1/\epsilon$ around the origin, up to an isometry of size $\epsilon$ or less. The definition below explains this more precisely.

**Definition 3.1.1** The distance $d(T, T')$ between two tilings of $\mathbb{E}^n$ in $\Omega$ is defined as the smaller of $\ln(\frac{3}{2})$ and $\ln\big(1 + 1/R(T, T')\big)$, that is:

$$d(T, T') = \inf \big( \ln(\frac{3}{2}), \ln(1 + \frac{1}{R(T, T')})\big).$$

To calculate $R(T, T')$, we use the metric $d_O$ on $Isom(\mathbb{E}^n)$ defined in the first Chapter:

$$R(T, T') = \sup_r \{\exists \phi, \psi \in Isom(\mathbb{E}^n) \text{ s.t. } d_O(\phi, id_{\mathbb{E}^n}), d_O(\psi, id_{\mathbb{E}^n}) < \frac{1}{2r} \text{ and}$$

$$[\phi(T)]_{B_r(O)} = [\psi(T')]_{B_r(O)}\}.$$

**Proposition 3.1.2** *$(\Omega, d)$ is a metric space. We call $d$ the* tiling metric*.*

PROOF We have to show that:

(i) $d(T, T') = d(T', T)$

(ii) $d(T, T') \geq 0$ and $d(T, T') = 0$ iff $T = T'$

(iii) $d(T, T'') \leq d(T, T') + d(T', T'')$

(i)+(ii) Clearly, by definition, $d$ is symmetric, positive, and $d(T, T) = 0$. Suppose $T \neq T'$, then there exists tiles $t \in T, t' \in T'$, such that $t \neq t'$, but the interiors $t^\circ, t'^\circ$ intersect. Therefore, for $\epsilon$ small enough and for all $\phi, \psi \in Isom(\mathbb{E}^n)$ with $d_O(\phi, id_{\mathbb{E}^n}), d_O(\psi, id_{\mathbb{E}^n}) < \epsilon$, we have

$$\phi(t) \neq \psi(t') \text{ and } (\phi(t))^\circ \cap (\psi(t'))^\circ \neq \emptyset.$$

Choose $r > \frac{1}{2\epsilon}$ (hence $\epsilon > \frac{1}{2r}$) such that $B_r(O) \cap t$ and $B_r(O) \cap t'$ are both non-empty. The argument above shows that whatever $\phi, \psi \in Isom(\mathbb{E}^n)$ with $d_O(\phi, id_{\mathbb{E}^n}), d_O(\psi, id_{\mathbb{E}^n}) < \frac{1}{2r}$, we choose:

$$[\phi(T)]_{B_r(O)} \neq [\psi(T')]_{B_r(O)} .$$

The same holds for all $r' \geq r$; hence, $R(T, T') \leq r$ and $d(T, T') \neq 0$.

(iii)We aim to prove triangle inequality. Suppose $R(T, T') > 2$, then there exist $2 < r \leq R(T, T')$ and $\phi, \psi \in Isom(\mathbb{E}^n)$, such that

$$d_O(\phi, id_{\mathbb{E}^n}), d_O(\psi, id_{\mathbb{E}^n}) < \frac{1}{2r}$$

and

$$[\phi(T)]_{B_r(O)} = [\psi(T')]_{B_r(O)}.$$

Similarly, suppose $R(T', T'') > 2$, then there exist $2 < r' \leq R(T', T'')$ and $\chi, \omega \in Isom(\mathbb{E}^n)$, such that

$$d_O(\chi, id_{\mathbb{E}^n}), d_O(\omega, id_{\mathbb{E}^n}) < \frac{1}{2r'}$$

and

$$[\chi(T')]_{B_{r'}(O)} = [\omega(T'')]_{B_{r'}(O)}.$$

Choose $r_0 = \frac{rr'}{r+r'}$. $[\phi(T)]_{B_r(O)} = [\psi(T')]_{B_r(O)}$, implies $[\chi(\phi(T))]_{\chi(B_r(O))} = [\chi(\psi(T'))]_{\chi(B_r(O))}$. Assume $\chi = \tau \cdot a$, where $a \in O(\mathbb{E}^n_O), \tau \in Trans(\mathbb{E}^n)$, then we have $a(B_r(O)) = B_r(O)$. Therefore, $[\chi(\phi(T))]_{\tau(B_r(O))} = [\chi(\psi(T'))]_{\tau(B_r(O))}$. By the choice of $r_0$, we see that $B_{r_0}(O) \subset \tau B_r(O)$, since:

$$\|\tau\| = d_O(\tau, 0) \leq d_O(\tau, 0) + d_O(a, id_{\mathbb{E}^n_O}) = d_O(\tau a, id_{\mathbb{E}^n}) < \frac{1}{2r'}$$

and

$$r - \frac{1}{2r'} = \frac{2rr' - 1}{2r'} > r_0 \ .$$

Hence,

$$(*) \qquad [\chi(\phi(T))]_{B_{r_0}(O)} = [\chi(\psi(T'))]_{B_{r_0}(O)}.$$

$[\chi(T')]_{B_{r'}(O)} = [\omega(T'')]_{B_{r'}(O)}$ implies $[\bar{\psi}\big(\chi(T')\big)]_{\bar{\psi}B_{r'}(O)} = [\bar{\psi}\big(\omega(T'')\big)]_{\bar{\psi}B_{r'}(O)}$, with $\bar{\psi} = \chi\psi\chi^{-1}$. Since by Lemma 1.2.2 (v) we have:

$$d_O(\bar{\psi}, id_{\mathbb{E}^n}) = d_O(\chi\psi\chi^{-1}, id_{\mathbb{E}^n}) \leq d_O(\psi, id_{\mathbb{E}^n})\big(1 + d_O(\chi, id_{\mathbb{E}^n})\big) < \frac{1}{2r}\big(1 + \frac{1}{2r'}\big),$$

by exchanging the roles of $r$ and $r'$, we find, as in the previous argument that:

$$\begin{aligned} r' - \frac{1}{2r}(1 + \frac{1}{2r'}) \geq r_o = \frac{rr'}{r + r'} &\iff (r + r')r' - (r + r') \cdot \frac{2r' + 1}{4rr'} \geq rr' \\ &\iff 4rr'\big(rr' + (r')^2\big) - (r + r')(2r' + 1) \geq 4r^2 r'^2 \\ &\iff 4r(r')^3 - (2rr' + r + 2r'^2 + r') \geq 0 \\ &\iff 4r(r')^3 \geq 2rr' + 2r'^2 + r + r' \\ &\iff 2(r')^2 \cdot 2r'r \geq (2r' + 1)(r + r'); \end{aligned} \tag{3.1}$$

as $(2r' + 1)r + (2r' + 1)r' = (2r' + 1)(r + r')$.
The last inequality in (3.1) holds because, if $r, r' > 2$, then:

$$\begin{aligned} 2(r')^2 = (r')^2 + (r')^2 \geq 2r' + 1 \\ 2rr' = rr' + rr' \geq r + r' \ . \end{aligned} \tag{3.2}$$

Therefore,

$$(**) \qquad [\chi\psi(T')]_{B_{r_o}(O)} = [\bar{\psi}\big(\chi(T')\big)]_{B_{r_o}(O)} = [\bar{\psi}\big(\omega(T'')\big)]_{B_{r_o}(O)}.$$

Since

$$\begin{aligned} d_O(\chi\phi, id_{\mathbb{E}^n}) &\leq d_O(\chi, id_{\mathbb{E}^n}) + d_O(\phi, id_{\mathbb{E}^n}) \\ &\leq \frac{1}{2r'} + \frac{1}{2r} \\ &= \frac{1}{2r_0} \end{aligned} \tag{3.3}$$

and

$$d_O(\chi\psi\chi^{-1}\omega, id_{\mathbb{E}^n}) \leq d_O(\chi\psi\chi^{-1}, id_{\mathbb{E}^n}) + d_O(\omega, id_{\mathbb{E}^n})$$
$$\leq \frac{1}{2r} + (\frac{1}{2r}\frac{1}{2r'}) + \frac{1}{2r'} \tag{3.4}$$
$$= \frac{1}{2r_o} + \frac{1}{2r} \cdot \frac{1}{2r'} .$$

(*) and (**) show that: $\frac{1}{r_0} + \frac{1}{2r}\frac{1}{r'} \geq \frac{1}{R(T,T'')}$, which implies

$$d(T,T'') = \ln\big(1 + \frac{1}{R(T,T'')}\big) \leq \ln\big(1 + \frac{1}{r_0} + \frac{1}{2r} \cdot \frac{1}{r'}\big)$$

i.e. $d(T,T'') \leq d(T,T') + d(T',T'')$, because $\ln\big(1 + \frac{1}{R(T,T'')}\big) \leq \ln(1 + \frac{1}{r}) + \ln(1 + \frac{1}{r'})$ (by taking exp for both sides). Hence, the triangle inequality follows.
If $R(T,T') \leq 2$ or $R(T',T'') \leq 2$, then the triangle inequality follows immediately from the definition of $d$. □

The distance $d$ on the space of tilings depends on the chosen origin $O$, but the topology on the space of tilings induced by this distance does not.

**Proposition 3.1.3** *For two different points $O, O' \in \mathbb{E}^n$, the metrics $d_O$ and $d_{O'}$ induce the same topology on a space of tilings of $\mathbb{E}^n$.*

Proof The underlying reason for the assertion to hold is that by Lemma 1.2.2 (vii), the metrics $d_O, d_{O'}$ on $\mathbb{E}^n$ are comparable, that is, there exists a constant $C > 1$ such that for all $\phi, \psi \in Isom(\mathbb{E}^n)$,

$$\frac{1}{C} \cdot d_{O'}(\phi,\psi) \leq d_O(\phi,\psi) \leq C \cdot d_{O'}(\phi,\psi).$$

Furthermore, we use that $B_r(O') \subset B_{2r}(O)$ for $r > \|\overrightarrow{OO'}\|_{Eucl}$, and vice versa.
In more details, assume that $R_O(T,T') > 2 \cdot \|\overrightarrow{OO'}\|_{Eucl}$ holds for two simple tilings $T, T'$ of $\mathbb{E}^n$. By definition there exist $r > \|\overrightarrow{OO'}\|_{Eucl}$ and $\phi, \psi \in Isom(\mathbb{E}^n)$ such that $d_O(\phi, id_{\mathbb{E}^n}), d_O(\psi, id_{\mathbb{E}^n}) < \frac{1}{4r}$ and $[\phi(T)]_{B_{2r}(O)} = [\psi(T')]_{B_{2r}(O)}$. Consequently,

$$d_{O'}(\phi, id_{\mathbb{E}^n}), d_{O'}(\psi, id_{\mathbb{E}^n}) < \frac{C}{4r} < \frac{C}{2r}$$

and

$$[\phi(T)]_{B_r(O')} = [\psi(T')]_{B_r(O')}.$$

$C > 1$ implies $B_{\frac{r}{C}}(O') \subset B_r(O')$, and hence $R_{O'}(T,T') \geq \frac{1}{2C} \cdot R_O(T,T')$. Reversing the roles of $T$ and $T'$ we conclude $R_{O'}(T,T') \leq 2C \cdot R_O(T,T')$ if $R_{O'}(T,T') > 2 \cdot \|\overrightarrow{OO'}\|_{Eucl}$. This shows the comparability of the metrics $d_O$ and $d_{O'}$ for small distances, hence the induced topologies are equal. □

We want to show that a tiling shifted by an isometry close to $id_{\mathbb{E}^n}$ is close to the original tiling in the sense of the metric $d$. For this purpose, we must be able to construct square roots of isometries. For translation $\tau \in Trans(\mathbb{E}^n)$ this is easy:
If we compose $\frac{\tau}{2}$ with itself, we obtain $\tau$. But for orthogonal maps $\alpha \in O(\mathbb{E}^n_O)$, more work needs to be done. The existence of a square root $\sqrt{\alpha}$ of $\alpha = \left(\sqrt{\alpha}\right)^2$ follows from a general decomposition theorem for orthogonal maps:

**Theorem 3.1.4** [[4], Description of isometries, pp.292-293]
*Suppose $V$ is a real inner-product space and $\alpha \in \mathcal{L}(V)$. Then $\alpha$ is an isometry if and only if there is an orthonormal basis of $V$ with respect to which $\alpha$ is a block diagonal matrix:*

$$\alpha = \begin{pmatrix} M_1 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & M_r \end{pmatrix};$$

*where each block $M_i$ on the diagonal is of the form: $M_i = (1)$ or $(-1)$ or $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, the rotational matrix with angle $\theta \in (0, \pi)$.*

As long as $M_i \neq (-1)$, there exists $N_i$ such that $M_i = \left(N_i\right)^2$. For $M_i = (1)$, we have $N_i = (1)$, and for $M_i = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, we can choose $N_i = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$.

If we set $\beta = \begin{pmatrix} N_1 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & N_r \end{pmatrix}$, we have $\beta^2 = \alpha$. Hence, $\beta$ is the square root of $\alpha$.

**Lemma 3.1.5** *For $\alpha$ and $\beta$ as above,*

$$\|\beta - id_{\mathbb{E}^n}\|_{op} \leq \|\alpha - id_{\mathbb{E}^n}\|_{op}.$$

PROOF Since the basis on which we can write $\alpha$ and $\beta$ have $\alpha - id_{\mathbb{E}^n}$ and $\beta - id_{\mathbb{E}^n}$ in block form is orthonormal, the square of the operator norms $\|\alpha - id_{\mathbb{E}^n}\|^2_{op}$ and $\|\beta - id_{\mathbb{E}^n}\|^2_{op}$ is the sum of the squares of the operator norms $\|M_i - id\|^2_{op}$ and $\|N_i - id\|^2_{op}$.
If $M_i = N_i = (1)$, then these operator norms are 0.
If $M_i = R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $N_i = R(\frac{\theta}{2})$, then:

$$\begin{aligned}
\|R(\theta) - id_{\mathbb{R}^2}\|_{op} &= max_{(x,y)\in S^1}\|R(\theta) - id_{\mathbb{R}^2})\begin{pmatrix} x \\ y \end{pmatrix}\|_{Eucl} \\
&= max_{(x,y)\in S^1}\|\begin{pmatrix} x(\cos\theta - 1) - y\sin\theta \\ x\sin\theta + y(\cos\theta - 1) \end{pmatrix}\|_{Eucl} \\
&= max_{(x,y)\in S^1}\sqrt{\big(x(\cos\theta - 1) - y\sin\theta\big)^2 + \big(xsin\theta + y(\cos\theta - 1)\big)^2} \\
&= max_{(x,y)\in S^1}\sqrt{x^2(\cos\theta - 1)^2 - 2xy(\cos\theta - 1)\sin\theta + y^2\sin^2\theta +} \\
&\overline{x^2\sin^2\theta + 2xysin\theta(\cos\theta - 1) + y^2(\cos\theta - 1)^2} \\
&= \sqrt{(\cos\theta - 1)^2 + \sin^2\theta} \\
&= \sqrt{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta} \\
&= \sqrt{2 - 2\cos\theta}.
\end{aligned} \tag{3.5}$$

Since $2 - 2cos\theta \geq 2 - 2cos\frac{\theta}{2} \geq 0$ if $\theta \in (0, \pi)$, we get:

$$\|\beta - id_{\mathbb{E}^n}\|_{op} \leq \|\alpha - id_{\mathbb{E}^n}\|_{op}.$$

□

**Proposition 3.1.6** *For a simple tiling $T$ and $\phi_k \in Isom(\mathbb{E}^n)$.*

$$\textit{If} \quad d_O(\phi_k, id_{\mathbb{E}^n}) \longrightarrow 0\ , \ \textit{then} \ \ d(\phi_k(T), T) \longrightarrow 0.$$

PROOF Put $\phi_k = \tau_k\alpha_k$ where $\tau_k \in Trans(\mathbb{E}^n)$ and $\alpha_k \in O(\mathbb{E}^n_O)$. Then, by the assumption:

$$\|\tau_k\|_{Eucl} = d_O(\tau_k, id_{\mathbb{E}^n}) \longrightarrow 0 \quad \text{and} \quad \|\alpha_k\|_{op} = d_O(\alpha_k, id_{\mathbb{E}^n}) \longrightarrow 0.$$

Notice that,

$$d\big(\phi_k(T), T\big) = d\big(\tau_k(\alpha_k(T)), T\big) \leq d\big(\tau_k(\alpha_k(T)), \alpha_k(T)\big) + d\big(\alpha_k(T), T\big).$$

Now, if we show that $d\big(\tau_k(\alpha_k(T)), \alpha_k(T)\big) \longrightarrow 0$, and $d\big(\alpha_k(T), T\big) \longrightarrow 0$, we are done.
First, notice that $d\big(\tau_k(\alpha_k(T)), \alpha_k(T)\big) \leq \ln\big(1 + d_O(\tau_k, id_{\mathbb{E}^n})\big)$:
Choose $k$ large enough such that $d_O(\tau_k, id_{\mathbb{E}^n}) = \|\tau_k\| < 2$, and set $\psi = -\frac{\tau_k}{2}, \chi = \frac{\tau_k}{2}$. For $r < \frac{1}{d_O(\tau_k, id_{\mathbb{E}^n})}$ we have $d_O(\psi, id_{\mathbb{E}^n}) = \frac{d_O(\tau_k, id_{\mathbb{E}^n})}{2} < \frac{1}{2r}$, $d_O(\chi, id_{\mathbb{E}^n}) = \frac{d_O(\tau_k, id_{\mathbb{E}^n})}{2} < \frac{1}{2r}$ and $[\psi\big(\tau_k(\alpha_k(T))\big)]_{B_r(O)} = [\chi\big(\alpha_k(T)\big)]_{B_r(O)}$. So, by definition of $R\big(\tau_k(\alpha_k(T)), \alpha_k(T)\big)$, we have:
$R\big(\tau_k(\alpha_k(T)), \alpha_k(T)\big) \geq \frac{1}{d_O(\tau_k, id_{\mathbb{E}^n})} > \frac{1}{2}$, that implies:

$$d\big(\tau_k(\alpha_k(T)), \alpha_k(T)\big) = \ln\Big(1 + \frac{1}{R\big(\tau_k(\alpha_k(T)), \alpha_k(T)\big)}\Big) \leq \ln\big(1 + d_O(\tau_k, id_{\mathbb{E}^n})\big) \longrightarrow 0;$$

when $k \longrightarrow \infty$.
Second, let $\beta_k$ be the square root of $\alpha_k$ in $O(\mathbb{E}^n_O)$ as in the construction in Theorem 3.1.4. Setting $\gamma_k := \beta_k^{-1}$, we have:

$\|\gamma_k - id_{\mathbb{E}^n}\|_{op} = \|\beta_k - id_{\mathbb{E}^n}\|_{op} \leq \|\alpha_k - id_{\mathbb{E}^n}\|_{op}$; by Lemma 1.2.2(ii) and Lemma 3.1.5.

Furthermore, for any $R > 0$,

$$\gamma_k\big(\alpha_k(T)\big) = \beta_k(T) \quad \text{implies} \quad [\gamma_k\big(\alpha_k(T)\big)]_{B_R(O)} = [\beta_k(T)]_{B_R(O)}.$$

Therefore,

$$R\big(\alpha_k(T), T\big) \longrightarrow \infty \ \text{ if } \ \|\alpha_k - id_{\mathbb{E}^n}\|_{op} \longrightarrow 0 \ \text{ for } \ k \longrightarrow \infty;$$

and hence, $d\big(\alpha_k(T), T\big) \longrightarrow 0$, as required. □

## 3.2 Tiling Spaces

We now know what simple tilings are. From each tiling $T$, we will construct a complete space $\Omega_T$ of tilings and study it.

**Definition 3.2.1** The *orbit* of a tiling $T$ is the set $O(T)$ of isometrically shifted copies of $T$. That is:

$$O(T) = \{\phi(T)|\ \phi \in Isom(\mathbb{E}^n)\}.$$

Note that the orbit $O(T)$ may not be complete. See [26] for more examples.

**Definition 3.2.2** A *tiling space* $\Omega$ is a set of simple tilings made up of the same set of prototiles (finitely many up to isometry), that is:

(i) closed under isometry, i.e., if $T \in \Omega$ then $\phi(T) \in \Omega$, for all $\phi \in Isom(\mathbb{E}^n)$.

(ii) complete in the tiling metric, i.e., every Cauchy sequence of tilings in $\Omega$ has a limit in $\Omega$.

**Lemma 3.2.3** *The space of all simple tilings made up of the same set of prototiles (finitely many up to isometry), together with the metric $d$ is a complete metric space.*

Proof Suppose $(T_k)_{k\in\mathbb{N}}$ is a Cauchy sequence of simple tilings. If necessary, pass to a subsequence such that $\ln(1+s_k) = d(T_k, T_{k+1})$ is decreasing and $\sum_1^\infty s_k < \infty$. It follows from the definition of $d$ that there exists $\phi_k, \phi'_k \in Isom(\mathbb{E}^n); d_O(\phi_k, id_{\mathbb{E}^n}), d_O(\phi'_k, id_{\mathbb{E}^n}) < \frac{1}{2}s_k$ such that:

$$(*) \qquad [\phi_k(T_k)]_{B_{\frac{1}{s_k}}(O)} = [\phi'_k(T_{k+1})]_{B_{\frac{1}{s_k}}(O)}$$

Notice that $[\phi_k(T_k)]_{B_{\frac{1}{s_k}}(O)} = \phi_k\big([T_k]_{B_{\frac{1}{s_k}}(\phi_k^{-1}(O))}\big)$. Hence, (*) becomes

$$\phi_k\big([T_k]_{B_{\frac{1}{s_k}}(\phi_k^{-1}(O))}\big) = \phi'_k\big([T_{k+1}]_{B_{\frac{1}{s_k}}(\phi_k'^{-1}(O))}\big),$$

$$\Longrightarrow\ \phi_k'^{-1}\phi_k\big([T_k]_{B_{\frac{1}{s_k}}(\phi_k^{-1}(O))}\big) = [T_{k+1}]_{B_{\frac{1}{s_k}}(\phi_k'^{-1}(O))}.$$

**Claim 1:** $B_{\frac{1}{s_k}}\big((\phi_k'^{-1})(O)\big) \subset B_{\frac{1}{s_{k+1}}}\big(\phi_{k+1}^{-1}(O)\big)$.
**Proof of claim 1:**

$$x \in B_{\frac{1}{s_k}}\big((\phi_k'^{-1})(O)\big) \Longleftrightarrow \|x - (\phi'_k)^{-1}(O)\|_{Eucl} < \frac{1}{s_k}\ . \qquad (3.6)$$

On the other hand,

$$x \in B_{\frac{1}{s_{k+1}}}\big(\phi_{k+1}^{-1}(O)\big) \Longleftrightarrow \|x-(\phi_{k+1})^{-1}(O)\|_{Eucl} < \frac{1}{s_{k+1}}\,. \tag{3.7}$$

By suitably choosing the sequence $(s_k)_{k\in\mathbb{N}}$, this implies together with:

$$\|(\phi'_k)^{-1}(O)-O\| = \|\phi'_k(O)-O\| < \frac{1}{2}s_k,$$

and

$$\|\phi_{k+1}^{-1}(O)-O\| = \|\phi_{k+1}(O)-O\| < \frac{1}{2}s_{k+1};$$

that

$$\|x-\phi_{k+1}^{-1}(O)\| \leq \|x-(\phi'_k)^{-1}(O)\|+\|(\phi'_k)^{-1}(O)-\phi_{k+1}^{-1}(O)\| < \frac{1}{s_k}+\frac{1}{2}s_k+\frac{1}{2}s_{k+1} < \frac{1}{s_{k+1}}.$$

Hence, $x \in B_{\frac{1}{s_{k+1}}}\big(\phi_{k+1}^{-1}(O)\big)$. □

Define $\delta_k := \prod_{l=k}^{\infty}(\phi'_l)^{-1}\phi_l$, where the terms with lower index are to the right. The infinite composition exists because $d_O(\phi_l, id_{\mathbb{E}^n}), d_O(\phi'_l, id_{\mathbb{E}^n}) < \frac{1}{2s_l}$ and $\sum_{l=k}^{\infty} s_l < \infty$. Then:

$$\begin{aligned}\delta_k\big([T_k]_{B_{\frac{1}{s_k}}(\phi_k^{-1}(O))}\big) &= \delta_{k+1}(\phi'_k)^{-1}\phi_k\big([T_k]_{B_{\frac{1}{s_k}}(\phi_k^{-1}(O))}\big)\\ &= \delta_{k+1}\big([T_{k+1}]_{B_{\frac{1}{s_k}}((\phi'_k)^{-1}(O))}\big)\\ &\subset \delta_{k+1}\big([T_{k+1}]_{B_{\frac{1}{s_{k+1}}}(\phi_{k+1}^{-1}(O))}\big) \quad \text{by claim 1.}\end{aligned} \tag{3.8}$$

Hence, $T = \bigcup_{k=1}^{\infty}\delta_k\big([T_k]_{B_{\frac{1}{s_k}}(\phi_k^{-1}(O))}\big)$ is a simple tiling made up of the same set of prototiles as all the $T_k$.

**Claim 2:** $d(T_k, T) \longrightarrow 0$.

**Proof of claim 2:** We need to find a sequence $(t_k)_{k\in\mathbb{N}} \longrightarrow 0$ such that $d_O(T_k, T) \leq ln(1+t_k)$. Notice that

$$\delta_k\big([T_k]_{B_{\frac{1}{s_k}}(\phi_k^{-1}(O))}\big) = [\delta_k(T_k)]_{B_{\frac{1}{s_k}}\left(\delta_k^{-1}(\phi_k^{-1}(O))\right)} = [T]_{B_{\frac{1}{s_k}}\left(\delta_k^{-1}(\phi_k^{-1}(O))\right)}.$$

So it is enough to choose $t_k$ such that $d_O(\delta_k, id_{\mathbb{E}^n}) < \frac{1}{2}t_k$ and

$$B_{\frac{1}{t_k}}(O) \subset B_{\frac{1}{s_k}}(\delta_k^{-1}(\phi_k^{-1}(O)))\,,$$

because then $[T]_{B_{\frac{1}{s_k}}(\delta_k^{-1}(\phi_k^{-1}(O)))} \supset [T]_{B_{\frac{1}{t_k}}(O)} = [\delta_k(T_k)]_{B_{\frac{1}{t_k}}(O)}$. We achieve that by choosing $\frac{1}{t_k} \leq \frac{1}{s_k} - d_O(\delta_k^{-1}\phi_k^{-1}, id_{\mathbb{E}^n})$. Since $d_O(\delta_k^{-1}\phi_k^{-1}, id_{\mathbb{E}^n}) \longrightarrow 0$ by the considerations above and also $s_k \longrightarrow 0, d_O(\delta_k, id_{\mathbb{E}^n}) \longrightarrow 0$, the $t_k$ can be chosen to converge to 0 as requested, too.

□

**Definition 3.2.4** The *hull* $\Omega_T$ of a tiling $T$ is the closure of $O(T)$ in the space of all simple tilings made up of the same set of prototiles as $T$.

To build our knowledge about hulls let us look at the hulls of some simple 1-dimensional tilings in the following two examples.

**Example 3.2.5** Let $T_0$ be a tiling with just one kind of tile "white tile of length one", as in figure 3.1. First of all we identify the orbit of $T_0$. Notice that

$$Isom(\mathbb{E}^1) = Trans(\mathbb{E}^1) \rtimes O(\mathbb{E}^1_o),$$

where $Trans(\mathbb{E}^1) = \mathbb{R}$ and $O(\mathbb{E}^1_o) = \pm 1 \subset GL(\mathbb{R}) = \{(a) : a \in \mathbb{R} - \{0\}\}$. Translations and reflections at any point transform all white tiles into all white tiles i.e. every element is determined by the image one boundary point of a white tile which means $O(T_0)$ is a circle. For the hull of $T_0$, reflections have norm 1 that means, isometries with distance $< 1$ from $id_{\mathbb{E}^1}$ must be translations. Therefore, the hull of $T_0$ is again $O(T_0)$ which is just a circle.

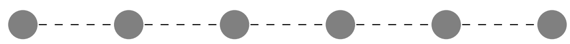


Figure 3.1: White tile; see [26].

**Example 3.2.6** Let $T_1$ be the "one black tile" tiling, as in figure 3.2. By repeating the same argument as above, we see that translations and reflections at any point will move the black tile on and on to the left and to the right, hence $O(T_1) = \mathbb{R}$. All white tilings of Example 3.2.5 are in the closure of $O(T_1)$, since every patch of this tiling can be found in a tiling of $O(T_1)$, both far enough to the right of the origin and far enough to the left.

Examples of hulls of higher-dimensional tilings will follow in Chapter 5.

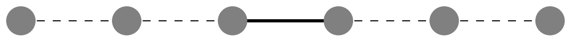


Figure 3.2: One black tile; [26].

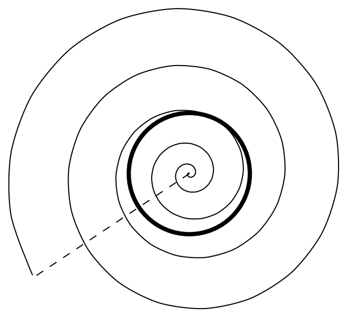


Figure 3.3: The hull of "one black tile"; see [26].

**Theorem 3.2.7** *[26]*
*If $T$ is a simple tiling, then $\Omega_T$ is compact.*

Proof We aim to show that every sequence in $\Omega_T$ has a convergent subsequence. As $T$ is a simple tiling, there is only a finite number of tile types up to isometry. Notice also that the number of ways in which tiles can abut are finite, i.e., for each radius $r$, there are only a finite number of possible patches, covering a ball $B_r(0)$ up to isometries $\tau \cdot \alpha$ with $\alpha \in O(\mathbb{E}^n_O)$ and $\tau$ a translation by a distance smaller than the diameter of the largest tile. Since $O(\mathbb{E}^n_O)$ is compact, this set of isometries is (relatively) compact. Hence, in any sequence of tilings in $\Omega_T$, there is a subsequence that converges on $B_r$. Using Cantor's Diagonalization argument, from the subsequence that converges on $B_1$, pick a subsequence that converges on $B_2$, a subsequence of that converges on $B_3$, a subsequence of that converges on $B_4$, and so on. Now, we take the first element of the sequence that converges on $B_1$, and the second element of the sequence that converges on $B_2$, and so on. This sequence of elements converges on every bounded set, and so, will form a Cauchy sequence in the tiling space. Hence, this sequence has a limit in $\Omega_T$ since $\Omega_T$ is complete.

□

# Chapter 4

# Equivalences of Tilings and Tiling Spaces

What does it mean when we say that two simple tilings or two simple tiling spaces are equivalent? There are several different notions that explain this, as we will see in this chapter.

Since we introduced a topology on tiling spaces in the previous Chapter, we have the notion of a continuous map $f : \Omega \to \Omega'$ between two tiling spaces $\Omega, \Omega'$. The map $f$ is a homeomorphism if $f$ is $1-1$, onto and $f^{-1}$ is also continuous.
For a homeomorphism between simple tiling hulls, we only need to check whether $f$ is continuous, $1-1$ and onto, since $f^{-1}$ is automatically continuous as $\Omega_T$ is compact according to Theorem 3.2.7.

Next, we want to consider continuous maps respectively homeomorphisms, which interact properly with the action of the isometry group on the tiling spaces.

## 4.1 Topological Isometric Conjugacy

Originally, such factor maps were defined only using translations. When requiring $f(\phi(T)) = \phi(f(T))$ for arbitrary isometries, many homeomorphisms, which are a topological conjugacy for translations, are no longer topological isometric-conjugacies. The reason is that general isometries do not commute with each other, but translations do.

**Example 4.1.1** Consider the map $f$ of a tiling space $\Omega$ onto itself, such that $f(T) = \phi(T)$, for $\phi \in Isom(\mathbb{E}^n)$, and $T \in \Omega$:
For example, if $T$ is the standard lattice in $\mathbb{E}^2$ and $\Omega_T$ its hull (with respect to isometries) as constructed in Example 5.1.5. Choose $\phi$ as a translation by $v$, that is, $f : T' \to T' + v$ for any $T' \in \Omega_T$. It is clear that $f$ is a factor map with respect to translations, since $f(T' + w) = T' + w + v = f(T') + w$. On the other hand, $f(\psi(T)) = \psi(T) + v \neq \psi(T + v)$, if $\psi$ is a rotation by 90° around (0,0), and $v = (\frac{1}{4}, \frac{1}{4})$: Then $\psi(T) = T$ implies that $\psi(T) + v = T + v$, but $\psi(T + v) \neq T + v$, which means that $f$ is not a factor map with respect to isometries.

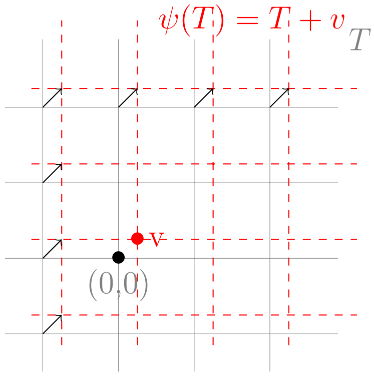



Figure 4.1: Translated standard lattice tilings of $\mathbb{E}^2$ .

**Remark 4.1.2** If we allow for an additional isometry on the affine space $\mathbb{E}^n$ (in our case, just $\phi$), then $\Omega_T$ and $\Omega_{T'}$ become topologically conjugated again (see also Proposition 4.2.7).

This motivates the following definition:

**Definition 4.1.3** For tiling spaces $\Omega$ and $\Omega'$, the continuous map $f : \Omega \to \Omega'$ is called an *isometric-factor map* ($\gamma$-i-factor map for short) if there exists $\gamma \in Isom(\mathbb{E}^n)$, such that for all $\phi \in Isom(\mathbb{E}^n)$, and for all $T \in \Omega$:

$$f\big(\phi(T)\big) = \big(\gamma\phi\gamma^{-1}\big)(T).$$

If $f$ is also a homeomorphism, then $f$ is called a *topological isometric-conjugacy* (topological $\gamma$-i-conjugacy for short).

**Remark 4.1.4** To distinguish between translationally and isometrically factor map/topological conjugacy, we add "translationally" and "isometrically".

If we only speak of factor map/topological conjugacy; we always mean "isometrically factor map/topological conjugacy ".
Similarly, for $\gamma$, we often leave $\gamma$ out if it is clear from the context what $\gamma$ is.

**Lemma 4.1.5** *The inverse map $f^{-1} : \Omega' \to \Omega$ of a topological i-conjugacy $f : \Omega \to \Omega'$ is also a topological i-conjugacy.*

Proof For $T' \in \Omega'$, there exists a unique $T \in \Omega$, such that $f(T) = T'$ (as $f$ is a homeomorphism). $f$ is a topological i-conjugacy; therefore, there exists $\gamma \in Isom(\mathbb{E}^n)$ such that

$$\begin{aligned} f\big(\phi'(T')\big) &= (\gamma^{-1}\phi'\gamma)(T') \ \ ;\forall \phi' \in Isom(\mathbb{E}^n) \\ &\Longleftrightarrow \phi'(T') = f^{-1}\big((\gamma^{-1}\phi'\gamma)(T')\big) \\ &\Longleftrightarrow \gamma\phi\gamma^{-1}(T') = f^{-1}\big(\phi(T')\big) \ \ ; \text{ as } \phi' = \gamma\phi\gamma^{-1} \ . \end{aligned} \tag{4.1}$$

By choosing $\gamma' = \gamma^{-1}$, we get:

$$f^{-1}\big(\phi(T')\big) = \big(\gamma'^{-1}\phi\gamma'(T')\big) \ ;\forall T' \in \Omega'.$$

Hence, $f^{-1}$ is a topological $\gamma^{-1}$-i-conjugacy. □

**Remark 4.1.6** If we define topological conjugacies using only translations, the definition reduces to:

$$\forall \tau \in Trans(\mathbb{E}^n), \quad f(T + \tau) = f(T) + \tau.$$

Since conjugating in the abelian group, $Trans(\mathbb{E}^n)$ is trivial. One could also allow for automorphisms of $Isom(\mathbb{E}^n)$ different from $\phi \to \gamma^{-1}\phi\gamma$ (if they exists at all). These notions of topological conjugacy on tiling spaces are different, as Example 4.1.1 shows.

**Example 4.1.7** Let $T$ be the standard lattice tiling constructed from tiles which are squares with vertices of the form $(n, m), (n+1, m), (n, m+1), (n+1, m+1)$; $n, m \in \mathbb{Z}$. and $T'$ the slanted lattice tiling obtained from the rhomb with vertices $(0, 0), (1, 0), (2, 1), (1, 1)$ and all its translations by vectors $(n, m) \in \mathbb{Z}^2$, see Figure 4.2.

There is a natural map $f : \Omega_{T'} \to \Omega_T$, such that $f$ is a 4-1 map.
Given a tiling $\bar{T}' \in \Omega_{T'}$, we obtain a tiling $\bar{T} \in \Omega_T$ by halving each of the rhombs in $\bar{T}'$ and uniting two of these halves to a square whenever they meet face-to-face along a diagonal edge.

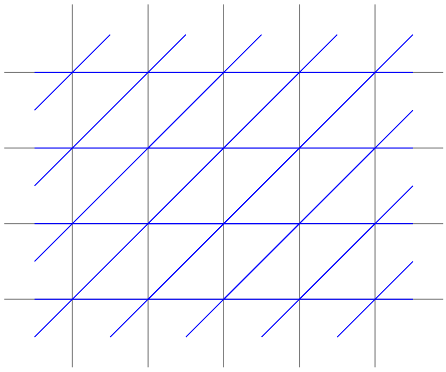


Figure 4.2: Standard lattice tiling and slanted lattice tiling of $\mathbb{E}^2$.

The following picture shows how we obtain the same standard lattice tiling from 4 different slanted lattice tilings:

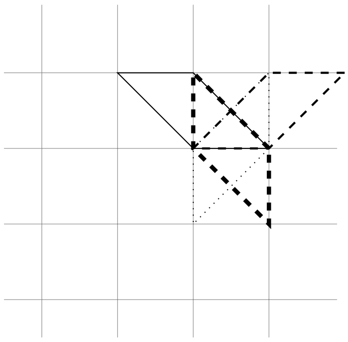


Figure 4.3: Standard lattice tile can be obtained from one of these four different slanted lattice tiles.

The four slanted lattice tiles in Figure 4.3, are mapped to each other by isometries in the cosets of $D_2$ in $D_4$.
Obviously, halving the rhombs commutes with applying an isometry on them, so that $f$ is an i-factor map (with $\gamma = id_{\mathbb{E}^2}$).

## 4.2 Mutual Local Derivability

The strongest notion of equivalence used in the literature is Mutual Local Derivability; (MLD for short).

To obtain our results on crystallographic tilings in the sense of Chapter 5; we have to extend the definition of MLD in [26] to isometrical MLD (Isometrical case) by imposing the taxicab metric $d_O$ on $Isom(\mathbb{E}^n)$ (as in section 1.2), then replacing translations by isometries in the original definition, as we will see.

### 4.2.1 MLD Tiling Spaces

**Definition 4.2.1** If $\Omega$ and $\Omega'$ are tiling spaces, we say that $\Omega'$ is *isometric locally derivable* ($\gamma$-i-LD for short) from $\Omega$ if there is a surjective $\gamma$-i-factor map $f : \Omega \to \Omega'$ that is defined locally. More precisely, there exists a radius $R$ such that, whenever two tilings $T_1, T_2 \in \Omega$ agree on a ball of radius $R$ around $x \in \mathbb{E}^n$, the tilings $f(T_1)$ and $f(T_2)$ in $\Omega'$ agree on the patches covering $\gamma(x)$, i.e.,

$$[T_1]_{B_R(x)} = [T_2]_{B_R(x)} \text{ then } [f(T_1)]_{\{\gamma(x)\}} = [f(T_2)]_{\{\gamma(x)\}} \,.$$

If these implications hold, $R$ will be called the *i-LD radius*.
If $f^{-1}$ is a topological $\gamma^{-1}$-i-conjugacy making $\Omega$ $\gamma^{-1}$- i-LD from $\Omega'$, then $\Omega$ and $\Omega'$ are called *i-MLD*.

**Remark 4.2.2** To distinguish between translationally and isometrically MLD, we add "translationally" and "isometrically".
If we only speak of MLD, we always mean "isometrically-MLD".
Similarly, for $\gamma$-LD, we often leave out $\gamma$ if it is clear from the context what $\gamma$ is.

**Example 4.2.3** Consider the standard lattice tiling $T$ in the Euclidean plane. By dividing the tiling $T$ into halves, we get a new tiling $T'$ (Figure 4.4). This gives a natural map $f$ between the tiling spaces $\Omega_T$ and $\Omega_{T'}$ by setting $f\big(\phi(T)\big) = \phi(T')$ for all $\phi \in Isom(\mathbb{E}^n)$, since the hull $\Omega_T$ of $T$ and $\Omega_{T'}$ of $T'$ are equal to the orbits of $T$ and $T'$ (see Example 5.1.5).
Clearly, $f$ is continuous and an i-factor map (using $\gamma = id_{\mathbb{E}^2}$), as its defining equations shows.
Also, $f$ is surjective, but is not a homeomorphism as it is not $1-1$, since a given tiling in $\Omega_{T'}$ can be obtained from two tilings $T, \bar{T}$ in $\Omega_T$ differing by a translation of length $\frac{1}{2}$ (see Figure 4.4). Therefore, $f(\phi(T)) = f(\psi(T))$ and

$\phi \neq \psi$. The i-LD property is satisfied because we can obtain the red tile by taking one of the black tiles and subdividing it, which is a local condition. Then, we can find a radius $R$ around $x \in \mathbb{E}^n$, such that:

$$[T]_{B_R(x)} = [T']_{B_R(x)} \implies [f(T)]_{\{x\}} = [f(T')]_{\{x\}}.$$

For example, $R > \sqrt{2}$ will do, because then, $B_R(x)$ will always contain a square of any tiling in $\Omega_T$, for any $x \in \mathbb{E}^2$.

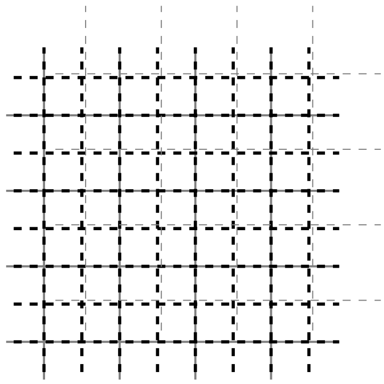


Figure 4.4: LD but not MLD lattice tilings ($T$=——, $T'$=- - -, $\bar{T}$=- - -).

**Lemma 4.2.4** *If $\Omega$ and $\Omega'$ are two i-MLD tiling spaces, with i-MLD radius $R$, then, for tilings $T_1, T_2 \in \Omega$,*

$$[T_1]_{B_{r+R}(x)} = [T_2]_{B_{r+R}(x)} \implies [f(T_1)]_{B_r(\gamma(x))} = [f(T_2)]_{B_r(\gamma(x))}; \quad \textit{for all } r \geq 0.$$

Proof By covering the ball $B_{r+R}(x)$ with balls $B_R(x')$ where $|x' - x| < r \iff x' \in B_r(x)$, we will have:

$$\begin{aligned} [T_1]_{B_{r+R}(x)} = [T_2]_{B_{r+R}(x)} &\implies [T_1]_{B_R(x')} = [T_2]_{B_R(x')} \ ; \quad \forall x' \in B_r(x) \\ &\implies [f(T_1)]_{\{x'\}} = [f(T_2)]_{\{x'\}}; \quad \forall x' \in B_r(\gamma(x)) \\ &\implies \bigcup_{x' \in B_r(\gamma(x))} [f(T_1)]_{\{x'\}} = \bigcup_{x' \in B_r(\gamma(x))} [f(T_2)]_{\{x'\}} \\ &\implies [f(T_1)]_{B_r(\gamma(x))} = [f(T_2)]_{B_r(\gamma(x))}; \end{aligned} \tag{4.2}$$

since $\bigcup_{x' \in B_r(x)} [f(T_1)]_{\{x'\}}$ covers $B_r(\gamma(x))$. □

**Lemma 4.2.5** *Being i-MLD has an equivalence relation on tiling spaces.*

PROOF Clearly, by definition, i-MLD is reflexive and symmetric. For transitivity, suppose $T_1, T_2 \in \Omega$ and $T_1' = f(T_1), T_2' = f(T_2), T_1'' = g(T_2')$,
$T_2'' = g(T_2')$, where $f$ is a topological $\gamma_f$-i-conjugacy with i-LD radius $R_f$, and $g$ is a topological $\gamma_g$-i-conjugacy with i-LD radius $R_g$.
If $[T_1]_{B_{R_f+R_g}(x)} = [T_2]_{B_{R_f+R_g}(x)}$, then, by using Lemma 4.2.4 we have:

$$[T_1']_{B_{R_g}(\gamma_f(x))} = [T_2']_{B_{R_g}(\gamma_f(x))}.$$

Hence, by Definition 4.2.1, we have:

$$[T_1'']_{\left\{\gamma_g\left(\gamma_f(x)\right)\right\}} = [T_2'']_{\left\{\gamma_g\left(\gamma_f(x)\right)\right\}};$$

and $g \circ f$ is a topological $(\gamma_g \circ \gamma_f)$-i-conjugacy.

□

**Remark 4.2.6** :

(i) It is enough to check the $\gamma$-i-LD property of a $\gamma$-i-factor map $f$ at a given point $x$:
Take an isometry $\phi$ mapping $y$ to $x$, i.e., $\phi(y) = x$, such that
$\forall\ T_1, T_2 \in \Omega$, we have:

$$\begin{aligned}
[T_1]_{B_R(y)} = [T_2]_{B_R(y)} &\implies [\phi(T_1)]_{B_R(x)} = [\phi(T_2)]_{B_R(x)} \quad ; \phi(T_1), \phi(T_2) \in \Omega \\
&\implies [f\big(\phi(T_1)\big)]_{\{\gamma(x)\}} = [f\big(\phi(T_2)\big)]_{\{\gamma(x)\}} \\
&\qquad\qquad \text{as } f \text{ is } \gamma\text{-i-LD} \\
&\implies [\gamma\phi\gamma^{-1}f(T_1)]_{\{\gamma(x)\}} = [\gamma\phi\gamma^{-1}f(T_2)]_{\{\gamma(x)\}}; \\
&\qquad\qquad \text{as } f \text{ is } \gamma\text{-i-factor map} \\
&\implies [\phi\gamma^{-1}f(T_1)]_{\{x\}} = [\phi\gamma^{-1}f(T_2)]_{\{x\}} \\
&\implies [f(T_1)]_{\{\gamma\phi^{-1}(x)\}} = [f(T_2)]_{\{\gamma\phi^{-1}(x)\}} \\
&\implies [f(T_1)]_{\gamma(y)} = [f(T_2)]_{\gamma(y)}.
\end{aligned} \tag{4.3}$$

□

(ii) The properties of $f(T)$ near the point $\gamma(x)$ are determined by the properties of $T$ on a ball around $x$, for all tilings $T$ in the tiling space $\Omega$.

(iii) Originally, $i-MLD$ was defined using only translations, and for tilings rather than for tiling spaces; (see Definition 4.2.9).

**Proposition 4.2.7** *The map $f : \Omega \to \Omega$ on a tiling space $\Omega$, defined by $f : T \to \phi(T)$ is a topological conjugacy defining an i-MLD homeomorphism; $\phi \in Isom(\mathbb{E}^n)$.*

PROOF To prove that $f$ is continuous, we use a metric on $\Omega$ constructed from the points $O$ and $\phi(O)$. Since, as per Proposition 3.1.3, the induced topologies are equal, it would suffice to show that:

$$\forall \epsilon > 0\ \exists \delta > 0 : d_O(T, T') < \delta \implies d_{\phi(O)}(f(T), f(T')) < \epsilon.$$

Choose $0 < \epsilon < ln(\frac{3}{2})$ and pick $\delta := \epsilon$. $d_O(T, T') < \delta$, which means that there exists $\psi, \rho \in Isom(\mathbb{E}^n)$, where $d_O(\psi, id_{\mathbb{E}^n}), d_O(\rho, id_{\mathbb{E}^n}) < \frac{1}{2r}$ with $r := \frac{1}{e^\delta - 1}$, such that:

$$\begin{aligned}
[\psi(T)]_{B_r(O)} &= [\rho(T')]_{B_r(O)} \\
\implies \phi([\psi(T)]_{B_r(O)}) &= \phi([\rho(T')]_{B_r(O)}) \\
\implies [\phi\psi(T)]_{B_r(\phi(O))} &= [\phi\rho(T')]_{B_r(\phi(O))} \\
\implies [\phi\psi\phi^{-1} f(T)]_{B_r(\phi(O))} &= [\phi\rho\phi^{-1} f(T')]_{B_r(\phi(O))} \\
\implies d_{\phi(O)}(f(T), f(T')) &< ln(1 + \frac{1}{r}) = \delta = \epsilon
\end{aligned}$$

since $d_{\phi(O)}(\phi\psi\phi^{-1}, id_{\mathbb{E}^n}) = d_O(\psi, id_{\mathbb{E}^n})$ and $d_{\phi(O)}(\phi\rho\phi^{-1}, id_{\mathbb{E}^n}) = d_O(\rho, id_{\mathbb{E}^n})$ by Lemma 1.2.2 (v). As required, this shows that $f$ is continuous. □
Clearly, $f$ is $1-1$, since if we map two different tilings by an isometry $\phi$, we obtain two different tilings; also, $f$ is onto since $f\big(\phi^{-1}(T)\big) = T$. Hence, $f$ is a homeomorphism.
$f$ is a topological $\phi$-i-conjugacy, since

$$f\big(\psi(T)\big) = \phi\psi(T) = \phi\psi\phi^{-1}\phi(T) = \phi\psi\phi^{-1} f(T);\ \psi \in Isom(\mathbb{E}^n).$$

Finally, we aim to show that $f$ is i-MLD for any MLD radius $R$. First, notice that $f$ is $\phi$-i-LD:

$$\begin{aligned}
[T_1]_{B_R(x)} = [T_2]_{B_R(x)}\ ; \forall T_1, T_2 \in \Omega &\implies [\phi(T_1)]_{B_R(\phi(x))} = [\phi(T_2)]_{B_R(\phi(x))} \\
&\implies [f(T_1)]_{\{\phi(x)\}} = [f(T_2)]_{\{\phi(x)\}}.
\end{aligned} \tag{4.4}$$

Similarly, $f^{-1}$ is also $\phi$-i-LD, since $f^{-1}$ is given by $\phi^{-1}$. Hence, $f$ is i-MLD. □

**Remark 4.2.8** There are tiling spaces which are topological i-conjugacies, but not i-MLD. See, for example, the usual Penrose tiling space and the rational Penrose tiling space in ([26], pp. 41-44).

### 4.2.2 MLD Tilings

**Definition 4.2.9** If $T$ and $T'$ are tilings, $T'$ is said to be *$\gamma$-i-locally derivable* from $T$, if for some finite radius $R$, the properties of $T'$ at each point $\gamma(x) \in \mathbb{E}^n$ are determined by the properties of $T$ in a ball of radius $R$ around $x$. More formally, $T'$ is $\gamma$-i-locally derivable from $T$ if there exists a radius $R$ such that, for $x \in \mathbb{E}^n$ and $\phi \in Isom(\mathbb{E}^n)$, we have the following property:

$$[T]_{B_R(x)} = [\phi(T)]_{B_R(x)} \text{ implies } [T']_{\{\gamma(x)\}} = [\gamma\phi\gamma^{-1}T']_{\{\gamma(x)\}}.$$

If these implications hold, then the radius $R$ will be called the *i-LD radius*. If $T$ is $\gamma^{-1}$-i-locally derivable from $T'$, and $T'$ is $\gamma$-i-locally derivable from $T$, then $T$ and $T'$ are called *i-MLD*.

Notice that Remark 4.2.2 is applied in this section as well.

Notice that S-MLD definition which was mentioned in [5] does not contain conjugation with $\gamma$ (this is necessary for $T$ and $\gamma(T)$ to be MLD); and does not discuss its properties. In particular its connection to MLD of the hulls of $T$ and $T'$, as we will see later on.

The following example shows that $T$ and $T'$ can be translationally-MLD, but **not** isometrically-MLD.

**Example 4.2.10** The standard lattice tiling $T$, and the slanted lattice tiling $T'$ are translationally MLD. $T'$ is translationally LD from $T$ because $[T]_{B_r(x)} = [T-y]_{B_r(x)}$ implies $y \in \mathbb{Z}^2$, as the vertices of $T$ are all in $\mathbb{Z}^2$. But $y \in \mathbb{Z}^2$ also implies $T' = T'-y$, hence we have an equality of patches $[T']_{\{x\}} = [T'-y]_{\{x\}}$. In the same way we can deduce that $T$ is translationally LD from $T'$. Hence, $T$ and $T'$ are translationally-MLD.
On the other hand $T'$ is not isometrically LD from $T$: Rotating $T$ by $90°$ around a midpoint $x$ of a tile of $T$ does not change $T$ but Figure 4.5 shows that the patches of $T'$ and the rotated tiling $T'$ covering $x$ are certainly different.

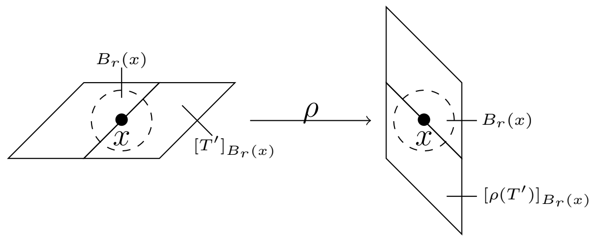



Figure 4.5: Rotation of tiles counterclockwise by 90°.

**Lemma 4.2.11** *If $T'$ is $\gamma$-i-locally derivable from $T$ with LD-radius $R$, then for every $r$:*

$$[T]_{B_{r+R}(x)} = [\phi(T)]_{B_{r+R}(x)} \implies [T']_{B_r(\gamma(x))} = [\gamma\phi\gamma^{-1}T']_{B_r(\gamma(x))}.$$

Proof By covering $B_{r+R}(x)$ with balls $B_R(x')$, and $B_{r+R}(y)$ with balls $B_R(y')$, where $|x'-x| < r$ and $|y'-y| < r \iff x' \in B_r(x)$ and $y' \in B_r(y)$, we will have:

$$\begin{aligned}
[T]_{B_{r+R}(x)} = [\phi(T)]_{B_{r+R}(x)} &\implies [T]_{B_R(x')} = [\phi(T)]_{B_R(x')} \; ; \quad \forall x' \in B_r(x) \\
&\implies [T']_{\{\gamma(x')\}} = [\gamma\phi\gamma^{-1}T']_{\{\gamma(x')\}} \; ; \quad \forall x' \in B_r(x) \\
&\implies \bigcup_{x' \in B_r(\gamma(x))} [T']_{\{\gamma(x')\}} = \bigcup_{x' \in B_r(\gamma(x))} [\gamma\phi\gamma^{-1}T']_{\{x'\}} \\
&\implies [T']_{B_r(\gamma(x))} = [\gamma\phi\gamma^{-1}T']_{B_r(\gamma(x))}.
\end{aligned} \tag{4.5}$$

□

**Lemma 4.2.12** *Being i-MLD has an equivalence relation on tilings.*

Proof A tiling is trivially i-MLD with itself. If $T$ and $T'$ are i-MLD tilings, then they are each locally derivable from the other; therefore, the property is symmetric. Finally, we show transitivity. Suppose $T, T'$ and $T''$ are tilings, such that $T'$ is $\gamma$-i-locally derivable from $T$ with i-LD radius $R$, and $T''$ is $\gamma'$-i-locally derivable from $T'$ with i-LD radius $R'$. If $[T]_{B_{R+R'}(x)} = [\phi(T)]_{B_{R+R'}(x)}$, then, by Lemma 4.2.11, $[T']_{B_{R'}(\gamma(x))} = [\gamma\phi\gamma^{-1}T']_{B_{R'}(\gamma(x))}$. Hence, by Definition 4.2.9, we have:

$$[T'']_{\{\gamma'\gamma(x)\}} = [\gamma'\gamma\phi\gamma^{-1}\gamma'^{-1}T'']_{\{\gamma'\gamma(x)\}}.$$

Hence, $T''$ is $(\gamma'\gamma)$-i-LD from $T$, as required to show transitivity.

□

**Proposition 4.2.13** *If a tiling $T_2$ is obtained by applying an isometry $\theta \in Isom(\mathbb{E}^n)$ on a tiling $T_1$, that is, $T_2 = \theta(T_1)$; then, $T_1$ and $T_2$ are i-MLD.*

PROOF Assume that $[T_1]_{B_r(x)} = [\phi(T_1)]_{B_r(x)}$:

$$\Longrightarrow [T_1]_{\{x\}} = [\phi(T_1)]_{\{x\}}$$
$$\Longrightarrow \theta\big([T_1]_{\{x\}}\big) = \theta\big([\phi(T_1)]_{\{x\}}\big)$$
$$\Longrightarrow [\theta(T_1)]_{\{\theta(x)\}} = [\theta\phi(T_1)]_{\{\theta(x)\}}$$
$$\Longrightarrow [T_2]_{\{\theta(x)\}} = [\theta\phi\theta^{-1}T_2]_{\{\theta(x)\}}.$$

Hence, $T_2$ is $\theta$-i-LD from $T_1$. Similarly, $T_1$ is $\theta^{-1}$-i-LD from $T_2$, and so, $T_1$ and $T_2$ are i-MLD.

□

**Lemma 4.2.14** *Let $T_n \longrightarrow T$ and $S_n \longrightarrow S$ convergent series of tilings, such that $[T_n]_{B_r(x)} = [S_n]_{B_r(x)}$ for some $r > 0$. Then,*

$$[T]_{\{x\}} = [S]_{\{x\}}.$$

PROOF Let $t \in [T]_{\{x\}}$ be a tile containing $x$. Since $T_n \longrightarrow T$, we see that there exist tiles $t_n \in [T_n]_{B_r(x)}$ such that $t_n \longrightarrow t$. This implies $t_n \in [S_n]_{B_r(x)}$ as $[T_n]_{B_r(x)} = [S_n]_{B_r(x)}$. Next, $t \in [S]_{\{x\}}$, since $S_n \longrightarrow S$. Hence, $[T]_{\{x\}} = [S]_{\{x\}}$.

□

**Lemma 4.2.15** *If $T$ and $T'$ are i-MLD tilings, then their associated hulls $\Omega_T$ and $\Omega_{T'}$ are i-MLD.*

PROOF Suppose $T'$ is $\gamma$-i-LD from $T$ with i-LD-radius $R$, and $T$ is $\gamma^{-1}$-i-LD from $T'$ with i-LD-radius $R$. Construct the map $f : \Omega_T \to \Omega_{T'}$ by setting $f(T) = T'$ and extend it to the orbit of $T$ by $f(\phi(T)) = \gamma\phi\gamma^{-1}(T')$.
**Claim 1:** The map $f : O(T) \to O(T')$ is continuous with respect to the topologies induced from the hulls $\Omega_T$ and $\Omega_{T'}$.
**Proof of claim 1:** Let $T_n \to \bar{T}$ be a convergent sequence in $O(T)$. Choose $\phi_n, \bar{\phi} \in Isom(\mathbb{E}^n)$ such that $T_n = \phi_n(T)$ and $\bar{T} = \bar{\phi}(T)$. Since the orbit $O(T)$ can be a nowhere closed dense subset of the hull $\Omega_T$, we cannot assume that $\phi_n \to \bar{\phi}$. Instead we combine the definition of the distance between tilings and that of local derivability.
$T_n \to \bar{T}$ tells us that there exists a large $R \gg 0$ and $\psi_n, \bar{\psi_n}$ tending to $id_{\mathbb{E}^n}$ such that

$$[\psi_n\phi_n(T)]_{B_R(x)} = [\bar{\psi_n}\bar{\phi}(T)]_{B_R(x)}.$$

This implies

$$[T]_{B_R(\phi_n^{-1}\psi_n^{-1}(x))} = [\phi_n^{-1}\psi_n^{-1}\bar{\psi}_n\bar{\phi}(T)]_{B_R(\phi_n^{-1}\psi_n^{-1}(x))}.$$

Since $R$ will eventually be much larger than the LD-radius of $T$ and $T'$ we may conclude

$$[T']_{B_{R'}(\gamma\phi_n^{-1}\psi_n^{-1}(x))} = [\gamma\phi_n^{-1}\psi_n^{-1}\bar{\psi}_n\bar{\phi}\gamma^{-1}(T')]_{B_{R'}(\gamma\phi_n^{-1}\psi_n^{-1}(x))} \ ,$$

for some $R' > R/2$. Hence

$$[\gamma\phi_n\gamma^{-1}(T')]_{B_{R'}(\gamma\psi_n^{-1}(x))} = [\gamma\psi_n^{-1}\bar{\psi}_n\bar{\phi}\gamma^{-1}(T')]_{B_{R'}(\gamma\psi_n^{-1}(x))}.$$

Setting $\bar{\psi}'_n := \gamma\psi_n^{-1}\bar{\psi}_n^{-1}\gamma^{-1}$ and possibly further reducing $R'$ by an arbitrarily small amount we obtain

$$[\gamma\phi_n\gamma^{-1}(T')]_{B_{R'}(\gamma(x))} = [\bar{\psi}'_n\gamma\bar{\phi}\gamma^{-1}(T')]_{B_{R'}(\gamma(x))}$$

since $d_O(\psi_n^{-1}, id_{\mathbb{E}^n}) \longrightarrow 0$. Then Lemma 1.2.2 (vii) implies

$$\gamma\phi_n\gamma^{-1}(T') \longrightarrow \gamma\bar{\phi}\gamma^{-1}(T').$$

□

**Claim 2:** $f$ is a $\gamma$-i-factor map, that is: $f\big(\phi(\bar{T})\big) = \gamma\phi\gamma^{-1}f(\bar{T}) \ \ \forall \bar{T} \in \Omega_T$.
**Proof of claim 2:** If $\bar{T} = lim_{n\to\infty}\phi_n(T)$, then:

$$\begin{aligned} f(\bar{T}) &= f\big(lim_{n\to\infty}\phi_n(T)\big) \\ &= lim_{n\to\infty} f\big(\phi_n(T)\big); \quad \text{by claim 1} \\ &= lim_{n\to\infty}\gamma\phi_n\gamma^{-1}f(T). \end{aligned} \tag{4.6}$$

Now, we have,

$$\begin{aligned} \gamma\phi\gamma^{-1}f(\bar{T}) &= lim_{n\to\infty}\gamma\phi\phi_n\gamma^{-1}f(T) \\ &= lim_{n\to\infty} f\big(\phi\phi_n T\big) \\ &= f\big(\phi\big(lim_{n\to\infty}\phi_n(T)\big)\big); \quad \text{by claim 1} \\ &= f\big(\phi(\bar{T})\big). \end{aligned} \tag{4.7}$$

□

**Claim 3:** $f^{-1}$ is a $\gamma^{-1}$-i-factor map.
**Proof of claim 3:** $f^{-1}$ can be defined by $f^{-1}(T') = T$ and extended to the orbit, and then to the hull as above. Hence, similar to the proof of claim 2, we will have:

$$f^{-1}\big(\phi(\bar{T}')\big) = \gamma^{-1}\phi\gamma f^{-1}(\bar{T}') \ \ \forall \bar{T}' \in \Omega_{T'}.$$

□

**Claim 4:** $f$ is $\gamma$-i-LD.
**Proof of claim 4:** We first consider the case that $T_1, T_2$ are in the orbit of $T$, with $T_1 = \phi(T)$ and $T_2 = \psi(T)$.

$$\begin{aligned}
[\phi(T)]_{B_R(x)} = [\psi(T)]_{B_R(x)} &\Longrightarrow \phi\big([T]_{B_R(\phi^{-1}(x))}\big) = [\psi(T)]_{B_R(x)} \\
&\Longrightarrow [T]_{B_R(\phi^{-1}(x))} = [\phi^{-1}\psi(T)]_{B_R(\phi^{-1}(x))} \\
&\Longrightarrow [T']_{\{\gamma(\phi^{-1}(x))\}} = [\gamma\phi^{-1}\psi\gamma^{-1}T']_{\{\gamma(\phi^{-1}(x))\}} \\
&\Longrightarrow [\phi\gamma^{-1}T']_{\{x\}} = [\psi\gamma^{-1}T']_{\{x\}} \\
&\Longrightarrow [\gamma\phi\gamma^{-1}T']_{\{\gamma(x)\}} = [\gamma\psi\gamma^{-1}T']_{\{\gamma(x)\}} \\
&\Longrightarrow [f\big(\phi(T)\big)]_{\{\gamma(x)\}} = [f\big(\psi(T)\big)]_{\{\gamma(x)\}}.
\end{aligned} \tag{4.8}$$

□

Now, we can prove the general case:
Assume that $T_1 = \lim_{n\to\infty} \phi_n(T), T_2 = \lim_{n\to\infty} \psi_n(T)$, and $[T_1]_{B_R(x)} = [T_2]_{B_R(x)}$. Furthermore, define the distance on $\Omega_T$ using the origin $x$ (see Definition 3.1.1), and the distance on $\Omega_{T'}$ using the origin $\gamma(x)$. Then, for a given $R' > R$, $d\big(T_1, \phi_n(T)\big) \longrightarrow 0$ implies that for $n \gg 0$, there exists $\phi'_n, \phi''_n \in Isom(\mathbb{E}^n)$, such that $d_x(\phi'_n, id_{\mathbb{E}^n}), d_x(\phi''_n, id_{\mathbb{E}^n})$ is arbitrarily small and

$$[\phi'_n T_1]_{B_{R'}(x)} = [\phi''_n \phi_n T]_{B_{R'}(x)}.$$

Similarly, for $n \gg 0$, there exists $\psi'_n, \psi''_n \in Isom(\mathbb{E}^n)$, such that $d_x(\psi'_n, id_{\mathbb{E}^n})$ and $d_x(\psi''_n, id_{\mathbb{E}^n})$ are arbitrarily small and

$$[\psi'_n T_2]_{B_{R'}(x)} = [\psi''_n \psi_n T]_{B_{R'}(x)}.$$

These two equalities imply that:

$$[T_1]_{B_{R'}(\phi'^{-1}_n(x))} = [\phi'^{-1}_n \phi''_n \phi_n T]_{B_{R'}(\phi'^{-1}_n(x))}$$

and

$$[T_2]_{B_{R'}(\psi'^{-1}_n(x))} = [\psi'^{-1}_n \psi''_n \psi_n T]_{B_{R'}(\psi'^{-1}_n(x))}.$$

Since $R' > R$ and $\phi'_n, \psi'_n$ are arbitrarily close to $id_{\mathbb{E}^n}$, we have $B_R(x) \subset B_{R'}\big(\phi'^{-1}_n(x)\big)$ respectively $B_{R'}\big(\psi'^{-1}_n(x)\big)$; hence,

$$[T_1]_{B_R(x)} = [\phi'^{-1}_n \phi''_n \phi_n T]_{B_R(x)}$$

and

$$[T_2]_{B_R(x)} = [\psi'^{-1}_n \psi''_n \psi_n T]_{B_R(x)}.$$

Therefore, by assumption, $\bar{\phi_n} := \phi'^{-1}_n \phi''_n \phi_n$ and $\bar{\psi_n} := \psi'^{-1}_n \psi''_n \psi_n$ are close to $\phi_n$ and $\psi_n$ such that:

$$[\bar{\phi_n} T]_{B_R(x)} = [\bar{\psi_n} T]_{B_R(x)}.$$

Now, using the special case in the calculation (4.8) in the beginning of the proof of claim 4, we conclude:

$$[f(\bar{\phi_n}(T)))]_{\{\gamma(x)\}} = [f(\bar{\psi_n}(T))]_{\{\gamma(x)\}}.$$

Notice that $\lim_{n\to\infty} \bar{\phi}_n T = T_1$ and $\lim_{n\to\infty} \bar{\psi}_n T = T_2$, hence, according to Lemma 4.2.14 we have,

$$[f(T_1)]_{\gamma(x)} = [f(T_2)]_{\gamma(x)}.$$

as claimed.
In a completely analogous way, we can show that the inverse $f^{-1}$ is $\gamma^{-1}$-i-LD.

□

**Lemma 4.2.16** *If $\Omega$ and $\Omega'$ are i-MLD tiling spaces with i-MLD homeomorphism $f : \Omega \to \Omega'$, then $T$ and $f(T)$ are i-MLD, for all tilings $T \in \Omega$.*

Proof Suppose $\Omega_T$ and $\Omega'_T$ are tiling spaces and $\gamma$-i-LD with the map $f : \Omega \to \Omega'$. If $T \in \Omega$, the properties of $f(T)$ near the point $\gamma(x)$ are determined by the properties of $T$ on some balls around $x$. Hence, the tilings $T$ and $f(T)$ are i-MLD. In more detail,

$$\begin{aligned}[T]_{B_R(x)} = [\phi(T)]_{B_R(x)} &\Longrightarrow [f(T)]_{\{\gamma(x)\}} = [f(\phi(T))]_{\{\gamma(x)\}} \text{ by Definition 4.2.1} \\ &\Longrightarrow [f(T)]_{\{\gamma(x)\}} = [\gamma\phi\gamma^{-1}f(T)]_{\{\gamma(x)\}}.\end{aligned} \tag{4.9}$$

Therefore, $T$ and $f(T)$ are i-MLD, for all tilings $T \in \Omega$.

□

# Chapter 5

# Crystallographic Tilings

## 5.1 Definitions and First Properties

**Definition 5.1.1** The *automorphism group* of a simple tiling $T \subset \mathbb{E}$ is defined as the set of all $\gamma \in Isom(\mathbb{E}^n)$, such that $\gamma(T) = T$, that is:

$$Aut(T) := \{\gamma \in Isom(\mathbb{E}^n) | \gamma(T) = T\}.$$

**Definition 5.1.2** An isometrically simple tiling $T \subset \mathbb{E}^n$ is crystallographic if its automorphism group $Aut(T) \in Isom(\mathbb{E}^n)$ is crystallographic.

**Example 5.1.3** The standard lattice tiling $T$ consists of tiles which are squares with vertices of the form $(n, m), (n+1, m), (n, m+1), (n+1, m+1)$, where $n, m \in \mathbb{Z}$.

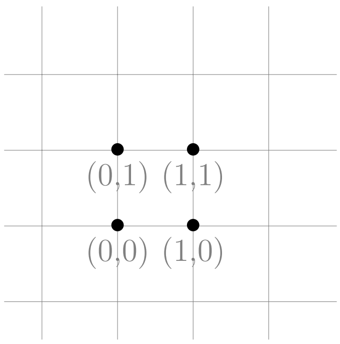



Figure 5.1: Standard lattice tilings of $\mathbb{E}^2$ .

We calculate the automorphism group of $T$, $Aut(T)$ as follows:
$\phi \in Aut(T)$ maps points $(n,m) \in \mathbb{Z}^2$ onto points $\phi(n,m) \in \mathbb{Z}^2$. We can write $\phi$ as a product of translation and orthogonal map, i.e., $\phi = \tau \cdot \alpha$; $\tau \in Trans(\mathbb{E}^2), \alpha \in O(\mathbb{E}^2_O)$. Notice that $(-\tau)\phi = \alpha$; $\alpha$ fixes the origin, therefore, $\phi(0,0) = (n_o, m_o)$; $n_o, m_o \in \mathbb{Z}$, implies, $\tau := (n_o, m_o)$. That means, $\pm\tau \in Aut(T)$, i.e., $\alpha \in Aut(T) \cap O(\mathbb{E}^2_O)$. We know that $\alpha$ maps points $(n,m) \in \mathbb{Z}^2$ to points $\alpha(n,m) \in \mathbb{Z}^2$. We also know that $\alpha$ is an orthogonal map, therefore, $det(\alpha) = \pm 1$, which means that $\alpha \in GL(2,\mathbb{Z}) \cap O(\mathbb{E}^2_O)$.
Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ denote the $2 \times 2$-matrix associated to $\alpha$. Then,
$A^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{det(A)} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \in M(2,\mathbb{Z})$, with
$det(A) = a_{11}a_{22} - a_{12}a_{21} = \pm 1$. Also, $A$ is orthogonal, hence,

$$A \cdot A^T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{12}^2 & a_{11}a_{21} + a_{12}a_{22} \\ a_{11}a_{21} + a_{21}a_{12} & a_{21}^2 + a_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore the following entries for the matrix $A$ are possible:

$$a_{11} = 1, a_{21} = 0, a_{12} = 0, a_{22} = \pm 1 \text{ or}$$
$$a_{11} = -1, a_{21} = 0, a_{12} = 0, a_{22} = \pm 1 \text{ or}$$
$$a_{11} = 0, a_{21} = 1, a_{12} = \pm 1, a_{22} = 0 \text{ or}$$
$$a_{11} = 0, a_{21} = -1, a_{12} = \pm 1, a_{22} = 0.$$

This gives us eight matrices of $Aut(T)$ which are orthogonal maps fixing the origin, i.e., $Aut(T) \cap O(\mathbb{E}^2_O)$ consists only of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

This is the dihedral group $D_4$, or the symmetry group of the square with vertices: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus, we have completely described the automorphism group of $T$ as:

$$Aut(T) = \mathbb{Z}^2 \rtimes \left(Aut(T) \cap O(\mathbb{E}^2_O)\right) \cong \mathbb{Z}^2 \rtimes D_4;$$

where $\mathbb{Z}^2$ are translations with integer coefficients and $Aut(T) \cap O(\mathbb{E}^2_O) \cong D_4$ is the point group of $Aut(T)$.
Note that $Isom(\mathbb{E}^n)/Aut(T)$ is compact, and therefore, $Aut(T)$ is a crystallographic group (by definition).

In [19] crystallographic point sets was defined. The authors did not use our notions; they defined periods of point sets (translations that transform the point set into itself). Then, if we take all these periods together they make up a group of translations. If this group is a lattice of full rank, then this group is called *Crystallographic Point set.*
This is the same as in our case since when we say the automorphism group has to be Crystallographic that means it contains a lattice of full rank; see [19] for more details on Crystallographic Point sets.

**Proposition 5.1.4** *If $T$ is a tiling such that its automorphism group $Aut(T)$ is a crystallographic group (a so-called crystallographic tiling (see Definition 5.1.2), then the hull $\Omega_T = Isom(\mathbb{E}^n)/Aut(T)$.*

PROOF By construction of the orbit of $T$, $O(T) = Isom(\mathbb{E}^n)/Aut(T)$. Since $Aut(T)$ is a crystallographic group, then, by definition of crystallographic group, the quotient $Isom(\mathbb{E}^n)/Aut(T)$ is compact, therefore closed, so $O(T) = \Omega_T$. Hence, $\Omega_T = Isom(\mathbb{E}^n)/Aut(T)$, as required.

□

**Example 5.1.5** According to Proposition 5.1.4, the hull of the standard lattice tiling $T$ is:

$$\Omega_T = Isom(\mathbb{E}^n)/\big(\mathbb{Z}^2 \rtimes (Aut(T) \cap O(\mathbb{E}^2_o))\big).$$

**Example 5.1.6** Let $T'$ be the slanted lattice tiling constructed in the same way as in Example 4.1.7. In a similar way as in Example 5.1.3; we can calculate $Aut(T') = \mathbb{Z}^2 \rtimes D_2$ where
$D_2 \cong Aut(T') \cap O(\mathbb{E}^2_o) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1, 0 \\ 0, -1 \end{pmatrix} \right\}$; that is a crystallographic group with point group $\cong D_2$, and $\Omega_{T'} = Isom(\mathbb{E}^2)/Aut(T')$ (by Proposition 5.1.4).

For a crystallographic group $\Gamma \subset Isom(\mathbb{E}^n)$, we have:

**Proposition 5.1.7** *For almost all $x \in \mathbb{E}^n$, the stabilizer $Stab_\Gamma(x) = \{id_{\mathbb{E}^n_O}\}$.*

PROOF For $\rho \in O(\mathbb{E}^n_O)$, we have $\|\rho \cdot x - O\| = \|x - O\|$ as the distance does not depend on $\rho$. Also, we have $\|-\tau \cdot x - O\| = \|x - \tau - O\|$, so for $\|\tau\|$ large enough, we get $\|x - \tau - O\| > \|x - O\|$. Consequently, if $\gamma = \tau \cdot \rho \in Stab_\Gamma(x)$, then $\tau$ can only be a finite number of translations.

Proposition 1.1.21 shows that there exists a crystallographic group $\Gamma^*$ containing $\Gamma$, such that $\Gamma^*/\Gamma^* \cap Trans(\mathbb{E}^n) \cong G$, and there exists a section $s : G \to \Gamma^*$ such that:

$$\Gamma^* = (\Gamma^* \cap Trans(\mathbb{E}^n)) \rtimes s(G) \subset Trans(\mathbb{E}^n) \rtimes O(\mathbb{E}^n_O) = Isom(\mathbb{E}^n).$$

Since $Stap_\Gamma(x) \subset Stab_{\Gamma^*}(x)$, we only need to show the claim for $\Gamma^*$.
$G$ is a finite group by Bieberbach's Theorem, hence there is also only a finite number of possible $\rho$'s. Therefore, $|Stab_{\Gamma^*}(x)| < \infty$.

□

**Remark 5.1.8** If $Stab_\Gamma(x) = \{id_{\mathbb{E}^n}\}$, then the tile of $VT(\Gamma \cdot x)$ containing $\gamma \cdot x$ is the image of the tile containing $x$ under $\gamma$ (and no other isometry of $\Gamma$).
For $\gamma_1, \gamma_2 \in Aut(T)$, such that $\gamma_1(x) = x = \gamma_2(x)$ we have $\gamma_2^{-1} \cdot \gamma_1 \cdot x = x$, hence by the definition of a stabilizer, $\gamma_2^{-1}\gamma_1 \in Stab(x) = \{id_{\mathbb{E}^n}\}$. This implies $\gamma_1 = \gamma_2$.

## 5.2 Equivalences Between Crystallographic Tilings

**Theorem 5.2.1** *For any two crystallographic tilings $T, T'$ of $\mathbb{E}^n$, $T'$ is $\gamma$-i-LD from $T$, if and only if*

$$\gamma \; Aut(T) \; \gamma^{-1} \subset Aut(T').$$

PROOF ($\Longrightarrow$) Assume that $T'$ is $\gamma$-i-LD from $T$. If we choose $R_o$ as an LD-radius, then all $R \geq R_o$ are also LD-radii. Now, we can choose $R$ such that, for any $x \in \mathbb{E}^n$, $\bigcup_{\sigma \in Aut(T) \cap Trans(\mathbb{E}^n)} \sigma\big(B_R(x)\big) = \mathbb{E}^n$, since $Aut(T)$ is a crystallographic group. So, $Aut(T) \cap Trans(\mathbb{E}^n)$ is a lattice of translations of full rank $n$, according to Bieberbach's Theorem. Assuming that $\rho \in Aut(T)$, we need to show $\gamma\rho\gamma^{-1}(t'_j) \in T'$ for all tiles $t'_j \in T'$:
Choose $\gamma(x) \in \gamma\rho\gamma^{-1}(t'_j)^o$. $\rho(T) = T$ as $\rho \in Aut(T)$. This implies that $[T]_{B_R(x)} = [\rho(T)]_{B_R(x)}$. Now, since $T'$ is $\gamma$-i-LD from $T$, we get:

$$[T']_{\{\gamma(x)\}} = [\gamma\rho\gamma^{-1}(T')]_{\{\gamma(x)\}} = \{\gamma\rho\gamma^{-1}(t'_j)\}.$$

Therefore, there exists $t'_k \in T'$ such that $t'_k = \gamma\rho\gamma^{-1}(t'_j)$ as required.
($\Longleftarrow$)Assume $\gamma \; Aut(T) \; \gamma^{-1} \subset Aut(T')$. Again, $Aut(T) \cap Trans(\mathbb{E}^n)$ is a lattice of translations of full rank $n$. Then, there exists $R > 0$, such that for all $x \in \mathbb{E}^n$:

$$(*) \qquad \bigcup_{\sigma \in Aut(T) \cap Trans(\mathbb{E}^n)} \sigma\big(B_R(x)\big) = \mathbb{E}^n.$$

Assume $[T]_{B_R(x)} = [\rho(T)]_{B_R(x)}$ for some $\rho \in Isom(\mathbb{E}^n)$.
**Claim:** $\rho \in Aut(T)$.
**Proof of the claim:** Notice that, $\forall \sigma \in Isom(\mathbb{E}^n), R > 0, x \in \mathbb{E}^n$:

$$(**) \qquad \sigma\big([T]_{B_R(x)}\big) = [\sigma(T)]_{B_R(\sigma(x))}.$$

It is enough to show: $\rho(t_i) \in T \;\; \forall$ tiles $t_i \in T$. From (*) we see that:

$$\exists \sigma \in Aut(T) \cap Trans(\mathbb{E}^n) \;\text{ s.t. }\; \sigma(t_i) \in [T]_{B_R(\rho^{-1}(x))}.$$

Also, from(**) and the assumption on $\rho$, we see:

$$\rho\big([T]_{B_R(\rho^{-1}(x))}\big) = [\rho(T)]_{B_R(x)} = [T]_{B_R(x)}.$$

Therefore, there exists a tile $t_j \in T$ such that:

$$t_j = \rho\big(\sigma(t_i)\big) = \big(\rho\sigma\rho^{-1}\big) \cdot \rho(t_i).$$

For $\sigma' = \rho\sigma\rho^{-1}$, we have: $\rho(t_i) = (\sigma')^{-1}(t_j) \in T$. Therefore, it is enough to show that: $\rho\sigma\rho^{-1} \in Aut(T)$ for all $\sigma \in Aut(T) \cap Trans(\mathbb{E}^n)$, equivalently:

$$\rho\big(Aut(T) \cap Trans(\mathbb{E}^n)\big)\rho^{-1} = Aut(T) \cap Trans(\mathbb{E}^n).$$

This is true for $\rho \in Trans(\mathbb{E}^n)$, since translations commute.
In the general case:
Choose generators $s_1, ..., s_n$ of $Aut(T) \cap Trans(\mathbb{E}^n)$, such that:

$$\rho\sigma\rho^{-1} = \rho(k_1 s_1 + ... + k_n s_n)\rho^{-1}.$$

It is enough to show that: $\rho s_i \rho^{-1} \in Aut(T) \cap Trans(\mathbb{E}^n); i = 1, ..., n$. Choose $R$ large enough, such that for an $R' < R$:

- (*) holds for $R'$.
- $R' + \max_{i=1,..,n} \|s_i\|_{Eucl} < R$.

Let $s \in \{s_1, ..., s_n\}$, then:

$$\begin{aligned}(\rho s\rho^{-1})\big([T]_{B_{R'}(x)}\big) &= (\rho s\rho^{-1})\big([\rho(T)]_{B_{R'}(x)}\big) \quad \text{by assumption on } R, R' \text{ and } \rho\\ &= (\rho s)\big([T]_{B_{R'}(\rho^{-1}(x))}\big) \quad \text{by (**)}\\ &= \rho\big([T]_{B_{R'}(s\rho^{-1}(x))}\big) \quad \text{by (**) and } s(T) = T\\ &= [\rho(T)]_{B_{R'}(\rho s\rho^{-1}(x))} = [T]_{B_{R'}(\rho s\rho^{-1}(x))} \quad \text{by (**).}\end{aligned} \tag{5.1}$$

Here, $B_{R'}\big(\rho s\rho^{-1}(x)\big) \subset B_R(x)$ because $R' + \|\rho s\rho^{-1}\|_{Eucl} = R' + \|s\|_{Eucl} < R$ (by assumption on $R$). From (5.1), we see that:

$$t_i \in [T]_{B_{R'}(x)} \implies (\rho s\rho^{-1})(t_i) \in T.$$

Let $\tau \in Aut(T) \cap Trans(\mathbb{E}^n)$, then:

$$\begin{aligned}(\rho s\rho^{-1})\big([T]_{B_{R'}(\tau(x))}\big) &= \rho s\rho^{-1}\big(\tau([T]_{B_{R'}(x)})\big) \quad \text{by (**)}\\ &= \tau\big(\rho s\rho^{-1}([T]_{B_{R'}(x)})\big) \quad \text{since translations commute}\\ &= \tau\big([T]_{B_{R'}(\rho s\rho^{-1}(x))}\big) \quad \text{by (5.1)}\\ &= [T]_{B_{R'}\big((\rho s\rho^{-1})(\tau(x))\big)} \quad \text{by (**).}\end{aligned} \tag{5.2}$$

Therefore,

$$\forall t_j \in [T]_{B_{R'}(\tau(x))}: \quad \rho s\rho^{-1} t_j \in T.$$

Since $\bigcup_{\tau \in Aut(T) \cap Trans(\mathbb{E}^n)} B_{R'}(\tau(x)) = \mathbb{E}^n$, every tile of $T$ lies in a patch $[T]_{B_{R'}(\tau(x))}$, i.e., every tile of $T$ is mapped to another tile of $T$ by $\rho s \rho^{-1}$. Hence, $\rho s \rho^{-1} \in Aut(T)$, as required. □

Consequently, from $\gamma \ Aut(T) \ \gamma^{-1} \subset Aut(T')$, we have $\gamma \rho \gamma^{-1} \in Aut(T')$, hence, $[T']_{\{\gamma(x)\}} = [\gamma \rho \gamma^{-1} T']_{\{\gamma(x)\}}$, as requested for $T'$ being $\gamma$-i-LD from $T$.

□

**Remark 5.2.2** Note that in the proof of direction " $\Longrightarrow$ ", we have not used that $Aut(T')$ is a crystallographic group. In particular, we have shown that if an arbitrary tiling $T'$ is $\gamma$-i-LD from the crystallographic Tiling $T$, then $T'$ is itself a crystallographic tiling.

As a direct consequence of this theorem, we obtain:

**Theorem 5.2.3** *Two crystallographic tilings are MLD if and only if their automorphism groups are conjugated by an isomorphism.*

**Remark 5.2.4** The two tilings of Example 4.1.7 have different automorphism groups, but are mutually locally derivable if one only uses translations for the definition, that is:
Assume

$$[T]_{B_R(x)} = [T + \tau]_{B_R(x)} \quad ; \text{ for } R \gg 0, \tau \in Trans(\mathbb{E}^n).$$

From the proof of Theorem 5.2.1, we see that:

$$\tau \in Aut(T) \cap Trans(\mathbb{E}^n) = Aut(T') \cap Trans(\mathbb{E}^n).$$

That implies,

$$[T']_{B_R(x)} = [T' + \tau]_{B_R(x)}.$$

Therefore, $T'$ is translationally-LD from $T$. Similarly, $T$ is translationally-LD from $T'$ and hence, $T$ and $T'$ are translationally-MLD.

## 5.3 Construction of Crystallographic Tilings

**Theorem 5.3.1** *For every crystallographic group* $\Gamma \subset Isom(\mathbb{E}^n)$*; there exists a simple tiling* $T$ *with* $Aut(T) = \Gamma$.

PROOF The strategy to construct a simple tiling $T$ with $Aut(T) = \Gamma$ is as follows:

(i) Choose $x \in \mathbb{E}^n$ such that $Stab_\Gamma(\{x\}) = \{id_{\mathbb{E}^n}\}$. Such an $x$ must exists according to Proposition 5.1.7.

(ii) Construct the Voronoi-cell tiling $VT(\Gamma \cdot x)$ of the orbit $\Gamma \cdot x$ of a point $x \in \mathbb{E}^n$ as in (i). This is a simple tiling by Theorem 2.3.1.

(iii) Let $t_x \in VT(\Gamma \cdot x)$ be the tile containing $x$. Subdivide each tile $\gamma \cdot t_x$ of $VT(\Gamma \cdot x); \gamma \in \Gamma$ by cones, each having a face of $\gamma(t_x)$ as its basis and the point $\gamma \cdot y$ as the vertex, where $y$ is a sufficiently general point in $t_x$.

This subdivision tiling $T_\Gamma$ is simple and will have automorphism group $\Gamma$, because the possibly existing additional automorphisms of $VT(\Gamma \cdot x)$ not lying in $\Gamma$, do not map the subdivision cones onto each other.
In more detail, choose $y$ such that the distances of $y$ to the vertices of $t_x$ are mutually distinct and are also different from all the lengths of edges of $t_x$. This is possible if we choose $y$ away from a finite number of spheres around the vertices of $t_x$, with radii equal to the edge lengths of $t_x$, and also away from the finite number of hyperplanes reflecting one vertex of $t_x$ to another. Obviously, $\Gamma \subset Aut(T_\Gamma)$. On the other hand, let $\delta \in Isom(\mathbb{E}^n)$ be an automorphism of $T_\Gamma$. Let $C_1 \cup C_2 \cup ... \cup C_r = t_x$ be the subdivision of $t_x$ into cones $C_i$. Notice that, $\delta(C_1)$ must be one of the subdivision cones in a tile $\gamma \cdot t_x \in VT(\Gamma \cdot x)$. Since the lengths of the edges to the vertex of the cone are all different by construction, $C_1' := \delta(C_1) = \gamma(C_1)$, and the vertex and the edges of $C_1'$ must be mapped to the vertex of $C_1'$ and the same edges by $\delta$ and $\gamma$. Since, vertices of $C_1$ span all of $\mathbb{E}^n$, $\delta$ and $\gamma$ are uniquely determined by the images of these vertices (as affine transformations of $\mathbb{E}^n$), and hence, are equal.

□

**Example 5.3.2** The subdivision of the Voronoi-cell tiling is necessary to kill additional automorphisms, as shown by the example of $\Gamma = \mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2$, where $e_1$ and $e_2$ are orthogonal standard basis vectors in $\mathbb{R}^2$:
Obviously, every point $x \in \mathbb{E}^2$ has a trivial stabilizer in $\Gamma$, but the Voronoi-cell tiling of $\Gamma \cdot x$ is the standard lattice tiling $T$ consisting of tiles which are

squares with vertices of the form $(n,m),(n+1,m),(n,m+1),(n+1,m+1)$; with $n,m \in \mathbb{Z}$, as in Figure 5.2.
In Example 5.1.5, we have calculated that:
$Aut(T) = (\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2) \rtimes D_4$.

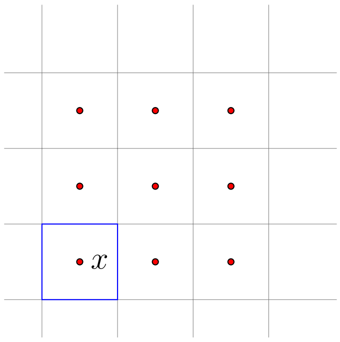



Figure 5.2: ——Voronoi-cell tiling of standard lattice orbit in $\mathbb{E}^2$ .

After the subdivision with $y$ being sufficiently general; we obtain a tiling with the automorphism group $\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2$:

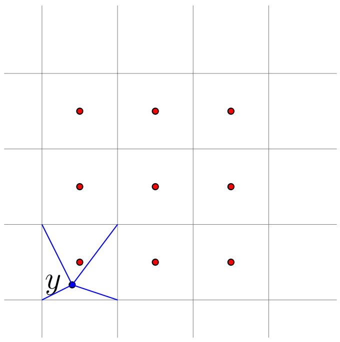



Figure 5.3: Tiling of $\mathbb{E}^2$ with automorphism group $\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2$.

# Chapter 6

# Cut-and-Project Method for Crystallographic Tilings

Known quasi-crystallographic tilings like the Penrose tiling can be obtained by projecting a subset of a point lattice onto a plane. We will describe a more general construction of cut-and-project tilings from an arbitrary given crystallographic tiling T, and not only lattices and given cut-and-project data (that is, projection subspace and window), in detail.

As a first step, points must be chosen in each prototile to obtain a Delone set. The points in one prototile should be invariant under the isometry group of the prototile, so it does not matter which isometry is applied on the prototile to obtain an actual tile in the tiling; we always choose the same points in the tile.

Then, the cut-and-project Delone set can be constructed using the cut-and-project data, and from this set, we can construct the Voronoi-cell tiling. One has to show that, the projected point set is also a Delone set, and that the associated Voronoi-cell tiling is simple.

## 6.1 Delone Sets from Crystallographic Tilings

Take $T$ to be a crystallographic tiling of $\mathbb{E}^n$, and construct a Delone set $X$ out of it. To this purpose, choose finite sets of points $X_i$ in each prototile $t_i$ fixed by the symmetry group of the prototile. This is possible as per Lemma 1.3.11.

**Example 6.1.1** An obvious set of points to choose would be the vertices of the prototiles since vertices must be mapped to vertices by isometries. On the other hand, vertices are not the only choice of the set of points. There are some cases where one can determine the symmetry centres of the prototiles. For example, if we take the standard lattice tiling in any dimension, there is no difference whether we choose vertices or symmetry centres, since the symmetry centres look like the shifted points of the vertices, as shown in Figure 6.1.

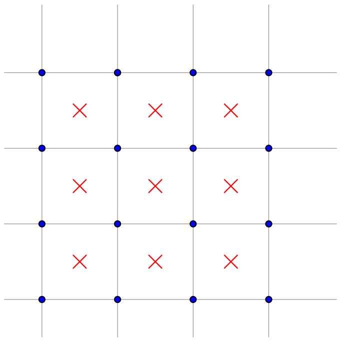


Figure 6.1: Standard lattice tiling with vertices ∘ and symmetry centers ×.

**Definition 6.1.2** A *point set data* $\{(X_i, t_i)_i\}$ of a tiling $T$ consists of a finite set of points $X_i$ for each prototile $t_i$ that is invariant under the isometry group of $t_i$.

**Proposition 6.1.3** *Given a point set data* $\{(X_i, t_i)_i\}$ *the point set*

$$X_T = \bigcup_{t \in T, \gamma(t_i)=t} \gamma(X_i) \ ; \ \gamma \in Isom(\mathbb{E}^n)$$

*is a Delone set.*

Proof Note that the union runs over all tiles $t \in T$, and for each $t$, we choose an isometry $\gamma \in Isom(\mathbb{E}^n)$ mapping the prototile $t_i$ behind $t$ to $t = \gamma(t_i)$. Now, $\bigcup_{t \in T} t = \mathbb{E}^n$ and $\exists R : t \subset B_R(x)$ for all points $x \in t$, where $R$ only depends on the prototile behind $t$. Since we only have a finite number of prototiles, there exists an $R$ working for all $t$ at once. This means that $\bigcup_{x \in X_T} B_R(x) = \mathbb{E}^n$, because for each $t$, $x \in t \cap X_T$. Hence, the covering radius of $X_T$ is less than or equal to $R$, in particular the covering radius of $X_T$ is finite.

If the packing radius of $X_T$ is $r$, then open balls of radius $r$ centered at the points of $X_T$ will be disjoint from each other, and each open ball centered at one of the points of $X_T$ with radius $2r$ will be disjoint from the rest of $X_T$. Now, for a given $R$ and all choices of $y \in \mathbb{E}^n$, there is only a finite number of patches $[T]_{B_R(y)}$ up to isometries. This was already used in the proof of compactness of hull (see Theorem 3.2.7). This means that there are only a finite number of intersection sets $B_R(y) \cap X_T$ up to isometries. Furthermore, for $y \in \mathbb{E}^n$, the set

$$\{d(x, x') : x \neq x' \in B_R(y) \cap X_T\}$$

is finite, as $B_R(y) \cap X_T$ only intersects a finite number of tiles and each tile intersects $X_T$ in a finite set of points. Since,

$$\{d(x, x') : x \neq x' \in B_R(y) \cap X_T\}$$

is invariant under isometries, we conclude that

$$r := \frac{1}{2} \inf\{d(x, x') : x \neq x' \in B_R(y) \cap X_T, y \in \mathbb{E}^n\} > 0.$$

Hence, from all what we have discussed, we have shown that $X_T$ is a Delone set.

□

**Remark 6.1.4** The important thing about choosing points in prototiles that are fixed under the symmetry group of $t_i$ is that, for the definition of $X_T$, we need to get the same points in tile $t$ independent of the isometry used to get from $t_i$ to $t$. In Example 6.1.1, $X_i$ is the symmetry centres of the prototiles $t_i$ in the standard lattice tiling $T$ where we can get the tile $t_1$ from the tile $t_1$ by a 90° rotation. It is clear that Example 6.1.1 satisfies Proposition 6.1.3.

**Proposition 6.1.5** *For a point set $X_T$ associated to a tiling $T$ as above, the Voronoi-cell tiling $T_{X_T}$ is $id_{\mathbb{E}^n}$-i-LD from $T$.*

PROOF $T_{X_T}$ is $id_{\mathbb{E}^n}$-i-LD from $T$ if there exists a radius $R$ such that, for $x \in \mathbb{E}^n$ and $\phi \in Isom(\mathbb{E}^n)$, we have:

$$[T]_{B_R(x)} = [\phi(T)]_{B_R(x)} \Longrightarrow [T_{X_T}]_{\{x\}} = [\phi(T_{X_T})]_{\{x\}}.$$

Now,

$$\begin{aligned} [T]_{B_R(x)} = [\phi(T)]_{B_R(x)} &\Longrightarrow [X_T]_{B_R(x)} = [\phi(X_T)]_{B_R(x)} \\ &\Longrightarrow [T_{X_T}]_{\{x\}} = [\phi(T_{X_T})]_{\{x\}}; \end{aligned} \tag{6.1}$$

for large enough $R$ independent of $x$ because by Proposition 2.2.5, the Voronoi-cell around a point $x \in X_T$ only depends on points of $X_T$ up to a distance of $x$ that is independent of $x$. □

There are many counterexamples of the other direction of Proposition 6.1.5. That is, $T$ is not LD from $T_{X_T}$. If $T$ is a crystallographic tiling this is equivalent to $Aut(T) \subsetneqq Aut(T_{X_T})$. We will also check this condition in the following examples.

**Example 6.1.6** Choose the symmetry centres as the set of points of the prototiles. The Voronoi-cell tiling we gain is just the shifted standard lattice tiling (see Figure 6.2). So, $T$ is LD from $T_{X_T}$ by Proposition 4.2.7.

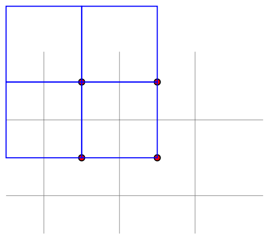


Figure 6.2: Standard lattice tiling (——) and Voronoi-cell tiling of its vertices (——).

**Example 6.1.7** For the slanted lattice tiling as in Example 4.1.7, if we choose the point set $X_T$ as the vertices of $T$, this set coincides with the vertices of the standard lattice tiling; if we choose the point set data as the vertices or the symmetry centres, both cases will give us Delone set as a standard lattice tiling, which has more automorphisms than the slanted tiling. Therefore, $Aut(T) \subsetneqq Aut(X_T) = Aut(T_{X_T})$, and $T$ is not LD from $T_{X_T}$.
If we take the standard lattice tiling and take points in the tiles that are close to all the vertices and invariant under the automorphism group of the square then from Figure 6.3 it is clear that $Aut(T_{X_T})$ contains horizontal and vertical translations by $\frac{1}{2}$. On the other hand, if we look at $X_T$, the horizontal translations by $\frac{1}{2}$ are not contained in $Aut(X_T)$. Hence, $Aut(T) = Aut(X_T) \subsetneqq Aut(T_{X_T})$, and $T$ is not LD from $T_{X_T}$.

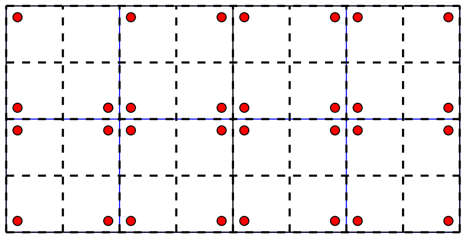


Figure 6.3: Standard lattice tiling, point set data and Voronoi-cell tiling II.

**Corollary 6.1.8**

$$Aut(T) \subset Aut(X_T) \subset Aut(T_{X_T}).$$

PROOF First of all, we will show that $Aut(T) \subset Aut(X_T)$. Let $\phi \in Aut(T)$, such that $\forall t \in T : \phi(t) \in T$. Now, by construction, if $\gamma(t_i) = t$ for the prototile $t_i$ and the isometry $\gamma$, we have:

$$\phi(t \cap X_T) = \phi\big(\gamma(X_i)\big);$$

since by construction $t \cap X_T = \gamma(X_i)$. This implies that:

$$\phi(X_T) = \bigcup_{t \in T; \gamma(t_i)=t} \phi\big(\gamma(X_i)\big) = \bigcup_{t \in T; \gamma(t_i)=t} \phi(t) \cap X_T = \bigcup_{t \in T; \gamma'(t_i)=t} \gamma'(X_i) = X_T,$$

since $\phi$ is an automorphism of $T$, hence, $\phi(T)$ runs over all tiles of $T$ if t does, and $\gamma' = \phi\gamma \in Aut(T)$. Therefore, $\phi \in Aut(X_T)$.
The second step now is to prove that $Aut(X_T) \subset Aut(T_{X_T})$. Assume $\phi \in Aut(X_T)$ and let $t \in T_{X_T}$. Notice that $t = t_x$ for some $x \in X_T$, where

$$t_x = \{y \in \mathbb{E}^n : d(y, x) \leq d(y, x') \forall x' \in X_T\} \tag{6.2}$$

This implies $d\big(\phi(y), \phi(x)\big) \leq d\big(\phi(y), \phi(x')\big)$ for all $x \in X_T$, and since $\phi(x')$ runs through all points of $X_T$ if $x'$ does, we have $\phi(tx) = t\phi(x)$ which means that $\phi \in Aut(T_{X_T})$.

□

## 6.2 General Cut-and-Project Construction

We will first define cut-and-project data for the Euclidean space $\mathbb{E}^n$.
Let $E \subset \mathbb{E}^n$ be an $m$-dimensional hyperplane, and $E^\perp \subset \mathbb{E}^n$ be an $(n-m)$-dimensional hyperplane orthogonal to $E$. Let $\Pi$ be the orthogonal projector onto $E$, and $\Pi^\perp$ the orthogonal projector onto $E^\perp$, that is $\Pi : \mathbb{E}^n \to E$ and $\Pi^\perp : \mathbb{E}^n \to E^\perp$.
Then, we fix a compact subset $K \subset E^\perp$ such that $K^\circ \neq \emptyset$ and $\overline{K}^\circ = K$. $K$ will be called the *window* for the projection, $E$ the *projection hyperplane*, $K \times E$ the *cylinder*, (see Figure 6.4), and $(K, E)$ *cut-and-project data* for $\mathbb{E}^n$.

**Construction in steps:**
Given a crystallographic tiling $T$ of $\mathbb{E}^n$, cut-and-project data $(K, E)$ for $\mathbb{E}^n$ can be used to construct a new tiling $T'$ of $E$ through the following steps:

(i) Choose point-set data $\{(X_i, t_i)_i\}$ of T as in section 6.1, where $t_1, t_2, ..., t_k$ are the prototiles of $T$.

(ii) Construct the Delone set $X_T = \bigcup_{t \in T; \gamma(t_i) = t} \gamma(X_i)$ from the point-set data, as in section 6.1.

(iii) Cut and project: Set $X_{T'} = \Pi\big(X_T \cap (K \times E)\big)$, where the cylinder $K \times E$ is defined with respect to the unique intersection point in $E^\perp \cap E$ as the origin. This step requires that $X_T \cap (K \times E)$ is not empty, which will be the case under the assumption on the window $K$ discussed later.

(iv) $T'$ is the Voronoi-cell tiling $VT(X_{T'})$ associated to $X_{T'}$.

**Remark 6.2.1** The intersection $X_T \cap (K \times E)$ could be empty as in Figure 6.5). Here the projection subspace $E$ has slope 1 with respect to the standard lattice, so never passes through one of the lattice points, and the minimal distance to the lattice points will even be positive. Therefore, if we choose a small enough $K \subset E^\perp$ window, the cylinder $K \times E$ will not contain any of the lattice points.

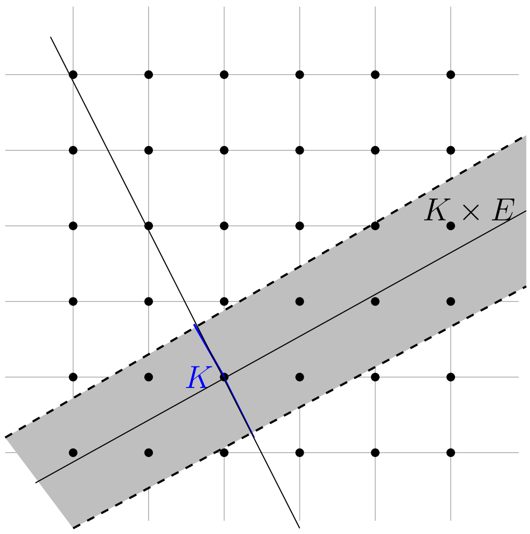



Figure 6.4: Cut-and-project method with projection subspace $E$ and window $K$.

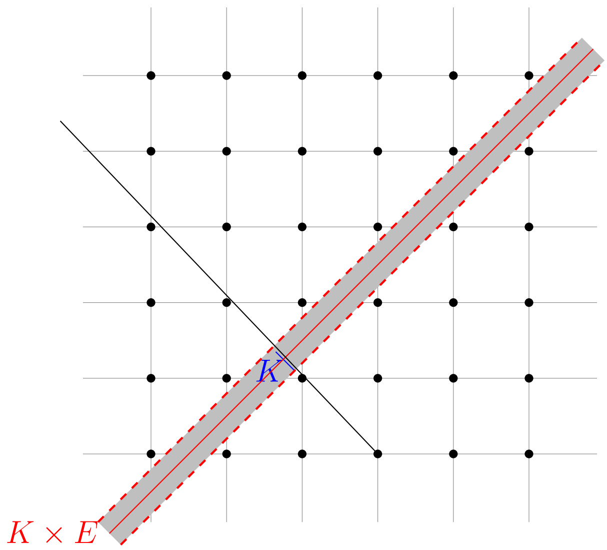



Figure 6.5: The projection result is empty.

**Assumption on window $K$:**

$$(*) \qquad K^\circ \cap \Pi^\perp(X_T) \neq \emptyset \ .$$

**Remark 6.2.2** Notice that:

(i) If the interior of a window $K$ does not intersect $\Pi^\perp(X_T)$, we can move $K$ with an isometry $\rho$ of $E^\perp$ to a window $\rho(K)$ with non-empty intersection with $\Pi^\perp(X_T)$.

(ii) If only the boundary of $K$ intersects $\Pi^\perp(X_T)$, there are lots of cases to distinguish, depending on whether components of the boundary contain enough image points of $X_T$.

Under this assumption with regard to the window $K$, $T'$ displays the following properties:

**Theorem 6.2.3** *$X_{T'}$ is a Delone set.*

**Theorem 6.2.4** *$T'$ is a simple tiling.*

The proof for these facts will occupy the rest of the section.

The main tool for the proof of Theorem 6.2.3 and Theorem 6.2.4 is Kronecker's Approximation Theorem (in several dimensions).

**Theorem 6.2.5** [[1], First form of Kronecker's Approximation Theorem] *If $\alpha_1, ..., \alpha_n$ are arbitrary real numbers, if $\theta_1, ..., \theta_n$ are $\mathbb{Z}$-linearly independent real numbers, and if $\epsilon > 0$ is arbitrary, then there exists a real number $t > 0$ and integers $h_1, ..., h_n$, such that:*

$$|t\theta_i - h_i - \alpha_i| < \epsilon \text{ for } i = 1, 2, ..., n.$$

**Remark 6.2.6** Kronecker's Approximation Theorem as in [1] only states that $t$ is a real number, but the proof goes through if one stays restricted to $t > 0$; this is what we need later.

Under additional assumptions on the $\alpha_i$ Kronecker's Approximation Theorem holds for arbitrary real numbers $\theta_1, ..., \theta_N$, irrespective of whether they are $\mathbb{Z}$-linearly independent or not.

**Corollary 6.2.7** If $\theta_1, ..., \theta_N$ are real numbers and $\alpha_1, ..., \alpha_N$ are real numbers satisfying the same $\mathbb{Z}$-linear relations as $\theta_1, ..., \theta_N$, then for every $\epsilon > 0$, there exists a real number $t > 0$ and integers $h_1, ..., h_N$, such that:

$$|t\theta_i - h_i - \alpha_i| < \epsilon \text{ for } i = 1, 2, ..., N.$$

PROOF By reordering, we can achieve that $\theta_1, ..., \theta_k$ are $\mathbb{Z}$-linearly independent, $1 \leq k \leq N$, and for each $\theta_i$; $i = k+1, ..., N$, there is a $\mathbb{Z}$-linear relation:

$$n_1^{(i)}\theta_1 + ... + n_k^{(i)}\theta_k - n_i\theta_i = 0; \quad n_1^{(i)}, ..., n_k^{(i)}, n_i \in \mathbb{Z}.$$

By multiplying with $\frac{\Pi_{j=k+1}^{N} n_j}{n_i}$ we can achieve $n_{k+1} = ... = n_n = n > 0$. By Kronecker's Approximation Theorem, for any $\epsilon' > 0$, there is a real number $t' > 0$, and there are integers $h_1', ..., h_k'$, such that:

$$|t'\theta_1 - h_1' - \frac{\alpha_1}{n}| < \epsilon', ..., |t'\theta_k - h_k' - \frac{\alpha_k}{n}| < \epsilon'.$$

This implies that:

$$|t'(n_1^{(i)}\theta_1 + ... + n_k^{(i)}\theta_k) - (n_1^{(i)}h_1' + ... + n_k^{(i)}h_k') - (n_1^{(i)}\alpha_1 + ... + n_k^{(i)}\alpha_k)| < |n_1^{(i)} + ... + n_k^{(i)}|\epsilon'$$

$$\iff |t'n\theta_i - h_i' - n\alpha_i| < |n_1^{(i)} + ... + n_k^{(i)}|\epsilon'; \text{ with } h_i' = n_1^{(i)}h_1' + ... + n_k^{(i)}h_k' \in \mathbb{Z}.$$

Now, if we multiply $|t'\theta_i - h_i - \frac{\alpha_i}{n}| < \epsilon'$ for $i = 1, ..., k$ with $n$, we get:

$$|t'n\theta_i - nh_i - \alpha_i| < |n|\epsilon'.$$

Hence, $\epsilon' := \min\{\frac{\epsilon}{n}, \frac{\epsilon}{|n_1^{(i)} + ... + n_k^{(i)}|}\}$, $t = t'n > 0$, $h_i := nh_i'$ for $i = 1, ..., k$, and $h_i = h_i'$ for $i = k+1, ..., N$ are the choices required for the claim.

□

The next theorem shows the relative denseness of $X_{T'}$ in a special case, where $dim(E) = 1$.

**Theorem 6.2.8** *The set* $X_{T'} = \Pi\big(X_T \cap (K \times E)\big)$ *is relatively dense in the case that* $dim(E) = 1$ *and* $dim(E^\perp) = N - 1$.

PROOF $Aut(T)$ is crystallographic, which means that there exists a lattice of full rank $\Lambda \subset Aut(T)$. Since $\Lambda \subset Aut(T) \subset Aut(X_T)$, the orbit $\Lambda \cdot x$ of a point $x \in X_T$ is contained in $X_T$. Choose $x \in X_T$ such that $\Pi^\perp(x) \in K^\circ$ (possible by Assumption (*) on the window $K$). Also, choose a basis of $\Lambda$

and let $x$ be the origin of the coordinate system on $\mathbb{E}^n$ given by this basis. In terms of this basis, $E = \mathbb{R} \cdot (\theta_1, ..., \theta_N)$. Permuting the coordinates, we can arrive at the conclusion that $\theta_1, ..., \theta_K$ are $\mathbb{Z}$-linearly independent and $\theta_{K+1}, ..., \theta_N \in \mathbb{Q}\theta_1 + ... + \mathbb{Q}\theta_K$. In particular, there is a $(N-K) \times K$-matrix $M$ with integer entries such that:

$$M' \cdot \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{pmatrix} = \left( \begin{array}{c|ccc} & d_{K+1} & & 0 \\ M & & \ddots & \\ & 0 & & d_N \end{array} \right) \cdot \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_K \\ \theta_{K+1} \\ \vdots \\ \theta_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Set $H = \left\{ (x_1, ..., x_N) : \left( \begin{array}{c|ccc} & d_{K+1} & & 0 \\ M & & \ddots & \\ & 0 & & d_N \end{array} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$. Then, $E \subset H$ and $dim\ H = K$.

**Claim 1:** $H \cap \Lambda$ is a lattice $\Lambda_H$ of full rank $K$.

**Proof of claim 1:** $H \cap \Lambda = ker\ \phi_{M'}$, where $\phi_{M'} : \mathbb{Z}^N \to \mathbb{Z}^{N-K}$ is defined by $\begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \to M' \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$. $M'$ has rank $N-K$ because of the $(N-K) \times (N-K)$ diagonal matrix on the right. Hence, the $\mathbb{Q}$-linear map $\phi_{M'} \otimes_{\mathbb{Z}} \mathbb{Q}$ has rank $N-K$, therefore, $dim\ \big(ker(\phi_{M'} \otimes_{\mathbb{Z}} \mathbb{Q})\big) = K$ and $dim\ \big(ker(\phi_{M'} \otimes_{\mathbb{Z}} \mathbb{Q})\big)$ is the rank of the torsion-free part of the finitely generated abelian group $ker\ \phi_{M'}$. Since $ker\ \phi_{M'}$ is torsion-free as a subgroup of $\mathbb{Z}^N$, we have $ker\ \phi_{M'} \cong \mathbb{Z}^K$.

$\square$

Now, choose a $\mathbb{Z}$-basis of $\Lambda_H$. This is also an $\mathbb{R}$-basis of $H$. In terms of this, write $E = \mathbb{R} \cdot (\theta_1^H, ..., \theta_K^H)$.

**Claim 2:** $\theta_1^H, ..., \theta_K^H \in \mathbb{R}$ are $\mathbb{Z}$-linearly independent.

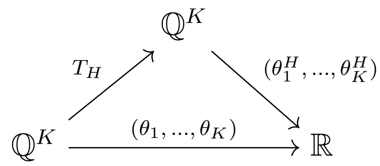
**Proof of claim 2:** It is enough to show that $\theta_1^H, ..., \theta_K^H$ are $\mathbb{Q}$-linearly independent. Let $\tau_i = (t_{1i}^H, ..., t_{Ni}^H) \in \mathbb{Z}^N$ , $i = 1, ..., K$ be the chosen $\mathbb{Z}$-basis vectors of $\Lambda_H \subset \mathbb{Z}^N$. Then,

$$T \cdot \begin{pmatrix} \theta_1^H \\ \theta_2^H \\ \vdots \\ \theta_K^H \end{pmatrix} = \begin{pmatrix} t_{11}^H & \dots & t_{1K}^H \\ \vdots & & \vdots \\ t_{N1}^H & \dots & t_{NK}^H \end{pmatrix} \cdot \begin{pmatrix} \theta_1^H \\ \theta_2^H \\ \vdots \\ \theta_K^H \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{pmatrix};$$

$\theta_1, ..., \theta_K$ are $\mathbb{Z}$-linearly independent, hence, $\mathbb{Q}$-linearly independent. Therefore, the homomorphism $\theta : \mathbb{Q}^K \xhookrightarrow{(\theta_1, ..., \theta_K)} \mathbb{R}$ given by $e_1 \mapsto \theta_1, ..., e_K \mapsto \theta_K$ is injective.

$\theta$ factorises by the map $\mathbb{Q}^K \xrightarrow{T_H} \mathbb{Q}^K$ given by the matrix $T_H = \begin{pmatrix} t_{11}^H & \dots & t_{1K}^H \\ \vdots & & \vdots \\ t_{K1}^H & \dots & t_{KK}^H \end{pmatrix}$

through $\theta_H : \mathbb{Q}^K \xrightarrow{(\theta_1^H, \dots, \theta_K^H)} \mathbb{R}$ given by $\theta_1^H, \dots, \theta_K^H$:




The matrix $T_H$ is of full rank since its rows are the first $K$ rows of the rank $K$ matrix $T$, and the last $N-K$ rows of $T$ are linear combinations of the first rows of $T$ because the columns of $T$ are in the kernel of the linear map described by the matrix $\left(\begin{array}{c|ccc} & d_{K+1} & & 0 \\ M & & \ddots & \\ & 0 & & d_N \end{array}\right)$. Hence, $\mathbb{Q}^K \xrightarrow{T_H} \mathbb{Q}^K$ is an isomorphism; therefore, $\mathbb{Q}^K \xrightarrow{(\theta_1^H, \dots, \theta_K^H)} \mathbb{R}$ is injective.

□

For the theorem, it is enough to show that:

$$\Pi\big(\Lambda_H \cdot x \cap ((K \cap H) \times E)\big)$$

is relatively dense on $E$ because $\Lambda_H \cdot x \subset \Lambda \cdot x \subset X_T$.
Setting $H := \mathbb{R}^N, E^\perp := E^\perp \cap H$ and $\Lambda \cdot x := \Lambda_H \cdot x$. We can reduce to the situation where $(\theta_1, \dots, \theta_N)$ consists of $\mathbb{Z}$-linearly independent coordinates. Choose basis vectors $\sigma_1, \dots, \sigma_{N-1}$ of $E^\perp$. The vectors $\theta, \sigma_1, \dots, \sigma_{N-1}$ are also a basis of $\mathbb{R}^N$. Therefore, we can use two basis of $\mathbb{R}^N$ to get two maximum norms on $\mathbb{R}^N$, denoted by $\|\cdot\|_\Lambda$ and $\|\cdot\|_{E,E^\perp}$. Since $\mathbb{R}^N$ is finite-dimensional, these two norms are comparable. Henceforth, we will use $\|\cdot\|_{E,E^\perp}$.
**Claim 3:**
$\forall \epsilon > 0\ \exists (n_1, \dots, n_N) \in \mathbb{Z}^N$ and $\exists t > 0 : \|t \cdot (\theta_1, .., \theta_N) - (n_1, \dots, n_N)\|_{E,E^\perp} < \epsilon$ and $\Pi^\perp(n_1, \dots, n_N) = \sum_{i=1}^{N-1} s_i \underline{\sigma_i}$ with $s_i \geq 0$.
**Proof of claim 3:** $\theta_1, \dots, \theta_N$ are $\mathbb{Z}$-linearly independent. As the metric is comparable, we can achieve claim 3 by applying Theorem 6.2.5 (first form of Kronecker's Approximation Theorem). Take $x' := x + \sum_{i=1}^{N-1} \frac{\epsilon}{2} \sigma_i$.

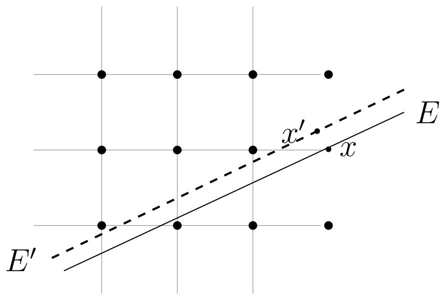



Figure 6.6: Construction of $E'$ through $x'$.

From the first form of Kronecker's Approximation Theorem there exists $t > 0$ and $(n_1, ..., n_N) \in \mathbb{Z}^N$, such that:

$$\|t \cdot (\theta_1, ..., \theta_N) + \sum_{i=1}^{N-1} \frac{\epsilon}{2}\sigma_i - (n_1, ..., n_N)\|_{E,E^\perp} \leq \frac{\epsilon}{2}.$$

Then,

$$\begin{aligned} \|t \cdot (\theta_1, ..., \theta_N) - (n_1, ..., n_N)\|_{E,E^\perp} &\leq \|t \cdot (\theta_1, ..., \theta_N) + \sum_{i=1}^{N-1} \frac{\epsilon}{2}\sigma_i - (n_1, ..., n_N)\|_{E,E^\perp} \\ &\quad + \|\frac{\epsilon}{2} \sum_{i=1}^{N-1} \sigma_i\|_{E,E^\perp} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{6.3}$$

Furthermore,

$$\begin{aligned} \|\Pi^\perp\big(\sum_{i=1}^{N-1} \frac{\epsilon}{2}\sigma_i - (n_1, ..., n_N)\big)\|_{E,E^\perp} &= \|\Pi^\perp\big(t \cdot (\theta_1, ..., \theta_N) + \sum_{i=1}^{N-1} \frac{\epsilon}{2}\sigma_i - (n_1, ..., n_N)\big)\|_{E,E^\perp} \\ &\leq \|t \cdot (\theta_1, ..., \theta_N) + \sum_{i=1}^{N-1} \frac{\epsilon}{2}\sigma_i - (n_1, ..., n_N)\|_{E,E^\perp} \\ &\leq \frac{\epsilon}{2}. \end{aligned} \tag{6.4}$$

This means that the coefficients $s_i$ of $\Pi^\perp(n_1, ..., n_N) = \sum_{i=1}^{N-1} s_i\sigma_i$ deviate at most by $\frac{\epsilon}{2}$ from $\frac{\epsilon}{2}$, so that $s_i \geq 0$. □

Now, we can use claim 3 to find a radius $R > 0$, such that

$$\forall y \in E, \quad B_R(y) \cap \Pi(\Lambda \cdot x) \cap (K^\circ \times E) \text{ is not empty .}$$

For each $\epsilon > 0$, claim 3 gives $2^{N-1}$ points $x_i = (n_1, ..., n_N) \in \mathbb{Z}^N, i = 1, ..., 2^{N-1}$ such that for each $x_i$ there exists $t_i > 0$ with

$$x_i = (n_1, ..., n_N) = t_i(\theta_1, ..., \theta_N) + ...$$

and the $s_{ji}$ run through all combinations of being positive and negative when $i$ runs from 1 to $2^{N-1}$.
The negative signs can be brought in by changing the relevant basis vectors $\sigma_j$ to $-\sigma_j$.
If we choose a small enough $\epsilon > 0$, all the points $x_i$ have an orthogonal projection $\Pi^\perp(x_i) = \sum s_{ji}\sigma_j \in K^\circ$.
**Claim 4:** For every $k \gg 0$, there exists $k_1, ..., k_{2^{N-1}} \geq 0$, such that $\sum_{i=1}^{2^{N-1}} k_i \geq k$ and $\sum_{i=1}^{2^{N-1}} k_i x_i \in K^\circ \times E$.
**Proof of claim 4:** Assume that

$$\|\Pi^\perp(\sum_{i=1}^{2^{N-1}} k_i x_i)\|_{E,E^\perp} = \|\sum_{i=1}^{2^{N-1}} \sum_{j=1}^{N-1} k_i s_{ji} \sigma_j)\|_{E,E^\perp} < \epsilon.$$

Since all $s_{ji}$ have absolute value less than $\epsilon$, adding $\sum_{j=1}^{N-1} s_{ji}\sigma_j$ to $\Pi^\perp(\sum_{i=1}^{2^{N-1}} k_i x_i)$ will not increase the norm of this vector if the $s_{ji}$ has the correct sign. Since all combinations of signs are achieved when $i$ runs from 1 to $2^{N-1}$, this shows that there exists an $i_\circ$ such that

$$\|\Pi^\perp\big(\sum_{i=1, i\neq i_\circ}^{2^{N-1}} k_i x_i + (k_{i_\circ} + 1)x_{i_\circ}\big)\|_{E,E^\perp} < \epsilon.$$

Repeating this argument will make $\sum_{i=1}^{2^{N-1}} k_i$ arbitrarily large.

□

The argument above also shows that, the points in $\Pi(\Lambda \cdot x) \cap (K^\circ \times E)$ are separated at most by $R = \max_{i=1,...,2^{N-1}} \|\Pi(x_i)\|_{E,E^\perp}$. Since $\|\Pi(\sum_{i=1}^{2^{N-1}} k_i x_i)\|_{E,E^\perp} = \sum_{i=1}^{2^{N-1}} k_i \|\Pi(x_i)\|_{E,E^\perp}$, claim 4 shows that the points in $\Pi(\Lambda \cdot x) \cap (K^\circ \times E)$ occur arbitrarily far away from $x$. Consequently,

$$B_R(y) \cap \Pi(\Lambda \cdot x) \cap (K^\circ \times E) \text{ always contains a point.}$$

□

The following more general theorem is a consequence of the proof of Theorem 6.2.8 above.

**Theorem 6.2.9** *If the dimension of the hyperplane $E$ is $n$, then the set*

$$X_{T'} = \Pi\big(X_T \cap (K \times E)\big)$$

*is relatively dense (if the window $K$ satisfies the assumption (*)).*

Proof Choose lines $E_1, ..., E_n \subset E$ through a point $x \in X_T$, whose spanning vectors $e_i$ are linearly independent. Then, construct points in $X_T$ arbitrarily close to lines $E_i$, as in claims 3 and 4 in the proof of Theorem 6.2.8 (using $E_i^\perp$ instead of $E^\perp$ and the preimage of the window $K$ in $E_i^\perp$, under the orthogonal projection $E_i^\perp \to E^\perp$). For $p \in E$, split up $p - x = \sum p_i e_i$. Choose points $x_i \in X_T$ approximating $E_i$ closest to $x + p_i e_i$. Then, the distance of $\sum_{i=1}^n x_i$ to $p$ is bounded independently of $p$.

□

**Theorem 6.2.10** *The set $X_{T'} = \Pi\big(X_T \cap (K \times E)\big)$ is uniformly discrete.*

Proof $Aut(X_T)$ is crystallographic, that is, a subgroup of a product of a lattice $\Lambda$ of translations of full rank and a finite point group, of finite index (see Proposition 1.1.21). For a fixed $R$, consider the following intersections:

$$B_R(y) \cap (K \times E) \cap X_T; \quad \forall y \in X_T \cap (K \times E)\ .$$

**Claim:** There are only a finite number of these bounded point sets, up to translations.

**Proof of the claim:** Take a fundamental domain $D \subset \mathbb{R}^N$ of $\Lambda$. Notice that $D$ is compact as $\Lambda$ is a lattice of full rank. Therefore, $D \cap X_T$ is finite, i.e., $D \cap X_T = \{x_1, ..., x_s\}$. Now, for all $x \in X_T$, there exists $\tau \in \Lambda$ such that $\tau(x) \in D$ and $\tau(x) = x_i$; therefore, $\bigcup_{i=1}^s \Lambda \cdot x_i = X_T$.

In particular, if $y = \tau(x_i)$ with $\tau \in \Lambda$, then, for any radius $R > 0$:

$$B_R(y) \cap X_T = \tau\big(B_R(x_i) \cap X_T\big),$$

as $\tau$ is an isometry in $Aut(X_T)$. Therefore, there are only finitely many point sets $B_R(y) \cap X_T$, up to translations in $Aut(X_T)$.

If $B_R(y) \cap X_T$ and $B_R(y') \cap X_T$ are mapped to each other by a translation, then $B_R(y) \cap X_T \cap (K \times E)$ and $B_R(y') \cap X_T \cap (K \times E)$ may not be mapped to each other by this translation because $B_R(y') \cap X_T \cap (K \times E)$

and $B_R(y') \cap X_T \cap \tau(K \times E)$ are different. On the other hand, there are only finitely many different point sets $B_R(y') \cap X_T \cap \tau(K \times E)$ for all $\tau \in Aut(X_T)$, because the number of points in $B_R(y') \cap X_T$ is finite. Hence, the claim follows. □

$X_{T'} \subset E$ is relatively dense, that is, $\exists R > 0$, such that $B_R(y) \cap X_{T'} \neq \emptyset$ for all $y \in E$. Choose points $\{y_i\}_{i \in I}$ such that $\bigcup B_R(y_i) = E$. For each $y_i$, choose $x_i \in B_R(y_i) \cap X_{T'}$ such that if we take the radius $2R$, then $B_R(y_i) \subset B_{2R}(x_i)$. This implies that $\bigcup_{i \in I} B_{2R}(x_i) = E$, $\bigcup_{i \in I} B_{2R}(x_i) = \Pi\big(\bigcup_{i \in I} B_{2R}(\overline{x}_i)\big)$ with $\overline{x}_i \in X_T \cap (K \times E)$ and $\Pi(\overline{x}_i) = x_i$.
From the claim, the number of distances of points in the sets

$$\Pi\big(B_{2R}(\overline{x}_i) \cap X_T \cap (K \times E)\big) = B_{2R}(x_i) \cap X_{T'} \ ; \forall x_i, i \in I$$

is finite, because projected translated points have the same distance as the projected points themselves.
Now, choose $0 < r < 2R$ as the minimum of these distances:
For $y, y' \in X_{T'}$, there is $x_i$, such that $y \in B_{2R}(x_i)$. If $y' \in B_{2R}(x_i)$, then $d(y, y') \geq r$ (by the choice of $r$). On the other hand, if $y' \notin B_{2R}(x_i)$, then $d(y, y') \geq 2R > r$, also by the choice of $r$. Hence, in both cases $d(y, y') > r$, so $X_{T'}$ is uniformly discrete.

□

**Theorem 6.2.11** *$X_{T'}$ is a Delone set.*

Proof Straightforward from Theorem 6.2.9 and Theorem 6.2.10.

□

**Theorem 6.2.12** *The Voronoi-cell tiling $VT(X_{T'})$ associated to the Delone set $X_{T'}$ is a simple tiling.*

Proof The claim made by Theorem 6.2.10 implies that there are only finitely many types of projections:

$$\Pi\big(B_R(y) \cap X_T \cap (K \times E)\big), \text{ up to translation in } E \text{ , for all } y \in X_T.$$

This is the case since translating, and then projecting to $E$ is the same as projecting first and then translating inside $E$.
If we decompose the translation $\tau$ as $\tau = \tau_E \oplus \tau_{E^\perp}$, then, $\Pi \circ \tau = \tau_E \circ \Pi$. Since projections of balls to $E$ are balls of the same radius in $E$, we have:

$$\Pi\big(B_R(y) \cap X_T \cap (K \times E)\big) = X_{T'} \cap B_R\big(\Pi(y)\big).$$

Consequently, there are only finitely many point sets of type $X_{T'} \cap B_R\big(\Pi(y)\big)$, up to translations in $E$.
By Proposition 2.2.5 on Delone sets, we know that for large enough $R \gg 0$, $X_{T'} \cap B_R\big(\Pi(y)\big)$ determines the Voronoi-cell of $\Pi(y)$ in $VT(X_{T'})$. Hence, there is only a finite number of tile types in $VT(X_{T'})$; up to translations in $E$, and so, $VT(X_{T'})$ is a simple tiling.

$\square$

# Bibliography


[1] Tom M. Apostol. *Modular Functions and Dirichlet series in Number Theory*. Graduate texts in Mathematics. Springer, Verlag, New York, 1997. Second edition.

[2] M. A. Armstrong. *Groups and Symmetry*. Undergraduate texts in Mathematics. Springer, United States of America, 1988.

[3] Louis Auslander. An account of the theory of crystallographic groups. *Proc. Amer. Math. Soc.*, 16(American Mathematical Society):1230–1236, 1965.

[4] Sheldon Axler. *Linear Algebra Done Right*. Undergraduate texts in Mathematics. Springer, United state of America, 1991. Undergraduate text in mathematics.

[5] Schlottmannt M. Baaket M. and JamisP.D. Quasiperiodic tilings with tenfold symmetry and equivalence with respect to local derivability. *J. Phys. A: Math. Gen*, 11(24):4637–4654, 1991.

[6] David Bailey. Pentagon-tilings. http://www.tess-elation.co.uk/pentagon-tilings.

[7] Solomyak Boris. Spectrum of dynamical systems arising from delone sets. *Fields institute Monographs, The Fields Institute For Research In Mathematical Sciences*, 10:265–291, 1998.

[8] Stanley N. Burris and H.P. *A Course in Universal Algebra*. A Series of Millennium edition. Springer, New york, 2011.

[9] Daniel R. Farkas. Crystallographic groups and their mathematics. *Rocky Mountain J. Math.*, 11(4):x+511, 1981.

[10] Alan Forrest, John Hunton, and Johannes Kellendonk. Topological invariants for projection method patterns. *Mem. Amer. Math. Soc.*, 159(758):x+120, 2002.

[11] Branko Grunbaum. *Convex Polytopes.* Graduate texts in Mathematics. A Division of Jhon Wiley and Sons, New York, 1967. Second edition.

[12] Branko Grünbaum and G. C. Shephard. *Tilings and patterns.* A Series of Books in the Mathematical Sciences. W. H. Freeman and Company, New York, 1989. An introduction.

[13] Paul R. Halmos. *Finite dimensional vector spaces.* Textbooks in Mathematics. Princeton University Press, Princeton, New Jersey, 1958. Second edition.

[14] I. Martin Isaacs. *Algebra: A graduate course.* Mathematics Series. Brooks/Cole, Pacific Grove, 1994.

[15] Rajaa Issa. Generative algorithms: (l-system) http://www.grasshopper3d.com/profiles/blogs/generative-algorithms.

[16] Bill Jacob. *Linear Algebra.* A Series of books in the mathematical sciences. W. H. Fraaman and Company, University of California, 1990. Second edition.

[17] Serge Lang. *Algebra.* Graduate Texts in Mathematics. Springer, New york, 2002. Third edition.

[18] Kathryn Lindsey. Aperiodic tilings. http://www.math.cornell.edu/ mec/2008-2009/kathrynlindsey/project/page5.htm.

[19] Baake M. and Grimm U. *Aperiodic order.* Encyclopedia Of Mathematics and Its Applications. Cambridge University Press, New york, 2013. Volume 1.

[20] Ramirez-Solano Maria. *Construction of the continuous hull for the combinatorics of a regular pentagonal tiling of the plane.* The center for Symmetry and Deformation.

[21] James R. Munkres. *Topology: a first course.* illustrated. Prentice-Hall Inc, the University of Michigan, 1974. Digitized 4 Sep 2010.

[22] R. Penrose. The role of aesthetics in pure and applied mathematical research. *Bull. Inst. Math. Appl.*, 10(1):266–271, 1974.

[23] P.M.Cohn. *Algebra 1.* Textbooks in Mathematics. Wiley, University of London, 1982. Second edition.

[24] Charles Radin. The pinwheel tilings of the plane. *Annals of Mathematics.*, 139(3):661–702, 1994.

[25] E. Arthur Robinson. Symbolic dynamics and tilings of $\mathbb{R}^d$. *Symbolic Dynamics and its Applications.*, 60:81–158, 2004.

[26] Lorenzo Sadun. *Topology of tiling spaces.* University Lecture Series. University Of Toronto Press, United state of America, 1997. Volume 46.

[27] Lorenzo Sadun. Tilings, tiling spaces and topology. *Philos. Mag.*, 86:875–881, 2006.

[28] Micheal Searcoid. *Metric spaces.* Springer, Verlag London, 2007. Undergraduate Mathematics Series.

[29] Marjorie Senechal. *Quasicrystals and geometry.* Undergraduate texts in Mathematics. The Press Syndicate of the University of Cambridge, 1996. New Ed edition.

[30] Mozes Shahar. Tilings, substitution systems and dynamical systems generated by them. *Journal of Analuse Mathematique*, 53(1):139–186, 1989.

[31] Bruce. Solomon. *Linear Algebra, Geometry and Transformation.* textbooks in Mathematics. CRC Press, United state, 2014.

[32] Tammo Tom Dieck. *Algebraic Topology.* Textbooks in Mathematics. European Mathematics Society, Germany, 2008.

[33] D. R. Wilkins. *Normed vector spaces and functional analysis.* A Series of Books in the Mathematical Sciences. University of Toronto Press, Toronto, 1997. Vol. XX, No. I.

**Index Of Definitions and Notations:**

List Of Symbols:

| | |
|---|---|
| $\mathbb{E}^n$ | Euclidean space |
| $Isom(\mathbb{E}^n)$ | group of isometries |
| $\mathbb{R}^n$ | real numbers in $n$ dimension |
| $Perm(\mathbb{E}^n)$ | group of 1-1 permutations |
| $Trans(\mathbb{E}^n)$ | group of translations |
| $\tau$ | refer to translation |
| $s$ | refer to translation |
| $O$ | fixed origin |
| $ev_O$ | evaluation map with fixed origin $O$ |
| $\mathbb{E}^n_O$ | Euclidean space with fixed origin $O$ |
| $\mathbb{E}^n_{O'}$ | Euclidean space with fixed origin $O'$ |
| $OO'$ | unique translation sends $O$ to $O'$ |
| $Aff(\mathbb{E}^n)$ | affine group |
| $ad\ f$ | map from translation to translation defined as $f\tau f^{-1}$ |
| $O(Trans(\mathbb{E}^n))$ | orthogonal group of translations |
| $O(\mathbb{E}^n_O)$ | orthogonal group of $\mathbb{E}^n_O$ |
| $\|\tau\|$ | the metric norm of $\tau$ |
| $B(\tau, \epsilon)$ | ball of radius $\epsilon$ centered at $\tau$ |
| $U_i$ | collection of open sets |
| $U^c$ | complement of open set |
| $\langle \cdot, \cdot \rangle_O$ | inner product on $\mathbb{E}^n_O$ |
| $B_O(\tau, \epsilon)$ | ball of radius $\epsilon$ centered at $\tau$ with respect to the fixed origin $O$ |

| | |
|---|---|
| $\tau_n$ | sequence of translations |
| $\longrightarrow_{pc}$ | point-wise convergence |
| $\longrightarrow_{uc}$ | uniform convergence |
| $\longrightarrow_{n}$ | norm convergence |
| $\rtimes$ | semi-direct product |
| $\Gamma$ | Crystallographic group |
| $Isom(\mathbb{E}^n)/\Gamma$ | quotient group |
| $End(\mathbb{E}^n_O)$ | group of endomorphisms with fixed origin |
| $d_\circ(\phi, \psi)$ | taxi-cab metric defined on $Isom(\mathbb{E}^n)$ |
| $\\| \cdot \\|_{op}$ | operator norm |
| $\\| \cdot \\|_{Eucl}$ | Euclidean norm |
| $id_{\mathbb{E}^n}$ | identity element in $\mathbb{E}^n$ |
| inf | the infimum of a set "greatest lower bound" |
| sup | the supremum of a set " least upper bound" |
| $B_r(x)$ | ball of radius $r$ around $x$ |
| $\\|a_{ij}\\|_\infty$ | max norm topology on $End(\mathbb{E}^n_O)$ |
| $det(A)$ | determinant of $A$ |
| $\bar{X}$ | closure of a set $X$ |
| $X^\circ$ | interior of a set $X$ |
| $T$ | Tiling |
| $t$ | refer to a tile in $T$ |
| $\{t_i\}_{i\in I}$ | set of tiles in $T$ |

| | |
|---|---|
| $d(T, T')$ | the tiling metric |
| $T + r$ | all tiles in $T$ shifted by $r \in \mathbb{R}^n$ |
| $\phi(T)$ | Tiling with all tiles in $T$ mapped by $\phi$ |
| $[T\ ]_A$ | patch of a tiling $T$ consists of all tiles intersect $A$ |
| $D$ | Delone set |
| $V_{pi}$ | Voronoi-cells |
| $VT(D)$ | Voronoi-cell tiling constructed by $D$ |
| $S$ | regular n-simplex |
| $VT_{\Gamma(p)}$ | Voronoi-tiling associated to $\Gamma$ |
| $Hp, q$ | half space |
| $L_{p,q}$ | hyper-plane |
| $d(T, T')$ | distance between two tilings $T, T'$ |
| $\Omega_T$ | tiling space |
| $(\Omega, d)$ | tiling metric space |
| $[\phi(T)\ ]_{B_r(0)}$ | patch of all tiles in $T$ mapped by $\phi$ with $B_r(0)$ |
| max | the maximum " largest value of a set" |
| $O(T)$ | the orbit of a tiling $T$ |
| $f\big(\phi(T)\big)$ | factor map for $T$ such that $f\big(\phi(T)\big) = \gamma\phi\gamma^{-1}(T)$ |
| $(T_n)_{n\in\mathbb{N}}$ | sequence of tilings |
| $T_n \longrightarrow T$ | convergent series of tilings |
| $Aut(T)$ | automorphism group of $T$ |
| $\Omega_T$ | the hull of a tiling $T$ |

| | |
|---|---|
| $\lim_{n\to\infty}$ | value that a sequence "approaches" when $n \to \infty$ |
| $GL(2,\mathbb{Z})$ | group of invertible $2\times 2$-matrices with entries in $\mathbb{Z}$ |
| $M(2,\mathbb{Z})$ | group of $2\times 2$-matrices in $\mathbb{Z}$ |
| $Stab_\Gamma(x)$ | stabilizer of $x$ in $\Gamma$ |
| $VT(\Gamma\cdot x)$ | Voronoi-cell tiling associated to the orbit $\Gamma\cdot x$ which is a Delone set |
| $\oplus$ | addition as a binary operation |
| $\otimes$ | multiplication as a binary operation |
| $D_2$ | abelian-dihedral group where $D_2 \cong \mathbb{Z}_2$ |
| $D_4$ | abelian-dihedral group where $D_4 \cong \mathbb{Z}_2\times\mathbb{Z}_2$ |
| $\{(X_i,t_i)_i\}$ | point set data |
| $X_T$ | point set of $\{(X_i,t_i)_i\}$ |
| $E$ | $m$-dimensional projection hyperplane |
| $E^\perp$ | $n-m$- dimensional projection hyperplane orthogonal to $E$ |
| $\Pi$ | orthogonal projector onto $E$ |
| $\Pi^\perp$ | orthogonal projector onto $E^\perp$ |
| $K$ | window for the projection |
| $K\times E$ | the cylinder |
| $(K,E)$ | cut-and-project data for $E^n$ |
| $dim(E)$ | the dimension of $E$ |