Hybrid uncertain analysis for exterior acoustic field prediction with interval random parameters

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Abstract: For exterior acoustic field problems that lack sufficient information to construct precise probability distributions，an interval random model is introduced to deal with the uncertain parameters. In the interval random model, the probability variables are employed to treat the uncertain parameters, whereas some distribution parameters of random variables are modeled as interval variables instead of precise values. Based on the interval random model, the interval random finite element equation for exterior acoustic fields is established and a hybrid uncertain analysis

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method is presented to solve the exterior acoustic field problem with interval random variables. In the presented method, by temporarily neglecting the uncertainties of interval variables, a first-order stochastic perturbation method is adopted to calculate the expectation and the variance of the response vector. According to the monotonicity of the expectation and variance of the response vector with respect to the interval variables, the lower and upper bounds of the expectation and variance of the response vector can be calculated by the vertex method. In addition, in order to ensure accuracy of the proposed method, the subinterval technique is introduced and investigated. The numerical example of a square flexible shell model is presented to demonstrate the effectiveness of the proposed method.

Key Words: Uncertain exterior acoustic field prediction; Interval random variable; Matrix perturbation method; Random moment method; Vertex method; subinterval technique;

1. Introduction

In the past decades, there has been an increasing interesting in studying predictive methods for the dynamic response of systems with uncertain parameters. The traditional analysis techniques to cope with the uncertainties are generally based on the probabilistic model, in which the random variables are used to model the uncertainties of parameters existing in the practical engineering problems [1-2]. In the probabilistic approaches, probability distributions of the random variables are defined unambiguously. The Monte Carlo method is still the most versatile probabilistic method for stochastic problems so far [3-6]. However, the accuracy of the Monte Carlo method strongly depends on the number of samples generated by a random number generator. Due to its excessive computational cost, the Monte Carlo method is not applicable to the large-scale stochastic engineering problems. Perturbation stochastic finite element method [7-10], spectral stochastic finite element method [11-13] and Bayesian approach [14-17] are alternative approaches for the random problems, and have acquired significant successes. Except the probabilistic approaches mentioned formerly, other modern stochastic finite element methods have also accomplished great progresses. By integrating with the finite element method, the joint diagonalization approach has been well applied to linear probabilistic systems [18-19]. The Fourier discretization scheme has been developed to deal with stochastic problems in order to improving the computational efficiency and accuracy of stochastic finite element methods [20-22]. By using probabilistic approaches, a large amount of statistical information or experimental data is required to construct precise probability distributions of uncertain parameters. Unfortunately, in many practical applications, the experimental samples to determine the probability distributions are not always available or sometimes very difficult to obtain. As a consequence, we have to make some suitable assumptions for probability distributions of random variables. However, these assumed probability distributions may be unreliable and the results obtained by the probabilistic approach based these assumed probability distributions may be incorrect [23].

To describe the uncertainties of parameters without enough information objectively, the interval random model is developed for uncertainty quantification. In the interval random model, the uncertain parameters are treated as random variables whose distribution parameters with limited information can only be expressed as interval variables instead of precise values. This uncertain model was firstly proposed to find the least favorable value of the mean-square response of a random vibration problem by Elishakoff and Colombi [24-25]. Subsequently, the interval random model was applied to the structural response analysis [26] and the structural reliability analysis [27-28]. Based on the classical probabilistic reliability theory and the interval analysis technique, the failure probabilistic interval of structures with interval random variables was evaluated by Qiu et al [29]. By combining the Monte Carlo simulation process into the interval analysis, Zhang et al. proposed an interval Monte Carlo method to investigate the interval failure probabilities of structures with interval random variables [30]. In order to improve the efficiency of the interval Monte Carlo method, an interval quasi-Monte Carlo method was proposed to calculate the variation ranges of the structural failure probabilities [31]. Recently, Xia et al. have proposed an interval random perturbation method to compute the bounds of expectations and variances of the responses of acoustic fields and structural-acoustic systems with interval random variables [32-33]. On the basis of the change-of-variable technique and interval perturbation technique, an inverse mapping hybrid perturbation method was proposed to estimate the variation ranges of the response probability distributions of acoustic fields with interval random variables [34]. Chen et al. have proposed a hybrid stochastic interval perturbation method for determining the energy flow in coupled vibrating systems with interval random variables [35]. As mentioned above, researches on the interval random models have achieved significance progress. However, from the overall perspective, studies on the interval random models are still in its preliminary stage and are limited to the special fields. Up to now, the application of the interval random model in the exterior acoustic problem is promising, yet hardly been exploited.

Traditional numerical analysis for the exterior acoustic problem with given parameters has received considerable attention in the last decades [36-37]. However, due to the effects of manufacturing tolerance, physical imperfection and system complex, uncertainties usually exist in material properties, geometric dimensions and boundary conditions. Without considering the uncertainties of the exterior acoustic problem, the results computed by the deterministic numerical approach may be unreliable. Some developments have been achieved in the analysis for the exterior acoustic field with uncertainties. Chen et al. proposed a hybrid perturbation method to calculate the bounds of expectations and variances of the responses of exterior acoustic fields with both random variables and interval variables [38]. Wang et al. proposed two interval analysis methods for the exterior acoustic field prediction with interval variables [39]. Recently, an efficient evidence-theory-based analysis method is proposed by Chen et al. for the response prediction of exterior acoustic fields with epistemic uncertainty [40].

This paper aims to solve the problem of an exterior acoustic field with interval random variables, in which the random variables are used to model the uncertain parameters, whereas some key distribution parameters of the random variables are expressed as interval variables but not precise values. Compared with the problem of an exterior acoustic field with both random variables and interval variables studied in literature [38], the interval random model in this work is more complicated. To some extent, the hybrid uncertain random and interval model is a special case of interval random model. The random variable can be interpreted as an interval random variable with deterministic distributional parameters, and the interval variable can be interpreted as an interval random variable whose standard variance is equal to zero. Based on the interval random model, the interval random dynamic equilibrium equation of the exterior acoustic field is established. Inspired by the way of handling the hybrid uncertain exterior acoustic field with both random variables and interval variables in literature [38], a similar analysis process is presented for the exterior acoustic field prediction with interval random variables. In the present method, by temporarily neglecting the uncertainties of interval variables, a first-order stochastic perturbation method is adopted to calculate the expectation and variance of the response vector. According to the linear monotonicity of the expectation and variance of the response vector with respect to interval variables, the lower and upper bounds of the expectation and the variance of the response vector can be calculated by a vertex method. Besides, in order to guarantee the accuracy of the presented method, the subinterval technique is introduced into the hybrid uncertain analysis for the exterior acoustic field with interval random variables.

The remainder of this paper is organized as follows. The equilibrium equation for the exterior acoustic field prediction is discussed in Section 2. In Section 3, an uncertain analysis method to calculate the bounds of the expectation and the variance of the response vector of the exterior acoustic field with interval random parameters is deduced. Section 4 provides a numerical example and Section 5 gives some conclusions.

2. Formulation of the exterior acoustic field prediction

The coupled finite element method/boundary element method (FEM/BEM) is widely used to handle the exterior acoustic field prediction because it only refers to structural surface discretization and solves exterior problems naturally. The acoustic medium is assumed to be inviscid and incompressible. Only the normal direction of the interaction between the acoustic field and the vibrating structure is considered.

2.1. FEM formulation for shell structure

In the frequency domain, the finite element equation of the shell structure with considering damping can be expressed as

$$(\boldsymbol{K}_s + i\omega\boldsymbol{C}_s - \omega^2\boldsymbol{M}_s)\boldsymbol{u}_s = \boldsymbol{F}_s \tag{1}$$

where $\boldsymbol{K}_s$, $\boldsymbol{M}_s$, $\boldsymbol{C}_s$, and $\boldsymbol{F}_s$ denote the structural stiffness matrix, structural mass matrix, structural damping matrix and exciting force vector, respectively; $\boldsymbol{u}_s$ denotes the displacement vector; ω is the angular frequency; and $i=\sqrt{-1}$ is an imaginary unit. The structural damping matrix $\boldsymbol{C}_s$ can be expressed as

$$\boldsymbol{C}_s = \alpha\boldsymbol{M}_s + \beta\boldsymbol{K}_s \tag{2}$$

where α and β are the damping coefficients of the damping material.

In the steady-state form, the relationship between the structural velocity vector $\boldsymbol{v}$

and displacement vector $\boldsymbol{u}_s$ can be expressed as following

$$\boldsymbol{v} = i\omega \boldsymbol{u_s} \tag{3}$$

The dynamic equilibrium equation about the structural vibration velocity can be obtained through multiplying both sides of Eq. (1) with $i\omega$

$$(\boldsymbol{K}_s + i\omega \boldsymbol{C}_s - \omega^2 \boldsymbol{M}_s)\boldsymbol{v_s} = i\omega \boldsymbol{F}_s \tag{4}$$

2.2. BEM formulation for acoustic field

The Helmholtz equation can be cast into a boundary integral equation as follows

$$C(P)p(P) = \int_S (i\rho_f \omega v_n G_0 + p\frac{\partial G_0}{\partial \boldsymbol{n}})ds \tag{5}$$

Here, the symbol P represents the objective field point where the sound pressure will be computed and C represents the interpolation coefficient. The integration of Eq. (5) is conducted along the structural surface S, where the symbol ρ_f and v_n denote the mass density and normal velocity of the acoustic medium, respectively; ω and c are the angular frequency and the sound speed, respectively. The symbol G_0 denotes the Green function which is the fundamental solution of the Helmholtz equation. The symbol $\boldsymbol{n}$ is the normal vector pointing to the acoustic domain. These variables are also illustrated in Fig. 1.

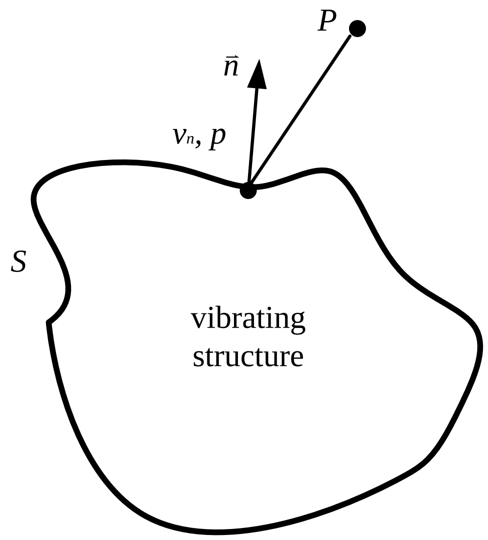


Fig. 1 Schematic illustrating of the boundary integral equation

Discretize the Helmholtz integral equation using the standard boundary element method, which gives the following system of algebraic equations

$$\boldsymbol{HP} = \boldsymbol{G}\boldsymbol{v}_n \tag{6}$$

$$\boldsymbol{H} = \sum_{l=1}^{N} C_l \delta_{lm} + \hat{H}_{lm}, \delta_{lm} = \begin{cases} 1 & l = m, \\ 0 & l \neq m, \end{cases} \tag{7}$$

$$\boldsymbol{G} = \sum_{m=1}^{N} G_{lm},\ \ \hat{H}_{lm} = \int_{S_l} \frac{\partial G_0}{\partial n} ds,\ \ G_{lm} = \int_{S_l} i \rho_f \omega G_0 ds \tag{8}$$

where S_l denotes the area of the lth element.

2.3. Coupled FEM/BEM for exterior acoustic field prediction

Combining the governing equation of the structure and acoustic domain described previously, the coupled governing equation of the structure-acoustic system can be expressed as follows

$$\boldsymbol{ZU} = \boldsymbol{F} \tag{9}$$

where $\boldsymbol{Z}$ is the structural–acoustic dynamic stiffness matrix; $\boldsymbol{U}$ is the frequency response vector; and $\boldsymbol{F}$ is the external excitation vector. They can be expressed as

$$\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{K}_s + i\omega \boldsymbol{C}_s - \omega^2 \boldsymbol{M}_s & -\boldsymbol{L} \\ \boldsymbol{GT} & -\boldsymbol{H} \end{bmatrix} \tag{10}$$

$$\boldsymbol{U} = \{\boldsymbol{v} \quad \boldsymbol{p}\}^{\mathrm{T}} \tag{11}$$

$$\boldsymbol{F} = \{i\omega \boldsymbol{F}_s \quad 0\}^{\mathrm{T}} \tag{12}$$

Here, the symbol $\boldsymbol{T}$ denotes the conversion matrix which transforms the velocity vector to the normal velocity vector; the symbol $\boldsymbol{v}$ and $\boldsymbol{p}$ are the structural velocity vector and the sound pressure vector on the interface, respectively; the symbol $\boldsymbol{L}$ denotes the structure-acoustic coupling matrix.

If Eq. (9) is solved, by substituting $\boldsymbol{v}$ and $\boldsymbol{p}$ into Eq. (5), the sound pressure P^c at any point c in the exterior field can be computed by

$$P^c = \boldsymbol{gTv} + \boldsymbol{hp} = [\boldsymbol{gT} \quad \boldsymbol{h}] \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{p} \end{bmatrix} = \boldsymbol{CU} = \boldsymbol{CZ}^{-1}\boldsymbol{F} \tag{13}$$

where $\boldsymbol{g}$, $\boldsymbol{h}$ and $\boldsymbol{C}$ are the interpolation coefficient row vectors in the exterior acoustic field

$$\boldsymbol{g} = \int_S i\rho_f \omega G_0 ds \tag{14}$$

$$\boldsymbol{h} = \int_S \frac{\partial G_0}{\partial \boldsymbol{n}} ds \tag{15}$$

$$\boldsymbol{C} = [\boldsymbol{gT} \quad \boldsymbol{h}] \tag{16}$$

3. Matrix perturbation analysis for the exterior acoustic field prediction with interval random parameters

3.1. Exterior acoustic field prediction with interval random parameters

Let $\boldsymbol{A}$ be the parameter vector of the exterior field problem consist of the structural properties, the acoustic properties and the external excitation. Due to the effects of model inaccuracies, physical imperfections, multiphase characteristics of materials and unpredictable environment factors, the parameter vector $\boldsymbol{A}$ is treated as interval random vector and denoted as $\boldsymbol{A}^R(\boldsymbol{a}^I)$. The interval random vector composed by all independent interval random parameters can be expressed as

$$\boldsymbol{A}^R(\boldsymbol{a}^I) = (A_1^{\ R}(\boldsymbol{a}_1^I),\ldots,A_n^{\ R}(\boldsymbol{a}_n^I),\ldots,A_{L_R}^{\ R}(\boldsymbol{a}_{L_R}^I)) \ , \quad n=1,\, 2,\, \ldots,\, L_R \tag{17}$$

Where L_R represents the number of interval random parameters; $A_n^{\ R}(\boldsymbol{a}_n^I)$ represents the n-th interval random parameters; $\boldsymbol{a}_n^I$ represents the interval vector related to the interval random parameter $A_n^{\ R}(\boldsymbol{a}_n^I)$ and can be expressed as

$$\boldsymbol{a}_n^I = \left[\boldsymbol{a}_n^L, \boldsymbol{a}_n^U\right] = \boldsymbol{a}_n^m + \Delta\boldsymbol{a}^I \tag{18}$$

$$\boldsymbol{a}_n^m = \frac{\boldsymbol{a}_n^U + \boldsymbol{a}_n^L}{2} \tag{19}$$

$$\Delta\boldsymbol{a}_n^I = \left[-\Delta\boldsymbol{a}_n, \Delta\boldsymbol{a}_n\right], \Delta\boldsymbol{a}_n = \frac{\boldsymbol{a}_n^U - \boldsymbol{a}_n^L}{2} \tag{20}$$

or in component forms

$$a_i^I = \left[a_i^L, a_i^U\right] = a_i^m + \Delta a_i^I, \quad \Delta a_i^I = \left[-\Delta a_i, \Delta a_i\right], \ i = 1, 2, \ldots, l_n \tag{21}$$

$$\Delta a_i = \frac{a_i^U - a_i^L}{2}, a_i^m = \frac{a_i^U + a_i^L}{2} \tag{22}$$

Where l_n denotes the number of interval variables related to $A_n^{\ R}(\boldsymbol{a}_n^I)$; $\boldsymbol{a}_n^L$ and $\boldsymbol{a}_n^U$ denote the lower and upper bounds of the interval vector $\boldsymbol{a}_n^I$; $\boldsymbol{a}_n^m$ denotes the of the midpoint interval vector $\boldsymbol{a}_n^I$; $\Delta\boldsymbol{a}_n$ denotes the maximum deviation width of the interval vector $\boldsymbol{a}_n^I$; $\Delta\boldsymbol{a}_n^I$ denotes the deviation interval of the interval vector $\boldsymbol{a}_n^I$; a_i^L and a_i^U denote the lower and upper bounds of the interval variable a_i^I; a_i^m denotes the midpoint of the interval variable a_i^I; Δa_i denotes the maximum deviation width of the interval variable a_i^I; Δa_i^I denotes the deviation interval of the interval variable a_i^I.

The expectation $\mu(\boldsymbol{A}^R(\boldsymbol{a}^I))$ and variance $\sigma^2(\boldsymbol{A}^R(\boldsymbol{a}^I))$ of the interval random vector $\boldsymbol{A}^R(\boldsymbol{a}^I)$ can be expressed as

$$\mu(\boldsymbol{A}^R(\boldsymbol{a}^I)) = \mu(A_1{}^R(\boldsymbol{a}_1^I),\ldots,A_n{}^R(\boldsymbol{a}_n^I),\ldots,A_L{}^R(\boldsymbol{a}_L^I)) \tag{23}$$

$$\sigma^2(\boldsymbol{A}^R(\boldsymbol{a}^I)) = \sigma^2(A_1{}^R(\boldsymbol{a}_1^I),\ldots,A_n{}^R(\boldsymbol{a}_n^I),\ldots,A_L{}^R(\boldsymbol{a}_L^I)) \tag{24}$$

With respect to the interval random vector $\boldsymbol{A}^R(\boldsymbol{a}^I)$, the structural–acoustic dynamic stiffness matrix $\boldsymbol{Z}$ is interval random matrix, the external excitation vector $\boldsymbol{F}$ and the interpolation coefficient vector $\boldsymbol{C}$ are interval random vectors. Thus, the frequency response of sound pressure $P^c(\boldsymbol{A}^R(\boldsymbol{a}^I))$ at any point c in the exterior field is also an interval random vector and can be expressed as

$$P^c(\boldsymbol{A}^R(\boldsymbol{a}^I)) = \boldsymbol{C}(\boldsymbol{A}^R(\boldsymbol{a}^I))\boldsymbol{Z}^{-1}(\boldsymbol{A}^R(\boldsymbol{a}^I))\boldsymbol{F}(\boldsymbol{A}^R(\boldsymbol{a}^I)) \tag{25}$$

3.2. Matrix perturbation analysis for the exterior acoustic field with interval random parameters

The interval variables related to the interval random parameters $\boldsymbol{A}^R(\boldsymbol{a}^I)$ are firstly regarded as constant. The first-order Taylor expansion of the interval random matrix $\boldsymbol{Z}(\boldsymbol{A}^R(\boldsymbol{a}^I))$ at the expectation of the interval random vector $\boldsymbol{A}^R(\boldsymbol{a}^I)$ can be expressed as

$$\boldsymbol{Z}(\boldsymbol{A}^R(\boldsymbol{a}^I)) = \boldsymbol{Z}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I))) + \sum_{n=1}^{L_R} \frac{\partial \boldsymbol{Z}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I)))}{\partial A_n^R}(A_n{}^R - \mu(A_n{}^R(\boldsymbol{a}_n^I))) = \boldsymbol{Z}^R + \Delta\boldsymbol{Z}^R \tag{26}$$

$$\boldsymbol{Z}^R = \boldsymbol{Z}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I))), \Delta\boldsymbol{Z}^R = \sum_{n=1}^{L_R} \frac{\partial \boldsymbol{Z}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I)))}{\partial A_n^R}(A_n{}^R - \mu(A_n{}^R(\boldsymbol{a}_n^I))) \tag{27}$$

where $\boldsymbol{Z}^R$ and $\Delta\boldsymbol{Z}^R$ are the mean value and deviation of the structural–acoustic dynamic stiffness matrix $\boldsymbol{Z}$ with considering the related interval variables of the interval random variable $\boldsymbol{A}^R(\boldsymbol{a}^I)$ are constant.

Similarly, the first-order Taylor expansion of the interval random external excitation vector $\boldsymbol{F}$ and the interpolation coefficient vector $\boldsymbol{C}$ can also be expressed as

$$\boldsymbol{F}(\boldsymbol{A}^R(\boldsymbol{a}^I)) = \boldsymbol{F}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I))) + \sum_{n=1}^{L_R} \frac{\partial \boldsymbol{F}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I)))}{\partial A_n^R}(A_n^R - \mu(A_n^R(\boldsymbol{a}_n^I))) = \boldsymbol{F}^R + \Delta \boldsymbol{F}^R \tag{28}$$

$$\boldsymbol{F}^R = \boldsymbol{F}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I))), \Delta \boldsymbol{F}^R = \sum_{n=1}^{L_R} \frac{\partial \boldsymbol{F}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I)))}{\partial A_n^R}(A_n^R - \mu(A_n^R(\boldsymbol{a}_n^I))) \tag{29}$$

$$\boldsymbol{C}(\boldsymbol{A}^R(\boldsymbol{a}^I)) = \boldsymbol{C}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I))) + \sum_{n=1}^{L_R} \frac{\partial \boldsymbol{C}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I)))}{\partial A_n^R}(A_n^R - \mu(A_n^R(\boldsymbol{a}_n^I))) = \boldsymbol{C}^R + \Delta \boldsymbol{C}^R \tag{30}$$

$$\boldsymbol{C}^R = \boldsymbol{C}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I))), \Delta \boldsymbol{C}^R = \sum_{n=1}^{L_R} \frac{\partial \boldsymbol{C}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I)))}{\partial A_n^R}(A_n^R - \mu(A_n^R(\boldsymbol{a}_n^I))) \tag{31}$$

where $\boldsymbol{F}^R$ and $\Delta\boldsymbol{F}^R$ are the mean value and deviation of $\boldsymbol{F}$ with considering the related interval variables of the interval random variable $\boldsymbol{A}^R(\boldsymbol{a}^I)$ are constant, $\boldsymbol{C}^R$ and $\Delta\boldsymbol{C}^R$ are the mean value and deviation of $\boldsymbol{C}$ with considering the related interval variables of the interval random variable $\boldsymbol{A}^R(\boldsymbol{a}^I)$ are constant.

Substituting Eqs. (26), (28) and (30) into Eq. (25), Eq. (25) can be transformed into

$$P^c(\boldsymbol{A}^R(\boldsymbol{a}^I)) = (\boldsymbol{C}^R + \Delta\boldsymbol{C}^R)(\boldsymbol{Z}^R + \Delta\boldsymbol{Z}^R)^{-1}(\boldsymbol{F}^R + \Delta\boldsymbol{F}^R) \tag{32}$$

$(\boldsymbol{Z}^R + \Delta\boldsymbol{Z}^R)^{-1}$ can be expanded through Neumann series if the spectral radius of $(\boldsymbol{Z}^R)^{-1}\Delta\boldsymbol{Z}^R$ is less than 1 [41].

$$(\boldsymbol{Z}^R + \Delta\boldsymbol{Z}^R)^{-1} = \left(\boldsymbol{Z}^R\right)^{-1} + \sum_{r=1}^{\infty}\left(\boldsymbol{Z}^R\right)^{-1}\left(-\Delta\boldsymbol{Z}^R\left(\boldsymbol{Z}^R\right)^{-1}\right)^r \tag{33}$$

By substituting Eq. (33) into Eq. (32), we can obtain

$$
\begin{aligned}
P^{c}(\boldsymbol{A}^{R}(\boldsymbol{a}^{I})) &= \boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\boldsymbol{F}^{R} + \boldsymbol{C}^{R}\sum_{r=1}^{\infty}\left(\boldsymbol{Z}^{R}\right)^{-1}\left(-\Delta\boldsymbol{Z}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\right)^{r}\boldsymbol{F}^{R} + \Delta\boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\boldsymbol{F}^{R} \\
&+ \Delta\boldsymbol{C}^{R}\sum_{r=1}^{\infty}\left(\boldsymbol{Z}^{R}\right)^{-1}\left(-\Delta\boldsymbol{Z}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\right)^{r}\boldsymbol{F}^{R} + \boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\Delta\boldsymbol{F}^{R} + \Delta\boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\Delta\boldsymbol{F}^{R} \\
&+ \boldsymbol{C}^{R}\sum_{r=1}^{\infty}\left(\boldsymbol{Z}^{R}\right)^{-1}\left(-\Delta\boldsymbol{Z}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\right)^{r}\Delta\boldsymbol{F}^{R} + \Delta\boldsymbol{C}^{R}\sum_{r=1}^{\infty}\left(\boldsymbol{Z}^{R}\right)^{-1}\left(-\Delta\boldsymbol{Z}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\right)^{r}\Delta\boldsymbol{F}^{R}
\end{aligned}
\tag{34}
$$

For most engineering problems, the improvement of accuracy through higher order perturbation terms is rather small when compared with the increase of the computational effort. Therefore, only the first-order perturbation term is reserved and Eq. (34) can be rewritten as

$$
\begin{aligned}
P^{c}(\boldsymbol{A}^{R}(\boldsymbol{a}^{I})) = \boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\boldsymbol{F}^{R} + \Delta\boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\boldsymbol{F}^{R} + \boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\Delta\boldsymbol{F}^{R} \\
-\boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\Delta\boldsymbol{Z}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\boldsymbol{F}^{R}
\end{aligned}
\tag{35}
$$

Substituting Eqs. (27), (29) and (31) into Eq. (35), we can obtain the frequency response vector

$$
\begin{aligned}
P^{c}(\boldsymbol{A}^{R}(\boldsymbol{a}^{I})) &= \boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\boldsymbol{F}^{R} + \sum_{n=1}^{L_R}\frac{\partial\boldsymbol{C}(\mu(\boldsymbol{A}^{R}(\boldsymbol{a}^{I})))}{\partial A_n^{R}}(A_n^{R} - \mu(A_n^{R}(\boldsymbol{a}_n^{I})))\left(\boldsymbol{Z}^{R}\right)^{-1}\boldsymbol{F}^{R} \\
&+ \boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\sum_{n=1}^{L_R}\frac{\partial\boldsymbol{F}(\mu(\boldsymbol{A}^{R}(\boldsymbol{a}^{I})))}{\partial A_n^{R}}(A_n^{R} - \mu(A_n^{R}(\boldsymbol{a}_n^{I}))) \\
&- \boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\sum_{n=1}^{L_R}\frac{\partial\boldsymbol{Z}(\mu(\boldsymbol{A}^{R}(\boldsymbol{a}^{I})))}{\partial A_n^{R}}(A_n^{R} - \mu(A_n^{R}(\boldsymbol{a}_n^{I})))\left(\boldsymbol{Z}^{R}\right)^{-1}\boldsymbol{F}^{R}
\end{aligned}
\tag{36}
$$

Based on the random moment method, the expectation $U(\boldsymbol{A}^{R}(\boldsymbol{a}^{I}))$ of the frequency response vector $P^{c}(\boldsymbol{A}^{R}(\boldsymbol{a}^{I}))$ can be expressed as

$$
U(\boldsymbol{A}^{R}(\boldsymbol{a}^{I})) = \mu(P^{c}(\boldsymbol{A}^{R}(\boldsymbol{a}^{I}))) = \boldsymbol{C}^{R}\left(\boldsymbol{Z}^{R}\right)^{-1}\boldsymbol{F}^{R} \tag{37}
$$

As all the random parameters are independent of each other, the covariance between different random parameters equate to zero. Therefore, the variance $V(\boldsymbol{A}^{R}(\boldsymbol{a}^{I}))$ of the frequency response vector $P^{c}(\boldsymbol{A}^{R}(\boldsymbol{a}^{I}))$ can be expressed as

$$
\begin{aligned}
V(\boldsymbol{A}^R(\boldsymbol{a}^I)) &= \sigma^2(P^c(\boldsymbol{A}^R(\boldsymbol{a}^I))) \\
&= \sum_{n=1}^{L_R}\left(\frac{\partial \boldsymbol{C}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I)))}{\partial A_n^R}\left(\boldsymbol{Z}^R\right)^{-1}\boldsymbol{F}^R + \boldsymbol{C}^R\left(\boldsymbol{Z}^R\right)^{-1}\frac{\partial \boldsymbol{F}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I)))}{\partial A_n^R}\right. \\
&\quad \left. -\boldsymbol{C}^R\left(\boldsymbol{Z}^R\right)^{-1}\frac{\partial \boldsymbol{Z}(\mu(\boldsymbol{A}^R(\boldsymbol{a}^I)))}{\partial A_n^R}\left(\boldsymbol{Z}^R\right)^{-1}\boldsymbol{F}^R\right)^2 \sigma^2(\boldsymbol{A}^R(\boldsymbol{a}^I))
\end{aligned} \tag{38}
$$

Considering the interval variables related to the interval random variables $\boldsymbol{A}^R(\boldsymbol{a}^I)$, the expectation $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$ and variance $V(\boldsymbol{A}^R(\boldsymbol{a}^I))$ are interval vectors. The expectation $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$ and the variance $V(\boldsymbol{A}^R(\boldsymbol{a}^I))$ can be expanded through the first-order Taylor series at the midpoint of the interval variables

$$
U(\boldsymbol{A}^R(\boldsymbol{a}^I)) = U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m))) + \sum_{l=1}^{l_n}\frac{\partial U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m)))}{\partial a_l^I}\Delta a_l^I \tag{39}
$$

$$
V(\boldsymbol{A}^R(\boldsymbol{a}^I)) = V(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m))) + \sum_{l=1}^{l_n}\frac{\partial V(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m)))}{\partial a_l^I}\Delta a_l^I \tag{40}
$$

As $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$ and $V(\boldsymbol{A}^R(\boldsymbol{a}^I))$ may be an implicit function of the interval variables a_n^I, the central difference method is adopted to compute the first derivative of $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$ and $V(\boldsymbol{A}^R(\boldsymbol{a}^I))$. The process to obtain the derivative of $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$ is simple, which can be expressed as

$$
\frac{\partial U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m)))}{\partial a_l^I} = \frac{U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m + \delta\boldsymbol{a}_l))) - U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m - \delta\boldsymbol{a}_l)))}{2\delta a_l} \tag{41}
$$

where $\delta a_l \le \Delta a_l$ is the variation of the interval variable a_l^I, $\delta\boldsymbol{a}_l$ is the variation vector associated with δa_l and can be expressed as

$$
\delta\boldsymbol{a}_l = (0, \ldots, \delta a_l, \ldots, 0) \tag{42}
$$

As the impact of neglecting the higher order terms of Taylor expansion is unpredictable, the first-order Taylor expansion is limited to the hybrid uncertain problems with small variation ranges of the interval variables. Thus, we just discuss the interval random variables whose distribution parameters are interval variables with small variation ranges. For interval variables with small variation ranges, δa_l can be replaced by Δa_l approximately and $\Delta \boldsymbol{a}_l = (0,\ldots,\Delta a_l,\ldots,0)$. Then, Eq. (41) can be rewritten as

$$\frac{\partial U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m)))}{\partial a_l^I} = \frac{U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m + \Delta \boldsymbol{a}_l))) - U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m - \Delta \boldsymbol{a}_l)))}{2\Delta a_l} \tag{43}$$

Substituting Eq. (43) into Eq. (39), one gets

$$U(\boldsymbol{A}^R(\boldsymbol{a}^I)) = U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m))) + \sum_{l=1}^{l_n} \frac{U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m + \Delta \boldsymbol{a}_l))) - U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m - \Delta \boldsymbol{a}_l)))}{2\Delta a_l} \Delta a_l^I \tag{44}$$

It is apparent that $\partial U(\boldsymbol{A}^R(\boldsymbol{a}^I)) / \partial \Delta a_l^I$ is a constant vector which is not involved with the interval variable Δa_l^I. Therefore, $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$ is a monotonic function associated with Δa_l^I .

The vertex $\Delta \hat{a}_l$ of the interval variable Δa_l^I can be defined as

$$\Delta \hat{a}_l = \left\{ \Delta \hat{a}_l \in \Delta a_l^I : \Delta \hat{a}_l = -\Delta a_l \text{ or } \Delta \hat{a}_l = \Delta a_l \right\},\ l = 1,\ldots,l_n \tag{45}$$

For each interval variable a_l^I , $\Delta \hat{a}_l = -\Delta a_l$ or $\Delta \hat{a}_l = \Delta a_l$ is determined by $\partial U(\boldsymbol{A}^R(\boldsymbol{a}^I)) / \partial \Delta a_l^I$. If the sign of $\partial U(\boldsymbol{A}^R(\boldsymbol{a}^I)) / \partial \Delta a_l^I$ is positive, $\Delta \hat{a}_l = \Delta a_l$ is used for obtaining the maximum value of the expectation $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$ and $\Delta \hat{a}_l = -\Delta a_l$ is used for calculating the minimum value of the expectation $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$, and vice versa. Through judging the sign of $\partial U(\boldsymbol{A}^R(\boldsymbol{a}^I)) / \partial \Delta a_l^I$, the vertices that lead to the maximum or the minimum values of the expectation $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$ can be determined. Here, let $\Delta \hat{\boldsymbol{a}}_{max}$ denotes the vertex combination that can be used to compute the maximum value of the expectation $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$, and $\Delta \hat{\boldsymbol{a}}_{min}$ denotes the vertex combination for calculating the minimum value of the expectation $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$.

Substituting $\Delta\hat{\boldsymbol{a}}_{min}$ and $\Delta\hat{\boldsymbol{a}}_{max}$ into Eq. (44), the lower and upper bounds of the expectation $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$ can be expressed as

$$U_{lower} = U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m))) + \sum_{l=1}^{l_n} \frac{U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m + \Delta\boldsymbol{a}_l))) - U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m - \Delta\boldsymbol{a}_l)))}{2\Delta a_l} \Delta\hat{a}_{l,min} \quad (46)$$

$$U_{upper} = U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m))) + \sum_{l=1}^{l_n} \frac{U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m + \Delta\boldsymbol{a}_l))) - U(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m - \Delta\boldsymbol{a}_l)))}{2\Delta a_l} \Delta\hat{a}_{l,max} \quad (47)$$

Similarly, the lower and upper bounds of the variance $V(\boldsymbol{A}^R(\boldsymbol{a}^I))$ can be expressed as

$$V_{lower} = V(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m))) + \sum_{l=1}^{l_n} \frac{V(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m + \Delta\boldsymbol{a}_l))) - V(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m - \Delta\boldsymbol{a}_l)))}{2\Delta a_l} \Delta\hat{a}_{l,min} \quad (48)$$

$$V_{upper} = V(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m))) + \sum_{l=1}^{l_n} \frac{V(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m + \Delta\boldsymbol{a}_l))) - V(\mu(\boldsymbol{A}^R(\boldsymbol{a}^m - \Delta\boldsymbol{a}_l)))}{2\Delta a_l} \Delta\hat{a}_{l,max} \quad (49)$$

On the basis of the lower and upper bounds of the variance $V(\boldsymbol{A}^R(\boldsymbol{a}^I))$, the interval of standard variance SD^I can be obtained as

$$SD^I = \left[(V_{lower})^{1/2}, (V_{upper})^{1/2} \right] \quad (50)$$

3.3. Derivation of integrating subinterval technique for the exterior acoustic field with interval random parameters

When comes to the situation that the interval range of the interval random uncertainty is not relative small, due to the complicated perturbation formulation, the high-order series Taylor expansions are hard to implement here. With regard to this issue, the subinterval technique [42-44] is herein introduced to guarantee the accuracy of the presented method. Assuming that the number of the subintervals for the interval parameter a_i^I is N_i, the subinterval can be defined as

$$a_{r_i,i}^I = \left[a_i^L + 2(r_i - 1)\Delta a_i / N_i, a_i^L + 2r_i \Delta a_i / N_i \right],\ r_i = 1, 2, \ldots, N_i \quad (51)$$

where $a^I_{r_i,i}$ (i=1, 2, …, L) is the r_ith subinterval of the ith interval parameter a^I_i; L is the total number of the intervals.

According to the permutation and combination theory, there are $N_1N_2\ldots N_L$ combinations of subinterval random variables produced. Each subinterval random variable combination can be expressed as

$$\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L}) = \boldsymbol{A}^R\left(a^I_{r_1,1},\ldots,a^I_{r_i,i},\ldots,a^I_{r_L,L}\right),\ r_i=1,2,\ldots,N_i,\ i=1,2,\ldots,L \tag{52}$$

For these subinterval random variable combinations, the subinterval random variable dynamic equilibrium equation of the exterior field can be rewritten as

$$P^c(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L})) = \boldsymbol{C}(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L}))\boldsymbol{Z}^{-1}(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L}))\boldsymbol{F}(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L})) \tag{53}$$

By applying the presented hybrid uncertain analysis method in these subinterval random variable combinations, the subinterval expectation vector $U(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L}))$ and subinterval variance vector $V(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L}))$ of the frequency response can be obtained.

By employing the following interval union operation, the interval expectation $U(\boldsymbol{A}^R(\boldsymbol{a}^I))$ and the interval variance $V(\boldsymbol{A}^R(\boldsymbol{a}^I))$ of the frequency response can be obtained.

$$\begin{aligned} U(\boldsymbol{A}^R(\boldsymbol{a}^I)) &= \bigcup_{r_i=1,\ldots N_i} U(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L})) \\ &= \left[\min_{r_i=1,\ldots N_i}\left(U_{lower}(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L}))\right),\ \max_{r_i=1,\ldots N_i}\left(U_{upper}(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L}))\right)\right] \end{aligned} \tag{54}$$

$$\begin{aligned} V(\boldsymbol{A}^R(\boldsymbol{a}^I)) &= \bigcup_{r_i=1,\ldots N_i} V(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L})) \\ &= \left[\min_{r_i=1,\ldots N_i}\left(V_{lower}(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L}))\right),\ \max_{r_i=1,\ldots N_i}\left(V_{upper}(\boldsymbol{A}^R(\boldsymbol{a}^I_{r_1\ldots r_i\ldots r_L}))\right)\right] \end{aligned} \tag{55}$$

where the subscript *lower* and *upper* stand for the lower and upper bounds of the interval, respectively.

The presented method is deduced in the standard coupled FEM/BEM framework. Thus, it can be effectively used to practical engineering problems whose analytical solutions could not be acquired. Obviously, the computational cost of the presented method is relevant to the number of interval variables, and it will increase exponentially when combining the subinterval technique to improve the accuracy. However, as the number of interval variables involved in the interval random vector of the exterior acoustic system is modest, the consumed time for the prediction of the exterior acoustic problem with interval random variables is acceptable. Furthermore, the computational burden of the presented method is very small compared with the Monte Carlo method. As a result, the proposed method can be considered as an efficient way to predict the frequency response of the exterior acoustic problem with interval random variables.

4. Numerical examples

A square flexible shell model of dimensions 0.4×0.4 m is depicted in Fig. 2. The shell is excited by a unit of normal harmonic point force at the middle point. The four vertices are set to be fixed. The number of elements of the shell structure is 36 and the acoustic interface is discretized by the same elements. The damping coefficients are assumed as α=0.5 and β=0.1. The density of air is ρ_f =1.184 kg/m^3, and the sound speed of air is c =343.4 m/s. The Poisson's ratio of the shell is ν =0.3.

Due to the manufacturing/construction tolerances, the Young's modulus, the density and the thickness of the shell are assumed as interval random variables and follow the normal distribution. The symbol α is used to denote the uncertain level of interval variables. The expectations of the density of the shell is $\mu(\rho_s)$ =2700 kg/m^3. The interval of the expectations of the Young's modulus and the thickness of the shell are $\mu(E)^I$ =71[(1-α), (1+α)] GPa and $\mu(t)^I$ =5[(1-α), (1+α)] mm, respectively. The interval of the standard variance of the Young's modulus, the density and the thickness of the shell are $\sigma(E)^I$ =1.42[(1-α), (1+α)] GPa, $\sigma(\rho_s)^I$ =54[(1-α), (1+α)] kg/m^3 and $\sigma(t)^I$ =0.1[(1-α), (1+α)] mm, respectively. Simulations of this square flexible shell are carried out by MATLAB R2009a on a 2.93GHz Core(TM)2 CPU E7500.

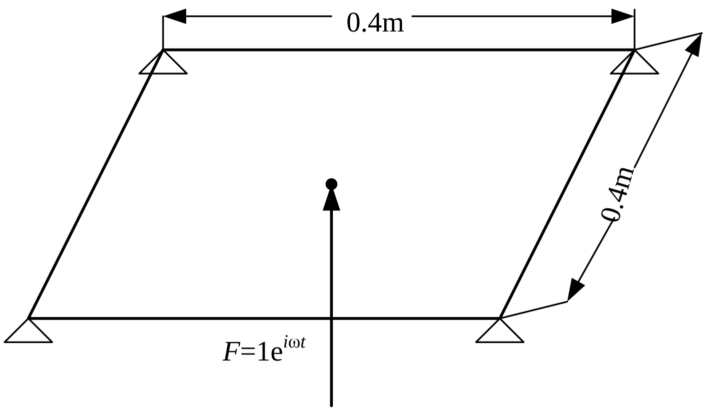
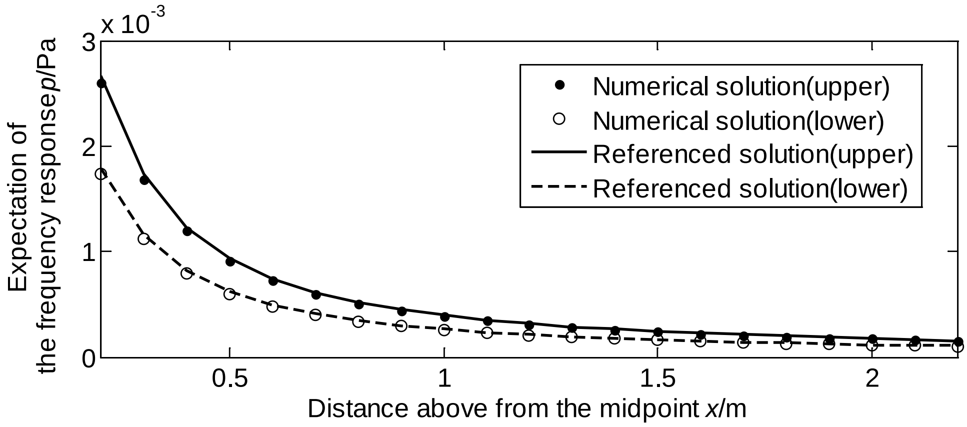
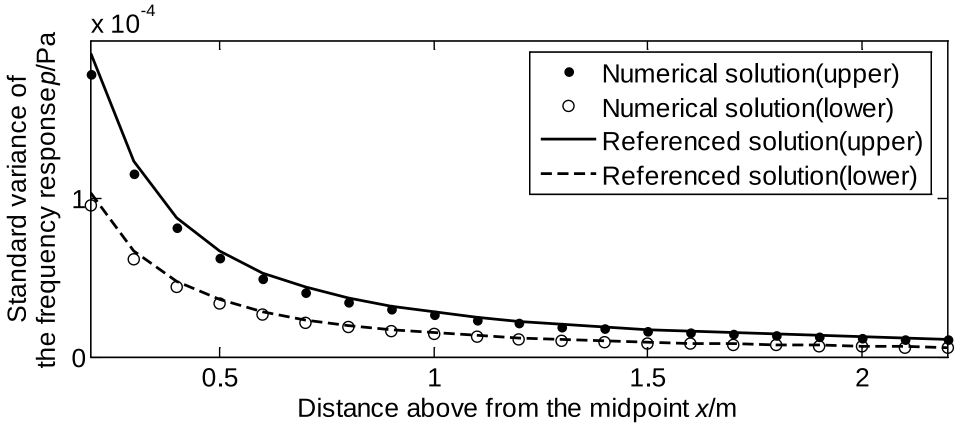


Fig. 2 A square flexible shell model

Assume that the uncertain level of interval variables are α =5%. The lower and upper bounds of expectation and standard variance of the frequency response amplitude at the points vertically above the midpoint of the squared shell obtained by the present method and Monte Carlo method are plotted in Fig. 3 for frequency 50 Hz, Fig. 4 for frequency 100 Hz and Fig. 5 for frequency 150 Hz. The vertical distances from these points to the midpoint of the squared shell are from 0.2 m to 2.2 m with a step of 0.1 m. The results obtained by the Monte Carlo method are used as the reference solutions. In the implement of Monte Carlo method, the total number of the samples is 10^7, in which the sample size of interval variables subjected to corresponding bound combination is 10^3 and the sample size of random parameters associated with normal distribution is 10^4. When the sample size is 10^7, the intervals of the expectation and the standard variance of the frequency response amplitude at the point vertically above the midpoint of the squared shell with the distance of 1 m at the frequency 100 Hz are [0.0005927, 0.0008884] and [3.4266×10^{-5}, 6.36254×10^{-5}], respectively. However, when the sample size is 10^9, in which the sample size of interval variables subjected to corresponding bound combination is 10^4 and the sample size of random parameters associated with normal distribution is 10^5, the intervals of the expectation and the standard variance are [0.0005930, 0.0008879] and [3.36543×10^{-5}, 6.29727E$\times10^{-5}$], respectively. It can be found out that the results of the Monte Carlo method have been only changed little with the sample size increasing exponentially. Thus, considering the computational cost, the sample size of Monte Carlo method is set to 10^7. From Figs. 3 to 5, we can see that the lower and upper bounds of expectations and standard variances of the frequency response amplitudes at the points vertically above the midpoint of the shell obtained by the present method match the bounds yielded by Monte Carlo method well. It indicates that the accuracy of the proposed method is good, when it is used to calculate the intervals of expectation and standard variance of the frequency response amplitude of the exterior acoustic system with interval random variables.

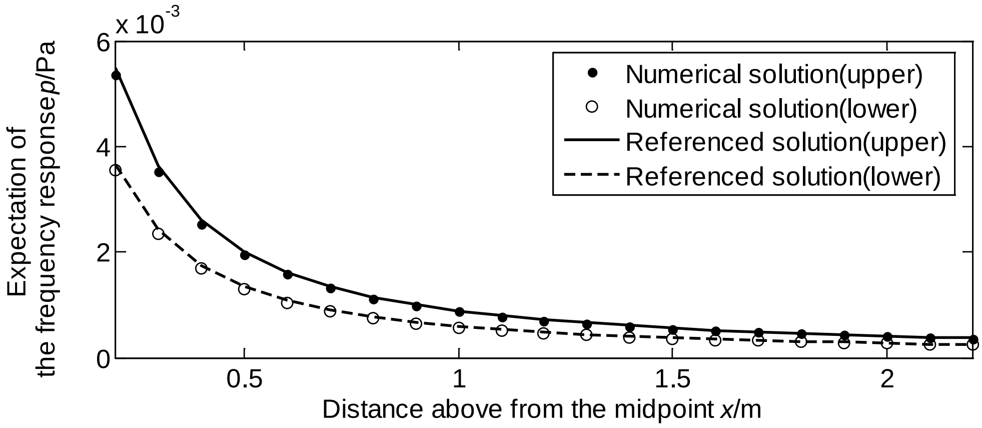


(a)

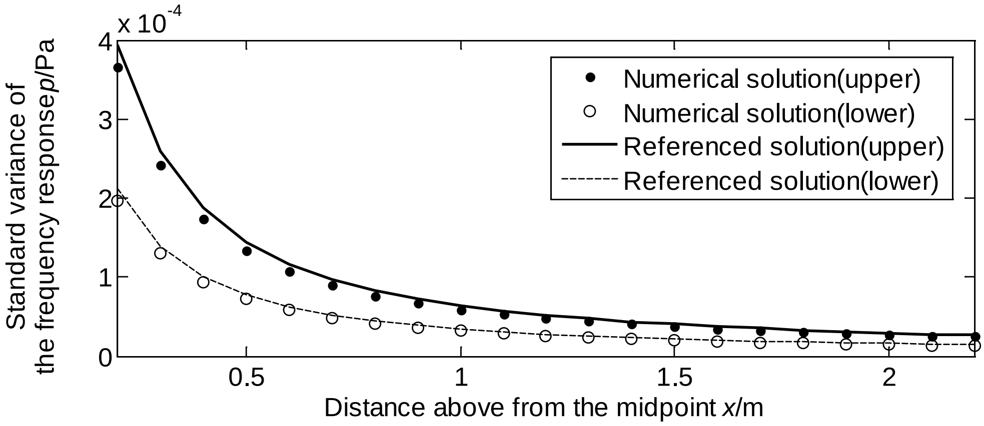


(b)

Fig. 3 The lower and upper bounds of expectation and standard variance of the frequency response amplitude vertically above from the midpoint of the shell (f = 50 Hz): (a) expectation and (b) standard variance.

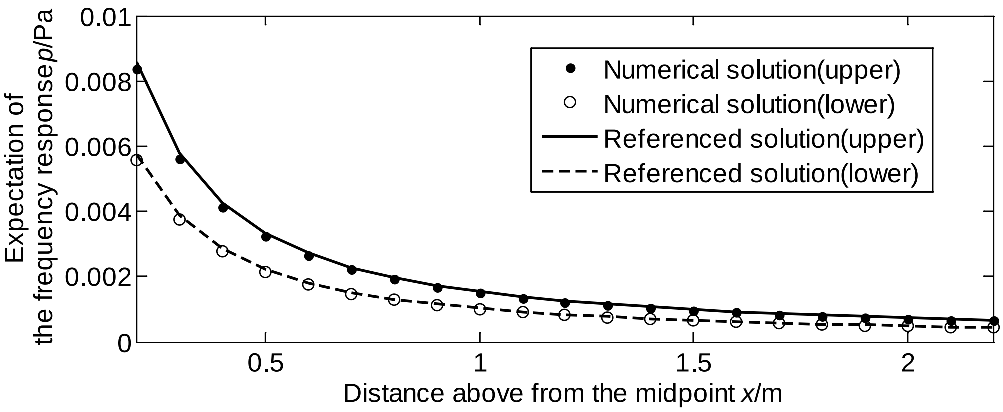


(a)

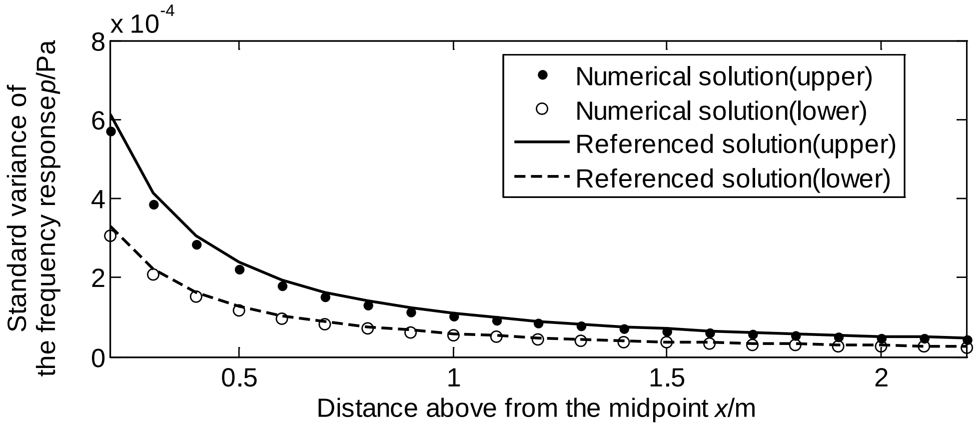


(b)

Fig. 4 The lower and upper bounds of expectation and standard variance of the frequency response amplitude vertically above from the midpoint of the shell (f = 100 Hz): (a) expectation and (b) standard variance.



(a)



(b)

Fig. 5 The lower and upper bounds of expectation and standard variance of the frequency response amplitude vertically above from the midpoint of the shell (f = 150 Hz): (a) expectation and (b) standard variance.

The relative errors of the lower and upper bounds of the expectation and standard variance of the frequency response amplitude at the points above the midpoint of the squared shell with the distances from 0.2 m to 2.2 m are shown in Table 1 for frequency 150 Hz. The symbols "LB" and "UB" denote the lower bound and upper bound, respectively. From Table 1, we can see that the relative errors of the proposed method are acceptable compared with the Monte Carlo method whose computational cost is excessive. The main cause of the relative errors is that the effects of neglecting the higher order terms of Taylor series and the higher order terms of Neumann series are unpredictable and uncontrollable. The other reason for the relative errors is the impacts derived from the central difference method for the non-linear function, in which the differential of interval variables is replaced by the maximum deviation width of interval variable approximately. Furthermore, we can find from Table 1 that the relative errors of the bounds of standard variances are larger than that of the bounds of expectations. The reason is that the number of interval variables associated with the standard variance is more than the number of interval variables associated with the expectation. In this case, the interval variables associated with the expectation are $\mu(E)^I$ and $\mu(t)^I$; whereas, the interval variables associated with the standard variance are $\mu(E)^I$, $\mu(t)^I$, $\sigma(E)^I$, $\sigma(\rho_s)^I$ and $\sigma(t)^I$. Spontaneously, the relative errors of standard variance with 5 interval variables are larger than the relative errors of expectation with 2 interval variables.

Table 1 Bounds of the expectation and standard variance of the frequency response amplitude vertically above from the midpoint of the shell (f = 150 Hz)

Distance above from the midpoint(m)	Bounds	Expectation (Pa)		Relative errors	Standard Variance (Pa)		Relative errors
		Referenced solution	Numerical solution		Referenced solution	Numerical solution	

0.2	LB	0.005726	0.005556	2.97%	0.0003309	0.0003065	7.37%
	UB	0.00858	0.008352	2.66%	0.0006142	0.0005719	6.89%
0.3	LB	0.003863	0.003748	2.98%	0.0002232	0.0002068	7.35%
	UB	0.005788	0.005634	2.66%	0.0004143	0.0003858	6.88%
0.4	LB	0.00284	0.002755	2.99%	0.0001641	0.000152	7.37%
	UB	0.004255	0.004141	2.68%	0.0003046	0.0002836	6.89%
0.5	LB	0.002218	0.002152	2.98%	0.0001282	0.0001187	7.41%
	UB	0.003324	0.003236	2.65%	0.000238	0.0002216	6.89%
0.6	LB	0.00181	0.001756	2.98%	0.0001046	0.00009687	7.39%
	UB	0.002712	0.00264	2.65%	0.0001941	0.0001808	6.85%
0.7	LB	0.001524	0.001478	3.02%	0.00008806	0.00008156	7.38%
	UB	0.002283	0.002222	2.67%	0.0001634	0.0001522	6.85%
0.8	LB	0.001314	0.001274	3.04%	0.00007591	0.0000703	7.39%
	UB	0.001968	0.001916	2.64%	0.0001409	0.0001312	6.88%
0.9	LB	0.001153	0.001119	2.95%	0.00006663	0.00006171	7.38%
	UB	0.001728	0.001682	2.66%	0.0001237	0.0001152	6.87%
1	LB	0.001027	0.0009963	2.99%	0.00005934	0.00005496	7.38%
	UB	0.001539	0.001498	2.66%	0.0001101	0.0001026	6.81%
1.1	LB	0.0009251	0.0008976	2.97%	0.00005346	0.00004951	7.39%
	UB	0.001386	0.001394	0.58%	0.00009923	0.00009239	6.89%
1.2	LB	0.0008415	0.0008165	2.97%	0.00004863	0.00004504	7.38%
	UB	0.001261	0.001227	2.70%	0.00009026	0.00008404	6.89%
1.3	LB	0.0007716	0.0007486	2.98%	0.00004459	0.0000413	7.38%
	UB	0.001156	0.001125	2.68%	0.00008276	0.00007706	6.89%
1.4	LB	0.0007123	0.0006911	2.98%	0.00004116	0.00003812	7.39%
	UB	0.001067	0.001039	2.62%	0.0000764	0.00007114	6.88%
1.5	LB	0.0006614	0.0006417	2.98%	0.00003822	0.0000354	7.38%
	UB	0.0009911	0.0009646	2.67%	0.00007094	0.00006605	6.89%
1.6	LB	0.0006172	0.0005989	2.97%	0.00003567	0.00003304	7.37%
	UB	0.0009249	0.0009002	2.67%	0.0000662	0.00006164	6.89%
1.7	LB	0.0005786	0.0005613	2.99%	0.00003343	0.00003097	7.36%
	UB	0.0008669	0.0008438	2.66%	0.00006205	0.00005778	6.88%
1.8	LB	0.0005444	0.0005282	2.98%	0.00003146	0.00002914	7.37%
	UB	0.0008158	0.000794	2.67%	0.00005839	0.00005437	6.88%
1.9	LB	0.000514	0.0004988	2.96%	0.00002971	0.00002751	7.40%
	UB	0.0007703	0.0007497	2.67%	0.00005514	0.00005134	6.89%
2.0	LB	0.0004869	0.0004724	2.98%	0.00002814	0.00002606	7.39%
	UB	0.0007296	0.0007101	2.67%	0.00005222	0.00004863	6.87%
2.1	LB	0.0004624	0.0004487	2.96%	0.00002672	0.00002475	7.37%
	UB	0.0006929	0.0006745	2.66%	0.0000496	0.00004618	6.90%
2.2	LB	0.0004403	0.0004272	2.98%	0.00002545	0.00002357	7.39%
	UB	0.0006598	0.0006422	2.67%	0.00004723	0.00004397	6.90%

Computational efficiency is an important factor to evaluate the performances of numerical methods. The computational time of the Monte Carlo method with 10^7 samples to calculate the bounds of the expectation and the standard variance of the frequency response amplitude at the points vertically above the midpoint of the squared shell is 13,527,394 s when the frequency is 150 Hz. Whereas, the computational time of the presented method is 179 s. Namely, the computational cost of the present method is much less than that of Monte Carlo method when they are used to predict the bounds of the expectation and the standard variance of the frequency response amplitude of the exterior acoustic system with interval random variables.

In order to investigate the effect of the uncertain levels of interval variables on the computational accuracy of the proposed method, the relative errors of the bounds of expectation and standard variance at the points vertically above the midpoint of the squared shell with the distance of 1m and 2m are listed in Tables 2 and 3. The considered uncertain levels of interval variables are 1%, 3%, 5% and 7%, respectively. The considered frequency is 100 Hz. From tables 2 and 3, we can see that the relative errors of the bounds of expectation and standard variance show an increasing trend with the increase in the uncertain levels. The relative errors of the proposed method are acceptable when the uncertain levels reach 5%. However, when the uncertain level exceed 5%, the bounds of standard variance obtained by the proposed method deviate far from the bounds obtained by Monte Carlo method, and the corresponding errors are unacceptable. Nonetheless, the present method can be used to predict the intervals of expectation and standard variance of the response of the exterior acoustic field with interval random variables when the uncertain levels of the interval variables are small.

Table 2 Bounds of the expectation and standard variance of the frequency response amplitude vertically above from the midpoint of the shell with the distance of 1m (f = 100 Hz)

Uncertain level	Bounds	Expectation (Pa)		Relative errors	Standard Variance (Pa)		Relative errors
		Referenced solution	Numerical solution		Referenced solution	Numerical solution	
1%	LB	0.0006923	0.0006912	0.16%	0.00004269	0.00004275	0.14%
	UB	0.000753	0.0007487	0.57%	0.00004974	0.00004821	3.08%
3%	LB	0.0006406	0.0006334	1.12%	0.00003776	0.00003727	1.30%
	UB	0.0008169	0.0008064	1.29%	0.00005661	0.00005369	5.16%
5%	LB	0.0005928	0.0005751	2.99%	0.00003427	0.00003173	7.41%
	UB	0.0008885	0.0008647	2.68%	0.00006363	0.00005924	6.90%
7%	LB	0.0005497	0.0005158	6.17%	0.00003051	0.00002607	14.55%
	UB	0.0009677	0.000924	4.52%	0.00007216	0.00006489	10.07%

Table 3 Bounds of the expectation and standard variance of the frequency response amplitude vertically above from the midpoint of the shell with the distance of 2m (f = 100 Hz)

Uncertain level	Bounds	Expectation (Pa)		Relative errors	Standard Variance (Pa)		Relative errors
		Referenced solution	Numerical solution		Referenced solution	Numerical solution	
1%	LB	0.0003184	0.0003179	0.16%	0.00001963	0.00001966	0.15%
	UB	0.0003463	0.0003443	0.58%	0.00002287	0.00002217	3.06%
3%	LB	0.0002946	0.0002913	1.12%	0.00001737	0.00001714	1.32%
	UB	0.0003757	0.0003709	1.28%	0.00002603	0.00002469	5.15%
5%	LB	0.0002726	0.0002645	2.97%	0.00001576	0.00001459	7.42%
	UB	0.0004086	0.0003977	2.67%	0.00002926	0.00002724	6.90%
7%	LB	0.0002528	0.0002372	6.17%	0.00001403	0.00001199	14.54%
	UB	0.000445	0.0004249	4.52%	0.00003319	0.00002984	10.09%

To guarantee the accuracy of the present method for the hybrid uncertain exterior acoustic problem, the sub-interval analysis technique is herein adopted. Through dividing the interval variable into several sub-intervals, the uncertain level of each sub-interval can be decreased, and the accuracy of the present method can be ensured. To demonstrate the effectiveness of combining the subinterval technique with the proposed method, cases with two subintervals and four subintervals are calculated.

The considered uncertain level of the interval variables is set to 10%. The considered frequency is 100 Hz. The relative errors of the bounds of expectation and standard variance at the points vertically above the midpoint of the squared shell with the distance of 1m and 2m are listed in Tables 4 and 5. The symbol N stands for the number of the subintervals and all of the interval random variables are divided into N subinterval random variables. It can be seen from tables 4 and 5 that the relative errors of the bounds of expectation and standard variance decrease with increasing subinterval number, which means that the accuracy of presented method can be improved by integrating it with the subinterval technique. Besides, the results obtained by integrating four subintervals match the reference solutions very well. Although the computational cost increase exponentially when combining the subinterval technique, but it is still relative small compared with the Monte Carlo method.

Table 4 Bounds of the expectation and standard variance of the frequency response amplitude vertically above from the midpoint of the shell with the distance of 1m with different subintervals

Number of subintervals	Bounds	Expectation (Pa)		Relative errors	Standard Variance (Pa)		Relative errors
		Referenced solution	Numerical solution		Referenced solution	Numerical solution	
N=0	LB	0.0004922	0.0004243	13.80%	0.00002569	0.00001727	32.78%
	UB	0.001104	0.001016	7.97%	0.00008749	0.00007370	15.76%
N=2	LB	0.0004922	0.0004790	2.68%	0.00002569	0.00002395	6.77%
	UB	0.001104	0.001071	2.99%	0.00008749	0.00008106	7.35%
N=4	LB	0.0004922	0.0004891	0.63%	0.00002569	0.00002510	2.30%
	UB	0.001104	0.001089	1.36%	0.00008749	0.00008363	4.41%

Table 5 Bounds of the expectation and standard variance of the frequency response amplitude vertically above from the midpoint of the shell with the distance of 2m with different subintervals

Number of subintervals	Bounds	Expectation (Pa)		Relative errors	Standard Variance (Pa)		Relative errors
		Referenced	Numerical		Referenced	Numerical	

		solution	solution		solution	solution	
N=0	LB	0.0002264	0.0001951	13.83%	0.00001182	0.00000794	32.83%
	UB	0.0005076	0.0004671	7.98%	0.00004024	0.00003389	15.78%
N=2	LB	0.0002264	0.0002203	2.69%	0.00001182	0.00001101	6.85%
	UB	0.0005076	0.0004925	2.97%	0.00004024	0.00003728	7.36%
N=4	LB	0.0002264	0.0002249	0.66%	0.00001182	0.00001154	2.37%
	UB	0.0005076	0.0005010	1.30%	0.00004024	0.00003846	4.42%

5. Conclusions

In this paper, a hybrid uncertain analysis technique is proposed for the hybrid uncertain exterior acoustic field prediction, in which the uncertain parameters are modeled as the random variables whose distribution parameters are expressed as interval variables instead of precise value due to lacking sufficient information. Since the limited information of the distribution parameters of random variables can be well reflected by the interval variables, the proposed method can be considered as a valuable alternative method of stochastic method for epistemic uncertain problems. In the proposed method, the expectation and variance of response vector can be obtained by using the matrix perturbation theory and the random moment method with considering the related interval variables of the interval random variable are constant. Afterward, on the basis of the linear monotonicity of the expectation and variance of the response vector with respect to the interval variables, the lower and upper bounds of expectation and variance of the response vector can be computed through the vertex method. The numerical results on a square flexible shell verify the effectiveness of the proposed method for the hybrid uncertain exterior acoustic field prediction with interval random parameters. Furthermore, the subinterval technique is herein introduced and investigated to guarantee the accuracy of the presented method.

As a result, good effects are achieved by integrating the subinterval technique with the presented method. Thus, the present approaches can be considered as an effective engineering method to quantify the effects of interval random uncertainty on the response of the exterior acoustic field.

Acknowledgments

The paper is supported by the National Natural Science Foundation of China (No.11572121), the Independent Research Project of the State Key Laboratory of Advanced Design and Manufacturing for Vehicle Body in Hunan University (Grant No.71375004) and Hunan Provincial Innovation Foundation for Postgraduate (Grant No. CX2014B147).

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